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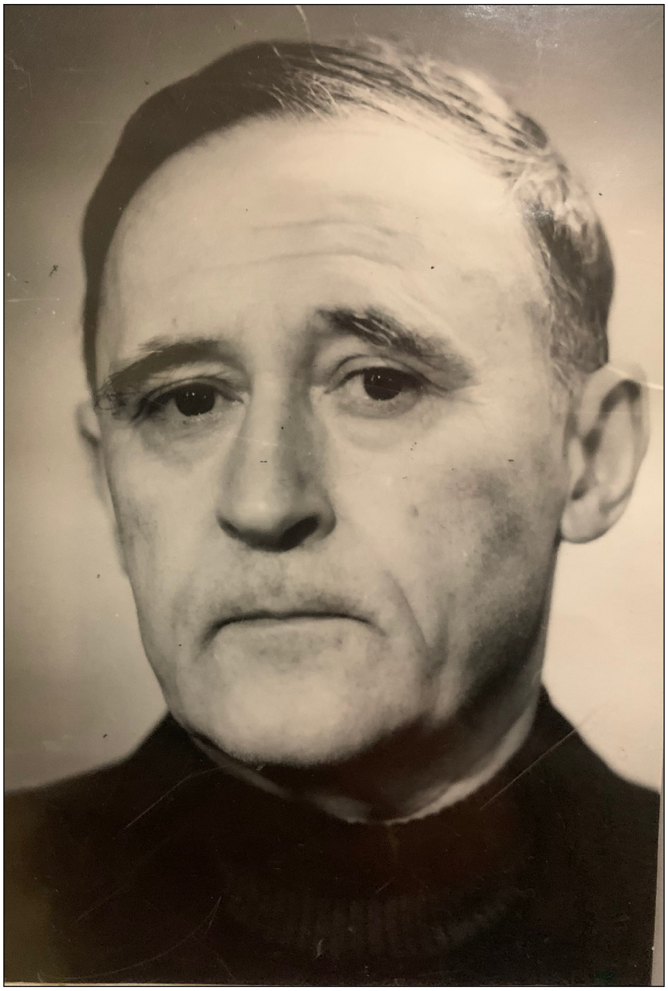
Naum Ilyitch Feldman

by Nikolay Moshchevitin



This volume gathers papers dedicated to the memory of Professor Naum Ilyitch Feldman (1918–1994). For many years he worked at Moscow Lomonosov State University, in the Department of Number Theory. I remember him as one of my favorite teachers at the University.

Naum Ilyitch Feldman was born in a Jewish family on November 26, 1918, in the city of Melitopol in the south of Russia. Those were difficult times for Russia. Toward the end of World War I a revolution befell the Russian Empire, followed by a civil war.

After graduating from secondary school in 1936, N. I. Feldman studied at Saint Petersburg University (called at that time Leningrad University) under the supervision of Professor Rodion Osievich Kuzmin.

From 1941 till the end of the World War II he served in the Soviet Army as an artillery officer. After the war he entered a PhD program at Moscow University. His scientific adviser was Alexandr Osipovich Gelfond. In 1949 he published his first papers, devoted to approximations to logarithms of algebraic numbers and got a doctoral degree. From 1961 till his death on April 20, 1994, he worked at Moscow Lomonosov State University. I remember him as an excellent lecturer, a widely educated mathematician and an extremely friendly person.

His research was mostly devoted to transcendence theory and its applications. I can say that he was a great admirer of numbers. He truly loved them. I will mention three of his best known results.

• Following earlier papers by Alexander Gelfond, and developing a method invented by Alan Baker, Naum Ilyitch Feldman was the first to obtain in 1967–68 effective polynomial lower bounds for linear forms in logarithms of algebraic numbers, that is, bounds of the form

$$|h_1 \log \alpha_1 + \ldots + h_d \log \alpha_d| \geq C_1 (\max_{1\leq j\leq d} |h_j|)^{-C_2}, \quad \forall\, (h_1, \ldots, h_d) \in \mathbb{Z}^d \setminus \{(0, \ldots, 0)\},$$

where $\alpha_1, \ldots, \alpha_d$ are algebraic numbers such that their logarithms are linearly independent over $\mathbb{Q}$. In fact, he obtained a more general result, namely that the restriction on the linear form coefficients can be relaxed: they can be assumed to be arbitrary algebraic numbers. A result of this type with more appropriate constants C_1, C_2 was later obtained by Eugenii Matveev, one of Feldman's students. Matveev's result has many applications, especially in Diophantine equations.

• N. I. Feldman was the first to obtain (in 1971) an effective improvement of Liouville's theorem, i.e., an inequality of the form

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C_1}{q^{n-C_2}}, \quad \forall\, \frac{p}{q} \in \mathbb{Q},$$

where α is an algebraic number of degree n and C_1, C_2 are effective positive constants.

• For many years Naum Ilyitch worked on bounds for the transcendence measure of π. His first result in this direction was published in 1951. His last result was announced in 1993 at a conference at Oberwolfach Mathematical Institute. It gives a lower bound

$$|P(\pi)| \geq e^{-C(d \log d + \log H) d (1+\log d)}$$

for an integer polynomial $P \in \mathbb{Z}[z]$ with $d = \deg P \geq d_0$, $C = 151$, and H denoting the height of P. In 1999, Yuri Aleksentsev reduced the constant 151 to 21.4708.

In June 2019 Moscow Institute of Physics and Technology in collaboration with Moscow Lomonosov State University organised a conference in the memory of Professor Feldman. The current issue of the Moscow Journal of Combinatorics and Number Theory comprises papers of the participants, as well as some other papers dedicated to Naum Ilyitch.

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Effective simultaneous rational approximation to pairs of real quadratic numbers

Yann Bugeaud

To the memory of Naum Ilich Feldman (1918–1994)

Let ξ, ζ be quadratic real numbers in distinct quadratic fields. We establish the existence of effectively computable, positive real numbers τ and c such that, for every integer q with $q > c$, we have

$$\max\{\|q\xi\|, \|q\zeta\|\} > q^{-1+\tau},$$

where $\|\cdot\|$ denotes the distance to the nearest integer.

1. Introduction and results

Let ξ be an irrational real number. The real number μ is an irrationality measure for ξ if there exists a positive real number $c(\xi)$ such that every rational number p/q with $q \geq 1$ satisfies

$$\left|\xi - \frac{p}{q}\right| > \frac{c(\xi)}{q^{\mu}}.$$

If, moreover, the constant $c(\xi)$ is effectively computable, then μ is an effective irrationality measure for ξ. We denote by $\mu(\xi)$ the infimum of the irrationality measures for ξ and call it the irrationality exponent of ξ and we denote by $\mu_{\text{eff}}(\xi)$ the infimum of the effective irrationality measures for ξ and call it the effective irrationality exponent of ξ. It follows from the theory of continued fractions that $\mu(\xi) \geq 2$ and an easy covering argument shows that equality holds for almost all ξ, with respect to the Lebesgue measure. Furthermore, if ξ is real algebraic of degree $d \geq 2$, then Liouville's inequality implies that $\mu_{\text{eff}}(\xi) \leq d$, while Roth's theorem asserts that $\mu(\xi) = 2$. To get better upper bounds for the effective irrationality exponents of algebraic numbers is a notoriously challenging problem.

The first result of this type was obtained by Alan Baker [1964], who established that $\mu_{\text{eff}}(\sqrt[3]{2}) \leq 2.955$, but his method applies only to a very restricted class of algebraic numbers. A few years later, Feldman [1971], by means of a refinement of the lower bounds for linear forms in logarithms of algebraic numbers established by Baker, proved that the effective irrationality exponent of an arbitrary real algebraic number of degree greater than 2 is strictly less than its degree; see also [Bilu and Bugeaud 2000] for a proof depending on lower bounds for linear forms in only two logarithms. Subsequently, Bombieri [1993; 2002] gave an alternative proof of Feldman's result, completely independent of the theory of linear forms in logarithms and based on the Thue–Siegel principle. Further results can be found in [Bugeaud 2018a]; see in particular Section 4.10.

MSC2010: primary 11J13; secondary 11D09, 11J86.
Keywords: simultaneous approximation, Pell equation, linear form in logarithms.

In this note, we are concerned with the simultaneous approximation to pairs of real numbers by rational numbers having the same denominator. We extend the above definition of (effective) irrationality exponent as follows. Let ξ, ζ be real numbers such that $1, \xi, \zeta$ are linearly independent over the rational numbers. The real number μ is a simultaneous irrationality measure for the pair (ξ, ζ) if there exists a positive real number $c(\xi, \zeta)$ such that, for every integer triple (p, q, r) with $q \geq 1$, we have

$$\max\left\{\left|\xi - \frac{p}{q}\right|, \left|\zeta - \frac{r}{q}\right|\right\} > \frac{c(\xi, \zeta)}{q^{\mu}}.$$

If, moreover, the constant $c(\xi, \zeta)$ is effectively computable, then μ is an effective irrationality measure for the pair (ξ, ζ). We denote by $\mu(\xi, \zeta)$ the infimum of the irrationality measures for the pair (ξ, ζ) and call it the irrationality exponent of the pair (ξ, ζ), and we denote by $\mu_{\text{eff}}(\xi, \zeta)$ the infimum of the effective irrationality measures for the pair (ξ, ζ) and call it the effective irrationality exponent of the pair (ξ, ζ).

Let ξ, ζ be real numbers such that $1, \xi, \zeta$ are linearly independent over the rational numbers. An easy application of Minkowski's theorem implies that $\mu(\xi, \zeta) \geq \frac{3}{2}$ and a covering lemma shows that equality holds for almost all pairs (ξ, ζ), with respect to the planar Lebesgue measure. Schmidt [1967] established that $\mu(\xi, \zeta) = \frac{3}{2}$ if ξ and ζ are both real and algebraic. His result is ineffective and gives no better information on $\mu_{\text{eff}}(\xi, \zeta)$ than the obvious inequality

$$\mu_{\text{eff}}(\xi, \zeta) \leq \max\{\mu_{\text{eff}}(\xi), \mu_{\text{eff}}(\zeta)\}.$$

The particular case where ξ and ζ are quadratic numbers in distinct number fields is of special interest. The obvious upper bound $\mu_{\text{eff}}(\xi, \zeta) \leq 2$ has been improved in some cases, in particular by Rickert [1993], who established among other results that

$$\max\left\{\left|\sqrt{2} - \frac{p}{q}\right|, \left|\sqrt{3} - \frac{r}{q}\right|\right\} > \frac{10^{-7}}{q^{1.913}} \quad \text{for integers } p, q, r \geq 1,$$

and subsequently by Bennett [1995; 1996]. The method used in these papers applies only to a very restricted class of pairs (ξ, ζ) of quadratic numbers.

The purpose of the present note is to show how the theory of linear forms in logarithms (or, alternatively, Bombieri's method) allows us to improve the trivial upper bound $\mu_{\text{eff}}(\xi, \zeta) \leq 2$ for all quadratic real numbers ξ and ζ in distinct quadratic fields.

Theorem 1.1. *Let ξ, ζ be real quadratic numbers in distinct quadratic fields. Let R_ξ and R_ζ denote the regulators of the fields $\mathbb{Q}(\xi)$ and $\mathbb{Q}(\zeta)$, respectively. There exists an absolute, positive, effectively computable real number c_1 such that*

$$\mu_{\text{eff}}(\xi, \zeta) \leq 2 - (c_1 R_\xi R_\zeta)^{-1}. \tag{1-1}$$

In particular, if a, b are positive integers such that none of a, b, and ab is a perfect square, then there exists an absolute, positive, effectively computable real number c_2 such that

$$\mu_{\text{eff}}(\sqrt{a}, \sqrt{b}) \leq 2 - (c_2 \sqrt{ab} (\log a)(\log b))^{-1}.$$

The last assertion of Theorem 1.1 is an immediate consequence of the first one, since for any squarefree integer $D \geq 2$ the regulator R_D of the quadratic field generated by $\sqrt{D}$ satisfies

$$R_D < \sqrt{D}(1 + \log \sqrt{D}); \tag{1-2}$$

see, e.g., Theorem 13.4 in [Hua 1982].

Theorem 1.1 is by no means surprising. It is ultimately a consequence of the quantity B', which has its origin in [Feldman 1968; 1971] and is the key tool for his effective improvement of Liouville's bound; see Theorem 2.1 and the discussion below it. Other consequences of the quantity B' can be found in [Bugeaud 2018a] and in the recent papers [Bugeaud 2018b; Bugeaud and Evertse 2017; Bugeaud et al. 2018].

We present a proof of Theorem 1.1 together with a proof of a slightly weaker version of it, with $R_\xi R_\zeta$ replaced by $R_\xi R_\zeta \log(R_\xi R_\zeta)$ in (1-1). For the latter result, we apply an estimate for linear forms in three logarithms, while the former is derived from a result of Bombieri [1993] (and can also be derived from an estimate for linear forms in only two logarithms). This is in accordance with the improvements on Liouville's bound obtained by these two methods. Namely, for an algebraic number ξ of degree d at least equal to 3, denoting by R_ξ the regulator of the number field generated by ξ, it follows from the theory of linear forms in logarithms and from Bombieri's method, respectively, that there exist absolute, effectively computable, positive real numbers c_3 and c_4 such that

$$\mu_{\mathrm{eff}}(\xi) \leq d - (c_3 R_\xi \log R_\xi)^{-1},$$
$$\mu_{\mathrm{eff}}(\xi) \leq d - (c_4 R_\xi)^{-1};$$

see, e.g., [Bugeaud 1998].

The last assertion of Theorem 1.1 is equivalent to the following statement on systems of Pellian equations.

Theorem 1.2. *Let a, b be positive integers such that none of a, b, and ab is a perfect square. Let u, v be nonzero integers. There exists an effectively computable, absolute real number c_5 such that all the solutions in positive integers x, y, z of the system of Pellian equations*

$$x^2 - ay^2 = u, \quad z^2 - by^2 = v$$

satisfy

$$\max\{x, y, z\} \leq (\max\{|u|, |v|, 2\})^{c_5 \sqrt{ab} (\log a)(\log b)}.$$

2. Auxiliary results

As usual, $h(\alpha)$ denotes the (logarithmic) Weil height of the algebraic number α. Our auxiliary result for the proof of (a slightly weaker version of) Theorems 1.1 and 1.2 is a particular case of Theorem 2.1 of [Bugeaud 2018a], which essentially reproduces a theorem of Waldschmidt [1993; 2000].

Theorem 2.1. *Let $n \geq 1$ be an integer. Let $\alpha_1, \dots, \alpha_n$ be nonzero algebraic numbers. Let $b_1, \dots, b_n$ be integers with $b_n \neq 0$. Let D be the degree over $\mathbb{Q}$ of the number field $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$. Let $A_1, \dots, A_n$ be real numbers with*

$$\log A_j \geq \max\left\{h(\alpha_j), \frac{\mathrm{e}}{D}|\log \alpha_j|, \frac{1}{D}\right\}, \quad 1 \leq j \leq n.$$

Let B' be a real number satisfying

$$B' \geq 3D, \quad B' \geq \max_{1 \leq j \leq n-1} \left\{ \frac{|b_n|}{\log A_j} + \frac{|b_j|}{\log A_n} \right\}.$$

If $b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n$ is nonzero, then we have

$$\log |b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| \geq -2^{n+26} n^{3n+9} D^{n+2} \log(3D) \log A_1 \cdots \log A_n \log B'.$$

The quantity B' in Theorem 2.1, which replaces the quantity

$$B = \max\{3D, |b_1|, \ldots, |b_n|\}$$

occurring in earlier estimates of Baker, originates in [Feldman 1968; 1971]. It is a consequence of the use of the functions $x \mapsto \binom{x}{k}$ instead of $x \mapsto x^k$ in the construction of the auxiliary function. The key point is the presence of the factor $\log A_n$ in the denominator in the definition of B'. It is of great interest when $b_n = 1$ and $\log A_n$ is large, since it then allows us, roughly speaking, to replace B by $B/(\log A_n)$.

The auxiliary result for the proof of Theorems 1.1 and 1.2 is a particular case of Theorem 2 of [Bombieri 1993]. Actually, since the dependence in the parameters d and κ occurring in this theorem has been improved in [Bugeaud 1998], we choose to quote below a particular case of Théorème 1 of that paper.

Theorem 2.2. *Let $\mathbb{K}$ be a real number field of degree d. Let Γ be a finitely generated subgroup of $\mathbb{K}^*$ and consider a system $\xi_1, \ldots, \xi_t$ of generators of Γ/tors. Let ξ in Γ, A in $\mathbb{K}^*$ and $\kappa > 0$ be such that $\kappa \leq 1$ and*

$$0 < |1 - A\xi| < \mathrm{e}^{-\kappa h(A\xi)} < 1.$$

Setting

$$C = 4.10^{19} d^4 \frac{(\log 3d)^7}{\kappa} \log^* \frac{d}{\kappa}, \quad Q = (2tC)^t \prod_{i=1}^{t} h(\xi_i),$$

we have the upper bound

$$h(\xi) \leq 10Q \max\{h(A), Q\}.$$

Bombieri's original proof of Theorem 2.2 (up to the dependence on d and κ) is independent of the theory of linear forms in logarithms. An alternative proof, given in [Bugeaud 1998], depends on lower estimates for linear forms in two logarithms (a careful reader can observe that, while the proof of Théorème 1 of [Bugeaud 1998] rests on estimates for linear forms in three logarithms, estimates for linear forms in two logarithms are enough to establish Theorem 2.2 above, and even with a better numerical constant, since we have assumed that $\mathbb{K}$ is a real number field) combined with a lemma of geometry of numbers from [Bombieri 1993]. To deduce Theorem 2.2 from estimates for linear forms in two logarithms, the crucial ingredient is ultimately the presence of the factor B' in these estimates.

3. Proofs

We start with the proof of (a slightly weaker version of) Theorem 1.2. Let a, b be positive integers such that $1, \sqrt{a}, \sqrt{b}$ are linearly independent over the rationals. Let u, v be nonzero integers and consider the system of Pellian equations

$$x^2 - ay^2 = u, \quad z^2 - by^2 = v \quad \text{in positive integers } x, y, z. \tag{3-1}$$

Set

$$U = \max\{|u|, |v|, 2\} \quad \text{and} \quad X = \max\{x, y, z\}.$$

It is well known [Baker and Davenport 1969; Pinch 1988] that the theory of linear forms in logarithms allows us to bound effectively X in terms of U. Our goal is to show that we can get a bound which is polynomial in U.

Let ε and η be the fundamental totally positive units of the rings of integers of the fields $\mathbb{Q}(\sqrt{a})$ and $\mathbb{Q}(\sqrt{b})$, respectively, normalized to be greater than 1. We note that ξ and η are at least equal to $(1+\sqrt{5})/2$.

Let x, y, and z be positive integers satisfying (3-1). Since the norm over $\mathbb{Q}$ of $x + y\sqrt{a}$ (resp. $z + y\sqrt{b}$) is u (resp. v), there exist nonnegative integers m, n and algebraic numbers α in $\mathbb{Q}(\sqrt{a})$ and β in $\mathbb{Q}(\sqrt{b})$ such that

$$\begin{gathered} \alpha \ge |\alpha^\sigma|, \quad \beta \ge |\beta^\sigma|, \quad \alpha\varepsilon^{-1} \le |\alpha^\sigma|\varepsilon, \quad \beta\eta^{-1} \le |\beta^\sigma|\eta, \\ x + y\sqrt{a} = \alpha\varepsilon^m, \quad \text{and} \quad z + y\sqrt{b} = \beta\eta^n, \end{gathered} \tag{3-2}$$

where the superscript σ denotes the Galois conjugacy.

Since $\varepsilon^\sigma = \varepsilon^{-1}$ and $\eta^\sigma = \eta^{-1}$, we have

$$\begin{aligned} 2y\sqrt{a} &= \alpha\varepsilon^m - \alpha^\sigma\varepsilon^{-m}, \\ 2y\sqrt{b} &= \beta\eta^n - \beta^\sigma\eta^{-n}. \end{aligned}$$

Set

$$\Lambda = \left|\alpha\beta^{-1}\sqrt{\frac{b}{a}}\,\varepsilon^m\eta^{-n} - 1\right| = \left|\alpha^\sigma\beta^{-1}\sqrt{\frac{b}{a}}\,\varepsilon^{-m}\eta^{-n} - \beta^\sigma\beta^{-1}\eta^{-2n}\right|. \tag{3-3}$$

Clearly, Λ is nonzero.

Set

$$U_0 = \max\{U, ab, \varepsilon^2, \eta^2\} \tag{3-4}$$

Observe that $\alpha = |u|/|\alpha^\sigma|$, $\beta = |v|/|\beta^\sigma|$, (3-2), and (3-4) imply

$$\alpha^2 \le |u|\varepsilon^2 \le U_0^2, \quad \beta^2 \le |v|\eta^2 \le U_0^2, \tag{3-5}$$

and

$$\begin{aligned} h\left(\alpha\beta^{-1}\sqrt{\frac{b}{a}}\right) &\le h(\alpha) + h(\beta) + h(\sqrt{a}) + h(\sqrt{b}) \\ &\le \log\alpha + \log\beta + \frac{\log a}{2} + \frac{\log b}{2} \le 3\log U_0. \end{aligned}$$

Assume first that

$$\max\{m\log\varepsilon, n\log\eta\} \ge 12\log U_0. \tag{3-6}$$

Observe that (3-3), (3-4), and (3-5) imply

$$\log\Lambda \le -n\log\eta + 2\log U_0, \tag{3-7}$$

and

$$|m\log\varepsilon - n\log\eta| \le 4\log U_0; \tag{3-8}$$

thus, by (3-6), we get

$$\log\Lambda \le -\max\{m\log\varepsilon, n\log\eta\} + 6\log U_0 \le -\frac{\max\{m\log\varepsilon, n\log\eta\}}{2}. \tag{3-9}$$

It then follows from Theorem 2.1 applied with $\alpha_1 = \varepsilon,\ \alpha_2 = \eta,\ \alpha_3 = \alpha\beta^{-1}\sqrt{b/a}$ that

$$\log\Lambda \gg -(\log U_0)(\log\varepsilon)(\log\eta)\log^*\frac{\max\{m,n\}}{\log U_0}, \tag{3-10}$$

where we write $\log^*$ for the function $\max\{1, \log\}$. Here and below, the numerical constant implied by $\ll$ is positive, absolute, and effectively computable.

The combination of (3-9) with (3-10) gives

$$\max\{m\log\varepsilon, n\log\eta\} \ll (\log U_0)(\log\varepsilon)(\log\eta)\log^*\frac{\max\{m,n\}}{\log U_0}.$$

We deduce that

$$X \ll \max\{m\log\varepsilon, n\log\eta\} \ll (\log\varepsilon)(\log\eta)\log^*(\max\{\log\varepsilon,\log\eta\})\log U_0,$$

while $X \ll \log U_0$ if (3-6) is not satisfied.

Consequently, no matter if (3-6) holds or not, there exist an effectively computable positive real number C_1, depending only on a and b, and an effectively computable positive, absolute real number c_6 such that

$$X \le C_1 U^{c_6(\log\varepsilon)(\log\eta)\log^*(\max\{\log\varepsilon,\log\eta\})}. \tag{3-11}$$

Combined with the upper bound (1-2), this gives Theorem 1.2 up to an extra logarithmic factor.

For the proof of (a slightly weaker version of) Theorem 1.1, without any loss of generality, we may assume that ξ, η are square roots of positive integers a, b as above. Then, keeping our notation, it follows from (3-11) that there exist effectively computable positive real numbers C_2 and C_3, depending only on a and b, such that

$$\begin{aligned}\max\left\{\left|\sqrt{a}-\frac{x}{y}\right|, \left|\sqrt{b}-\frac{z}{y}\right|\right\} &\ge \frac{C_2}{y^2}\max\{|x^2-ay^2|, |z^2-by^2|\}\\ &\ge \frac{C_2}{2y^2}\left(\frac{X}{C_1}\right)^{1/(c_6(\log\varepsilon)(\log\eta)\log^*(\max\{\log\varepsilon,\log\eta\}))}\\ &\ge \frac{C_3}{y^{2-1/(c_6(\log\varepsilon)(\log\eta)\log^*(\max\{\log\varepsilon,\log\eta\}))}}.\end{aligned}$$

Combined with (1-2), this completes the proof of Theorem 1.1 up to an extra logarithmic factor.

It remains for us to explain how to deduce Theorems 1.1 and 1.2 from Theorem 2.2, applied with Γ being the subgroup generated by ε and η,

$$A = \alpha\beta^{-1}\sqrt{b/a}, \quad \xi_1 = \varepsilon, \quad \xi_2 = \eta, \quad \text{and} \quad \xi = \varepsilon^m\eta^{-n}.$$

Note that

$$h(A\xi) \le h(A) + m\log\varepsilon + n\log\eta \le 3\log U_0 + m\log\varepsilon + n\log\eta. \tag{3-12}$$

Assume that (3-6) holds. By combining (3-6), (3-7), (3-8), and (3-12) we get

$$\log \Lambda \ll -\log U_0 - m \log \varepsilon - n \log \eta \ll -h(A\xi).$$

It then follows from Theorem 2.2 that

$$h(\xi) \ll ((\log \varepsilon)(\log \eta) h(A) + (\log \varepsilon)^2 (\log \eta)^2).$$

Since $h(A) \le 3 \log U_0$ and

$$X \ll \max\{m \log \varepsilon, n \log \eta\} \le 4h(\xi),$$

there exist an effectively computable positive real number C_4, depending only on a and b, and an effectively computable positive, absolute real number c_7 such that

$$X \le C_4 U^{c_7 (\log \varepsilon)(\log \eta)}. \tag{3-13}$$

By increasing c_7 and C_4 if necessary, we see that (3-13) also holds if (3-6) is not satisfied. Then, proceeding as below (3-11), we establish Theorems 1.1 and 1.2.

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Algebraic integers close to the unit circle

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We show that for each $d \geq 3$ there is a monic integer polynomial P of degree d which is irreducible over $\mathbb{Q}$ and has two complex conjugate roots as close to the unit circle as is allowed by the Liouville-type inequality.

1. Introduction

For a polynomial

$$P(x) := a(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x], \quad a \neq 0, \tag{1}$$

of degree $d \geq 2$ let $H(P)$ be its *height*, i.e., the maximum modulus of its coefficients, and let

$$M(P) := |a| \prod_{k=1}^{d} \max\{1, |\alpha_k|\}$$

be its *Mahler measure*. A polynomial root separation problem is to find the smallest possible nonzero distance between the roots of $P \in \mathbb{Z}[x]$ of fixed degree d in terms of $H(P)$ (or $M(P)$), when $H(P)$ (resp. $M(P)$) tends to infinity. By a result of [Mahler 1964], the smallest possible such distance must be at least $\sqrt{3}d^{-d/2-1}M(P)^{1-d}$. The exponent $1-d$ for $M(P)$ is the best possible if $d=2$ (trivially) and also if $d=3$, as proved in [Evertse 2004; Schönhage 2006]. The question of whether the exponent $1-d$ is the best possible for $d \geq 4$ is still open; see [Bugeaud and Dujella 2011; 2014; Bugeaud and Mignotte 2004; 2010; Dujella and Pejković 2011; Herman et al. 2018].

Recently, in [Bugeaud et al. 2017] it was shown that the modulus of the sum $\alpha_i + \alpha_j$, where α_i, α_j are real roots of P, is either 0 or bounded below by $c_1(d)M(P)^{1-d}$, where $c_1(d)$ depends on d only, and, moreover, that the exponent $1-d$ is the best possible; that is, there exists a monic irreducible polynomial $P \in \mathbb{Z}[x]$ of degree d whose two real roots α_i, α_j satisfy $0 < |\alpha_i + \alpha_j| < c_2(d)M(P)^{1-d}$.

The above-mentioned root separation problem is essentially an estimation of $\min_{i \neq j} |\alpha_i/\alpha_j - 1|$. As observed in [Dubickas 2013], this quantity has an advantage over the standard root separation $\operatorname{sep}(P) := \min_{i \neq j} |\alpha_i - \alpha_j|$ because it remains the same if we replace P by its reciprocal polynomial $P^*(x) = x^d P(1/x)$. In this context, it seems natural to consider a kind of symmetric separation $\min_{1 \leq i,j \leq d} |\alpha_i - 1/\alpha_j|$ or

$$\operatorname{symsep}(P) := \min_{\substack{1 \leq i,j \leq d \\ \alpha_i\alpha_j \neq 1}} |\alpha_i\alpha_j - 1|.$$

We claim that

$$\operatorname{symsep}(P) \geq 2^{1-d(d-1)/2} M(P)^{1-d} \tag{2}$$

for $d \geq 3$.

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We will prove (2) by a Liouville-type argument essentially the same as that in [Feldman 1981, Theorem 5.3] and [Waldschmidt 2000, Lemma 3.14]. (It seems that those results do not imply (2) directly when α_i, α_j are both real.)

Fix a pair of indices (i, j), where $1 \le i, j \le d$ and $i \ne j$. Then, for the number $\gamma = \alpha_i\alpha_j \ne 1$ of degree, say n, we have $M(\gamma) \le M(P)^{d-1}$, where P is defined in (1), since the minimal polynomial of γ, say $Q(x) = c\prod_{\ell=1}^{n}(x-\gamma_\ell) \in \mathbb{Z}[x]$, divides the polynomial

$$a^{d-1}\prod_{1\le k<l\le d}(x-\alpha_k\alpha_l) \in \mathbb{Z}[x].$$

Next, in view of $\gamma \ne 1$ we have $1 \le |Q(1)| = |c|\prod_{\ell=1}^{n}|1-\gamma_\ell|$. Using the estimates $|1-\gamma_\ell| \le 2\max\{1, \gamma_\ell\}$ for every $\ell \ge 2$ we derive that

$$1 \le |\gamma-1|2^{n-1}M(\gamma) \le |\gamma-1|2^{n-1}M(P)^{d-1}. \tag{3}$$

Note that $n = \deg(\alpha_i\alpha_j) \le d(d-1)/2$, with equality if and only if either $d = 2$ or (when $d \ge 3$) P is irreducible over $\mathbb{Q}$ and the Galois group of its splitting field is 2-transitive. In particular, if P is reducible, $d \ge 3$, and α_i, α_j are the roots of its distinct irreducible factors then

$$\deg(\alpha_i\alpha_j) \le \deg\alpha_i \deg\alpha_j \le \deg\alpha_i(d-\deg\alpha_i) < d(d-1)/2.$$

Therefore, as $\gamma = \alpha_i\alpha_j$, for each $d \ge 2$ from (3) it follows that

$$|\alpha_i\alpha_j - 1| \ge 2^{1-d(d-1)/2}M(P)^{1-d} \tag{4}$$

for any pair of indices $i \ne j$ from the set $\{1, \dots, d\}$ such that $\alpha_i\alpha_j \ne 1$.

We next claim that for each $d \ge 2$ and each $i \in \{1, \dots, d\}$

$$|\alpha_i^2 - 1| \ge 2^{1-d}M(P)^{-1} \tag{5}$$

if $\alpha_i \ne \pm 1$. Indeed, if $|\alpha_i^2 - 1| \ge 1$ then (5) holds trivially. Otherwise, one has either $|\alpha_i - 1| < 1$ or $|\alpha_i + 1| < 1$. Without loss of generality we may assume that $|\alpha_i - 1| < 1$. Then, by $|\alpha_i - 1| \ge 2^{1-d}M(P)^{-1}$ and

$$|\alpha_i + 1| = |2 + \alpha_i - 1| \ge 2 - |\alpha_i - 1| \ge 2 - 1 = 1,$$

we find that $|\alpha_i^2 - 1| = |\alpha_i - 1||\alpha_i + 1| \ge 2^{1-d}M(P)^{-1}$, which is (5).

Since the right-hand side of (4) does not exceed that of (5) for $d \ge 3$, the combination of (4) and (5) yields (2) for each $d \ge 3$. For $d = 2$ the right-hand side of (5) is smaller than that of (4), so $\operatorname{symsep}(P) \ge 2^{-1}M(P)^{-1}$.

As in the above-mentioned problem [Bugeaud et al. 2017], we will show that the exponent $1-d$ in (2) is the best possible and give some other properties of the polynomial with two roots very close to the unit circle.

Theorem 1. *For each $d \ge 3$ and each sufficiently large positive integer H there is a monic, irreducible over $\mathbb{Q}$ polynomial $P(x) = \prod_{j=1}^{d}(x-\beta_{j,H}) \in \mathbb{Z}[x]$ with $M(P) = H(P) = H$ whose two roots $\beta_{d-1,H}, \beta_{d,H}$ with smallest moduli are complex conjugate and satisfy*

$$\operatorname{symsep}(P) = |\beta_{d-1,H}\beta_{d,H} - 1| = |\beta_{d,H}|^2 - 1 \sim H^{1-d} \quad \textit{as } H \to \infty. \tag{6}$$

For the resultant of this polynomial P and its reciprocal $P^*(x) = x^d P(1/x)$ we have

$$|\mathrm{Res}(P, P^*)| = |P(1)P(-1)| = 3H^2 + u_d H + v_d, \tag{7}$$

where $u_d, v_d \in \mathbb{Z}$, and the roots of P satisfy

$$\prod_{1 \le k < l \le d} |\beta_{k,H}\beta_{l,H} - 1| = 1. \tag{8}$$

Note that for $d = 2$ we can take $P(x) = (H-1)x^2 - H$, where $H \ge 2$. Then, $M(P) = H(P) = H$ and $\mathrm{symsep}(P) = (H-1)^{-1}$, so (6) holds for $d = 2$ too, but in this example P is not monic.

In the next section we will present some lemmas. The proof of Theorem 1 is given in Section 3.

2. Lemmas for polynomials of the form $f(x) - H(x^2 + x + 1)$

Consider the sequence of monic integer polynomials $g_d(x)$, $d = 1, 2, 3, \ldots$, defined by $g_1(x) := x + 1$ and

$$g_d(x) := \frac{g_{d-1}(x) - g_{d-1}(0)}{x}(1 + x + x^2) + 1 \tag{9}$$

for $d = 2, 3, 4, \ldots$. Then, step by step, we find that

$$\begin{aligned}
g_2(x) &= x^2 + x + 2,\\
g_3(x) &= x^3 + 2x^2 + 2x + 2,\\
g_4(x) &= x^4 + 3x^3 + 5x^2 + 4x + 3,\\
g_5(x) &= x^5 + 4x^4 + 9x^3 + 12x^2 + 9x + 5,\\
g_6(x) &= x^6 + 5x^5 + 14x^4 + 25x^3 + 30x^2 + 21x + 10,\\
g_7(x) &= x^7 + 6x^6 + 20x^5 + 44x^4 + 69x^3 + 76x^2 + 51x + 22,
\end{aligned}$$

etc.

Although it is defined (and written) in a different way, one can verify that it is the same sequence of polynomials as that introduced in Sections 4.1 and 4.2 of [Uray 2019]. See the formulas (4.1), (4.2) (and a table below them) and the actual construction of polynomials in (4.8) of [Uray 2019]. One can formally check that the recurrence relations (4.1), (4.2) hold for the coefficients of the polynomials $xg_{d-1}(x)$, where g_d is introduced in (9). Our polynomials (9) are monic (not with leading coefficients -1), so in order to get the polynomials (4.8) as in Section 4.2 of [Uray 2019] with minus sign, we do not add but subtract from the polynomial $xg_{d-1}(x)$ of degree d the quadratic polynomial $Hx^2 + Hx + H$.

The next lemma is Theorem 2.4 of [Uray 2019].

Lemma 2. *For each $d \ge 3$ and each sufficiently large positive integer H the polynomial*

$$xg_{d-1}(x) - H(x^2 + x + 1)$$

has two complex conjugate roots, say $\beta_{d-1,H}$ and $\beta_{d,H} = \overline{\beta_{d-1,H}}$, whose moduli satisfy

$$(|\beta_{d-1,H}| - 1)H^{d-1} = (|\beta_{d,H}| - 1)H^{d-1} \sim \tfrac{1}{2} \quad \text{as } H \to \infty. \tag{10}$$

Note that, by (9), $xg_{d-1}(x)$ is a monic integer polynomial of degree d with zero constant term whose other coefficients are positive. The proof of Theorem 2.4 in [Uray 2019] is based on some manipulation with a certain matrix obtained from the coefficients of the polynomial $xg_{d-1}(x) - H(x^2+x+1)$; see Lemma 4.2 in [Uray 2019]. In particular, one needs to show that this polynomial has all its roots outside the unit circle for H large enough. We will also prove such a statement but in a different way and in a more general setting. More precisely, in the next lemma we describe the location of the roots of polynomials of the form $f(x) - H(x^2+x+1)$, where $f \in \mathbb{Z}[x]$ is a fixed monic polynomial, and in addition show the irreducibility of such polynomials over $\mathbb{Q}$.

Lemma 3. *Let $d \geq 3$ and let*

$$f(x) := x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$$

be a monic polynomial, not divisible by x^2+x+1. Then, for each sufficiently large positive integer H the polynomial

$$P(x) := f(x) - H(x^2+x+1) \tag{11}$$

is irreducible over $\mathbb{Q}$ and its roots $\alpha_{1,H}, \dots, \alpha_{d,H}$ can be labeled so that

$$|\alpha_{j,H} - e^{2\pi i j/(d-2)} H^{1/(d-2)}| < \frac{d(1+|a_{d-1}-1|)}{d-2} \tag{12}$$

for $j = 1, \dots, d-2$ and

$$|\alpha_{d-1,H} - e^{2\pi i/3}| = |\alpha_{d,H} - e^{-2\pi i/3}| < \frac{dL(f)}{H}, \tag{13}$$

where $L(f) := 1 + \sum_{k=0}^{d-1} |a_k|$.

Proof. Let $\xi \in \mathbb{C}$ be any complex number satisfying $P'(\xi) \neq 0$. We will show that if α is the root of P nearest to ξ then

$$|\alpha - \xi| \leq \frac{d|P(\xi)|}{|P'(\xi)|}. \tag{14}$$

The result is trivial if ξ is a root of P. If not, we have $P'(\xi)P(\xi) \neq 0$, and so

$$\frac{P'(\xi)}{P(\xi)} = \sum_{j=1}^{d} \frac{1}{\xi - \alpha_{j,H}}.$$

This yields $|P'(\xi)/P(\xi)| \leq d/|\xi - \alpha|$, which implies (14).

We first will apply (14) to $\xi = e^{2\pi i/3}$. According to (11), we obtain

$$|P(e^{2\pi i/3})| = |f(e^{2\pi i/3})| \leq 1 + \sum_{k=0}^{d-1} |a_k| = L(f).$$

Also,

$$P'(e^{2\pi i/3}) = f'(e^{2\pi i/3}) - H(2e^{2\pi i/3}+1) = f'(e^{2\pi i/3}) - i\sqrt{3}H,$$

so $|P'(e^{2\pi i/3})| \sim H\sqrt{3}$ as $H \to \infty$. Consequently, for each sufficiently large H we have

$$\frac{d|P(e^{2\pi i/3})|}{|P'(e^{2\pi i/3})|} \leq \frac{0.9dL(f)}{H}.$$

Thus, by (14), for each sufficiently large H the circle with center at $e^{2\pi i/3}$ and radius $0.9dL(f)/H$ contains a root of P. This, combined with the same argument for $\xi = e^{-2\pi i/3}$, implies the existence of two roots of P satisfying (13). Equality in (13) holds, because the roots $\alpha_{d-1,H}$ and $\alpha_{d,H}$ must be complex conjugate by (12).

