

KÄHLER–RICCI SHRINKERS AND FANO FIBRATIONS, I

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We prove that a Kähler–Ricci shrinker admits a polarized Fano fibration structure and hence is naturally a quasiprojective variety. This establishes a foundational connection between Kähler–Ricci shrinkers and algebraic geometry.

1. Introduction

A Kähler–Ricci shrinker (i.e., a complete shrinking gradient Kähler–Ricci soliton) consists of the data (X, J, ω, f) , where (X, J, ω) is a complete Kähler manifold and f is a real-valued function on X such that $J\nabla f$ is a holomorphic Killing field and

$$\mathrm{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial}f = \omega. \quad (1-1)$$

The vector field $J\nabla f$ is referred to as the *soliton vector field*.

A compact Kähler–Ricci shrinker lives on a Fano manifold, which is projective. In the noncompact case, there is no immediate restriction on the underlying complex manifold; a priori, it may even have infinite Betti numbers. The goal of this paper is to prove the following general result, which is a first step towards connecting Kähler–Ricci shrinkers and algebraic geometry.

Theorem 1.1. *A Kähler–Ricci shrinker (X, J, ω, f) naturally defines a polarized Fano fibration. In particular, X is quasiprojective.*

For the definition of polarized Fano fibration, see Definition 2.3. When the curvature tensor has quadratic decay, the above result was proved in [Conlon et al. 2024]; in this case, the shrinker is known to be asymptotically conical [Kotschwar and Wang 2015]. Theorem 1.1 is also known in complex dimension 2, where it follows from the classification of Kähler–Ricci shrinkers [Conlon et al. 2024; Cifarelli et al. 2026; 2024; Bamler et al. 2024; Li and Wang 2026]. On the other hand, Theorem 1.1 can be used to give a new proof of the main result in [Li and Wang 2026], see [Conlon et al. 2025, Section 4].

Theorem 1.1 answers an open question in [Conlon et al. 2024, Section 7.2]. Unlike conventional approaches to similar compactification problems that rely on producing holomorphic functions or

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holomorphic sections via analysis, which often require extra geometric assumptions at infinity, our proof instead makes an unexpected use of birational algebraic geometry, specifically Birkar's boundedness theorem [2023] for Fano-type varieties.

Theorem 1.1 establishes a foundational link between Kähler–Ricci shrinkers and algebraic geometry, inspiring many related questions in birational geometry. These topics are discussed in more detail in [Sun and Zhang 2024]; see also the discussion at the end of this paper.

2. Polarized Fano fibrations

In this section, we introduce the notions of a Fano fibration and a polarized Fano fibration. These provide the algebrogeometric setup for our work.

2.1. Fano fibrations. In this paper, a variety is always a quasiprojective integral scheme over $\mathbb{C}$. A *fibration* is by definition a surjective projective morphism $\pi : X \rightarrow Y$ between varieties such that $\pi_* \mathcal{O}_X = \mathcal{O}_Y$.

Definition 2.1 (Fano fibration). A Fano fibration is a fibration $\pi : X \rightarrow Y$ where X and Y are normal varieties, X has klt singularities, and $-K_X$ is a relatively ample $\mathbb{Q}$ -Cartier divisor.

This definition agrees with the one introduced in [Birkar 2024], where the study of Fano fibrations is motivated by the minimal model program. We emphasize, however, that we do not require the base Y to be projective, and we allow π to be birational. In the usual projective setting, these morphisms are precisely the K_X -negative extremal contractions arising in Mori theory: a Fano fibration contracts a K_X -negative extremal face, and it is the contraction of an extremal ray when the relative Picard number $\rho(X/Y)$ equals 1; see [Mori 1982; Kawamata et al. 1987; Kollár and Mori 1998].

2.2. Polarized Fano fibrations. *Polarized* Fano fibrations are Fano fibrations endowed with an equivariant torus action. We first recall the definition of *polarized affine cones* in [Collins and Székelyhidi 2019], which provides the algebrogeometric framework for studying canonical metrics in Sasaki geometry. Throughout this paper, $\mathbb{T}$ denotes a real compact torus whose Lie algebra is denoted by $\text{Lie}(\mathbb{T})$.

