

A new explicit formula for Bernoulli numbers involving the Euler number

Sumit Kumar Jha

We derive a new explicit formula for Bernoulli numbers in terms of the Stirling numbers of the second kind and the Euler numbers. As a corollary of our result, we obtain an explicit formula for the even Euler numbers in terms of the Stirling numbers of the second kind.

Definition 1. The *Bernoulli numbers* B_n can be defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n t^n}{n!},$$

where $|t| < 2\pi$.

Definition 2. A *Stirling number of the second kind*, denoted by $S(n, m)$, is the number of ways of partitioning a set of n elements into m nonempty sets.

There are many known explicit formulas known for the Bernoulli numbers [Gould 1972; Jha 2019]. The following formulas express the Bernoulli numbers explicitly in terms of the Stirling numbers of the second kind:

$$B_r = \sum_{k=1}^r (-1)^k \cdot k! \frac{S(r, k)}{k+1},$$

$$(-1)^{r-1} B_r = \sum_{k=1}^r (-1)^k \frac{S(r, k)}{k+1} \cdot (k-1)!,$$

$$B_{r+1} = \frac{(-1)^r \cdot (r+1) \cdot 2^r}{2^{r+1} - 1} \sum_{k=1}^r \frac{S(r, k)}{k+1} (-1)^k 2^{-2k} \frac{(2k-1)!}{(k-1)!}.$$

Definition 3. The *Euler numbers* are a sequence of integers, denoted by E_n , which can be defined by the Taylor series expansion

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} \cdot t^n,$$

where $\cosh t$ is the hyperbolic cosine.

We prove the following.

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Theorem 4. *We have*

$$B_{r+1} = -\frac{r+1}{4(1+2^{-(r+1)}(1-2^{-r}))} \left(\sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)} + 4^{-r} E_r \right), \quad (1)$$

where $S(r, k)$ denotes the Stirling numbers of the second kind, $x^{(n)} = (x)(x+1)\cdots(x+n-1)$ denotes the rising factorial, and E_r denotes the Euler number.

Proof. We begin with the result

$$\frac{\sin n\pi}{\pi} \int_0^\infty x^{n-1} \frac{\text{Li}_s(-x)}{1+x} dx = \zeta(s) - \zeta(s, 1-n),$$

where $\text{Li}_s(-x)$ denotes the polylogarithm function, $\zeta(s)$ is the Riemann zeta function, and $\zeta(s, 1-n)$ is the Hurwitz zeta function. The integral above is valid for all $s \in \mathbb{C} \setminus \{1\}$ and $0 < n < 1$. This integral can be obtained from formula 3.2.1.6 in [Brychkov et al. 2019].

Plugging $n = \frac{3}{4}$ and $s = -r$, a negative integer, into the integral above we get

$$\int_0^\infty x^{-1/4} \frac{\text{Li}_{-r}(-x)}{1+x} dx = \sqrt{2}\pi \left(\frac{B_{r+1}(\frac{1}{4}) - B_{r+1}}{r+1} \right).$$

Now, we use the representation from [Landsburg 2009]

$$\text{Li}_{-r}(-x) = \sum_{k=1}^r k! S(r, k) \left(\frac{1}{1+x} \right)^{k+1} (-x)^k,$$

which can be easily proved using induction on r .

As a result, we have

$$\begin{aligned} \int_0^\infty x^{-1/4} \frac{\text{Li}_{-r}(-x)}{1+x} dx &= \sum_{k=1}^r (-1)^k \cdot k! S(r, k) \int_0^\infty \frac{x^{k-1/4}}{(1+x)^{k+2}} dx \\ &= \sum_{k=1}^r (-1)^k \cdot k! S(r, k) \cdot \frac{\Gamma(k + \frac{3}{4}) \Gamma(\frac{5}{4})}{\Gamma(k+2)} \\ &= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \Gamma(k + \frac{3}{4}) \Gamma(\frac{5}{4}) \\ &= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}+1\right) \cdots \left(\frac{3}{4}+k-1\right) \Gamma(\frac{3}{4}) \Gamma(\frac{5}{4}) \\ &= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}+1\right) \cdots \left(\frac{3}{4}+k-1\right) \frac{1}{2\sqrt{2}}\pi \\ &= \sum_{k=1}^r (-1)^k \cdot \frac{S(r, k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)} \frac{1}{2\sqrt{2}}\pi, \end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function.

But, from [Weisstein], we have

$$\frac{B_{r+1}\left(\frac{1}{4}\right) - B_{r+1}}{r+1} = \frac{(-2^{-(r+1)}(1-2^{-r})B_{r+1} - 4^{-(r+1)}(r+1)E_r - B_{r+1})}{r+1}.$$

Thus, we have

$$B_{r+1} = -\frac{r+1}{4(1+2^{-(r+1)}(1-2^{-r}))} \left(\sum_{k=1}^r (-1)^k \cdot \frac{S(r,k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)} + 4^{-r} E_r \right).$$

□

If we let $r = 2l$, an even integer, in (1) we immediately obtain:

Corollary 5.
$$E_{2l} = -4^{2l} \sum_{k=1}^{2l} (-1)^k \cdot \frac{S(2l,k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)}.$$

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SUMIT KUMAR JHA:

kumarjha.sumit@research.iiit.ac.in

International Institute of Information Technology, Hyderabad, India

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