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ON PUTATIVE SELF-SIMILARITY FOR INCOMPRESSIBLE 3D EULER

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We consider hypothetical solutions of 3D Euler which blow up in finite time in a self-similar fashion. We prove that, if the initial data has finite kinetic energy, then the similarity exponent γ which governs the rate of zooming in must be larger than $\frac{2}{5}$. If a smooth globally self-similar blowup profile exists, and this profile satisfies an outgoing property, we prove that $\gamma \geq \frac{1}{2}$. For axisymmetric solutions, we establish the bound $\gamma \geq \frac{1}{2}$ under the sole assumption that the velocity profile is C^2 smooth.

1. Introduction

Self-similarity [Barenblatt 1996] is a powerful idea: it says that things look the same at different scales if you just know how to zoom in or out. It is a particularly useful tool for identifying singular solutions of nonlinear PDEs via a reduction to ODEs [Sedov 1959; Eggers and Fontelos 2015]. Self-similarity is found in compressible fluids and in many other physical systems when dynamics are determined by local interactions. Compressible Euler equations for instance exhibit self-similar explosions [Sedov 1946; von Neumann 1947; Taylor 1950], implosions [Guderley 1942; Hunter 1960; Merle et al. 2022a; Buckmaster et al. 2025; Shao et al. 2025] and shocks [Buckmaster et al. 2022; 2023].

Incompressible fluids are not local: the outside matters. This makes the link between possible singularities and local self-similar behavior rather tenuous. Several recent works have reported self-similar incompressible singularities, either as rigorous proofs or in computational studies. In all of them there is a remnant of compression due to either the presence of boundaries [Chen and Hou 2022; 2025b; 2025a; Luo and Hou 2014; Wang et al. 2023; 2025a; 2025b; 2025c], or the lack of smoothness of vorticity [Elgindi 2021; Chen and Hou 2021; Elgindi et al. 2021; Elgindi and Pasqualotto 2023; Chen 2024; Córdoba et al. 2025]. The goal of this paper is to analyze constraints on putative self-similar singularities for the incompressible 3D Euler equations with smooth initial data, in the absence of boundaries.

We recall the vorticity formulation of incompressible 3D Euler equations

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u, \quad \nabla \cdot u = 0, \quad \omega = \nabla \times u. \tag{1-1}$$

The smooth and localized initial data ω_0 for the Cauchy problem associated to (1-1) is specified at time $t = 0$, and the equations are posed on the whole space ($x \in \mathbb{R}^3$). It is well known that singularities of any kind cannot arise in finite time from smooth and localized initial data unless the vorticity becomes

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infinite, so that its maximum magnitude is not time integrable. This is the well-known Beale–Kato–Majda criterion [Beale et al. 1984]. We say that (1-1) exhibits a self-similar singularity if there exists a first time $T_* > 0$, a space location $x_* \in \mathbb{R}^3$, a similarity exponent $\gamma > 0$, and a vorticity profile Ω such that

$$\omega(x,t) = \frac{1}{T_* - t}\Omega\left(\frac{x - x_*}{(T_* - t)^\gamma}\right) + \text{l.o.t.} \quad \text{as } t \to T_*^-. \tag{1-2}$$

The factor $(T_* - t)^{-1}$ appearing in (1-2) is unavoidable, and consistent with [loc. cit.]. A more precise meaning of a self-similar singularity for (1-1) will be given later in the paper; see Section 2 for an asymptotically self-similar blowup, and Section 3 for a globally self-similar blowup.

If a true self-similar singularity forms in (1-1), the value of the exponent γ is an important and powerful dynamic property, and needs to be understood. Finding a profile Ω is a well-known challenge. Because access to self-similar solutions is fraught with numerical and theoretical difficulties, it is useful to establish strict mathematical guardrails and to provide computational benchmarks.

The Euler equations have a great variety of different types of solutions. Blowup of smooth solutions with infinite kinetic energy has been proved [Childress et al. 1989; Constantin 2000; Ohkitani and Gibbon 2000; Stuart 1988]. We prove that, if the initial data of (1-1) has finite kinetic energy, then a solution behaving as in (1-2) must have $\gamma \geq \frac{2}{5}$; see Theorem 2.1.

Beyond a general intrinsic interest in lower bounds for γ as a benchmark for ongoing computational and analytical studies, in the second half of the paper we focus on the hypothetical case when γ lies below the parabolic threshold of $\frac{1}{2}$, relevant for incompressible 3D Navier–Stokes. At $\gamma = \frac{1}{2}$, the advection operator $\partial_t + u \cdot \nabla$ and the dissipative operator $-\Delta$ are in exact balance when acting on vorticities in the form (1-2). Liouville theorems [Nečas et al. 1996; Tsai 1998; Seregin 2005; Chae and Wolf 2017] rule out certain globally self-similar solutions for the 3D incompressible Navier–Stokes equations with $\gamma = \frac{1}{2}$. Under reasonable assumptions, $\gamma > \frac{1}{2}$ cannot be the exponent of an approximate self-similar blowup for 3D incompressible Navier–Stokes equations (see Proposition 3.1). For self-similar scaling with $\gamma < \frac{1}{2}$, the viscous term yields a vanishingly small force competing with the Euler nonlinearity. If self-similar solutions of Euler equations with $\gamma < \frac{1}{2}$ are found, they can be used to obtain 3D Navier–Stokes blowup as a perturbation of the 3D Euler one, allowing the inviscid blowup to be used to find a viscous one. This is indeed a rigorously proven fact in the realm of compressible flows: globally self-similar implosion singularities for 3D compressible isentropic Euler equations can be used to provide asymptotically self-similar implosion singularities for 3D compressible barotropic Navier–Stokes equations with constant viscosity coefficients; see [Merle et al. 2022a; 2022b; Buckmaster et al. 2025; Shao et al. 2025]. In the process of establishing the blowup [Merle et al. 2022b; Buckmaster et al. 2025; Shao et al. 2025], in addition to having $\gamma < \frac{1}{2}$ the authors prove and use that the inviscid similarity profile has infinite regularity (real-analyticity) in a neighborhood of all sonic points (stagnation points on the self-similar fast-acoustic characteristics).

An analogous approach to establish blowup for incompressible viscous flows hinges on finding self-similar blowup of (1-1) with $\gamma < \frac{1}{2}$. This is advocated in [Wang et al. 2023; 2025a].

Elgindi [2025] identifies an outgoing property (see (3-36)) as being an essential ingredient for the existence of self-similar profiles. This outgoing property quantifies the statement that all self-similar Lagrangian trajectories that originate from nonzero labels must escape to infinity as (self-similar) time diverges. In particular, this condition implies that the self-similar Lagrangian flow has no stagnation points, except for the trivial one at the origin.

We prove that, if a globally self-similar smooth solution to incompressible 3D Euler equations exists, and it satisfies a local version of the outgoing property, then we must have $\gamma \geq \frac{1}{2}$; see Theorems 3.8 and 3.9. Thus, in this case, the approach proposed to produce incompressible Navier–Stokes singularities via Euler self-similar solutions must fail.

Independently of the outgoing property, if a globally self-similar vorticity profile Ω exists, then it cannot be too small, as measured by a certain scaling-invariant norm; see Theorem 3.4.

When restricting our analysis to axisymmetric similarity profiles, we prove a much stronger result. If the velocity field is C^2 smooth, then $\gamma \geq \frac{1}{2}$; see Theorem 4.5. Additionally, if the velocity has nonzero swirl at a stagnation point of the self-similar Lagrangian flow map, then $\gamma = \frac{1}{2}$; see Theorem 4.3.

Our proofs are based on basic, well-known properties of incompressible Euler and Navier–Stokes equations. We chose to present rigorous results, with explicit minimal assumptions. The fact that $\gamma \geq \frac{2}{5}$ for finite kinetic energy flows is such a result. The relevance of the exponent $\gamma = \frac{1}{2}$ in Euler flows is one of the main messages of this paper. This exponent, which is natural for Navier–Stokes self-similarity, is natural for Euler equations as well. This is due to the invariance of circulation, a quantity with dimension of kinematic viscosity.

2. On asymptotically self-similar blowup with finite kinetic energy

A 3D incompressible Euler singularity cannot be just a vorticity magnitude singularity — the spatial gradient of vorticity must also blow up fast enough. For $\mu \in (0, 1)$, let

$$\ell_\mu(t) := \min\left\{L_0; \left(\frac{[\omega(\cdot, t)]_\mu}{\|u(\cdot, t)\|_{L^2}}\right)^{-2/(2\mu+5)}\right\}$$

be the length scale formed with the vorticity Hölder seminorm

$$[\omega(\cdot, t)]_\mu := \sup_{0<|x-y|\leq L_0} \frac{|\omega(x, t) - \omega(y, t)|}{|x - y|^\mu}$$

and the L^2 norm of velocity, where L_0 is an arbitrary reference length scale. Constantin [1994, Theorem 1, p. 38] showed that, if

$$\int_0^{T_*} \ell_\mu(t)^{-5/2}\, dt < \infty,$$

then no singularities can occur from smooth and localized data at time T_*. This result follows easily from the representation of vorticity and from the conservation of kinetic energy.

Constantin [1994] also remarks as a consequence that, if the vorticity is assumed to have a globally self-similar blowup, namely

$$\omega(x,t) = \frac{1}{T_*-t}\Omega\left(\frac{x}{L(t)}\right),$$

then we must have

$$\int_0^{T_*} L(t)^{-5/2}\,dt = \infty.$$

In particular, if $L(t) = L_0(1-t/T_*)^\gamma$ as in (1-2), then $\gamma \geq \frac{2}{5}$ is necessary for blowup. Here we revisit this result, without assuming the existence of a globally self-similar profile; instead, we assume only local behavior consistent with self-similarity. Furthermore, we link the putative exponent γ to the behavior of the velocity away from the local self-similar behavior. Chae [2007] showed that self-similar blowup cannot occur if the vorticity profile decays at infinity fast enough.[1] This is not what we do here; our assumed behavior does not invoke a profile, but, if a profile existed, it would have required only that the profile's first derivatives be bounded in a fixed ball. We make assumptions about the nature of the true solution of (1-1), which would be automatically satisfied if a self-similar C^1 smooth vorticity profile existed.

