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Using a new integral construction which combines the idea of symmetry suggested by V. Salikhov in 2007 and the integral introduced by Marcovecchio in 2009, we obtain a new bound for approximation to π by numbers from the field $\mathbb{Q}(\sqrt{3})$.

1. Introduction. Integral construction. Arithmetic part.

We continue our research initiated in [Androsenko and Salikhov 2015] and [Luchin and Salikhov 2018]. In this paper we prove the following result.

Theorem 1. *Let* $\mu > 10.2209$; $p_1,\ p_2,\ p_3,\ p_4 \in \mathbb{Z}$, $(p_3,\ p_4) \neq (0, 0)$, $P = \max_{1\le i\le 4} |p_i|$, *and* $P > P_0(\mu)$. *Then*

$$\left|\pi - \frac{p_1\sqrt{3}+p_2}{p_3\sqrt{3}+p_4}\right| > P^{-\mu}. \tag{1}$$

The first inequality of this type was proven in [Amoroso and Viola 2001]:

$$\left|\pi - \frac{a+b\sqrt{3}}{c+d\sqrt{3}}\right| > \mathit{constant}\cdot \max\{|a|, |b|, |c|, |d|\}^{-46.9075\ldots},$$

where $a, b, c, d \in \mathbb{Z}$, $(c, d) \neq (0, 0)$. This result was improved in [Tomashevskaya 2008], with the value $10.3567\ldots$ for μ.

The proof of the new bound (1) is related to the application of the following integral construction. Let $h, j, k, l, m, q \in \mathbb{Z}^+$, $h+j+q = k+l+m$, $h+j-k \ge 0$, $k+l-j \ge 0$, $k+m-h \ge 0$; $x \in \mathbb{C}$, $\operatorname{Re} x > 0$, $x \neq 1$. Consider the integral

$$J = \frac{1}{2\pi i}\int_0^{-\infty} ds \int_{-i\infty}^{i\infty} \frac{s^h t^j\, dt}{\sqrt{\frac{s}{s-1}}(1-s)^{k+l-j+1}(s-t)^{h+j-k+1}(t-x)^{k+m-h+1}}. \tag{2}$$

The result of Theorem 1 is obtained by taking

$$x = \frac{2+\sqrt{3}}{4}, \tag{3}$$

$$h = 11n,\quad j = 37n,\quad k = 16n,\quad l = 27n,\quad m = 37n,\quad q = 32n,\quad n \in \mathbb{N},\quad n \to \infty. \tag{4}$$

The only thing that distinguishes the integral (2) from the one in [Marcovecchio 2009, (5), p. 148]

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is the factor $\sqrt{s/(s-1)}$ in the denominator of the integrand. For the first time the integral (2) was considered in [Androsenko and Salikhov 2015]. Using this integral, Androsenko [2015] improved the estimate for the irrationality measure of the number $\pi/\sqrt{3}$. In [Luchin and Salikhov 2018], thanks to the integral (2), it became possible to obtain a new bound for the approximation of $\ln 2$ by numbers from the field $\mathbb{Q}(\sqrt{2})$. In our argument below we substantially apply the method developed in that work.

In [Luchin and Salikhov 2018] (equalities (7)–(9)) it was shown that the integral (2) can be represented in the form

$$J = -\int_0^1 R(z)\,dz, \tag{5}$$

where

$$R(z) = 2(-1)^{j-k} \sum_{l_1=\max(0,q-l)}^{k+m-h} (-1)^{l_1} \binom{j}{k+m-h-l_1} \cdot \frac{x^{l-q+l_1}}{(x-1)^{h+j-k+l_1+1}} \binom{h+j-k+l_1}{l_1} R_{l_1}(z), \tag{6}$$

$$R_{l_1}(z) = \frac{z^{2h}(1-z^2)^{l+l_1}}{\left(\frac{x}{x-1} - z^2\right)^{h+j-k+l_1+1}} = \frac{z^{2h}(1-z^2)^{l+l_1}}{(-(2+\sqrt{3})^2 - z^2)^{h+j-k+l_1+1}}. \tag{7}$$

Here we use notation from [Luchin and Salikhov 2018]. As in that article, we write

$$\omega(l_1) = \binom{h+j-k+l_1}{l_1} \int_0^1 R_{l_1}(z)\,dz. \tag{8}$$

Let K be the ring of numbers of the form $a+b\sqrt{3}$, where $a, b \in \mathbb{Z}$, and for positive integers $M \in \mathbb{N}$ we put $q_M = \operatorname{lcm}(1, 2, \ldots, M)$ and $q_0 = 1$.

Lemma 1. *Let $M_0 = \max(2k+2l-2j, h+j-k, k+m-h)$, $m \geq q$. For all $l_1 \leq k+m-h$, one has*

$$q_{M_0}\omega(l_1) = \frac{1}{48} \cdot 2^{2q-2m} \cdot (a(l_1)\pi + b(l_1)), \tag{9}$$

where $a(l_1), b(l_1)$ belong to $\mathbb{K}$.

Proof. For $N \in \mathbb{Z}^+$ we write

$$D_N(f(z)) = \frac{1}{N!} \cdot f^{(N)}(2i + i\sqrt{3}).$$

Since the integrand (7) of the integral in (8) is even, we have expansion into a sum of simplest fractions:

$$\begin{aligned} R_{l_1}(z) &= \frac{(-1)^{q-m-1} z^{2h}(z^2-1)^{l+l_1}}{(z^2-(2i+i\sqrt{3})^2)^{h+j-k+l_1+1}} \\ &= P(z) + \sum_{\nu=1}^{h+j-k+l_1+1} \left(\frac{(-1)^\nu k_\nu}{(z-2i-i\sqrt{3})^\nu} + \frac{k_\nu}{(z+2i+i\sqrt{3})^\nu} \right), \end{aligned} \tag{10}$$

where $P(z) \in \mathbb{K}[z]$ and $\deg P(z) = 2(k+l-j-1)$;

$$(-1)^\nu k_\nu = D_{h+j-k+l_1+1-\nu}(R_{l_1}(z)(z-2i-i\sqrt{3})^{h+j-k+l_1+1}).$$

