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A new explicit formula for Bernoulli numbers involving the Euler number

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We derive a new explicit formula for Bernoulli numbers in terms of the Stirling numbers of the second kind and the Euler numbers. As a corollary of our result, we obtain an explicit formula for the even Euler numbers in terms of the Stirling numbers of the second kind.

Definition 1. The *Bernoulli numbers* B_n can be defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n t^n}{n!},$$

where $|t| < 2\pi$.

Definition 2. A *Stirling number of the second kind*, denoted by $S(n, m)$, is the number of ways of partitioning a set of n elements into m nonempty sets.

There are many known explicit formulas known for the Bernoulli numbers [Gould 1972; Jha 2019]. The following formulas express the Bernoulli numbers explicitly in terms of the Stirling numbers of the second kind:

$$B_r = \sum_{k=1}^{r} (-1)^k \cdot k! \, \frac{S(r, k)}{k+1},$$

$$(-1)^{r-1} B_r = \sum_{k=1}^{r} (-1)^k \frac{S(r, k)}{k+1} \cdot (k-1)!\,,$$

$$B_{r+1} = \frac{(-1)^r \cdot (r+1) \cdot 2^r}{2^{r+1} - 1} \sum_{k=1}^{r} \frac{S(r, k)}{k+1} (-1)^k 2^{-2k} \frac{(2k-1)!}{(k-1)!}.$$

Definition 3. The *Euler numbers* are a sequence of integers, denoted by E_n, which can be defined by the Taylor series expansion

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} \cdot t^n,$$

where $\cosh t$ is the hyperbolic cosine.

We prove the following.

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Theorem 4. *We have*

$$B_{r+1} = -\frac{r+1}{4(1+2^{-(r+1)}(1-2^{-r}))}\biggl(\sum_{k=1}^{r}(-1)^k \cdot \frac{S(r,k)}{k+1} \cdot \bigl(\tfrac{3}{4}\bigr)^{(k)} + 4^{-r}E_r\biggr), \tag{1}$$

where $S(r,k)$ denotes the Stirling numbers of the second kind, $x^{(n)} = (x)(x+1)\cdots(x+n-1)$ denotes the rising factorial, and E_r denotes the Euler number.

Proof. We begin with the result

$$\frac{\sin n\pi}{\pi}\int_0^\infty x^{n-1}\frac{\mathrm{Li}_s(-x)}{1+x}\,dx = \zeta(s) - \zeta(s, 1-n),$$

where $\mathrm{Li}_s(-x)$ denotes the polylogarithm function, $\zeta(s)$ is the Riemann zeta function, and $\zeta(s,1-n)$ is the Hurwitz zeta function. The integral above is valid for all $s \in \mathbb{C}\setminus\{1\}$ and $0 < n < 1$. This integral can be obtained from formula 3.2.1.6 in [Brychkov et al. 2019].

Plugging $n = \frac{3}{4}$ and $s = -r$, a negative integer, into the integral above we get

$$\int_0^\infty x^{-1/4}\frac{\mathrm{Li}_{-r}(-x)}{1+x}\,dx = \sqrt{2}\pi\biggl(\frac{B_{r+1}\bigl(\frac{1}{4}\bigr) - B_{r+1}}{r+1}\biggr).$$

Now, we use the representation from [Landsburg 2009]

$$\mathrm{Li}_{-r}(-x) = \sum_{k=1}^{r} k!\,S(r,k)\biggl(\frac{1}{1+x}\biggr)^{k+1}(-x)^k,$$

which can be easily proved using induction on r.

As a result, we have

$$\begin{aligned}
\int_0^\infty x^{-1/4}\frac{\mathrm{Li}_{-r}(-x)}{1+x}\,dx &= \sum_{k=1}^{r}(-1)^k \cdot k!\,S(r,k)\int_0^\infty \frac{x^{k-1/4}}{(1+x)^{k+2}}\,dx \\
&= \sum_{k=1}^{r}(-1)^k \cdot k!\,S(r,k)\cdot\frac{\Gamma\bigl(k+\frac{3}{4}\bigr)\,\Gamma\bigl(\frac{5}{4}\bigr)}{\Gamma(k+2)} \\
&= \sum_{k=1}^{r}(-1)^k \cdot \frac{S(r,k)}{k+1}\cdot\Gamma\bigl(k+\tfrac{3}{4}\bigr)\,\Gamma\bigl(\tfrac{5}{4}\bigr) \\
&= \sum_{k=1}^{r}(-1)^k \cdot \frac{S(r,k)}{k+1}\cdot\bigl(\tfrac{3}{4}\bigr)\cdot\bigl(\tfrac{3}{4}+1\bigr)\cdots\bigl(\tfrac{3}{4}+k-1\bigr)\,\Gamma\bigl(\tfrac{3}{4}\bigr)\,\Gamma\bigl(\tfrac{5}{4}\bigr) \\
&= \sum_{k=1}^{r}(-1)^k \cdot \frac{S(r,k)}{k+1}\cdot\bigl(\tfrac{3}{4}\bigr)\cdot\bigl(\tfrac{3}{4}+1\bigr)\cdots\bigl(\tfrac{3}{4}+k-1\bigr)\,\tfrac{1}{2\sqrt{2}}\pi \\
&= \sum_{k=1}^{r}(-1)^k \cdot \frac{S(r,k)}{k+1}\cdot\bigl(\tfrac{3}{4}\bigr)^{(k)}\,\tfrac{1}{2\sqrt{2}}\pi,
\end{aligned}$$

where $\Gamma(\,\cdot\,)$ is the Gamma function.

But, from [Weisstein], we have

$$\frac{B_{r+1}\left(\frac{1}{4}\right)-B_{r+1}}{r+1}=\frac{(-2^{-(r+1)}(1-2^{-r})B_{r+1}-4^{-(r+1)}(r+1)E_r-B_{r+1})}{r+1}.$$

Thus, we have

$$B_{r+1}=-\frac{r+1}{4(1+2^{-(r+1)}(1-2^{-r}))}\left(\sum_{k=1}^{r}(-1)^k\cdot\frac{S(r,k)}{k+1}\cdot\left(\tfrac{3}{4}\right)^{(k)}+4^{-r}E_r\right).$$

□

If we let $r=2l$, an even integer, in (1) we immediately obtain:

Corollary 5.
$$E_{2l}=-4^{2l}\sum_{k=1}^{2l}(-1)^k\cdot\frac{S(2l,k)}{k+1}\cdot\left(\tfrac{3}{4}\right)^{(k)}.$$

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