Theorem 2.1. *Assume the 3D Euler equation (1-1) has initial data $\omega_0 \in C^1(\mathbb{R}^3)$ with finite kinetic energy, i.e., $u_0 = (-\Delta)^{-1}\nabla\times\omega_0 \in L^2(\mathbb{R}^3)$. Assume that the resulting local-in-time smooth solution blows up at some finite time $T_* > 0$. If there exists $\gamma > 0$ such that*

$$\sup_{t\in[0,T_*)} (T_*-t)^{1+\gamma}\|\nabla\omega(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} < \infty, \tag{2-1}$$

then $\gamma \geq \frac{2}{5}$.

Remark 2.2. A few comments concerning Theorem 2.1 are in order:

(a) For a self-similar blowup (see (1-2)) with profile Ω, assumption (2-1) is automatic if $\nabla\Omega \in L^\infty$.

(b) The proof of Theorem 2.1 does not use the full assumption (2-1). If the blowup occurs at some point $x_* \in \mathbb{R}^3$, we only use that $\sup_{t\in[0,T_*)}(T_*-t)^{1+\gamma}\|\nabla\omega(\cdot,t)\|_{L^\infty(B(t))} < \infty$, where $B(t)$ is the ball of radius $(T_*-t)^{2(1+\gamma)/7}$ centered at x_*.

(c) If in addition to (2-1) we have

$$\sup_{t\in[0,T_*)} \|u(\cdot,t)\|_{L^p(\mathbb{R}^3)} < \infty \tag{2-2}$$

[1] From a complementary viewpoint, for locally self-similar profiles in the corresponding range of exponents, Bronzi and Shvydkoy [2015] showed (see also [Shvydkoy 2013]) that a nontrivial finite-energy blowup must carry a positive amount of energy, in the sense that the local energy average admits a lower bound matching the upper bound dictated by finite kinetic energy.

for some $p \geq 2$, then the lower bound on γ becomes

$$\gamma \geq \frac{p}{p+3}.$$

In particular, if $u \in L^\infty(0, T_*; L^3(\mathbb{R}^3))$, then $\gamma \geq \frac{1}{2}$.

Proof of Theorem 2.1. We recall from [Constantin 1994] (see also [Constantin and Fefferman 1993; Constantin 2017]) that the magnitude of vorticity satisfies

$$\partial_t|\omega| + u \cdot \nabla|\omega| = \alpha|\omega|, \tag{2-3}$$

where the stretching factor α is defined as

$$\alpha(x,t) := \frac{3}{4\pi}\,\text{p.v.}\int_{\mathbb{R}^3} (\hat{y}\cdot\xi(x,t))\big(\hat{y}\cdot(\omega(x+y,t)\times\xi(x,t))\big)\frac{dy}{|y|^3}, \tag{2-4}$$

where $\hat{y} = y/|y| \in \mathbb{S}^2$ and $\xi(x,t) = \omega(x,t)/|\omega(x,t)| \in \mathbb{S}^2$.

We consider a smooth cutoff function $\chi : \mathbb{R}_+ \to \mathbb{R}$, with $\chi \equiv 1$ on $[0,1]$, and $\chi \equiv 0$ on $[2,\infty)$. Let $R(t) > 0$, to be determined later. We decompose $\alpha = \alpha_{\text{in}} + \alpha_{\text{out}}$, where

$$\begin{aligned}\alpha_{\text{in}}(x,t) &:= \frac{3}{4\pi}\,\text{p.v.}\int_{\mathbb{R}^3} (\hat{y}\cdot\xi(x,t))\big(\hat{y}\cdot(\omega(x+y,t)\times\xi(x,t))\big)\chi\left(\frac{|y|}{R(t)}\right)\frac{dy}{|y|^3}\\ &= \frac{3}{4\pi}\int_{\mathbb{R}^3} (\hat{y}\cdot\xi(x,t))\big(\hat{y}\cdot\big((\omega(x+y,t)-\omega(x,t))\times\xi(x,t)\big)\big)\chi\left(\frac{|y|}{R(t)}\right)\frac{dy}{|y|^3}\end{aligned} \tag{2-5}$$

and

$$\begin{aligned}\alpha_{\text{out}}(x,t) &:= \frac{3}{4\pi}\,\text{p.v.}\int_{\mathbb{R}^3} (\hat{y}\cdot\xi(x,t))\big(\hat{y}\cdot(\omega(x+y,t)\times\xi(x,t))\big)\left(1-\chi\left(\frac{|y|}{R(t)}\right)\right)\frac{dy}{|y|^3}\\ &= \frac{3}{4\pi}\int_{\mathbb{R}^3} u(x+y,t)\cdot\big((5(\hat{y}\cdot\xi)^2-1)\hat{y}-2(\hat{y}\cdot\xi)\xi\big)\left(1-\chi\left(\frac{|y|}{R(t)}\right)\right)\frac{dy}{|y|^4}\\ &\quad - \frac{3}{4\pi R(t)}\int_{\mathbb{R}^3}(\hat{y}\cdot\xi)u(x+y,t)\cdot(\hat{y}\times(\xi\times\hat{y}))\chi'\left(\frac{|y|}{R(t)}\right)\frac{dy}{|y|^3}.\end{aligned} \tag{2-6}$$

In the inner integral we used that $\omega(x,t)\times\xi(x,t) = 0$, while in the outer integral we have integrated by parts with respect to curl_y. We make two claims:

- There exists a constant $C_{\text{in}} > 0$, depending only on χ, such that

$$|\alpha_{\text{in}}(x,t)| \leq C_{\text{in}} R(t)\|\nabla\omega(\cdot,t)\|_{L^\infty(B_{2R(t)}(x))}. \tag{2-7}$$

  This estimate follows directly from (2-5) upon noting that $|\hat{y}| = 1 = |\xi(x,t)|$, and by bounding $|\omega(x+y,t)-\omega(x,t)| \leq |y|\|\nabla\omega(\cdot,t)\|_{L^\infty(B_{2R(t)})}$.

- For any $p \geq 1$, there exists a constant $C_{\text{out}} > 0$, depending only on p and χ, such that

$$|\alpha_{\text{out}}(x,t)| \leq C_{\text{out}} R(t)^{-1-3/p}\|u(\cdot,t)\|_{L^p(\mathbb{R}^3)}. \tag{2-8}$$

  This estimate follows directly from (2-6), using Hölder's inequality and $|\hat{y}| = 1 = |\xi(x,t)|$.

The theorem now immediately follows from (2-7)–(2-8), by letting $p=2$. Indeed, classical solutions of the 3D Euler equations conserve their kinetic energy, $\|u(\cdot,t)\|_{L^2(\mathbb{R}^3)} = \|u_0\|_{L^2(\mathbb{R}^3)}$, and hence we have an a priori bound for $\mathsf{RHS}_{(2\text{-}8)}$. Moreover, assumption (2-1) implies an upper bound for $\mathsf{RHS}_{(2\text{-}7)}$. Optimizing in $R(t)$ and using (2-1), we obtain the pointwise bound

$$|\alpha(x,t)| \le C(T_*-t)^{-5(1+\gamma)/7}\Big(\sup_{t\in[0,T_*)}(T_*-t)^{1+\gamma}\|\nabla\omega(\cdot,t)\|_{L^\infty(B_{2R(t)}(x))}\Big)^{5/7}\|u_0\|_{L^2(\mathbb{R}^3)}^{2/7},$$

where $C=C(C_{\text{in}},C_{\text{out}})>0$. Thus, if $\frac{5}{7}(1+\gamma)<1$ — which is equivalent to $\gamma<\frac{2}{5}$ — then we have $\int_0^{T_*}\|\alpha(\cdot,t)\|_{L^\infty}\,dt<\infty$, and so, by (2-3) and the Beale–Kato–Majda criterion, no blowup can occur at time T_*. Therefore, a singularity necessitates $\gamma\ge\frac{2}{5}$.

If in addition to (2-1) we also know that (2-2) holds, using (2-8) we may analogously prove

$$|\alpha(x,t)| \le C(T_*-t)^{-(p+3)(1+\gamma)/(2p+3)}\Big(\sup_{t\in[0,T_*)}(T_*-t)^{1+\gamma}\|\nabla\omega(\cdot,t)\|_{L^\infty(B_{2R(t)}(x))}\Big)^{(p+3)/(2p+3)} \cdot \sup_{t\in[0,T_*)}\|u(\cdot,t)\|_{L^p(\mathbb{R}^3)}^{p/(2p+3)},$$

where $C=C(C_{\text{in}},C_{\text{out}},p)>0$. Thus, if $(p+3)(1+\gamma)/(2p+3)<1$ — which is equivalent to $\gamma<p/(p+3)$ — then $\int_0^{T_*}\|\alpha(\cdot,t)\|_{L^\infty}\,dt<\infty$, and no blowup can occur at time T_*. Therefore, a singularity necessitates $\gamma\ge p/(p+3)$, as claimed in Remark 2.2(c). □

3. On globally self-similar blowup

In order to obtain better lower bounds on the similarity exponent $\gamma\ge\frac{2}{5}$, we analyze the hypothetical case in which the 3D incompressible Euler equations (1-1), written in velocity form as

$$\partial_t u + u\cdot\nabla u + \nabla p = 0, \qquad \nabla\cdot u = 0, \tag{3-1}$$

admit a globally self-similar singularity at a time $T_*>0$, at a single point $x_*\in\mathbb{R}^3$.

3.1. ***The self-similar ansatz.*** Due to Galilean symmetry, we assume without loss of generality that the singularity occurs at the space location $x_*=0$, and that the similarity profile for the velocity vanishes at this point. Due to time-rescaling symmetry, we take without loss of generality $T_*=1$. That is, we investigate the globally self-similar ansatz

$$\begin{aligned} u(x,t) &= (1-t)^{\gamma-1}U(y), \quad & p(x,t) &= (1-t)^{2(\gamma-1)}P(y),\\ \omega(x,t) &= \frac{1}{1-t}\Omega(y), & y &= \frac{x}{(1-t)^\gamma}. \end{aligned} \tag{3-2}$$

This ansatz is less restrictive than it seems. A more general form, $\omega(x,t)=A(t)\widetilde{\Omega}\big((x-x(t))/\ell(t),\tau(t)\big)$, leads to the form above. Indeed, writing the Euler equations in vorticity form and using the fact that the velocity and vorticity are related by the linear relation $\omega=\nabla\times u$, we see that the nonlinear term $u\cdot\nabla\omega-\omega\cdot\nabla u$ scales with amplitude A^2 under the general ansatz above. We thus arrive at the ODE

$\dot{A} = A^2$, where $\dot{A}$ is a time derivative.[2] This leads inevitably to $A(t) = (T-t)^{-1}$. The rest follows on dividing the Euler equation by A^2 and taking the simplest situation, when the ensuing coefficients are constant. Then the constant coefficient in front of $y \cdot \nabla\Omega$ leads to $\ell(t) = \ell_0(1 - t/T)^\gamma$. The term due to $x(t)$ is integrated, because A and ℓ are known. It follows that $x(t) = x_* - (c/\gamma)\ell(t)$, where c is a constant, and this is dealt with by translating $\widetilde{\Omega}$.

