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Two integral transformations related to $\zeta(2)$

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To the memory of Naum Ilyitch Feldman (1918–1994)

We prove two integral transformations that relate different constructions of rational approximations to $\zeta(2)$. The first one relates a double integral over the unit square and a Barnes-type integral. The second one relates two Barnes-type integrals and was discovered and proved by W. Zudilin using an automated proof method. Here we offer a proof based on more classical means such as contiguous relations, the second Barnes lemma and the duplication formula for the gamma function.

1. Introduction

A few years ago, W. Zudilin [2014] refined a long-standing record for the upper bound of the irrationality measure of $\zeta(2)$, let us call it $\mu(\zeta(2))$, by proving that $\mu(\zeta(2)) \leqslant 5.09541178\cdots = \mu_0$, say. This simply means that for every $\mu > \mu_0$ the inequality

$$\left|\zeta(2) - \frac{p}{q}\right| < q^{-\mu}$$

has only finitely many solutions $(p, q) \in \mathbb{Z}^2$. The previously known upper bound for $\mu(\zeta(2))$ was established by G. Rhin and C. Viola [1996], who introduced their permutation-group method and proved that $\mu(\zeta(2)) \leqslant 5.44124250\ldots$. A slightly different proof of their result was presented in [Marcovecchio 2013]. The construction in [Rhin and Viola 1996] of a suitable sequence of rational approximations to $\zeta(2)$ relies on a certain family of double integrals over the unit square that is more general than those in all previous papers, such as, e.g., [Beukers 1979; Hata 1995], while in [Zudilin 2014] two different complex integrals are employed (*first* and *second tales*), and an identity between these integrals is established (the *interlude*). In connection to this last identity, we provide the reader with two more references: [Nassrallah and Rahman 1986, Equation (3.17)] and [Verma and Jain 1992, Equation (4.8)]. They are closely related, though it seems to be not straightforward to fit [Zudilin 2014, Equation (19)] in one, or both, of them.

One purpose of the present paper is to present a bridge between those different approaches. The existence of such a bridge, though in a context that seemingly does not cover the general construction in [Zudilin 2014], is made explicit in [Zudilin 2007]. Here we resort to the concept of *multiple Legendre polynomials* introduced in [Marcovecchio 2012] and developed in [Marcovecchio 2014]. This is detailed in Section 2 below, leading to the integral transformation (2), and offers a new viewpoint to the first tale in [Zudilin 2014]. The second aim of the paper is to give a *human-generated* proof of [Zudilin 2014, Proposition 2], i.e., the interlude. This is the subject of Section 3 below. Our proofs are self-contained

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and rely heavily on contiguous relations, in the spirit of the treatment of the real double integrals made in [Rhin and Viola 1996], In the first part we also employ a little bit of the traditional machinery of iterated partial integration usually involved in similar context where Legendre-type polynomials play a crucial role. In the second part, the second Barnes lemma and the duplication formula for the gamma function come into play.

2. The first transformation

2A. ***Overview on multiple Legendre polynomials.*** We recall the definition and a few facts about the so-called multiple Legendre polynomials we introduced in [Marcovecchio 2014]. For any $n \geqslant 1$, let $p_1, \dots, p_n, q_1, \dots, q_n \geqslant 0$ be integers, and let

$$\mathcal{L}_n(p_1, q_1; \dots; p_n, q_n; z) \in \mathbb{Z}[z] \tag{1}$$

be the polynomials recursively defined by

$$\mathcal{L}_n(p_1, q_1; \dots; p_n, q_n; z) = z^{q_n}(1-z)^{p_n} D_{p_n+q_n}(z^{p_n}(1-z)^{q_n}\mathcal{L}_{n-1}(p_1, q_1; \dots; p_{n-1}, q_{n-1}; z)),$$

where by agreement $\mathcal{L}_0(z) \equiv 1$. Here and in the sequel

$$D_m(f(u)) = \frac{1}{m!}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^m f(u).$$

We emphasize the main property of $\mathcal{L}_n(p_1, q_1; \dots; p_n, q_n; z)$ in the following proposition. Let $\mathfrak{S}_n$ be the symmetric group of n elements.

Proposition 2.1. *For τ, σ in the symmetric group $\mathfrak{S}_n$, the polynomial*

$$(p_{\tau(1)} + q_{\sigma(1)})! \cdots (p_{\tau(n)} + q_{\sigma(n)})!\, \mathcal{L}_n(p_{\tau(1)}, q_{\sigma(1)}; \dots; p_{\tau(n)}, q_{\sigma(n)}; z)$$

does not depend on τ and σ. Briefly, we say that it is $\langle p_1, \dots, p_n\rangle$-stable and $\langle q_1, \dots, q_n\rangle$-stable.

In particular, for σ in $\mathfrak{S}_n$,

$$\mathcal{L}_n(p_{\sigma(1)}, q_{\sigma(1)}; \dots; p_{\sigma(n)}, q_{\sigma(n)}; z)$$

is independent of σ. We briefly say that it is $\langle (p_1, q_1), \dots, (p_n, q_n)\rangle$-stable.

For a proof, we refer the reader to [Marcovecchio 2014, p. 1834]. The above proposition summarizes the *raison d'être* of (1), and is crucial in the proof of (2) below.