By Leibniz's formula, we see from (10) that

$$\begin{aligned} k_\nu &= (-1)^{q-m+\nu-1} D_{h+j-k+l_1+1-\nu}\left(\frac{z^{2h}(z-1)^{l+l_1}(z+1)^{l+l_1}}{(z+2i+i\sqrt{3})^{h+j-k+l_1+1}}\right) \\ &= (-1)^{q-m+\nu-1}\sum_{\overline{m}\in M_\nu} D_{m_1}(z^{2h})\, D_{m_2}((z-1)^{l+l_1})\, D_{m_3}((z+1)^{l+l_1})\, D_{m_4}\big((z+2i+i\sqrt{3})^{-(h+j-k+l_1+1)}\big), \end{aligned}$$

where we have set $\overline{m} = (m_1, m_2, m_3, m_4)$ and

$$M_\nu = \{\overline{m} \in (\mathbb{Z}^+)^4 \mid m_1+m_2+m_3+m_4 = h+j-k+l_1+1-\nu;\ m_1 \le 2h;\ m_2, m_3 \le l+l_1\}.$$

So

$$\begin{aligned} k_\nu = (-1)^{q-m+\nu-1}\sum_{\overline{m}\in M_\nu} &\binom{2h}{m_1}\binom{l+l_1}{m_2}\binom{l+l_1}{m_3}\binom{h+j-k+l_1+m_4}{m_4}\cdot(-1)^{m_4}\cdot \\ &\cdot (2i+i\sqrt{3})^{2h-m_1}\cdot(-1+2i+i\sqrt{3})^{l+l_1-m_2}\cdot(1+2i+i\sqrt{3})^{l+l_1-m_3}\cdot \\ &\cdot (2(2i+i\sqrt{3}))^{-(h+j-k+l_1+m_4+1)} \end{aligned}$$

For $N \in \mathbb{N}$ we have

$$(-1+2i+i\sqrt{3})^N = (2i+2e^{i\frac{2\pi}{3}})^N = 2^{N-1}(i+e^{i\frac{2\pi}{3}})^N \cdot 2.$$

But $2\cdot(i+e^{i\frac{2\pi}{3}})^N \in \mathbb{K}[i]$ and so $(-1+2i+i\sqrt{3})^N = 2^{N-1}\cdot k'_N$, where $k'_N \in \mathbb{K}[i]$. Similarly we have $(1+2i+i\sqrt{3})^N = 2^{N-1}\cdot k''_N$, where $k''_N \in \mathbb{K}[i]$.

So

$$k_\nu = \sum_{\overline{m}\in M_\nu} k_\nu(\overline{m})\cdot 2^{l+l_1-m_2-1}\cdot 2^{l+l_1-m_3-1}\cdot 2^{-(h+j-k+l_1+m_4+1)},$$

where all $k_\nu(\overline{m}) \in \mathbb{K}[i]$.

Moreover $m_2+m_3+m_4 \le h+j-k+l_1+1-\nu$, so

$$\begin{aligned} l+l_1-m_2-1+l+l_1-m_3-1-(h+j-k+l_1+1+m_4) &\ge 2(l+l_1-h-j+k-l_1)+\nu-4 \\ &= 2q-2m+\nu-4. \end{aligned}$$

This gives

$$k_\nu = 2^{2q-2m+\nu-4}\cdot\tilde{k}_\nu,\ \tilde{k}_\nu \in \mathbb{K}[i],\ \nu = 1,\dots,h+j-k+l_1+1. \tag{11}$$

From (10) we have

$$\begin{aligned} \int_0^1 R_{l_1}(z)\,dz = \int_0^1 P(z)\,dz + \sum_{\nu=2}^{h+j-k+l_1+1}\bigg(\frac{k_\nu}{\nu-1}\bigg(\frac{1}{(2i+i\sqrt{3}-1)^{\nu-1}} - \frac{1}{(2i+i\sqrt{3}+1)^{\nu-1}}\bigg)\bigg) \\ + k_1 \ln\frac{2i+i\sqrt{3}+z}{2i+i\sqrt{3}-z}\bigg|_0^1. \end{aligned} \tag{12}$$

Obviously $2+\sqrt{3}=\tan\frac{5\pi}{12}$. Let $\ln z=\ln|z|+i\varphi$ where $\varphi\in(-\pi;\pi]$. Then

$$\ln\frac{2i+i\sqrt{3}+z}{2i+i\sqrt{3}-z}\bigg|_0^1=\ln\frac{1+2i+i\sqrt{3}}{2i+i\sqrt{3}-1}=\ln\frac{1+i\tan\frac{5\pi}{12}}{i\tan\frac{5\pi}{12}-1}=\ln\frac{e^{i\frac{5\pi}{12}}}{e^{i\frac{7\pi}{12}}}=-i\frac{\pi}{6};$$

$$\frac{1}{2i+i\sqrt{3}+1}=\frac{1-i\sqrt{3}-2i}{1+(2+\sqrt{3})^2}=\frac{e^{-i\frac{\pi}{3}}-i}{2(2+\sqrt{3})},\qquad \frac{1}{(2i+i\sqrt{3}+1)^{\nu-1}}=\frac{2(e^{-i\frac{\pi}{3}}-i)^{\nu-1}(2-\sqrt{3})^{\nu-1}}{2^\nu}.$$

As before we have $2(e^{-i\frac{\pi}{3}}-i)^{\nu-1}\in\mathbb{K}[i]$ and so

$$\frac{1}{(2i+i\sqrt{3}+1)^{\nu-1}}=2^{-\nu}x'_\nu,,\qquad \frac{1}{(2i+i\sqrt{3}-1)^{\nu-1}}=2^{-\nu}x''_\nu,$$

where x'_ν, $x''_\nu\in\mathbb{K}[i]$. Thus from (11) and (12) we have

$$\int_0^1 R_{l_1}(z)\,dz=\int_0^1 P(z)\,dz+\sum_{\nu=2}^{h+j-k+l_1+1}\frac{1}{\nu-1}\cdot 2^{2q-2m-4}\cdot\tilde{k}_\nu+\frac{2^{2q-2m-3}}{6}\cdot\tilde{k}_1\pi, \tag{13}$$

where $\tilde{k}_\nu\in\mathbb{K}[i]$ for all ν.