Definition 2.2 (polarized affine cone [Collins and Székelyhidi 2019]). A polarized affine cone $(Y, \xi, \mathbb{T})$ consists of a normal affine variety $Y = \text{Spec}(R)$ equipped with a $\mathbb{T}$ -action with a unique fixed point and an element $\xi \in \text{Lie}(\mathbb{T})$ in the Reeb cone. The Reeb cone of the $\mathbb{T}$ -action is defined as the set of all $\eta \in \text{Lie}(\mathbb{T})$ such that, under the weight decomposition

$$R = \bigoplus_{\alpha \in \text{Lie}(\mathbb{T})^*} R_\alpha,$$

we have $\langle \alpha, \eta \rangle > 0$ for all nonzero α with $R_\alpha \neq \{0\}$.

Definition 2.3 (polarized Fano fibration). A polarized Fano fibration $(\pi : X \rightarrow Y, \xi)$ consists of a Fano fibration $\pi : X \rightarrow Y$ equipped with an equivariant torus $\mathbb{T}$ -action and an element $\xi \in \text{Lie}(\mathbb{T})$ such that $(Y, \mathbb{T}, \xi)$ is a polarized affine cone.

We also call ξ the Reeb vector field. The Reeb cone of the polarized Fano fibration is defined to be the Reeb cone of Y .

The terminology “polarization” used here is related to, but different from, its standard meaning in algebraic geometry. Instead of an ample line bundle, a polarization here refers to a torus action together with a choice of Reeb vector field $\xi \in \text{Lie}(\mathbb{T})$, whose induced grading assigns strictly positive weights to all nonconstant regular functions on the affine cone. The connection to the usual meaning of a polarization in algebraic geometry is seen from the fact that a projective variety X with a choice of an ample line bundle L naturally generates a polarized affine cone by taking the Spec of the section ring of L .

Notice that Y is determined by X as $\text{Spec}(R_X)$, where R_X denotes the ring of regular functions on X . By definition, it also follows that there are $\mathbb{T}$ -equivariant embeddings of Y into $\mathbb{C}^{N_1}$ and X into $\mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$, for some linear actions of $\mathbb{T}$ on $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, such that π is induced by the projection $\pi_1 : \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \rightarrow \mathbb{C}^{N_1}$, and $\pi_2^* \mathcal{O}(1)|_X$ gives $-mK_X$ for some positive integer m .

As we shall see, the notion of a polarized Fano fibration provides the appropriate framework for studying general Kähler–Ricci shrinkers. It also incorporates many settings introduced previously. First, when Y is a point, by definition the Reeb cone is the entire $\text{Lie}(\mathbb{T})$. In this case, a polarized Fano fibration is simply a Fano variety X endowed with a holomorphic vector field ξ , which we call a *polarized Fano variety*. Again, we emphasize that the meaning of a polarization here is different from its standard meaning in algebraic geometry. When $\xi = 0$, this is the setting for studying positive Kähler–Einstein metrics. When $\xi \neq 0$, this is the setting for studying compact Kähler–Ricci shrinkers with a prescribed soliton vector field. Secondly, when π is the identity map, a polarized Fano fibration is the same as a polarized affine cone with klt singularities. This is usually referred to as a *Fano cone*. This is the setting for studying the existence of Ricci-flat Kähler cone metrics with a given Reeb vector field.

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Let (X, J, ω, f) be a noncompact Kähler–Ricci shrinker. In this case, by [Cao and Zhou 2010], the function f is bounded below and proper. The vector field $\xi = J\nabla f$ is a holomorphic Killing field on X and $\xi \lrcorner \omega = -df$. By the Bochner formula, one can see that

$$\bar{\partial}(\Delta_\omega f + f + \frac{1}{2}(J\xi) \cdot f) = 0.$$

Since f is real-valued, we may normalize f so that

$$\Delta_\omega f + f + \frac{1}{2}(J\xi) \cdot f = 0. \tag{3-1}$$

In particular, $\inf_X f < 0$.

The vector field ξ generates a holomorphic isometric action of a compact (real) torus $\mathbb{T}$ on X . In general, $\mathbb{T}$ may have rank bigger than 1. We first perturb ξ to a rational element η in $\text{Lie}(\mathbb{T})$ which generates an S^1 -action with a proper moment map. As is explained in [Conlon et al. 2024, Proposition 5.14], the $\mathbb{T}$ -action has a moment map $\mu : X \rightarrow \text{Lie}(\mathbb{T})^*$ such that each $\eta \in \text{Lie}(\mathbb{T})$, viewed as a holomorphic vector field on X , satisfies $\eta \lrcorner \omega = -du_\eta$ for $u_\eta := \langle \mu, \eta \rangle$. Moreover, u_η can be normalized by requiring

$$\Delta_\omega u_\eta + u_\eta + \frac{1}{2}(J\eta) \cdot f = 0. \quad (3-2)$$

By definition, $f = u_\xi$. Since f is proper and bounded below, μ is also a proper map. By the generalized Atiyah–Guillemin–Sternberg convexity theorem in the noncompact setting due to Hilgert, Neeb and Plank [Hilgert et al. 1994], the image $\mu(X)$ is a closed, convex and locally polyhedral set in $\text{Lie}(\mathbb{T})^*$.