Since $\mathcal{L}_1(0, 0; z) = 1$ and

$$\mathcal{L}_2(0, q; p, 0; z) = (-1)^q \binom{p+q}{p} z^q (1-z)^p = \binom{p+q}{p} \mathcal{L}_1(p, q; z),$$

up to changing n into some $n' < n$ we may essentially suppose that $p_i + q_j > 0$ for any $i, j = 1, \dots, n$, i.e., either $p_1, \dots, p_n > 0$ or $q_1, \dots, q_n > 0$.

We remark that

$$\deg \mathcal{L}_n(p_1, q_1; \dots; p_n, q_n; z) = p_1 + q_1 + \cdots + p_n + q_n,$$
$$\operatorname*{ord}_{z=0} \mathcal{L}_n(p_1, q_1; \dots; p_n, q_n; z) = \max\{q_1, \dots, q_n\}.$$

In [Marcovecchio 2013] we explained how to recover the irrationality measure of $\zeta(2)$ proven in [Rhin and Viola 1996] by using double integrals of Beukers' type involving the polynomial $\mathcal{L}_2(p_1, q_1; p_2, q_2; z)$.

In [Marcovecchio 2014] the polynomials (1) with $n > 2$ were shown to be a significant tool in connection to diophantine properties of other constants. For example, we obtained an irrationality measure of $\log 2$ with $n = 3$ (already proved in an earlier paper, through a different method), a new nonquadraticity measure of $\log 2$, a new noncubicity measure of $\log\left(\frac{5}{4}\right)$ with $n = 4$, and a new nonquarticity measure of $\log\left(\frac{20}{19}\right)$ with $n = 5$: all those results are the best known. Here we obtain, for $n = 3$, the same rational approximations to $\zeta(2)$ produced in [Zudilin 2014].

2B. ***Double integrals over the unit square.*** Let $n \geqslant 2$ be an integer and $L, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n \geqslant 0$ be integers such that $L \geqslant q_0$. We introduce the double integral (sometimes abbreviated as $\mathcal{I}_n$)

$$\mathcal{I}_n(L; q_0; p_1, q_1; \ldots; p_n, q_n) = (-1)^{q_0} \int_0^1\int_0^1 \mathcal{L}_n(p_1, q_1; \ldots; p_n, q_n; x) y^{L-q_0}(1-y)^{q_0} \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}.$$

By the symmetry properties of $\mathcal{L}_n(p_1, q_1; \ldots; p_n, q_n; x)$ we deduce that if $L \geqslant q_j$ for all $j = 0, 1, \ldots, n$, then

$$(p_1+q_1)! \cdots (p_n+q_n)!\, \mathcal{I}_n(L; p_1, \ldots, p_n; q_0, q_1, \ldots, q_n)$$

is $\langle p_1, \ldots, p_n\rangle$-stable and $\langle q_1, \ldots, q_n\rangle$-stable. By performing a (p_n+q_n)-fold integration by parts with respect to x, taking into account that

$$D_{p_n+q_n}\left(\frac{x^{q_n}(1-x)^{p_n}}{1-xy}\right) = \frac{y^{-q_n}(1-y^{-1})^{p_n} y^{p_n+q_n}}{(1-xy)^{p_n+q_n+1}},$$

we write

$$\mathcal{I}_n(L; q_0; p_1, q_1; \ldots; p_n, q_n) \\ = (-1)^{q_n+q_{n+1}} \int_0^1\int_0^1 \frac{x^{p_n}(1-x)^{q_n}\mathcal{L}_{n-1}(p_1, q_1; \ldots; p_{n-1}, q_{n-1}; x) y^{L-q_0}(1-y)^{p_n+q_0}}{(1-xy)^{p_n+q_n+1}}\,\mathrm{d}x\,\mathrm{d}y.$$

By performing a second (p_n+q_n)-fold integration by parts, this time with respect to y, when $L \geqslant \max\{q_0, q_n\}$, the integral $\mathcal{I}_n(L; q_0; p_1, q_1; \ldots; p_n, q_n)$ also equals

$$(-1)^{p_n+\max\{q_0,q_n\}} \int_0^1\int_0^1 x^{p_n}(1-x)^{\min\{q_0,q_n\}}\mathcal{L}_{n-1}(p_1, q_1; \ldots; p_{n-1}, q_{n-1}; x) \\ \times y^{p_n+\min\{q_0,q_n\}}(1-y)^{\max\{0,q_n-q_0\}} D_{p_n+q_n}(y^{L-q_0}(1-y)^{p_n+q_0}) \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}.$$

We have

$$D_{p_n+q_n}(y^{L-q_0}(1-y)^{p_n+q_0}) = \frac{(L-q_0)!\,(p_n+q_0)!}{(L-q_n)!\,(p_n+q_n)!}(1-y)^{q_0-q_n} D_{p_n+q_0}(y^{L-q_n}(1-y)^{p_n+q_n});$$

see, e.g., [Marcovecchio 2013, Equation (8)]. By combining this with Proposition 2.1, we get the following:

Proposition 2.2. *If $L \geqslant \max\{q_0, q_1, \ldots, q_n\}$, then the function*

$$\frac{(p_1+q_1)! \cdots (p_n+q_n)!}{(L-q_0)!} \mathcal{I}_n(L; q_0; p_1, q_1; \ldots; p_n, q_n)$$

is $\langle p_1, \ldots, p_n\rangle$-stable and $\langle q_0, q_1, \ldots, q_n\rangle$-stable.