It follows from the definition of M_0 that $A_1:=q_{M_0}\int_0^1 P(z)\,dz$ lies in $\mathbb{K}$. It is also easy to check that $q_{M_0}\binom{h+j-k+l_1}{l_1}\cdot\frac{1}{\nu-1}=:A_\nu$ lies in $\mathbb{N}$ for all $\nu=2,\dots,h+j-k+l_1+1$. Then it follows from (8) and (13) that

$$\begin{aligned}q_{M_0}\omega(l_1)&=q_{M_0}\binom{h+j-k+l_1}{l_1}\cdot\int_0^1 R_{l_1}(z)\,dz\\&=\binom{h+j-k+l_1}{l_1}A_1+2^{2q-2m-4}\cdot\sum_{\nu=2}^{h+j-k+l_1+1}A_\nu\tilde{k}_\nu+\frac{1}{3}\cdot 2^{2q-2m-4}\cdot\tilde{k}'_1\pi,\end{aligned}$$

whence, since $m\ge q$, we get equality (9), where $a(l_1)$, $b(l_1)\in\mathbb{K}[i]$. But, obviously, $a(l_1)$, $b(l_1)\in\mathbb{R}$. Therefore $a(l_1)$, $b(l_1)\in\mathbb{K}$. This completes the proof of lemma. □

Corollary 1. *The integral* (2) *for $m\ge q$ admits the representation*

$$6\cdot 2^{-2q}q_{M_0}J=a\pi+b,\ a,b\in\mathbb{K}. \tag{14}$$

Proof. For $x=\frac{2+\sqrt{3}}{4}$ we have

$$\frac{x^{l-q+l_1}}{(x-1)^{h+j-k+l_1+1}}=\frac{(2+\sqrt{3})^{l-q+l_1}4^{h+j-k+l_1+1}}{4^{l-q+l_1}\left(\sqrt{3}-2\right)^{h+j-k+l_1+1}}=4^{m+1}\cdot C(l_1),$$

where $C(l_1)\in\mathbb{K}$.

Therefore from (5), (6), (8) and (9) we have

$$q_{M_0}J=\frac{1}{6}\cdot 4^m\cdot 4^{q-m}\sum_{l_1=\max(0,q-l)}^{k+m-h}d(l_1)c(l_1)\left(a(l_1)\pi+b(l_1)\right),$$

where all $d(l_1)\in\mathbb{Z}$, and this implies (14). □

Together with the family of parameters (4), we should consider a more general choice of parameters

$$(h, j, k, l, m, q) = n(h', j', k', l', m', q'), \tag{15}$$

where $h', j', k', l', m', q' \in \mathbb{Z}^+$.

It is convenient to denote the integral (2) for parameters of the form (15) and for x of the form (3) as

$$J := J_n = J_n(h', j', k', l', m', q'). \tag{16}$$

For the family of parameters (15) we write

$$Mn = \max\{2(k+l-j), 2h, 2k, h+j-k, k+m-h, l, m, j, q\}. \tag{17}$$

Let p be a prime, $p > \sqrt{Mn}$ and $\omega = \{\frac{n}{p}\}$ be the fractional part of the number $\frac{n}{p}$. Consider the inequalities

$$\begin{aligned}
&[2k'\omega]+[(l'+k'-j')\omega]+[m'\omega]+[l'\omega]-[k'\omega]-[2(l'+k'-j')\omega]-[(h'+j'-k')\omega]-[(k'+m'-h')\omega] > 0,\\
&[2h'\omega]+[(l'+k'-j')\omega]+[j'\omega]+[q'\omega]-[h'\omega]-[2(l'+k'-j')\omega]-[(h'+j'-k')\omega]-[(k'+m'-h')\omega] > 0,\\
&[j'\omega]+[m'\omega]-[(h'+j'-k')\omega]-[(k'+m'-h')\omega] > 0,\\
&[2k'\omega]+[(l'+k'-j')\omega]+[q'\omega]-[k'\omega]-[2(l'+k'-j')\omega]-[(k'+m'-h')\omega] > 0,\\
&[2h'\omega]+[(l'+k'-j')\omega]+[l'\omega]-[h'\omega]-[2(l'+k'-j')\omega]-[(h'+j'-k')\omega] > 0.
\end{aligned} \tag{18}$$

These inequalities were first studied in detail in [Androsenko and Salikhov 2015, (11), p. 491] and later applied in [Luchin and Salikhov 2018]. They are slightly different from those considered for the same purpose in [Marcovecchio 2009, (31)].

By Δ_n we denote the product of all primes $p > \sqrt{Mn}$ for which $\omega = \{\frac{n}{p}\}$ satisfies at least one of the inequalities (18). The following lemma sharpens the result obtained in Corollary 1.

Lemma 2. *When $m \geq q$ the integral* (16) *admits the representation*

$$6 \cdot 2^{-2q} \cdot \frac{q_{Mn}}{\Delta_n} \cdot J_n = A_n \pi + B_n, \tag{19}$$

where $A_n, B_n \in \mathbb{K}$, $n \in \mathbb{N}$.

Proof. The representation (19) follows from (14) due to a standard procedure of refining the denominator (see, for example, Lemma 3 in [Androsenko 2015]). □

The following lemma, similar to Lemma 4 from [Luchin and Salikhov 2018], plays an important role in the proof of Theorem 1.