3.1.1. *The stationary self-similar PDE.* The similarity profiles U, P, Ω appearing in (3-2) satisfy the (stationary) self-similar Euler equation in velocity form

$$(1-\gamma)U + \gamma(y\cdot\nabla)U + (U\cdot\nabla)U + \nabla P = 0, \quad \nabla\cdot U = 0, \tag{3-3}$$

or, equivalently, the (stationary) self-similar Euler equation in vorticity form

$$\Omega + \gamma(y\cdot\nabla)\Omega + (U\cdot\nabla)\Omega = (\Omega\cdot\nabla)U, \quad \Omega = \nabla\times U, \quad \nabla\cdot U = 0. \tag{3-4}$$

As mentioned earlier, due to Galilean symmetry, we assume throughout this section that the self-similar velocity profile U vanishes at the origin. We also only consider profiles U which are sublinear at infinity.[3] We summarize these properties as

$$U(0) = 0, \quad \lim_{|y|\to\infty} |y|^{-1}|U(y)| = 0. \tag{3-5}$$

3.1.2. *Space rescaling and normalization.* Note that, if U and Ω give a solution of (3-4) on $\mathbb{R}^3$, then $U_\lambda(y) = \lambda U(y/\lambda)$ and $\Omega_\lambda(y) = \Omega(y/\lambda)$, also give a solution of (3-4) on $\mathbb{R}^3$. Under this rescaling, we have[4]

$$\begin{aligned} \|\nabla^2 U_\lambda\|_{L^\infty(\mathbb{R}^3)} &= \frac{1}{\lambda}\|\nabla^2 U\|_{L^\infty(\mathbb{R}^3)}, \\ \|\nabla\Omega_\lambda\|_{L^\infty(\mathbb{R}^3)} &= \frac{1}{\lambda}\|\nabla\Omega\|_{L^\infty(\mathbb{R}^3)}, \\ \|\Omega_\lambda\|_{L^p(\mathbb{R}^3)} &= \lambda^{3/p}\|\Omega\|_{L^p(\mathbb{R}^3)}. \end{aligned} \tag{3-6}$$

In order to fix units, throughout this section we consider profiles which are normalized via (3-6) to satisfy

$$\|\nabla^2 U\|_{L^\infty(\mathbb{R}^3)} = 1. \tag{3-7}$$

3.1.3. *Behavior of the profiles at space infinity.* While the assumption of sublinearity of U at infinity (see (3-5)) is necessary for any reasonable self-similar profile, the self-similar PDEs (3-3) and (3-4) impose more stringent assumptions. Indeed, (3-5) dictates that the term $\gamma y\cdot\nabla$ is stronger than the term $U\cdot\nabla$ for large values of $|y|$. Thus, the behavior of U and Ω as $|y|\to\infty$ must respect the kernel of the operators $(1-\gamma) + \gamma y\cdot\nabla$ (for U) and $1 + \gamma y\cdot\nabla$ (for Ω). We deduce that the leading-order behavior of the

[2] This time derivative can be with respect to an arbitrary internal clock, that is: $\frac{d}{dt}$ can be replaced by $(1/f(t))\frac{d}{dt}$, where $f(t)$ is an arbitrary nonvanishing function.

[3] These are the profiles for which we may (typically) employ a localization or cutoff procedure at large values of y, to ensure that the velocity field u in original (x,t) coordinates has finite kinetic energy.

[4] As we shall see later (see (3-8)), Ω is only expected to lie in $L^p(\mathbb{R}^3)$ when $p > 3\gamma$.

self-similar profiles in the far field is given by

$$|U| \sim |y|^{(\gamma-1)/\gamma} \quad \text{and} \quad |\Omega| \sim |y|^{-1/\gamma} \quad \text{as } |y| \to \infty.$$

It is convenient to quantify the above asymptotic descriptions. Recalling that $U(0) = 0$, we assume that there exists a constant $\mathsf{C}_\flat > 0$ such that

$$|U(y)| \le \mathsf{C}_\flat |y| \langle y \rangle^{-1/\gamma} \quad \text{and} \quad |\Omega(y)| + |\nabla U(y)| \le \mathsf{C}_\flat \langle y \rangle^{-1/\gamma} \quad \text{for all } y \in \mathbb{R}^3. \tag{3-8}$$

3.2. ***A remark for the viscous case.*** We consider here hypothetical blowup for the 3D incompressible Navier–Stokes equation (set $\mathsf{RHS}_{(3\text{-}1)} = \Delta u$ instead of 0), where the vorticity $\omega(x,t)$ is approximately self-similar.

Proposition 3.1. *Let $\omega(x,t)$ be a solution of the 3D incompressible Navier–Stokes equation, which can be written as*

$$\omega(x,t) = \omega_{\text{in}}(x,t) + \omega_{\text{out}}(x,t),$$

where ω_{out} is regular, in the sense that

$$\int_0^1 \|\omega_{\text{out}}(\cdot,t)\|^4_{L^2(\mathbb{R}^3)} dt \le \Gamma < \infty, \tag{3-9}$$

and where ω_{in} is supported in $B_{\ell(t)}(x(t))$ with $\ell(t) \sim (1-t)^\gamma$, and satisfies

$$\|\omega_{\text{in}}(\cdot,t)\|_{L^2(B_{\ell(t)}(x(t)))} \le C(1-t)^{-1+3\gamma/2} \tag{3-10}$$

for some $C < \infty$. Then, if the solution blows up at time $t = 1$, it follows that $\gamma \le \frac{1}{2}$.

Remark 3.2. The assumption (3-10) is satisfied if

$$\omega_{\text{in}}(x,t) \sim \frac{1}{1-t}\Omega\left(\frac{x - x(t)}{(1-t)^\gamma}\right) \quad \text{in } B_{\ell(t)}(x(t)), \text{ as } t \to 1^-,$$

with $|\Omega(y)| \le C$ for $|y| \le 1$, and $\ell(t) \sim (1-t)^\gamma$. The assumption (3-9) is satisfied, for instance, if $\omega_{\text{out}}(x,t) = 0$ for $|x - x(t)| \le \ell(t)$,

$$|\omega_{\text{out}}(x,t)| \le C_1 |x - x(t)|^{-1/\gamma} \quad \text{for } \ell(t) \le |x - x(t)| \le 1,$$

or, more generally, if

$$\int_0^1 \|\omega_{\text{out}}(t)\|^4_{L^2(B_1(x(t)))}\, dt \le C_2 < \infty, \tag{3-11}$$

and if, for instance,

$$\sup_{|x-x(t)|\ge 1} |\omega_{\text{out}}(x,t)| \le C_3.$$

Indeed, because $\omega_{\text{out}}(x,t) = \omega(x,t)$ for all $|x - x(t)| \ge 1$, we have $|\omega_{\text{out}}(x,t)|^2 \le C_3|\omega(x,t)|$ for $|x - x(t)| \ge 1$. Then $\|\omega_{\text{out}}(t)\|^2_{L^2(\mathbb{R}^3 \setminus B(x(t),1))} \le C_3 \|\omega(t)\|_{L^1(\mathbb{R}^3)}$. As is well known, $\sup_t \|\omega(t)\|_{L^1(\mathbb{R}^3)}$ is bounded in terms of the initial data [Constantin 1990], so the assumptions imply that $\omega_{\text{out}} \in L^4(dt; L^2(\mathbb{R}^3))$.

Thus, the assumption (3-9) is satisfied if ω is a smooth vorticity matched to an inner self-similar blowup ansatz with asymptotic behavior at infinity of the type (3-8). The main point is that both the inner blowup profile and the matching vorticity magnitude majorant $z(x,t) = |x - x(t)|^{-1/\gamma}\mathbf{1}_{\{\ell(t)\le|x-x(t)|\le 1\}}$ have the property that they belong to $L^4(dt; L^2(\mathbb{R}^3))$ if $\gamma > \frac{1}{2}$, which is a class of nonblowup.

Proof of Proposition 3.1. We add the contributions from the inner (3-10) and outer (3-9) pieces and deduce

$$\int_0^1 \|\omega(\cdot,t)\|_{L^2(\mathbb{R}^3)}^4 dt \le 16\Big(\Gamma + \frac{C^4}{6\gamma - 3}\Big).$$

If $\gamma > \frac{1}{2}$, the right-hand side is finite, and there is no blowup because of a well-known regularity criterion [Constantin and Foias 1988]. □

3.3. ***Self-similar vortex stretching.*** We revisit aspects of the proof of Theorem 2.1, under the additional assumption that a globally self-similar profile exists (as specified in Section 3.1).

Denote the vorticity direction vector as

$$\Xi(y) = \frac{\Omega(y)}{|\Omega(y)|},$$

and define the self-similar vortex stretching factor by

$$\mathsf{A}(y) := \Xi^i(y)\partial_i U^j(y)\Xi^j(y).$$

Taking the dot product of (3-4) with $\Omega|\Omega|^{p-2}$, we deduce the identity

$$|\Omega|^p + \frac{1}{p}(\gamma y + U)\cdot\nabla(|\Omega|^p) = \mathsf{A}|\Omega|^p. \tag{3-12}$$

In view of (3-8), we expect $|\Omega|$ to be integrable at infinity only for $p > 3\gamma$, which is why we stated (3-12) with general p, as opposed to the standard $p = 2$ formulation.