It is worth noticing that the complementary transformation

$$D_{p_n+q_n}(y^{L-q_0}(1-y)^{p_n+q_0}) = \frac{(L-q_0)!\,(p_n+q_0)!}{(L-q_n)!\,(p_n+q_n)!}(-y)^{L-p_n-q_n-q_0} D_{L-q_0}(y^{p_n+q_n}(1-y)^{L-q_n})$$

from [Marcovecchio 2013, Equation (7)] is compatible with the symmetry properties in Proposition 2.1 only when $n=2$ and $L=q_0+q_1+q_2$. This case was examined in [Marcovecchio 2013, Section 2], and the results therein are the same as in [Rhin and Viola 1996].

2C. ***Relation with Zudilin's integral.*** By [Hata 1995, Lemma 1.1]

$$\mathcal{I}_n(L;\,q_0;\,p_1,q_1;\,\dots;\,p_n,q_n) = \mathcal{J}_n(L;\,q_0;\,p_1,q_1;\,\dots;\,p_n,q_n)\zeta(2) - \mathcal{A}_n(L;\,q_0;\,p_1,q_1;\,\dots;\,p_n,q_n),$$

with $\mathcal{A}_n(L;\,q_0;\,p_1,q_1;\,\dots;\,p_n,q_n)\in\mathbb{Q}$ and

$$\mathcal{J}_n(L;\,q_0;\,p_1,q_1;\,\dots;\,p_n,q_n) = \frac{(-1)^{q_0}}{2\pi i}\oint_{|x|=\varrho}\mathcal{L}_n(p_1,q_1;\,\dots;\,p_n,q_n;\,x)\frac{1}{x^{L-q_0}}\left(1-\frac{1}{x}\right)^{q_0}\frac{\mathrm{d}x}{x}\in\mathbb{Z}$$

for any small positive ϱ. We now perform the change of variable $x=u/(u-1)$, and take into account

$$\mathcal{L}_n(p_1,q_1;\,\dots;\,p_n,q_n;\,x) = (1-u)\mathcal{S}_n(p_1,q_1;\,\dots;\,p_n,q_n;\,u),$$

where

$$\mathcal{S}_n(p_1,q_1;\,\dots;\,p_n,q_n;\,u) = \sum_{k\geqslant 0}\binom{k+p_1}{p_1+q_1}\cdots\binom{k+p_n}{p_n+q_n}u^k;$$

see [Marcovecchio 2014, Equation (7)]. Also, when x runs along a small closed path around 0 in the positive direction, $1-x$ moves around 1 in the negative direction, and so does $1-u=(1-x)^{-1}$. Hence $u=x/(x-1)$ spans a small closed path around 0 in the positive direction, and

$$\mathcal{J}_n(L;\,q_0;\,p_1,q_1;\,\dots;\,p_n,q_n) = \frac{(-1)^{q_0}}{2\pi i}\oint\mathcal{S}_n(p_1,q_1;\,\dots;\,p_n,q_n;\,u)\left(1-\frac{1}{u}\right)^{L-q_0}\frac{1}{u^{q_0}}\frac{\mathrm{d}u}{u}.$$

By applying the binomial theorem to $(1-1/u)^{L-q_0}$ we infer that

$$\mathcal{J}_n(L;\,q_0;\,p_1,q_1;\,\dots;\,p_n,q_n) = \sum_{j=0}^{L-q_0}(-1)^{j+q_0}\binom{L-q_0}{j}\binom{j+p_1+q_0}{p_1+q_1}\cdots\binom{j+p_n+q_0}{p_n+q_n}.$$

This proves that $\mathcal{J}_3(L;\,q_0;\,p_1,q_1;\,p_2,q_2;\,p_3,q_3)$ coincides, up to the sign, with the coefficient of $\zeta(2)$ in [Zudilin 2014, Proposition 1], where the parameters $(a_1,\dots,a_4)$ and $(b_1,\dots,b_4)$ are chosen to be

$$(a_1,a_2,a_3,a_4)=(q_1,q_2,q_3,q_0)\quad\text{and}\quad(b_1,b_2,b_3,b_4)=(-p_1,-p_2,-p_3,L+1),$$

and therefore suggests that the linear forms obtained through the two constructions coincide as well. More precisely, let

$$R_n(L;\,q_0;\,p_1,q_1;\,\dots;\,p_n,q_n;\,t) := (-1)^{q_0+p_1+q_1+\cdots+p_n+q_n}\frac{(L-q_0)!}{(t+q_0)_{L-q_0+1}}\prod_{j=1}^{n}\frac{(t-p_j)_{p_j+q_j}}{(p_j+q_j)!}$$

and

$$\widetilde{\mathcal{I}}_n(L; q_0; p_1, q_1; \dots; p_n, q_n) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} R_n(L; q_0; p_1, q_1; \dots; p_n, q_n; t)\left(\frac{\pi}{\sin \pi t}\right)^2 \mathrm{d}t.$$

In $\widetilde{\mathcal{I}}_n$, and in all similar integrals hereafter, the integration path is chosen so that it separates two sequences of consecutive poles of the integrand.