Lemma 3.1. *For η sufficiently close to ξ , the function u_η is proper and bounded below.*

Proof. For $\eta \in \text{Lie}(\mathbb{T})$ we denote by L_η the associated linear function on $\text{Lie}(\mathbb{T})^*$. Since $\inf_X f < 0$, we can find a point $w_0 \in \mu(X)$ with $L_\xi(w_0) = 0$. Now, for any $w \in \mu(X)$ with $L_\xi(w) > 1$, by the convexity of $\mu(X)$ we may write

$$w = L_\xi(w)w' + (1 - L_\xi(w))w_0$$

for some $w' \in \mu(X)$ with $L_\xi(w') = 1$. Since f is proper, the level set $\{L_\xi(w) = 1\} \cap \mu(X)$ is compact. Fix a norm $\|\cdot\|$ on $\text{Lie}(\mathbb{T})$; then we can find $\delta > 0$ such that, if $\|\eta - \xi\| \leq \delta$, then $\frac{3}{2} > L_\eta(w' - w_0) > \frac{1}{2}$ for all $w' \in \mu(X)$ with $L_\xi(w') = 1$. For such $\eta \in \text{Lie}(\mathbb{T})$, we then have

$$L_\eta(w) = L_\eta(w' - w_0)L_\xi(w) + L_\eta(w_0) \geq \frac{1}{2}L_\xi(w) + L_\eta(w_0).$$

In particular, L_η is proper and bounded below on $\mu(X)$. The conclusion follows. $\square$

We fix an element η given in the above lemma such that it generates an effective S^1 -action and denote $u := u_\eta$. Then $\eta = J\nabla u$. It is well known that u is a Morse–Bott function with only even indices. Since u is proper, we then conclude that each of its level sets is connected [Hilgert et al. 1994, Theorem 4.1]. For a regular value z of u , we do Kähler reduction at $u = z$. The result is a Kähler metric with orbifold singularities ω_z on a polarized log pair (X_z, D_z, L_z) . Here X_z is a normal complex-analytic variety with quotient singularities; the effective divisor D_z on X_z is given by

$$D_z = \sum_{\alpha} \left(1 - \frac{1}{r_{z,\alpha}}\right) D_{z,\alpha}, \quad (3-3)$$

where the divisors $D_{z,\alpha}$ correspond to complex codimension 2 loci (in X) of the set of points with nontrivial stabilizer, and $r_{z,\alpha}$ is the order of the isotropy along a generic point of $D_{z,\alpha}$; ω_z is a Kähler metric on X_z with orbifold singularities along the singular set of X_z and with cone angle $2\pi/r_{z,\alpha}$ along $D_{z,\alpha}$. In particular, (X_z, D_z) is a klt pair. There is also an orbifold Hermitian line bundle L_z over X_z associated

with the S^1 -action, which can also be viewed as a $\mathbb{Q}$ -Cartier divisor on X_z [Boyer and Galicki 2008, Chapter 4]. The induced hermitian metric on L_z has Chern curvature form $-\sqrt{-1}\partial\bar{\partial}\omega_z$.

Lemma 3.2. *In the sense of currents,*

$$\text{Ric}(\omega_z) = 2\pi\delta_{D_z} + \omega_z - z\partial_z\omega_z + \sqrt{-1}\partial\bar{\partial}(-f + \log|\nabla u|^2), \quad (3-4)$$

where δ_{D_z} denotes the $(1, 1)$ -current defined by integration along D_z .

Proof. This follows from a direct computation (see for example [Burns and Guillemin 2004], or by following [Sun and Zhang 2019, Section 2.1]). For the convenience of readers, we include a proof here. Since the result is local, it suffices to consider the locus where the S^1 -action is free. Denote by $\{w_1, \dots, w_{n-1}\}$ the local holomorphic coordinates on the complex quotient R . Then they can be viewed as local holomorphic functions on X . Let t be an arbitrary local function on X with $\eta(t) = 1$. Then $\{z = u_\eta, t, w_1, \dots, w_{n-1}\}$ gives a local coordinate system on X and we have $\eta = \partial_t$.