By the very definition of A, the fact that $|\Xi(y)| = 1$, and assumption (3-8), we have $|\mathsf{A}(y)| \le \mathsf{C}_\flat\langle y\rangle^{-1/\gamma}$ for all $y \in \mathbb{R}^3$. We aim to obtain a direct bound for the vortex stretching term; from (2-4), we have

$$\mathsf{A}(y) = \frac{3}{4\pi}\,\text{p.v.}\int_{\mathbb{R}^3}(\hat z\cdot\Xi(y))\big[\hat z\cdot(\Omega(y+z)\times\Xi(y))\big]\frac{dz}{|z|^3}, \tag{3-13}$$

where $\hat z = z/|z|$. Let $\chi = \chi(|z|)$ be a smooth radially decreasing cutoff function, with $\chi \equiv 1$ in B_1 and $\chi \equiv 0$ in $B_2^{\complement}$. Let $L > 0$ be a length scale to be optimized later. Then, from (3-13), and (3-8), we obtain that, for $p > 3\gamma$,

$$\Big|\mathsf{A}(y) - \frac{3}{4\pi}\int_{\mathbb{R}^3}(\hat z\cdot\Xi(y))\big[\hat z\cdot(\Omega(y+z)\times\Xi(y))\big]\chi\Big(\frac{|z|}{L}\Big)\frac{dz}{|z|^3}\Big| \le \frac{3^{1/p}(p-1)^{(p-1)/p}}{(4\pi)^{1/p}}\|\Omega\|_{L^p(\mathbb{R}^3)}L^{-3/p}. \tag{3-14}$$

On the other hand, since $\nabla\Omega \in L^\infty(\mathbb{R}^3)$, using the cancellation property $\Omega(y)\times\Xi(y)=0$ and Taylor's theorem, we obtain the estimate

$$\frac{3}{4\pi}\left|\int_{\mathbb{R}^3}(\hat{z}\cdot\Xi(y))\big[\hat{z}\cdot\big((\Omega(y+z)-\Omega(y))\times\Xi(y)\big)\big]\chi\Big(\frac{|z|}{L}\Big)\frac{dz}{|z|^3}\right| \leq 6\|\nabla\Omega\|_{L^\infty(\mathbb{R}^3)}L. \tag{3-15}$$

Optimizing the choice of L in (3-14) and (3-15), we deduce the pointwise bound

$$|\mathsf{A}(y)| \leq \mathsf{C}_p\|\nabla\Omega\|_{L^\infty(\mathbb{R}^3)}^{3/(p+3)}\|\Omega\|_{L^p(\mathbb{R}^3)}^{p/(p+3)}, \quad \mathsf{C}_p := 2\cdot 6^{3/(p+3)}\Big(\frac{3}{4\pi}\Big)^{1/(p+3)}(p-1)^{(p-1)/(p+3)} \tag{3-16}$$

for all $y\in\mathbb{R}^3$. We note that the quantity present in (3-16), namely $\|\nabla\Omega\|_{L^\infty(\mathbb{R}^3)}^{3/(p+3)}\|\Omega\|_{L^p(\mathbb{R}^3)}^{p/(p+3)}$, is scaling-invariant under the natural scaling from (3-6).

We wish to pair the pointwise upper bound for A in (3-16) with a lower bound. In order to achieve this, we evaluate (3-12) at a global maximum of $|\Omega|$, to directly obtain:

Proposition 3.3. *If y_* is a (self-similar) space location at which $|\Omega|\not\equiv 0$ attains its global maximum, then $\mathsf{A}(y_*)=1$.*

By combining the bound (3-16) with Proposition 3.3, we have thus established:

Theorem 3.4. *Let (U,Ω) be any nontrivial solution of (3-4) which satisfies the bound (3-8) for some $\gamma>0$. Then, the vorticity profile cannot be too small, in the sense Ω must obey the scaling-invariant lower bound*

$$\|\nabla\Omega\|_{L^\infty(\mathbb{R}^3)}^{3/(p+3)}\|\Omega\|_{L^p(\mathbb{R}^3)}^{p/(p+3)} \geq \mathsf{C}_p^{-1} \tag{3-17}$$

for any $p>\max\{3\gamma,1\}$, where the constant $\mathsf{C}_p>0$ is as defined in (3-16).

Proof. Since (3-16) must hold at $y_*=\operatorname{argmax}_{y\in\mathbb{R}^3}|\Omega|$, the bound (3-17) follows from $\mathsf{A}(y_*)=1$; see Proposition 3.3. □

Remark 3.5. Under the normalization (3-7), and using the definition of C_p, we obtain from (3-17) the requirement

$$\|\Omega\|_{L^p(\mathbb{R}^3)} \geq 2^{-1-3/p}\big(\tfrac{\pi}{1296}\big)^{1/p}(p-1)^{-1+1/p} \tag{3-18}$$

whenever $p>3\gamma$.

3.4. ***Self-similar Lagrangian particle trajectories.*** It is convenient to denote the total transport velocity appearing in (3-3) and (3-4) as

$$V(y) := \gamma y + U(y). \tag{3-19}$$

The self-similar Lagrangian trajectories are defined as solutions of the autonomous ODE

$$\frac{d}{d\tau}Y(a,\tau) = V(Y(a,\tau)) \tag{3-20}$$

with $Y(a,0)=a$. Let $X=X(a,t)$ denote the Lagrangian particle trajectories associated to the velocity field $u=u(x,t)$; that is, $\frac{d}{dt}X(a,t)=u(X(a,t),t)$ for all $t\in(0,1)$, with initial datum $X(a,0)=a$. The self-similar Lagrangian particle trajectories are related to X via

$$Y(a,\tau)=\frac{X(a,t)}{(1-t)^{\gamma}} \quad \text{with } \tau=-\log(1-t). \tag{3-21}$$

We note that, for $a=0$, we have $Y(0,\tau)=0$ for all $\tau\in\mathbb{R}$; the origin is a fixed point of the dynamical system induced by (3-20).

3.4.1. *The self-similar Cauchy formula and the Weber formula.* The Cauchy vorticity (vector) transport formula $\omega(X(a,t),t)=(\nabla_a X)(a,t)\omega_0(a)$ transformed into self-similar coordinates becomes

$$\Omega(Y(a,\tau))=e^{-(\gamma+1)\tau}(\nabla_a Y)(a,\tau)\Omega(a) \tag{3-22}$$

for all $a\in\mathbb{R}^3$ and all $\tau\in\mathbb{R}$. The difference between (3-22) and the classical Cauchy formula is that $\det(\nabla_a Y)(a,\tau)=e^{3\gamma\tau}$ (instead of $=1$), because the vector field $\gamma y+U(y)$ is not divergence-free; we have $\nabla\cdot(\gamma y+U(y))=3\gamma$. The classical Weber formula for Euler equations is

$$u^j(X(a,t),t)\partial_i X^j(a,t)-u^i(a,0)=\partial_i\pi(a,t) \tag{3-23}$$

and is obtained by checking that $\partial_t\big((\nabla X(a,t))^T u(X(a,t),t)\big)$ is a gradient. The self-similar Weber formula is

$$e^{(1-2\gamma)\tau}U^j(Y(a,\tau))\partial_i Y^j(a,\tau)-U^i(a)=\partial_i q(a,\tau) \tag{3-24}$$

for an appropriate scalar function q. The proof of the Weber formula (3-24) follows either from changing variables in (3-23), or directly in similarity variables from the observation stemming from (3-3) that

$$\partial_\tau((\nabla Y)^T U(Y))+(1-2\gamma)(\nabla Y)^T U(Y)+\nabla\big(P(Y)-\tfrac{1}{2}|U(Y)|^2\big)=0$$

holds at each a. Thus,

$$e^{(1-2\gamma)\tau}((\nabla Y)^T U(Y))(a,\tau)-U(a)=-\nabla\int_0^\tau e^{(1-2\gamma)s}\big(P(Y)-\tfrac{1}{2}|U(Y)|^2\big)\,ds$$

is a gradient.

3.4.2. *The self-similar Kelvin circulation theorem.* Let $C(0)\subset\mathbb{R}^3$ be a simple closed curve parametrized as $C(0)=\{a(\lambda):\lambda\in\mathbb{T}\}$. The circulation of U on a loop $C(0)$ is the integral of the 1-form $U(y)\,dy$ on that loop, which is computed as

$$\Gamma_{\rm ss}(0)=\oint_{C(0)}U\,dy=\int_{\mathbb{T}}U^j(a(\lambda))\dot{a}^j(\lambda)\,d\lambda, \tag{3-25}$$

where $\dot{a}^j=\frac{d}{d\lambda}a^j$. We note that V (recall (3-19)) and U have the same circulation on any loop because the 1-forms $V(y)\,dy$ and $U(y)\,dy$ differ by an exact 1-form, $d\big(\frac{1}{2}\gamma|y|^2\big)$.

The self-similar Weber formula (3-24) says that the pullback of the 1-form $e^{(1-2\gamma)\tau}U(y)\,dy$ under the diffeomorphism $Y(a,\tau)$ is the sum of the form $U(a)\,da$ and an exact 1-form, dq. The circulation of U on $C(\tau)$, where $C(\tau)=Y(C(0),\tau)$ is the push forward (image) of the loop $C(0)$, is

$$\Gamma_{\rm ss}(\tau)=\oint_{C(\tau)}U(y)\,dy. \tag{3-26}$$

The self-similar Weber formula (3-24) implies

$$e^{(1-2\gamma)\tau}\oint_{C(\tau)}U(y)\,dy=\oint_{C(0)}U(y)\,dy. \tag{3-27}$$

Indeed, multiplying (3-24) by $\dot a^i(\lambda)$ and integrating $d\lambda$ on $\mathbb{T}$, we have

$$\begin{aligned} e^{(1-2\gamma)\tau}\Gamma_{\rm ss}(\tau)&=e^{(1-2\gamma)\tau}\oint_{C(\tau)}U(y)\,dy\\ &=e^{(1-2\gamma)\tau}\int_{\mathbb{T}}U^j\big(Y(a(\lambda),\tau)\big)(\partial_iY^j)(a(\lambda),\tau)\dot a^i(\lambda)\,d\lambda\\ &=\Gamma_{\rm ss}(0). \end{aligned} \tag{3-28}$$

In the original variables, the Weber formula says that the pullback of the 1-form $u(x,t)\,dx$ under the Lagrangian flow $X(a,t)$ is the sum of the time-independent 1-form $u(a,0)\,da$ and an exact 1-form. As a consequence, Kelvin's circulation theorem states that the circulation $\Gamma(t)$ around the material loop $C(t)=\{X(a(\lambda),t):\lambda\in\mathbb{T}\}$ is conserved. That is, we have $\frac{d}{dt}\Gamma(t)=0$, where

$$\Gamma(t)=\oint_{C(t)}u(x,t)\,dx=\int_{\mathbb{T}}u^j\big(X(a(\lambda),t),t\big)(\partial_iX^j)(a(\lambda),t)\dot a^i(\lambda)\,d\lambda.$$

Changing to self-similar variables using (3-21) and time $\tau=-\log(1-t)$, we note that

$$\Gamma(t)=e^{(1-2\gamma)\tau}\Gamma_{\rm ss}(\tau).$$

Remark 3.6. Because $e^{(1-2\gamma)\tau}\Gamma_{\rm ss}(\tau)$ is constant in time, if the integral $\int_{\mathbb{T}}(\,\cdot\,)\,d\lambda$ appearing in (3-28) is nonzero, and remains bounded away from zero as $\tau\to\infty$, then we must have $\gamma=\frac12$ in order to avoid a vanishing/blowing-up exponential factor. Thus, the self-similar Kelvin circulation theorem identifies the similarity exponent $\gamma=\frac12$ as being distinguished for 3D Euler, without knowledge of any viscous regularization. It also indicates that fixed points of (3-20) — and, by extension, points on the stable manifold of these fixed points — play an important role in the analysis. A result in this direction is offered in Theorem 4.3 below.