Theorem 2.3. *For all nonnegative integers $p_1, \dots, p_n, q_0, q_1, \dots, q_n$ and for all integers $L \geqslant q_0$*

$$\mathcal{I}_n(L; q_0; p_1, q_1; \dots; p_n, q_n) = \widetilde{\mathcal{I}}_n(L; q_0; p_1, q_1; \dots; p_n, q_n). \tag{2}$$

Proof. In the double integral defining $\mathcal{I}_n(L; q_0; p_1, q_1; \dots; p_n, q_n)$, we apply the binomial theorem to obtain

$$y^{L-q_0}(1-y)^{q_0} = \sum_{j=0}^{L-q_0}(-1)^j\binom{L-q_0}{j}(1-y)^{q_0+j},$$

whence

$$\mathcal{I}_n(L; q_0; p_1, q_1; \dots; p_n, q_n) = \sum_{j=0}^{L-q_0}\binom{L-q_0}{j}\mathcal{I}_n(q_0+j; q_0+j; p_1, q_1; \dots; p_n, q_n).$$

Since

$$\frac{(L-q_0)!}{(t+q_0)_{L-q_0+1}} = \sum_{j=0}^{L-q_0}(-1)^j\binom{L-q_0}{j}\frac{1}{t+q_0+j},$$

we also have

$$\widetilde{\mathcal{I}}_n(L; q_0; p_1, q_1; \dots; p_n, q_n) = \sum_{j=0}^{L-q_0}\binom{L-q_0}{j}\widetilde{\mathcal{I}}_n(j+q_0; j+q_0; p_1, q_1; \dots; p_n, q_n).$$

Hence we may assume that $L = q_0$.

We now argue by induction on n. Let $n = 0$. We have, see [Rhin and Viola 1996],

$$\int_0^1\int_0^1 (1-y)^{q_0}\frac{\mathrm{d}x\,\mathrm{d}y}{1-xy} = \int_0^1\int_0^1 (xy)^{q_0}\frac{\mathrm{d}x\,\mathrm{d}y}{1-xy} = \zeta(2) - \sum_{j=1}^{q_0}\frac{1}{j^2}.$$

Also, by [Zudilin 2014, Lemma 2]

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\left(\frac{\pi}{\sin\pi t}\right)^2\frac{\mathrm{d}t}{t+q_0} = \zeta(2) - \sum_{j=1}^{q_0}\frac{1}{j^2}.$$

This achieves the case $n = 0$.

Let $n > 0$. If $q_1 > 0$, then by $t + q_1 - 1 = (t - p_1 - 1) + (p_1 + q_1)$ we have

$$\frac{(t-p_1)_{p_1+q_1}}{(p_1+q_1)!} = \frac{(t-p_1-1)_{p_1+q_1}}{(p_1+q_1)!} + \frac{(t-p_1)_{p_1+q_1-1}}{(p_1+q_1-1)!}.$$

It follows that

$$\widetilde{\mathcal{I}}_n(L,q_0;\,p_1,q_1;\dots)=\widetilde{\mathcal{I}}_n(L,q_0;\,p_1+1,q_1-1;\dots)-\widetilde{\mathcal{I}}_n(L,q_0;\,p_1,q_1-1;\dots).$$

Similarly, since $\mathcal{L}_1(p_1,q_1;x)=(-x)^{q_1}(1-x)^{p_1}=(-x)^{q_1-1}(1-x)^{p_1+1}-(-x)^{q_1-1}(1-x)^{p_1}$, we have

$$\mathcal{L}_n(p_1,q_1;\dots)=\mathcal{L}_n(p_1+1,q_1-1;\dots)-\mathcal{L}_n(p_1,q_1-1;\dots).$$

Therefore

$$\mathcal{I}_n(L,q_0;\,p_1,q_1;\dots)=\mathcal{I}_n(L,q_0;\,p_1+1,q_1-1;\dots)-\mathcal{I}_n(L,q_0;\,p_1,q_1-1;\dots).$$

By repeating this argument finitely many times, we eventually write $\mathcal{I}_n$ and $\widetilde{\mathcal{I}}_n$ as sums of integrals with $q_1=0$, which we temporarily assume (let us recall that $L=q_0$).

If $n>1$, by interchanging q_1 and q_2 we may assume that $q_2=0$, instead of $q_1=0$. If $n=1$, after interchanging q_0 and q_1 we have no longer $L=q_0$, but $L=q_1$ and $q_0=0$ instead. For technical reasons the case $n=1$ will be dealt with separately.

If $p_1>0$, from $t-p_1=(t+q_1)-(p_1+q_1)$ we have

$$\frac{(t-p_1)_{p_1+q_1}}{(p_1+q_1)!}=\frac{(t-p_1+1)_{p_1+q_1}}{(p_1+q_1)!}-\frac{(t-p_1+1)_{p_1-1+q_1}}{(p_1-1+q_1)!}.$$

We may repeat the same argument as before, with the roles of p_1 and q_1 interchanged, and write

$$\mathcal{I}_n(L,q_0;\,p_1,q_1;\dots)=\mathcal{I}_n(L,q_0;\,p_1-1,q_1+1;\dots)-\mathcal{I}_n(L,q_0;\,p_1-1,q_1;\dots),$$

and similarly for $\widetilde{\mathcal{I}}_n$. We eventually write $\mathcal{I}_n$ and $\widetilde{\mathcal{I}}_n$ as sums of integrals with $p_1=q_0=0$, when $n>1$. If $n=1$, the decomposition of $\mathcal{I}_1$ and $\widetilde{\mathcal{I}}_1$ starts under the conditions $q_1=L$ and $q_0=0$, and terminates when either $q_1=L$ and $p_1=q_0=0$, or $q_1=L+1$ and $q_0=0$.