In order to prove (12) we fix $j \in \{1, \dots, d-2\}$ and apply (14) to the number $\xi_j := e^{2\pi ij/(d-2)} H^{1/(d-2)}$. From $\xi_j^{d-2} = H$ and

$$P(x) = x^2(x^{d-2} - H) + x(a_{d-1}x^{d-2} - H) + (a_{d-2}x^{d-2} - H) + \sum_{k=0}^{d-3} a_k x^k,$$

it follows that

$$P(\xi_j) = e^{2\pi ij/(d-2)}(a_{d-1} - 1)H^{1+/(d-2)} + (a_{d-2} - 1)H + \sum_{k=0}^{d-3} a_k \xi_j^k.$$

The modulus of the last sum $\sum_{k=0}^{d-3} a_k \xi_j^k$ is clearly less than H for H large enough. Hence,

$$|P(\xi_j)| < \left(\tfrac{1}{2} + |a_{d-1} - 1|\right) H^{1+1/(d-2)} \tag{15}$$

for each sufficiently large H. Similarly, from

$$P'(x) = x(dx^{d-2} - 2H) + (d-1)a_{d-1}x^{d-2} - H + \sum_{k=1}^{d-2} k a_k x^{k-1},$$

it follows that $|P'(\xi_j)| \sim (d-2)H^{1+1/(d-2)}$ as $H \to \infty$. Combining this with (15), we find that

$$\frac{d|P(\xi_j)|}{|P'(\xi_j)|} < \frac{d(1 + |a_{d-1} - 1|)}{d-2}$$

for H large enough. Hence, by (14), we derive the existence of the root of P, say $\alpha_{j,H}$, satisfying (12).

It remains to prove that P is irreducible. For a contradiction, suppose P is reducible, that is, $P(x) = P_1(x)P_2(x)$, where $P_1, P_2 \in \mathbb{Z}[x]$ are of degrees say $d_1 \geq 1$ and $d_2 \geq 1$, respectively. Since $\alpha_{d-1,H}$ and $\alpha_{d,H}$ are the roots of the same irreducible factor of P, without restriction of generality we may assume that they both are roots of P_1. Inserting $x = e^{2\pi i/3}$ and $x = e^{-2\pi i/3}$ into $P(x)$, by (11), we find that

$$\begin{aligned} f(e^{2\pi i/3}) f(e^{-2\pi i/3}) &= P(e^{2\pi i/3}) P(e^{-2\pi i/3}) \\ &= P_1(e^{2\pi i/3}) P_1(e^{-2\pi i/3}) P_2(e^{2\pi i/3}) P_2(e^{-2\pi i/3}). \end{aligned}$$

Here, as $f(x)$ is not divisible by $x^2 + x + 1$, the modulus of the left-hand side is a nonzero integer, which is bounded above by $L(f)^2$. By (12), we get $|P_2(e^{\pm 2\pi i/3})| \sim H^{d_2/(d-2)}$ as $H \to \infty$. Hence,

$$0 < |P_1(e^{2\pi i/3}) P_1(e^{-2\pi i/3})| = \frac{|f(e^{2\pi i/3}) f(e^{-2\pi i/3})|}{|P_2(e^{2\pi i/3}) P_2(e^{-2\pi i/3})|} < \frac{2L(f)^2}{H^{2d_2/(d-2)}}$$

for each sufficiently large H. However, $P_1(e^{2\pi i/3}) P_1(e^{-2\pi i/3})$ is a nonzero integer due to the fact that $e^{2\pi i/3}$, $e^{-2\pi i/3}$ are quadratic conjugate algebraic integers. Selecting H so large that $2L(f)^2 < H^{2d_2/(d-2)}$ we conclude that there is a rational integer greater than 0 and smaller than 1, which is absurd. □

Now, we will express the moduli of two smallest roots of the polynomial P described in Lemma 3 in terms the resultant of P and its reciprocal P^*, and show that this resultant, as a polynomial in H, has degree at least 2.

Lemma 4. *For any monic polynomial $P \in \mathbb{Z}[x]$ of degree d satisfying $P(0) \neq 0$ and $P(\pm 1) \neq 0$ the resultant* $\operatorname{Res}(P, P^*)$ *of P and its reciprocal polynomial $P^*(x) = x^d P(1/x)$ is divisible by $P(1)P(-1)$.*

In particular, the resultant $\operatorname{Res}(P, P^*)$ *of any polynomial P defined in Lemma 3 and its reciprocal polynomial P^* is divisible by $(3H - f(1))(H - f(-1))$ if $H \in \mathbb{N} \setminus \{f(-1)\}$. Furthermore, the roots $\alpha_{d-1,H}$ and $\alpha_{d,H}$ of this P (as they are defined in Lemma 3) satisfy*

$$||\alpha_{d-1,H}| - 1| = ||\alpha_{d,H}| - 1| \sim \frac{\sqrt{|\operatorname{Res}(P, P^*)|}}{2\sqrt{3}H^d} \quad \textit{as } H \to \infty. \tag{16}$$

Proof. If $P(x) = \prod_{i=1}^{d}(x - \beta_i)$ and $b = P(0)$ then

$$\begin{aligned}|\operatorname{Res}(P, P^*)| &= |b|^d \prod_{k,l=1}^{d} |\beta_k - \beta_l^{-1}| = \prod_{k,l=1}^{d} |\beta_k\beta_l - 1| \\ &= \prod_{k=1}^{d} |\beta_k^2 - 1| \prod_{1 \le k < l \le d} |\beta_k\beta_l - 1|^2 = |P(1)P(-1)| \prod_{1 \le k < l \le d} |\beta_k\beta_l - 1|^2.\end{aligned}$$

Since the product $\prod_{1 \le k < l \le d}(\beta_k\beta_l - 1)$ is a symmetric function in $\beta_1, \dots, \beta_d$, it is a rational integer, which implies that $P(1)P(-1)$ divides $\operatorname{Res}(P, P^*)$.

The second claim for P defined in (11) follows by $P(1) = f(1) - 3H$ and $P(-1) = f(-1) - H$. Note that $f(1) \neq 3H$, since $f(x) = xg_{d-1}(x)$ and so, by (9), we deduce that

$$f(1) = g_{d-1}(1) = 3(g_{d-2}(1) - g_{d-2}(0)) + 1,$$

which implies that $f(1)$ is not divisible by 3 for each $d \ge 2$.

Finally, to prove (16) for P defined in Lemma 3 we first observe that

$$\begin{aligned}|\operatorname{Res}(P, P^*)| &= H^d \prod_{k,l=1}^{d} |\alpha_{k,H} - \alpha_{l,H}^{-1}| = H^d \prod_{k,l=1}^{d} |\alpha_{k,H} - \overline{\alpha_{l,H}}^{-1}| = \prod_{k,l=1}^{d} |\alpha_{k,H}\overline{\alpha_{l,H}} - 1| \\ &= \prod_{k=1}^{d} ||\alpha_{k,H}|^2 - 1| \prod_{k \ne l} |\alpha_{k,H}\overline{\alpha_{l,H}} - 1| = ||\alpha_{d-1,H}|^2 - 1|^2 \prod_{k=1}^{d-2} ||\alpha_{k,H}|^2 - 1| \prod_{1 \le k < l \le d} |\alpha_{k,H}\overline{\alpha_{l,H}} - 1|^2.\end{aligned}$$

Hence,

$$||\alpha_{d-1,H}|^2 - 1| = \frac{\sqrt{|\operatorname{Res}(P, P^*)|}}{\prod_{k=1}^{d-2}(|\alpha_{k,H}|^2 - 1)^{1/2} \prod_{1 \le k < l \le d} |\alpha_{k,H}\overline{\alpha_{l,H}} - 1|}. \tag{17}$$

By (12), we have

$$\prod_{k=1}^{d-2}(|\alpha_{k,H}|^2 - 1)^{1/2} \sim H \quad \text{as } H \to \infty. \tag{18}$$

Similarly, by (12), $|\alpha_{k,H}\overline{\alpha_{l,H}} - 1| \sim H^{2/(d-2)}$ as $H \to \infty$ when $l \le d-2$. Likewise, by (12) and (13), $|\alpha_{k,H}\overline{\alpha_{l,H}} - 1| \sim H^{1/(d-2)}$ as $H \to \infty$ when $k \le d-2$ and $l \in \{d-1, d\}$. It is also clear that

$$|\alpha_{d-1,H}\overline{\alpha_{d,H}} - 1| = |\alpha_{d-1,H}^2 - 1| \sim \sqrt{3}$$

as $H \to \infty$ by (13). Therefore, in view of

$$\binom{d-2}{2}\frac{2}{d-2} + \frac{2(d-2)}{d-2} = d-1$$

we obtain

$$\prod_{1 \le k < l \le d} |\alpha_{k,H}\overline{\alpha_{l,H}} - 1| \sim H^{d-1}\sqrt{3} \quad \text{as } H \to \infty. \tag{19}$$

On substituting (18) and (19) into (17) we obtain

$$||\alpha_{d-1,H}|^2 - 1| \sim \frac{\sqrt{|\mathrm{Res}(P, P^*)|}}{\sqrt{3}H^d} \quad \text{as } H \to \infty.$$

This yields (16), since $|\alpha_{d-1,H}| + 1 \sim 2$ as $H \to \infty$ by (13). □

3. Proof of Theorem 1

Let P be the polynomial $xg_{d-1}(x) - H(x^2 + x + 1)$ which is defined in the beginning of Section 2. By (9), the polynomial $f(x) = xg_{d-1}(x)$ modulo $x^2 + x + 1$ is x for each $d \ge 3$, so it is not divisible by $x^2 + x + 1$. Hence, P is a polynomial of the form as considered in Lemma 3. Thus, by Lemmas 2 and 3, we see that for H large enough $P \in \mathbb{Z}[x]$ is a monic irreducible polynomial satisfying

$$\mathrm{symsep}(P) = |\beta_{d-1,H}\beta_{d,H} - 1| = ||\beta_{d-1,H}|^2 - 1| \sim 2||\beta_{d-1,H}| - 1| \sim H^{1-d}$$

as $H \to \infty$. This proves (6). In particular, by Lemmas 2 and 3, all the roots of P lie outside the unit circle for each sufficiently large H, and hence $M(P) = H(P) = H$.

Next, by (10) and (16), we obtain $|\mathrm{Res}(P, P^*)| \sim 3H^2$ as $H \to \infty$. On the other hand, by Lemma 4, for any positive integer $H \ne f(-1)$ the resultant $\mathrm{Res}(P, P^*)$ is divisible by $(3H - f(1))(H - f(-1))$. Note that $\mathrm{Res}(P, P^*)$ is a polynomial in H with integer coefficients, since it can be written as a polynomial of the Sylvester matrix. This clearly forces

$$\mathrm{Res}(P, P^*) = \theta_d(3H - f(1))(H - f(-1)),$$

where $\theta_d \in \{-1, 1\}$ for each $d \ge 3$, which proves (7). In particular, a simple calculation shows that $\theta_2 = -1$, since the resultant of the polynomials

$$x^3 + (1-H)x^2 + (2-H)x - H \quad \text{and} \quad -Hx^3 + (2-H)x^2 + (1-H)x + 1$$

equals $-3H^2 - 2H + 8$, and similar calculations imply $\theta_3 = 1$.

Finally, since

$$\prod_{1 \le k < l \le d} |\beta_{k,H}\beta_{l,H} - 1|^2 = \frac{|\mathrm{Res}(P, P^*)|}{|P(1)P(-1)|}$$

(which was already proved in Lemma 4), (7) implies (8).

4. Some examples

By (10), the roots $\beta_{d-1,H}$ and $\beta_{d,H}$ of P are outside the unit circle. In fact, two smallest roots of P can also be inside the unit circle and close to the unit circle. For example, for $d=3$ take $f(x)=x^3+x^2+bx$. (Here, $b=2$ corresponds to the case considered in Lemma 2.) Then, the resultant of the polynomial $P(x)=x^3+(1-H)x^2+(b-H)x-H$ and its reciprocal P^* equals

$$-3(b-1)^2H^2-2(b-1)^3H+b^4-3b^2+2b.$$

Selecting $b=0$ we find that $\operatorname{Res}(P, P^*)=-3H^2+2H$ and $P(H)=-H<0$. So, the largest root of P, $\alpha_{1,H}$, is greater than H, and hence the two smallest roots $\alpha_{2,H}, \alpha_{3,H}$ that are complex conjugate by (13) and satisfy (16) must be inside the unit circle. Therefore, by Lemma 4,

$$1-|\alpha_{2,H}|=1-|\alpha_{3,H}|\sim\frac{1}{2H^2}\quad\text{as } H\to\infty.$$

In particular, as the reciprocal polynomial P^* has exactly two roots outside the unit circle, this shows that the results estimating how close the roots of an integer polynomial can be to the unit circle, e.g., [Dimitrov and Habegger 2019, Lemma 4.3] and [Dubickas 1997, Theorem 2], cannot be improved by much. For example, the upper bound $4d\log(2dM(P))$ on the quantity $\sum_j \log^+(1/||\alpha_j|-1|)$, where the sum taken is over the roots of a degree-d polynomial $P\in\mathbb{Z}[x]$ with $|\alpha_j|\neq 1$, (as obtained in [Dimitrov and Habegger 2019, Lemma 4.3]) cannot be made smaller than $(2d-2)\log M(P)+\log 4$ by (6).

For $d=4$ let us consider $f(x)=x^4+2x^3+2x^2+bx$. (Again, $b=2$ corresponds to the case considered in Lemma 2.) The resultant of the polynomial $P(x)=x^4+2x^3+(2-H)x^2+(b-H)x-H$ and its reciprocal P^* is equal to

$$3(b-1)^4H^2+2(b^5-8b^4+22b^3-28b^2+17b-4)H-b^6+15b^4-40b^3+45b^2-24b+5.$$

Selecting $b=0$ we find that $\operatorname{Res}(P, P^*)=3H^2-8H+5$. This time, one can verify that two smallest roots of $P(x)=x^4+2x^2+(2-H)x^2-Hx-H$ are outside the unit circle and we have

$$|\alpha_{3,H}|-1=|\alpha_{4,H}|-1\sim\frac{1}{2H^2}$$

as $H\to\infty$ by (16).

Finally, consider the polynomial $P(x)=x^3-Hx^2+2x-H$ (which is of a different type, since only the coefficients for x^2 and the constant term are "large"). Due to $P(H-1/H)=-1/H^3<0$ and $P(H)=H>0$, there is a root $\alpha_{1,H}$ of P in the interval $(H-1/H, H)$, so $\alpha_{1,H}\sim H$ as $H\to\infty$. Therefore,

$$H-\alpha_{1,H}=\frac{2\alpha_{1,H}-H}{\alpha_{1,H}^2}\sim\frac{1}{H}\quad\text{as } H\to\infty.$$

Two other roots are complex conjugate numbers $\alpha_{2,H}$ and $\alpha_{3,H}$, which tend to i and $-i$, respectively, as $H\to\infty$. From

$$|\alpha_{2,H}|^2-1=|\alpha_{3,H}|^2-1=\alpha_{2,H}\alpha_{3,H}-1=\frac{H-\alpha_{1,H}}{\alpha_{1,H}}\sim\frac{1}{H\alpha_{1,H}}\sim\frac{1}{H^2}$$

as $H \to \infty$ we obtain

$$|\alpha_{2,H}| - 1 = |\alpha_{3,H}| - 1 \sim \frac{1}{2H^2} \quad \text{as } H \to \infty.$$

So, although the polynomial $x^3 - Hx^2 + 2x - H$ is different from that considered in Lemma 2 (i.e., $xg_2(x) - H(x^2 + x + 1) = x^3 + (1 - H)x^2 + (2 - H)x - H$), for its roots (10) holds as well. (Its irreducibility over $\mathbb{Q}$ for each sufficiently large positive integer H can be shown by the same argument as that in Lemma 3.)

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On transcendental entire functions with infinitely many derivatives taking integer values at several points

Michel Waldschmidt

Let $s_0, s_1, \dots, s_{m-1}$ be complex numbers and $r_0, \dots, r_{m-1}$ rational integers in the range $0 \le r_j \le m-1$. Our first goal is to prove that if an entire function f of sufficiently small exponential type satisfies $f^{(mn+r_j)}(s_j) \in \mathbb{Z}$ for $0 \le j \le m-1$ and all sufficiently large n, then f is a polynomial. Under suitable assumptions on $s_0, s_1, \dots, s_{m-1}$ and $r_0, \dots, r_{m-1}$, we introduce interpolation polynomials Λ_{nj} ($n \ge 0$, $0 \le j \le m-1$) satisfying

$$\Lambda_{nj}^{(mk+r_\ell)}(s_\ell) = \delta_{j\ell}\delta_{nk} \quad \text{for } n, k \ge 0 \text{ and } 0 \le j, \ell \le m-1,$$

and we show that any entire function f of sufficiently small exponential type has a convergent expansion

$$f(z) = \sum_{n\ge 0}\sum_{j=0}^{m-1} f^{(mn+r_j)}(s_j)\Lambda_{nj}(z).$$

The case $r_j = j$ for $0 \le j \le m-1$ involves successive derivatives $f^{(n)}(w_n)$ of f evaluated at points of a periodic sequence $\boldsymbol{w} = (w_n)_{n\ge 0}$ of complex numbers, where $w_{mh+j} = s_j$ ($h \ge 0$, $0 \le j \le m$). More generally, given a bounded (not necessarily periodic) sequence $\boldsymbol{w} = (w_n)_{n\ge 0}$ of complex numbers, we consider similar interpolation formulae

$$f(z) = \sum_{n\ge 0} f^{(n)}(w_n)\Omega_{\boldsymbol{w},n}(z)$$

involving polynomials $\Omega_{\boldsymbol{w},n}(z)$ which were introduced by W. Gontcharoff in 1930. Under suitable assumptions, we show that the hypothesis $f^{(n)}(w_n) \in \mathbb{Z}$ for all sufficiently large n implies that f is a polynomial.

1. Introduction

Given a finite set of points S in the complex plane and an infinite subset $\mathscr{S}$ of $S \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers, we ask for a lower bound on the order of growth of a transcendental entire function f such that $f^{(n)}(s) \in \mathbb{Z}$ for all $(s, n) \in \mathscr{S}$. In [Waldschmidt 2019], we discussed the case $S = \{s_0, s_1\}$ using interpolation polynomials of Lidstone, Whittaker and Gontcharoff, together with results of Schoenberg and Macintyre.

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Here we introduce generalizations of these interpolation polynomials to several points and we deduce lower bounds for the growth of transcendental entire functions with corresponding integral values of their derivatives. We first consider periodic sequences: given complex numbers $s_0, s_1, \dots, s_{m-1}$ and rational integers $r_0, \dots, r_{m-1}$ in the range $0 \le r_j \le m-1$, we set

$$\mathscr{S} = \{(s_j, mn + r_j) \mid n \ge 0,\ 0 \le j \le m-1\};$$

under suitable assumptions, we give a lower bound for the growth order of a transcendental entire function f satisfying $f^{(mn+r_j)}(s_j) \in \mathbb{Z}$ for $0 \le j \le m-1$ and all sufficiently large n (Theorem 2). That some assumption is necessary is obvious from the example $m = 2$, $s_0 = s_1 = r_0 = r_1 = 0$: given any transcendental entire function g, say of order 0, the function $f(z) = zg(z^2)$ is a transcendental entire function of the same order satisfying $f^{(2n)}(s_0) = 0$ for all $n \ge 0$.

Next, we consider a sequence $(w_n)_{n\ge 0}$ of elements in S and we prove that an entire function of sufficiently small exponential type satisfying $f^{(n)}(w_n) \in \mathbb{Z}$ for all sufficiently large n is a polynomial (Theorem 5(a)).

In Section 4, we show how to interpolate entire functions of sufficiently small exponential type with respect to periodic subsets of $\{s_0, s_1, \dots, s_{m-1}\} \times \mathbb{N}$. Our approach requires that some determinant $D(s_0, s_1, \dots, s_{m-1})$ (depending also on $r_0, \dots, r_{m-1}$) does not vanish; this assumption cannot be omitted (it could be weakened, but we do not address this issue here).

In Section 5, we introduce interpolation polynomials attached to a sequence of elements belonging to $\{s_0, s_1, \dots, s_{m-1}\}$. We deduce that if f is an entire function of sufficiently small exponential type such that, for all sufficiently large n, at least one of the $2^m - 1$ nonempty products of elements $f^{(n)}(s_0), f^{(n)}(s_1), \dots, f^{(n)}(s_{m-1})$ is in $\mathbb{Z}$, then f is a polynomial (Theorem 5(b)).

2. Notation and auxiliary results

We denote by δ_{ij} the Kronecker symbol,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j, \end{cases}$$

and by $f^{(n)}$ the n-th derivative $(\mathrm{d}^n/\mathrm{d}z^n)f$ of an analytic function $f(z)$.

The order of an entire function f is

$$\varrho(f) = \limsup_{r\to\infty} \frac{\log\log |f|_r}{\log r}, \quad \text{where } |f|_r = \sup_{|z|=r} |f(z)|,$$

and the exponential type is

$$\tau(f) = \limsup_{r\to\infty} \frac{\log |f|_r}{r}\cdot$$

For each $z_0 \in \mathbb{C}$, we have

$$\limsup_{n\to\infty} |f^{(n)}(z_0)|^{1/n} = \tau(f). \tag{2-1}$$

Cauchy's inequalities

$$\frac{|f^{(n)}(z_0)|}{n!} r^n \le |f|_{r+|z_0|} \tag{2-2}$$

are valid for any entire function f and all $z_0 \in \mathbb{C}$, $n \geq 0$ and $r > 0$. We will also use Stirling's formula: for all $N \geq 1$, we have

$$N^N \mathrm{e}^{-N}\sqrt{2\pi N} < N! < N^N \mathrm{e}^{-N}\sqrt{2\pi N}\mathrm{e}^{1/(12N)}. \tag{2-3}$$

For the arithmetical applications, our main assumption on the growth of our functions f is

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r < \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\max\{|s_0|,|s_1|,\dots,|s_{m-1}|\}}. \tag{2-4}$$

This condition arises from the following auxiliary result, based on Cauchy's upper bound for the derivatives and Stirling approximation formula for $n!$ [Waldschmidt 2019, Proposition 12]:

Proposition 1. *Let f be an entire function and let $A \geq 0$. Assume*

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r < \frac{\mathrm{e}^{-A}}{\sqrt{2\pi}}. \tag{2-5}$$

Then there exists $n_0 > 0$ such that, for $n \geq n_0$ and for all $z \in \mathbb{C}$ in the disc $|z| \leq A$, we have

$$|f^{(n)}(z)| < 1.$$

3. Integer values of derivatives of entire functions

3A. ***Periodic sequences.*** Let $s_0, s_1, \dots, s_{m-1}$ be complex numbers, not necessarily distinct. We write $\boldsymbol{s}$ for the tuple $(s_0, s_1, \dots, s_{m-1})$. Let ζ be a primitive m-th root of unity and let $r_0, \dots, r_{m-1}$ be m integers satisfying $0 \leq r_j \leq j$ $(0 \leq j \leq m-1)$. Our main assumption is that the determinant

$$\mathrm{D}(\boldsymbol{s}) = \det\left(\frac{k!}{(k-r_j)!}s_j^{k-r_j}\right)_{0\leq j,k\leq m-1}$$

does not vanish. Here, $a!/(a-b)!$ is understood to be 0 for $a < b$. This assumption means that the linear map

$$\begin{aligned} \mathbb{C}[z]_{\leq m-1} &\longrightarrow \mathbb{C}^m, \\ L(z) &\longmapsto (L^{(r_j)}(s_j))_{0\leq j\leq m-1}, \end{aligned} \tag{3-1}$$

is an isomorphism of $\mathbb{C}$-vector spaces, $\mathbb{C}[z]_{\leq m-1}$ being the space of polynomials of degree $\leq m-1$.

For $t \in \mathbb{C}$, consider the $m \times m$ matrix

$$M(t) = (\zeta^{kr_\ell}\mathrm{e}^{\zeta^k t s_\ell})_{0\leq k,\ell\leq m-1}$$

and its determinant $\Delta(t)$,

$$\Delta(t) = \det\begin{pmatrix} \mathrm{e}^{ts_0} & \mathrm{e}^{ts_1} & \cdots & \mathrm{e}^{ts_{m-1}} \\ \zeta^{r_0}\mathrm{e}^{\zeta ts_0} & \zeta^{r_1}\mathrm{e}^{\zeta ts_1} & \cdots & \zeta^{r_{m-1}}\mathrm{e}^{\zeta ts_{m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta^{(m-1)r_0}\mathrm{e}^{\zeta^{m-1}ts_0} & \zeta^{(m-1)r_1}\mathrm{e}^{\zeta^{m-1}ts_1} & \cdots & \zeta^{(m-1)r_{m-1}}\mathrm{e}^{\zeta^{m-1}ts_{m-1}} \end{pmatrix}.$$

We will show (Lemma 9) that the exponential polynomial $\Delta(t)$ is not the zero function.

Theorem 2. *Assume* $\mathrm{D}(\boldsymbol{s}) \neq 0$. *Let* $\tau > 0$ *be such that* $\Delta(t)$ *does not vanish for* $0 < |t| < \tau$. *Let* f *be an entire function of exponential type* $< \tau$ *which satisfies* (2-4) *and also*, *for each* n *sufficiently large*,

$$f^{(mn+r_j)}(s_j) \in \mathbb{Z} \quad \text{for } j = 0, \dots, m-1.$$

Then f *is a polynomial.*

In the case $m = 1$, we can take $\tau = 1$ and the assumption that the exponential type is < 1 can be replaced by the weaker condition (2-5) with $A = 0$, according to a classical result of Pólya on Hurwitz functions; see [Waldschmidt 2019, §2].

Let us give two further examples. Proofs will be given in Section 4A.

Our first example is with $r_0 = r_1 = \cdots = r_{m-1} = 0$. In this case, the assumption $\mathrm{D}(\boldsymbol{s}) \neq 0$ is satisfied if and only if $s_0, s_1, \dots, s_{m-1}$ are pairwise distinct (Section 4A, Example 1).

Corollary 3. *Assume that* $s_0, s_1, \dots, s_{m-1}$ *are pairwise distinct. An entire function of sufficiently small exponential type*, *satisfying*

$$f^{(mn)}(s_j) \in \mathbb{Z}$$

for $j = 0, \dots, m-1$ *and for all sufficiently large* n, *is a polynomial.*

For $m = 2$ (Lidstone interpolation), with $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$, Corollary 3 follows also from [Waldschmidt 2019, Corollary 2], where the assumption on the exponential type $\tau(f)$ of f is

$$\tau(f) < \min\left\{1, \frac{\pi}{|s_0 - s_1|}\right\},$$

and this assumption is best possible. Indeed:

- The function

$$f(z) = \frac{\sinh(z - s_1)}{\sinh(s_0 - s_1)}$$

has exponential type 1 and satisfies $f^{(2n)}(s_0) = 1$ and $f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

- The function

$$f(z) = \sin\left(\pi \frac{z - s_0}{s_1 - s_0}\right)$$

has exponential type $\pi/|s_1 - s_0|$ and satisfies $f^{(2n)}(s_0) = f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

Our second example is $r_j = j$ for $j = 0, 1, \dots, m-1$. The assumption $\mathrm{D}(\boldsymbol{s}) \neq 0$ is always satisfied (Section 4A, Example 2).

Corollary 4. *An entire function of sufficiently small exponential type satisfying*

$$f^{(mn+j)}(s_j) \in \mathbb{Z}$$

for $j = 0, \dots, m-1$ *and for all sufficiently large* n *is a polynomial.*

In the case $m = 2$ (Whittaker interpolation), with $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$, Corollary 4 also follows from [Waldschmidt 2019, Corollary 6] (after permutation of s_0 and s_1), where the assumption is

$$\tau(f) < \min\left\{1, \frac{\pi}{2|s_0 - s_1|}\right\},$$

and this assumption is best possible. Indeed:

- The function
$$f(z) = \frac{\sinh(z - s_0)}{\cosh(s_1 - s_0)}$$
has exponential type 1 and satisfies $f^{(2n)}(s_0) = 0$ and $f^{(2n+1)}(s_1) = 1$ for all $n \geq 0$.

- The function
$$f(z) = \cos\left(\frac{\pi}{2}\frac{z - s_1}{s_1 - s_0}\right)$$
has exponential type $\pi/(2|s_1 - s_0|)$ and satisfies $f^{(2n)}(s_0) = f^{(2n+1)}(s_1) = 0$ for all $n \geq 0$.

3B. ***Sequence of derivatives at finitely many points.*** The next result deals with a situation more general than Corollary 4.

Theorem 5. *Let $A > 0$, let f be an entire function satisfying* (2-5), *and let the exponential type $\tau(f)$ of f satisfy*

$$\tau(f) < \frac{\log 2}{A}.$$

(a) *Assume that for all sufficiently large integers n, there exists $w_n \in \mathbb{C}$ with $|w_n| < A$ such that $f^{(n)}(w_n) \in \mathbb{Z}$. Then f is a polynomial.*

(b) *Let $s_0, s_1, \dots, s_{m-1}$ be m complex numbers, not necessarily distinct, satisfying*

$$\max_{0 \leq j \leq m-1} |s_j| < A.$$

Assume that, for all sufficiently large n, there exists a nonempty subset I_n of $\{0, 1, \dots, m-1\}$ such that the product

$$\prod_{j \in I_n} f^{(n)}(s_j)$$

is in $\mathbb{Z}$. Then f is a polynomial.

The case $m = 2$ in part (b) of Theorem 5 is [Waldschmidt 2019, Theorem 8].

3C. ***Content.*** In Section 4 we deal with periodic subsets of $S \times \mathbb{N}$: we generalize the construction of Lidstone polynomials to several points and we prove Theorem 2 and Corollaries 3 and 4. In Section 5, we introduce and study interpolation polynomials associated with a sequence of elements in S and we prove Theorem 5.

4. Periodic case

Let $s_0, s_1, \dots, s_{m-1}$ be distinct complex numbers and $r_0, \dots, r_{m-1}$ rational integers satisfying $0 \leq r_0 \leq r_1 \leq \cdots \leq r_{m-1} \leq m - 1$.

4A. ***The determinant*** **D(z)*: proofs of Corollaries 3 and 4.*** Let $z_0, z_1, \dots, z_{m-1}$ be independent variables. Write z for $(z_0, z_1, \dots, z_{m-1})$. Let K be the field $\mathbb{Q}(z_0, z_1, \dots, z_{m-1})$ and D(z) be the determinant

$$\det\left(\frac{k!}{(k-r_j)!} z_j^{k-r_j}\right)_{0\le j,k\le m-1} \in \mathbb{Q}[z] \subset K.$$

Recall $a!/(a-b)! = 0$ for $a < b$.

For $j = 0, 1, \dots, m-1$, the row vector

$$\begin{aligned} v_j &= \left(\frac{k!}{(k-r_j)!} z_j^{k-r_j}\right)_{k=0,1,\dots,m-1} \\ &= \left(0, 0, \dots, 0, r_j!, \frac{(r_j+1)!}{1!} z_j, \frac{(r_j+2)!}{2!} z_j^2, \dots, \frac{(m-1)!}{(m-1-r_j)!} z_j^{m-1-r_j}\right) \end{aligned}$$

belongs to $\{0\}^{r_j} \times K^{m-r_j}$. If $r_j > j$ for some $j \in \{0, 1, \dots, m-1\}$, then the $m-j$ vectors $v_j, v_{j+1}, \dots, v_{m-1}$ all belong to the subspace $\{0\}^{j+1} \times K^{m-j-1}$ of K^m, the dimension of which is $m-j-1$; hence the determinant D(z) vanishes.

Assume $r_j \le j$ for $0 \le j \le m-1$. For the degree given by the lexicographic order, the leading term of the polynomial D(z) is the product of the elements on the diagonal. The degree in z_j of D(z) is $\le m-1-r_j$. For $k = 0, 1, \dots, m-1$, define

$$\mathcal{E}(k) = \{(i, j) \mid 0 \le i < j \le m-1,\ r_i = r_j\}.$$

In the ring $\mathbb{Q}[z_0, z_1, \dots, z_{m-1}]$, the polynomial D($z$) is divisible by

$$\prod_{(i,j)\in\mathcal{E}(k)} (z_j - z_i).$$

If there is no extra nonconstant factor, the only zeros of D(z) are given by $z_i = z_j$ with $r_i = r_j$ and $i < j$. But extra factors may occur.

Examples. (1) [Poritsky 1932], quoted by [Macintyre 1954, §3; Buck 1955]:

$$r_0 = r_1 = \cdots = r_{m-1} = 0.$$

The Vandermonde determinant

$$D(s) = \det(s_j^k)_{0\le j,k\le m-1} = \det\begin{pmatrix} 1 & s_0 & s_0^2 & \cdots & s_0^{m-1} \\ 1 & s_1 & s_1^2 & \cdots & s_1^{m-1} \\ 1 & s_2 & s_2^2 & \cdots & s_2^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_{m-1} & s_{m-1}^2 & \cdots & s_{m-1}^{m-1} \end{pmatrix} = \prod_{0\le j<\ell\le m-1} (s_\ell - s_j)$$

does not vanish if and only if $s_0, s_1, \dots, s_{m-1}$ are pairwise distinct.

(2) [Gontcharoff 1930], quoted by [Macintyre 1954, §4; Buck 1955]:

$$r_j = j \quad \text{for } j = 0, 1, \dots, m-1.$$

Then

$$D(\boldsymbol{s}) = \det\left(\frac{k!}{(k-j)!} s_j^{k-j}\right)_{0\le j,k\le m-1}$$

$$= \det\begin{pmatrix} 1 & s_0 & s_0^2 & s_0^3 & \cdots & s_0^{m-2} & s_0^{m-1} \\ 0 & 1 & 2s_1 & 3s_1^2 & \cdots & (m-2)s_1^{m-3} & (m-1)s_1^{m-2} \\ 0 & 0 & 2 & 6s_2 & \cdots & (m-2)(m-3)s_3^{m-4} & (m-1)(m-2)s_2^{m-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (m-2)! & (m-1)!s_{m-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & (m-1)! \end{pmatrix} = \prod_{j=0}^{m-1} j!$$

does not vanish.

(3) Take $m=3,\ r_0=r_1=0,\ r_2=1$. Then

$$\mathrm{D}(z_0,z_1,z_2) = \begin{vmatrix} 1 & z_0 & z_0^2 \\ 1 & z_1 & z_1^2 \\ 0 & 1 & 2z_2 \end{vmatrix} = (z_1-z_0)(2z_2-z_1-z_0).$$

A polynomial of degree 2 vanishing at s and $-s$ with $s\ne 0$ has a zero derivative at the origin. For the study of entire functions f satisfying

$$f^{(3n)}(s_0)\in\mathbb{Z},\qquad f^{(3n)}(s_1)\in\mathbb{Z},\qquad f^{(3n+1)}(s_2)\in\mathbb{Z}\quad\text{for } n\ge 0,$$

the assumption $\mathrm{D}(\boldsymbol{s})\ne 0$ amounts to $2s_2\ne s_1+s_0$.

4B. *Interpolation polynomials.* The following interpolation polynomials generalize the sequences of polynomials introduced by Lidstone, Whittaker, Poritsky, Gontcharoff and others.

Proposition 6. *Assume* $\mathrm{D}(\boldsymbol{s})\ne 0$*. Then there exists a unique family of polynomials* $(\Lambda_{nj}(z))_{n\ge 0,0\le j\le m-1}$ *satisfying*

$$\Lambda_{nj}^{(mk+r_\ell)}(s_\ell)=\delta_{j\ell}\delta_{nk}\quad \textit{for } n,k\ge 0 \textit{ and } 0\le j,\ell\le m-1. \tag{4-1}$$

For $n\ge 0$ *and* $0\le j\le m-1$ *the polynomial* Λ_{nj} *has degree* $\le mn+m-1$.

This result plays a main role in our paper; we give two proofs of it.

First proof of Proposition 6. Assuming $\mathrm{D}(\boldsymbol{s})\ne 0$, we prove by induction on n that the linear map

$$\psi_n : \mathbb{C}[z]_{\le m(n+1)-1} \longrightarrow \mathbb{C}^{m(n+1)},$$
$$L(z)\longmapsto (L^{(mk+r_\ell)}(s_\ell))_{0\le \ell\le m-1 0\le k\le n},$$

is an isomorphism of $\mathbb{C}$–vector spaces. For $n=0$ this is the assumption (3-1). Assume ψ_{n-1} is injective for some $n\ge 1$. Let $L\in\ker\psi_n$. Then $L^{(m)}\in\ker\psi_{n-1}$; hence $L^{(m)}=0$, which means that L has degree $<m$. From (3-1) we conclude $L=0$.

The fact that ψ_n is injective for all n implies that if a polynomial $f\in\mathbb{C}[z]$ satisfies $f^{(mk+r_\ell)}(s_\ell)=0$ for all $k\ge 0$ and all ℓ with $0\le\ell\le m-1$, then $f=0$. This shows the unicity of the solution Λ_{nj} of (4-1).

Since ψ_n is injective, it is an isomorphism, and hence surjective: for $0\le j\le n-1$ there exists a unique polynomial $\Lambda_{nj}\in\mathbb{C}[z]_{\le m(n+1)-1}$ such that $\Lambda_{nj}^{(mk+r_\ell)}(s_\ell)=\delta_{j\ell}\delta_{nk}$ for $0\le j,\ell\le m-1$. These conditions show that the set of polynomials Λ_{kj} for $0 \le k \le n,\ 0 \le j \le m-1$, is a basis of $\mathbb{C}[z]_{\le m(n+1)-1}$: any polynomial $f \in \mathbb{C}[z]$ of degree $\le m(n+1)-1$ can be written in a unique way

$$f(z) = \sum_{j=0}^{m-1} \sum_{k=0}^{n} a_{kj} \Lambda_{kj}(z),$$

and therefore the coefficients are given by $a_{kj} = f^{(mk+r_j)}(s_j)$. □

Second proof of Proposition 6. The conditions (4-1) mean that any polynomial $f \in \mathbb{C}[z]$ has an expansion

$$f(z) = \sum_{j=0}^{m-1} \sum_{n \ge 0} f^{(mn+r_j)}(s_j) \Lambda_{nj}(z), \tag{4-2}$$

where only finitely many terms on the right-hand side are nonzero.

Assuming $\mathrm{D}(\boldsymbol{s}) \ne 0$, we first prove the unicity of such an expansion by induction on the degree of f. The assumption $\mathrm{D}(\boldsymbol{s}) \ne 0$ shows that there is no nonzero polynomial of degree $< m$ satisfying $f^{(mn+r_j)}(s_j) = 0$ for all (n, j) with $0 \le n,\ j \le m-1$. Now if f is a polynomial satisfying $f^{(mn+r_j)}(s_j) = 0$ for all (n, j) with $n \ge 0$ and $0 \le j \le m-1$, then $f^{(m)}$ satisfies the same conditions and has a degree less than the degree of f. By the induction hypothesis we deduce $f^{(m)} = 0$, which means that f has degree $< m$; hence $f = 0$. This proves the unicity.

For the existence, let us show that, under the assumption $\mathrm{D}(\boldsymbol{s}) \ne 0$, the recurrence relations

$$\Lambda_{nj}^{(m)} = \Lambda_{n-1,j}, \qquad \Lambda_{nj}^{(r_\ell)}(s_\ell) = 0 \quad \text{for } n \ge 1, \qquad \Lambda_{0j}^{(r_\ell)}(s_\ell) = \delta_{j\ell} \quad \text{for } 0 \le j, \ell \le m-1$$

have a unique solution given by polynomials $\Lambda_{nj}(z)$ $(n \ge 0,\ j = 0, \dots, m-1)$, where Λ_{nj} has degree $\le mn + m - 1$. Clearly, these polynomials will satisfy (4-1).