3.4.3. *The self-similar Bernoulli function.* Note that $\nabla\times V=\nabla\times U=\Omega$ since $\nabla\times y=0$. In (3-3) we rewrite the first three terms as

$$\begin{aligned} (1-\gamma)U+(\gamma y+U)\cdot\nabla U&=(1-\gamma)(V-\gamma y)-\gamma V+V\cdot\nabla V\\ &=-\gamma(1-\gamma)\nabla\big(\tfrac12|y|^2\big)+(1-2\gamma)V+\Omega\times V+\nabla\big(\tfrac12|V|^2\big). \end{aligned}$$

After rearranging, we obtain that (3-3) is equivalent to

$$(1-2\gamma)V+\Omega\times V+\nabla\mathscr{H}=0, \tag{3-29}$$

where the *self-similar Bernoulli function*[5] is defined by

$$\mathscr{H}(y):=\tfrac{1}{2}|\gamma y+U(y)|^2+P(y)+\tfrac{1}{2}\gamma(\gamma-1)|y|^2. \tag{3-30}$$

Dotting (3-29) with V yields the transport identity

$$V\cdot\nabla\mathscr{H}=(2\gamma-1)|V|^2, \tag{3-31}$$

which, restated along self-similar Lagrangian trajectories (3-20), is

$$\frac{d}{d\tau}\mathscr{H}(Y(a,\tau))=(2\gamma-1)|V(Y(a,\tau))|^2. \tag{3-32}$$

Two immediate consequences follow. First, for $\gamma<\frac{1}{2}$ the Bernoulli function is *strictly decreasing* along any nonstationary trajectory, i.e.,

$$\gamma<\tfrac{1}{2}\implies\frac{d}{d\tau}\mathscr{H}(Y(a,\tau))\le 0, \tag{3-33}$$

with equality if and only if $V(Y(a,\tau))=0$. Second, since $|V(y)|\sim\gamma|y|$ for $|y|\gg 1$ (see (3-8)) and for $\gamma<1$ we have $P(y)=o(1)$ as $|y|\to\infty$,[6] the far-field behavior of the Bernoulli function is

$$\mathscr{H}(y)=\tfrac{1}{2}\gamma(2\gamma-1)|y|^2+o(|y|^2)\quad\text{as }|y|\to\infty. \tag{3-34}$$

In particular, for $\gamma<\frac{1}{2}$ we have $\mathscr{H}(y)\to-\infty$ as $|y|\to\infty$.

3.5. ***An outgoing property?*** We note that all trajectories with sufficiently large labels escape to infinity. Indeed, the bound (3-8) implies that, for all $y\in\mathbb{R}^3$ such that

$$|y|\ge R_\flat:=\max\left\{1,\left(\frac{2C_\flat}{\gamma}\right)^\gamma\right\}, \tag{3-35}$$

we have $|U(y)\cdot y|\le\frac{1}{2}\gamma|y|^2$, and therefore $(\gamma y+U(y))\cdot y\ge\frac{1}{2}\gamma|y|^2$. From (3-20) we then immediately obtain that, if $|a|\ge R_\flat$, then $|Y(a,\tau)|$ is a strictly increasing function of τ, and we have the bounds $|a|e^{\gamma\tau/2}\le|Y(a,\tau)|\le|a|e^{3\gamma\tau/2}$ and $e^{\gamma\tau/2}\le|\nabla_aY(a,\tau)|\le e^{3\gamma\tau/2}$ for all $\tau\ge 0$.

The exponential escape to infinity of self-similar Lagrangian flow maps (as exhibited above) is well known to play a fundamental role in proving both the existence and the stability of self-similar profiles. In the context of the compressible Euler equations, see [Buckmaster et al. 2023; 2025; Merle et al. 2022a;

[5] See [Nečas et al. 1996] for use of the analogous Bernoulli function in the case of 3D incompressible Navier–Stokes.

[6] A bound for the pressure may be obtained as in [Nečas et al. 1996]. Taking the divergence of (3-3), we have $P=P_h+\mathcal{R}_i\mathcal{R}_j(U^iU^j)$, where P_h is harmonic and $\mathcal{R}$ is the vector of Riesz-transforms. For $\gamma<1$, the second term in this expression decays to 0 as $|y|\to\infty$. From (3-3) and (3-8) we also have that $|\nabla P|$ (and hence also $|\nabla P_h|$) decays to 0 as $|y|\to\infty$ when $\gamma<1$, showing that P_h is a constant. Since the pressure is defined only up to a constant, $P_h\equiv 0$.

Chen et al. 2024]. In our incompressible 3D Euler setting, we only know that trajectories $Y(a,\tau)$ for which $|a|$ is sufficiently large escape to infinity, and that the trajectory emanating from $a=0$ is frozen at the origin.

A priori, we cannot rule out the existence of points $y\in\mathbb{R}^3\setminus\{0\}$ for which $V(y)=\gamma y+U(y)=0$. In the compressible setting, such points exist — *sonic points* — and they are well known to cause tremendous headaches, such as severe instabilities, even within the class of radially symmetric solutions [Guderley 1942; Merle et al. 2022a; Buckmaster et al. 2025]. For incompressible 3D Euler equations, a *global outgoing property*, quantified as a lower bound of the type

$$V(y)\cdot y=(\gamma y+U(y))\cdot y\geq c_*|y|^2\quad\text{for all }\ y\in\mathbb{R}^3,\quad\text{for some }\ c_*>0, \tag{3-36}$$

has been recently identified by Elgindi [2025] as one of the two fundamental assumptions needed to prove the existence of solutions to (3-4); see [ibid., Hypothesis 6.7]. Assumption (3-36) implies that the only zero of V occurs at $y=0$ (since $c_*>0$). In this section we want to allow for a generalized outgoing property, which allows for the degenerate case $c_*=0$, allows the function V to vanish away from the origin, and only makes assumptions that are local, near the nodal set of V.

Definition 3.7. We say that the field $V(y)=\gamma y+U(y)$ satisfies the *local outgoing property* if the nodal set

$$\mathcal{N}_V:=\{y_*\in\mathbb{R}^3: V(y_*)=0\}$$

is finite,[7] and if there exists $c_*\geq 0$ and $\varepsilon_*>0$ such that, for all $y_*\in\mathcal{N}_V$, we have

$$V(y)\cdot(y-y_*)\geq c_*|y-y_*|^2\quad\text{when }\ |y-y_*|\leq\varepsilon_*. \tag{3-37}$$

A few remarks concerning Definition 3.7 are in order:

- Note that $\mathcal{N}_V\neq\varnothing$ because $0\in\mathcal{N}_V$ always.
- The global outgoing property (3-36) implies the local outgoing property of Definition 3.7. Indeed, upon evaluating (3-36) at any $y_*\in\mathcal{N}_V$, we deduce that $0=V(y_*)\cdot y_*\geq c_*|y_*|^2$; this shows $y_*=0$ since $c_*>0$. Thus, $\mathcal{N}_V=\{0\}$. Condition (3-37) thus holds for any $\varepsilon_*>0$.
- When $\mathcal{N}_V=\{0\}$, (3-37) generalizes (3-36) not just in the local-in-y nature of the inequality; when $c_*=0$ it also allows for $V(y)\cdot y=\gamma|y|^2+y\cdot U(y)$ to vanish as $|y|^{2n}$ for $n\geq 2$ as $|y|\to 0$.

Our next result points out a significant consequence of assumption (3-37).

Theorem 3.8. *Let U be a C^2 smooth globally self-similar velocity profile for 3D incompressible Euler. If there exists $y_*\in\mathcal{N}_V$ such that $\Omega(y_*)\neq 0$, and if the local outgoing property (3-37) holds* ***in an open neighborhood*** *of $y=y_*$ for some $c_*\geq 0$, then*

$$\gamma\geq\tfrac{1}{2}+c_*.$$

[7] As discussed below (3-35), the bound (3-8) implies $\mathcal{N}_V\subset B_{R_0}(0)$. Hence, if we assume that $\mathcal{N}_V$ has *no accumulation points*, then this set must automatically be finite.

Proof. We rewrite the self-similar vorticity equation (3-4) as

$$\Omega + V \cdot \nabla \Omega = \mathbb{S}\Omega, \tag{3-38}$$

where $\mathbb{S} = \frac{1}{2}((\nabla U) + (\nabla U)^{\intercal})$ is the rate-of-strain matrix. If $y_* \in \mathbb{R}^3$ is a zero of V, restricting (3-38) to $y = y_*$, we deduce that the matrix

$$\mathbb{S}_{y_*} := \mathbb{S}(y_*)$$

has 1 as an eigenvalue, with associated eigenvector $\Omega(y_*) \neq 0$. Now $\mathbb{S}_{y_*}$ is a real, symmetric, traceless matrix ($\nabla \cdot U = 0$) and we denote its eigenvalues (which are real) as $1, \lambda_2, \lambda_3$, with $\lambda_2 + \lambda_3 = -1$.