Lemma 3. *Let $n, d \in \mathbb{N}$, $\theta \in \mathbb{R}$, $\sqrt{d} \notin N$, and $L_n = (\Lambda_1(n)\sqrt{d} + \Lambda_2(n))\theta + \Lambda_3(n)\sqrt{d} + \Lambda_4(n)$, where each $\Lambda_i(n)$ belongs to $\mathbb{Z}$, and let $\Lambda(n) = \max_{1\leq i\leq 4} |\Lambda_i(n)|$. Let $\lim_{n\to\infty} \frac{1}{n} \ln |\Lambda_1(n)\sqrt{d} + \Lambda_2(n)| = \gamma_1$, $\lim_{n\to\infty} \sup \frac{1}{n} \ln |\Lambda(n)| \leq \gamma_2$. Suppose that for some constant $\gamma_3 > \gamma_2$ and for every $\varepsilon_1, \varepsilon_2 > 0$ there exists $N = N(\varepsilon_1, \varepsilon_2)$ such that the inequalities*

$$e^{-(\gamma_3+\varepsilon_1)m} \leq |L_m| \leq e^{-(\gamma_3-\varepsilon_2)m} \tag{20}$$

hold for any $n \geq N$ and at least one of the values $m \in \{n, n+1\}$. Further, let $\gamma_1 + \gamma_2 > 0$, $\mu > \frac{2(\gamma_1+\gamma_3)}{\gamma_3-\gamma_2}$; $p_1, p_2, p_3, p_4 \in \mathbb{Z}$,$(p_3, p_4) \neq (0, 0)$, $P = \max_{1\leq i\leq 4} |p_i|$ and $P > P_0(\mu)$. Then

$$\left|\theta - \frac{p_1\sqrt{d} + p_2}{p_3\sqrt{d} + p_4}\right| > P^{-\mu}. \tag{21}$$

Remark. Assumptions similar to those from Lemma 3 were used in [Amoroso and Viola 2001; Salnikova 2008; Hata 2000].

We prove Theorem 1 by applying Lemma 3 to the linear form

$$L_n = (2-\sqrt{3})^{128n} \cdot 4^{-32n} \cdot \frac{q_{Mn}}{\Delta_n} J_n = (\Lambda_1(n)\sqrt{3} + \Lambda_2(n))\pi + \Lambda_3(n)\sqrt{3} + \Lambda_4(n), \tag{22}$$

where each $\Lambda_i(n)$ is an integer, J_n is the integral (16) for the family of parameters (4), and Mn is defined by equality (17) for the family of parameters (4).

The corresponding constants γ_1 and γ_3 will be calculated in the next Section 2, and the constant γ_2 in Section 3.

2. Asymptotics

The argument of this part is almost completely analogous to those from [Luchin and Salikhov 2018, §2].

Everywhere in the sequel (see (2) and (16)) we write

$$J_n := J_n(11, 37, 16, 27, 37, 32) = \frac{1}{2\pi i}\int_0^{-\infty} ds \int_{-i\infty}^{i\infty} G(s, t)\, dt, \tag{23}$$

where

$$G(s, t) = \varphi(s, t)(f(s, t))^n, \tag{24}$$

with

$$f(s, t) = \frac{s^{11}t^{37}}{(1-s)^6(s-t)^{32}(t-x)^{42}}, \quad \varphi(s, t) = \frac{1}{\sqrt{\frac{s}{s-1}}(1-s)(s-t)(t-x)}, \quad x = \frac{2+\sqrt{3}}{4}.$$

The saddle points are the solutions of the system $f'_s(s, t) = 0$, $f'_t(s, t) = 0$ that differ from the zeros of the function $f(s, t)$. In [Androsenko and Salikhov 2015] (see p. 492, equations (12)) this system was solved in the general case for the integral (16). For the function $f(s, t)$ considered above we have three saddle points:

$$(s_1, t_1) \approx (0.994847; 0.967621), \tag{25}$$

$$(s_2, t_2) \approx (0.324712 + 0.292582i, -0.637736 - 0.207638i), \tag{26}$$

and $(s_3, t_3) = (\bar{s}_2, \bar{t}_2)$, the complex conjugate of (s_2, t_2). We write $\xi = (s, t) \in \mathbb{C}^2$.

Lemma 4. *Let ξ^0 be a nondegenerate saddle point of the function $S(\xi)$, let γ be a two-dimensional smooth complex manifold with boundary, let ξ^0 be an interior point of γ, let the functions $\varphi(\xi)$ and $S(\xi)$*

be holomorphic at the point ξ^0, *and let also* $\max_{\xi\in\gamma} \operatorname{Re} S(\xi)$ *be attained only at the point* ξ^0. *Let*

$$F(\lambda) = \int_\gamma \varphi(\xi)\exp(\lambda S(\xi))d\xi$$

and $S''_{\xi\xi}(\xi^0) = \left(\begin{pmatrix} S''_{ss}(\xi^0) \\ S''_{st}(\xi^0)\end{pmatrix} \begin{pmatrix} S''_{st}(\xi^0) \\ S''_{tt}(\xi^0)\end{pmatrix}\right)$ *be the Hesse matrix, and suppose that* $\det S''_{\xi\xi}(\xi^0) \neq 0$. *Then*

$$F(\lambda) = \frac{2\pi}{\lambda}\exp(\lambda S(\xi^0)) \cdot (\det S''_{\xi\xi}(\xi^0))^{-\frac{1}{2}}(\varphi(\xi^0) + O(\lambda^{-1})) \tag{27}$$

as $\lambda \to +\infty$.

Proof. This statement is proved in the [Fedoryuk 1977], p. 259, Proposition 1.1. □

Lemma 5. *For the linear form* (22) *we have the equation*

$$\gamma_1 := \lim_{n\to\infty}\frac{1}{n}\ln|\Lambda_1(n)\sqrt{3}+\Lambda_2(n)| = 128\ln(2-\sqrt{3}) - 32\ln 4 + M_1 + \ln|f(s_1,t_1)| \approx 85.303863, \tag{28}$$

where the value

$$M_1 = M - \lim_{n\to\infty}\frac{1}{n}\ln\Delta_n \approx 11.313066 \tag{29}$$

is calculated using inequalities (18) *for the set of parameters* (4) *and*

$$\ln|f(s_1,t_1)| \approx 286.922828.$$

Proof. Let the integral (23) be written in the form

$$J_n = A'_n\pi + B_n,$$

where A'_n, $B'_n \in \mathbb{Q}[\sqrt{3}]$ (see (19)). Consider the circles $L_1 = \{t : |t| = t_1\}$ and $L_2 = \{s : |s| = s_1\}$. Obviously, $\max_{(s,t)\in L_2^*\times L_1^*}\ln|f(s,t)|$ is attained only at the point (s_1, t_1). As in Lemma 6 from [Luchin and Salikhov 2018], we have

$$A'_n = \frac{1}{2(2\pi i)^2}\int_{L_2} ds \int_{L_1} G(s,t)\,dt,$$

where the function $G(s,t)$ was defined in (23). Here we used the inequalities $x < t_1 < s_1 < 1$.