From the assumption $\mathrm{D}(\boldsymbol{s}) \ne 0$ we deduce that, for $0 \le j \le m-1$, there is a unique polynomial Λ_{0j} of degree $< m$ satisfying

$$\Lambda_{0j}^{(r_\ell)}(s_\ell) = \delta_{j\ell} \quad \text{for } 0 \le \ell \le m-1.$$

By induction, given $n \ge 1$ and $j \in \{0, 1, \dots, m-1\}$, once we know $\Lambda_{n-1,j}(z)$, we choose a solution L of the differential equation $L^{(m)} = \Lambda_{n-1,j}$; using again the assumption $\mathrm{D}(\boldsymbol{s}) \ne 0$, we deduce that there is a unique polynomial $\tilde{L}$ of degree $< m$ satisfying $\tilde{L}^{(r_\ell)}(s_\ell) = L^{(r_\ell)}(s_\ell)$ for $0 \le \ell \le m-1$; then the solution is given by $\Lambda_{nj} = L - \tilde{L}$. □

Remark. The following converse of Proposition 6 is plain: if there exists a unique tuple

$$(\Lambda_{00}(z), \Lambda_{01}(z), \dots, \Lambda_{0,m-1}(z))$$

of polynomials of degree $\le m-1$ satisfying

$$\Lambda_{0j}^{(r_\ell)}(s_\ell) = \delta_{j\ell} \quad \text{for } 0 \le j, \ell \le m-1,$$

then $\mathrm{D}(\boldsymbol{s}) \ne 0$.

Examples. Special cases of Proposition 6 have already been introduced in the literature.

(1) Lidstone polynomials with $\{0, 1\}$ [Waldschmidt 2019, §3.1]:

$$m = 2, \quad s_0 = 0, \quad s_1 = 1, \quad r_0 = r_1 = 0, \quad \Lambda_{n0}(z) = \Lambda_n(1-z), \quad \Lambda_{n1}(z) = \Lambda_n(z).$$

(2) Lidstone polynomials with $\{s_0, s_1\}$ and $s_0 \neq s_1$; with the notation of [Waldschmidt 2019, §3.2]:

$$m = 2, \quad r_0 = r_1 = 0, \quad \Lambda_{n0}(z) = -\widetilde{\Lambda}_n(z - s_1), \quad \Lambda_{n1}(z) = \widetilde{\Lambda}_n(z - s_0).$$

(3) Whittaker polynomials with $\{0, 1\}$; with the notation of [Waldschmidt 2019, §5.1]:

$$m = 2, \quad s_0 = 1, \quad s_1 = 0, \quad r_0 = 0, \quad r_1 = 1, \quad \Lambda_{n0}(z) = M_n(z), \quad \Lambda_{n1}(z) = M'_{n+1}(z - 1).$$

(4) Whittaker polynomials with $\{s_0, s_1\}$; with the notation of [Waldschmidt 2019, §5.2] (beware that this reference deals with the even derivatives at s_0 and the odd derivatives at s_1, while here we impose $r_0 \leq r_1$):

$$m = 2, \quad r_0 = 0, \quad r_1 = 1, \quad \Lambda_{n0}(z) = \widetilde{M}_n(z - s_1), \quad \Lambda_{n1}(z) = \widetilde{M}'_{n+1}(z - s_0).$$

(5) [Poritsky 1932], quoted by [Macintyre 1954, §3; Buck 1955] (see also [Gelfond 1971, Chapter 3, §4.3]): assuming $s_0, s_1, \dots, s_{m-1}$ are pairwise distinct,

$$r_0 = r_1 = \cdots = r_{m-1} = 0.$$

(6) [Gontcharoff 1930], quoted by [Macintyre 1954, §4; Buck 1955] (see also [Gelfond 1971, Chapter 3, §4.2]):

$$r_j = j \quad \text{for } j = 0, 1, \dots, m - 1.$$

4C. ***Exponential sums, following D. Roy.*** This section is due to D. Roy (private communication).

Given complex numbers $a_0, a_1, \dots$ and nonnegative real numbers $c_0, c_1, \dots,$ we write

$$\sum_{i \geq 0} a_i z^i \preceq_z \sum_{i \geq 0} c_i z^i$$

if $|a_i| \leq c_i$ for all $i \geq 0$. In the same way, given two power series $\sum_{i \geq 0, j \geq 0} a_{ij} t^i z^j$ and $\sum_{i \geq 0, j \geq 0} c_{ij} t^i z^j$ with $a_{ij} \in \mathbb{C}$ and $c_{ij} \in \mathbb{R}_{\geq 0}$, we write

$$\sum_{i \geq 0} \sum_{j \geq 0} a_{ij} t^i z^j \preceq_{t,z} \sum_{i \geq 0} \sum_{j \geq 0} c_i t^i z^j$$

if $|a_{ij}| \leq c_{ij}$ for all i, j.

We first give a quantitative version of Proposition 6.

Lemma 7. *There exists a constant $\Theta > 0$ such that*

$$\Lambda_{nj}(z) \preceq_z \sum_{i=0}^{m(n+1)-1} \frac{\Theta^{m(n+1)-i}}{i!} z^i$$

for all $n \geq 0$ and $j = 0, 1, \dots, m - 1$.

Proof. We proceed by induction. For $n = 0$ it suffices to choose $\Theta > 0$ sufficiently large so that

$$\Lambda_{0j}(z) \preceq_z \sum_{i=0}^{m-1} \frac{\Theta^{m-i}}{i!} z^i$$

for $j = 0, 1, \dots, m-1$. Assume

$$\Lambda_{n-1,j}(z) \preceq_z \sum_{i=0}^{mn-1} \frac{\Theta^{mn-i}}{i!} z^i$$

for some integer $n \geq 1$ and for $j = 0, 1, \dots, m-1$. Fix an integer j and let $L(z) \in \mathbb{C}[z]$ be the polynomial satisfying

$$L^{(m)}(z) = \Lambda_{n-1,j}(z) \quad \text{and} \quad L(0) = L'(0) = \cdots = L^{(m-1)}(0) = 0.$$

We have

$$L(z) \preceq_z \sum_{i=0}^{mn-1} \frac{\Theta^{mn-i}}{(i+m)!} z^{i+m} = \sum_{i=m}^{m(n+1)-1} \frac{\Theta^{m(n+1)-i}}{i!} z^i.$$

Set $A = \max\{1, |s_0|, \dots, |s_{m-1}|\}$. For $\ell = 0, 1, \dots, m-1$, we have

$$|L^{(r_\ell)}(s_\ell)| \leq \sum_{i=0}^{mn-1} \frac{\Theta^{mn-i} A^{i+m-r_\ell}}{(i+m-r_\ell)!} = \Theta^{mn} A^{m-r_\ell} \sum_{i=0}^{mn-1} \frac{(A/\Theta)^i}{(i+m-r_\ell)!} \leq \Theta^{mn} A^m \exp(A/\Theta).$$

From the isomorphism (3-1) it follows that there is a constant $B > 0$ such that, for any polynomial $\tilde{L}(z) \in \mathbb{C}[z]_{\leq m-1}$,

$$\tilde{L}(z) \preceq_z \bigl(\max_{0 \leq \ell \leq m-1} |\tilde{L}^{(r_\ell)}(s_\ell)|\bigr) B \sum_{i=0}^{m-1} \frac{z^i}{i!}.$$

Choosing $\tilde{L}(z)$ such that

$$\tilde{L}^{(r_\ell)}(s_\ell) = L^{(r_\ell)}(s_\ell)$$

for $\ell = 0, 1, \dots, m-1$ and assuming $\Theta \geq 1$ sufficiently large so that

$$\Theta \geq B A^m \exp(A/\Theta),$$

we get

$$\tilde{L}(z) \preceq_z \Theta^{mn+1} \sum_{i=0}^{m-1} \frac{z^i}{i!} \preceq_z \sum_{i=0}^{m-1} \frac{\Theta^{m(n+1)-i}}{i!} z^i;$$

hence

$$\Lambda_{nj}(z) = L(z) - \tilde{L}(z) \preceq_z \sum_{i=0}^{m(n+1)-1} \frac{\Theta^{m(n+1)-i}}{i!} z^i. \qquad \square$$

For $j = 0, 1, \dots, m-1$ and $z \in \mathbb{C}$, consider the power series $\varphi_j(t, z) \in \mathbb{C}[[t]]$ defined by

$$\varphi_j(t, z) = \sum_{n \geq 0} t^{mn+r_j} \Lambda_{nj}(z). \tag{4-3}$$

From Lemma 7 it follows that we have

$$\varphi_j(t, z) \preceq_{t,z} \sum_{n \geq 0} \sum_{i=0}^{m(n+1)-1} \frac{\Theta^{m(n+1)-i}}{i!} t^{mn+r_j} z^i,$$

and therefore the function of two complex variables $(t, z) \mapsto \varphi_j(t, z)$ is analytic in the domain $|t| < 1/\Theta$, $z \in \mathbb{C}$.

Lemma 8. *For* $|t| < 1/\Theta$ *and* $z \in \mathbb{C}$, *we have*

$$e^{tz} = \sum_{j=0}^{m-1} e^{ts_j} \varphi_j(t, z).$$

Proof. Define, for $|t| < 1/\Theta$ and $z \in \mathbb{C}$,

$$F(t, z) = \sum_{j=0}^{m-1} e^{ts_j} \varphi_j(t, z) - e^{tz}.$$

We have

$$F(t, z) = \sum_{j=0}^{m-1} e^{ts_j} \sum_{n\geq 0} t^{mn+r_j} \Lambda_{nj}(z) - e^{tz} = \sum_{n\geq 0} a_n(z) t^n,$$

where $a_n(z) \in \mathbb{C}[z]_{\leq n+m-1}$ for all $n \geq 0$. We obtain, for all $k \geq 0$ and $\ell = 0, 1, \ldots, m-1$,

$$\left(\frac{\partial}{\partial z}\right)^{mk+r_\ell} F(t, z)\bigg|_{z=s_\ell} = \sum_{j=0}^{m-1} e^{ts_j} \sum_{n\geq 0} t^{mn+r_j} \Lambda_{nj}^{(mk+r_\ell)}(s_\ell) - t^{mk+r_\ell} e^{ts_\ell} = 0;$$

hence

$$\sum_{n\geq 0} a_n^{(mk+r_\ell)}(s_\ell) t^n = 0$$

for $|t| < 1/\Theta$. Therefore $a_n^{(mk+r_\ell)}(s_\ell) = 0$ for all $k \geq 0$, $n \geq 0$ and $\ell = 0, 1, \ldots, m-1$. We conclude $a_n(z) = 0$ for all $n \geq 0$, which proves Lemma 8. □

For $0 < |t| < 1/\Theta$ and $j = 0, 1, \ldots, m-1$, we have

$$\left(\frac{\partial}{\partial z}\right)^{m} \varphi_j(t, z) = \sum_{n\geq 0} t^{mn+r_j} \Lambda_{nj}^{(m)}(z) = \sum_{n\geq 1} t^{mn+r_j} \Lambda_{n-1,j}(z) = t^m \varphi_j(t, z).$$

The functions $\varphi_0(t, z), \varphi_1(t, z), \ldots, \varphi_{m-1}(t, z)$ are the solutions of the differential equation

$$f^{(m)}(z) = t^m f(z)$$

with the initial conditions

$$\left(\frac{\partial}{\partial z}\right)^{r_\ell} \varphi_j(t, s_\ell) = t^{r_\ell} \delta_{j\ell} \quad \text{for } 0 \leq j, \ell \leq m-1. \tag{4-4}$$

Recall that ζ is a primitive m-th root of unity. The general solution of this differential equation is a linear combination of the functions $e^{\zeta^k tz}$ ($k = 0, 1, \ldots, m-1$) with coefficients depending on t. Hence for $0 < |t| < 1/\Theta$ there exist complex numbers $c_{jk}(t)$ ($j, k = 0, 1, \ldots, m-1$) such that

$$\varphi_j(t, z) = \sum_{k=0}^{m-1} c_{jk}(t) e^{\zeta^k tz}. \tag{4-5}$$

For $\ell = 0, 1, \ldots, m-1$, this yields

$$\sum_{k=0}^{m-1} c_{jk}(t)(\zeta^k t)^\ell = \left(\frac{\partial}{\partial z}\right)^{\ell} \varphi_j(t, z)\bigg|_{z=0} = \sum_{n\geq 0} t^{mn+r_j} \Lambda_{nj}^{(\ell)}(0),$$

and thus we deduce that

$$t^{\ell}\sum_{k=0}^{m-1} c_{jk}(t)\zeta^{k\ell} \preceq_t \sum_{n\geq 0}\Theta^{m(n+1)-\ell}t^{mn+r_j}.$$

Since the matrix $(\zeta^{k\ell})_{0\leq k,\ell\leq m-1}$ is invertible, this shows that the functions $c_{jk}(t)$ are meromorphic for $|t|<1/\Theta$ with at most a pole at $t=0$ of order $\leq m-1$.

4D. ***Analytic continuation of $\varphi_j(t,z)$.*** From (4-4) and (4-5) we deduce that for $j=0,1,\dots,m-1$ and $0<|t|<1/\Theta$, we have

$$\sum_{k=0}^{m-1} c_{jk}(t)\zeta^{kr_\ell}\mathrm{e}^{\zeta^k t s_\ell} = \delta_{j\ell} \quad (0\leq \ell\leq m-1).$$

Hence for $|t|<1/\Theta$ the matrix $(c_{jk}(t))_{0\leq j,k\leq m-1}$ is the inverse of the matrix $M(t)$. Recall (Section 3A) that $\Delta(t)$ is the determinant of the matrix $M(t)=(\zeta^{kr_\ell}\mathrm{e}^{\zeta^k t s_\ell})_{0\leq k,\ell\leq m-1}$. We deduce:

Lemma 9. *The determinant $\Delta(t)$ does not vanish for $0<|t|<1/\Theta\cdot$*

The determinant $\Delta(t)$ defines a nonzero entire function in $\mathbb{C}$. We extend the definition of $c_{jk}(t)$ to meromorphic functions in $\mathbb{C}$ by the condition that the matrix $(c_{jk}(t))_{0\leq j,k\leq m-1}$ is the inverse of the matrix $M(t)$. From the assumption in Theorem 2 that $\Delta(t)$ does not vanish for $0<|t|<\tau$, we infer that $c_{jk}(t)$ is analytic in the domain $0<|t|<\tau$. By means of (4-5), this defines $\varphi_j(t,z)$ for all $z\in\mathbb{C}$ and for all t with $\Delta(t)\neq 0$. In particular the function of two variables $t\mapsto\varphi_j(t,z)$ is analytic in the domain $|t|<\tau,\ z\in\mathbb{C}$, and (4-3) is valid in this domain.

Lemma 10. *Let ϱ satisfy $0<\varrho<\tau$. For $z\in\mathbb{C}$ and $0\leq j\leq m-1$ we have*

$$|\Lambda_{nj}(z)|\leq \varrho^{-mn-r_j}\sup_{|t|=\varrho}|\varphi_j(t,z)|.$$

Proof. The Taylor expansion at the origin of the meromorphic function $t\mapsto\varphi_j(t,z)$ is given by the formula (4-3), which is therefore valid for $|t|<\tau$. Hence

$$\Lambda_{nj}(z)=\frac{1}{2i\pi}\int_{|t|=\varrho}\varphi_j(t,z)t^{-mn-r_j-1}\,\mathrm{d}t.$$

Lemma 10 follows. □

Examples. (1) Lidstone [Waldschmidt 2019, §3.1]: $m=2$, $s_0=0$, $s_1=1$, $r_0=r_1=0$,

$$\varphi_0(t,z)=\frac{\sinh((1-z)t)}{\sinh(t)},\quad \varphi_1(t,z)=\frac{\sinh(tz)}{\sinh(t)}\cdot$$

(2) Whittaker [Waldschmidt 2019, §5.1]: $m=2$, $s_0=1$, $s_1=0$, $r_0=0$, $r_1=1$,

$$\varphi_0(t,z)=\frac{\cosh(tz)}{\cosh(t)},\quad \varphi_1(t,z)=\frac{\sinh((z-1)t)}{\cosh(t)}\cdot$$

(3) Poritsky interpolation; see [Macintyre 1954, §3]: $r_0=r_1=\cdots=r_{m-1}=0$. The condition $\mathrm{D}(\boldsymbol{s})=0$ means that $s_0,s_1,\dots,s_{m-1}$ are pairwise distinct. The coefficient of $t^{m(m-1)/2}$ in the Taylor expansion at

the origin of $\Delta(t)$ is given by the following formula involving two Vandermonde determinants:

$$\frac{1}{1!\,2!\cdots(m-1)!}\det\begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & \zeta & \cdots & \zeta^{m-1}\\ 1 & \zeta^2 & \cdots & \zeta^{2(m-1)}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \zeta^{m-1} & \cdots & \zeta^{(m-1)^2}\end{pmatrix}\det\begin{pmatrix}1 & 1 & \cdots & 1\\ s_0 & s_1 & \cdots & s_{m-1}\\ s_0^2 & s_1^2 & \cdots & s_{m-1}^2\\ \vdots & \vdots & \ddots & \vdots\\ s_0^{m-1} & s_1^{m-1} & \cdots & s_{m-1}^{m-1}\end{pmatrix}.$$

Hence $\Delta(t)$ has a zero at the origin of multiplicity $m(m-1)/2$.

For $0 \le j \le m-1$, the order of the zero at $t=0$ of $\Delta(t)\varphi_j(t,z)$ is at least $m(m-1)/2$.

(4) Gontcharoff interpolation; see [Macintyre 1954, §4]: $r_j = j$ for $j = 0, 1, \ldots, m-1$. In this case $\Delta(0)$ is the Vandermonde determinant

$$\det\begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & \zeta & \cdots & \zeta^{m-1}\\ 1 & \zeta^2 & \cdots & \zeta^{2(m-1)}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \zeta^{m-1} & \cdots & \zeta^{(m-1)^2}\end{pmatrix},$$

and hence is not zero.

4E. *Laplace transform.* The main tool for the proof of Theorem 2 is the following result.

Proposition 11. *Assume* $\mathrm{D}(s) \ne 0$ *and* $\Delta(t) \ne 0$ *for* $0 < |t| < \tau$. *Then any entire function* f *of exponential type* $< \tau$ *has an expansion of the form* (4-2), *where the series in the right-hand side is absolutely and uniformly convergent for* z *on any compact space in* $\mathbb{C}$.

As a consequence:

Corollary 12. *Under the assumptions of Proposition 11, if an entire function* f *has exponential type* $< \tau$ *and satisfies*

$$f^{(mn+r_j)}(s_j) = 0 \quad \text{for } j = 0, \ldots, m-1 \text{ and all sufficiently large } n,$$

then f *is a polynomial.*

The bound for the exponential type is sharp: if $\alpha \ne 0$ is a zero of Δ, then there exists a transcendental entire function of exponential type $|\alpha|$ satisfying the vanishing conditions of Corollary 12; for the proof, see Proposition 9(a) of [Waldschmidt 2019].

The strategy for the proof of Proposition 11 will be to check that for $|t| < \tau$ the function $f_t(z) = \mathrm{e}^{tz}$ admits the expansion (4-2), and then to deduce the general case by means of the Laplace transform, which is called the method of the kernel expansion in [Buck 1955; Boas and Buck 1964, Chapter I, §3; Macintyre 1954, §1].

We have $f_t^{(m)} = t^m f_t$ and

$$f_t^{(r_j)}(s_j) = t^{r_j}\,\mathrm{e}^{ts_j}.$$

Proof of Proposition 11. Let

$$f(z) = \sum_{n\ge 0} \frac{a_n}{n!} z^n$$

be an entire function of exponential type $\tau(f)$. Using (2-1), we deduce that the Laplace transform of f,

$$F(t)=\sum_{n\geq 0} a_n t^{-n-1},$$

is analytic in the domain $|t|>\tau(f)$. From Cauchy's residue theorem, it follows that for $\varrho>\tau(f)$ we have

$$f(z)=\frac{1}{2\pi i}\int_{|t|=\varrho} \mathrm{e}^{tz}F(t)\,\mathrm{d}t.$$

Hence

$$f^{(mn+r_j)}(z)=\frac{1}{2\pi i}\int_{|t|=\varrho} t^{mn+r_j}\,\mathrm{e}^{tz}F(t)\,\mathrm{d}t.$$

Assume $\tau(f)<\tau$. Let ϱ satisfy $\tau(f)<\varrho<\tau$. We deduce from (4-3) (which is valid for $|t|<\tau$) and Lemma 8 that, for $|t|=\varrho$, we have

$$\mathrm{e}^{tz}=\sum_{j=0}^{m-1}\mathrm{e}^{ts_j}\varphi_j(t,z)=\sum_{n\geq 0}\sum_{j=0}^{m-1}\mathrm{e}^{ts_j}t^{mn+r_j}\Lambda_{nj}(z),$$

which is the expansion (4-2) for the function $f_t(z)=\mathrm{e}^{tz}$.

We now use Lemma 10 and permute the integral and the series to deduce

$$f(z)=\sum_{n\geq 0}\sum_{j=0}^{m-1}\left(\frac{1}{2\pi i}\int_{|t|=\varrho} t^{mn+r_j}\,\mathrm{e}^{ts_j}F(t)\,\mathrm{d}t\right)\Lambda_{nj}(z)=\sum_{n\geq 0} f^{(mn+r_j)}(s_j)\Lambda_{nj}(z).$$

Using again Lemma 10 together with (2-1), we check that the last series is absolutely and uniformly convergent for z on any compact space in $\mathbb{C}$. □

Proof of Theorem 2. Let f be an entire function satisfying the assumptions of Theorem 2. From the assumption (2-4) and Proposition 1, we deduce that for n sufficiently large, we have

$$f^{(mn+r_j)}(s_j)=0 \quad \text{for } j=0,\ldots,m-1.$$

Since the exponential type of f is $<\tau$, we deduce from Corollary 12 that f is a polynomial. □

5. Sequence of derivatives at several points

Given a sequence $\boldsymbol{w}=(w_n)_{n\geq 0}$ of complex numbers, we investigate the entire functions f such that the numbers $f^{(n)}(w_n)$ are in $\mathbb{Z}$. Under suitable assumptions, we reduce this question to the case where these numbers all vanish.

5A. ***Abel–Gontcharoff interpolation.*** We start with any sequence $\boldsymbol{w}=(w_n)_{n\geq 0}$ of complex numbers. Following [Gontcharoff 1930] (see also [Evgrafov 1954; Popov 2002]), we define a sequence of polynomials $(\Omega_{w_0,w_1,\ldots,w_{n-1}})_{n\geq 0}$ in $\mathbb{C}[z]$ as follows: we set $\Omega_\varnothing=1$, $\Omega_{w_0}(z)=z-w_0$, and, for $n\geq 1$, we define $\Omega_{w_0,w_1,w_2,\ldots,w_n}(z)$ as the polynomial of degree $n+1$ which is the primitive of $\Omega_{w_1,w_2,\ldots,w_n}$ vanishing at w_0. For $n\geq 0$, we write $\Omega_{n;\boldsymbol{w}}$ for $\Omega_{w_0,w_1,\ldots,w_{n-1}}$, a polynomial of degree n which depends only on the first n terms of the sequence $\boldsymbol{w}$. The leading term of $\Omega_{n;\boldsymbol{w}}$ is $(1/n!)z^n$. An equivalent definition is

$$\Omega_{n;\boldsymbol{w}}^{(k)}(w_k)=\delta_{kn}$$

for $n \geq 0$ and $k \geq 0$. As a consequence, any polynomial P can be written as a finite sum

$$P(z) = \sum_{n \geq 0} P^{(n)}(w_n)\Omega_{n;\boldsymbol{w}}(z).$$

In particular, for $N \geq 0$ we have

$$\frac{z^N}{N!} = \sum_{n=0}^{N} \frac{1}{(N-n)!} w_n^{N-n} \Omega_{n;\boldsymbol{w}}(z).$$

This gives an inductive formula defining $\Omega_{N;\boldsymbol{w}}$: for $N \geq 0$,

$$\Omega_{N;\boldsymbol{w}}(z) = \frac{z^N}{N!} - \sum_{n=0}^{N-1} \frac{1}{(N-n)!} w_n^{N-n} \Omega_{n;\boldsymbol{w}}(z). \tag{5-1}$$

We also have

$$\Omega_{w_0,w_1,\ldots,w_n}(z) = \Omega_{0,w_1-w_0,w_2-w_0,\ldots,w_n-w_0}(z-w_0).$$

With $w_0 = 0$, the first polynomials are given by

$$\begin{aligned}
2!\,\Omega_{0,w_1}(z) &= (z-w_1)^2 - w_1^2,\\
3!\,\Omega_{0,w_1,w_2}(z) &= (z-w_2)^3 - 3(w_1-w_2)^2 z + w_2^3,\\
4!\,\Omega_{0,w_1,w_2,w_3}(z) &= (z-w_3)^4 - 6(w_2-w_3)^2(z-w_1)^2 - 4(w_1-w_3)^3 z + 6w_1^2(w_2-w_3)^2 - w_3^4.
\end{aligned}$$

From the definition we deduce the following formula, involving iterated integrals:

$$\Omega_{w_0,w_1,\ldots,w_{n-1}}(z) = \int_{w_0}^{z} \mathrm{d}t_1 \int_{w_1}^{t_1} \mathrm{d}t_2 \cdots \int_{w_{n-1}}^{t_{n-1}} \mathrm{d}t_n.$$

These polynomials are also given by a determinant [Gontcharoff 1930, p. 7]:

$$\Omega_{w_0,w_1,\ldots,w_{n-1}}(z) = (-1)^n \begin{vmatrix}
1 & \frac{z}{1!} & \frac{z^2}{2!} & \cdots & \frac{z^{n-1}}{(n-1)!} & \frac{z^n}{n!}\\
1 & \frac{w_0}{1!} & \frac{w_0^2}{2!} & \cdots & \frac{w_0^{n-1}}{(n-1)!} & \frac{w_0^n}{n!}\\
0 & 1 & \frac{w_1}{1!} & \cdots & \frac{w_1^{n-2}}{(n-2)!} & \frac{w_1^{n-1}}{(n-1)!}\\
0 & 0 & 1 & \cdots & \frac{w_2^{n-3}}{(n-3)!} & \frac{w_2^{n-2}}{(n-2)!}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 1 & \frac{w_{n-1}}{1!}
\end{vmatrix}.$$

With the sequence $\boldsymbol{w} = (1, 0, 1, 0, \ldots, 0, 1, \ldots)$, we recover the Whittaker polynomials [Waldschmidt 2019, §5]

$$\Omega_{2n;\boldsymbol{w}}(z) = M_n(z), \quad \Omega_{2n+1,\boldsymbol{w}}(z) = M'_{n+1}(z-1).$$

Another example, considered by N. Abel (see [Halphén 1882; Gontcharoff 1930, p. 7; Buck 1948, §7]), is the arithmetic progression $\boldsymbol{w} = (a+nt)_{n\geq 0}$ with a in $\mathbb{C}$ and t in $\mathbb{C} \setminus \{0\}$, where

$$\Omega_{n;\boldsymbol{w}}(z) = \frac{1}{n!}(z-a)(z-a-nt)^{n-1}$$

for $n \geq 1$, which satisfies

$$\Omega'_{n;\boldsymbol{w}}(z) = \Omega_{n-1;\boldsymbol{w}}(z-t).$$

Theorem III in [Gontcharoff 1930, p. 29] gives sufficient conditions on the sequence $(w_n)_{n\geq 0}$ so that an entire function f satisfying some growth condition has an expansion

$$f(z) = \sum_{n\geq 0} f^{(n)}(w_n)\Omega_{n;\boldsymbol{w}}(z).$$

In the case that we are going to consider where the sequence $(|w_n|)_{n\geq 0}$ is bounded, say $|w_n - w_0| \leq r$, the condition [Gontcharoff 1930, (31′), p. 33] reduces to $\tau < 1/(er)$. See also [Whittaker 1934, §10] for an improvement in the case $m = 2$.

From now on we assume that the sequence $(|w_n|)_{n\geq 0}$ is bounded. Let $A > \sup_{n\geq 0} |w_n|$.

Proposition 13. *Let $\kappa > 1/\log 2$. For n sufficiently large, we have, for all $r \geq |A|$,*

$$|\Omega_{n;\boldsymbol{w}}|_r \leq (\kappa r)^n.$$

Proof. Let $c_0, c_1, c_2, \ldots$ be the sequence of positive numbers defined by induction as follows: $c_0 = 1$ and, for $n \geq 1$,

$$c_n = \frac{1}{n!} + \frac{c_0}{n!} + \frac{c_1}{(n-1)!} + \cdots + \frac{c_{n-2}}{2!} + c_{n-1}.$$

From (5-1) we deduce by induction, for $|z| \leq r$ and all $n \geq 0$,

$$|\Omega_{n;\boldsymbol{w}}(z)| \leq c_n r^n.$$

Let κ_1 satisfy $1/\log 2 < \kappa_1 < \kappa$ and let $A > 0$ satisfy

$$A \geq (2 - \mathrm{e}^{1/\kappa_1})^{-1} \max_{n\geq 0} \frac{1}{\kappa_1^n n!}\cdot$$

One checks by induction $c_n \leq A\kappa_1^n$ for all $n \geq 0$ thanks to the upper bound

$$\frac{1}{n!} + A\kappa_1^n(\mathrm{e}^{1/\kappa_1} - 1) \leq A\kappa_1^n.$$

Therefore, for sufficiently large n, we have $c_n < \kappa^n$. □

In the case $m = 2$ and $w_n \in \{0, 1\}$ for all $n \geq 0$, a sharper estimate has been achieved in [Whittaker 1934, §10], namely

$$|\Omega_{n;\boldsymbol{w}}(z)| \leq \tfrac{1}{2}\mathrm{e}^2\left(\tfrac{1}{2} + R\right)^n$$

for $\left|z - \frac{1}{2}\right| = R$. The proof relies on explicit formulae for the polynomials $\Omega_{n;\boldsymbol{w}}(z)$.

From Proposition 13 we deduce the following interpolation formula:

Proposition 14. *Let f be an entire function of exponential type $\tau(f)$ satisfying*

$$\tau(f) < \frac{\log 2}{A}\cdot$$

Let r be a real number in the range

$$A \leq r < \frac{\log 2}{\tau(f)}\cdot$$

Then

$$f(z) = \sum_{n\geq 0} f^{(n)}(w_n)\Omega_{n;\boldsymbol{w}}(z),$$

where the series on the right-hand side is absolutely and uniformly convergent in the disk $|z| \leq r$.

Proof. Let κ and τ satisfy

$$\kappa > \frac{1}{\log 2}, \qquad \tau(f) < \tau < \frac{1}{\kappa r}\cdot$$

Write the Taylor expansion of f at the origin:

$$f(z) = \sum_{N\geq 0} a_N \frac{z^N}{N!}\cdot$$

From (2-1) we deduce that there exists a constant $c > 0$ such that, for all $N \geq 0$, we have $|a_N| \leq c\tau^N$. For $|z| \leq r$, we have

$$|a_N| \sum_{n=0}^{N} \left| \frac{1}{(N-n)!} w_n^{N-n} \Omega_{n;\boldsymbol{w}}(z) \right| \leq c\tau^N \sum_{n=0}^{N} \frac{A^{N-n}(\kappa r)^n}{(N-n)!} \leq c\mathrm{e}^{A/\kappa r}(\tau\kappa r)^N,$$

which is the general term of a convergent series, since $\tau\kappa r < 1$. Hence

$$\begin{aligned} f(z) &= \sum_{N\geq 0} a_N \sum_{n=0}^{N} \frac{1}{(N-n)!} w_n^{N-n} \Omega_{n;\boldsymbol{w}}(z) \\ &= \sum_{n\geq 0} \Omega_{n;\boldsymbol{w}}(z) \sum_{N\geq n} a_N \frac{1}{(N-n)!} w_n^{N-n} = \sum_{n\geq 0} \Omega_{n;\boldsymbol{w}}(z) f^{(n)}(w_n). \end{aligned}$$

□

Remark. Notice that here the expansions are valid in a bounded domain of $\mathbb{C}$, not in the entire complex plane as in Section 4E for instance.

Corollary 15. *If an entire function f of exponential type $\tau(f) < \log 2/A$ satisfies $f^{(n)}(w_n) = 0$ for all sufficiently large n, then f is a polynomial.*

Replacing z by Az, one may assume $A = 1$, and then Corollary 15 is [Whittaker 1964, Theorem 8], a special case of one of Takenaka's theorems.

In the special case where the set $\{w_0, w_1, w_2, \dots\}$ is finite, say $S = \{s_0, s_1, \dots, s_{m-1}\}$ with

$$\max\{|s_0|, |s_1|, \dots, |s_{m-1}|\} < A,$$

Corollary 15 reduces to the following statement:

Corollary 16. *If an entire function f of exponential type $\tau(f) < \log 2/A$ satisfies*

$$\prod_{j=0}^{m-1} f^{(n)}(s_j) = 0$$

for all sufficiently large n, then f is a polynomial.

5B. *Sequence of elements in S.*

Proof of Theorem 5. Denote by $\tau(f)$ the exponential type of f. Since f satisfies the hypothesis (2-5) of Proposition 1, for n sufficiently large we have $|f^{(n)}(z)| < 1$ for all $|z| < A$.

Under the assumption (a) of Theorem 5, for n sufficiently large we have $f^{(n)}(w_n) = 0$. Corollary 15 implies that f is a polynomial.

For each sufficiently large n, the product $\prod_{j\in I_n} f^{(n)}(s_j)$ is an integer of absolute value less than 1, and hence it vanishes. Part (b) of Theorem 5 follows from Corollary 16. □

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Can polylogarithms at algebraic points be linearly independent?

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Dedicated to the memory of Professor Naum Ilyitch Feldman

Let r, m be positive integers. Let $0 \le x < 1$ be a rational number. We denote by $\Phi_s(x, z)$ the s-th Lerch function

$$\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+x+1)^s},$$

with $s = 1, 2, \ldots, r$. When $x = 0$, this is the polylogarithmic function. Let $\alpha_1, \ldots, \alpha_m$ be pairwise distinct algebraic numbers with $0 < |\alpha_j| < 1$ $(1 \le j \le m)$. We state a linear independence criterion over algebraic number fields of all the $rm + 1$ numbers: $\Phi_1(x, \alpha_1)$, $\Phi_2(x, \alpha_1)$, $\ldots$, $\Phi_r(x, \alpha_1)$, $\Phi_1(x, \alpha_2)$, $\Phi_2(x, \alpha_2)$, $\ldots$, $\Phi_r(x, \alpha_2)$, $\ldots$, $\Phi_1(x, \alpha_m)$, $\Phi_2(x, \alpha_m)$, $\ldots$, $\Phi_r(x, \alpha_m)$ and 1. We obtain an explicit sufficient condition for the linear independence of values of the r Lerch functions $\Phi_1(x, z), \ldots, \Phi_r(x, z)$ at *m distinct points in an algebraic number field of arbitrary finite degree* without any assumptions on r and m. When $x = 0$, our result implies the linear independence of polylogarithms of distinct algebraic numbers of arbitrary degree, subject to a metric condition. We give an outline of our proof together with concrete examples of linearly independent polylogarithms.

1. Introduction

Let s be a nonnegative integer and $0 \le x < 1$ be a rational number. We study the linear independence of values of the s-th Lerch function defined by

$$\Phi_s(x, z) = \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+x+1)^s}, \quad z \in \mathbb{C},\ |z| < 1.$$

The s-th Lerch function $\Phi_s(x, z)$ satisfies the inhomogeneous differential equation

$$\frac{d}{dz}\Phi_s(x, z) = \frac{1}{z}\Phi_{s-1}(x, z) - \frac{x}{z}\Phi_s(x, z), \quad s \ge 1. \tag{1}$$

Then the s-th Lerch function is a G-function in the sense of Siegel [1929]; see also [Feldman and Nesterenko 1998].

Note that in the case of $x = 0$, we have $\Phi_s(0, z) = \mathrm{Li}_s(z)$, where

$$\mathrm{Li}_s(z) = \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)^s}, \quad z \in \mathbb{C},\ |z| < 1,$$

is the s-th polylogarithmic function.

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Let r, m be positive integers and K be an algebraic number field. Consider $\alpha_1, \dots, \alpha_m \in K \setminus \{0\}$ with $\alpha_{i_1} \neq \alpha_{i_2}$ for $1 \le i_1 < i_2 \le m$ and $0 \le x \in \mathbb{Q}$.

We define the vector of formal power series $\vec{\Phi}$ by

$$\vec{\Phi} := {}^t(1, \Phi_1(x, \alpha_1 z), \dots, \Phi_r(x, \alpha_1 z), \dots, \Phi_1(x, \alpha_m z), \dots, \Phi_r(x, \alpha_m z)) \in K[[z]]^{rm+1},$$

the vector of rational functions $\vec{A}(\alpha_i) := {}^t(\alpha_i/(1-\alpha_i z), 0, \dots, 0) \in K(z)^r$ and an $r \times r$ matrix $A(x)$ by

$$A(x) := \begin{pmatrix} -x/z & 0 & \cdots & 0 \\ 1/z & -x/z & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1/z & -x/z \end{pmatrix} \quad \text{if } r \ge 2, \qquad A(x) = \left(-\frac{x}{z}\right) \quad \text{if } r = 1.$$

Then, taking the differential equation (1) into account, the vector $\vec{\Phi}$ satisfies the following system of differential equations in $\vec{y}$:

$$\frac{d}{dz}\vec{y} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vec{A}(\alpha_1) & A(x) & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ \vec{A}(\alpha_m) & O & \cdots & A(x) \end{pmatrix} \vec{y}. \tag{2}$$

We see that (2) is indeed a system of *homogenous* differential equations in $\vec{y}$.

We consider r Lerch functions $\Phi_s(x, z)$, $1 \le s \le r$. The linear independence of $\mathrm{Li}_s(\alpha)$ at one rational number α, with $1 \le s \le r$, was studied by E. M. Nikishin [1979a]. It was generalized to the Lerch function by Kawashima [2014] and to algebraic cases by M. Hirose, M. Kawashima and N. Sato [Hirose et al. 2017]. See also [Hirata-Kohno et al. 2017] for examples. M. Hata [1990] adapted generalized Legendre polynomials modifying Padé-type constructions of G. V. Chudnovsky [1979; 1982; 1983], see also [Chudnovsky and Chudnovsky 1985], to obtain the linear independence of $\mathrm{Li}_s(\alpha)$ (indeed of the Lerch transcendent function) for different s but at one rational number α. His result implies the irrationality of $\mathrm{Li}_2(1/q)$ with q integer, $q \ge 12$, whereas it was announced in [Chudnovsky 1979] the irrationality of $\mathrm{Li}_2(1/q)$ with $q \ge 14$. Later, Hata [1993] gave the irrationality of the value of $\mathrm{Li}_2(1/q)$ with q integer, $q \ge 7$ or $q \le -5$.

Rhin and C. Viola [2005] adapted their permutation group method, established in [Rhin and Viola 1996], to get the irrationality of $\mathrm{Li}_2(\alpha)$ for certain $\alpha \in \mathbb{Q}$, involving the irrationality $\mathrm{Li}_2(1/q)$, with $q \ge 6$, $q \in \mathbb{Z}$, in qualitative and quantitative forms. More recently, Viola and W. Zudilin [2018] extended the permutation group method with constructions to establish the linear independence of 1, $\mathrm{Li}_1(1/q)$, $\mathrm{Li}_2(1/q)$, $\mathrm{Li}_2(1/(1-q))$ over $\mathbb{Q}$ with an integer $q \ge 9$ or $q \le -8$ and more generally, that of 1, $\mathrm{Li}_1(\alpha)$, $\mathrm{Li}_2(\alpha)$, $\mathrm{Li}_2(\alpha/(\alpha-1))$ for certain $\alpha \in \mathbb{Q}$. See also important related works [Fischler et al. 2019; Marcovecchio 2006; Miladi 2001; Rivoal 2003; Zudilin 1996].

With respect to logarithms, G. Rhin and P. Toffin [1986] created a system of Padé approximants to show the linear independence of the natural logarithms of *distinct* $\alpha_1, \dots, \alpha_m$, either rational or quadratic imaginary numbers, under a *metric* condition requiring the points $\alpha_1, \dots, \alpha_m$ to be very close to the origin 0. This method provides a refinement of previous lower bounds for linear forms in logarithms, especially for effective bounds obtained by A. Baker [1975] and an essential improvement due to N. I. Feldman [1968], valid under the above-stated metric condition. This proof in [Rhin and Toffin 1986] opened a new path, albeit unexplored systematically, during the next decades to show the linear independence of logarithms over $\mathbb{Q}$ at *distinct* $\alpha \in \mathbb{Q}$, relying only on Padé approximations.