For $n>1$, by interchanging q_1 and q_2 again, we may instead assume that $p_1=q_1=0$. Then

$$\mathcal{L}_n(0,0;\,p_2,q_2;\dots;\,p_n,q_n;\,x)=\mathcal{L}_{n-1}(p_2,q_2;\dots;\,p_n,q_n;\,x),$$

whence

$$\mathcal{I}_n(L,q_0;\,0,0;\,,\,p_2,q_2;\dots;\,p_n,q_n)=\mathcal{I}_{n-1}(L,q_0;\,p_2,q_2;\dots;\,p_n,q_n).$$

Similarly

$$R_n(L,q_0;\,0,0;\,p_2,q_2;\dots;\,p_n,q_n;\,t)=R_n(L,q_0;\,p_2,q_2;\dots;\,p_n,q_n;\,t);$$

hence

$$\widetilde{\mathcal{I}}_n(L,q_0;\,0,0;\,p_2,q_2;\dots;\,p_n,q_n)=\widetilde{\mathcal{I}}_{n-1}(L,q_0;\,p_2,q_2;\dots;\,p_n,q_n).$$

By the induction hypothesis

$$\mathcal{I}_{n-1}(L,q_0;\,p_2,q_2;\dots;\,p_n,q_n)=\widetilde{\mathcal{I}}_{n-1}(L,q_0;\,p_2,q_2;\dots;\,p_n,q_n),$$

which proves (2).

For $n=1$, if $p_1=q_0=0$ and $L=q_1$ we can still interchange q_0 and q_1, so that $p_1=q_1=0$ and $L=q_0$, and then apply the previous reduction to $\mathcal{I}_1(L,q_0;0,0)$ and $\widetilde{\mathcal{I}}_1(L,q_0;0,0)$. In the case $q_1=L+1$ and $q_0=0$, we have

$$R_1(L,0;\,p_1,L+1)=(-1)^{p_1+L+1}\frac{L!}{(t)_{L+1}}\frac{(t-p_1)_{p_1+L+1}}{(p_1+L+1)!}=(-1)^{p_1+L+1}\frac{L!\,(t-p_1)_{p_1}}{(p_1+L+1)!}.$$

By [Zudilin 2014, Lemma 1],

$$\widetilde{\mathcal{I}}_1(L,0;\,p_1,L+1)=(-1)^{p_1+L+1}\frac{L!\,p_1!}{(p_1+L+1)!}\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{(t-p_1)_{p_1}}{p_1!}\left(\frac{\pi}{\sin\pi t}\right)^2\mathrm{d}t$$
$$=(-1)^{L+1}\frac{L!\,p_1!}{(p_1+1)(p_1+L+1)!}.$$

On the other hand, see again [Rhin and Viola 1996],

$$(-1)^{L+1}\mathcal{I}_1(L,0;\,p_1,L+1)=\int_0^1\int_0^1 x^{L+1}(1-x)^{p_1}y^L\frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}$$
$$=\int_0^1\int_0^1 x^{p_1}(1-x)^L y^{p_1}\,\mathrm{d}x\,\mathrm{d}y=\frac{L!\,p_1!}{(p_1+1)(p_1+L+1)!}.$$

Therefore $\mathcal{I}_1(L,0;\,p_1,L+1)=\widetilde{\mathcal{I}}_1(L,0;\,p_1,L+1)$. □

We conclude this section by considering some instances of the integrals $\widetilde{\mathcal{I}}_n$.

Proposition 2.4. *For all nonnegative integers p_1, p_2, q_1, q_2 we have*

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{(t-p_1)_{p_1+q_1}}{(p_1+q_1)!}\frac{(t-p_2)_{p_2+q_2}}{(p_2+q_2)!}\left(\frac{\pi}{\sin\pi t}\right)^2\mathrm{d}t=(-1)^{p_1+p_2}\frac{(p_1+q_2)!\,(p_2+q_1)!}{(1+p_1+p_2+q_1+q_2)!}. \quad (3)$$

The above identity is a slight generalization of [Zudilin 2014, Lemma 1], and its easy proof by the first Barnes lemma is omitted for brevity.

Proposition 2.5. *For all nonnegative integers p_1, p_2, q_0, q_1, q_2 we have*

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{(t-p_1)_{p_1+q_1}}{(p_1+q_1)!}\frac{(t-p_2)_{p_2+q_2}}{(p_2+q_2)!}\frac{(L-q_0)!}{(t+q_0)_{L-q_0+1}}\left(\frac{\pi}{\sin\pi t}\right)^2\mathrm{d}t$$
$$=(-1)^{p_1+p_2}\frac{(p_1+q_0)!\,(p_1+q_2)!\,(p_2+q_0)!\,(p_2+q_1)!}{(L-q_1)!\,(L-q_2)!}, \quad (4)$$

where $L=1+p_1+p_2+q_0+q_1+q_2$.

Clearly, (4) is a particular case of the second Barnes lemma. It seems intriguing that (4) is, up to a sign, the product of two instances of (3), namely with (p_1,p_2,q_0,q_2) and with (p_1,p_2,q_1,q_0). After all, that is not too surprising, for the first and the second Barnes lemmas are the integral analogues of the Chu–Vandermonde and the Pfaff–Saalschütz formulas, respectively.