We apply Lemma 4 for the function $S(s,t) = \ln f(s,t) = \ln|f(s,t)| + ih(s,t)$ (a certain branch of the logarithm defined on the set $\gamma = \gamma_2 \times \gamma_1$, where γ_2 is a small arc of the circle L_2 including the point $s_1 + 0i$ and γ_1 is a small arc of L_1 including $t_1 + 0i$. In our case for the Hessian we have $\det S''_{\xi\xi}(s_1,t_1) \approx 1.92\times 10^{10} \neq 0$.

Using equality (27) of Lemma 4, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\ln A'_n = \ln|f(s,t)| \approx 286.922828. \tag{30}$$

Let us now evaluate the constant M_1. For the family of parameters (4) from (17) we obtain

$$Mn = \max(12n, 22n, 32n, 32n, 42n, 27n, 37n, 37n, 32n) = 42n.$$

Inequalities (18) for the family of parameters (4) have the form

$$\begin{aligned} [32\omega]+[6\omega]+[37\omega]+[27\omega]-[16\omega]-[12\omega]-[32\omega]-[42\omega]&>0,\\ [22\omega]+[6\omega]+[37\omega]+[32\omega]-[11\omega]-[12\omega]-[32\omega]-[42\omega]&>0,\\ [37\omega]+[37\omega]-[32\omega]-[42\omega]&>0,\\ [32\omega]+[6\omega]+[32\omega]-[16\omega]-[12\omega]-[42\omega]&>0,\\ [22\omega]+[6\omega]+[27\omega]-[11\omega]-[12\omega]-[32\omega]&>0. \end{aligned} \tag{31}$$

The set E of numbers $\omega\in[0;1)$ satisfying at least one of the inequalities (31) has the form

$$\begin{aligned} E=&\big[\tfrac{1}{37};\tfrac{1}{14}\big)\cup\big[\tfrac{2}{27};\tfrac{2}{21}\big)\cup\big[\tfrac{4}{37};\tfrac{5}{42}\big)\cup\big[\tfrac{5}{37};\tfrac{1}{7}\big)\cup\big[\tfrac{4}{27};\tfrac{1}{6}\big)\cup\big[\tfrac{5}{27};\tfrac{4}{21}\big)\cup\big[\tfrac{8}{37};\tfrac{1}{4}\big)\\ &\cup\big[\tfrac{10}{37};\tfrac{2}{7}\big)\cup\big[\tfrac{11}{37};\tfrac{13}{42}\big)\cup\big[\tfrac{12}{37};\tfrac{5}{14}\big)\cup\big[\tfrac{10}{27};\tfrac{8}{21}\big)\cup\big[\tfrac{15}{37};\tfrac{5}{12}\big)\cup\big[\tfrac{16}{37};\tfrac{7}{16}\big)\cup\big[\tfrac{17}{37};\tfrac{10}{21}\big)\\ &\cup\big[\tfrac{18}{37};\tfrac{1}{2}\big)\cup\big[\tfrac{19}{37};\tfrac{23}{42}\big)\cup\big[\tfrac{5}{9};\tfrac{4}{7}\big)\cup\big[\tfrac{16}{27};\tfrac{25}{42}\big)\cup\big[\tfrac{23}{37};\tfrac{5}{8}\big)\cup\big[\tfrac{24}{37};\tfrac{2}{3}\big)\cup\big[\tfrac{25}{37};\tfrac{29}{42}\big)\\ &\cup\big[\tfrac{26}{37};\tfrac{31}{42}\big)\cup\big[\tfrac{20}{27};\tfrac{3}{4}\big)\cup\big[\tfrac{28}{37};\tfrac{16}{21}\big)\cup\big[\tfrac{7}{9};\tfrac{11}{14}\big)\cup\big[\tfrac{30}{37};\tfrac{13}{16}\big)\cup\big[\tfrac{31}{37};\tfrac{6}{7}\big)\cup\big[\tfrac{19}{22};\tfrac{37}{42}\big)\\ &\cup\big[\tfrac{8}{9};\tfrac{11}{12}\big)\cup\big[\tfrac{34}{37};\tfrac{13}{14}\big)\cup\big[\tfrac{35}{37};\tfrac{20}{21}\big)\cup\big[\tfrac{26}{27};\tfrac{41}{42}\big). \end{aligned}$$

Let $\psi(x)=\Gamma'(x)/\Gamma(x)$, where $\Gamma(x)$ stands for the gamma function. Then, in a standard way (see Lemma 6 in [Nesterenko 2010]) we obtain

$$\Delta=\lim_{n\to\infty}\frac{1}{n}\ln\Delta_n=\big(\psi\big(\tfrac{1}{14}\big)-\psi\big(\tfrac{1}{37}\big)\big)+\big(\psi\big(\tfrac{2}{21}\big)-\psi\big(\tfrac{2}{27}\big)\big)+\cdots+\big(\psi\big(\tfrac{41}{42}\big)-\psi\big(\tfrac{26}{27}\big)\big)\approx 30.686934.$$

Finally,

$$\lim_{n\to\infty}\frac{1}{n}\cdot\ln\frac{q_{Mn}}{\Delta_n}=42-\Delta=:M_1\approx 11.313066.$$

It follows from (19) and (22) that $\Lambda_1(n)\sqrt{3}+\Lambda_2(n)=6(2-\sqrt{3})^{128n}\cdot 4^{-32n}\cdot(q_{Mn}/\Delta_n)\cdot A'_n$, and from (30) we obtain the statement of the lemma. □

Lemma 6. *The value of γ_3 for the linear form* (22) *satisfies the equality*

$$\gamma_3=32\ln 4-128\ln(2-\sqrt{3})-M_1-\ln|f(s_2,t_2)|\approx 245.593134, \tag{32}$$

Proof. The argument here is similar to the proof of Lemma 7 in [Luchin and Salikhov 2018]. In our case for the value of the Hessian we have $\det S''_{\xi\xi}(s_2;t_2)\approx -6702+4059i\neq 0$. Note that if $h(s,t)=\operatorname{Im}\ln s,t$, then $h(s_2,t_2)=:\omega\approx -1.833$. Let $\omega_0=\frac{1}{2}(\pi+\omega)$ (earlier in [Luchin and Salikhov 2018] the corresponding values were $\omega\approx 1.9062$, $\omega_0=\frac{1}{2}(\pi-\omega)$). The end of the proof of Lemma 6 is identical to Lemma 7 from [Luchin and Salikhov 2018]. □

We note that, by Lemma 5, we have $M_1\approx 11.313066$ and $\ln|f(s_2,t_2)|\approx -43.974169$. So we obtain the equality (32).