Since $\mathrm{Li}_1(z)$ coincides with the usual natural logarithm, the Rhin–Toffin method suggests how to adapt Padé approximations to deal with the linear independence of polylogarithms at distinct points $\alpha_1, \dots, \alpha_m$.

We give a new criterion to show the linear independence of all the $rm+1$ numbers: $\Phi_1(x,\alpha_1)$, $\Phi_2(x,\alpha_1)$, $\dots$, $\Phi_r(x,\alpha_1)$, $\Phi_1(x,\alpha_2)$, $\Phi_2(x,\alpha_2)$, $\dots$, $\Phi_r(x,\alpha_2)$, $\dots$, $\Phi_1(x,\alpha_m)$, $\Phi_2(x,\alpha_m)$, $\dots$, $\Phi_r(x,\alpha_m)$ and 1, over an algebraic number field K, supposing $\alpha_1, \dots, \alpha_m$ pairwise distinct in K, assumed to be sufficiently close to the origin, which we will make precise later. We also give an outline of our proof with basic ideas.

Our linear independence criterion for the values of the Lerch functions, including the case of polylogarithmic functions, at *distinct points in an algebraic number field of arbitrary finite degree*, is not covered by the previous criterion in [Galochkin 1974; 1975], as is explained below in Remarks 1.1, 1.3 and 1.4 and Example 6.3.

Remark 1.1. Let us describe here previous linear independence results concerning with values of the Lerch functions, at distinct rational or imaginary quadratic points, due to A. I. Galochkin [1974; 1975], Y. Z. Flicker [1977] K. Väänänen [1980], together with a result by K. Väänänen and G. Xu [Väänänen and Xu 1988]. First, we introduce the result of Galochkin [1974, Theorem 1]. All notation and conventions are those of the above-mentioned article, pages 385-388; see also [Nurmagomedov 1971].

Theorem 1.2 [Galochkin 1974, Theorem 1]. *Let I be $\mathbb{Q}$ or an imaginary quadratic field and K be a finite extension of I with $[K:I]=\kappa<\infty$. For $1\le s\in\mathbb{Z}$, consider $f_1(z),\dots,f_s(z)\in K[\![z]\!]$ which belong to the subclass $G(K,C_0,Q,\Lambda)$ with $C_0Q\ge 2$, $C=\max(1,C_0)$* (see [Galochkin 1974, Definitions 1, 2]). *Assume that the functions are not connected by any nonzero polynomial in s variables, of degree not exceeding N, with coefficients in $\mathbb{C}(z)$. Let $1\le d\in\mathbb{Z}$ and*

$$u := \binom{N+s}{s} + \kappa\binom{N-d+s}{s} - \kappa\binom{N+s}{s},$$

with $N\ge d$.

Suppose now

$$u>0. \tag{3}$$

Then there exists an explicit constant $c_0>0$ which depends on N, d and $f_1(z),\dots,f_s(z)$, satisfying the following property: for any integer with $|q|>c_0$ and a nonzero polynomial $P(x_1,\dots,x_s)\in\mathbb{Z}[x_1,\dots,x_s]$ of degree $d\le N$, we have

$$P(f_1(1/q),\dots,f_s(1/q))\neq 0.$$

In particular, when $d=1$, we have $su=\binom{N-1+s}{N}\{N+s(1-\kappa)\}$. Thus, under the condition that $N>s(\kappa-1)$ together with the assumption of the algebraic independence of the functions $f_1(z),\dots,f_s(z)$ over $\mathbb{C}(z)$, the linear independence of values of these s functions over K at the point $1/q$ follows.

It is worth noting that Flicker [1977] proved a p-adic analogue of Galochkin's theorem. Building on both Galochkin's and Flicker's work, Väänänen [1980] refined the above-mentioned results and generalized to a system of differential equations, both in the complex and the p-adic cases and also proved a Baker-type lower bound for linear combinations of classical logarithms and polylogarithms, also subject to a metric condition as above.

For these results to work, one needs that the G-functions belong to the subclass $G(K, C_0, Q, \Lambda)$ with $C_0 Q \geq 2$, that is, roughly speaking, a set of particular G-functions satisfying a system of linear differential equations, under a hypothesis called the Galochkin condition or (G, C)-condition, given in Definition 2 of [Galochkin 1974] (this is the same as (G, C)-function condition in [Chudnovsky 1984] and the (G, C)-assumption in [Chudnovsky and Chudnovsky 1985]).

More significant progress was made by Chudnovsky [1984], who proved that, for G-functions satisfying a differential equation system as in (2), Galochkin's condition automatically holds.

Summing up, thanks to the above-mentioned results, as soon as we can show that the considered G-functions satisfy a linear system of differential equations as in (2), as well as that the functions are *linearly independent* over $\mathbb{C}(z)$, we get the linear independence of the special values *provided* condition (3) is satisfied. Condition (3) comes from the use of Siegel's lemma to construct Padé approximants (whereas we avoid using Siegel's lemma in the present article).

We are now in a position to compare our results with the above-mentioned series of results. Restraining ourselves to the functions 1, $\Phi_s(x, \alpha_i z)$, with $1 \leq i \leq m$, $1 \leq s \leq r$, one can check they are *linearly independent* over $\mathbb{C}(z)$, in a similar way to [Väänänen 1980, pp. 292, 293]; see [David et al. $\geq$ 2020a] (it may be worth noting that Galochkin's condition can be checked by hand in this special case, and thus one can also proceed without using Chudnovsky's observation). Hence, we are in the case $N = 1$; thus necessarily $d = 1$.

However, for $N = d = 1$, condition (3) reads $u = s(1 - \kappa) + 1 \leq -s + 1 < 0$ if $\kappa \geq 2$; hence the assumption $u > 0$ of Galochkin's theorem never holds when $N = 1$ as soon as the base field considered is *not contained* in an imaginary quadratic field.

On the contrary, our criterion covers also such a case, since the base field can be an arbitrary number field. Namely our result gives the linear independence of values of the Lerch functions, when $N = 1$, applying our explicit construction of Padé approximations of 1, $\Phi_s(x, \alpha_i/z)$ that is done around infinity, not around the origin (this is one of the reasons why our corresponding assumption is much weaker than that of Galochkin's theorem). Nevertheless, as we see in Example 6.3 below, our linear independence result for the values of Lerch functions is valid for algebraic points in K *of arbitrary degree*, to which neither Galochkin's [1974; 1975] nor Väänänen's [1980] results apply.

Remark 1.3. It is also worth noting that our result (see [David et al. $\geq$ 2020a] for details) is quantitative, with totally explicit constants, which is not the case of previous results.

Remark 1.4. A result of [Väänänen and Xu 1988] actually deals with general base fields as in our case. However, this is not applicable in our situation, because of the degenerate nature of the system (2).

The new ingredient in the article relies on a few points. First and foremost, we introduce a systematic construction of Padé approximants, which heavily relies on the computations made by past authors. Our modifications and generalizations of the method Nikishin [1979a; 1979b] developed, as well as of the Rhin–Toffin method [1986], supply *a formally regulated construction* of Padé approximants. Secondly an irrationality criterion, combined with the metric property provided for by Padé approximation, leads to the irrationality for the values of the Lerch functions at points sufficiently close to the origin (the precise sufficient condition, which we explain later, comes from the coupling of the criteria with Padé approximation). This strategy works only if one can ensure *the injectivity of evaluation maps* defined by systems of Padé approximation, which can be now interpreted as a nonvanishing property of a Hermite-type determinant, which we succeed in proving. Our criterion also gives much more relaxed assumptions than the previous results in [Galochkin 1974; 1975], since we rely on our new *formal* construction of explicit Padé approximants, by avoiding the use of Siegel's lemma.

2. Notation and main results

We fix an algebraic closure of $\mathbb{Q}$ and denote it by $\overline{\mathbb{Q}}$. For a finite subset $S \subset \overline{\mathbb{Q}}$, we define the denominator of S by

$$\mathrm{den}(S) := \min\{0 < n \in \mathbb{Z} \mid n\alpha \text{ is an algebraic integer for any } \alpha \in S\}.$$

Let $\mathbb{N}$ be the set of strictly positive integers. Let $m, r \in \mathbb{N}$ and K be an algebraic number field of finite degree over $\mathbb{Q}$. We denote the ring of integers of K by $\mathcal{O}_K$ and the completion of K with respect to the fixed embedding $\iota_\infty : K \hookrightarrow \mathbb{C}$ by K_∞. Then $[K_\infty : \mathbb{R}] = 1$ if $K_\infty \subset \mathbb{R}$, and $[K_\infty : \mathbb{R}] = 2$ otherwise.

Let $x \in \mathbb{Q} \cap [0, 1)$. Put

$$\mu(x) := \mathrm{den}(x) \prod_{q:\text{prime},\, q \mid \mathrm{den}(x)} q^{1/(q-1)}.$$

Consider $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_m) \in (K \setminus \{0\})^m$, with $\alpha_i \neq \alpha_j$ for all $1 \le i < j \le m$. For $1 \le g \le [K : \mathbb{Q}]$, we denote by $\alpha^{(g)}$ the g-th conjugate of $\alpha \in K$ over $\mathbb{Q}$.

Let $\beta \in K \setminus \{0\}$ with $\max_{1 \le i \le m}(|\alpha_i|) < |\beta|$. We put

$$D(\boldsymbol{\alpha}, \beta) := \mathrm{den}(\alpha_1, \dots, \alpha_m, \beta).$$

We also define

$$\begin{aligned}
\mathbb{A}(\boldsymbol{\alpha}, \beta, x) &:= \log|\beta| - (rm+1)\log\max_i(|\alpha_i|) \\
&\qquad - \Big\{rm\big(\log D(\boldsymbol{\alpha}, \beta) + r\big[\mathrm{den}(x) + \log\tfrac{5}{2}\big]\big) + r(\log 3 + \log\mu(x))\Big\},\\
\mathcal{A}^{(g)}(\boldsymbol{\alpha}, \beta, x) &:= rm(\log D(\boldsymbol{\alpha}, \beta) + \log\max(1, \min(|\alpha_i^{(g)}|)^{-1} \cdot |\beta^{(g)}|) + r[\mathrm{den}(x) - \log 2]) \\
&\qquad + r\left(\log\mu(x) + \sum_{i=1}^{m} \log(2^r|\alpha_i| + 3^r \max(|\alpha_i^{(g)}|, |\beta^{(g)}|))\right) + \log 3 \quad \text{for } 1 \le g \le [K : \mathbb{Q}],\\
V(\boldsymbol{\alpha}, \beta, x) &:= \mathbb{A}(\boldsymbol{\alpha}, \beta, x) + \mathcal{A}^{(1)}(\boldsymbol{\alpha}, \beta, x) - \frac{\sum_{g=1}^{[K:\mathbb{Q}]} \mathcal{A}^{(g)}(\boldsymbol{\alpha}, \beta, x)}{[K_\infty : \mathbb{R}]}.
\end{aligned}$$

We then obtain the following statement.

Theorem 2.1. *Assume $V(\boldsymbol{\alpha}, \beta, x) > 0$. Then the $rm+1$ numbers*

$$1, \Phi_1(x, \alpha_1/\beta), \dots, \Phi_r(x, \alpha_1/\beta), \dots, \Phi_1(x, \alpha_m/\beta), \dots, \Phi_r(x, \alpha_m/\beta)$$

are linearly independent over K.

In the special case where K equals $\mathbb{Q}$ or an imaginary quadratic field, Corollary 6 in [Väänänen 1980] gives an analogous quantitative result for polylogarithms, but the needed condition there is not so explicit as ours, $V(\alpha, \beta, 0) > 0$. For a general number field K, Theorem 2.1 is the first result to give the linear independence of the values of the Lerch function, even in the case of polylogarithms, at distinct algebraic numbers.

3. Construction of Padé approximants

We now explain how we construct Padé approximants of the Lerch functions. Since the full proof is long, it will be provided, with all relevant details, in the forthcoming articles [David et al. $\geq$ 2020a; $\geq$ 2020b], with a p-adic analogue as well as quantitative measures of linear independence.

First we recall the definition of Padé approximants of formal Laurent series. In the rest of this section, we denote by L a unique factorization domain of characteristic 0. We define the order function ord_∞ at $z=\infty$ by

$$\mathrm{ord}_\infty : L[z][[1/z]] \to \mathbb{Z}\cup\{\infty\}, \quad \sum_k a_k\cdot\frac{1}{z^k} \mapsto \min\{k\in\mathbb{Z} \mid a_k\neq 0\}.$$

Lemma 3.1. *Let r be a positive integer, $f_1(z),\dots,f_r(z)\in 1/z\cdot L[[1/z]]$ and $\boldsymbol{n}:=(n_1,\dots,n_r)\in\mathbb{N}^r$. Put $N:=\sum_{i=1}^r n_i$. Let M be a positive integer with $M\geq N$. Then there exists a family of polynomials $(P_0(z), P_1(z),\dots,P_r(z))\in L[z]^{r+1}\setminus\{\boldsymbol{0}\}$ satisfying the following conditions*:

(i) $\deg P_0(z)\leq M$.

(ii) $\mathrm{ord}_\infty\, P_0(z)f_j(z)-P_j(z)\geq n_j+1$ *for* $1\leq j\leq r$.

Definition 3.2. Using the notation of Lemma 3.1, we call a family of polynomials $(P_0(z),P_1(z),\dots,P_r(z))\in L[z]^{r+1}$ satisfying the properties (i) and (ii) *Padé-type approximants of* $(f_1,\dots,f_r)$ *of weight* $\boldsymbol{n}$ *and of degree* M.

For the Padé-type approximants $(P_0(z), P_1(z),\dots,P_r(z))$, of $(f_1,\dots,f_r)$ of weight $\boldsymbol{n}$, we call the family of formal Laurent series $(P_0(z)f_j(z)-P_j(z))_{1\leq j\leq r}$ *Padé-type approximation systems of* $(f_1,\dots,f_r)$, *of weight* $\boldsymbol{n}$ *and of degree* M.

In the sequel, we take $x\in L\setminus\mathbb{Z}_{<0}$ and assume $x+k$ are invertible in L for any $k\in\mathbb{N}$.

We now introduce notation for formal primitive, derivation, and evaluation maps. Let I be a finite set; we assume that L contains $K[X_i, 1/X_i]_{X_i\in I}$, where K is a number field. In the sequel, it will be convenient to work formally and thus to treat as many quantities as variables as is useful, and we shall freely extend the set I as need arises.

Notation 3.3. (i) For $\alpha\in L$, we denote by Eval_α the linear evaluation map $L[T]\to L,\ P\mapsto P(\alpha)$.

(ii) For $P\in L[T]$, we denote by $[P]$ the multiplication by P ($Q\mapsto PQ$).

(iii) We also denote by Prim_x (formal primitive) the linear operator $L[T]\to L[T]$, defined by

$$P\mapsto \frac{1}{T^{1+x}}\int_0^T \xi^x P(\xi)\,d\xi.$$

(iv) We denote by Deri_x the derivative map

$$P\mapsto T^{-x}\frac{d}{dT}(T^{x+1}P(T)),$$

and by $S_{n,x}$ for $n\geq 1$ the map taking

$$T^k\mapsto \frac{(k+x+1)_n}{n!}T^k,$$

where $(k+x+1)_n := (k+x+1)\cdots(k+x+n)$, that is, the divided derivative mapping

$$P \mapsto \frac{1}{n!}T^{-x}\frac{d^n}{dT^n}(T^{n+x}P) = \frac{1}{n!}\left(\frac{d}{dT}+\frac{x}{T}\right)^n T^n(P),$$

so that $\mathrm{Deri}_x = S_{1,x}$.

(v) If φ is an L-automorphism of an L-module M and k an integer, we define

$$\varphi^{(k)} := \begin{cases} \overbrace{\varphi\circ\cdots\circ\varphi}^{k\text{ times}} & \text{if } k>0, \\ \mathrm{id}_M & \text{if } k=0, \\ \underbrace{\varphi^{-1}\circ\cdots\circ\varphi^{-1}}_{-k\text{ times}} & \text{if } k<0. \end{cases}$$

For a given $l \in \mathbb{Z}$, we define the linear map $\varphi_{\alpha,x,l}$ as follows.

Notation 3.4.
$$\varphi_{\alpha,x,l} := [\alpha]\circ \mathrm{Eval}_\alpha \circ \mathrm{Prim}_x^{(l)}.$$

For any nonnegative integers k, note that $\varphi_{\alpha,x,s}(T^k)$ is a formal analogue of

$$\frac{1}{(s-1)!}\int_0^\alpha T^{k+x}\log^{s-1}\frac{1}{T}\,dT.$$

For convenience, we collect below the following elementary facts.

Facts 3.5. (i) The map Prim_x is an isomorphism and its inverse is Deri_x for $x \in L\setminus\mathbb{Z}_{<0}$. Hence $\varphi_{\alpha,x,s}$ is well-defined for $s \le -1$.

(ii) For any integers $n_1 \ge 0$, $n_2 \ge 0$ and $x \in L\setminus\mathbb{Z}_{<0}$ with $x+k$ invertible in L for any $k\in\mathbb{N}$, the divided derivatives $S_{n_1,x}$ and $S_{n_2,x}$ commute; namely $S_{n_1,x}\circ S_{n_2,x} = S_{n_2,x}\circ S_{n_1,x}$.

(iii) For any integer $s\in\mathbb{Z}$ and any $\alpha\in L$, we have $\varphi_{\alpha,x,s}\circ\mathrm{Deri}_x = \varphi_{\alpha,x,s-1}$.

(iv) By continuity, all the above-mentioned maps extend to $L[[T]]$ with respect to the natural valuation.

(v) The kernel of the map $\varphi_{\alpha,x,0}$ is the ideal $(T-\alpha)$ for any $x\in L\setminus\mathbb{Z}_{<0}$.

Using Facts 3.5(iv), the classical Lerch function is indeed expressed as a natural image by $\varphi_{\alpha,x,s}$, with $s\ge 1$, by

$$\varphi_{\alpha,x,s}\left(\frac{1}{z-T}\right) = \Phi_s\left(x,\frac{\alpha}{z}\right). \tag{4}$$

Consider $\boldsymbol{\alpha} := (\alpha_1,\dots,\alpha_m)\in(L\setminus\{0\})^m$ with $\alpha_i\ne\alpha_j$ for $i\ne j$. We study Padé approximants of type II of the functions $(\Phi_s(x,\alpha_i/z))_{1\le i\le m,\,1\le s\le r}$.

Let l be a nonnegative integer with $0\le l\le rm$. For a positive integer n, we define the family of polynomials

$$P_{n,l}(\boldsymbol{\alpha},x\mid z) := \mathrm{Eval}_z\circ S_{n,x}^{(r)}\left(T^l\prod_{i=1}^m (T-\alpha_i)^{rn}\right), \tag{5}$$

$$P_{n,l,i,s}(\boldsymbol{\alpha},x\mid z) := \varphi_{\alpha_i,x,s}\left(\frac{P_{n,l}(\boldsymbol{\alpha},x\mid z)-P_{n,l}(\boldsymbol{\alpha},x\mid T)}{z-T}\right)\quad \text{for } 1\le i\le m,\ 1\le s\le r. \tag{6}$$

Under the notation above, we obtain the following theorem.

Theorem 3.6. *For each* $0 \leq l \leq rm$, *the family of polynomials* $(P_{n,l}(\boldsymbol{\alpha}, x \mid z),\, P_{n,l,i,s}(\boldsymbol{\alpha}, x \mid z))_{1\leq i\leq m,\, 1\leq s\leq r}$ *forms a Padé-type approximants system of* $(\Phi_s(x, \alpha_i/z))_{1\leq i\leq m,\, 1\leq s\leq r}$ *of weight* $(n, \dots, n) \in \mathbb{N}^{rm}$ *and of degree* $rmn + l$.

Proof. By the definition of $P_{n,l}(\boldsymbol{\alpha}, x \mid z)$, we have

$$\deg P_{n,l}(\boldsymbol{\alpha}, x \mid z) = rmn + l.$$

Hence the condition on the degree is verified. We only need to check the condition on the valuation.

Put

$$R_{n,l,i,s}(\boldsymbol{\alpha}, x \mid z) := P_{n,l}(\boldsymbol{\alpha}, x \mid z)\Phi_s(x, \alpha_i/z) - P_{n,l,i,s}(\boldsymbol{\alpha}, x \mid z).$$

Then, by the definition of $R_{n,l,i,s}(\boldsymbol{\alpha}, x \mid z)$ with property (4), we obtain

$$\begin{aligned} R_{n,l,i,s}(\boldsymbol{\alpha}, x \mid z) &= P_{n,l}(\boldsymbol{\alpha}, x \mid z)\varphi_{\alpha_i,x,s}\left(\frac{1}{z-T}\right) - P_{n,l,i,s}(\boldsymbol{\alpha}, x \mid z) \\ &= \varphi_{\alpha_i,x,s}\left(\frac{P_{n,l}(\boldsymbol{\alpha}, x \mid T)}{z-T}\right) = \sum_{k=0}^{\infty} \varphi_{\alpha_i,x,s}(T^k P_{n,l}(\boldsymbol{\alpha}, x \mid T))\frac{1}{z^{k+1}}. \end{aligned} \tag{7}$$

Note that in $\mathrm{End}_K(K[T])$ we have the identities

$$\begin{aligned} S_{n,x} &= \frac{1}{n!} S_{1,x} \circ \cdots \circ (S_{1,x} + n - 1) &&\text{for } n \in \mathbb{N}, \\ [T^k] \circ S_{1,x} &= (S_{1,x} - k) \circ [T^k] &&\text{for } k \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

By the definition of $P_{n,l}(\boldsymbol{\alpha}, x \mid T)$ and the identities above, for each $1 \leq s \leq r,\ 0 \leq k \leq n-1$, there exists a polynomial $U_{s,k}(X) \in \mathbb{Q}[X]$ of $\deg U_{s,k} = nr - s$, satisfying

$$T^k P_{n,l}(\boldsymbol{\alpha}, x \mid T) = S_{1,x}^{(s)} \circ U_{s,k}(S_{1,x})\left(T^{k+l}\prod_{i=1}^{m}(T-\alpha_i)^{rn}\right).$$

By the Leibniz rule, we obtain that $U_{s,k}(S_{1,x})\big(T^{k+l}\prod_{i=1}^{m}(T-\alpha_i)^{rn}\big)$ belongs to the ideal $(T-\alpha_i)$ for each $1 \leq i \leq m$. Hence we get

$$\varphi_{\alpha_i,x,s}(T^k P_{n,l}(\boldsymbol{\alpha}, x \mid T)) = \varphi_{\alpha_i,x,0} \circ U_{s,k}(S_{1,x})\left(T^{k+l}\prod_{i=1}^{m}(T-\alpha_i)^{rn}\right) = 0$$

for $1 \leq i \leq m,\ 1 \leq s \leq r$ and $0 \leq k \leq n-1$.

Consequently, by the expansion above of $R_{n,l,i,s}(\boldsymbol{\alpha}, x \mid z)$, we obtain

$$\mathrm{ord}_{\infty} R_{n,l,i,s}(\boldsymbol{\alpha}, x \mid z) \geq n+1 \quad \text{for } 1 \leq i \leq m,\ 1 \leq s \leq r.$$

Then Theorem 3.6 follows. □

4. Metric approximations and linear independence criteria

We now give a few of the estimates associated with the Padé approximation we just constructed. They do not need involved arguments to be proven; however, due to the technical nature of the construction, computations are somewhat heavy and we skip them to keep in line with the spirit of this article.

The estimates in Lemma 4.1 can be combined with an appropriate linear independence criterion to provide for a measure.

Lemma 4.1. *Let n be a positive integer, x a rational number with $0 \le x < 1$ and $\beta \in K \setminus \{0\}$. Then for any $1 \le g \le [K:\mathbb{Q}]$, we have*

$$\max_{\substack{0\le l\le rm\\1\le i\le m\\1\le s\le r}} |P^{(g)}_{n,l,i,s}(\boldsymbol{\alpha}, x \mid \beta)| \le \max(|\alpha_i^{(g)}|)^{rm}\left(\frac{3}{2}\right)^{r^2m+r}\left(\frac{3}{2^{rm}}\prod_{j=1}^{m}[2^r|\alpha_j^{(g)}|+3^r\max(|\alpha_i^{(g)}|)]\right)^{rn} \times \begin{cases} \dfrac{(\min(|\alpha_i^{(g)}|)^{-1}|\beta^{(g)}|)^{rm(n+1)}}{\min(|\alpha_i^{(g)}|)^{-1}|\beta^{(g)}|-1} & \text{if } \min(|\alpha_i^{(g)}|)^{-1}|\beta^{(g)}|>1,\\ rm(n+1) & \text{if } \min(|\alpha_i^{(g)}|)^{-1}|\beta^{(g)}|\le 1\end{cases}$$

for $1 \le i \le m$.

For the error term, we have

$$\max_{0\le l\le rm}|R_{n,l,i,s}(\boldsymbol{\alpha}, x\mid\beta)| \le \frac{\max_{1\le i\le m}(1,|\alpha_i|)^{rm+1}}{|\beta|-\max_j(|\alpha_j|)}\left(\frac{3}{2}\right)^{r^2m+r}\left(\left(\frac{\max_j(|\alpha_j|)^{rm+1}}{|\beta|}\right)^n\left(3\left(\frac{5}{2}\right)^{rm}\right)\right)^{rn}.$$

We give here an outline of the proof. By (5) and (6), we have

$$P_{n,l}(\boldsymbol{\alpha}, x\mid z)=\sum_{k=0}^{rmn}\Bigg[\sum_{\substack{1\le i\le m\\0\le k_i\le rn\\\sum_i k_i=k}}\left(\prod_{i=1}^{m}\binom{rn}{k_i}(-\alpha_i)^{rn-k_i}\right)\Bigg]\left(\frac{(k+l+x+1)_n}{n!}\right)^r z^{k+l},$$

$$P_{n,l,i,s}(\boldsymbol{\alpha}, x\mid z)=\sum_{u=\max(l,1)-1}^{rmn+l-1}\Bigg[\sum_{k=u+1}^{rmn+l}\Bigg(\sum_{\substack{1\le i'\le m\\0\le k_{i'}\le rn\\\sum_{i'}k_{i'}=k-l}}\prod_{i'=1}^{m}\binom{rn}{k_{i'}}(-\alpha_{i'})^{rn-k_{i'}}\Bigg)\left(\frac{(1+k+x)_n}{n!}\right)^r\frac{\alpha_i^{k-u}}{(k-u)^s}\Bigg]z^u.$$

By the above equalities together with the triangle inequality, we obtain the upper bounds for $|P^{(g)}_{n,l,i,s}(\boldsymbol{\alpha}, x\mid\beta)|$ and $|P_{n,l}(\boldsymbol{\alpha}, x\mid\beta)|$.

For the term $|R_{n,l,i,s}(\boldsymbol{\alpha}, x\mid\beta)|$, we use (7).

We then state a general linear independence criterion:

Proposition 4.2. *Let K be an algebraic number field of finite degree over $\mathbb{Q}$. We denote the completion of K with respect to the fixed embedding ι_∞ by K_∞. Let $m \in \mathbb{N}$ and $\theta_0 := 1, \theta_1, \dots, \theta_m \in \mathbb{C}\setminus\{0\}$.*

Suppose that there exists a set of matrices

$$\{(A_{n,l,j})_{0\le l,j\le m}\}_{n\in\mathbb{N}} \subset \mathrm{M}_{m+1}(\mathcal{O}_K)\cap \mathrm{GL}_{m+1}(K).$$

Assume further that there exist positive real numbers

$$\{\mathcal{A}^{(g)}\}_{1\le g\le[K:\mathbb{Q}]}$$

and a positive real number $\mathbb{A}$ *satisfying the conditions*

$$\max_{0\le l,j\le m}|A^{(g)}_{n,l,j}|\le e^{\mathcal{A}^{(g)}\cdot n+o(n)}\quad \text{for}\ \ 1\le g\le[K:\mathbb{Q}]\ (n\to\infty), \tag{8}$$

$$\max_{\substack{0\le l\le m\\ 1\le j\le m}}|A_{n,l,0}\cdot\theta_j-A_{n,l,j}|\le e^{-\mathbb{A}\cdot n+o(n)}\qquad (n\to\infty). \tag{9}$$

We put

$$V:=\mathbb{A}+\mathcal{A}^{(1)}-\frac{\sum_{g=1}^{[K:\mathbb{Q}]}\mathcal{A}^{(g)}}{[K_\infty:\mathbb{R}]}.$$

If $V>0$, *then the numbers* $\theta_0,\dots,\theta_m$ *are linearly independent over* K.

Proof. Assume that there exists a vector $\boldsymbol{\beta}:=(\beta_0,\dots,\beta_m)\in\mathcal{O}_K\setminus\{\mathbf{0}\}$ satisfying $\Lambda(\boldsymbol{\beta},\boldsymbol{\theta}):=\sum_{i=0}^m\beta_i\theta_i=0$. For $n\in\mathbb{N}$, as we have $\det(A_{n,l,j})_{0\le l,j\le m}\neq 0$, there exists $0\le l_n\le m$ satisfying

$$B_{l_n}:=\sum_{j=0}^m A_{n,l_n,j}\beta_j\neq 0. \tag{10}$$

Put $R_{n,l,j}=A_{n,l,0}\theta_j-A_{n,l,j}$ for $1\le j\le m$ and $0\le l\le m$. Then by the definitions of $\Lambda(\boldsymbol{\beta},\boldsymbol{\theta})$, B_{l_n}, and $R_{n,l,j}$, we obtain

$$0=A_{n,l_n,0}\Lambda(\boldsymbol{\beta},\boldsymbol{\theta})=B_{l_n}+\sum_{j=1}^m R_{n,l_n,j}\beta_j.$$

Using the product formula for $B_{l_n}\in\mathcal{O}_K\setminus\{0\}$, it follows that

$$1\le{\prod_g}'|B^{(g)}_{l_n}|\times|B_{l_n}|^{[K_\infty:\mathbb{R}]}={\prod_g}'|B^{(g)}_{l_n}|\times\left|\sum_{j=1}^m R_{n,l_n,j}\beta_j\right|^{[K_\infty:\mathbb{R}]}. \tag{11}$$

Here $'$ in $\prod'_g$ means that g satisfies $2\le g\le[K:\mathbb{Q}]$ if $K_\infty=\mathbb{R}$ and $3\le g\le[K:\mathbb{Q}]$ if $K_\infty=\mathbb{C}$. Firstly, we look for an upper bound of $|B^{(g)}_{l_n}|$ for $g\neq 1$ if $K_\infty=\mathbb{R}$ and $g\neq 1,2$ if $K_\infty=\mathbb{C}$. Using inequality (8), we have

$$|B^{(g)}_{l_n}|\le e^{\mathcal{A}^{(g)}n+o(n)}\quad (n\to\infty). \tag{12}$$

Secondly, we give an upper bound for $\left|\sum_{j=1}^m R_{n,l_n,j}\beta_j\right|$. By (9), we get

$$\left|\sum_{j=1}^m R_{n,l_n,j}\beta_j\right|\le e^{-\mathbb{A}n+o(n)}\quad (n\to\infty). \tag{13}$$

Substituting the inequalities (12) and (13) into inequality (11), taking the $1/[K_\infty:\mathbb{R}]$-th power of the inequality, we obtain

$$1\le e^{-Vn+o(n)}\quad (n\to\infty).$$

Since $V>0$, we arrive at a contradiction for this inequality for all sufficiently large $n\in\mathbb{N}$. □

Theorem 3.6 gives us the sequence of matrices. The growth control of the size of the matrices to carry out the approximations is provided for in Lemma 4.1. However, the matrices do not always have algebraic integer entries. This is not a big deal. The defect of integrality comes from our operators Prim_x, Deri_x and it is corrected by multiplying by a suitable power of the least common multiple $d_n := \mathrm{lcm}(1, \dots, n)$, which is standard in the theory.

Plugging in these estimates in Proposition 4.2 leads us to the proof of the main theorem. The metric condition requiring the numbers to be sufficiently close to the origin is translated to the condition $V > 0$ in the linear independence criterion Proposition 4.2.

However, there is still a significant step to be performed. Now we need to prove that the matrices coming from the Padé approximation are indeed invertible. We describe this main step in the next section.

5. Nonvanishing of a determinant and the final step of the proof

In this section, we use the following notation. Let m, r be positive integers and K be a field with characteristic 0. We assume that $\alpha_1, \dots, \alpha_m, z, T$ all belong to the set of variables I, so our ring L contains $K[\alpha_i, z, T, 1/\alpha_i, 1/z, 1/T]$. Put $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_m)$.

For a positive integer l with $0 \le l \le rm$, and for $x \in K$, we put

$$P_{n,l}(z) := P_{n,l}(\boldsymbol{\alpha}, x \mid z),$$
$$P_{n,l,i,s}(z) := P_{n,l,i,s}(\boldsymbol{\alpha}, x \mid z) \quad \text{for } 1 \le i \le m,\ 1 \le s \le r.$$

The polynomials in the right-hand sides above have been already defined in (5) and (6) respectively.

We define a column vector $\vec{p}_{n,l}(z) \in K[z]^{rm+1}$ by

$$\vec{p}_{n,l}(z) := {}^t(P_{n,l}(z), P_{n,l,1,1}(z), \dots, P_{n,l,1,r}(z), \dots, P_{n,l,m,1}(z), \dots, P_{n,l,m,r}(z)).$$

Proposition 5.1. *We use the same notation as above. For any positive integer n, we have*

$$\Delta_n(z) := \det(\vec{p}_{n,0}(z) \cdots \vec{p}_{n,rm}(z)) \in K(\alpha_1, \alpha_2, \dots, \alpha_m) \setminus \{0\}.$$

To prove this, we firstly prove that the determinant $\Delta_n = \Delta_n(z)$ is a constant, i.e., is independent of z. Secondly, we regard Δ_n as an element of $K(\alpha_1, \dots, \alpha_m)$ viewing $\alpha_1, \dots, \alpha_m$ as indeterminates, and factor it up to a constant depending only on n, m, r. We finally show that this absolute constant Δ_n is nonzero. For this last step, we identify this determinant with a certain real integral to show that it does not vanish.

We shall prove

$$\Delta_n(z) \in K(\alpha_1, \dots, \alpha_m) \quad \text{for all } n \in \mathbb{N}.$$

For this, denote $P_{n,l}(z)\Phi_s(x, \alpha_i/z) - P_{n,l,i,s}(z)$ by $R_{n,l,i,s}(z)$ as above ($0 \le l \le rm,\ 1 \le i \le m,\ 1 \le s \le r$).

In the matrix giving the determinant $\Delta_n(z)$, we add, the first row multiplied by the $\Phi_s(x, \alpha_i/z)$, to the $((i-1)r + s + 1)$-th row ($1 \le i \le m$, $1 \le s \le r$), to obtain

$$\Delta_n(z) = (-1)^{rm} \det \begin{pmatrix} P_{n,0}(z) & \dots & P_{n,rm}(z) \\ R_{n,0,1,1}(z) & \dots & R_{n,rm,1,1}(z) \\ \vdots & \ddots & \vdots \\ R_{n,0,1,r}(z) & \dots & R_{n,rm,1,r}(z) \\ \vdots & \ddots & \vdots \\ R_{n,0,m,1}(z) & \dots & R_{n,rm,m,1}(z) \\ \vdots & \ddots & \vdots \\ R_{n,0,m,r}(z) & \dots & R_{n,rm,m,r}(z) \end{pmatrix}.$$

We denote by $\Delta_{n,s,t}(z)$, the (s,t)-th cofactor of the matrix in the right-hand side of the identity above. Then we have, developing along the first row

$$\Delta_n(z) = (-1)^{rm}\left(\sum_{l=0}^{rm} P_{n,l}(z)\Delta_{n,1,l+1}(z)\right). \tag{14}$$

Since we have

$$\mathrm{ord}_\infty R_{n,l,i,s}(z) \geq n+1 \quad \text{for } 0 \leq l \leq rm,\ 1 \leq i \leq m \text{ and } 1 \leq s \leq r,$$

we get

$$\mathrm{ord}_\infty \Delta_{n,1,l+1}(z) \geq (n+1)rm.$$

Combining the fact $\deg P_{n,l}(z) = rmn + l$ with the lower bound of $\mathrm{ord}_\infty \Delta_{n,1,l+1}(z)$ above, we obtain

$$P_{n,l}(z)\Delta_{n,1,l+1}(z) \in 1/z \cdot K[\![1/z]\!] \quad \text{for } 0 \leq l \leq rm-1,$$
$$P_{n,rm}(z)\Delta_{n,1,rm+1}(z) \in K[\![1/z]\!].$$

Note that in the relation above, the constant term of $P_{n,rm}(z)\Delta_{n,1,rm+1}(z)$ is

$$(\text{coefficient of } z^{rm(n+1)} \text{of } P_{n,rm}(z)) \times (\text{coefficient of } 1/z^{rm(n+1)} \text{of } \Delta_{n,1,rm+1}(z)).$$

Thus by (14), the determinant $\Delta_n(z)$ is a polynomial in z with nonpositive valuation with respect to ord_∞; consequently it turns to be a constant. Moreover, the terms of strictly negative valuation should be canceled out. Hence we have

$$\begin{aligned}\Delta_n = \Delta_n(z) &= (-1)^{rm}\left(\sum_{l=0}^{rm} P_{n,l}(z)\Delta_{n,1,l+1}(z)\right)\\ &= (-1)^{rm}(\text{constant term of } P_{n,rm}(z)\Delta_{n,1,rm+1}(z)) \in K. \end{aligned} \tag{15}$$

We now need to rewrite Δ_n as a rational function of $\alpha_1, \dots, \alpha_m$ in a workable way. We further extend the set of variables and assume that the set I contains the rm variables $t_{i,s},\ 1 \leq i \leq m,\ 1 \leq s \leq r$, so that L contains

$$K[\alpha_1, \dots, \alpha_m, z, T, 1/\alpha_1, \dots, 1/\alpha_m, 1/z, 1/T][t_{i,s}].$$

For each variable $t_{i,s}$ and any integer l, we have a well-defined map for $\alpha \in L$,

$$\varphi_{\alpha,t_{i,s},x,l} : L[t_{i,s}]_{1\leq i\leq m, 1\leq s\leq r} \to L[t_{i',s'}]_{(i',s')\neq(i,s)}, \quad t_{i,s}^k \mapsto \frac{\alpha^{k+1}}{(k+x+1)^l}$$

since $L[t_{i,s}]_{1\leq i\leq m,1\leq s\leq r}$ can be regarded as a polynomial ring in one variable $L'[t_{i,s}]$ over $L' = L[t_{i',s'}]_{(i',s')\neq(i,s)}$.

Now for a positive integer n and an integer l with $0 \leq l \leq rm$, we put

$$A_{n,l}(T) := T^l \prod_{i=1}^m (T-\alpha_i)^{rn}.$$

By the definition of $A_{n,l}(T)$, we have $P_{n,l}(z) = \mathrm{Eval}_z \circ S_{n,x}^{(r)}(A_{n,l}(T))$.

Let us define a column vector $\vec{r}_{n,l} \in L^{rm}$ by

$$\vec{r}_{n,l} := {}^t\big(\varphi_{\alpha_1,t_{1,1},x,1}(t_{1,1}^n A_{n,l}(t_{1,1})), \dots, \varphi_{\alpha_1,t_{1,r},x,r}(t_{1,r}^n A_{n,l}(t_{1,r})), \dots, \varphi_{\alpha_m,t_{m,1},x,1}(t_{m,1}^n A_{n,l}(t_{m,1})), \dots, \varphi_{\alpha_m,t_{m,r},x,r}(t_{m,r}^n A_{n,l}(t_{m,r}))\big).$$

Lemma 5.2. *Under the notation above, we obtain the identity*

$$\Delta_n = (-1)^{rmn}\left(\frac{(1+rmn+rm+x)_n}{n!}\right)^r \det(\vec{r}_{n,0}\cdots\vec{r}_{n,rm-1}).$$

Proof. Using (15), we calculate constant term of $P_{n,rm}(z)\Delta_{n,1,rm+1}(z) \in K[[1/z]]$.