3. The second transformation

In the special case $n=3$, the integral in (2) is the integral introduced in [Zudilin 2014, Equation (10)]. Under the conditions

$$q_1-p_1=q_2-p_2=q_3-p_3,$$

by Proposition 2.2 we have

$$\mathcal{I}_3(L;\,q_0;\,p_2,q_1;\,p_3,q_2;\,p_1,q_3)=\mathcal{I}_3(L;\,q_0;\,p_3,q_1;\,p_1,q_2;\,p_2,q_3)$$

With the notation in [Zudilin 2014, Section 5],

$$
\begin{aligned}
&(-1)^{p_1+p_2+p_3}\mathcal{I}_3(L; q_0; p_2, q_1; p_3, q_2; p_1, q_3)\\
&\quad = (-1)^{a+b+e+f}\frac{\Gamma(g-b)}{\Gamma(e)\,\Gamma(f)\,\Gamma(e+f-a)}\\
&\qquad\qquad \times \frac{1}{2\pi i}\int_{-1/2-p_1-i\infty}^{-1/2-p_1+i\infty}\frac{\Gamma(a+t)\,\Gamma(b+t)\,\Gamma(e+t)\,\Gamma(f+t)}{\Gamma(1+t)\,\Gamma(1+a-e+t)\,\Gamma(1+a-f+t)\,\Gamma(g+t)}\Big(\frac{\pi}{\sin\pi t}\Big)^2\,dt,
\end{aligned}
$$

where

$$
\begin{gathered}
a = p_1+q_1+1, \quad b = p_1+q_0+1, \quad g = p_1+L+2,\\
e = p_1+q_2+1 = p_2+q_1+1, \quad f = p_1+q_3+1 = p_3+q_1+1.
\end{gathered}
$$

With regard to this integral, Zudilin stated the following conjecture, see [Zudilin 2014, Equation (19)]: *For a “sufficiently generic” choice of integral parameters a, b, e, f, g the following identity holds*:

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(e+t)\Gamma(f+t)}{\Gamma(1+t)\Gamma(1+a-e+t)\Gamma(1+a-f+t)\Gamma(g+t)}\Big(\frac{\pi}{\sin\pi t}\Big)^2\,dt\\
&\quad = (-1)^{a+b+e+f}\frac{\Gamma(e)\Gamma(f)\Gamma(e+f-a)}{\Gamma(g-b)}\\
&\qquad\qquad \times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a-b+g+2t)\,\Gamma(a+t)\,\Gamma(e+t)\,\Gamma(f+t)}{\Gamma(1+a+2t)\,\Gamma(1+a-b+t)\,\Gamma(e+f+t)\,\Gamma(g+t)}\,\frac{\pi}{\sin 2\pi t}\,dt. \qquad (5)
\end{aligned}
$$

In [Zudilin 2014, Proposition 2] the identity (5) is obtained in the particular case $a = 8n+1$, $b = 5n+1$, $e = 6n+1$, $f = 7n+1$ and $g = 14n+2$. The proof is based on the Gosper–Zeilberger algorithm of creative telescoping. This approach can be applied to any sufficiently generic, but concretely given sequences of vectors (a, b, e, f, g) corresponding to a fixed vector, not necessarily (8, 5, 6, 7, 14). In [Zudilin 2014] it is also conjectured that (5) is a specialization of a more general identity involving *complex* parameters a, b, e, f, g, and, intriguingly, it is speculated that Bailey could have possessed such an identity.

Towards (5) we now deduce two functional equations for both integrals in (5). Let

$$
\alpha(b, g; t) = \frac{\Gamma(g-b)\,\Gamma(b+t)}{\Gamma(g+t)}, \qquad \beta(a; b, g; t) = (-1)^b\frac{\Gamma(a-b+g+2t)}{\Gamma(1+a-b+t)\,\Gamma(g+t)}.
$$

We remark that

$$
\begin{gathered}
\alpha(b, g; t) = \alpha(b, g-1; t) - \alpha(b+1, g; t),\\
\beta(a; b, g; t) = \beta(a; b, g-1; t) - \beta(a; b+1, g; t),
\end{gathered}
$$

as is easily seen by using $\Gamma(1+z) = z\Gamma(z)$ and the trivial identities

$$
\begin{gathered}
g-b-1 = (g-t-1) - (b+t),\\
a-b+g+2t-1 = (g+t-1) + (a-b+t).
\end{gathered}
$$

We introduce

$$\mathcal{K}(a,b,e,f,g) = \frac{\Gamma(g-b)}{\Gamma(e)\Gamma(f)\Gamma(e+f-a)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+t)\Gamma(b+t)\Gamma(e+t)\Gamma(f+t)}{\Gamma(1+t)\Gamma(1+a-e+t)\Gamma(1+a-f+t)\Gamma(g+t)} \left(\frac{\pi}{\sin \pi t}\right)^2 \mathrm{d}t$$

and

$$\widetilde{\mathcal{K}}(a,b,e,f,g) = (-1)^{a+b+e+f} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a-b+g+2t)\,\Gamma(a+t)\,\Gamma(e+t)\,\Gamma(f+t)}{\Gamma(1+a+2t)\,\Gamma(1+a-b+t)\,\Gamma(e+f+t)\,\Gamma(g+t)} \frac{\pi}{\sin 2\pi t} \mathrm{d}t.$$

We have

$$\mathcal{K}(a,b,e,f,g+1) = \mathcal{K}(a,b,e,f,g) - \mathcal{K}(a,b+1,e,f,g+1), \tag{6}$$

$$\widetilde{\mathcal{K}}(a,b,e,f,g+1) = \widetilde{\mathcal{K}}(a,b,e,f,g) - \widetilde{\mathcal{K}}(a,b+1,e,f,g+1). \tag{7}$$