Since $V(y_*) = 0$, for $|y - y_*| \ll 1$ we have

$$\begin{aligned} V(y) \cdot (y - y_*) &= (y - y_*) \otimes (y - y_*) : (\nabla V)(y_*) + \mathcal{O}(|y - y_*|^3) \\ &= (y - y_*) \otimes (y - y_*) : (\gamma \,\mathrm{Id} + \mathbb{S}_{y_*}) + \mathcal{O}(|y - y_*|^3). \end{aligned}$$

Thus, if the outgoing condition (3-37) holds in an open neighborhood of y_*, then, for all $|y - y_*| \ll 1$, we have $\gamma + \hat{z} \cdot \mathbb{S}_{y_*} \hat{z} + \mathcal{O}(|y - y_*|) \geq c_*$, where $\hat{z} = (y - y_*)/|y - y_*|$. Passing $y \to y_*$, we deduce that the smallest eigenvalue of the matrix $\mathbb{S}_{y_*}$ satisfies $\lambda_{\min}(\mathbb{S}_{y_*}) \geq c_* - \gamma$. Thus, the sum of the two smallest eigenvalues of $\mathbb{S}_{y_*}$ is $\geq 2(c_* - \gamma)$, and, due to the zero trace property, the largest eigenvalue of the matrix $\mathbb{S}_{y_*}$ satisfies $\lambda_{\max}(\mathbb{S}_{y_*}) \leq 2(\gamma - c_*)$. We have thus shown that the eigenvalues of $\mathbb{S}_{y_*}$ satisfy

$$[\lambda_{\min}(\mathbb{S}_{y_*}), \lambda_{\max}(\mathbb{S}_{y_*})] \subseteq [c_* - \gamma, 2(\gamma - c_*)].$$

In particular, because the eigenvalue 1 lies in this interval, we deduce $1 \leq 2(\gamma - c_*)$. The claimed lower bound $\gamma \geq c_* + \frac{1}{2}$ now follows. □

Our next result considers the complementary case, when the vorticity vanishes on $\mathcal{N}_V$.

Theorem 3.9. *Let U be a nontrivial C^2 smooth globally self-similar velocity profile for 3D incompressible Euler equations. Assume that the local outgoing property of Definition 3.7 holds. Then*

$$\gamma \geq \tfrac{1}{2}.$$

Proof. By contradiction, assume that $\gamma < \frac{1}{2}$. We enumerate the nodal set of V as $\mathcal{N}_V = \{y_*^{(\ell)} : 0 \leq \ell \leq N\}$ for some $N \in \mathbb{N}$ and define

$$\delta_* := \tfrac{1}{6} \min\big(\tfrac{1}{2} - \gamma, \min_{0 \leq k \neq \ell \leq N} |y_*^{(\ell)} - y_*^{(k)}|\big) > 0. \tag{3-39}$$

We also define the sets

$$\mathcal{D} := \bigcup_{\ell=0}^{N} B_{\delta_*}(y_*^{(\ell)}), \quad \mathcal{H} := \mathcal{D}^{\complement} \cap \overline{B_{R_\flat}(0)},$$

where $R_\flat$ is as defined in (3-35). The set $\mathcal{H}$ is compact.

We claim that, for all $y_0 \in \mathbb{R}^3$, there exists a unique $y_* = y_*(y_0) \in \mathcal{N}_V$ and a maximal time $\tau_0 = \tau_0(y_0) \leq 0$ such that the backwards-in-time flow map satisfies

$$Y(y_0, \tau) \in \overline{B_{2\delta_*}(y_*)} \quad \text{for all } \tau \leq \tau_0. \tag{3-40}$$

In order to prove this claim, we first note that, due to the discussion in the first paragraph of Section 3.5, there exists[8] $\tau_0' \leq 0$ such that $Y(y_0, \tau) \in \overline{B_{R_\flat}(0)}$ for all $\tau \leq \tau_0'$. Define the set of times

$$J := \{\tau \in (-\infty, \tau_0'] : Y(y_0, \tau) \in \mathcal{H}\}.$$

Since all the zeros of $V(y) = \gamma y + U(y)$ lie in the interior of $\mathcal{D}$, V is continuous, and $\mathcal{H}$ is compact, there exists $c_\flat \in (0, 1)$ such that

$$c_\flat \leq |V(y)| \leq c_\flat^{-1} \quad \text{for all } y \in \mathcal{H}.$$

Therefore, since $Y(y_0, \tau) \in \overline{B_{R_\flat}(0)}$ for all $\tau \leq \tau_0'$, since the Bernoulli function $\mathcal{H}$ is continuous and $\gamma < \frac{1}{2}$, upon integrating (3-32) on $[\tau, \tau_0']$, by the definition of the set J we obtain

$$\begin{aligned}
(1-2\gamma)c_\flat^2 |J \cap [\tau, \tau_0']| &\leq (1-2\gamma) \int_\tau^{\tau_0'} \mathbf{1}_{\tau' \in J} |V(Y(y_0, \tau'))|^2 \, d\tau' \\
&\leq (1-2\gamma) \int_\tau^{\tau_0'} |V(Y(y_0, \tau'))|^2 \, d\tau' \\
&= \mathcal{H}(Y(y_0, \tau)) - \mathcal{H}(Y(y_0, \tau_0')) \\
&\leq \max_{\overline{B_{R_\flat}(0)}} \mathcal{H} - \min_{\overline{B_{R_\flat}(0)}} \mathcal{H} =: \mathsf{H}_\flat < \infty
\end{aligned}$$

for any $\tau \leq \tau_0'$. Letting $\tau \to -\infty$, we deduce that the set J has finite Lebesgue measure, with $|J| \leq \mathsf{H}_\flat c_\flat^{-2}(1-2\gamma)^{-1}$. Thus, the trajectory $\{Y(y_0, \tau)\}_{\tau \leq \tau_0'}$ spends a finite amount of time in $\mathcal{H}$. Note however that this trajectory spends all time, i.e., the entire interval $(-\infty, \tau_0']$, in $\overline{B_{R_\flat}(0)}$.

Next, for each $0 \leq \ell \leq N$ we consider the closed annulus $\mathcal{A}_\ell := \{y \in \mathbb{R}^3 : \delta_* \leq |y - y_*^{(\ell)}| \leq 2\delta_*\}$. By the definition of δ_* in (3-39), for every $y \in \mathcal{A}_\ell$ and every $k \neq \ell$, we have $|y - y_*^{(k)}| \geq |y_*^{(k)} - y_*^{(\ell)}| - |y - y_*^{(\ell)}| \geq 6\delta_* - 2\delta_* = 4\delta_* > \delta_*$, and therefore $\mathcal{A}_\ell \subset \mathcal{D}^{\complement}$. In particular, if $\tau' \leq \tau_0'$ is such that $Y(y_0, \tau') \in \mathcal{A}_\ell$, then $\tau' \in J$. Let us call a *crossing of* $\mathcal{A}_\ell$ a time interval $[\tau_2, \tau_1] \subset J$ such that $Y(y_0, \tau') \in \mathcal{A}_\ell$ for all $\tau' \in [\tau_2, \tau_1]$, and such that $\{|Y(y_0, \tau_2) - y_*^{(\ell)}|, |Y(y_0, \tau_1) - y_*^{(\ell)}|\} = \{\delta_*, 2\delta_*\}$. During a crossing $[\tau_2, \tau_1]$, the trajectory travels a distance of at least δ_* while remaining in $\mathcal{H}$, and hence $\tau_1 - \tau_2 \geq \delta_* c_\flat$. Due to our previously established bound on $|J|$, the total number of possible crossings of the annuli $\{\mathcal{A}_\ell\}_{\ell=0}^N$ with pairwise disjoint interiors has cardinality at most $\mathsf{H}_\flat(1-2\gamma)^{-1}c_\flat^{-3}\delta_*^{-1}$. Therefore, there exists $\tau_0'' \leq \tau_0'$ such that no crossing of any annulus $\{\mathcal{A}_\ell\}_{\ell=0}^N$ is possible on the time interval $(-\infty, \tau_0'']$.

[8] This is because, when $|y_0| > R_\flat$, we have $|Y(y_0, \tau)| \leq |y_0| e^{\gamma\tau/2}$ as long as $|Y(y_0, \tau)| \geq R_\flat$. Thus, there exists a finite, sufficiently negative τ_0' such that $|Y(y_0, \tau_0')| = R_\flat$. Once a backwards-in-time trajectory has entered the compact set $\overline{B_{R_\flat}(0)}$, it cannot leave it, by the very definition of $R_\flat$.

Since the trajectory $\{Y(y_0,\tau)\}_{\tau\le\tau_0''}$ only spends a finite amount of time in $\mathcal{H}$, there exists $\tau_0\le\tau_0''$ such that $Y(y_0,\tau_0)\in\mathcal{D}$; say $Y(y_0,\tau_0)\in B_{\delta_*}(y_*^{(\ell)})$ for some $0\le\ell\le N$. We let $y_*:=y_*^{(\ell)}$ and claim that $Y(y_0,\tau)\in\overline{B_{2\delta_*}(y_*)}$ for all $\tau\le\tau_0$. Otherwise, by the intermediate value theorem, this continuous trajectory would need to cross the annulus $\mathcal{A}_\ell$ on a time interval $[\tau_2,\tau_1]\subset(-\infty,\tau_0]\subset(-\infty,\tau_0'']$, a contradiction.

Finally, (3-39) implies that distinct points of $\mathcal{N}_V$ lie at distance at least $6\delta_*$ from one another, and hence the point y_* is uniquely determined. Since $Y(y_0,\cdot)$ is continuous and $\overline{B_{2\delta_*}(y_*)}$ is closed, the supremum of the admissible times τ_0 is attained, so $\tau_0(y_0)$ may indeed be taken to be maximal. This proves (3-40).