We need to deal with the noncommutativity of the multiplication by $[T]$ and the morphisms $S_{n,x}^{(k)}$. The defect of the commutativity is given by the following identity: there exists a set of rational numbers $\{e_{n,k}\}_{0\le k\le rn} \subset \mathbb{Q}$ with $e_{n,0} = (-1)^{rn}$ and

$$[T^n]\circ S_{n,x}^{(r)} = \sum_{k=0}^{rn} e_{n,k} S_{1,x}^{(k)} \circ [T^n].$$

Then we obtain

$$\begin{aligned}\varphi_{\alpha_i,x,s}(T^n P_{n,l}(T)) &= \sum_{k=0}^{rn} e_{n,k}\varphi_{\alpha_i,x,s}\circ S_{1,x}^{(k)}\circ [T^n](A_{n,l}(T))\\ &= \sum_{k=0}^{s-1} e_{n,k}\varphi_{\alpha_{i,x,s-k}}\circ [T^n](A_{n,l}(T)) + \sum_{k=s}^{rn} e_{n,k}\varphi_{\alpha_i,x,0}\circ S_{1,x}^{(k-s)}\circ [T^n](A_{n,l}(T))\\ &= \sum_{k=0}^{s-1} e_{n,k}\varphi_{\alpha_{i,x,s-k}}(T^n A_{n,l}(T))\end{aligned}$$

for $1 \le i \le m$ and $1 \le s \le r$; the conclusion follows, interpreting the above relations as linear manipulations of lines and columns leaving the determinant unchanged. □

Now, for nonnegative integers u, n, we put

$$P_{u,n}(t_{i,s}) = \prod_{i=1}^m \prod_{s=1}^r \left[t_{i,s}^u \prod_{j=1}^m (t_{i,s}-\alpha_j)^{rn}\right] \prod_{(i_1,s_1)<(i_2,s_2)} (t_{i_2,s_2} - t_{i_1,s_1}),$$

where the order $(i_1, s_1) < (i_2, s_2)$ follows lexicographically.

By $\bigcirc$, we denote the composite of morphisms. When no confusion is deemed to occur, we omit the subscripts $\underline{\alpha} = (\alpha_1, \dots, \alpha_m)$ and write

$$\psi = \psi_{\underline{\alpha}} := \bigcirc_{i=1}^m \bigcirc_{s=1}^r \varphi_{\alpha_i,t_{i,s},x,s}.$$

Note that, by definition of $\det(\vec{r}_{n,0}\cdots\vec{r}_{n,rm-1})$, we have

$$\det(\vec{r}_{n,0}\cdots\vec{r}_{n,rm-1}) = \psi(P_{n,n}).$$

Let u be a nonnegative integer. We are going to study the value

$$C_{n,u,m} := \psi(P_{u,n}).$$

By induction, we obtain the following proposition.

Proposition 5.3. *There exists a nonzero constant $c_{n,u,m} \in K$ satisfying*

$$C_{n,u,m} = c_{n,u,m} \prod_{i=1}^{m} \alpha_i^{r(u+1)+r^2n+\binom{r}{2}} \prod_{1 \le i_1 < i_2 \le m} (\alpha_{i_2} - \alpha_{i_1})^{(2n+1)r^2},$$

with $\binom{r}{2} = 0$ if $r = 1$.

We write the details of the proof of the proposition in the forthcoming articles [David et al. $\ge$ 2020a; $\ge$ 2020b]; however, we describe here our basic idea. Indeed, we prove the proposition by reducing to the case $m = 2$ and showing:

(i) $C_{n,u,2}$ is homogeneous of degree $2r(u+1) + 2r^2n + 2\binom{r}{2} + (2n+1)r^2$.

(ii) $(\alpha_1\alpha_2)^{r(u+1)+r^2n+\binom{r}{2}}$ divides $C_{n,u,2}$.

(iii) $(\alpha_1 - \alpha_2)^{(2n+1)r^2}$ divides $C_{n,u,2}$.

Here, we explain how the constant $c_{n,u,m}$ in Proposition 5.3 becomes nonzero. Whenever it is shown, then the determinant does not vanish.

We use the same notation as in Proposition 5.3. Define

$$D_{n,u,m} := \frac{C_{n,u,m}}{\prod_{i=1}^{m} \alpha_i^{r(u+1)+r^2n+\binom{r}{2}}} = c_{n,u,m} \times \prod_{1 \le i_1 < i_2 \le m} (\alpha_{i_2} - \alpha_{i_1})^{(2n+1)r^2}.$$

A straightforward calculation of an integral gives us

$$D_{n,u,m} = \bigcirc_{i'=1}^{m} \bigcirc_{s'=1}^{r} \varphi_{1,t_{i',s'},x,s'} \Biggl(\prod_{i=1}^{m} \prod_{s=1}^{r} \Biggl[t_{i,s}^{u} \cdot (t_{i,s}-1)^{rn} \cdot \prod_{\substack{\tilde{i} \ne i \\ 1 \le \tilde{i} \le m}} (\alpha_i t_{i,s} - \alpha_{\tilde{i}})^{rn} \Biggr] \times \prod_{i=1}^{m} \Biggl(\prod_{s_1 < s_2} (t_{i,s_2} - t_{i,s_1}) \Biggr) \prod_{i_1 < i_2} \prod_{1 \le s_1, s_2 \le r} (\alpha_{i_2} t_{i_2,s_2} - \alpha_{i_1} t_{i_1,s_1}) \Biggr).$$

We substitute $\alpha_m = 0$ in $D_{n,u,m}$; then we have

$$\begin{aligned} D_{n,u,m}|_{\alpha_m=0} &= c_{n,u,m} \prod_{i=1}^{m-1} (-\alpha_i)^{(2n+1)r^2} \prod_{1 \le i_1 < i_2 \le m-1} (\alpha_{i_2} - \alpha_{i_1})^{(2n+1)r^2} \\ &= \pm \prod_{i=1}^{m-1} \alpha_i^{(2n+1)r^2} \bigcirc_{s'=1}^{r} \varphi_{1,t_{m,s'},x,s'} \Biggl(\prod_{s=1}^{r} [t_{m,s}^{u} \cdot (t_{m,s}-1)^{rn}] \times \prod_{1 \le s_1 < s_2 \le r} (t_{m,s_2} - t_{m,s_1}) \Biggr) \\ &\quad \times \bigcirc_{i'=1}^{m-1} \bigcirc_{s'=1}^{r} \varphi_{1,t_{i',s'},x,s'} \Biggl(\prod_{i=1}^{m-1} \prod_{s=1}^{r} \Biggl[t_{i,s}^{u+r(n+1)} \cdot (t_{i,s}-1)^{rn} \cdot \prod_{\substack{\tilde{i} \ne i \\ 1 \le \tilde{i} \le m-1}} (\alpha_i t_{i,s} - \alpha_{\tilde{i}})^{rn} \Biggr] \\ &\quad \times \prod_{i=1}^{m-1} \Biggl(\prod_{1 \le s_1 < s_2 \le r} (t_{i,s_2} - t_{i,s_1}) \Biggr) \times \prod_{1 \le i_1 < i_2 \le m-1} \prod_{1 \le s_1, s_2 \le r} (\alpha_{i_2} t_{i_2,s_2} - \alpha_{i_1} t_{i_1,s_1}) \Biggr) \\ &= \pm \prod_{i=1}^{m-1} \alpha_i^{(2n+1)r^2} \bigcirc_{s'=1}^{r} \varphi_{1,t_{m,s'},x,s'} \Biggl(\prod_{s=1}^{r} [t_{m,s}^{u} \cdot (t_{m,s}-1)^{rn}] \times \prod_{1 \le s_1 < s_2 \le r} (t_{m,s_2} - t_{m,s_1}) \Biggr) \\ &\quad \times c_{n,u+r(n+1),m-1} \prod_{1 \le i_1 < i_2 \le m-1} (\alpha_{i_2} - \alpha_{i_1})^{(2n+1)r^2}. \end{aligned}$$

Thus we obtain

$$c_{n,u,m} = \pm \bigcirc_{s'=1}^{r} \varphi_{1,t_{m,s'},x,s'} \Bigg(\prod_{s=1}^{r} [t_{m,s}^{u} \cdot (t_{m,s} - 1)^{rn}] \times \prod_{1 \le s_1 < s_2 \le r} (t_{m,s_2} - t_{m,s_1}) \Bigg) c_{n,u+r(n+1),m-1}$$
$$= \pm \prod_{i=1}^{m} \Bigg(\bigcirc_{s'=1}^{r} \varphi_{1,t_{s'},x,s'} \Bigg(\prod_{s=1}^{r} [t_s^{u+(i-1)r(n+1)} \cdot (t_s - 1)^{rn}] \times \prod_{1 \le s_1 < s_2 \le r} (t_{s_2} - t_{s_1}) \Bigg) \Bigg).$$

We are then in a position to conclude. Indeed, using the definition of the operators $\varphi_{1,t_s,x,s}$, the composition of these operators is nothing but an integral over $[0, 1]^r$. More precisely,

$$\bigcirc_{s'=1}^{r} \varphi_{1,t_{s'},x,s'} \Bigg(\prod_{s=1}^{r} [t_s^{u} \cdot (t_s - 1)^{rn}] \cdot \prod_{1 \le s_1 < s_2 \le r} (t_{s_2} - t_{s_1}) \Bigg)$$
$$= \prod_{s'=1}^{r} \frac{1}{(s'-1)!} \int_0^1 \cdots \int_0^1 \prod_{s=1}^{r} \Bigg[t_s^{u+x} (t_s - 1)^{rn} \log^{s-1} \frac{1}{t_s} \Bigg] \cdot \prod_{1 \le s_1 < s_2 \le r} (t_{s_2} - t_{s_1}) \prod_{s=1}^{r} dt_s;$$

then a direct computation enables us to show this last integral does not vanish, which yields Proposition 5.1.

The statement of Theorem 2.1 now follows from Proposition 4.2, since the determinant is a nonvanishing algebraic constant.

6. Examples

We show here three examples of linearly independent polylogarithms, which are shown by our criterion.

Example 6.1. Put $r = m = 10$ and $x = 0$. Take $\boldsymbol{\alpha} := \big(1, \frac{1}{2}, \ldots, \frac{1}{10}\big)$ and $\beta = b$ with $|b| \ge e^{2715}$. Then we have $D(\boldsymbol{\alpha}, b) = d_{10} = 2520$. Since we have the inequalities

$$\log 2520 < 7.84, \quad \log 3 < 1.10, \quad \log \tfrac{5}{2} < 0.92,$$

we have

$$\log |b| > 100\big(10 + \log 2520 + 10 \log \tfrac{5}{2}\big) + 10 \log 3.$$

Then the $10^2 + 1$ numbers

$$1,\ \mathrm{Li}_1(1/b), \ldots, \mathrm{Li}_{10}(1/b), \ldots, \mathrm{Li}_1(1/(10b)), \ldots, \mathrm{Li}_{10}(1/(10b))$$

are linearly independent over $\mathbb{Q}$.

Example 6.2. Let $k \ge 2$ be an integer, set $r = m = 10^k$, $x = 0$. Take $\boldsymbol{\alpha} := (j)_{1 \le j \le 10^k}$ and $\beta = b \in \mathbb{Z}$ with $|b| \ge \exp(2 \cdot 10^{3k})$. Since $k \ge 2$, we can easily verify

$$\log |b| > (rm + 1) \log 10^k + \big(r^2 m \big(1 + \log \tfrac{5}{2}\big) + r \log 3\big)$$
$$= k(10^{2k} + 1) \log 10 + \big(10^{3k} \big(1 + \log \tfrac{5}{2}\big) + 10^k \log 3\big).$$

Then the $10^{2k} + 1$ numbers

$$1,\ \mathrm{Li}_1(1/b), \ldots, \mathrm{Li}_{10^k}(1/b), \ldots, \mathrm{Li}_1(10^k/b), \ldots, \mathrm{Li}_{10^k}(10^k/b)$$

are linearly independent over $\mathbb{Q}$. For instance, we take $r = m = 10^4$ and $b = 3^{2\cdot 10^{12}}$ then the $10^8 + 1$ numbers

$$1, \mathrm{Li}_1(1/3^{2\cdot 10^{12}}), \dots, \mathrm{Li}_{10^4}(1/3^{2\cdot 10^{12}}), \dots, \mathrm{Li}_1(10^4/3^{2\cdot 10^{12}}), \dots, \mathrm{Li}_{10^4}(10^4/3^{2\cdot 10^{12}}),$$

are all linearly independent over $\mathbb{Q}$.

Example 6.3. Let $M \geq 5$ be a natural number. Define the polynomial

$$f_M(X) := \Big(2 + \frac{1}{M}\Big)X^2 - 2X + \frac{2}{M}.$$

Then $X = (M \pm \sqrt{M^2 - 4M - 2})/(2M+1)$ are roots of $f_M(X)$.

Put

$$\beta := \frac{2M+1}{M - \sqrt{M^2 - 4M - 2}},$$

$K := \mathbb{Q}(\beta)$ and $\delta := e^{7908}$. We take $r = m = 10$, $\boldsymbol{\alpha} := \big(1, \frac{1}{2}, \dots, \frac{1}{10}\big)$ and

$$M > \frac{2\delta^2 - \delta + 1 + \sqrt{4\delta^4 + 4\delta^3 - 3\delta^2 - 6\delta + 5}}{4\delta - 4}.$$

Then we have

$$V(\boldsymbol{\alpha}, \beta, 0) = \mathbb{A}(\boldsymbol{\alpha}, \beta, 0) - \mathcal{A}^{(2)}(\boldsymbol{\alpha}, \beta, 0) > \log|\beta| - 7908 > 0.$$

Thus by Theorem 2.1, the $10^2 + 1$ numbers

$$1,\ \mathrm{Li}_1(1/\beta), \dots, \mathrm{Li}_{10^k}(1/\beta), \dots, \mathrm{Li}_1(1/10\beta), \dots, \mathrm{Li}_{10^k}(1/10\beta),$$

are linearly independent over K. For example,we take $M \geq e^{15817}$, the $10^2 + 1$ numbers

$$1,\ \mathrm{Li}_1(1/\beta), \dots, \mathrm{Li}_{10^k}(1/\beta), \dots, \mathrm{Li}_1(1/10\beta), \dots, \mathrm{Li}_{10^k}(1/10\beta),$$

are linearly independent over K.

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The irrationality measure of π is at most 7.103205334137 . . .

Doron Zeilberger and Wadim Zudilin

In memory of Naum Il'ich Feldman (1918–1994)

We use a variant of Salikhov's ingenious proof that the irrationality measure of π is at most 7.606308 . . . to prove that, in fact, it is at most 7.103205334137 . . . .

Introduction: How irrational is π?

Every number that is not rational (a quotient of integers) is *irrational*, but not all irrational numbers are born equal. To measure "how irrational" is a given number x, we define (see [Weisstein 2019]) the *irrationality measure* μ (also called the *irrationality exponent*) as the smallest number μ such that

$$\left| x - \frac{p}{q} \right| > \frac{1}{q^{\mu+\epsilon}}$$

holds for any $\epsilon > 0$ and all integers p and q with sufficiently large q.

It is not hard to see that the irrationality measure of e is 2, but the exact irrationality measure of π is unknown. It became a *competitive sport* to find lower and lower upper bounds for the irrationality measure of π. The first upper bound, of 42, was proved by Kurt Mahler [1953]. This record has been subsequently improved by Maurice Mignotte [1974], Gregory Chudnovsky [1982], and in three better-and-better articles, by Masayoshi Hata [1990; 1993a; 1993b]. The current "world record" is due to Vladislav Khasanovich Salikhov who proved the upper bound of 7.606308. This was announced in [Salikhov 2008] and published in [Salikhov 2010]. In this article we tweak Salikhov's method to beat his more than ten-year-old record to set a new world record of 7.103205334137 . . . .

The aim of our paper is not *just* to state and prove yet another record that would most likely be broken again sooner or later (we hope not that soon, unless it is by ourselves. . .), but to also explain our "experimental mathematics" methodology that pointed the way to the ultimate human-generated formal proof, to be given in Part II.[1] We also describe a fully rigorous, and fully computer-generated, proof of a coarser upper bound that is much better than many of the previous world records. This will be done in Part I. Readers who are not interested in the process of discovery, or computer proofs, can go straight to Part II, which is a self-contained human-generated and human-readable proof.

MSC2010: primary 11J82; secondary 11Y60, 33F10, 33C60.
Keywords: π, irrationality measure, experimental mathematics, Almkvist–Zeilberger algorithm.

[1] Accompanying Maple package: while this article has a fully rigorous human-made and human-readable proof of the claim in the title, it was *discovered* thanks to the Maple package `SALIKOHVpi.txt` available from http://sites.math.rutgers.edu/ zeilberg/mamarim/mamarimhtml/pimeas.html.

We are grateful to Vladislav Salikhov for pointing out a mistake in the previous version of our Lemma 2 below. Fixing the gap required employing new techniques, so that in the end this manuscript is more than just tweaking the construction in [Salikhov 2010].

Part I. The experimental mathematics way

General strategy. A good way to gain immortality, and become a *famous* **person**, is to be the first one to prove that a *famous* ***constant***, let's call it x, is irrational. One way to achieve this is to come up with two sequences of positive *integers* $\{a_n\}$ and $\{b_n\}$, and a *positive*, explicit real number δ such that there is a constant C, independent of n, such that, for all $n > 0$,

$$\left| x - \frac{a_n}{b_n} \right| \le \frac{C}{b_n^{1+\delta}}.$$

This immediately implies the irrationality of x and at the same time establishes an upper bound, namely $1 + 1/\delta$, for the irrationality measure of x.

This is exactly how, in 1978, the 64-year old Roger Apéry became immortal by doing the above with $x = \zeta(3)$ $\bigl($i.e., $\sum_{n=1}^{\infty} n^{-3}\bigr)$; see Alf van der Poorten's classic exposition [1979].

Shortly after, Frits Beukers [1979] gave a much simpler rendition of Apéry's construction by introducing a certain explicit triple integral

$$I(n) = \int_0^1\!\int_0^1\!\int_0^1 \left(\frac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z} \right)^n \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-(1-xy)z},$$

and pointing out that

(i) $I(n)$ is small and can be explicitly bounded,

(ii) $I(n) = A(n) + B(n)\zeta(3)$ for certain sequences of rational numbers $A(n)$, $B(n)$ that can be explicitly bounded, and

(iii) $A(n)\,\mathrm{lcm}(1, 2, \dots, n)^3$ and $B(n)$ are integers.

Since thanks to the prime number theorem $\mathrm{lcm}(1, 2, \dots, n)$ grows like $e^{n+o(n)}$ as $n \to \infty$, everything followed.

Shortly after, Krishna Alladi and Michael Robinson [1980] used one-dimensional analogs to reprove the irrationality of $\log 2$, and established an upper bound of 4.63 for its irrationality measure (subsequently improved, see [Weisstein 2019]) by considering the simple integral

$$I(n) = \int_0^1 \left(\frac{x(1-x)}{1+x} \right)^n \frac{\mathrm{d}x}{1+x}.$$

Our manuscript [Zeilberger and Zudilin 2019] is dedicated to further exploration of this theme.

An experimental mathematics reduction of Salikhov's approach. Salikhov [2010] essentially uses the same strategy, but with the far more complicated integral

$$I(n) = -i \int_{4-2i}^{4+2i} \left(\frac{(x-4+2i)^6(x-4-2i)^6(x-5)^6(x-6+2i)^6(x-6-2i)^6}{x^{10}(x-10)^{10}} \right)^n \frac{\mathrm{d}x}{x(x-10)}.$$

He then used *partial fractions* to claim that

$$I(n) = A(n) + B(n)\pi$$

for some sequences of $\{A(n)\}$, $\{B(n)\}$ of *rational numbers*.

Using the *saddle-point method*, he bounded $I(n)$ and $A(n)$, $B(n)$.

He then proved that if one sets

$$A'(n) = \operatorname{lcm}(1, 2, \dots, 10n)\left(\frac{25}{32}\right)^n A(n) \quad \text{and} \quad B'(n) = \operatorname{lcm}(1, 2, \dots, 10n)\left(\frac{25}{32}\right)^n B(n),$$

then $A'(n)$ and $B'(n)$ are *integer sequences*, and defining

$$I'(n) = \operatorname{lcm}(1, 2, \dots, 10n)\left(\frac{25}{32}\right)^n I(n),$$

using $\operatorname{lcm}(1, 2, \dots, 10n) = O(e^{10n+o(n)})$, one can explicitly bound $A'(n)$, $B'(n)$, $I'(n)$, and $I'(n)$ being small and $B'(n)$ being big, and one can get a crude upper bound for the irrationality measure, using the fact that $A'(n)/B'(n)$ approximate π.

Finally, the hard part was coming up with "additional saving", a sequence of integers $F(n)$ such that $A''(n) = A'(n)/F(n)$ and $B''(n) = B'(n)/F(n)$ are still integers. Setting $I''(n) = I'(n)/F(n)$ he squeezed more juice out of it, getting a larger δ and hence a smaller irrationality measure $1 + 1/\delta$, setting the current record of 7.606308 . . . .

Our approach is different. We do not use partial fractions, but rather the fact that, thanks to the Almkvist–Zeilberger algorithm [1990], there exists a third-order linear recurrence equation of the form

$$p_0(n)I(n) + p_1(n)I(n+1) + p_2(n)I(n+2) + p_3(n)I(n+3) = 0$$

for some explicit polynomials $p_0(n)$, $p_1(n)$, $p_2(n)$, $p_3(n)$. To save space, we do not reproduce it here, but refer the reader to http://sites.math.rutgers.edu/∼zeilberg/tokhniot/oSALIKHOVpi2.txt.

That webpage gives a new, computer-generated proof of the crude upper bound, only using the recurrence and the so-called Poincaré lemma that gives the asymptotics of $A(n)$, $B(n)$ and $I(n)$ from which it is immediate to bound $A'(n)$, $B'(n)$, and $I'(n)$. The only nonrigorous part in our approach is the study of the extra divisor $F(n)$, whose growth we estimate empirically.

For details see the above-mentioned computer-generated article.

Tweaking Salikhov's integral*: *looking where to dig. Looking at Salikhov's integral, it is natural to consider the more general integral

$$I_{A,B}(n) = -i\int_{4-2i}^{4+2i} \frac{(x-4+2i)^{2An}(x-4-2i)^{2An}(x-5)^{2An}(x-6+2i)^{2An}(x-6-2i)^{2An}}{x^{2Bn+1}(x-10)^{2Bn+1}}\,dx,$$

where Salikhov's integral is the special case $I_{3,5}(n)$. Perhaps we can do better? But before we invest time and energy, trying out many choices of A and B, it makes sense to do things *empirically*, crank out, say, 300 terms of the examined sequence and see whether it yields good "deltas".

Alas, even Maple and Mathematica will start to complain if we use the definition for, say, $n = 300$. Luckily, for each specific A and B, Shalosh B. Ekhad can quicky use the Almkvist–Zeilberger algorithm [1990] to crank out many terms, and thereby get very good estimates for the "deltas". This initial *reconnaissance* is very fast and gives you an indication on *where to dig*.

This is implemented in procedure `BestAB` in the Maple package `SALIKOHVpi.txt` mentioned above. Typing `BestAB(10,300);` gives the computer-generated article http://sites.math.rutgers.edu/~zeilberg/tokhniot/oSALIKHOVpi4.txt.

Most of the choices of (A, B) give negative, useless, deltas, but — *surprise*! — the choice of $A = 2$, $B = 3$ yields that the smallest δ in the range $290 \le n \le 300$ is 0.16605428729395818514. This beats the analogous value for the $A = 3$, $B = 5$ case, which equals 0.15727140930557009691. The "bronze medal" was won by $A = 5$, $B = 8$, which was almost as good, 0.15701995819256081077, followed by $A = 8$, $B = 13$, which gave the respectable 0.15586354092162189848. Next in line was a non-Fibonacci $A = 7$, $B = 10$ that placed fifth, with 0.12451550531454231901. For all the other "empirical deltas" see the above output file.

Once we found out that $A = 2$, $B = 3$ was a good gamble, we had another pleasant surprise. We can replace n by $n/2$ and still get combinations of 1 and π (in the original case $A = 3$, $B = 5$ of Salikhov, the odd indices n give combinations of 1 and $\arctan \frac{1}{7}$). This simplifies the recurrence, and a fully rigorous proof of the cruder upper bound of 10.747747465671804677... can be found at http://sites.math.rutgers.edu/~zeilberg/tokhniot/oSALIKHOVpi3.txt.

In order to get the more refined upper bound, we had to resort to nonrigorous estimates.

Luckily it was possible to make everything fully rigorous, and this brings us to Part II.

Part II. A fully rigorous (human-generated) proof of the claimed upper bound for the irrationality measure of π

Test bunny. For $n = 0, 1, 2, \ldots$, our integrals in question are

$$\begin{aligned} I_n &= 5i \int_{4-2i}^{4+2i} \frac{(x-5)^{2n}(x-4+2i)^{2n}(x-4-2i)^{2n}(x-6+2i)^{2n}(x-6-2i)^{2n}}{x^{3n+1}(x-10)^{3n+1}}\,\mathrm{d}x \\ &= i(-1)^{n+1} \int_{-1-2i}^{-1+2i} \frac{5x^{2n}(x+1+2i)^{2n}(x+1-2i)^{2n}(x-1+2i)^{2n}(x-1-2i)^{2n}}{(5+x)^{3n+1}(5-x)^{3n+1}}\,\mathrm{d}x. \end{aligned} \tag{1}$$

These are from the winning family in Part I.

Arithmetic. The integrand

$$R(x) = \frac{5x^{2n}(x+1+2i)^{2n}(x+1-2i)^{2n}(x-1+2i)^{2n}(x-1-2i)^{2n}}{(5+x)^{3n+1}(5-x)^{3n+1}} = \frac{5x^{2n}(x^4+6x^2+25)^{2n}}{(5+x)^{3n+1}(5-x)^{3n+1}} \tag{2}$$

possesses the symmetry $R(-x) = R(x)$ and therefore can be written as

$$R(x) = P(x) + \sum_{j=0}^{3n} \left(\frac{A_j}{(5+x)^{j+1}} + \frac{A_j}{(5-x)^{j+1}} \right) \tag{3}$$

for some rational A_j and a polynomial $P(x) \in \mathbb{Z}[x^2]$ of degree $4n - 2$.

Lemma 1. *The coefficients A_j in the partial-fraction expansion* (3) *satisfy*

$$2^{-\lfloor (5n+3j)/2\rfloor+1}5^{-j}A_j \in \mathbb{Z} \quad \text{for } j=0,1,\dots,3n. \tag{4}$$

In particular, they are integers.

Proof. To compute A_j, introduce linear operators

$$D_m\colon\ f(x)\mapsto \frac{1}{m!}\frac{\mathrm{d}^m f(x)}{\mathrm{d}x^m}\bigg|_{x=-5}.$$

Then with the help of Leibniz's formula we deduce

$$\begin{aligned}
A_j &= D_{3n-j}((x+5)^{3n+1}R(x))\\
&= 5\sum_{\substack{m_0,m_1,\dots,m_5\ge 0\\ m_1,\dots,m_5\le 2n\\ m_0+m_1+\cdots+m_5=3n-j}} D_{m_0}(5-x)^{-3n-1}D_{m_1}x^{2n}D_{m_2}(x+1+2i)^{2n}D_{m_3}(x+1-2i)^{2n}\\
&\qquad\qquad\times D_{m_4}(x-1+2i)^{2n}D_{m_5}(x-1-2i)^{2n}\\
&= 5\sum_{\boldsymbol{m}\in\mathcal{M}_j}(-1)^{m_1+\cdots+m_5}T(\boldsymbol{m})10^{-3n-1-m_0}5^{2n-m_1}(4-2i)^{2n-m_2}(4+2i)^{2n-m_3}(6-2i)^{2n-m_4}(6+2i)^{2n-m_5}\\
&= \sum_{\boldsymbol{m}\in\mathcal{M}_j}(-1)^{m_1+\cdots+m_5}T(\boldsymbol{m})2^{4n-1+j+m_1}(1-i)^{-m_4}(1+i)^{-m_5}5^j(2+i)^{m_2+m_5}(2-i)^{m_3+m_4}
\end{aligned}\tag{5}$$

for $j=0,\dots,3n$, where the summation is over the multi-indices $\boldsymbol{m}=(m_0,\dots,m_5)$ from the set

$$\mathcal{M}_j=\{(m_0,m_1,\dots,m_5): m_0,m_1,\dots,m_5\ge 0,\ m_1,\dots,m_5\le 2n,\ m_0+m_1+\cdots+m_5=3n-j\}\subset\mathbb{Z}^6_{\ge 0}$$

and

$$T(\boldsymbol{m})=T(m_0,m_1,\dots,m_5)=\binom{3n+m_0}{m_0}\prod_{\ell=1}^{5}\binom{2n}{m_\ell}\in\mathbb{Z}.$$

Now $m_4+m_5\le 3n-j$ and $(1\pm i)^2=\pm 2i$; hence

$$2^{\lceil (3n-j)/2\rceil}\times(1-i)^{-m_4}(1+i)^{-m_5}\in\mathbb{Z}[i]$$

and

$$2^{-\lfloor (5n+3j)/2\rfloor+1}\times 2^{4n-1+j+m_1}(1-i)^{-m_4}(1+i)^{-m_5}\in\mathbb{Z}[i].$$

Therefore, $2^{-\lfloor (5n+3j)/2\rfloor+1}5^{-j}A_j\in\mathbb{Z}[i]$ and the result follows from using the fact that $A_j\in\mathbb{Q}$. □

Formula (5) for the coefficients A_j makes sense for *any* integer $j\le 3n$; it generates the coefficients in the Laurent series expansion of $R(x)$ at $x=-5$. More precisely,

$$R(x)=\sum_{k=-3n}^{\infty}A_{-k}(x+5)^{k-1}=\sum_{j=0}^{3n}\frac{A_j}{(x+5)^{j+1}}+\sum_{k=1}^{\infty}A_{-k}(x+5)^{k-1}.$$

Note that A_j produced by formula (5) are not necessarily integral for *negative* j but at least they satisfy $10^{-j}A_j\in\mathbb{Z}$ for $j=-1,-2,\dots,-(4n-1)$ on the grounds of the formula, and we also have $10^{-j}A_j\in\mathbb{Z}$

for $j = 0, 1, 2, \dots, 3n$ in accordance with Lemma 1. Furthermore,

$$\sum_{j=0}^{3n} \frac{A_j}{(5-x)^{j+1}} = \sum_{j=0}^{3n} \frac{A_j}{(10-(x+5))^{j+1}} = \sum_{j=0}^{3n} A_j \sum_{k=1}^{\infty} \binom{j+k-1}{j} \frac{(x+5)^{k-1}}{10^{j+k}};$$

comparing the last two expansions with (3) we find

$$P(x) = R(x) - \sum_{j=0}^{3n}\left(\frac{A_j}{(5+x)^{j+1}} + \frac{A_j}{(5-x)^{j+1}}\right) = \sum_{k=1}^{\infty}\left(A_{-k} - \sum_{j=0}^{3n}\binom{j+k-1}{j}\frac{A_j}{10^{j+k}}\right)(x+5)^{k-1}.$$

On the other hand, $P(x)$ is a *polynomial* of degree $4n-2$; hence

$$P(x) = \sum_{k=1}^{4n-1}\left(A_{-k} - \sum_{j=0}^{3n}\binom{j+k-1}{j}\frac{A_j}{10^{j+k}}\right)(x+5)^{k-1}. \tag{6}$$

Lemma 2. *Any prime from the set*

$$\mathcal{P}_n = \left\{p > \max\{5, \sqrt{3n}\} : \frac{1}{2} \le \left\{\frac{n}{p}\right\} < \frac{2}{3}\right\} \subset \{p \text{ prime} : 5 < p \le 2n\}$$

satisfies the following property: if $p \mid j$ for $j \in \{-4n+1, -4n+2, \dots, 3n\}$, then $A_j \equiv 0 \pmod p$ (in other words, $p \mid 10^{-j} A_j$). (Here $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of the number.)

Proof. In order to establish the claim, we will cast the coefficients A_j in (5) differently. Observe that

$$\begin{aligned} R(x) &= \frac{5x^{2n}(x^2+(3-4i))^{2n}(x^2+(3+4i))^{2n}}{(25-x^2)^{3n+1}} \\ &= 5\sum_{n_1,n_2\ge 0}\binom{2n}{n_1}\binom{2n}{n_2}(3-4i)^{2n-n_1}(3+4i)^{2n-n_2}\frac{x^{2(n+n_1+n_2)}}{(25-x^2)^{3n+1}} \end{aligned}$$

and

$$\begin{aligned} \frac{x^{2m}}{(25-x^2)^{3n+1}} &= \frac{(5-(x+5))^{2m}}{(x+5)^{3n+1}(10-(x+5))^{3n+1}} \\ &= \frac{5^{2m}10^{-3n-1}}{(x+5)^{3n+1}}\sum_{k=0}^{\infty}\frac{(x+5)^k}{10^k}\sum_{n_0\ge 0}(-2)^{n_0}\binom{2m}{n_0}\binom{3n+k-n_0}{3n}; \end{aligned}$$

hence

$$A_j = \sum_{n_1,n_2\ge 0}(3-4i)^{2n-n_1}(3+4i)^{2n-n_2}5^{2n+2n_1+2n_2+1}10^{-(6n-j)-1}\binom{2n}{n_1}\binom{2n}{n_2}Z(n, n_1+n_2, j),$$

where

$$Z(n,m,j) = \sum_{n_0\ge 0}(-2)^{n_0}\binom{2n+2m}{n_0}\binom{6n-j-n_0}{3n}.$$

This means that our lemma is a consequence of the following divisibility property: if a prime $p \in \mathcal{P}_n$ divides j, then it also divides

$$\widehat{T}(n, n_1, n_2) = \binom{2n}{n_1}\binom{2n}{n_2}Z(n, n_1+n_2, j)$$

for any $n_1, n_2 \ge 0$.

From now on, we will repeatedly use the fact that the p-adic order of $N!$ satisfies $\operatorname{ord}_p N! = \lfloor N/p \rfloor = N/p - \{N/p\}$ when $p > \sqrt{N}$. In particular,

$$\operatorname{ord}_p \binom{2n}{n_\ell} = \lfloor 2\omega \rfloor - \lfloor 2\omega - \omega_\ell \rfloor - \lfloor \omega_\ell \rfloor = \lfloor 2\omega \rfloor - \lfloor 2\omega - \omega_\ell \rfloor \quad \text{for } \ell = 1, 2, \tag{7}$$

where the fractional parts $\omega = \{n/p\}$, $\omega_1 = \{n_1/p\}$ and $\omega_2 = \{n_2/p\}$ all belong to the interval $[0, 1)$. Since $p \in \mathcal{P}_n$, we have $\omega \in \left[\frac{1}{2}, \frac{2}{3}\right)$, so that $\lfloor 2\omega \rfloor = \lfloor 3\omega \rfloor = 1$. If at least one of the p-adic orders in (7) is positive then immediately $\operatorname{ord}_p \widehat{T}(n, n_1, n_2) \geq 1$, establishing the required divisibility; therefore, it remains to analyze the remaining situations assuming $\lfloor 2\omega - \omega_\ell \rfloor = \lfloor 2\omega \rfloor = 1$ for $\ell = 1, 2$, in other words, assuming

$$2\omega - \omega_1 \geq 1 \quad \text{and} \quad 2\omega - \omega_2 \geq 1.$$

The binomial sums $Z(n, m, j)$ can be realized as a terminating ${}_2F_1$ hypergeometric function, to which several classical transformations can be applied. For example, it can be transformed into

$$\begin{aligned} Z(n, m, j) &= \sum_{n_0 \geq 0} (-1)^{n_0} \binom{2n+2m}{n_0} \binom{6n-2(n+m)-j}{3n-j-n_0} \\ &= (-1)^{n+m} \sum_{k \in \mathbb{Z}} (-1)^k \binom{2n+2m}{n+m+k} \binom{4n-2m-j}{2n-m-k}. \end{aligned}$$

Though the expression does not possess a closed form in general, its particular instance $j = 0$ reduces to the *super Catalan numbers*

$$\frac{(2N)!\,(2M)!}{N!\,(N+M)!\,M!} = \sum_{k=-\infty}^{\infty} (-1)^k \binom{2N}{N+k} \binom{2M}{M+k};$$

see [Gessel 1992] also for the historical reference of this identity due to K. von Szily (1894). The argument in [Gessel 1992, Section 6] shows that the more general sum

$$\sum_{k=-\infty}^{\infty} (-1)^k \binom{2N}{N+k} \binom{2M-j}{M+k}$$

is the coefficient of t^{2N} in the polynomial

$$(-1)^N \frac{(2N)!\,(2M-j)!}{(N+M)!\,(N+M-j)!} (1+t)^{N+M} (1-t)^{N+M-j}.$$

In our situation $N = n + m$, $M = 2n - m$ with $m = n_1 + n_2$, the factorial-ratio factor

$$\frac{(2N)!\,(2M-j)!}{(N+M)!\,(N+M-j)!} = \frac{(2n+2n_1+2n_2)!\,(4n-2n_1-2n_2-j)!}{(3n)!\,(3n-j)!}$$

has the nonnegative p-adic order

$$\begin{aligned} &\lfloor 2\omega + 2\omega_1 + 2\omega_2 \rfloor + \lfloor 4\omega - 2\omega_1 - 2\omega_2 - j/p \rfloor - \lfloor 3\omega \rfloor - \lfloor 3\omega - j/p \rfloor \\ &\quad = \lfloor 2\omega + 2\omega_1 + 2\omega_2 \rfloor + \lfloor 4\omega - 2\omega_1 - 2\omega_2 \rfloor - 2\lfloor 3\omega \rfloor \end{aligned}$$

(we use $j/p \in \mathbb{Z}$), because $\lfloor 3\omega \rfloor = 1$,

$$2\omega + 2\omega_1 + 2\omega_2 \ge 2\omega \ge 1 \quad \text{and} \quad 4\omega - 2\omega_1 - 2\omega_2 \ge 4\omega - 4(2\omega - 1) = 4(1-\omega) > \tfrac{4}{3}.$$

Moreover, if this p-adic order is *positive* then $Z(n, n_1+n_2, j)$ is divisible by p; hence the divisibility of $\widehat{T}(n, n_1, n_2)$ follows. Thus, we are left with the situation when this order is zero,

$$\lfloor 2\omega + 2\omega_1 + 2\omega_2 \rfloor = \lfloor 4\omega - 2\omega_1 - 2\omega_2 \rfloor = 1,$$

meaning that

$$2\omega + 2\omega_1 + 2\omega_2 < 2 \quad \text{and} \quad 4\omega - 2\omega_1 - 2\omega_2 < 2. \tag{8}$$

We have to show that the coefficient of t^{2N} in $(1+t)^{N+M}(1-t)^{N+M-j}$ is divisible by p.