One can write $Jdz = h^{-1}\Theta = h^{-1}(dt - \theta)$, where $h > 0$ is a local function depending on $z, w_1, \dots, w_{n-1}$, and θ is a family of local 1-forms on R parametrized by z . One can see that $h^{-1} = \omega(\eta, J\eta) = |\nabla u|^2$.

We can write the Kähler form ω on X as

$$\omega = dz \wedge \Theta + \omega_z, \quad (3-5)$$

where ω_z is a family of $(1, 1)$ -forms on R parametrized by z . The condition that ω is Kähler is equivalent to

$$\partial_z^2\omega_z + d_R d_R^c h = 0, \quad d\Theta = \partial_z\omega_z - dz \wedge d_R^c h, \quad (3-6)$$

where d_R denotes the differential along R . Locally, we can write

$$\omega_z^{n-1} = V(\sqrt{-1})^{(n-1)^2} \Omega_R \wedge \bar{\Omega}_R,$$

where $\Omega_R = dw_1 \wedge \dots \wedge dw_{n-1}$ is the holomorphic volume form on R and V is a positive function. Then

$$\text{Ric}(\omega_z) = -\frac{1}{2}d_R d_R^c \log V. \quad (3-7)$$

On X , we have a local holomorphic volume form $\Omega = \kappa \wedge \Omega_R$, where κ is the dual $(1, 0)$ -form to the holomorphic vector field $\eta^{1,0} := \frac{1}{2}(\eta - \sqrt{-1}J\eta)$. One can see that $\kappa = (\Theta - \sqrt{-1}hdz + \kappa')$, where κ' is a form on R . Then

$$\frac{\omega^n}{h^{-1}V\Omega \wedge \bar{\Omega}} \equiv \text{const} \quad \text{and} \quad \text{Ric}(\omega) = -\frac{1}{2}dd^c \log(h^{-1}V). \quad (3-8)$$

Denote $F = f - \log(h^{-1}V)$; then, by (3-8) and (1-1), we obtain

$$\omega = \frac{1}{2}dd^c(F) = \frac{1}{2}(d_R d_R^c F + dz \wedge d_R^c(\partial_z F) + d(h^{-1}\partial_z F) \wedge \Theta + h^{-1}\partial_z F d\Theta). \quad (3-9)$$

Then, by (3-5), (3-6) and (3-7), we arrive at

$$2\omega_z = d_R d_R^c(f + \log h) + 2 \operatorname{Ric}(\omega_z) + h^{-1} \partial_z F \partial_z \omega_z \quad (3-10)$$

and

$$\partial_z(h^{-1} \partial_z F) = 2. \quad (3-11)$$

Hence, for some constant C , we have

$$\operatorname{Ric}(\omega_z) = \omega_z - (z + C) \partial_z \omega_z + \sqrt{-1} \partial \bar{\partial}(-f + \log |\nabla u|^2). \quad (3-12)$$

It suffices to show $C = 0$. Notice that

$$\mathcal{L}_{J\eta} F = -h^{-1} F_z = -2(z + C). \quad (3-13)$$

On the other hand, by the definition of F ,

$$\mathcal{L}_{J\eta} F = J\eta \cdot f - \mathcal{L}_{J\eta} \log(h^{-1} V). \quad (3-14)$$

Since $\mathcal{L}_{J\eta} \Omega = \mathcal{L}_\eta \Omega = 0$, we have

$$\mathcal{L}_{J\eta}(\log(h^{-1} V)) = \mathcal{L}_{J\eta} \left(\log \frac{\omega^n}{\Omega \wedge \bar{\Omega}} \right) = \frac{\mathcal{L}_{J\eta} \omega^n}{\omega^n} = -2\Delta_\omega u_\eta. \quad (3-15)$$

Combining (3-13), (3-14), (3-15) and (3-2), we conclude that $C = 0$. $\square$

So, as a log pair, we have

$$-(K_{X_z} + D_z) + zL_z > 0. \quad (3-16)$$

It is well known that the zero set Q of η is a disjoint union of smooth, totally geodesic complex submanifolds of X . Since u is proper, each component of Q is compact. The following is a crucial result.