Next, we estimate the size of $|(\nabla_a Y)(y_0,\tau)|$ for all $\tau\le\tau_0=\tau_0(y_0)$. Fix $y_*=y_*(y_0)$ as in (3-40). Differentiating (3-20) with respect to labels, and then contracting with $\nabla_a Y$, we obtain

$$\frac12\frac{d}{d\tau}|\nabla_a Y(y_0,\tau)|^2=\gamma|\nabla_a Y(y_0,\tau)|^2+\partial_{a_j}Y^i(y_0,\tau)\partial_{a_j}Y^k(y_0,\tau)(\partial_k U^i)(Y(y_0,\tau)).$$

For $\tau\le\tau_0$, we have $|Y(y_0,\tau)-y_*|\le 2\delta_*$, and thus, by the mean value theorem and the normalization $\|\nabla^2U\|_{L^\infty}=1$ (see (3-7)), we deduce

$$|(\nabla U)(Y(y_0,\tau))-(\nabla U)(y_*)|\le 2\delta_*\quad\text{for all }\ \tau\le\tau_0.$$

Since we have assumed $\gamma<\frac12$, by Theorem 3.8, for every $y_*\in\mathcal{N}_V$, we must have $\Omega(y_*)=0$, and hence $\nabla U(y_*)=\mathbb{S}_{y_*}$, where we recall that $\mathbb{S}_{y_*}=\frac12((\nabla U)(y_*)+(\nabla U)^\intercal(y_*))$ is the symmetric part of ∇U evaluated at y_*. Moreover, as in the proof of Theorem 3.8, from the local outgoing property at y_* and $c_*\ge0$ we deduce that $[\lambda_{\min}(\mathbb{S}_{y_*}),\lambda_{\max}(\mathbb{S}_{y_*})]\subseteq[c_*-\gamma,2(\gamma-c_*)]\subseteq[-\gamma,2\gamma]$. Combining the above estimates, we deduce that

$$\begin{aligned}\frac12\frac{d}{d\tau}|\nabla_a Y(y_0,\tau)|^2&\ge\gamma|\nabla_a Y(y_0,\tau)|^2+\partial_{a_j}Y^i(y_0,\tau)\partial_{a_j}Y^k(y_0,\tau)\mathbb{S}^{ik}_{y_*}-2\delta_*|\nabla_a Y(y_0,\tau)|^2\\&\ge-2\delta_*|\nabla_a Y(y_0,\tau)|^2\end{aligned}$$

for all $\tau\le\tau_0$. Integrating this inequality in time, it follows that

$$|(\nabla_a Y)(y_0,\tau)|\le|(\nabla_a Y)(y_0,\tau_0)|e^{2\delta_*(\tau_0-\tau)}\quad\text{for all }\ \tau\le\tau_0.\tag{3-41}$$

To conclude the proof, we appeal to the self-similar Cauchy formula (3-22), which states that

$$e^{(\gamma+1)\tau}\Omega(Y(y_0,\tau))=(\nabla_a Y)(y_0,\tau)\Omega(y_0)$$

for all $y_0\in\mathbb{R}^3$ and all $\tau\in\mathbb{R}$. We multiply from the left with the inverse matrix $(\nabla_a Y)^{-1}(y_0,\tau)$ and deduce

$$\Omega(y_0)=e^{(\gamma+1)\tau}(\nabla_a Y)^{-1}(y_0,\tau)\Omega(Y(y_0,\tau)).$$

Noting that $\det(\nabla_a Y)(y_0,\tau)=e^{3\gamma\tau}\neq0$ (so this matrix is indeed invertible), and using that, for a 3×3 matrix, the cofactor matrix is quadratic in the entries of the original matrix, we obtain that

$$|(\nabla_a Y)^{-1}(y_0,\tau)|\le18e^{-3\gamma\tau}|(\nabla_a Y)(y_0,\tau)|^2\le18e^{-3\gamma\tau}|(\nabla_a Y)(y_0,\tau_0)|^2e^{4\delta_*(\tau_0-\tau)}.$$

Moreover, since $\Omega(y_*)=0$, by the mean value theorem and the bound $\|\nabla\Omega\|_{L^\infty}\le 2$ (which follows from the normalization (3-7)), we have

$$|\Omega(Y(y_0,\tau))|\le 2|Y(y_0,\tau)-y_*|\le 4\delta_*$$

for all $\tau\le\tau_0$. By combining the three displays above, we deduce that, for all $\tau\le\tau_0$, we have

$$\begin{aligned}|\Omega(y_0)|&\le e^{(\gamma+1)\tau}\cdot 18e^{-3\gamma\tau}|(\nabla_a Y)(y_0,\tau_0)|^2 e^{4\delta_*(\tau_0-\tau)}\cdot 4\delta_*\\&=\big(72\delta_*|(\nabla_a Y)(y_0,\tau_0)|^2e^{4\delta_*\tau_0}\big)e^{(1-2\gamma-4\delta_*)\tau}.\end{aligned}$$

By the definition of δ_* in (3-39), we have $1-2\gamma-4\delta_*\ge 8\delta_*>0$. Thus, upon passing $\tau\to-\infty$ in the above expression, we deduce that $\Omega(y_0)=0$.

Since $y_0\in\mathbb{R}^3$ was arbitrary, we deduce $\Omega\equiv 0$. The velocity profile U is thus irrotational, incompressible, and decays at infinity (due to (3-8)), a contradiction to the assumption that U is nontrivial. Thus, $\gamma\ge\frac12$ must hold. □

4. Axisymmetric global self-similarity

For smooth initial conditions (for instance C^1 vorticity), whether the 3D Euler equations (1-1) exhibit axisymmetric singularities remains an open problem. As such, it is natural to revisit the discussion of hypothetical globally self-similar solutions from Section 3, and make the additional assumption that the self-similar profile $U(y)$ is axisymmetric around the $\vec{e}_3$ axis.

The focus on axisymmetric profiles is also motivated by ongoing computational studies (see, e.g., [Pumir and Siggia 1992; Luo and Hou 2014; Hou and Huang 2022; Hou 2023; Wang et al. 2023]), in which solutions U of the full (stationary) self-similar Euler equation (3-3) are sought within this symmetry class, because this dramatically reduces the computational space domain: from $y\in\mathbb{R}^3$ to $(r,z)\in\mathbb{R}_+\times\mathbb{R}$.

We adopt self-similar cylindrical coordinates (r,z,θ), where $r=\sqrt{y_1^2+y_2^2}$ and $z=y_3$. The self-similar velocity profile U can be written in this frame as

$$U(y)=U_r(r,z)\vec{e}_r+U_\theta(r,z)\vec{e}_\theta+U_z(r,z)\vec{e}_z.$$

The pressure is $P=P(r,z)$. With this notation, (3-3) becomes

$$\begin{aligned}(1-\gamma)U_r+(\gamma r+U_r)\partial_r U_r+(\gamma z+U_z)\partial_z U_r+\partial_r P&=\frac{1}{r}U_\theta^2,\\(1-\gamma)U_z+(\gamma r+U_r)\partial_r U_z+(\gamma z+U_z)\partial_z U_z+\partial_z P&=0,\\(1-\gamma)U_\theta+(\gamma r+U_r)\partial_r U_\theta+(\gamma z+U_z)\partial_z U_\theta&=-\frac{1}{r}U_rU_\theta,\\\partial_r U_r+\frac{1}{r}U_r+\partial_z U_z&=0.\end{aligned}\tag{4-1}$$

Remark 4.1. Because we consider the self-similar collapse of initially smooth solutions, the swirl component of velocity cannot vanish identically: $U_\theta\not\equiv 0$.

The vorticity vector $\Omega = \nabla \times U$ may be written in components as

$$\Omega(y) = \Omega_r(r,z)\vec{e}_r + \Omega_\theta(r,z)\vec{e}_\theta + \Omega_z(r,z)\vec{e}_z,$$

where

$$\Omega_r = -\partial_z U_\theta, \quad \Omega_z = \partial_r U_\theta + \frac{1}{r}U_\theta, \quad \Omega_\theta = \partial_z U_r - \partial_r U_z. \tag{4-2}$$

With this notation, (3-4) becomes

$$\begin{aligned}
&\Omega_r + (\gamma r + U_r)\partial_r\Omega_r + (\gamma z + U_z)\partial_z\Omega_r = \Omega_r\partial_r U_r + \Omega_z\partial_z U_r,\\
&\Omega_z + (\gamma r + U_r)\partial_r\Omega_z + (\gamma z + U_z)\partial_z\Omega_z = \Omega_r\partial_r U_z + \Omega_z\partial_z U_z,\\
&\Omega_\theta + (\gamma r + U_r)\partial_r\Omega_\theta + (\gamma z + U_z)\partial_z\Omega_\theta = \frac{1}{r}(U_r\Omega_\theta - 2U_\theta\Omega_r).
\end{aligned} \tag{4-3}$$

Remark 4.2. Recall that, if $\Omega(y)$ is assumed to be continuous, then, on the axis of symmetry $\{r = 0\}$, we have $\Omega_r(0,z) = \Omega_\theta(0,z) = 0$. The same reasoning applies for a continuous velocity: $U_r(0,z) = U_\theta(0,z) = 0$. This leaves open the possibility that $\Omega_z(0,z)$ or $U_z(0,z)$ do not vanish identically on the axis of symmetry.

4.1. ***Lagrangian trajectories in the meridional plane.*** The ODE system (3-20) for the three-dimensional flow map $Y(a,\tau)$ decomposes into three components

$$\begin{aligned}
\frac{d}{d\tau}R(r,z,\tau) &= \gamma R(r,z,\tau) + U_r(R(r,z,\tau), Z(r,z,\tau)),\\
\frac{d}{d\tau}Z(r,z,\tau) &= \gamma Z(r,z,\tau) + U_z(R(r,z,\tau), Z(r,z,\tau))
\end{aligned} \tag{4-4}$$

and

$$\frac{d}{d\tau}\Theta(r,z,\tau) = \frac{1}{R(r,z,\tau)}U_\theta(R(r,z,\tau), Z(r,z,\tau)), \tag{4-5}$$

with initial conditions $R(r,z,0) = r$, $Z(r,z,0) = z$ and $\Theta(r,z,0) = 0$. The dynamics of R and Z are decoupled from that of Θ, reducing the 3D Lagrangian evolution to a 2D autonomous system (4-4) in the meridional plane (R, Z). The full 3D trajectory is this 2D path spun around the $\vec{e}_3$-axis according to (4-5).