Setting $r = -j/p \in \mathbb{Z}$ and using the "freshman's dream identity" $(1-t)^p \equiv 1 - t^p \pmod p$ in the ring $\mathbb{Z}[\![t]\!]$, we find

$$\begin{aligned}(1+t)^{N+M}(1-t)^{N+M-j} &= (1-t^2)^{N+M}(1-t)^{-j} \\ &\equiv (1-t^2)^{N+M}(1-t^p)^r = \sum_{k_1 \ge 0}(-1)^{k_1}\binom{N+M}{k_1}t^{2k_1}\sum_{k_2\ge 0}(-1)^{k_2}\binom{r}{k_2}t^{pk_2};\end{aligned}$$

hence the coefficient of t^{2N} is congruent to

$$\sum_{k=0}^{\lfloor N/p \rfloor}(-1)^{k+N}\binom{N+M}{N-kp}\binom{r}{2k}$$

modulo p. The p-adic order of the *nonzero* binomial coefficients $\binom{N+M}{N-kp}$ does not depend on k:

$$\operatorname{ord}_p\binom{N+M}{N-kp} = -\left\{\frac{N}{p}+\frac{M}{p}\right\} + \left\{\frac{N}{p}-k\right\} + \left\{\frac{M}{p}+k\right\} = -\left\{\frac{N}{p}+\frac{M}{p}\right\} + \left\{\frac{N}{p}\right\} + \left\{\frac{M}{p}\right\}.$$

Recalling that $N = n+m$, $M = 2n-m$ with $m = n_1+n_2$ the latter quantity reads

$$\operatorname{ord}_p\binom{3n}{n+n_1+n_2} = \lfloor 3\omega \rfloor - \lfloor \omega + \omega_1 + \omega_2 \rfloor - \lfloor 2\omega - \omega_1 - \omega_2 \rfloor = 1,$$

where we employed (8) to get

$$\lfloor \omega + \omega_1 + \omega_2 \rfloor = \lfloor 2\omega - \omega_1 - \omega_2 \rfloor = 0.$$

This means that all binomial coefficients $\binom{N+M}{N-kp}$ are divisible by p, thus completing our proof of the divisibility of $\widehat{T}(n, n_1, n_2)$ by p, and of the lemma. □

Remark. An earlier version of Lemma 2 claimed that any prime $p \in \mathcal{P}_n$, satisfying $p \mid j$ for $j \in \{-4n, -4n+1, \dots, 3n\}$, divides $T(\boldsymbol{m})$ for all $\boldsymbol{m} \in \mathcal{M}_j$; this would clearly imply the present statement in view of formula (5). However, the claim about the divisibility properties of $T(\boldsymbol{m})$ was false.

Lemma 3. *Define* $\Phi = \Phi_n = \prod_{p \in \mathcal{P}_n} p$ *and*

$$L_n = \frac{\operatorname{lcm}(1, 2, \dots, 4n)}{\Phi_n} \in \mathbb{Z}.$$

Then

$$L_n \times \frac{10^{-j} A_j}{j} \in \mathbb{Z} \quad \textit{for } j \in \{-4n, -4n+1, \dots, 3n-1, 3n\},\ j \neq 0, \tag{9}$$

and $\Phi_n^{-1} \times A_0 \in \mathbb{Z}$.

Asymptotically,

$$\lim_{n\to\infty} \frac{\log \Phi_n}{n} = \frac{\Gamma'\bigl(\frac{2}{3}\bigr)}{\Gamma\bigl(\frac{2}{3}\bigr)} - \frac{\Gamma'\bigl(\frac{1}{2}\bigr)}{\Gamma\bigl(\frac{1}{2}\bigr)} = \frac{\pi}{2\sqrt{3}} - \log\frac{3\sqrt{3}}{4} = 0.64527561\dots \tag{10}$$

(see [Hata 1993b, Lemma 2.2]).

Proof. Note that

$$\operatorname{lcm}(1, 2, \dots, 4n) \times \frac{1}{j} \in \mathbb{Z} \quad \text{for } j \in \{-4n, -4n+1, \dots, 3n\},\ j \neq 0,$$

implying, for all such j,

$$L_n \cdot \frac{1}{j/p} \in \mathbb{Z} \quad \text{if } p \mid j,\ p \in \mathcal{P}_n. \tag{11}$$

On the other hand, it follows from formula (5) and Lemma 2 that

$$\frac{10^{-j} A_j}{p} \in \mathbb{Z} \quad \text{if } p \mid j,\ p \in \mathcal{P}_n. \tag{12}$$

Combining (11) and (12) results in claim (9). □

Lemma 4. *Write the polynomial $P(x) \in \mathbb{Z}[x]$ in the decomposition* (3) *as*

$$P(x) = \sum_{k=0}^{4n-2} B_k (x+1+2i)^k, \quad \textit{with } B_k \in \mathbb{Z}[i] \ \textit{for } k = 0, 1, \dots, 4n-2. \tag{13}$$

Then

$$2^{-\lfloor 5n/2 \rfloor + \lceil 3k/2 \rceil + 2} \times B_k \in \mathbb{Z}[i] \quad \textit{for } k = 0, 1, \dots, 4n-2. \tag{14}$$

Proof. If $k \geq 2n$ then $-\lfloor 5n/2 \rfloor + \lceil 3k/2 \rceil + 2 \geq 0$ and the inclusion in (14) follows from $B_k \in \mathbb{Z}[i]$. Therefore, we only need to verify (14) for $k < 2n$; since $R(x)$ from (2) has a zero of order $2n$ at $x = -1-2i$, we deduce from (3) that

$$\begin{aligned}
B_k &= -\frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \sum_{j=0}^{3n} \biggl(\frac{A_j}{(5+x)^{j+1}} + \frac{A_j}{(5-x)^{j+1}} \biggr) \bigg|_{x=-1-2i} \\
&= -\sum_{j=0}^{3n} (-1)^k \binom{j+k}{k} \biggl(\frac{A_j}{(5+x)^{j+k+1}} + (-1)^{j+1} \frac{A_j}{(x-5)^{j+k+1}} \biggr) \bigg|_{x=-1-2i} \\
&= -\sum_{j=0}^{3n} \binom{j+k}{k} \biggl(\frac{(-1)^k A_j}{(2(2-i))^{j+k+1}} + \frac{A_j}{(2(1+i)(2-i))^{j+k+1}} \biggr)
\end{aligned}$$

for $k=0,1,\dots,2n-1$. It follows then from (4) that

$$2^{-\lfloor 5n/2\rfloor+\lceil 3k/2\rceil+2}(2-i)^{3n+k+1}\times B_k\in\mathbb{Z}[i]$$

and, again, we recall $B_k\in\mathbb{Z}[i]$ to conclude with (14) for $k<2n$. □

Lemma 5. *For the polynomial $P(x)$ in the decomposition* (3), *we have*

$$2^{-\lfloor 5n/2\rfloor}L_n\times i\int_{-1-2i}^{-1+2i}P(x)\,\mathrm{d}x\in\mathbb{Z}. \tag{15}$$

Proof. We first compute the integral using representation (13),

$$\begin{aligned} i\int_{-1-2i}^{-1+2i}P(x)\,\mathrm{d}x&=i\sum_{k=0}^{4n-2}B_k\int_{-1-2i}^{-1+2i}(x+1+2i)^k\,\mathrm{d}x\\ &=i\sum_{k=0}^{4n-2}\frac{B_k}{k+1}(4i)^{k+1}=-\sum_{k=0}^{4n-2}\frac{2^{2k+2}B_k}{k+1}\,i^k \end{aligned}$$

implying

$$2^{-\lfloor 5n/2\rfloor}\operatorname{lcm}(1,2,\dots,4n)\times i\int_{-1-2i}^{-1+2i}P(x)\,\mathrm{d}x\in\mathbb{Z}[i] \tag{16}$$

on the basis of Lemma 4. On the other hand, if representation (6) is applied then

$$\begin{aligned} i\int_{-1-2i}^{-1+2i}P(x)\,\mathrm{d}x&=i\sum_{k=1}^{4n-1}\biggl(A_{-k}-\sum_{j=0}^{3n}\binom{j+k-1}{j}\frac{A_j}{10^{j+k}}\biggr)\int_{-1-2i}^{-1+2i}(x+5)^{k-1}\,\mathrm{d}x\\ &=i\sum_{k=1}^{4n-1}\biggl(\frac{A_{-k}}{k}-\sum_{j=0}^{3n}\binom{j+k-1}{j}\frac{A_j}{k\,10^{j+k}}\biggr)((4+2i)^k-(4-2i)^k)\\ &=\sum_{k=1}^{4n-1}\biggl(\frac{A_{-k}}{k}-\frac{A_0}{k}\frac{1}{10^k}-\sum_{j=1}^{3n}\binom{j+k-1}{j-1}\frac{A_j}{j}\frac{1}{10^{j+k}}\biggr)2^{k+1}\sum_{\substack{\ell=0\\ \ell\ \text{odd}}}^{k}\binom{k}{\ell}(-1)^{(\ell+1)/2}2^{k-\ell} \end{aligned}$$

is a rational number satisfying

$$\frac{\operatorname{lcm}(1,2,\dots,4n)}{\Phi_n}\times i\int_{-1-2i}^{-1+2i}P(x)\,\mathrm{d}x\in 10^{-4n}\mathbb{Z} \tag{17}$$

on the basis of Lemma 3. Finally, the two inclusions (16) and (17) combine into (15). □

Lemma 6. *For the partial-fraction part in* (3) (*without the $j=0$ term*), *we have*

$$2^{-\lfloor 5n/2\rfloor+1}L_n\times i\int_{-1-2i}^{-1+2i}\sum_{j=1}^{3n}A_j\biggl(\frac{1}{(5+x)^{j+1}}+\frac{1}{(5-x)^{j+1}}\biggr)\,\mathrm{d}x\in\mathbb{Z}.$$

Proof. This follows from

$$\begin{aligned} i\sum_{j=1}^{3n} A_j \int_{-1-2i}^{-1+2i} \left(\frac{1}{(5+x)^{j+1}} + \frac{1}{(5-x)^{j+1}}\right) \mathrm{d}x \\ &= i\sum_{j=1}^{3n} \frac{A_j}{j}\left(\frac{1}{(4-2i)^j} - \frac{1}{(4+2i)^j} - \frac{1}{(6+2i)^j} + \frac{1}{(6-2i)^j}\right) \\ &= i\sum_{j=1}^{3n} \frac{A_j}{j}\left(\frac{(2+i)^j}{2^j5^j} - \frac{(2-i)^j}{2^j5^j} - \frac{(2+i)^j}{2^j(1+i)^j5^j} + \frac{(2-i)^j}{2^j(1-i)^j5^j}\right) \in \mathbb{Q} \end{aligned}$$

and the inclusions of Lemma 1 and 3. □

Lemma 1 and the integrality of L_n imply that $2^{-\lfloor 5n/2\rfloor+1}L_n \times A_0 \in \mathbb{Z}$; together with the calculation

$$\int_{-1-2i}^{-1+2i} \left(\frac{1}{5+x} + \frac{1}{5-x}\right) \mathrm{d}x = \log(4+2i) - \log(4-2i) - \log(6-2i) + \log(6+2i) = \frac{\pi i}{2}$$

and Lemmas 5, 6 we are thus led to the following statement.

Proposition 7. *For the integrals I_n in* (1), *we have*

$$2^{-\lfloor 5n/2\rfloor+2}L_n \times I_n \in \mathbb{Z} + \mathbb{Z}\pi.$$

Asymptotics. By now we have legally settled that $I_n = a_n + b_n\pi$ for some *rational* a_n and b_n.

Proposition 8. *The asymptotics of the integrals I_n and the coefficients b_n in the representation $I_n = a_n + b_n\pi$ are as follows*:

$$\limsup_{n\to\infty} |I_n|^{1/n} = |N_1| = 0.029458495928\ldots \quad \textit{and} \quad \lim_{n\to\infty} b_n^{1/n} = N_3 = 21851.691396\ldots,$$

where

$$N_{1,2} = 0.02930189\ldots \pm i\,0.00303351\ldots, \qquad N_3 = 21851.691396\ldots$$

are the zeros of polynomial

$$108N^3 - 2359989N^2 + 138304N - 2048. \tag{18}$$

Proof. This rigorously follows from the Poincaré lemma supplied by the rigorously produced — thanks to the Almkvist–Zeilberger method [1990] — difference equation for the integrals I_n (hence also for a_n, b_n), whose indicial polynomial (more precisely, the indicial polynomial of its "constant-coefficients approximation") is exacty (18). Observe that $|I_n| \le 1$ follows from integrating over the *line* interval $[-1-2i, -1+2i]$ and trivially bounding the absolute value of the integrand on it. However, those who prefer traditional analytical methods can have fun going through the glorious details of the saddle-point method, at least after the change of variables $y = x^2$ is performed in (1). For that, one deals with the function

$$\widetilde{R}(y) = \frac{5g(y)^n}{y-25}, \quad \text{where } g(y) = \frac{y(y^2+6y+25)^2}{(y-25)^3},$$

and with the zeros

$$y_{1,2} = -1.91975076\ldots \mp i\,1.01250889\ldots, \qquad y_3 = 66.33950152\ldots$$

of

$$\frac{g'(y)}{g(y)} = \frac{2y^3 - 125y^2 - 500y - 625}{y(y^2+6y+25)(y-25)}.$$

Then $N_j = g(y_j)$ for $j = 1, 2, 3$. The remaining part is performing a suitable deformation of path in (1) to pass through the saddle points $\sqrt{y_1}$ and $\sqrt{y_2}$ (with the choice of branch such that the real parts of the roots are negative) and writing a Cauchy integral for b_n over a closed contour passing through the saddle points $\pm\sqrt{y_3}$. □

World record. It follows from Propositions 7 and 8 that the forms

$$I_n' = 2^{-\lfloor 5n/2\rfloor+2} L_n I_n = a_n' + b_n'\pi, \quad \text{where } n = 0, 1, 2, \ldots,$$

all have integral coefficients a_n', b_n' and the asymptotics

$$\limsup_{n\to\infty} \frac{\log|I_n'|}{n} = \log|N_1| - \frac{5}{2}\log 2 + 4 - \frac{\pi}{2\sqrt{3}} + \log\frac{3\sqrt{3}}{4} = -1.90291648559998\ldots$$

and

$$\lim_{n\to\infty} \frac{\log b_n'}{n} = \log N_3 - \frac{5}{2}\log 2 + 4 - \frac{\pi}{2\sqrt{3}} + \log\frac{3\sqrt{3}}{4} = 11.613890045331\ldots$$

(the asymptotics of L_n follows from the prime number theorem and (10)). This implies (see, e.g., [Salikhov 2010, Lemma 1]) that the irrationality measure of π is bounded above by

$$1 + \frac{11.613890045331\ldots}{1.90291648559998\ldots} = 7.10320533413700172750577342281\ldots.$$

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Approximating π by numbers in the field $\mathbb{Q}(\sqrt{3})$

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Using a new integral construction which combines the idea of symmetry suggested by V. Salikhov in 2007 and the integral introduced by Marcovecchio in 2009, we obtain a new bound for approximation to π by numbers from the field $\mathbb{Q}(\sqrt{3})$.

1. Introduction. Integral construction. Arithmetic part.

We continue our research initiated in [Androsenko and Salikhov 2015] and [Luchin and Salikhov 2018]. In this paper we prove the following result.

Theorem 1. *Let* $\mu > 10.2209$; $p_1,\ p_2,\ p_3,\ p_4 \in \mathbb{Z}$, $(p_3,\ p_4) \neq (0, 0)$, $P = \max_{1\le i\le 4} |p_i|$, *and* $P > P_0(\mu)$. *Then*

$$\left|\pi - \frac{p_1\sqrt{3}+p_2}{p_3\sqrt{3}+p_4}\right| > P^{-\mu}. \tag{1}$$

The first inequality of this type was proven in [Amoroso and Viola 2001]:

$$\left|\pi - \frac{a+b\sqrt{3}}{c+d\sqrt{3}}\right| > \mathit{constant}\cdot \max\{|a|, |b|, |c|, |d|\}^{-46.9075\ldots},$$

where $a, b, c, d \in \mathbb{Z}$, $(c, d) \neq (0, 0)$. This result was improved in [Tomashevskaya 2008], with the value $10.3567\ldots$ for μ.

The proof of the new bound (1) is related to the application of the following integral construction. Let $h, j, k, l, m, q \in \mathbb{Z}^+$, $h+j+q = k+l+m$, $h+j-k \ge 0$, $k+l-j \ge 0$, $k+m-h \ge 0$; $x \in \mathbb{C}$, $\operatorname{Re} x > 0$, $x \neq 1$. Consider the integral

$$J = \frac{1}{2\pi i}\int_0^{-\infty} ds \int_{-i\infty}^{i\infty} \frac{s^h t^j\, dt}{\sqrt{\frac{s}{s-1}}(1-s)^{k+l-j+1}(s-t)^{h+j-k+1}(t-x)^{k+m-h+1}}. \tag{2}$$

The result of Theorem 1 is obtained by taking

$$x = \frac{2+\sqrt{3}}{4}, \tag{3}$$

$$h = 11n,\quad j = 37n,\quad k = 16n,\quad l = 27n,\quad m = 37n,\quad q = 32n,\quad n \in \mathbb{N},\quad n \to \infty. \tag{4}$$

The only thing that distinguishes the integral (2) from the one in [Marcovecchio 2009, (5), p. 148]

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is the factor $\sqrt{s/(s-1)}$ in the denominator of the integrand. For the first time the integral (2) was considered in [Androsenko and Salikhov 2015]. Using this integral, Androsenko [2015] improved the estimate for the irrationality measure of the number $\pi/\sqrt{3}$. In [Luchin and Salikhov 2018], thanks to the integral (2), it became possible to obtain a new bound for the approximation of $\ln 2$ by numbers from the field $\mathbb{Q}(\sqrt{2})$. In our argument below we substantially apply the method developed in that work.

In [Luchin and Salikhov 2018] (equalities (7)–(9)) it was shown that the integral (2) can be represented in the form

$$J = -\int_0^1 R(z)\,dz, \tag{5}$$

where

$$R(z) = 2(-1)^{j-k} \sum_{l_1=\max(0,q-l)}^{k+m-h} (-1)^{l_1} \binom{j}{k+m-h-l_1} \cdot \frac{x^{l-q+l_1}}{(x-1)^{h+j-k+l_1+1}} \binom{h+j-k+l_1}{l_1} R_{l_1}(z), \tag{6}$$

$$R_{l_1}(z) = \frac{z^{2h}(1-z^2)^{l+l_1}}{\left(\dfrac{x}{x-1} - z^2\right)^{h+j-k+l_1+1}} = \frac{z^{2h}(1-z^2)^{l+l_1}}{(-(2+\sqrt{3})^2 - z^2)^{h+j-k+l_1+1}}\,. \tag{7}$$

Here we use notation from [Luchin and Salikhov 2018]. As in that article, we write

$$\omega(l_1) = \binom{h+j-k+l_1}{l_1} \int_0^1 R_{l_1}(z)\,dz. \tag{8}$$

Let K be the ring of numbers of the form $a+b\sqrt{3}$, where $a, b \in \mathbb{Z}$, and for positive integers $M \in \mathbb{N}$ we put $q_M = \operatorname{lcm}(1, 2, \ldots, M)$ and $q_0 = 1$.

Lemma 1. *Let* $M_0 = \max(2k+2l-2j, h+j-k, k+m-h)$, $m \geq q$. *For all* $l_1 \leq k+m-h$, *one has*

$$q_{M_0}\omega(l_1) = \frac{1}{48} \cdot 2^{2q-2m} \cdot (a(l_1)\pi + b(l_1)), \tag{9}$$

where $a(l_1), b(l_1)$ *belong to* $\mathbb{K}$.

Proof. For $N \in \mathbb{Z}^+$ we write

$$D_N(f(z)) = \frac{1}{N!} \cdot f^{(N)}(2i + i\sqrt{3}).$$

Since the integrand (7) of the integral in (8) is even, we have expansion into a sum of simplest fractions:

$$\begin{aligned} R_{l_1}(z) &= \frac{(-1)^{q-m-1} z^{2h}(z^2-1)^{l+l_1}}{(z^2-(2i+i\sqrt{3})^2)^{h+j-k+l_1+1}} \\ &= P(z) + \sum_{\nu=1}^{h+j-k+l_1+1} \left(\frac{(-1)^{\nu} k_{\nu}}{(z-2i-i\sqrt{3})^{\nu}} + \frac{k_{\nu}}{(z+2i+i\sqrt{3})^{\nu}} \right), \end{aligned} \tag{10}$$

where $P(z) \in \mathbb{K}[z]$ and $\deg P(z) = 2(k+l-j-1)$;

$$(-1)^{\nu} k_{\nu} = D_{h+j-k+l_1+1-\nu}(R_{l_1}(z)(z-2i-i\sqrt{3})^{h+j-k+l_1+1}).$$

By Leibniz's formula, we see from (10) that

$$k_\nu = (-1)^{q-m+\nu-1} D_{h+j-k+l_1+1-\nu}\left(\frac{z^{2h}(z-1)^{l+l_1}(z+1)^{l+l_1}}{(z+2i+i\sqrt{3})^{h+j-k+l_1+1}}\right)$$
$$= (-1)^{q-m+\nu-1}\sum_{\overline{m}\in M_\nu} D_{m_1}(z^{2h})\, D_{m_2}((z-1)^{l+l_1})\, D_{m_3}((z+1)^{l+l_1})\, D_{m_4}\big((z+2i+i\sqrt{3})^{-(h+j-k+l_1+1)}\big),$$

where we have set $\overline{m}=(m_1,m_2,m_3,m_4)$ and

$$M_\nu = \{\overline{m}\in(\mathbb{Z}^+)^4 \mid m_1+m_2+m_3+m_4 = h+j-k+l_1+1-\nu;\ m_1\le 2h;\ m_2,m_3\le l+l_1\}.$$

So

$$k_\nu = (-1)^{q-m+\nu-1}\sum_{\overline{m}\in M_\nu}\binom{2h}{m_1}\binom{l+l_1}{m_2}\binom{l+l_1}{m_3}\binom{h+j-k+l_1+m_4}{m_4}\cdot(-1)^{m_4}\cdot$$
$$\cdot(2i+i\sqrt{3})^{2h-m_1}\cdot(-1+2i+i\sqrt{3})^{l+l_1-m_2}\cdot(1+2i+i\sqrt{3})^{l+l_1-m_3}\cdot$$
$$\cdot(2(2i+i\sqrt{3}))^{-(h+j-k+l_1+m_4+1)}$$

For $N\in\mathbb{N}$ we have

$$(-1+2i+i\sqrt{3})^N = (2i+2e^{i\frac{2\pi}{3}})^N = 2^{N-1}(i+e^{i\frac{2\pi}{3}})^N\cdot 2.$$

But $2\cdot(i+e^{i\frac{2\pi}{3}})^N\in\mathbb{K}[i]$ and so $(-1+2i+i\sqrt{3})^N = 2^{N-1}\cdot k'_N$, where $k'_N\in\mathbb{K}[i]$. Similarly we have $(1+2i+i\sqrt{3})^N = 2^{N-1}\cdot k''_N$, where $k''_N\in\mathbb{K}[i]$.

So

$$k_\nu = \sum_{\overline{m}\in M_\nu} k_\nu(\overline{m})\cdot 2^{l+l_1-m_2-1}\cdot 2^{l+l_1-m_3-1}\cdot 2^{-(h+j-k+l_1+m_4+1)},$$

where all $k_\nu(\overline{m})\in\mathbb{K}[i]$.

Moreover $m_2+m_3+m_4\le h+j-k+l_1+1-\nu$, so

$$l+l_1-m_2-1+l+l_1-m_3-1-(h+j-k+l_1+1+m_4)\ge 2(l+l_1-h-j+k-l_1)+\nu-4$$
$$= 2q-2m+\nu-4.$$

This gives

$$k_\nu = 2^{2q-2m+\nu-4}\cdot\tilde{k}_\nu,\ \tilde{k}_\nu\in\mathbb{K}[i],\ \nu=1,\ldots,h+j-k+l_1+1. \tag{11}$$

From (10) we have

$$\int_0^1 R_{l_1}(z)\,dz = \int_0^1 P(z)\,dz + \sum_{\nu=2}^{h+j-k+l_1+1}\left(\frac{k_\nu}{\nu-1}\left(\frac{1}{(2i+i\sqrt{3}-1)^{\nu-1}}-\frac{1}{(2i+i\sqrt{3}+1)^{\nu-1}}\right)\right)$$
$$+ k_1\ln\frac{2i+i\sqrt{3}+z}{2i+i\sqrt{3}-z}\bigg|_0^1. \tag{12}$$

Obviously $2+\sqrt{3}=\tan\frac{5\pi}{12}$. Let $\ln z=\ln|z|+i\varphi$ where $\varphi\in(-\pi;\pi]$. Then

$$\ln\frac{2i+i\sqrt{3}+z}{2i+i\sqrt{3}-z}\Big|_0^1=\ln\frac{1+2i+i\sqrt{3}}{2i+i\sqrt{3}-1}=\ln\frac{1+i\tan\frac{5\pi}{12}}{i\tan\frac{5\pi}{12}-1}=\ln\frac{e^{i\frac{5\pi}{12}}}{e^{i\frac{7\pi}{12}}}=-i\frac{\pi}{6};$$

$$\frac{1}{2i+i\sqrt{3}+1}=\frac{1-i\sqrt{3}-2i}{1+(2+\sqrt{3})^2}=\frac{e^{-i\frac{\pi}{3}}-i}{2(2+\sqrt{3})},\quad \frac{1}{(2i+i\sqrt{3}+1)^{\nu-1}}=\frac{2(e^{-i\frac{\pi}{3}}-i)^{\nu-1}(2-\sqrt{3})^{\nu-1}}{2^{\nu}}.$$

As before we have $2(e^{-i\frac{\pi}{3}}-i)^{\nu-1}\in\mathbb{K}[i]$ and so

$$\frac{1}{(2i+i\sqrt{3}+1)^{\nu-1}}=2^{-\nu}x'_\nu,,\quad \frac{1}{(2i+i\sqrt{3}-1)^{\nu-1}}=2^{-\nu}x''_\nu,$$

where x'_ν, $x''_\nu\in\mathbb{K}[i]$. Thus from (11) and (12) we have

$$\int_0^1 R_{l_1}(z)\,dz=\int_0^1 P(z)\,dz+\sum_{\nu=2}^{h+j-k+l_1+1}\frac{1}{\nu-1}\cdot 2^{2q-2m-4}\cdot\tilde{k}_\nu+\frac{2^{2q-2m-3}}{6}\cdot\tilde{k}_1\pi, \tag{13}$$

where $\tilde{k}_\nu\in\mathbb{K}[i]$ for all ν.

It follows from the definition of M_0 that $A_1:=q_{M_0}\int_0^1 P(z)\,dz$ lies in $\mathbb{K}$. It is also easy to check that $q_{M_0}\binom{h+j-k+l_1}{l_1}\cdot\frac{1}{\nu-1}=:A_\nu$ lies in $\mathbb{N}$ for all $\nu=2,\ldots,h+j-k+l_1+1$. Then it follows from (8) and (13) that

$$\begin{aligned} q_{M_0}\omega(l_1)&=q_{M_0}\binom{h+j-k+l_1}{l_1}\cdot\int_0^1 R_{l_1}(z)\,dz\\ &=\binom{h+j-k+l_1}{l_1}A_1+2^{2q-2m-4}\cdot\sum_{\nu=2}^{h+j-k+l_1+1}A_\nu\tilde{k}_\nu+\frac{1}{3}\cdot 2^{2q-2m-4}\cdot\tilde{k}'_1\pi,\end{aligned}$$

whence, since $m\ge q$, we get equality (9), where $a(l_1)$, $b(l_1)\in\mathbb{K}[i]$. But, obviously, $a(l_1)$, $b(l_1)\in\mathbb{R}$. Therefore $a(l_1)$, $b(l_1)\in\mathbb{K}$. This completes the proof of lemma. □

Corollary 1. *The integral* (2) *for $m\ge q$ admits the representation*

$$6\cdot 2^{-2q}q_{M_0}J=a\pi+b,\ a,b\in\mathbb{K}. \tag{14}$$

Proof. For $x=\dfrac{2+\sqrt{3}}{4}$ we have

$$\frac{x^{l-q+l_1}}{(x-1)^{h+j-k+l_1+1}}=\frac{(2+\sqrt{3})^{l-q+l_1}4^{h+j-k+l_1+1}}{4^{l-q+l_1}\left(\sqrt{3}-2\right)^{h+j-k+l_1+1}}=4^{m+1}\cdot C(l_1),$$

where $C(l_1)\in\mathbb{K}$.

Therefore from (5), (6), (8) and (9) we have

$$q_{M_0}J=\frac{1}{6}\cdot 4^m\cdot 4^{q-m}\sum_{l_1=\max(0,q-l)}^{k+m-h}d(l_1)c(l_1)\left(a(l_1)\pi+b(l_1)\right),$$

where all $d(l_1)\in\mathbb{Z}$, and this implies (14). □

Together with the family of parameters (4), we should consider a more general choice of parameters

$$(h, j, k, l, m, q) = n(h', j', k', l', m', q'), \tag{15}$$

where $h', j', k', l', m', q' \in \mathbb{Z}^+$.

It is convenient to denote the integral (2) for parameters of the form (15) and for x of the form (3) as

$$J := J_n = J_n(h', j', k', l', m', q'). \tag{16}$$

For the family of parameters (15) we write

$$Mn = \max\{2(k+l-j), 2h, 2k, h+j-k, k+m-h, l, m, j, q\}. \tag{17}$$

Let p be a prime, $p > \sqrt{Mn}$ and $\omega = \{\frac{n}{p}\}$ be the fractional part of the number $\frac{n}{p}$. Consider the inequalities

$$\begin{gathered}
[2k'\omega]+[(l'+k'-j')\omega]+[m'\omega]+[l'\omega]-[k'\omega]-[2(l'+k'-j')\omega]-[(h'+j'-k')\omega]-[(k'+m'-h')\omega] > 0,\\
[2h'\omega]+[(l'+k'-j')\omega]+[j'\omega]+[q'\omega]-[h'\omega]-[2(l'+k'-j')\omega]-[(h'+j'-k')\omega]-[(k'+m'-h')\omega] > 0,\\
[j'\omega]+[m'\omega]-[(h'+j'-k')\omega]-[(k'+m'-h')\omega] > 0,\\
[2k'\omega]+[(l'+k'-j')\omega]+[q'\omega]-[k'\omega]-[2(l'+k'-j')\omega]-[(k'+m'-h')\omega] > 0,\\
[2h'\omega]+[(l'+k'-j')\omega]+[l'\omega]-[h'\omega]-[2(l'+k'-j')\omega]-[(h'+j'-k')\omega] > 0.
\end{gathered} \tag{18}$$

These inequalities were first studied in detail in [Androsenko and Salikhov 2015, (11), p. 491] and later applied in [Luchin and Salikhov 2018]. They are slightly different from those considered for the same purpose in [Marcovecchio 2009, (31)].

By Δ_n we denote the product of all primes $p > \sqrt{Mn}$ for which $\omega = \{\frac{n}{p}\}$ satisfies at least one of the inequalities (18). The following lemma sharpens the result obtained in Corollary 1.

Lemma 2. *When $m \geq q$ the integral* (16) *admits the representation*

$$6 \cdot 2^{-2q} \cdot \frac{q_{Mn}}{\Delta_n} \cdot J_n = A_n\pi + B_n, \tag{19}$$

where $A_n, B_n \in \mathbb{K}$, $n \in \mathbb{N}$.

Proof. The representation (19) follows from (14) due to a standard procedure of refining the denominator (see, for example, Lemma 3 in [Androsenko 2015]). □

The following lemma, similar to Lemma 4 from [Luchin and Salikhov 2018], plays an important role in the proof of Theorem 1.

Lemma 3. *Let $n, d \in \mathbb{N}$, $\theta \in \mathbb{R}$, $\sqrt{d} \notin N$, and $L_n = (\Lambda_1(n)\sqrt{d} + \Lambda_2(n))\theta + \Lambda_3(n)\sqrt{d} + \Lambda_4(n)$, where each $\Lambda_i(n)$ belongs to $\mathbb{Z}$, and let $\Lambda(n) = \max_{1\leq i\leq 4} |\Lambda_i(n)|$. Let $\lim_{n\to\infty} \frac{1}{n} \ln |\Lambda_1(n)\sqrt{d} + \Lambda_2(n)| = \gamma_1$, $\lim_{n\to\infty} \sup \frac{1}{n} \ln |\Lambda(n)| \leq \gamma_2$. Suppose that for some constant $\gamma_3 > \gamma_2$ and for every $\varepsilon_1, \varepsilon_2 > 0$ there exists $N = N(\varepsilon_1, \varepsilon_2)$ such that the inequalities*

$$e^{-(\gamma_3+\varepsilon_1)m} \leq |L_m| \leq e^{-(\gamma_3-\varepsilon_2)m} \tag{20}$$

hold for any $n \geq N$ *and at least one of the values* $m \in \{n, n+1\}$. *Further, let* $\gamma_1 + \gamma_2 > 0$, $\mu > \frac{2(\gamma_1+\gamma_3)}{\gamma_3-\gamma_2}$; $p_1, p_2, p_3, p_4 \in \mathbb{Z}$,$(p_3, p_4) \neq (0, 0)$, $P = \max_{1\leq i\leq 4} |p_i|$ *and* $P > P_0(\mu)$. *Then*

$$\left|\theta - \frac{p_1\sqrt{d} + p_2}{p_3\sqrt{d} + p_4}\right| > P^{-\mu}. \tag{21}$$

Remark. Assumptions similar to those from Lemma 3 were used in [Amoroso and Viola 2001; Salnikova 2008; Hata 2000].

We prove Theorem 1 by applying Lemma 3 to the linear form

$$L_n = (2-\sqrt{3})^{128n} \cdot 4^{-32n} \cdot \frac{q_{Mn}}{\Delta_n} J_n = (\Lambda_1(n)\sqrt{3} + \Lambda_2(n))\pi + \Lambda_3(n)\sqrt{3} + \Lambda_4(n), \tag{22}$$

where each $\Lambda_i(n)$ is an integer, J_n is the integral (16) for the family of parameters (4), and Mn is defined by equality (17) for the family of parameters (4).

The corresponding constants γ_1 and γ_3 will be calculated in the next Section 2, and the constant γ_2 in Section 3.

2. Asymptotics

The argument of this part is almost completely analogous to those from [Luchin and Salikhov 2018, §2].

Everywhere in the sequel (see (2) and (16)) we write

$$J_n := J_n(11, 37, 16, 27, 37, 32) = \frac{1}{2\pi i}\int_0^{-\infty} ds \int_{-i\infty}^{i\infty} G(s,t)\,dt, \tag{23}$$

where

$$G(s,t) = \varphi(s,t)(f(s,t))^n, \tag{24}$$

with

$$f(s,t) = \frac{s^{11}t^{37}}{(1-s)^6(s-t)^{32}(t-x)^{42}}, \quad \varphi(s,t) = \frac{1}{\sqrt{\frac{s}{s-1}(1-s)(s-t)(t-x)}}, \quad x = \frac{2+\sqrt{3}}{4}.$$

The saddle points are the solutions of the system $f'_s(s,t) = 0$, $f'_t(s,t) = 0$ that differ from the zeros of the function $f(s,t)$. In [Androsenko and Salikhov 2015] (see p. 492, equations (12)) this system was solved in the general case for the integral (16). For the function $f(s,t)$ considered above we have three saddle points:

$$(s_1, t_1) \approx (0.994847; 0.967621), \tag{25}$$

$$(s_2, t_2) \approx (0.324712 + 0.292582i, -0.637736 - 0.207638i), \tag{26}$$

and $(s_3, t_3) = (\bar{s}_2, \bar{t}_2)$, the complex conjugate of (s_2, t_2). We write $\xi = (s,t) \in \mathbb{C}^2$.

Lemma 4. *Let* ξ^0 *be a nondegenerate saddle point of the function* $S(\xi)$, *let* γ *be a two-dimensional smooth complex manifold with boundary, let* ξ^0 *be an interior point of* γ, *let the functions* $\varphi(\xi)$ *and* $S(\xi)$

be holomorphic at the point ξ^0, *and let also* $\max_{\xi\in\gamma} \operatorname{Re} S(\xi)$ *be attained only at the point* ξ^0. *Let*

$$F(\lambda) = \int_\gamma \varphi(\xi) \exp(\lambda S(\xi)) d\xi$$

and $S''_{\xi\xi}(\xi^0) = \left(\begin{pmatrix} S''_{ss}(\xi^0) \\ S''_{st}(\xi^0) \end{pmatrix} \begin{pmatrix} S''_{st}(\xi^0) \\ S''_{tt}(\xi^0) \end{pmatrix} \right)$ *be the Hesse matrix*, *and suppose that* $\det S''_{\xi\xi}(\xi^0) \neq 0$. *Then*

$$F(\lambda) = \frac{2\pi}{\lambda} \exp(\lambda S(\xi^0)) \cdot (\det S''_{\xi\xi}(\xi^0))^{-\frac{1}{2}} (\varphi(\xi^0) + O(\lambda^{-1})) \tag{27}$$

as $\lambda \to +\infty$.

Proof. This statement is proved in the [Fedoryuk 1977], p. 259, Proposition 1.1. ☐

Lemma 5. *For the linear form* (22) *we have the equation*

$$\gamma_1 := \lim_{n\to\infty} \frac{1}{n} \ln |\Lambda_1(n)\sqrt{3} + \Lambda_2(n)| = 128 \ln (2 - \sqrt{3}) - 32 \ln 4 + M_1 + \ln |f(s_1, t_1)| \approx 85.303863, \tag{28}$$

where the value

$$M_1 = M - \lim_{n\to\infty} \frac{1}{n} \ln \Delta_n \approx 11.313066 \tag{29}$$

is calculated using inequalities (18) *for the set of parameters* (4) *and*

$$\ln |f(s_1, t_1)| \approx 286.922828.$$

Proof. Let the integral (23) be written in the form

$$J_n = A'_n \pi + B_n,$$

where A'_n, $B'_n \in \mathbb{Q}[\sqrt{3}]$ (see (19)). Consider the circles $L_1 = \{t : |t| = t_1\}$ and $L_2 = \{s : |s| = s_1\}$. Obviously, $\max_{(s,t)\in L_2^*\times L_1^*} \ln |f(s, t)|$ is attained only at the point (s_1, t_1). As in Lemma 6 from [Luchin and Salikhov 2018], we have

$$A'_n = \frac{1}{2(2\pi i)^2} \int_{L_2} ds \int_{L_1} G(s, t)\, dt,$$

where the function $G(s, t)$ was defined in (23). Here we used the inequalities $x < t_1 < s_1 < 1$.

We apply Lemma 4 for the function $S(s, t) = \ln f(s, t) = \ln |f(s, t)| + ih(s, t)$ (a certain branch of the logarithm defined on the set $\gamma = \gamma_2 \times \gamma_1$, where γ_2 is a small arc of the circle L_2 including the point $s_1 + 0i$ and γ_1 is a small arc of L_1 including $t_1 + 0i$. In our case for the Hessian we have $\det S''_{\xi\xi}(s_1, t_1) \approx 1.92 \times 10^{10} \neq 0$.