3. Evaluation of the constant γ_2. End of the proof of Theorem 1.

The argument of §3 in [Luchin and Salikhov 2018] with minor changes should be repeated here. Therefore, we restrict ourselves to the statement of results and some comments.

In this section we put $D_N(f(x)) = \frac{1}{N!} f^{(N)}(x)$, where $N \in \mathbb{Z}^+$, and consider the operator $T = D_{k+m-h} \cdot x^j \cdot D_{h+j-k} = D_{42n} x^{37n} D_{32n}$. This operator is analogous to those considered in [Luchin and Salikhov 2018]. It should be mentioned that operators like T where used in [Marcovecchio 2009; Sorokin 1991; Marcovecchio 2014] and many other papers.

Lemma 7. *Let $l \le j$. The integral* (2) *satisfies the equality*

$$J = 2(-1)^{k+l-j} \cdot T\left(\sum_{\nu=0}^{k+l-j-1} \frac{1}{2(k+l-j-\nu)-1} \cdot \frac{x^{m-q+\nu}}{(x-1)^{\nu+1}} + \frac{x^{h-0.5}(-1)^{k+l-j}}{(1-x)^{k+l-j+0.5}} \arctan\sqrt{\frac{1-x}{x}} \right). \quad (33)$$

Proof. It is necessary to repeat the argument of Lemma 10 from [Luchin and Salikhov 2018] with the only change related to the case

$$x = \frac{2+\sqrt{3}}{4} < 1.$$

We obtain

$$\int_0^1 \frac{dz}{z^2 - \frac{x}{x-1}} = \int_0^1 \frac{dz}{z^2 + \frac{x}{1-x}} = \frac{\sqrt{1-x}}{\sqrt{x}} \arctan\sqrt{\frac{1-x}{x}}.$$

A similar integral was considered in [Luchin and Salikhov 2018, Lemma 10]:

$$\int_0^1 \frac{dz}{z^2 - \frac{x}{x-1}} = -\frac{\sqrt{x-1}}{\sqrt{x}} \ln\left(\sqrt{x} + \sqrt{x-1}\right),$$

This is the only difference between (33) and the similar equality (53) from [Luchin and Salikhov 2018]. □

Lemma 8. *Let $M \in \mathbb{N}$, $a, b \in \mathbb{R}$. Then*

$$D_M\left(\frac{x^a}{(1-x)^b}\right) = \sum_{r=0}^{M} \binom{a}{r} \binom{b-a+M-1}{M-r} \frac{x^{a-r}}{(1-x)^{b+M}}.$$

Proof. A similar statement was proven in [Luchin and Salikhov 2018, Lemma 11] for $(1-x) \to (x-1)$. To use that lemma it is enough to choose the branch of the logarithm such that $\ln(-z) = \ln|z| + i\pi$ is satisfied for $z \in \mathbb{R}$, $z > 0$, $\ln z \in \mathbb{R}$. Then, since $x < 1$, we have

$$(1-x)^b = (x-1)^b \cdot e^{i\pi b}, \quad (1-x)^{b+M} = (x-1)^{b+M} \cdot e^{i\pi b}(-1)^M,$$

and the statement of Lemma 8 follows from (54) from [Luchin and Salikhov 2018]. □

Lemma 9 [Luchin and Salikhov 2018, Lemma 12]. *For every $N \in \mathbb{N}$ and for arbitrary analytic functions $u = u(x)$ and $\vartheta = \vartheta(x)$ one has*

$$D_N(u\vartheta) = \vartheta \cdot D_N(u) + \sum_{\lambda=0}^{N-1} \frac{\lambda!\,(N-1-\lambda)!}{N!} D_{N-1-\lambda}(D_\lambda(u) \cdot \vartheta').$$

For $x \in \mathbb{R}$ we introduce the function

$$x^* = \begin{cases} x \ln x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ x \ln(-x) & \text{if } x < 0. \end{cases} \tag{34}$$

Obviously, the function x^* is odd.

Lemma 10 [Luchin and Salikhov 2018, Lemma 13]. *Let $n \in \mathbb{N}$, $n \to +\infty$, $b = b_0 n + O(1)$, $r = r_0 n + O(1)$, $b_0, r_0 \in \mathbb{R}$, $r \in \mathbb{Z}^+$ and $\binom{b}{r} \neq 0$. Then one has*

$$\lim_{n\to\infty} \frac{1}{n} \ln \left| \binom{b}{r} \right| = b_0^* - r_0^* - (b_0 - r_0)^*. \tag{35}$$

Now we apply the results obtained above to the linear form (22).