4.2. ***Fixed points of the meridional flow with nonzero swirl.*** Here we explore constraints imposed by the existence of a fixed point $(r,z) \in \mathbb{R}_+ \times \mathbb{R}$ for the dynamics (4-4) in the meridional plane, i.e., points such that

$$\gamma r + U_r(r,z) = 0 \quad \text{and} \quad \gamma z + U_z(r,z) = 0. \tag{4-6}$$

In this case, we automatically deduce $R(r,z,\tau) = r$, $Z(r,z,\tau) = z$ for all $\tau \in \mathbb{R}$, and $\Theta(r,z,\tau) = (\tau/r)U_\theta(r,z) \mod 2\pi$. If we additionally know that $U_\theta(r,z) \neq 0$,[9] we have thus obtained a periodic orbit for the full dynamics of the Lagrangian flow map $Y(\cdot,\tau)$ with nonzero circulation. This leads to the following result:

[9]The condition $U_\theta(r,z) \neq 0$ implies $r > 0$, due to Remark 4.2.

Theorem 4.3. *Assume that U is a C^2 smooth, axisymmetric, globally self-similar velocity profile for 3D incompressible Euler equations. Assume there exists a fixed point (r, z) for the dynamics in the meridional plane, i.e., (4-6) holds. If $U_\theta(r, z) \neq 0$, then the similarity exponent satisfies $\gamma = \frac{1}{2}$.*

Remark 4.4. If (r, z) is a solution of (4-6) with $U_\theta(r, z) = 0$, then also $\Omega_\theta(r, z) = 0$. This follows from the swirl component of the vorticity equation (4-3).

Proof of Theorem 4.3. We consider the loop $C(0)$ parametrized by (r, z, θ), with $\theta \in [0, 2\pi)$. Due to (4-6), this loop is an invariant set of the flow: $C(\tau) = C(0)$ because the Lagrangian flow (4-4)–(4-5) merely advects points along the loop. Combining (3-28) and (4-6), after a short computation we deduce that

$$\Gamma_{ss}(0) = e^{(1-2\gamma)\tau}\Gamma_{ss}(\tau) = e^{(1-2\gamma)\tau}\int_{\mathbb{T}} U_\theta(r, z) r\, d\lambda = e^{(1-2\gamma)\tau}\cdot 2\pi r U_\theta(r, z).$$

Since we have assumed $U_\theta(r, z) \neq 0$, we must have $r > 0$, and we deduce that $\gamma = \frac{1}{2}$ is the only exponent which conserves the circulation along this loop. □

4.3. ***No assumptions on the meridional flow.*** The main result of this section shows that solutions of (4-1) which have C^2 smooth velocity profiles automatically must satisfy $\gamma \geq \frac{1}{2}$, with *no further assumptions* on the meridional flow.

Theorem 4.5. *Let U be a nontrivial C^2 smooth axisymmetric globally self-similar velocity profile for the 3D incompressible Euler equations (4-1), satisfying (3-5) and (3-8). Then $\gamma \geq \frac{1}{2}$.*

Proof. Throughout this proof we work with trajectories of the meridional flow; that is, we analyze the solution $(R(\tau), Z(\tau)) = (R(r_0, z_0, \tau), Z(r_0, z_0, \tau))_{\tau\in\mathbb{R}}$ of the ODE (4-4), with initial condition (r_0, z_0) at $\tau = 0$.

Since $U_r(0, z) = 0$ for all $z \in \mathbb{R}$ (see Remark 4.2) and the ODE (4-4) is driven by a locally Lipschitz autonomous vector field, the axis of symmetry $\{r = 0\}$ is invariant under the meridional flow. Moreover, a trajectory originating at (r_0, z_0) with $r_0 > 0$ cannot touch the axis $\{r = 0\}$ in finite time; if it did so, it would have to coincide with a trajectory confined to the axis for all time.

Recall from (3-35) that, if $r^2 + z^2 \geq R_\flat^2$, then $|rU_r(r, z) + zU_z(r, z)| \leq \frac{1}{2}\gamma(r^2 + z^2)$. This bound implies that, for any fixed $(r_0, z_0) \in \mathbb{R}_+ \times \mathbb{R}$, the trajectory $(R(\tau), Z(\tau)) = (R(r_0, z_0, \tau), Z(r_0, z_0, \tau))$ satisfies

$$R(\tau)^2 + Z(\tau)^2 \leq \max\{r_0^2 + z_0^2, R_\flat^2\} \quad \text{for all } \tau \leq 0. \tag{4-7}$$

In order to prove (4-7), note that, by (4-4), the function $\rho(\tau) = \sqrt{R(\tau)^2 + Z(\tau)^2}$ satisfies $\frac{1}{2}\frac{d}{d\tau}\rho^2 = \gamma\rho^2 + (rU_r + zU_z)(R(\tau), Z(\tau))$. Thus, if $\rho \geq R_\flat$, then $\frac{d}{d\tau}\rho^2 \geq \gamma\rho^2 \geq \gamma R_\flat^2 > 0$, which proves (4-7) via a standard continuity argument.

Combining the two statements above, we have thus established that, for any $r_0 > 0$ and $z_0 \in \mathbb{R}$, the backwards-in-time trajectory $(R(\tau), Z(\tau)) = (R(r_0, z_0, \tau), Z(r_0, z_0, \tau))_{\tau\leq 0}$ is confined to the compact set $K_0 := \{(r, z) \in \mathbb{R}_+ \times \mathbb{R} : r^2 + z^2 \leq \max\{r_0^2 + z_0^2, R_\flat^2\}\}$, and $R(r_0, z_0, \tau) > 0$ for all $\tau \leq 0$.

For the remainder of the proof, assume *by contradiction* that $\gamma < \frac{1}{2}$.

We note that the U_θ equation in (4-1) gives

$$(\gamma r + U_r)\partial_r(rU_\theta) + (\gamma z + U_z)\partial_z(rU_\theta) + (1-2\gamma)(rU_\theta) = 0.$$

For a fixed $(r_0, z_0) \in \mathbb{R}_+ \times \mathbb{R}$ with $r_0 > 0$, we integrate this equation along the backwards trajectory $(R(\tau), Z(\tau))_{\tau \le 0}$ to obtain

$$e^{(1-2\gamma)\tau} R(\tau) U_\theta(R(\tau), Z(\tau)) = r_0 U_\theta(r_0, z_0) \quad \text{for all } \tau \le 0.$$

Since $\gamma < \frac{1}{2}$, we have $e^{(1-2\gamma)\tau} \to 0$ as $\tau \to -\infty$. Moreover, we previously showed that $(R(\tau), Z(\tau)) \in K_0$ for all $\tau \le 0$, and the continuous function $rU_\theta(r, z)$ is bounded on the compact set K_0. Therefore, upon passing $\tau \to -\infty$ in the above display, and using that $r_0 > 0$, we deduce that $U_\theta(r_0, z_0) = 0$; this fact is a manifestation of the conservation of circulation (3-27). With Remark 4.2, it follows that $U_\theta \equiv 0$ on $\mathbb{R}_+ \times \mathbb{R}$. Since U is C^1 smooth, from (4-2) we also obtain $\Omega_r \equiv \Omega_z \equiv 0$ on $\mathbb{R}_+ \times \mathbb{R}$.

To conclude, we use the Ω_θ equation in (4-3) and the fact that $U_\theta \equiv 0$ to deduce

$$(\gamma r + U_r)\partial_r\Big(\frac{1}{r}\Omega_\theta\Big) + (\gamma z + U_z)\partial_z\Big(\frac{1}{r}\Omega_\theta\Big) + (1+\gamma)\Big(\frac{1}{r}\Omega_\theta\Big) = 0.$$

For a fixed $(r_0, z_0) \in \mathbb{R}_+ \times \mathbb{R}$ with $r_0 > 0$, we integrate this equation along the backwards trajectory $(R(\tau), Z(\tau))_{\tau \le 0}$ to obtain

$$e^{(1+\gamma)\tau} \frac{1}{R(\tau)} \Omega_\theta(R(\tau), Z(\tau)) = \frac{1}{r_0}\Omega_\theta(r_0, z_0).$$

Since $\gamma > 0$, we have $e^{(1+\gamma)\tau} \to 0$ as $\tau \to -\infty$. Moreover, we previously showed that $(R(\tau), Z(\tau)) \in K_0$ for all $\tau \le 0$, and the function $(1/r)\Omega_\theta(r, z)$ is bounded on the compact set K_0.[10] Upon passing $\tau \to -\infty$ in the above display and using that $r_0 > 0$, we deduce $\Omega_\theta(r_0, z_0) = 0$. With Remark 4.2, we also know that Ω_θ vanishes on the axis of symmetry, and hence $\Omega_\theta \equiv 0$ on $\mathbb{R}_+ \times \mathbb{R}$.

With $\Omega_r \equiv \Omega_z \equiv \Omega_\theta \equiv 0$ on $\mathbb{R}_+ \times \mathbb{R}$, by axisymmetry and continuity we deduce that U is irrotational and incompressible on $\mathbb{R}^3$. Note however that (3-8) implies $|\nabla U(y)| \to 0$ as $|y| \to \infty$ for any $\gamma > 0$, and thus U must be a constant. Our normalization $U(0) = 0$ (see (3-5)) then proves U is trivial, which contradicts the assumption that U is nontrivial. Hence, the contradiction ansatz fails, and we must have $\gamma \ge \frac{1}{2}$. □

Remark 4.6. Inspecting the proof of Theorem 4.5, we note that the contradiction ansatz $\gamma < \frac{1}{2}$ was used *only once*: we appealed to $\gamma < \frac{1}{2}$ in the form $e^{(1-2\gamma)\tau} \to 0$ as $\tau \to -\infty$, in order to conclude $U_\theta \equiv 0$. The remainder of the proof only uses $\gamma > 0$. Thus, we have proven that there is *no swirl-free*, C^2 smooth, axisymmetric, globally self-similar velocity profile U for the 3D incompressible Euler equations (4-1) satisfying (3-5) and (3-8) for *any* $\gamma > 0$.

[10] Since Ω is C^1 smooth and $\Omega_\theta(0, z) = 0$, we have $(1/r)\Omega_\theta(r, z) = (1/r)\int_0^r (\partial_r \Omega_\theta)(r', z)\, dr'$. For any $(r, z) \in K_0$, the whole line segment $(r', z)_{0 \le r' \le r}$ lies in K_0, and therefore $(1/r)|\Omega_\theta(r, z)| \le \sup_{K_0} |\partial_r \Omega_\theta| < \infty$.

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