Using equality (27) of Lemma 4, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \ln A'_n = \ln |f(s, t)| \approx 286.922828. \tag{30}$$

Let us now evaluate the constant M_1. For the family of parameters (4) from (17) we obtain

$$Mn = \max (12n, 22n, 32n, 32n, 42n, 27n, 37n, 37n, 32n) = 42n.$$

Inequalities (18) for the family of parameters (4) have the form

$$
\begin{aligned}
[32\omega]+[6\omega]+[37\omega]+[27\omega]-[16\omega]-[12\omega]-[32\omega]-[42\omega]&>0,\\
[22\omega]+[6\omega]+[37\omega]+[32\omega]-[11\omega]-[12\omega]-[32\omega]-[42\omega]&>0,\\
[37\omega]+[37\omega]-[32\omega]-[42\omega]&>0,\\
[32\omega]+[6\omega]+[32\omega]-[16\omega]-[12\omega]-[42\omega]&>0,\\
[22\omega]+[6\omega]+[27\omega]-[11\omega]-[12\omega]-[32\omega]&>0.
\end{aligned}
\tag{31}
$$

The set E of numbers $\omega\in[0;1)$ satisfying at least one of the inequalities (31) has the form

$$
\begin{aligned}
E=&\left[\tfrac{1}{37};\tfrac{1}{14}\right)\cup\left[\tfrac{2}{27};\tfrac{2}{21}\right)\cup\left[\tfrac{4}{37};\tfrac{5}{42}\right)\cup\left[\tfrac{5}{37};\tfrac{1}{7}\right)\cup\left[\tfrac{4}{27};\tfrac{1}{6}\right)\cup\left[\tfrac{5}{27};\tfrac{4}{21}\right)\cup\left[\tfrac{8}{37};\tfrac{1}{4}\right)\\
&\cup\left[\tfrac{10}{37};\tfrac{2}{7}\right)\cup\left[\tfrac{11}{37};\tfrac{13}{42}\right)\cup\left[\tfrac{12}{37};\tfrac{5}{14}\right)\cup\left[\tfrac{10}{27};\tfrac{8}{21}\right)\cup\left[\tfrac{15}{37};\tfrac{5}{12}\right)\cup\left[\tfrac{16}{37};\tfrac{7}{16}\right)\cup\left[\tfrac{17}{37};\tfrac{10}{21}\right)\\
&\cup\left[\tfrac{18}{37};\tfrac{1}{2}\right)\cup\left[\tfrac{19}{37};\tfrac{23}{42}\right)\cup\left[\tfrac{5}{9};\tfrac{4}{7}\right)\cup\left[\tfrac{16}{27};\tfrac{25}{42}\right)\cup\left[\tfrac{23}{37};\tfrac{5}{8}\right)\cup\left[\tfrac{24}{37};\tfrac{2}{3}\right)\cup\left[\tfrac{25}{37};\tfrac{29}{42}\right)\\
&\cup\left[\tfrac{26}{37};\tfrac{31}{42}\right)\cup\left[\tfrac{20}{27};\tfrac{3}{4}\right)\cup\left[\tfrac{28}{37};\tfrac{16}{21}\right)\cup\left[\tfrac{7}{9};\tfrac{11}{14}\right)\cup\left[\tfrac{30}{37};\tfrac{13}{16}\right)\cup\left[\tfrac{31}{37};\tfrac{6}{7}\right)\cup\left[\tfrac{19}{22};\tfrac{37}{42}\right)\\
&\cup\left[\tfrac{8}{9};\tfrac{11}{12}\right)\cup\left[\tfrac{34}{37};\tfrac{13}{14}\right)\cup\left[\tfrac{35}{37};\tfrac{20}{21}\right)\cup\left[\tfrac{26}{27};\tfrac{41}{42}\right).
\end{aligned}
$$

Let $\psi(x)=\Gamma'(x)/\Gamma(x)$, where $\Gamma(x)$ stands for the gamma function. Then, in a standard way (see Lemma 6 in [Nesterenko 2010]) we obtain

$$\Delta=\lim_{n\to\infty}\frac{1}{n}\ln\Delta_n=\left(\psi\left(\tfrac{1}{14}\right)-\psi\left(\tfrac{1}{37}\right)\right)+\left(\psi\left(\tfrac{2}{21}\right)-\psi\left(\tfrac{2}{27}\right)\right)+\cdots+\left(\psi\left(\tfrac{41}{42}\right)-\psi\left(\tfrac{26}{27}\right)\right)\approx 30.686934.$$

Finally,

$$\lim_{n\to\infty}\frac{1}{n}\cdot\ln\frac{q_{Mn}}{\Delta_n}=42-\Delta=:M_1\approx 11.313066.$$

It follows from (19) and (22) that $\Lambda_1(n)\sqrt{3}+\Lambda_2(n)=6(2-\sqrt{3})^{128n}\cdot 4^{-32n}\cdot(q_{Mn}/\Delta_n)\cdot A'_n$, and from (30) we obtain the statement of the lemma. □

Lemma 6. *The value of γ_3 for the linear form* (22) *satisfies the equality*

$$\gamma_3=32\ln 4-128\ln(2-\sqrt{3})-M_1-\ln|f(s_2,t_2)|\approx 245.593134, \tag{32}$$

Proof. The argument here is similar to the proof of Lemma 7 in [Luchin and Salikhov 2018]. In our case for the value of the Hessian we have $\det S''_{\xi\xi}(s_2;t_2)\approx -6702+4059i\neq 0$. Note that if $h(s,t)=\operatorname{Im}\ln s,t$, then $h(s_2,t_2)=:\omega\approx -1.833$. Let $\omega_0=\frac{1}{2}(\pi+\omega)$ (earlier in [Luchin and Salikhov 2018] the corresponding values were $\omega\approx 1.9062$, $\omega_0=\frac{1}{2}(\pi-\omega)$). The end of the proof of Lemma 6 is identical to Lemma 7 from [Luchin and Salikhov 2018]. □

We note that, by Lemma 5, we have $M_1\approx 11.313066$ and $\ln|f(s_2,t_2)|\approx -43.974169$. So we obtain the equality (32).

3. Evaluation of the constant γ_2. End of the proof of Theorem 1.

The argument of §3 in [Luchin and Salikhov 2018] with minor changes should be repeated here. Therefore, we restrict ourselves to the statement of results and some comments.

In this section we put $D_N(f(x)) = \frac{1}{N!} f^{(N)}(x)$, where $N \in \mathbb{Z}^+$, and consider the operator $T = D_{k+m-h} \cdot x^j \cdot D_{h+j-k} = D_{42n} x^{37n} D_{32n}$. This operator is analogous to those considered in [Luchin and Salikhov 2018]. It should be mentioned that operators like T where used in [Marcovecchio 2009; Sorokin 1991; Marcovecchio 2014] and many other papers.

Lemma 7. *Let $l \le j$. The integral* (2) *satisfies the equality*

$$J = 2(-1)^{k+l-j} \cdot T\left(\sum_{\nu=0}^{k+l-j-1} \frac{1}{2(k+l-j-\nu)-1} \cdot \frac{x^{m-q+\nu}}{(x-1)^{\nu+1}} + \frac{x^{h-0.5}(-1)^{k+l-j}}{(1-x)^{k+l-j+0.5}} \arctan\sqrt{\frac{1-x}{x}} \right). \quad (33)$$

Proof. It is necessary to repeat the argument of Lemma 10 from [Luchin and Salikhov 2018] with the only change related to the case

$$x = \frac{2+\sqrt{3}}{4} < 1.$$

We obtain

$$\int_0^1 \frac{dz}{z^2 - \frac{x}{x-1}} = \int_0^1 \frac{dz}{z^2 + \frac{x}{1-x}} = \frac{\sqrt{1-x}}{\sqrt{x}} \arctan\sqrt{\frac{1-x}{x}}.$$

A similar integral was considered in [Luchin and Salikhov 2018, Lemma 10]:

$$\int_0^1 \frac{dz}{z^2 - \frac{x}{x-1}} = -\frac{\sqrt{x-1}}{\sqrt{x}} \ln\left(\sqrt{x} + \sqrt{x-1}\right),$$

This is the only difference between (33) and the similar equality (53) from [Luchin and Salikhov 2018]. □

Lemma 8. *Let $M \in \mathbb{N}$, $a, b \in \mathbb{R}$. Then*

$$D_M\left(\frac{x^a}{(1-x)^b}\right) = \sum_{r=0}^{M} \binom{a}{r} \binom{b-a+M-1}{M-r} \frac{x^{a-r}}{(1-x)^{b+M}}.$$

Proof. A similar statement was proven in [Luchin and Salikhov 2018, Lemma 11] for $(1-x) \to (x-1)$. To use that lemma it is enough to choose the branch of the logarithm such that $\ln(-z) = \ln|z| + i\pi$ is satisfied for $z \in \mathbb{R}$, $z > 0$, $\ln z \in \mathbb{R}$. Then, since $x < 1$, we have

$$(1-x)^b = (x-1)^b \cdot e^{i\pi b}, \quad (1-x)^{b+M} = (x-1)^{b+M} \cdot e^{i\pi b}(-1)^M,$$

and the statement of Lemma 8 follows from (54) from [Luchin and Salikhov 2018]. □

Lemma 9 [Luchin and Salikhov 2018, Lemma 12]. *For every $N \in \mathbb{N}$ and for arbitrary analytic functions $u = u(x)$ and $\vartheta = \vartheta(x)$ one has*

$$D_N(u\,\vartheta) = \vartheta \cdot D_N(u) + \sum_{\lambda=0}^{N-1} \frac{\lambda!\,(N-1-\lambda)!}{N!} D_{N-1-\lambda}(D_\lambda(u) \cdot \vartheta').$$

For $x \in \mathbb{R}$ we introduce the function

$$x^* = \begin{cases} x \ln x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ x \ln(-x) & \text{if } x < 0. \end{cases} \tag{34}$$

Obviously, the function x^* is odd.

Lemma 10 [Luchin and Salikhov 2018, Lemma 13]. *Let $n \in \mathbb{N}$, $n \to +\infty$, $b = b_0 n + O(1)$, $r = r_0 n + O(1)$, $b_0, r_0 \in \mathbb{R}$, $r \in \mathbb{Z}^+$ and $\binom{b}{r} \neq 0$. Then one has*

$$\lim_{n\to\infty} \frac{1}{n} \ln \left| \binom{b}{r} \right| = b_0^* - r_0^* - (b_0 - r_0)^*. \tag{35}$$

Now we apply the results obtained above to the linear form (22).

Relation (33) for the family of parameters (4) can be rewritten as

$$J_n = \sum_{\nu=0}^{6n-1} \frac{2(-1)^{\nu+1}}{12n - 2\nu - 1} \Sigma_{1,\nu} + 2\Sigma_2, \tag{36}$$

where

$$\Sigma_{1,\nu} = D_{42n}\left(x^{37n} D_{32n} \left(\frac{x^{5n+\nu}}{(1-x)^{\nu+1}} \right) \right), \tag{37}$$

$$\Sigma_2 = D_{42n}\left(x^{37n} D_{32n}\left(\frac{x^{11n-\frac{1}{2}}}{(1-x)^{6n+\frac{1}{2}}} \arctan \sqrt{\frac{1-x}{x}} \right) \right). \tag{38}$$

For example, we calculate a simpler function (37). Applying Lemma 8 for $\nu = 0, 1, \ldots, 6n-1$, we obtain

$$D_{32n}\left(\frac{x^{5n+\nu}}{(1-x)^{\nu+1}} \right) = \sum_{r=5n}^{5n+\nu} \binom{5n+\nu}{r} \binom{27n}{32n-r} \frac{x^{5n-\nu-r}}{(1-x)^{32n+1+\nu}}.$$

In a similar way we get

$$D_{42n}\left(x^{37n} \frac{x^{5n+\nu-r}}{(1-x)^{32n+\nu+1}} \right) = D_{42n}\left(\frac{x^{42n+\nu-r}}{(1-x)^{32n+\nu+1}} \right) = \sum_{\rho=0}^{42n} \binom{42n+\nu-r}{\rho} \binom{32n+r}{42n-\rho} \frac{x^{42n+\nu-r-\rho}}{(1-x)^{74n+\nu+1}}.$$

For $x = \dfrac{2+\sqrt{3}}{4}$ we obtain

$$\begin{aligned} \frac{x^{42n+\nu-r-\rho}}{(1-x)^{74n+\nu+1}} &= \frac{(2+\sqrt{3})^{42n+\nu-r-\rho}}{4^{42n+\nu-r-\rho}} \cdot 4^{74n+\nu+1} \cdot (2+\sqrt{3})^{74n+\nu+1} \\ &= 4^{32n+r+\rho+1} \cdot (2+\sqrt{3})^{116n+2\nu-r-\rho+1} \\ &= A \cdot 4^{r+\rho+1} (2-\sqrt{3})^{12n-2\nu-1+r+\rho}, \end{aligned}$$

where

$$A = 4^{32n} (2+\sqrt{3})^{128n}. \tag{39}$$

Moreover our parameters should necessarily satisfy $\rho \le 42n + \nu - r$ and $42n - \rho \le 32n + r$, i.e., $\rho \ge 10n - r$. Thus, for $\nu = 0, 1, \dots, 6n-1$, we have $\Sigma_{1,\nu} = A \cdot \Sigma'_{1,\nu}$ and

$$\Sigma'_{1,\nu} = \sum_{(r,\rho)\in B} \binom{5n+\nu}{r}\binom{27n}{32n-r}\binom{42n+\nu-r}{\rho}\binom{32n+r}{42n-\rho} \cdot 4^{r+\rho+1} \cdot (2-\sqrt{3})^{12n-2\nu-1+r+\rho}, \quad (40)$$

where $B = \{(r,\rho) \mid r \in [5n; 5n+\nu],\ \rho \in [\max(0; 10n-r); \min(42n; 42n+\nu-r)]\}$.

Let us calculate the function Σ_2 from (38), applying Lemma 9 for

$$N = 32n, \quad u = \frac{x^{11n-0.5}}{(1-x)^{6n+0.5}}, \quad \vartheta = \arctan\sqrt{\frac{1-x}{x}}, \quad \vartheta' = -\frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

We have

$$D_{32n}\left(\frac{x^{11n-\frac12}}{(1-x)^{6n+\frac12}} \arctan\sqrt{\frac{1-x}{x}}\right) = \arctan\sqrt{\frac{1-x}{x}} \cdot D_{32n}\left(\frac{x^{11n-\frac12}}{(1-x)^{6n+\frac12}}\right)$$
$$- \frac12 \sum_{\lambda=0}^{32n-1} \frac{\lambda!(32n-1-\lambda)!}{(32n)!} D_{32n-1-\lambda}\left(D_\lambda\left(\frac{x^{11n-\frac12}}{(1-x)^{6n+\frac12}}\right) \cdot \frac{1}{\sqrt{x}\sqrt{1-x}}\right).$$

Applying Lemma 9 to the first summand of this sum again, we obtain from (38) (when $N = 42n$, $u = x^{37n} D_{32n}\left(x^{11n-\frac12}/(1-x)^{6n+\frac12}\right)$ and $\vartheta = \arctan\sqrt{(1-x)/x}$) the equality

$$\Sigma_2 = \arctan\sqrt{\frac{1-x}{x}} D_{42n}\left(x^{37n} D_{32n}\left(\frac{x^{11n-\frac12}}{(1-x)^{6n+\frac12}}\right)\right)$$
$$- \frac12 \sum_{\lambda_1=0}^{42n-1} \frac{\lambda_1!(42n-1-\lambda_1)!}{(42n)!} D_{42n-1-\lambda_1}\left(D_{\lambda_1}\left(x^{37n} D_{32n}\left(\frac{x^{11n-\frac12}}{(1-x)^{6n+\frac12}}\right)\right)\frac{1}{\sqrt{x}\sqrt{1-x}}\right)$$
$$- \frac12 \sum_{\lambda=0}^{32n-1} \frac{\lambda!(32n-1-\lambda)!}{(32n)!} D_{42n}\left(x^{37n} D_{32n-1-\lambda}\left(D_\lambda\left(\frac{x^{11n-\frac12}}{(1-x)^{6n+\frac12}}\right)\frac{1}{\sqrt{x}\sqrt{1-x}}\right)\right). \quad (41)$$

We note that for $x = \frac{2+\sqrt{3}}{4}$ one has

$$\sqrt{x} \cdot \sqrt{1-x} = \frac14, \quad \arctan\sqrt{\frac{1-x}{x}} = \arctan(2-\sqrt{3}) = \frac{\pi}{12}.$$

Let us write

$$D_{42n}\left(x^{37n} D_{32n}\left(\frac{x^{11n-\frac12}}{(1-x)^{6n+\frac12}}\right)\right) =: A \cdot \Sigma'_2, \quad (42)$$

$$D_{42n-1-\lambda_1}\left(D_{\lambda_1}\left(x^{37n} D_{32n}\left(\frac{x^{11n-\frac12}}{(1-x)^{6n+\frac12}}\right)\right)\frac{1}{\sqrt{x}\sqrt{1-x}}\right) =: A \cdot \Sigma_{2,1}(\lambda_1), \quad (43)$$

$$D_{42n}\left(x^{37n} D_{32n-1-\lambda}\left(D_\lambda\left(\frac{x^{11n-\frac12}}{(1-x)^{6n+\frac12}}\right)\frac{1}{\sqrt{x}\sqrt{1-x}}\right)\right) =: A \cdot \Sigma_{2,2}(\lambda). \quad (44)$$

Functions (42)–(44) are calculated using Lemma 8 in a standard way (see (40), for example). We thus restrict ourselves to presenting the final results, namely

$$\Sigma_2' = \sum_{(r,\rho)\in B} \binom{11n-\frac{1}{2}}{r}\binom{27n}{32n-r}\binom{48n-r-\frac{1}{2}}{\rho}\binom{32n+r}{42n-\rho}\cdot 4^{r+\rho+1}\cdot(2-\sqrt{3})^{r+\rho}, \tag{45}$$

where $B = \{(r,\rho) \mid r \in [5n; 32n],\ \rho \in [\max(0; 10n-r); 42n]\}$;

$$\Sigma_{2,1}'(\lambda_1) = \sum_{(r_1,r_2,\rho)\in B_1} \binom{11n-\frac{1}{2}}{r_1}\binom{27n}{32n-r_1}\binom{42n-r_1-\frac{1}{2}}{r_2}\binom{r_1+\lambda_1-10n}{\lambda_1-r_2} \cdot\binom{48n-r_1-r_2-1}{\rho}\binom{32n+r_1+r_2}{42n-1-\lambda-\rho}\cdot 4^{r_1+r_2+\rho+1}\cdot(2-\sqrt{3})^{r_1+r_2+\rho+1}, \tag{46}$$

where $B_1 = \big\{(r_1,r_2,\rho) \in (\mathbb{Z}^+)^3 \;\big|\; r_1 \in [5n;32n], r_2 \le \lambda_1;$ if $r_1+\lambda_1 \ge 10n$, then $r_1+r_2 \ge 10n$; $\rho \le 42n-1-\lambda_1$, $\lambda_1+r_1+r_2+\rho \ge 10n-1$; if $r_1+r_2 < 48n-1$, then $\rho \le 48n-r_1-r_2-1\big\}$; and

$$\Sigma_{2,2}'(\lambda) = \frac{(\lambda-\Lambda)!\Lambda!(32n-1-\lambda)!}{(32n)!} \cdot \sum_{(r_1,r_2,r_3,\rho)\in B_2} \binom{11n-\frac{1}{2}}{r_1}\binom{6n+\Lambda-r_1-\frac{1}{2}}{\Lambda-r_1}\binom{11n-r_1-\frac{1}{2}}{r_2}\binom{\lambda-5n}{\lambda-\Lambda-r_2}\binom{11n-r_1-r_2}{r_3}\binom{27n+r_2}{32n-r_3-\lambda-1} \cdot\binom{48n-r_1-r_2-r_3-1}{\rho}\binom{32n+r_2+r_3}{42n-\rho}\cdot 4^{r_2+r_3+\rho+1}\cdot(2-\sqrt{3})^{2r_1+r_2+r_3+\rho+1}, \tag{47}$$

where $\lambda \in [0; 32n-1]$, $\Lambda \in [0;\lambda]$, $B_2 = \big\{(r_1,r_2,r_3,\rho) \in (\mathbb{Z}^+)^4 \;\big|\; r_1 \in [0;\Lambda], r_2 \in [0;\lambda-\Lambda]$, if $\lambda > 5n$, then also $r_2 \ge 5n-\Lambda$; $r_3 \in [\max(0, 5n-\lambda-r_2-1); 32n-1-\lambda]$, if $r_1+r_2 < 11n-1$, then also $r_3 \le 11n-r_1-r_2-1$; $\rho \in [0; \min(42n, 48n-1-r_1-r_2-r_3)]\big\}$.

Finally from (36)–(44) we obtain $J_n = A\cdot J_n^*$, where

$$J_n^* = \sum_{\nu=0}^{6n-1} \frac{2(-1)^{\nu+1}}{12n-2\nu-1}\Sigma_{1,\nu}' + 2\Sigma_2' - \sum_{\lambda_1=0}^{42n-1} \frac{\lambda_1!(42n-1-\lambda_1)!}{(42n)!}\Sigma_{2,1}'(\lambda_1) - \sum_{\lambda=0}^{32n-1}\frac{\lambda!(32n-1-\lambda)!}{(32n)!}\Sigma_{2,2}'(\lambda). \tag{48}$$

Then from (22) and (39) it follows that $(2-\sqrt{3})^{128n}4^{-32n}A = 1$. Thus,

$$L_n = \frac{q_{42n}}{\Delta_n} J_n^*.$$

All the summands in (48) were calculated in (40) and (45)–(47). The asymptotic q_{42n}/Δ_n was calculated in the proof of Lemma 5, so

$$\lim_{n\to\infty} \frac{1}{n}\frac{q_{42n}}{\Delta_n} = M_1 \approx 11.313066.$$

The final step of calculating γ_2 for the linear form (22) is similar to the proof from §3 in [Luchin and Salikhov 2018]. In particular, it is necessary to replace $(2-\sqrt{3})$ by $(2+\sqrt{3})$ in the sums (40) and (45)–(47). The asymptotic behavior of the binomial coefficients included in the summands of these sums is calculated using Lemma 10. Computer calculations show that the corresponding maximal summand is attained in the sum Σ_2' for the values of parameters $r \approx 10.256n$, $\rho \approx 31.431n$. The corresponding value of γ_2 is $\gamma_2 = 169.531 + 11.313066 = 180.844066$.

Then, by Lemma 3, the inequality (1) holds for

$$\mu = \frac{2(\gamma_1 + \gamma_3)}{\gamma_3 - \gamma_2} \approx 10.2209.$$

This completes the proof of Theorem 1.

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On approximations of solutions of the equation $P(z, \ln z) = 0$ by algebraic numbers

Alexander Galochkin and Anastasia Godunova

The paper is devoted to studying how well solutions of an equation $P(z, \ln z) = 0$, where $P(x, y) \in \mathbb{Z}[x, y]$, can be approximated with algebraic numbers. We prove a new bound with the help of a construction due to K. Mahler.

The length of a polynomial is the sum of the absolute values of its coefficients. The length of an algebraic number is the length of its canonical polynomial. Let $\ln z$ be an arbitrary branch of the logarithm. The main result of this paper is the following theorem.

Theorem 1. *Suppose*

$$P(x, y) \in \mathbb{Z}[x, y], \quad \deg_x P = d_1, \quad \deg_y P = d_2, \quad d_1 d_2 \neq 0,$$
$$\zeta \in \mathbb{C}, \quad P(\zeta, y) \not\equiv 0, \quad P(\zeta, \ln \zeta) = 0.$$

Then, for every $\varepsilon > 0$, the inequality

$$|\zeta - \theta| < \exp\left(-(4+\varepsilon) d_1 d_2 \frac{\ln^2 L(\theta)}{\ln \ln L(\theta)}\right) \tag{1}$$

admits only finitely many solutions in algebraic θ such that

$$\varkappa = \deg \theta = o(\ln \ln L(\theta)), \quad \textit{i.e.}, \quad \varkappa < \alpha(L) \cdot \ln \ln L, \quad \lim_{L \to \infty} \alpha(L) = 0. \tag{2}$$

The length of θ can be replaced in (1) by its height $H(\theta)$, as

$$L(\theta) \le (\varkappa + 1) H(\theta).$$

N. I. Feldman [1964] proved a theorem on approximations of the solutions of the equation $P(z, e^z) = 0$ by algebraic numbers. His result was improved in [Galochkin 1972]. A result similar to (1) can be obtained from [Nesterenko and Waldschmidt 1996, Theorem 5] but with a constant greater than 4 in the exponent. Our proof is based on Mahler's construction [1932a; 1932b; 1967] with a special choice of parameters.

Lemma 2. *Suppose $P(x) \in \mathbb{Z}[x]$, θ is an algebraic number, and $P(\theta) \neq 0$. Then*

$$|P(\theta)| \ge L(P)^{1-\deg \theta} L(\theta)^{-\deg P},$$

where $L(P)$ and $L(\theta)$ are the lengths of P and θ respectively.

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The proof can be found, for instance, in [Feldman 1981].

Lemma 3. *Let m, n be positive integers. For each $k = \overline{0, n}$ set*

$$\Phi_k(t) = (t-m)^{k+1} \prod_{j=0}^{m-1} (t-j)^{n+1}, \quad R_k(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{e^{tz}}{\Phi_k(t)}\, dt, \tag{3}$$

where Γ is the circle $|t| = m(n+1)+m$. Then

$$\operatorname{ord}_{z=0} R_k(z) = m(n+1)+k, \tag{4}$$

$$|R_k(z)| < (m(n+1)+m)e^{(m(n+1)+m)|z|}(m(n+1))^{-m(n+1)}, \tag{5}$$

$$R_k(z) = P_{k0}(z) + P_{k1}(z)e^z + \cdots + P_{km}(z)e^{mz}, \quad P_{kj}(z) \in \mathbb{Q}[z], \tag{6}$$

$$n_{kj} = \deg P_{kj} = (\operatorname{ord}_{t=j} \Phi_k(t)) - 1 = \begin{cases} n & \text{for } j = \overline{0, m-1}, \\ k & \text{for } j = m,\ k = \overline{0, n}. \end{cases} \tag{7}$$

Set $p_{kj}(z) = b^n n!\,(m!)^{n+1} P_{kj}(z)$, where $b = \operatorname{lcm}(1, 2, \ldots, m)$. Then

$$p_{kj}(z) \in \mathbb{Z}[z], \quad L(p_{kj}) < e^{\gamma_1 mn} n!, \quad \text{where } \gamma_1 \text{ is an absolute constant.} \tag{8}$$

Proof. We have

$$R_k(z) = \sum_{s=0}^{\infty} \frac{a_{ks}}{s!} z^s, \quad a_{ks} = \frac{1}{2\pi i} \oint_\Gamma \frac{t^s}{\Phi_k(t)}\, dt,$$

$a_{ks} = 0$ for $s < n_k$, $a_{k,n_k} \neq 0$, $n_k = m(n+1)+k$, which proves (4).

Inequality (5) follows from the estimate

$$|\Phi_k(t)| \geq (m(n+1))^{m(n+1)}$$

and an obvious estimate on the integral in (3).

We have

$$R_k(z) = \sum_{j=0}^{m} I_{kj}, \quad I_{kj} = \frac{1}{2\pi i} e^{jz} \oint_{|t-j|=1/2} \frac{e^{(t-j)z}}{\Phi_k(t)}\, dt = e^{jz} \sum_{s=0}^{n_{kj}} \frac{a_{kjs}}{s!} z^s,$$

where

$$a_{kjs} = \frac{1}{2\pi i} \oint_{|t-j|=1/2} \frac{(t-j)^s}{\Phi_k(t)}\, dt, \quad a_{k,j,n_{kj}} \neq 0, \quad P_{kj}(z) = \sum_{s=0}^{n_{kj}} \frac{a_{kjs}}{s!} z^s. \tag{9}$$

This proves both (6) and (7).

Since $|t-j| = \frac{1}{2}$, we have

$$|\Phi_k(t)| > (m!)^{n+1} e^{-\gamma_2 mn}.$$

Thus, by (9)

$$|a_{kjs}| < (m!)^{-n-1} e^{\gamma_3 mn}. \tag{10}$$

Let us use the substitution $t - j = bu$, where $b = \operatorname{lcm}(1, 2, \dots, m) = e^{O(m)}$, in order to transform the integral in (9). Then for $l \neq j$

$$t - l = bu + j - l = (j - l)\left(1 - \frac{bu}{l - j}\right).$$

This substitution gives $a_{kjs} = A_{kjs} B_{kjs}$, where

$$A_{kjs} = b^{s-n_{kj}} \prod_{l=0,\, l\neq j}^{m} (j - l)^{-n_{kl}-1},$$

$$B_{kjs} = \oint_{|u|=(2b)^{-1}} \prod_{l=0,\, l\neq j}^{m} \left(1 - \frac{bu}{l - j}\right)^{-n_{kl}-1} u^{s-n_{kj}-1}\, du,$$

with n_{lj} defined by (7).

The coefficients of the series

$$\left(1 - \frac{bu}{l - j}\right)^{-1} = \sum_{r=0}^{\infty} \left(\frac{b}{l - j}\right)^{r} u^{r}$$

are integers; hence $B_{kjs} \in \mathbb{Z}$ and $b^n (m!)^{n+1} A_{kjs} \in \mathbb{Z}$. Taking into account (9) and (10), we get (8). □

Lemma 4. *There exist polynomials $B_{ks}(u)$ which, together with the corresponding form*

$$V_k(u) = \sum_{s=0}^{n} B_{ks}(u)(\ln u)^s,$$

enjoy the properties

$$B_{ks}(u) \in \mathbb{Z}[u], \quad k = \overline{0, n}, \tag{11}$$

$$\deg B_{ks} = \begin{cases} m & \text{for } k \leq s, \\ m - 1 & \text{for } k > s, \end{cases} \tag{12}$$

$$\Delta(u) = \det |B_{ks}|_{k,s=\overline{0,n}} = \lambda (u - 1)^{m(n+1)}, \quad \lambda \neq 0, \tag{13}$$

$$L(B_{ks}(u)) < e^{\gamma_4 mn} n!\,, \tag{14}$$

$$|V_k(u)| < e^{\gamma_5 |u| mn} n^{-mn}. \tag{15}$$

Proof. Let us set

$$V_k(u) = b^n (n!)(m!)^{n+1} R_k(\ln u) = \sum_{j=0}^{m} p_{kj}(\ln u) u^j = \sum_{s=0}^{n} B_{ks}(u)(\ln u)^s. \tag{16}$$

Statements (11), (12), (14), and (15) follow from Lemma 3. Thus, it remains to prove (13).

First, let us assume that $|u - 1| < 1$ and that $\ln 1 = 0$. In this case we have by (4)

$$R_k(z) = z^{m(n+1)+k} T_k(z), \quad T_k(0) \neq 0,$$

whence, taking into account that $\ln 1 = 0$, we get

$$V_k(u) = (\ln u)^{m(n+1)+k} F_k(u) = (u - 1)^{m(n+1)+k} G_k(u), \quad G_k(1) \neq 0.$$

It follows from (12) that $\Delta(u) \not\equiv 0$ and that

$$\deg \Delta(u) = m(n+1).$$

Replacing the first column with the one consisting of $V_0(u), V_1(u), \ldots, V_n(u)$ preserves the determinant. Hence

$$\operatorname{ord}_{u=1} \Delta(u) \geq m(n+1),$$

which implies (13).

Moving along a path around the origin changes $\ln u$, but it does not change $B_{ks}(u)$. Therefore, it does not change $\Delta(u)$. Thus, (13) holds for every branch of the logarithm. □

Theorem 5. *Suppose*

$$P(x, y) \in \mathbb{Z}[x, y], \quad \deg_x P = d_1, \quad \deg_y P = d_2, \quad d_1 d_2 \neq 0.$$

Then, for every $\varepsilon > 0$ *and every* $r > 0$, *the inequality*

$$|P(\theta, \ln\theta)| < \exp\left(-(4+\varepsilon) d_1 d_2 \frac{\ln^2 L(\theta)}{\ln\ln L(\theta)}\right) \tag{17}$$

admits only finitely many solutions in algebraic θ *such that*

$$|\theta| < r \qquad \text{and} \qquad \varkappa = \deg\theta = o(\ln\ln L(\theta)) \quad \text{as } L(\theta) \to \infty. \tag{18}$$

Note that Theorem 1 follows from Theorem 5. Indeed, for all but finitely many θ Theorem 5 provides

$$|P(\theta, \ln\theta)| \geq \exp\left(-(4+\varepsilon) d_1 d_2 \frac{\ln^2 L(\theta)}{\ln\ln L(\theta)}\right).$$

Hence

$$|P(\theta, \ln\theta)| = |P(\zeta, \ln\zeta) - P(\theta, \ln\theta)| = \left|\int_\theta^\zeta P'(t, \ln t)\, dt\right| < \gamma_6 |\zeta - \theta|,$$

and we can assume that $|\zeta - \theta| < 1$, $r = |\zeta| + 1$. Thus, it remains to prove Theorem 5.

Proof of Theorem 5. Let us take

$$m = \left[\frac{d_1}{d_2} n\right], \quad n > d_2. \tag{19}$$

Then by (14) and (15) we have

$$L(B_{ks}(u)) < e^{\gamma_4 mn} n! < e^{\gamma_7 n^2}, \quad |V_k(u)| < e^{\gamma_7 n^2} n^{-d_1 d_2^{-1} n^2}. \tag{20}$$

Let θ be an algebraic number, $\varkappa = \deg\theta$, $L = L(\theta)$. We may assume that

$$\theta \neq 0, \quad \theta \neq 1, \quad P(\theta, y) \not\equiv 0.$$

This excludes finitely many values of θ. The values

$$W_k(\theta) = (\ln\theta)^k P(\theta, \ln\theta) = \sum_{s=0}^{n} A_{ks}(\theta)(\ln\theta)^s, \quad k = \overline{0, v}, \ v = n - d_2, \tag{21}$$

of the corresponding forms at $1, \ln\theta, \ldots, (\ln\theta)^n$ are linearly independent. Moreover, we have $|A_{ks}(\theta)| < e^{\gamma_8 n}$. Hence we can choose d_2 values among $V_0(\theta), \ldots, V_n(\theta)$ (say, $V_1(\theta), \ldots, V_{d_2}(\theta)$) which are linearly independent with the values from (21) and such that

$$D(\theta) = \begin{vmatrix} A_{00}(\theta) & A_{01}(\theta) & \cdots & A_{0n}(\theta) \\ \vdots & \vdots & & \vdots \\ A_{v0}(\theta) & A_{v1}(\theta) & \cdots & A_{vn}(\theta) \\ B_{10}(\theta) & B_{11}(\theta) & \cdots & B_{1n}(\theta) \\ \vdots & \vdots & & \vdots \\ B_{d_2,0}(\theta) & B_{d_2,1}(\theta) & \cdots & B_{d_2,n}(\theta) \end{vmatrix} \neq 0. \tag{22}$$

Consider the determinant

$$D(u) = \begin{vmatrix} A_{00}(u) & A_{01}(u) & \cdots & A_{0n}(u) \\ \vdots & \vdots & & \vdots \\ A_{v0}(u) & A_{v1}(u) & \cdots & A_{vn}(u) \\ B_{10}(u) & B_{11}(u) & \cdots & B_{1n}(u) \\ \vdots & \vdots & & \vdots \\ B_{d_2,0}(u) & B_{d_2,1}(u) & \cdots & B_{d_2,n}(u) \end{vmatrix}$$

as a polynomial of u. By (11), (12), (19), (20), and (22),

$$D(u) \in \mathbb{Z}[u], \quad D(\theta) \neq 0, \quad \deg D(u) \leq nd_1 + md_2 \leq 2nd_1, \quad L(D(u)) < e^{\gamma_9 n^2}. \tag{23}$$

By Lemma 2,

$$|D(\theta)| \geq e^{(1-\varkappa)\gamma_9 n^2} L^{-2d_1 n} > e^{-\gamma_9 \varkappa n^2} L^{-2d_1 n}, \quad L = L(\theta). \tag{24}$$

On the other hand,

$$D(\theta) = \begin{vmatrix} W_0(\theta) & A_{01}(\theta) & \cdots & A_{0n}(\theta) \\ \vdots & \vdots & & \vdots \\ W_v(\theta) & A_{v1}(\theta) & \cdots & A_{vn}(\theta) \\ V_1(\theta) & B_{11}(\theta) & \cdots & B_{1n}(\theta) \\ \vdots & \vdots & & \vdots \\ V_{d_2}(\theta) & B_{d_2,1}(\theta) & \cdots & B_{d_2,n}(\theta) \end{vmatrix};$$

i.e.,

$$D(\theta) = \sum_{k=0}^{v} W_k(\theta) M_k(\theta) + \sum_{l=1}^{d_2} V_l(\theta) N_l(\theta),$$

where $M_k(\theta)$ and $N_l(\theta)$ are the cofactors of the first column of $D(\theta)$. It follows from (20), (21), and (23) that

$$|W_k(\theta)| < e^{\gamma_8 n} |P(\theta, \ln\theta)|, \quad |V_l(\theta)| < e^{\gamma_7 n^2} n^{-d_1 d_2^{-1} n^2},$$
$$|M_k(\theta)| < e^{\gamma_{10} n^2}, \quad |N_l(\theta)| < e^{\gamma_{10} n^2}.$$

Hence

$$|D(\theta)| < e^{\gamma_{11} n^2} |P(\theta, \ln\theta)| + e^{\gamma_{11} n^2} n^{-d_1 d_2^{-1} n^2}.$$

Taking into account (24), we get

$$e^{-2d_1 n \ln L} < e^{\gamma_{12} \varkappa n^2} |P(\theta, \ln\theta)| + e^{\gamma_{12} \varkappa n^2 - d_1 d_2^{-1} n^2 \ln n}. \tag{25}$$

Given an arbitrary $\varepsilon > 0$, let us set

$$n = \left[\left(2 + \frac{\varepsilon}{4}\right) d_2 \frac{\ln L}{\ln \ln L}\right].$$

Then

$$2d_1 n \ln L \sim \left(4 + \frac{\varepsilon}{2}\right) d_1 d_2 \frac{\ln^2 L}{\ln \ln L} \qquad \text{as } L \to \infty,$$

$$d_1 d_2^{-1} n^2 \ln n \sim \left(4 + \varepsilon + \frac{\varepsilon^2}{16}\right) d_1 d_2 \frac{\ln^2 L}{\ln \ln L} \quad \text{as } L \to \infty.$$

Hence due to restrictions (18),

$$\gamma_{12} \varkappa n^2 = o\left(\frac{\ln^2 L}{\ln \ln L}\right).$$

Thus, for L large enough we have $e^{\gamma_{12}\varkappa n^2 - d_1 d_2^{-1} n^2 \ln n} < \frac{1}{2} e^{-2d_1 n \ln L}$. Combining this with (25), we get

$$|P(\theta, \ln \theta)| > \tfrac{1}{2} e^{-\gamma_{12}\varkappa n^2 - 2d_1 n \ln L} > \exp\left(-(4 + \varepsilon) d_1 d_2 \frac{\ln^2 L}{\ln \ln L}\right),$$

which implies that inequality (17) has finitely many solutions.

Theorems 5 and 1 are proved. □

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Two integral transformations related to $\zeta(2)$

Raffaele Marcovecchio

To the memory of Naum Ilyitch Feldman (1918–1994)

We prove two integral transformations that relate different constructions of rational approximations to $\zeta(2)$. The first one relates a double integral over the unit square and a Barnes-type integral. The second one relates two Barnes-type integrals and was discovered and proved by W. Zudilin using an automated proof method. Here we offer a proof based on more classical means such as contiguous relations, the second Barnes lemma and the duplication formula for the gamma function.

1. Introduction

A few years ago, W. Zudilin [2014] refined a long-standing record for the upper bound of the irrationality measure of $\zeta(2)$, let us call it $\mu(\zeta(2))$, by proving that $\mu(\zeta(2)) \leqslant 5.09541178\cdots = \mu_0$, say. This simply means that for every $\mu > \mu_0$ the inequality

$$\left|\zeta(2) - \frac{p}{q}\right| < q^{-\mu}$$

has only finitely many solutions $(p, q) \in \mathbb{Z}^2$. The previously known upper bound for $\mu(\zeta(2))$ was established by G. Rhin and C. Viola [1996], who introduced their permutation-group method and proved that $\mu(\zeta(2)) \leqslant 5.44124250\ldots$. A slightly different proof of their result was presented in [Marcovecchio 2013]. The construction in [Rhin and Viola 1996] of a suitable sequence of rational approximations to $\zeta(2)$ relies on a certain family of double integrals over the unit square that is more general than those in all previous papers, such as, e.g., [Beukers 1979; Hata 1995], while in [Zudilin 2014] two different complex integrals are employed (*first* and *second tales*), and an identity between these integrals is established (the *interlude*). In connection to this last identity, we provide the reader with two more references: [Nassrallah and Rahman 1986, Equation (3.17)] and [Verma and Jain 1992, Equation (4.8)]. They are closely related, though it seems to be not straightforward to fit [Zudilin 2014, Equation (19)] in one, or both, of them.