The second functional equation is obtained as follows. We consider the rational function

$$\gamma(a,e,f;t) = \frac{\Gamma(f+t)}{\Gamma(f)\Gamma(1+t)} \frac{\Gamma(a+t)}{\Gamma(e)\Gamma(1+a-e+t)} \frac{\Gamma(e+t)}{\Gamma(e+f-a)\Gamma(1+a-f+t)}.$$

From the identity

$$1 = xyz - (x-1)(y-1)(z-1) - (y-1)z - (z-1)x - (x-1)y$$

in the form

$$1 = \frac{f+t}{f}\frac{a+t}{e}\frac{e+t}{e+f-a} - \frac{t}{f}\frac{a-e+t}{e}\frac{a-f+t}{e+f-a} - \frac{a-e+t}{e}\frac{e+t}{e+f-a} - \frac{f+t}{f}\frac{a-f+t}{e+f-a} - \frac{t}{f}\frac{a+t}{e},$$

we obtain, after multiplication by $\gamma(a,e,f;t)$, that

$$\gamma(a,e,f;t) = \gamma(a+1,e+1,f+1;t) - \gamma(a+1,e+1,f+1;t-1) - \gamma(a,e+1,f;t) - \gamma(a,e,f+1;t) - \gamma(a+2,e+1,f+1;t-1).$$

By writing $b+t = (b+1)+(t-1)$ and $g+t = (g+1)+(t-1)$ where it suits, we obtain

$$\mathcal{K}(a,b,e,f,g) = \mathcal{K}(a+1,b,e+1,f+1,g) - \mathcal{K}(a+1,b+1,e+1,f+1,g+1) - \mathcal{K}(a,b,e+1,f,g) - \mathcal{K}(a,b,e,f+1,g) - \mathcal{K}(a+2,b+1,e+1,f+1,g+1).$$

Using (6) inside the last identity, after some elementary manipulation we get

$$\mathcal{K}(a+1,b,e+1,f+1,g+1) = \mathcal{K}(a,b,e,f,g) + \mathcal{K}(a,b,e+1,f,g) + \mathcal{K}(a,b,e,f+1,g) + \mathcal{K}(a+2,b+1,e+1,f+1,g+1). \tag{8}$$

Let

$$\delta(a,e,f;t) = \frac{\Gamma(a+t)\Gamma(e+t)\Gamma(f+t)}{\Gamma(1+a+2t)\Gamma(e+f+t)}.$$

Since

$$1 = \frac{e+t}{e+f+t} + \frac{f+t}{e+f+t} + \frac{a+t}{e+f+t} - \frac{a+2t}{e+f+t},$$

we have

$$\delta(a,e,f;t)=\delta(a,e+1,f;t)+\delta(a,e,f+1;t)+\delta(a+2,e+1,f+1;t-1)-\delta(a+1,e+1,f+1;t-1).$$

We easily deduce the analogue of (8) for $\widetilde{\mathcal{K}}$:

$$\widetilde{\mathcal{K}}(a+1,b,e+1,f+1,g+1) \\ =\widetilde{\mathcal{K}}(a,b,e,f,g)+\widetilde{\mathcal{K}}(a,b,e+1,f,g)+\widetilde{\mathcal{K}}(a,b,e,f+1,g)+\widetilde{\mathcal{K}}(a+2,b+1,e+1,f+1,g+1). \quad (9)$$

We assume that a, b, e, f are integers, and define

$$\Delta = a+b+e+f-1-(1+a-e)-(1+a-f)-g = b+2e+2f-a-g-3.$$

Furthermore, we assume that $2e-a$ and $2f-a$ are positive, $g-b$ is not a nonpositive integer, and that it is always possible to separate the two groups of poles in the integrands in (5).

Lemma 3.1. *If $a=1$ and $\Delta=-2$, then $\mathcal{K}(a,b,e,f,g)=\widetilde{\mathcal{K}}(a,b,e,f,g)$ and therefore (5) holds.*

Proof. We have

$$\mathcal{K}(1,b,e,f,b+2e+2f-2) \\ = \frac{\Gamma(2e+2f-2)}{\Gamma(e)\Gamma(f)\Gamma(e+f-1)}\frac{(-1)^{e+f}}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(b+t)\Gamma(e+t)\Gamma(f+t)\Gamma(e-1-t)\Gamma(f-1-t)}{\Gamma(g+t)}\,\mathrm{d}t.$$

This integral can be computed with the help of the second Barnes lemma; see (4).

On the other hand,

$$\widetilde{\mathcal{K}}(1,b,e,f,b+2e+2f-2) \\ = (-1)^{1+b+e+f}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(2e+2f-1+2t)\,\Gamma(1+t)\,\Gamma(e+t)\,\Gamma(f+t)}{\Gamma(2+2t)\,\Gamma(2-b+t)\,\Gamma(e+f+t)\,\Gamma(b+2e+2f-2+t)}\,\frac{\pi}{\sin 2\pi t}\,\mathrm{d}t.$$

