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A convexity criterion for unique ergodicity of interval exchange transformations

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A criterion for unique ergodicity for points of a curve in the space of interval exchange transformation is given.

There is a metaconjecture in metric number theory that states that any Diophantine property that holds for generic vectors in $\mathbb{R}^n$ should hold for generic vectors on nondegenerate submanifolds; see [Kleinbock 2001, Section 4]. Mahler asked for example whether almost all points on the curve $s \mapsto (s, s^2, \dots, s^d)$ are very well approximable. This has been answered affirmatively by Sprindzuk. It is believed that an analogue of this phenomenon holds also for the unique ergodicity property for interval exchange transformations. Minsky and Weiss [2014] provided a general condition for unique ergodicity to hold. In this note we provide an easy-to-check criterion for their condition to be satisfied.

Let σ denote a permutation of d elements. Let Ω denote the antisymmetric matrix

$$\Omega_{ij} = \begin{cases} 1 & i > j,\ \sigma(i) < \sigma(j), \\ -1 & i < j,\ \sigma(i) > \sigma(j), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\boldsymbol{a} = (a_1, \dots, a_d) \in \mathbb{R}^d_+$ be a row vector with positive entries $a_i > 0$ and the associated interval be $I_{\boldsymbol{a}} = \big[0, \sum a_i\big)$, which is divided into d subintervals $I_i = [x_{i-1}, x_i)$, where $x_i = \sum_{j \le i} a_j$ are called discontinuities. Also introduce $x'_i = \sum_{j \le i} a_{\sigma^{-1}(j)}$. An *interval exchange transformation* $T : I_{\boldsymbol{a}} \to I_{\boldsymbol{a}}$ defined by the data $(\sigma, \boldsymbol{a})$ is the map

$$T(x) = x + (\boldsymbol{a}\Omega)_j = x - x_j + x'_{\sigma(j)} \quad \text{for } x \in I_j.$$

In words, T permutes the intervals I_j of length a_j according to σ. The form Ω_{ij} captures the exchange of two intervals I_i, I_j relative to each other.

We shall always assume that the permutation σ is irreducible in the sense that if $\{1, \dots, k\} \subset \mathcal{A} = \{1, \dots, d\}$ is invariant under σ then $k = d$.

Masur [1982] and Veech [1982] proved independently that for almost all $\boldsymbol{a} \in \mathbb{R}^d_+$, the interval exchange transformation T associated to $(\sigma, \boldsymbol{a})$ is uniquely ergodic; that is, the only T-invariant probability measure on I is Lebesgue measure.

Motivated by a conjecture of Mahler in the theory of Diophantine approximation, Minsky and Weiss [2014] proved the following theorem.

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Theorem 1 (Minsky–Weiss). *Let $\sigma = (d, \dots, 1)$ and $\boldsymbol{a}(s) = (s, s^2, \dots, s^d)$. Then for Lebesgue almost all $s > 0$, the interval exchange transformation associated to $(\sigma, \boldsymbol{a}(s))$ is uniquely ergodic.*

We extend the theorem of Minsky and Weiss to arbitrary permutations by means of a simple convexity criterion.

We first recall the theorem from which Theorem 1 is deduced, which requires us to introduce more definitions. A connection of T is a triple (m, x_i, x_j) for which $T^m(x_i) = x_j$. As noted by Keane, if the coordinates a_k of $\boldsymbol{a}$ are rationally independent then T has no connections. We shall restrict to curves $\boldsymbol{a}(s)$ for which this is the case for almost all s. This implies that there are no T-invariant atomic probability measures.

Let $\boldsymbol{b} = (b_1, \dots, b_d) \in \mathbb{R}^d$ be a row vector, which we will take to be $\boldsymbol{b} = \dot{\boldsymbol{a}}$, the derivative of a curve $\boldsymbol{a}(s)$. Define $y_i = \sum_{j \le i} b_i$ and $y'_i = \sum_{j \le i} b_{\sigma^{-1}(j)}$. We put

$$L(x) = (\Omega \boldsymbol{b}^T)_i = y_i - y'_{\sigma(i)} \quad \text{for } x \in I_i.$$

We call $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^d_+ \times \mathbb{R}^d$ a positive pair if $\mu(L) > 0$ for any T-invariant probability measure μ. The following is a simplified statement of Theorem 6.2 in [Minsky and Weiss 2014].