Relation (33) for the family of parameters (4) can be rewritten as

$$J_n = \sum_{\nu=0}^{6n-1} \frac{2(-1)^{\nu+1}}{12n - 2\nu - 1} \Sigma_{1,\nu} + 2\Sigma_2, \tag{36}$$

where

$$\Sigma_{1,\nu} = D_{42n}\left(x^{37n} D_{32n} \left(\frac{x^{5n+\nu}}{(1-x)^{\nu+1}} \right) \right), \tag{37}$$

$$\Sigma_2 = D_{42n}\left(x^{37n} D_{32n} \left(\frac{x^{11n-\frac{1}{2}}}{(1-x)^{6n+\frac{1}{2}}} \arctan \sqrt{\frac{1-x}{x}} \right) \right). \tag{38}$$

For example, we calculate a simpler function (37). Applying Lemma 8 for $\nu = 0, 1, \dots, 6n-1$, we obtain

$$D_{32n}\left(\frac{x^{5n+\nu}}{(1-x)^{\nu+1}} \right) = \sum_{r=5n}^{5n+\nu} \binom{5n+\nu}{r} \binom{27n}{32n-r} \frac{x^{5n-\nu-r}}{(1-x)^{32n+1+\nu}}.$$

In a similar way we get

$$D_{42n}\left(x^{37n} \frac{x^{5n+\nu-r}}{(1-x)^{32n+\nu+1}} \right) = D_{42n}\left(\frac{x^{42n+\nu-r}}{(1-x)^{32n+\nu+1}} \right) = \sum_{\rho=0}^{42n} \binom{42n+\nu-r}{\rho} \binom{32n+r}{42n-\rho} \frac{x^{42n+\nu-r-\rho}}{(1-x)^{74n+\nu+1}}.$$

For $x = \dfrac{2+\sqrt{3}}{4}$ we obtain

$$\begin{aligned} \frac{x^{42n+\nu-r-\rho}}{(1-x)^{74n+\nu+1}} &= \frac{(2+\sqrt{3})^{42n+\nu-r-\rho}}{4^{42n+\nu-r-\rho}} \cdot 4^{74n+\nu+1} \cdot (2+\sqrt{3})^{74n+\nu+1} \\ &= 4^{32n+r+\rho+1} \cdot (2+\sqrt{3})^{116n+2\nu-r-\rho+1} \\ &= A \cdot 4^{r+\rho+1} (2-\sqrt{3})^{12n-2\nu-1+r+\rho}, \end{aligned}$$

where

$$A = 4^{32n} (2+\sqrt{3})^{128n}. \tag{39}$$

Moreover our parameters should necessarily satisfy $\rho \le 42n + \nu - r$ and $42n - \rho \le 32n + r$, i.e., $\rho \ge 10n - r$. Thus, for $\nu = 0, 1, \ldots, 6n-1$, we have $\Sigma_{1,\nu} = A \cdot \Sigma'_{1,\nu}$ and

$$\Sigma'_{1,\nu} = \sum_{(r,\rho)\in B} \binom{5n+\nu}{r}\binom{27n}{32n-r}\binom{42n+\nu-r}{\rho}\binom{32n+r}{42n-\rho} \cdot 4^{r+\rho+1} \cdot (2-\sqrt{3})^{12n-2\nu-1+r+\rho}, \qquad (40)$$

where $B = \{(r,\rho) \mid r \in [5n; 5n+\nu],\ \rho \in [\max(0; 10n-r); \min(42n; 42n+\nu-r)]\}$.

Let us calculate the function Σ_2 from (38), applying Lemma 9 for

$$N = 32n, \quad u = \frac{x^{11n-0.5}}{(1-x)^{6n+0.5}}, \quad \vartheta = \arctan\sqrt{\frac{1-x}{x}}, \quad \vartheta' = -\frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

We have

$$D_{32n}\left(\frac{x^{11n-\frac{1}{2}}}{(1-x)^{6n+\frac{1}{2}}} \arctan\sqrt{\frac{1-x}{x}}\right) = \arctan\sqrt{\frac{1-x}{x}} \cdot D_{32n}\left(\frac{x^{11n-\frac{1}{2}}}{(1-x)^{6n+\frac{1}{2}}}\right) - \frac{1}{2}\sum_{\lambda=0}^{32n-1} \frac{\lambda!(32n-1-\lambda)!}{(32n)!} D_{32n-1-\lambda}\left(D_\lambda\left(\frac{x^{11n-\frac{1}{2}}}{(1-x)^{6n+\frac{1}{2}}}\right) \cdot \frac{1}{\sqrt{x}\sqrt{1-x}}\right).$$

Applying Lemma 9 to the first summand of this sum again, we obtain from (38) (when $N = 42n$, $u = x^{37n} D_{32n}\big(x^{11n-\frac{1}{2}}/(1-x)^{6n+\frac{1}{2}}\big)$ and $\vartheta = \arctan\sqrt{(1-x)/x}$) the equality

$$\begin{aligned}\Sigma_2 = {}& \arctan\sqrt{\frac{1-x}{x}} D_{42n}\left(x^{37n} D_{32n}\left(\frac{x^{11n-\frac{1}{2}}}{(1-x)^{6n+\frac{1}{2}}}\right)\right)\\ & - \frac{1}{2}\sum_{\lambda_1=0}^{42n-1} \frac{\lambda_1!(42n-1-\lambda_1)!}{(42n)!} D_{42n-1-\lambda_1}\left(D_{\lambda_1}\left(x^{37n} D_{32n}\left(\frac{x^{11n-\frac{1}{2}}}{(1-x)^{6n+\frac{1}{2}}}\right)\right)\frac{1}{\sqrt{x}\sqrt{1-x}}\right)\\ & - \frac{1}{2}\sum_{\lambda=0}^{32n-1} \frac{\lambda!(32n-1-\lambda)!}{(32n)!} D_{42n}\left(x^{37n} D_{32n-1-\lambda}\left(D_\lambda\left(\frac{x^{11n-\frac{1}{2}}}{(1-x)^{6n+\frac{1}{2}}}\right)\frac{1}{\sqrt{x}\sqrt{1-x}}\right)\right).\end{aligned} \qquad (41)$$

We note that for $x = \frac{2+\sqrt{3}}{4}$ one has

$$\sqrt{x}\cdot\sqrt{1-x} = \frac{1}{4}, \quad \arctan\sqrt{\frac{1-x}{x}} = \arctan(2-\sqrt{3}) = \frac{\pi}{12}.$$