One purpose of the present paper is to present a bridge between those different approaches. The existence of such a bridge, though in a context that seemingly does not cover the general construction in [Zudilin 2014], is made explicit in [Zudilin 2007]. Here we resort to the concept of *multiple Legendre polynomials* introduced in [Marcovecchio 2012] and developed in [Marcovecchio 2014]. This is detailed in Section 2 below, leading to the integral transformation (2), and offers a new viewpoint to the first tale in [Zudilin 2014]. The second aim of the paper is to give a *human-generated* proof of [Zudilin 2014, Proposition 2], i.e., the interlude. This is the subject of Section 3 below. Our proofs are self-contained

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and rely heavily on contiguous relations, in the spirit of the treatment of the real double integrals made in [Rhin and Viola 1996], In the first part we also employ a little bit of the traditional machinery of iterated partial integration usually involved in similar context where Legendre-type polynomials play a crucial role. In the second part, the second Barnes lemma and the duplication formula for the gamma function come into play.

2. The first transformation

2A. ***Overview on multiple Legendre polynomials.*** We recall the definition and a few facts about the so-called multiple Legendre polynomials we introduced in [Marcovecchio 2014]. For any $n \geqslant 1$, let $p_1, \dots, p_n, q_1, \dots, q_n \geqslant 0$ be integers, and let

$$\mathcal{L}_n(p_1, q_1; \dots; p_n, q_n; z) \in \mathbb{Z}[z] \tag{1}$$

be the polynomials recursively defined by

$$\mathcal{L}_n(p_1, q_1; \dots; p_n, q_n; z) = z^{q_n}(1-z)^{p_n} D_{p_n+q_n}(z^{p_n}(1-z)^{q_n}\mathcal{L}_{n-1}(p_1, q_1; \dots; p_{n-1}, q_{n-1}; z)),$$

where by agreement $\mathcal{L}_0(z) \equiv 1$. Here and in the sequel

$$D_m(f(u)) = \frac{1}{m!}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^m f(u).$$

We emphasize the main property of $\mathcal{L}_n(p_1, q_1; \dots; p_n, q_n; z)$ in the following proposition. Let $\mathfrak{S}_n$ be the symmetric group of n elements.

Proposition 2.1. *For τ, σ in the symmetric group $\mathfrak{S}_n$, the polynomial*

$$(p_{\tau(1)} + q_{\sigma(1)})! \cdots (p_{\tau(n)} + q_{\sigma(n)})!\, \mathcal{L}_n(p_{\tau(1)}, q_{\sigma(1)}; \dots; p_{\tau(n)}, q_{\sigma(n)}; z)$$

does not depend on τ and σ. Briefly, we say that it is $\langle p_1, \dots, p_n\rangle$-stable and $\langle q_1, \dots, q_n\rangle$-stable.

In particular, for σ in $\mathfrak{S}_n$,

$$\mathcal{L}_n(p_{\sigma(1)}, q_{\sigma(1)}; \dots; p_{\sigma(n)}, q_{\sigma(n)}; z)$$

is independent of σ. We briefly say that it is $\langle (p_1, q_1), \dots, (p_n, q_n)\rangle$-stable.

For a proof, we refer the reader to [Marcovecchio 2014, p. 1834]. The above proposition summarizes the *raison d'être* of (1), and is crucial in the proof of (2) below.

Since $\mathcal{L}_1(0, 0; z) = 1$ and

$$\mathcal{L}_2(0, q; p, 0; z) = (-1)^q \binom{p+q}{p} z^q (1-z)^p = \binom{p+q}{p} \mathcal{L}_1(p, q; z),$$

up to changing n into some $n' < n$ we may essentially suppose that $p_i + q_j > 0$ for any $i, j = 1, \dots, n$, i.e., either $p_1, \dots, p_n > 0$ or $q_1, \dots, q_n > 0$.

We remark that

$$\deg \mathcal{L}_n(p_1, q_1; \dots; p_n, q_n; z) = p_1 + q_1 + \dots + p_n + q_n,$$
$$\operatorname*{ord}_{z=0} \mathcal{L}_n(p_1, q_1; \dots; p_n, q_n; z) = \max\{q_1, \dots, q_n\}.$$

In [Marcovecchio 2013] we explained how to recover the irrationality measure of $\zeta(2)$ proven in [Rhin and Viola 1996] by using double integrals of Beukers' type involving the polynomial $\mathcal{L}_2(p_1, q_1; p_2, q_2; z)$.

In [Marcovecchio 2014] the polynomials (1) with $n > 2$ were shown to be a significant tool in connection to diophantine properties of other constants. For example, we obtained an irrationality measure of $\log 2$ with $n = 3$ (already proved in an earlier paper, through a different method), a new nonquadraticity measure of $\log 2$, a new noncubicity measure of $\log\left(\frac{5}{4}\right)$ with $n = 4$, and a new nonquarticity measure of $\log\left(\frac{20}{19}\right)$ with $n = 5$: all those results are the best known. Here we obtain, for $n = 3$, the same rational approximations to $\zeta(2)$ produced in [Zudilin 2014].

2B. ***Double integrals over the unit square.*** Let $n \geqslant 2$ be an integer and $L, p_1, \dots, p_n, q_0, q_1, \dots, q_n \geqslant 0$ be integers such that $L \geqslant q_0$. We introduce the double integral (sometimes abbreviated as $\mathcal{I}_n$)

$$\mathcal{I}_n(L; q_0; p_1, q_1; \dots; p_n, q_n) = (-1)^{q_0} \int_0^1 \int_0^1 \mathcal{L}_n(p_1, q_1; \dots; p_n, q_n; x) y^{L-q_0} (1-y)^{q_0} \frac{\mathrm{d}x\, \mathrm{d}y}{1-xy}.$$

By the symmetry properties of $\mathcal{L}_n(p_1, q_1; \dots; p_n, q_n; x)$ we deduce that if $L \geqslant q_j$ for all $j = 0, 1, \dots, n$, then

$$(p_1+q_1)! \cdots (p_n+q_n)!\, \mathcal{I}_n(L; p_1, \dots, p_n; q_0, q_1, \dots, q_n)$$

is $\langle p_1, \dots, p_n\rangle$-stable and $\langle q_1, \dots, q_n\rangle$-stable. By performing a (p_n+q_n)-fold integration by parts with respect to x, taking into account that

$$D_{p_n+q_n}\left(\frac{x^{q_n}(1-x)^{p_n}}{1-xy}\right) = \frac{y^{-q_n}(1-y^{-1})^{p_n} y^{p_n+q_n}}{(1-xy)^{p_n+q_n+1}},$$

we write

$$\begin{aligned}
&\mathcal{I}_n(L; q_0; p_1, q_1; \dots; p_n, q_n) \\
&\quad = (-1)^{q_n+q_{n+1}} \int_0^1 \int_0^1 \frac{x^{p_n}(1-x)^{q_n} \mathcal{L}_{n-1}(p_1, q_1; \dots; p_{n-1}, q_{n-1}; x) y^{L-q_0}(1-y)^{p_n+q_0}}{(1-xy)^{p_n+q_n+1}}\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned}$$

By performing a second (p_n+q_n)-fold integration by parts, this time with respect to y, when $L \geqslant \max\{q_0, q_n\}$, the integral $\mathcal{I}_n(L; q_0; p_1, q_1; \dots; p_n, q_n)$ also equals

$$\begin{aligned}
&(-1)^{p_n+\max\{q_0,q_n\}} \int_0^1 \int_0^1 x^{p_n}(1-x)^{\min\{q_0,q_n\}} \mathcal{L}_{n-1}(p_1, q_1; \dots; p_{n-1}, q_{n-1}; x) \\
&\qquad \times y^{p_n+\min\{q_0,q_n\}}(1-y)^{\max\{0,q_n-q_0\}} D_{p_n+q_n}(y^{L-q_0}(1-y)^{p_n+q_0}) \frac{\mathrm{d}x\, \mathrm{d}y}{1-xy}.
\end{aligned}$$

We have

$$D_{p_n+q_n}(y^{L-q_0}(1-y)^{p_n+q_0}) = \frac{(L-q_0)!\,(p_n+q_0)!}{(L-q_n)!\,(p_n+q_n)!}(1-y)^{q_0-q_n} D_{p_n+q_0}(y^{L-q_n}(1-y)^{p_n+q_n});$$

see, e.g., [Marcovecchio 2013, Equation (8)]. By combining this with Proposition 2.1, we get the following:

Proposition 2.2. *If $L \geqslant \max\{q_0, q_1, \dots, q_n\}$, then the function*

$$\frac{(p_1+q_1)! \cdots (p_n+q_n)!}{(L-q_0)!} \mathcal{I}_n(L; q_0; p_1, q_1; \dots; p_n, q_n)$$

is $\langle p_1, \dots, p_n\rangle$-stable and $\langle q_0, q_1, \dots, q_n\rangle$-stable.

It is worth noticing that the complementary transformation

$$D_{p_n+q_n}(y^{L-q_0}(1-y)^{p_n+q_0}) = \frac{(L-q_0)!\,(p_n+q_0)!}{(L-q_n)!\,(p_n+q_n)!}(-y)^{L-p_n-q_n-q_0}D_{L-q_0}(y^{p_n+q_n}(1-y)^{L-q_n})$$

from [Marcovecchio 2013, Equation (7)] is compatible with the symmetry properties in Proposition 2.1 only when $n=2$ and $L=q_0+q_1+q_2$. This case was examined in [Marcovecchio 2013, Section 2], and the results therein are the same as in [Rhin and Viola 1996].

2C. ***Relation with Zudilin's integral.*** By [Hata 1995, Lemma 1.1]

$$\mathcal{I}_n(L;q_0;p_1,q_1;\ldots;p_n,q_n) = \mathcal{J}_n(L;q_0;p_1,q_1;\ldots;p_n,q_n)\zeta(2) - \mathcal{A}_n(L;q_0;p_1,q_1;\ldots;p_n,q_n),$$

with $\mathcal{A}_n(L;q_0;p_1,q_1;\ldots;p_n,q_n)\in\mathbb{Q}$ and

$$\mathcal{J}_n(L;q_0;p_1,q_1;\ldots;p_n,q_n) = \frac{(-1)^{q_0}}{2\pi i}\oint_{|x|=\varrho}\mathcal{L}_n(p_1,q_1;\ldots;p_n,q_n;x)\frac{1}{x^{L-q_0}}\left(1-\frac{1}{x}\right)^{q_0}\frac{\mathrm{d}x}{x}\in\mathbb{Z}$$

for any small positive ϱ. We now perform the change of variable $x=u/(u-1)$, and take into account

$$\mathcal{L}_n(p_1,q_1;\ldots;p_n,q_n;x) = (1-u)\mathcal{S}_n(p_1,q_1;\ldots;p_n,q_n;u),$$

where

$$\mathcal{S}_n(p_1,q_1;\ldots;p_n,q_n;u) = \sum_{k\geqslant 0}\binom{k+p_1}{p_1+q_1}\cdots\binom{k+p_n}{p_n+q_n}u^k;$$

see [Marcovecchio 2014, Equation (7)]. Also, when x runs along a small closed path around 0 in the positive direction, $1-x$ moves around 1 in the negative direction, and so does $1-u=(1-x)^{-1}$. Hence $u=x/(x-1)$ spans a small closed path around 0 in the positive direction, and

$$\mathcal{J}_n(L;q_0;p_1,q_1;\ldots;p_n,q_n) = \frac{(-1)^{q_0}}{2\pi i}\oint\mathcal{S}_n(p_1,q_1;\ldots;p_n,q_n;u)\left(1-\frac{1}{u}\right)^{L-q_0}\frac{1}{u^{q_0}}\frac{\mathrm{d}u}{u}.$$

By applying the binomial theorem to $(1-1/u)^{L-q_0}$ we infer that

$$\mathcal{J}_n(L;q_0;p_1,q_1;\ldots;p_n,q_n) = \sum_{j=0}^{L-q_0}(-1)^{j+q_0}\binom{L-q_0}{j}\binom{j+p_1+q_0}{p_1+q_1}\cdots\binom{j+p_n+q_0}{p_n+q_n}.$$

This proves that $\mathcal{J}_3(L;q_0;p_1,q_1;p_2,q_2;p_3,q_3)$ coincides, up to the sign, with the coefficient of $\zeta(2)$ in [Zudilin 2014, Proposition 1], where the parameters $(a_1,\ldots,a_4)$ and $(b_1,\ldots,b_4)$ are chosen to be

$$(a_1,a_2,a_3,a_4) = (q_1,q_2,q_3,q_0) \quad\text{and}\quad (b_1,b_2,b_3,b_4) = (-p_1,-p_2,-p_3,L+1),$$

and therefore suggests that the linear forms obtained through the two constructions coincide as well. More precisely, let

$$R_n(L;q_0;p_1,q_1;\ldots;p_n,q_n;t) := (-1)^{q_0+p_1+q_1+\cdots+p_n+q_n}\frac{(L-q_0)!}{(t+q_0)_{L-q_0+1}}\prod_{j=1}^{n}\frac{(t-p_j)_{p_j+q_j}}{(p_j+q_j)!}$$

and

$$\widetilde{\mathcal{I}}_n(L; q_0; p_1, q_1; \dots; p_n, q_n) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} R_n(L; q_0; p_1, q_1; \dots; p_n, q_n; t)\left(\frac{\pi}{\sin \pi t}\right)^2 \mathrm{d}t.$$

In $\widetilde{\mathcal{I}}_n$, and in all similar integrals hereafter, the integration path is chosen so that it separates two sequences of consecutive poles of the integrand.

Theorem 2.3. *For all nonnegative integers $p_1, \dots, p_n, q_0, q_1, \dots, q_n$ and for all integers $L \geqslant q_0$*

$$\mathcal{I}_n(L; q_0; p_1, q_1; \dots; p_n, q_n) = \widetilde{\mathcal{I}}_n(L; q_0; p_1, q_1; \dots; p_n, q_n). \tag{2}$$

Proof. In the double integral defining $\mathcal{I}_n(L; q_0; p_1, q_1; \dots; p_n, q_n)$, we apply the binomial theorem to obtain

$$y^{L-q_0}(1-y)^{q_0} = \sum_{j=0}^{L-q_0} (-1)^j \binom{L-q_0}{j} (1-y)^{q_0+j},$$

whence

$$\mathcal{I}_n(L; q_0; p_1, q_1; \dots; p_n, q_n) = \sum_{j=0}^{L-q_0} \binom{L-q_0}{j} \mathcal{I}_n(q_0+j; q_0+j; p_1, q_1; \dots; p_n, q_n).$$

Since

$$\frac{(L-q_0)!}{(t+q_0)_{L-q_0+1}} = \sum_{j=0}^{L-q_0} (-1)^j \binom{L-q_0}{j} \frac{1}{t+q_0+j},$$

we also have

$$\widetilde{\mathcal{I}}_n(L; q_0; p_1, q_1; \dots; p_n, q_n) = \sum_{j=0}^{L-q_0} \binom{L-q_0}{j} \widetilde{\mathcal{I}}_n(j+q_0; j+q_0; p_1, q_1; \dots; p_n, q_n).$$

Hence we may assume that $L = q_0$.

We now argue by induction on n. Let $n = 0$. We have, see [Rhin and Viola 1996],

$$\int_0^1\int_0^1 (1-y)^{q_0} \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy} = \int_0^1\int_0^1 (xy)^{q_0} \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy} = \zeta(2) - \sum_{j=1}^{q_0} \frac{1}{j^2}.$$

Also, by [Zudilin 2014, Lemma 2]

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \left(\frac{\pi}{\sin \pi t}\right)^2 \frac{\mathrm{d}t}{t+q_0} = \zeta(2) - \sum_{j=1}^{q_0} \frac{1}{j^2}.$$

This achieves the case $n = 0$.

Let $n > 0$. If $q_1 > 0$, then by $t + q_1 - 1 = (t - p_1 - 1) + (p_1 + q_1)$ we have

$$\frac{(t-p_1)_{p_1+q_1}}{(p_1+q_1)!} = \frac{(t-p_1-1)_{p_1+q_1}}{(p_1+q_1)!} + \frac{(t-p_1)_{p_1+q_1-1}}{(p_1+q_1-1)!}.$$

It follows that

$$\widetilde{\mathcal{I}}_n(L,q_0;\,p_1,q_1;\dots)=\widetilde{\mathcal{I}}_n(L,q_0;\,p_1+1,q_1-1;\dots)-\widetilde{\mathcal{I}}_n(L,q_0;\,p_1,q_1-1;\dots).$$

Similarly, since $\mathcal{L}_1(p_1,q_1;x)=(-x)^{q_1}(1-x)^{p_1}=(-x)^{q_1-1}(1-x)^{p_1+1}-(-x)^{q_1-1}(1-x)^{p_1}$, we have

$$\mathcal{L}_n(p_1,q_1;\dots)=\mathcal{L}_n(p_1+1,q_1-1;\dots)-\mathcal{L}_n(p_1,q_1-1;\dots).$$

Therefore

$$\mathcal{I}_n(L,q_0;\,p_1,q_1;\dots)=\mathcal{I}_n(L,q_0;\,p_1+1,q_1-1;\dots)-\mathcal{I}_n(L,q_0;\,p_1,q_1-1;\dots).$$

By repeating this argument finitely many times, we eventually write $\mathcal{I}_n$ and $\widetilde{\mathcal{I}}_n$ as sums of integrals with $q_1=0$, which we temporarily assume (let us recall that $L=q_0$).

If $n>1$, by interchanging q_1 and q_2 we may assume that $q_2=0$, instead of $q_1=0$. If $n=1$, after interchanging q_0 and q_1 we have no longer $L=q_0$, but $L=q_1$ and $q_0=0$ instead. For technical reasons the case $n=1$ will be dealt with separately.

If $p_1>0$, from $t-p_1=(t+q_1)-(p_1+q_1)$ we have

$$\frac{(t-p_1)_{p_1+q_1}}{(p_1+q_1)!}=\frac{(t-p_1+1)_{p_1+q_1}}{(p_1+q_1)!}-\frac{(t-p_1+1)_{p_1-1+q_1}}{(p_1-1+q_1)!}.$$

We may repeat the same argument as before, with the roles of p_1 and q_1 interchanged, and write

$$\mathcal{I}_n(L,q_0;\,p_1,q_1;\dots)=\mathcal{I}_n(L,q_0;\,p_1-1,q_1+1;\dots)-\mathcal{I}_n(L,q_0;\,p_1-1,q_1;\dots),$$

and similarly for $\widetilde{\mathcal{I}}_n$. We eventually write $\mathcal{I}_n$ and $\widetilde{\mathcal{I}}_n$ as sums of integrals with $p_1=q_0=0$, when $n>1$. If $n=1$, the decomposition of $\mathcal{I}_1$ and $\widetilde{\mathcal{I}}_1$ starts under the conditions $q_1=L$ and $q_0=0$, and terminates when either $q_1=L$ and $p_1=q_0=0$, or $q_1=L+1$ and $q_0=0$.

For $n>1$, by interchanging q_1 and q_2 again, we may instead assume that $p_1=q_1=0$. Then

$$\mathcal{L}_n(0,0;\,p_2,q_2;\dots;\,p_n,q_n;x)=\mathcal{L}_{n-1}(p_2,q_2;\dots;\,p_n,q_n;x),$$

whence

$$\mathcal{I}_n(L,q_0;0,0;\,,p_2,q_2;\dots;\,p_n,q_n)=\mathcal{I}_{n-1}(L,q_0;\,p_2,q_2;\dots;\,p_n,q_n).$$

Similarly

$$R_n(L,q_0;0,0;\,p_2,q_2;\dots;\,p_n,q_n;t)=R_n(L,q_0;\,p_2,q_2;\dots;\,p_n,q_n;t);$$

hence

$$\widetilde{\mathcal{I}}_n(L,q_0;0,0;\,p_2,q_2;\dots;\,p_n,q_n)=\widetilde{\mathcal{I}}_{n-1}(L,q_0;\,p_2,q_2;\dots;\,p_n,q_n).$$

By the induction hypothesis

$$\mathcal{I}_{n-1}(L,q_0;\,p_2,q_2;\dots;\,p_n,q_n)=\widetilde{\mathcal{I}}_{n-1}(L,q_0;\,p_2,q_2;\dots;\,p_n,q_n),$$

which proves (2).

For $n=1$, if $p_1=q_0=0$ and $L=q_1$ we can still interchange q_0 and q_1, so that $p_1=q_1=0$ and $L=q_0$, and then apply the previous reduction to $\mathcal{I}_1(L,q_0;0,0)$ and $\widetilde{\mathcal{I}}_1(L,q_0;0,0)$. In the case $q_1=L+1$ and $q_0=0$, we have

$$R_1(L,0;\,p_1,L+1)=(-1)^{p_1+L+1}\frac{L!}{(t)_{L+1}}\frac{(t-p_1)_{p_1+L+1}}{(p_1+L+1)!}=(-1)^{p_1+L+1}\frac{L!\,(t-p_1)_{p_1}}{(p_1+L+1)!}.$$

By [Zudilin 2014, Lemma 1],

$$\begin{aligned}\widetilde{\mathcal{I}}_1(L,0;\,p_1,L+1) &= (-1)^{p_1+L+1}\frac{L!\,p_1!}{(p_1+L+1)!}\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{(t-p_1)_{p_1}}{p_1!}\left(\frac{\pi}{\sin\pi t}\right)^2 \mathrm{d}t\\ &= (-1)^{L+1}\frac{L!\,p_1!}{(p_1+1)(p_1+L+1)!}.\end{aligned}$$

On the other hand, see again [Rhin and Viola 1996],

$$\begin{aligned}(-1)^{L+1}\mathcal{I}_1(L,0;\,p_1,L+1) &= \int_0^1\int_0^1 x^{L+1}(1-x)^{p_1}y^L\frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}\\ &= \int_0^1\int_0^1 x^{p_1}(1-x)^L y^{p_1}\,\mathrm{d}x\,\mathrm{d}y = \frac{L!\,p_1!}{(p_1+1)(p_1+L+1)!}.\end{aligned}$$

Therefore $\mathcal{I}_1(L,0;\,p_1,L+1)=\widetilde{\mathcal{I}}_1(L,0;\,p_1,L+1)$. □

We conclude this section by considering some instances of the integrals $\widetilde{\mathcal{I}}_n$.

Proposition 2.4. *For all nonnegative integers p_1, p_2, q_1, q_2 we have*

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{(t-p_1)_{p_1+q_1}}{(p_1+q_1)!}\frac{(t-p_2)_{p_2+q_2}}{(p_2+q_2)!}\left(\frac{\pi}{\sin\pi t}\right)^2\mathrm{d}t = (-1)^{p_1+p_2}\frac{(p_1+q_2)!\,(p_2+q_1)!}{(1+p_1+p_2+q_1+q_2)!}. \tag{3}$$

The above identity is a slight generalization of [Zudilin 2014, Lemma 1], and its easy proof by the first Barnes lemma is omitted for brevity.

Proposition 2.5. *For all nonnegative integers p_1, p_2, q_0, q_1, q_2 we have*

$$\begin{aligned}\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{(t-p_1)_{p_1+q_1}}{(p_1+q_1)!}\frac{(t-p_2)_{p_2+q_2}}{(p_2+q_2)!}\frac{(L-q_0)!}{(t+q_0)_{L-q_0+1}}\left(\frac{\pi}{\sin\pi t}\right)^2\mathrm{d}t&\\ = (-1)^{p_1+p_2}\frac{(p_1+q_0)!\,(p_1+q_2)!\,(p_2+q_0)!\,(p_2+q_1)!}{(L-q_1)!\,(L-q_2)!}&,\end{aligned}\tag{4}$$

where $L=1+p_1+p_2+q_0+q_1+q_2$.

Clearly, (4) is a particular case of the second Barnes lemma. It seems intriguing that (4) is, up to a sign, the product of two instances of (3), namely with (p_1,p_2,q_0,q_2) and with (p_1,p_2,q_1,q_0). After all, that is not too surprising, for the first and the second Barnes lemmas are the integral analogues of the Chu–Vandermonde and the Pfaff–Saalschütz formulas, respectively.

3. The second transformation

In the special case $n=3$, the integral in (2) is the integral introduced in [Zudilin 2014, Equation (10)]. Under the conditions

$$q_1-p_1=q_2-p_2=q_3-p_3,$$

by Proposition 2.2 we have

$$\mathcal{I}_3(L;\,q_0;\,p_2,q_1;\,p_3,q_2;\,p_1,q_3)=\mathcal{I}_3(L;\,q_0;\,p_3,q_1;\,p_1,q_2;\,p_2,q_3)$$

With the notation in [Zudilin 2014, Section 5],

$$(-1)^{p_1+p_2+p_3}\mathcal{I}_3(L; q_0; p_2, q_1; p_3, q_2; p_1, q_3)$$
$$= (-1)^{a+b+e+f}\frac{\Gamma(g-b)}{\Gamma(e)\,\Gamma(f)\,\Gamma(e+f-a)}$$
$$\times\frac{1}{2\pi i}\int_{-1/2-p_1-i\infty}^{-1/2-p_1+i\infty}\frac{\Gamma(a+t)\,\Gamma(b+t)\,\Gamma(e+t)\,\Gamma(f+t)}{\Gamma(1+t)\,\Gamma(1+a-e+t)\,\Gamma(1+a-f+t)\,\Gamma(g+t)}\left(\frac{\pi}{\sin\pi t}\right)^2\mathrm{d}t,$$

where

$$a = p_1+q_1+1,\quad b = p_1+q_0+1,\quad g = p_1+L+2,$$
$$e = p_1+q_2+1 = p_2+q_1+1,\quad f = p_1+q_3+1 = p_3+q_1+1.$$

With regard to this integral, Zudilin stated the following conjecture, see [Zudilin 2014, Equation (19)]: *For a "sufficiently generic" choice of integral parameters a, b, e, f, g the following identity holds*:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(e+t)\Gamma(f+t)}{\Gamma(1+t)\Gamma(1+a-e+t)\Gamma(1+a-f+t)\Gamma(g+t)}\left(\frac{\pi}{\sin\pi t}\right)^2\mathrm{d}t$$
$$= (-1)^{a+b+e+f}\frac{\Gamma(e)\Gamma(f)\Gamma(e+f-a)}{\Gamma(g-b)}$$
$$\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a-b+g+2t)\,\Gamma(a+t)\,\Gamma(e+t)\,\Gamma(f+t)}{\Gamma(1+a+2t)\,\Gamma(1+a-b+t)\,\Gamma(e+f+t)\,\Gamma(g+t)}\,\frac{\pi}{\sin 2\pi t}\,\mathrm{d}t.\quad (5)$$

In [Zudilin 2014, Proposition 2] the identity (5) is obtained in the particular case $a = 8n+1$, $b = 5n+1$, $e = 6n+1$, $f = 7n+1$ and $g = 14n+2$. The proof is based on the Gosper–Zeilberger algorithm of creative telescoping. This approach can be applied to any sufficiently generic, but concretely given sequences of vectors (a, b, e, f, g) corresponding to a fixed vector, not necessarily $(8, 5, 6, 7, 14)$. In [Zudilin 2014] it is also conjectured that (5) is a specialization of a more general identity involving *complex* parameters a, b, e, f, g, and, intriguingly, it is speculated that Bailey could have possessed such an identity.

Towards (5) we now deduce two functional equations for both integrals in (5). Let

$$\alpha(b, g; t) = \frac{\Gamma(g-b)\,\Gamma(b+t)}{\Gamma(g+t)},\quad \beta(a; b, g; t) = (-1)^b\frac{\Gamma(a-b+g+2t)}{\Gamma(1+a-b+t)\,\Gamma(g+t)}.$$

We remark that

$$\alpha(b, g; t) = \alpha(b, g-1; t) - \alpha(b+1, g; t),$$
$$\beta(a; b, g; t) = \beta(a; b, g-1; t) - \beta(a; b+1, g; t),$$

as is easily seen by using $\Gamma(1+z) = z\Gamma(z)$ and the trivial identities

$$g-b-1 = (g-t-1)-(b+t),$$
$$a-b+g+2t-1 = (g+t-1)+(a-b+t).$$

We introduce

$$\begin{aligned}\mathcal{K}(a,b,e,f,g)&\\ =\frac{\Gamma(g-b)}{\Gamma(e)\Gamma(f)\Gamma(e+f-a)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(e+t)\Gamma(f+t)}{\Gamma(1+t)\Gamma(1+a-e+t)\Gamma(1+a-f+t)\Gamma(g+t)}\left(\frac{\pi}{\sin\pi t}\right)^2\mathrm{d}t\end{aligned}$$

and

$$\widetilde{\mathcal{K}}(a,b,e,f,g)=(-1)^{a+b+e+f}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a-b+g+2t)\,\Gamma(a+t)\,\Gamma(e+t)\,\Gamma(f+t)}{\Gamma(1+a+2t)\,\Gamma(1+a-b+t)\,\Gamma(e+f+t)\,\Gamma(g+t)}\,\frac{\pi}{\sin 2\pi t}\,\mathrm{d}t.$$

We have

$$\mathcal{K}(a,b,e,f,g+1)=\mathcal{K}(a,b,e,f,g)-\mathcal{K}(a,b+1,e,f,g+1), \tag{6}$$

$$\widetilde{\mathcal{K}}(a,b,e,f,g+1)=\widetilde{\mathcal{K}}(a,b,e,f,g)-\widetilde{\mathcal{K}}(a,b+1,e,f,g+1). \tag{7}$$

The second functional equation is obtained as follows. We consider the rational function

$$\gamma(a,e,f;t)=\frac{\Gamma(f+t)}{\Gamma(f)\Gamma(1+t)}\frac{\Gamma(a+t)}{\Gamma(e)\Gamma(1+a-e+t)}\frac{\Gamma(e+t)}{\Gamma(e+f-a)\Gamma(1+a-f+t)}.$$

From the identity

$$1=xyz-(x-1)(y-1)(z-1)-(y-1)z-(z-1)x-(x-1)y$$

in the form

$$1=\frac{f+t}{f}\frac{a+t}{e}\frac{e+t}{e+f-a}-\frac{t}{f}\frac{a-e+t}{e}\frac{a-f+t}{e+f-a}-\frac{a-e+t}{e}\frac{e+t}{e+f-a}-\frac{f+t}{f}\frac{a-f+t}{e+f-a}-\frac{t}{f}\frac{a+t}{e},$$

we obtain, after multiplication by $\gamma(a,e,f;t)$, that

$$\begin{aligned}\gamma(a,e,f;t)=\gamma(a+1,e+1,f+1;t)-\gamma(a+1,e+1,f+1;t-1)&\\ -\gamma(a,e+1,f;t)-\gamma(a,e,f+1;t)-\gamma(a+2,e+1,f+1;t-1).&\end{aligned}$$

By writing $b+t=(b+1)+(t-1)$ and $g+t=(g+1)+(t-1)$ where it suits, we obtain

$$\begin{aligned}\mathcal{K}(a,b,e,f,g)=\mathcal{K}(a+1,b,e+1,f+1,g)-\mathcal{K}(a+1,b+1,e+1,f+1,g+1)&\\ -\mathcal{K}(a,b,e+1,f,g)-\mathcal{K}(a,b,e,f+1,g)-\mathcal{K}(a+2,b+1,e+1,f+1,g+1).&\end{aligned}$$

Using (6) inside the last identity, after some elementary manipulation we get

$$\begin{aligned}&\mathcal{K}(a+1,b,e+1,f+1,g+1)\\ &=\mathcal{K}(a,b,e,f,g)+\mathcal{K}(a,b,e+1,f,g)+\mathcal{K}(a,b,e,f+1,g)+\mathcal{K}(a+2,b+1,e+1,f+1,g+1).\end{aligned} \tag{8}$$

Let

$$\delta(a,e,f;t)=\frac{\Gamma(a+t)\Gamma(e+t)\Gamma(f+t)}{\Gamma(1+a+2t)\Gamma(e+f+t)}.$$

Since

$$1=\frac{e+t}{e+f+t}+\frac{f+t}{e+f+t}+\frac{a+t}{e+f+t}-\frac{a+2t}{e+f+t},$$

we have

$$\delta(a, e, f; t) = \delta(a, e+1, f; t) + \delta(a, e, f+1; t) + \delta(a+2, e+1, f+1; t-1) - \delta(a+1, e+1, f+1; t-1).$$

We easily deduce the analogue of (8) for $\widetilde{\mathcal{K}}$:

$$\begin{aligned}&\widetilde{\mathcal{K}}(a+1, b, e+1, f+1, g+1)\\ &= \widetilde{\mathcal{K}}(a, b, e, f, g) + \widetilde{\mathcal{K}}(a, b, e+1, f, g) + \widetilde{\mathcal{K}}(a, b, e, f+1, g) + \widetilde{\mathcal{K}}(a+2, b+1, e+1, f+1, g+1). \quad (9)\end{aligned}$$

We assume that a, b, e, f are integers, and define

$$\Delta = a+b+e+f-1-(1+a-e)-(1+a-f)-g = b+2e+2f-a-g-3.$$

Furthermore, we assume that $2e-a$ and $2f-a$ are positive, $g-b$ is not a nonpositive integer, and that it is always possible to separate the two groups of poles in the integrands in (5).

Lemma 3.1. *If $a = 1$ and $\Delta = -2$, then $\mathcal{K}(a, b, e, f, g) = \widetilde{\mathcal{K}}(a, b, e, f, g)$ and therefore* (5) *holds.*

Proof. We have

$$\begin{aligned}&\mathcal{K}(1, b, e, f, b+2e+2f-2)\\ &\quad = \frac{\Gamma(2e+2f-2)}{\Gamma(e)\Gamma(f)\Gamma(e+f-1)} \frac{(-1)^{e+f}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(b+t)\Gamma(e+t)\Gamma(f+t)\Gamma(e-1-t)\Gamma(f-1-t)}{\Gamma(g+t)}\, \mathrm{d}t.\end{aligned}$$

This integral can be computed with the help of the second Barnes lemma; see (4).

On the other hand,

$$\begin{aligned}&\widetilde{\mathcal{K}}(1, b, e, f, b+2e+2f-2)\\ &\quad = (-1)^{1+b+e+f} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(2e+2f-1+2t)\,\Gamma(1+t)\,\Gamma(e+t)\,\Gamma(f+t)}{\Gamma(2+2t)\,\Gamma(2-b+t)\,\Gamma(e+f+t)\,\Gamma(b+2e+2f-2+t)} \frac{\pi}{\sin 2\pi t}\, \mathrm{d}t.\end{aligned}$$

Applying the duplication formula

$$\Gamma(2z) = \frac{2^{2z}\Gamma(z)\Gamma\bigl(z+\frac{1}{2}\bigr)}{2\sqrt{\pi}}$$

to $\Gamma(2e+2f-1+2t)$ and $\Gamma(2+2t)$, the factors $\Gamma(1+t)$ and $\Gamma(e+f+t)$ cancel out, so that

$$\begin{aligned}&\widetilde{\mathcal{K}}(1, b, e, f, b+2e+2f-2)\\ &\quad = (-1)^{e+f} 2^{2e+2f-3} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(e+f-1/2+t)\,\Gamma(e+t)\,\Gamma(f+t)\,\Gamma(-1/2-t)\,\Gamma(b-1-t)}{\Gamma(b+2e+2f-2+t)}\, \mathrm{d}t.\end{aligned}$$

This also can be computed with the help of the second Barnes lemma. After a few simplifications using the duplication formula again, we arrive at the desired equality. □

Lemma 3.2. *If $a = 1$ and Δ is a negative integer $\leqslant -2$, then $\mathcal{K}(a, b, e, f, g) = \widetilde{\mathcal{K}}(a, b, e, f, g)$ and therefore* (5) *holds.*

Proof. This follows from the previous lemma by induction, using (6) and (7). □

Lemma 3.3. *If $a = 1$ (and $g \in \mathbb{C}$), then $\mathcal{K}(a, b, e, f, g) = \widetilde{\mathcal{K}}(a, b, e, f, g)$ and therefore* (5) *holds.*

Proof. This follows from the last lemma by Carlson's theorem; see [Bailey 1964, Section 5.3]. □

Lemma 3.4. *If* $1+a=2e$, *then* $\mathcal{K}(a,b,e,f,g)=\widetilde{\mathcal{K}}(a,b,e,f,g)$ *and therefore* (5) *holds.*

Proof. This can be proved in three steps, just like the lemma above. We start with the special case when the additional condition $\Delta=-2$ holds, and in that case we use the second Barnes lemma. Then we use (6) and (7) to deal with the case where Δ is an integer $\leqslant -2$. Finally we use Carlson's theorem to deal with the general case. □

Lemma 3.5. *If* $1+a=2f$, *then* $\mathcal{K}(a,b,e,f,g)=\widetilde{\mathcal{K}}(a,b,e,f,g)$ *and therefore* (5) *holds.*

Proof. This follows from the previous lemma and the symmetry properties $\mathcal{K}(a,b,e,f,g)=\mathcal{K}(a,b,f,e,g)$ and $\widetilde{\mathcal{K}}(a,b,e,f,g)=\widetilde{\mathcal{K}}(a,b,f,e,g)$. □

We are going to use (8) and (9) in an iterative fashion, similar to the proof of Theorem 2.1 in [Rhin and Viola 1996], to prove the following:

Theorem 3.6. *Let* a, e, f *be positive integers such that* $2e-a$ *and* $2f-a$ *are positive, and that* $a \geqslant \max\{e, f\}$. *Let* $b\in\mathbb{Z}$ *and* $g\in\mathbb{C}$ *such that* $g-b$ *is not a nonpositive integer. Then* (5) *holds.*

Proof. Note that $e+f-a$ is also positive, because $2(e+f-a)=(2e-a)+(2f-a)$. By continuity we may assume that $g\notin\mathbb{Z}$. We put $(a,b,e,f,g)=(a_0-1,b_0,e_0-1,f_0-1,g_0-1)$ in (8), say. The integral $\mathcal{K}(a_0,b_0,e_0,f_0,g_0)$ is decomposed into the sum of four integrals $\mathcal{K}(a_j,b_j,e_j,f_j,g_j)$, with $j=1,\dots,4$, such that for each $j=1,\dots,4$ at least two among a_j, $2e_j-a_j$ and $2f_j-a_j$ equal the corresponding integer with $j=0$ decreased by 1, while their sum $2e_j+2f_j-a_j$ is decreased either by 1 or by 3. Therefore the iteration must terminate with at least one of the following three conditions fulfilled: $a=1$, $2e-a=1$ or $2f-a=1$. However, we need to keep e, f and $e+f-a$ positive throughout the process. To this end, we remark that if $e_j=e_0-1$ then $2e_j-a_j=2e_0-a_j-1$, and we start the process with $e\geqslant 2e-a$. Similarly, if $f_j=f_0-1$ then $2f_j-a_j=2f_0-a_j-1$, and $f\geqslant 2f-a$. Also, if $e_j+f_j-a_j=e_0+f_0-a_0-1$, then $2e_j-a_j=2e_0-a_j-1$ and $2f_j-a_j=2f_0-a_j-1$, and we obviously have $e+f-a\geqslant\min\{2e-a,2f-a\}$. □

For a bit more generality, we show how to dispense with the somewhat artificial assumption $a\geqslant\max\{e,f\}$, to obtain:

Theorem 3.7. *Let* a, e, f *be positive integers such that* $2e-a$ *and* $2f-a$ *are positive. Let* $b\in\mathbb{Z}$ *and* $g\in\mathbb{C}$ *such that* $g-b$ *is not a nonpositive integer. Then* (5) *holds.*

Proof. By the theorem above, we only need to deal with the case $\max\{a,e,f\}=\max\{e,f\}=e$, say. With the change of variable from t to s by putting $t=s+e-a$, we obtain

$$\mathcal{K}(a,b,e,f,g)=\mathcal{K}(2e-a,b+e-a,e,e+f-a,g+e-a),$$
$$\widetilde{\mathcal{K}}(a,b,e,f,g)=\widetilde{\mathcal{K}}(2e-a,b+e-a,e,e+f-a,g+e-a).$$

Since $2e-a\geqslant e$ and $2e-a\geqslant e+f-a$, the identity (5) follows from the case in the theorem above. □

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