Theorem 2 (Minsky–Weiss). *If $\boldsymbol{a} : A \to \mathbb{R}^d_+$ is a C^2-curve defined on an interval $A \subset \mathbb{R}$ and σ is a permutation for which $(\boldsymbol{a}(s), \dot{\boldsymbol{a}}(s))$ is positive for Lebesgue almost all $s \in A$ then T associated to $(\sigma, \boldsymbol{a}(s))$ is uniquely ergodic for almost all $s \in A$.*

While the condition seems hard to check — involving all T-invariant μ's (from which we want to deduce that there is only one!) — we see however that a sufficient criterion for positivity is $L(x) > 0$ pointwise for every $x \in I$.

Let us now explain a suspension construction of Masur for an interval exchange transformation T associated to $(\sigma, \boldsymbol{a})$. Let $\boldsymbol{b} \in \mathbb{R}^d$ be a "height vector" (associated to the "length vector" $\boldsymbol{a}$) and define ζ_i to be $(a_i, b_i) \in \mathbb{R}^2$ and their slopes to be $\kappa_i = b_i / a_i$.

Let Γ_t be the curve obtained by connecting the points

$$C_0 = C^t_0 = (0, 0), \quad C^t_1 = \zeta_1, \quad C^t_2 = \zeta_1 + \zeta_2, \quad \dots, \quad C_d = C^t_d = \sum_{i=1}^{d} \zeta_i$$

and Γ_b be the curve obtained by connecting

$$C_0 = C^b_0, \quad C^b_1 = \zeta_{\sigma^{-1}(1)}, \quad C^b_2 = \zeta_{\sigma^{-1}(1)} + \zeta_{\sigma^{-1}(2)}, \quad \dots, \quad C_d = C^b_d.$$

If ζ_1 lies above $\zeta_{\sigma^{-1}(1)}$, i.e., if $\kappa_1 > \kappa_{\sigma^{-1}(d)}$, then we call Γ_t the top curve and Γ_b the bottom curve. They have common endpoints C_0 and C_d. We denote their union by Γ. If there are no further intersections, Γ bounds a polygon. In this case, we are identifying the line segment $[C^t_{k-1}, C^t_k]$ of Γ_t with segment $[C^b_{j-1}, C^b_j]$ in Γ_b, where $k = \sigma^{-1}(j)$. One obtains a closed topological surface S which outside of the corners of the polygon inherits a flat structure from $\mathbb{R}^2$. This means that there is an atlas of charts $\{(U, \psi)\}$ with U open and $\psi : U \to \mathbb{R}^2$ continuous such that for any two charts $\psi_i : U_i \to \mathbb{R}^2$ over a common point p, we find a translation $v \in \mathbb{R}^2$ such that $\psi_1 = \psi_2 + v$ for all points around p. It is possible to complement to an atlas defined on all of M by considering the complex multiplication on $\mathbb{R}^2 = \mathbb{C}$ with maps of the form $\psi = \phi^{\alpha+1}$ for a homeomorphism $\phi : U \to \mathbb{C}$ that contains 0 in its range. The points with $\alpha > 0$ are called the singularities of M.

M is endowed with a dynamical system, the vertical straight line flow that preserves the natural area form coming from the flat metric. We note that the interval I embeds into M as a horizontal line starting from the origin. The vertical straight line flow defines a suspension of the interval exchange transformation T by considering the induced transformation on I, namely the first return map of $I \to I$. We can now understand the meaning of $L(x)$: it is the return time of x to I, and as such positive.

Self-intersections of the curves Γ_t and Γ_b give rise to a "nonsensical picture"; see depictions on page 247 of [Minsky and Weiss 2014]. We observe here that one can make sense of the picture even if the curve Γ has self-intersection, by attaching a half-translation structure to it, but we have not tried to follow up on this direction. Instead, we shall restrict ourselves to a criterion that avoids self-intersections.

Lemma 3. *Let $\boldsymbol{a} \in \mathbb{R}^d$ be a length vector, $\boldsymbol{b} \in \mathbb{R}^d$ be a height vector and $\Gamma_t : I \to \mathbb{R}^2$ be the top curve constructed by concatenating the vectors $\zeta_i = (a_i, b_i) \in \mathbb{R}^2$, i.e., $\Gamma_t\big(\sum_{i \le j} a_i\big) = C^t_j$. Suppose the slopes $\kappa_i = b_i/a_i$ of ζ_i are strictly monotonically decreasing so that Γ_t is convex. Then for any irreducible permutation σ and bottom curve Γ_b constructed from vectors $\zeta_{\sigma^{-1}(1)}, \dots, \zeta_{\sigma^{-1}(d)}$, the closed curve $\Gamma = \Gamma_t \cup \Gamma_b$ has no self-intersections. In particular, $(\boldsymbol{a}, \boldsymbol{b})$ defines a positive pair if connection-free.*

Proof. We shall argue by induction on the number of symbols d. The base case is on two elements $d = 2$. By monotonicity $\kappa_1 > \kappa_2$ and by irreducibility $\sigma = (2, 1)$. Then $\Gamma_t \cup \Gamma_b$ bounds a parallelogram.