Let us write

$$D_{42n}\left(x^{37n} D_{32n}\left(\frac{x^{11n-\frac{1}{2}}}{(1-x)^{6n+\frac{1}{2}}}\right)\right) =: A \cdot \Sigma'_2, \qquad (42)$$

$$D_{42n-1-\lambda_1}\left(D_{\lambda_1}\left(x^{37n} D_{32n}\left(\frac{x^{11n-\frac{1}{2}}}{(1-x)^{6n+\frac{1}{2}}}\right)\right)\frac{1}{\sqrt{x}\sqrt{1-x}}\right) =: A \cdot \Sigma_{2,1}(\lambda_1), \qquad (43)$$

$$D_{42n}\left(x^{37n} D_{32n-1-\lambda}\left(D_\lambda\left(\frac{x^{11n-\frac{1}{2}}}{(1-x)^{6n+\frac{1}{2}}}\right)\frac{1}{\sqrt{x}\sqrt{1-x}}\right)\right) =: A \cdot \Sigma_{2,2}(\lambda). \qquad (44)$$

Functions (42)–(44) are calculated using Lemma 8 in a standard way (see (40), for example). We thus restrict ourselves to presenting the final results, namely

$$\Sigma_2' = \sum_{(r,\rho)\in B} \binom{11n-\frac{1}{2}}{r}\binom{27n}{32n-r}\binom{48n-r-\frac{1}{2}}{\rho}\binom{32n+r}{42n-\rho}\cdot 4^{r+\rho+1}\cdot(2-\sqrt{3})^{r+\rho}, \tag{45}$$

where $B = \{(r,\rho)\,|\, r\in[5n;32n],\ \rho\in[\max(0;10n-r);42n]\}$;

$$\Sigma_{2,1}'(\lambda_1) = \sum_{(r_1,r_2,\rho)\in B_1} \binom{11n-\frac{1}{2}}{r_1}\binom{27n}{32n-r_1}\binom{42n-r_1-\frac{1}{2}}{r_2}\binom{r_1+\lambda_1-10n}{\lambda_1-r_2} \cdot\binom{48n-r_1-r_2-1}{\rho}\binom{32n+r_1+r_2}{42n-1-\lambda-\rho}\cdot 4^{r_1+r_2+\rho+1}\cdot(2-\sqrt{3})^{r_1+r_2+\rho+1}, \tag{46}$$

where $B_1 = \{(r_1,r_2,\rho)\in(\mathbb{Z}^+)^3 \mid r_1\in[5n;32n], r_2\le\lambda_1;$ if $r_1+\lambda_1\ge 10n$, then $r_1+r_2\ge 10n$; $\rho\le 42n-1-\lambda_1$, $\lambda_1+r_1+r_2+\rho\ge 10n-1$; if $r_1+r_2<48n-1$, then $\rho\le 48n-r_1-r_2-1\}$; and

$$\Sigma_{2,2}'(\lambda) = \frac{(\lambda-\Lambda)!\Lambda!(32n-1-\lambda)!}{(32n)!} \cdot \sum_{(r_1,r_2,r_3,\rho)\in B_2}\binom{11n-\frac{1}{2}}{r_1}\binom{6n+\Lambda-r_1-\frac{1}{2}}{\Lambda-r_1}\binom{11n-r_1-\frac{1}{2}}{r_2}\binom{\lambda-5n}{\lambda-\Lambda-r_2}\binom{11n-r_1-r_2}{r_3}\binom{27n+r_2}{32n-r_3-\lambda-1} \cdot\binom{48n-r_1-r_2-r_3-1}{\rho}\binom{32n+r_2+r_3}{42n-\rho}\cdot 4^{r_2+r_3+\rho+1}\cdot(2-\sqrt{3})^{2r_1+r_2+r_3+\rho+1}, \tag{47}$$

where $\lambda\in[0;32n-1]$, $\Lambda\in[0;\lambda]$, $B_2 = \{(r_1,r_2,r_3,\rho)\in(\mathbb{Z}^+)^4 \mid r_1\in[0;\Lambda], r_2\in[0;\lambda-\Lambda]$, if $\lambda>5n$, then also $r_2\ge 5n-\Lambda$; $r_3\in[\max(0,5n-\lambda-r_2-1);32n-1-\lambda]$, if $r_1+r_2<11n-1$, then also $r_3\le 11n-r_1-r_2-1$; $\rho\in[0;\min(42n,48n-1-r_1-r_2-r_3)]\}$.

Finally from (36)–(44) we obtain $J_n = A\cdot J_n^*$, where

$$J_n^* = \sum_{\nu=0}^{6n-1}\frac{2(-1)^{\nu+1}}{12n-2\nu-1}\Sigma_{1,\nu}' + 2\Sigma_2' - \sum_{\lambda_1=0}^{42n-1}\frac{\lambda_1!(42n-1-\lambda_1)!}{(42n)!}\Sigma_{2,1}'(\lambda_1) - \sum_{\lambda=0}^{32n-1}\frac{\lambda!(32n-1-\lambda)!}{(32n)!}\Sigma_{2,2}'(\lambda). \tag{48}$$

Then from (22) and (39) it follows that $(2-\sqrt{3})^{128n}4^{-32n}A = 1$. Thus,

$$L_n = \frac{q_{42n}}{\Delta_n}J_n^*.$$

All the summands in (48) were calculated in (40) and (45)–(47). The asymptotic q_{42n}/Δ_n was calculated in the proof of Lemma 5, so

$$\lim_{n\to\infty}\frac{1}{n}\frac{q_{42n}}{\Delta_n} = M_1 \approx 11.313066.$$

The final step of calculating γ_2 for the linear form (22) is similar to the proof from §3 in [Luchin and Salikhov 2018]. In particular, it is necessary to replace $(2-\sqrt{3})$ by $(2+\sqrt{3})$ in the sums (40) and (45)–(47). The asymptotic behavior of the binomial coefficients included in the summands of these sums is calculated using Lemma 10. Computer calculations show that the corresponding maximal summand is attained in the sum Σ_2' for the values of parameters $r \approx 10.256n$, $\rho \approx 31.431n$. The corresponding value of γ_2 is $\gamma_2 = 169.531 + 11.313066 = 180.844066$.

Then, by Lemma 3, the inequality (1) holds for

$$\mu = \frac{2(\gamma_1 + \gamma_3)}{\gamma_3 - \gamma_2} \approx 10.2209.$$

This completes the proof of Theorem 1.

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In memoriam
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