Applying the duplication formula

$$\Gamma(2z)=\frac{2^{2z}\Gamma(z)\Gamma\bigl(z+\frac{1}{2}\bigr)}{2\sqrt{\pi}}$$

to $\Gamma(2e+2f-1+2t)$ and $\Gamma(2+2t)$, the factors $\Gamma(1+t)$ and $\Gamma(e+f+t)$ cancel out, so that

$$\widetilde{\mathcal{K}}(1,b,e,f,b+2e+2f-2) \\ = (-1)^{e+f}2^{2e+2f-3}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(e+f-1/2+t)\,\Gamma(e+t)\,\Gamma(f+t)\,\Gamma(-1/2-t)\,\Gamma(b-1-t)}{\Gamma(b+2e+2f-2+t)}\,\mathrm{d}t.$$

This also can be computed with the help of the second Barnes lemma. After a few simplifications using the duplication formula again, we arrive at the desired equality. □

Lemma 3.2. *If $a=1$ and Δ is a negative integer $\leqslant -2$, then $\mathcal{K}(a,b,e,f,g)=\widetilde{\mathcal{K}}(a,b,e,f,g)$ and therefore (5) holds.*

Proof. This follows from the previous lemma by induction, using (6) and (7). □

Lemma 3.3. *If $a=1$ (and $g\in\mathbb{C}$), then $\mathcal{K}(a,b,e,f,g)=\widetilde{\mathcal{K}}(a,b,e,f,g)$ and therefore (5) holds.*

Proof. This follows from the last lemma by Carlson's theorem; see [Bailey 1964, Section 5.3]. □

Lemma 3.4. *If* $1+a=2e$, *then* $\mathcal{K}(a,b,e,f,g)=\widetilde{\mathcal{K}}(a,b,e,f,g)$ *and therefore* (5) *holds.*

Proof. This can be proved in three steps, just like the lemma above. We start with the special case when the additional condition $\Delta=-2$ holds, and in that case we use the second Barnes lemma. Then we use (6) and (7) to deal with the case where Δ is an integer $\leqslant -2$. Finally we use Carlson's theorem to deal with the general case. □

Lemma 3.5. *If* $1+a=2f$, *then* $\mathcal{K}(a,b,e,f,g)=\widetilde{\mathcal{K}}(a,b,e,f,g)$ *and therefore* (5) *holds.*

Proof. This follows from the previous lemma and the symmetry properties $\mathcal{K}(a,b,e,f,g)=\mathcal{K}(a,b,f,e,g)$ and $\widetilde{\mathcal{K}}(a,b,e,f,g)=\widetilde{\mathcal{K}}(a,b,f,e,g)$. □

We are going to use (8) and (9) in an iterative fashion, similar to the proof of Theorem 2.1 in [Rhin and Viola 1996], to prove the following:

Theorem 3.6. *Let* a,e,f *be positive integers such that* $2e-a$ *and* $2f-a$ *are positive, and that* $a\geqslant\max\{e,f\}$. *Let* $b\in\mathbb{Z}$ *and* $g\in\mathbb{C}$ *such that* $g-b$ *is not a nonpositive integer. Then* (5) *holds.*

Proof. Note that $e+f-a$ is also positive, because $2(e+f-a)=(2e-a)+(2f-a)$. By continuity we may assume that $g\notin\mathbb{Z}$. We put $(a,b,e,f,g)=(a_0-1,b_0,e_0-1,f_0-1,g_0-1)$ in (8), say. The integral $\mathcal{K}(a_0,b_0,e_0,f_0,g_0)$ is decomposed into the sum of four integrals $\mathcal{K}(a_j,b_j,e_j,f_j,g_j)$, with $j=1,\dots,4$, such that for each $j=1,\dots,4$ at least two among a_j, $2e_j-a_j$ and $2f_j-a_j$ equal the corresponding integer with $j=0$ decreased by 1, while their sum $2e_j+2f_j-a_j$ is decreased either by 1 or by 3. Therefore the iteration must terminate with at least one of the following three conditions fulfilled: $a=1$, $2e-a=1$ or $2f-a=1$. However, we need to keep e, f and $e+f-a$ positive throughout the process. To this end, we remark that if $e_j=e_0-1$ then $2e_j-a_j=2e_0-a_j-1$, and we start the process with $e\geqslant 2e-a$. Similarly, if $f_j=f_0-1$ then $2f_j-a_j=2f_0-a_j-1$, and $f\geqslant 2f-a$. Also, if $e_j+f_j-a_j=e_0+f_0-a_0-1$, then $2e_j-a_j=2e_0-a_j-1$ and $2f_j-a_j=2f_0-a_j-1$, and we obviously have $e+f-a\geqslant\min\{2e-a,2f-a\}$. □

For a bit more generality, we show how to dispense with the somewhat artificial assumption $a\geqslant\max\{e,f\}$, to obtain:

Theorem 3.7. *Let* a,e,f *be positive integers such that* $2e-a$ *and* $2f-a$ *are positive. Let* $b\in\mathbb{Z}$ *and* $g\in\mathbb{C}$ *such that* $g-b$ *is not a nonpositive integer. Then* (5) *holds.*

Proof. By the theorem above, we only need to deal with the case $\max\{a,e,f\}=\max\{e,f\}=e$, say. With the change of variable from t to s by putting $t=s+e-a$, we obtain

$$\mathcal{K}(a,b,e,f,g)=\mathcal{K}(2e-a,b+e-a,e,e+f-a,g+e-a),$$
$$\widetilde{\mathcal{K}}(a,b,e,f,g)=\widetilde{\mathcal{K}}(2e-a,b+e-a,e,e+f-a,g+e-a).$$

Since $2e-a\geqslant e$ and $2e-a\geqslant e+f-a$, the identity (5) follows from the case in the theorem above. □

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