Assume now that for all $d' < d$ the lemma is true. Let $\Gamma_{b,j}$ the curve from concatenating $C_0, C^b_1, \dots, C^b_j$ from left to right, i.e., restricting $\Gamma_b : I \to \mathbb{R}^2$ to $\bigcup_{i=1}^{j} I_{\sigma^{-1}(i)}$. We now start another induction and assume that for all $j' < j$, $\Gamma_{b,j'}$ does not intersect Γ_t. For the base of the induction $j = 1$, there is nothing to check.

If $\Gamma_{b,j}$ intersects Γ_t then by the induction hypothesis it does so with its final line segment $[C^b_{j-1}, C^b_j]$. We put $k = \sigma^{-1}(j)$ such that $C^b_{j-1} + \zeta_k = C^b_j$, intersecting, say, the i-th segment $[C^t_{i-1}, C^t_i]$ of Γ_t. Then $\kappa_k > \kappa_i$. By monotonicity, ζ_k has to appear to the left of ζ_i in Γ_t; i.e., $k < i$.

We now describe a procedure of removing $[C^b_{j-1}, C^b_j]$ to obtain a smaller permutation to apply the induction hypothesis on d'.

Observe that if $K \subset \sigma^{-1}(\{1, \dots, j\})$, we can define the curves $\Gamma_{t,K}, \Gamma_{b,j,K}$ that one obtains from taking Γ_t and removing the line segments $[C^t_{k-1}, C^t_k]$ for $k \in K$ from Γ_t and taking $\Gamma_{b,j}$ and removing the line segments $[C^b_{\sigma^{-1}(j'-1)}, C^b_{\sigma^{-1}(j')}]$ from $\Gamma_{b,j}$ for $\sigma^{-1}(j') \in K$. Below, we shall have the additional property that $K \subset \{1, \dots, i-1\}$. We obtain a new permutation σ_K obtained by removing the symbols $k \in K$. If it is no longer irreducible then the maximal invariant subset $\{1, \dots, \ell\}$ must be contained in $\sigma^{-1}(\{1, \dots, j-1\})$ (or else σ is already reducible). By removing the subpermutation on $(1, \dots, \ell)$ from σ_K, we can allow ourselves to only consider the irreducible component σ' of σ_K containing i.

We now choose $K = \{k' = \sigma^{-1}(j') : j' \le j \text{ and } \kappa_{k'} \ge \kappa_k\}$. We note that $k' \in K$ implies $k' \le k = \sigma^{-1}(j)$ and that $i \notin K$. Hence the curve $\Gamma_{t,K}$ is only changed to the left of its line segment $[C^t_{i-1}, C^t_i]$, and most importantly, the curve $\Gamma_{b,j,K}$ still intersects $[C^t_{i-1}, C^t_i]$. To see this, divide the plane into two half-planes with boundary ∂ containing ζ_k attached to the right endpoint of $\Gamma_{b,j,K}$, and we see that $\Gamma_{b,j,K}$ stays to the upper-left half-plane. Since $[C^t_{i-1}, C^t_i]$ intersects ∂, it also intersects $\Gamma_{b,j,K}$ as claimed.

If σ_K is no longer irreducible then we proceed with the irreducible restriction σ' as described above, supported on, say, $\mathcal{B} \subset \mathcal{A}$. Consider the associated pair $(\boldsymbol{a}', \boldsymbol{b}')$, where $\boldsymbol{a}', \boldsymbol{b}' \in \mathbb{R}^{|\mathcal{B}|}$ by restricting to the support of σ'. These give still monotone slopes, and the induction hypothesis on $d' < d$ applies; i.e., there are no self-intersections. By construction, however, the curve defined by $\boldsymbol{a}', \boldsymbol{b}'$ and σ' has at least one self-intersection. □

Remark 4. We have an analogous criterion if κ_i are increasing in i, in which case $\Gamma_b\big(\sum_{i\leq j} a_j\big) = C_j^t$, and we apply the argument of Lemma 3 with roles of Γ_t and Γ_b exchanged.

Remark 5. Barak Weiss has informed us of a topological proof of Lemma 3, which we invite the reader to find herself.

Corollary 6. *Theorem 1 holds for any irreducible permutation.*

Proof. The slopes associated to $\boldsymbol{a}(s) = (s, s^2, \dots, s^d)$ are $\kappa_i = is^{i-1}/s^i = i/s$, monotone in i. □

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