Proposition 3.3. *Q is compact.*

We first assume Proposition 3.3. Then there exists a $z_* > 0$ such that the set $X_{\geq z_*} := u^{-1}([z_*, \infty))$ does not intersect Q . So $[z_*, \infty)$ does not contain a critical value. For $z \geq z_*$, we consider the Kähler reduction $(X_z, D_z, L_z, \omega_z)$ at $u = z$. We may identify (X_z, D_z, L_z) holomorphically with a fixed polarized log pair (M, D, L) for all $z \geq z_*$, and ω_z is a family of conical Kähler metrics on M parametrized by $z \geq z_*$. We have $\partial_z[\omega_z] = 2\pi c_1(L)$. It follows from (3-16) that $-(K_M + D) + zL$ is ample on M for all $z \geq z_*$. Choosing a rational z , we see that M is projective. It also follows that L is nef.

By the Kawamata basepoint-free theorem, L is semiample; that is, there is a surjective morphism $p : M \rightarrow M'$ and a very ample line bundle L' over M' such that $p^*L' = L^k$ for some positive integer k . Let Y' be the affine cone associated to the very ample line bundle L' . Denote by $\mathcal{L}$ the total space of the line bundle L^{-k} over M . By construction and properties of the Kähler reduction, we have natural holomorphic maps $\pi' : X_{>z_*} \rightarrow \mathcal{L} \rightarrow Y'$, and the $\mathbb{C}^*$ -action generated by η and $J\eta$ descends to a $\mathbb{C}^*$ -action

on Y' , so π' is $\mathbb{C}^*$ -equivariant. Using the flow of $-J\eta = \nabla u$, we see π' extends to the complement of an analytic subset in X , and, using the fact that Y' is affine, π' extends to a global proper holomorphic map $X \rightarrow Y'$. From (1-1), the line bundle $-K_X$ admits a hermitian metric with positive curvature form. Since Y' is an affine cone, we can find a small neighborhood U' of the vertex such that $\pi'^{-1}(U')$ is weakly pseudoconvex [Demailly 1997, Chapter VIII]. Then we can apply Hörmander's L^2 method to construct holomorphic sections of $-mK_X$, and, for some $U \subseteq U'$, we get an embedding

$$F : \pi'^{-1}(U) \hookrightarrow U \times \mathbb{CP}^N. \quad (3-17)$$

Notice that we have an induced S^1 -action on $-K_X$, and, by Fourier expansion, we may assume the holomorphic sections used to construct the embedding F in (3-17) are homogeneous with respect to the S^1 -action. Then we may assume F is S^1 -equivariant with respect to the natural S^1 -actions on $\pi'^{-1}(U)$ and U , and some linear S^1 -action on $\mathbb{CP}^N$. Now the S^1 -action naturally complexifies to a holomorphic $\mathbb{C}^*$ -action, and then we obtain a global proper holomorphic embedding

$$X \hookrightarrow Y' \times \mathbb{CP}^N. \quad (3-18)$$

Then, using the $\mathbb{C}^*$ -action and the standard Fourier expansion argument, as in [Donaldson and Sun 2014, Section 2.3], we know that the image of X under this proper holomorphic embedding is algebraic and therefore X admits a quasiprojective variety structure and π' is a projective morphism. Let $\pi : X \rightarrow Y$ be the Stein factorization of π' . Then Y is also an affine variety and the ring of regular functions on Y coincides with that of X . Therefore, X has no bounded holomorphic functions and then we can also easily show that $-J\xi$ has only positive weights on nonconstant holomorphic functions. The $\mathbb{T}$ -action generated by the soliton vector field ξ naturally descends to Y , making it into a polarized affine cone, so π is a polarized Fano fibration. This completes the proof of Theorem 1.1.

Now we prove Proposition 3.3. We list the components of Q as $Q_1, Q_2, \dots$ so that the corresponding critical values $z_1, z_2, \dots$ are in increasing order. It follows from (3-2) that the minimum value z_1 is negative.

Since (z_{j-1}, z_j) contains a rational number, one sees immediately that X_z is a projective variety for all regular values z . For $z \in (z_{j-1}, z_j)$, the flow of $\nabla u = J\eta$ naturally identifies (X_z, D_z, L_z) with a fixed polarized log pair. We need to investigate the birational transformation on (X_z, D_z, L_z) when z crosses a critical value. This fits into the general picture of variation of GIT (see [Dolgachev and Hu 1998; Thaddeus 1996]). In our situation, since X is only a complex manifold, we cannot directly appeal to the algebraic geometry. Instead, we work out the details explicitly.

Given any component of Q on which $u = z_j$, we have an induced linear S^1 -action on the holomorphic normal bundle N . Suppose it has complex index m and coindex l . On the fibers of N , the S^1 -action has positive integer weights $p_1, \dots, p_l$ and negative integer weights $-q_1, \dots, -q_m$. Denote $r := \gcd(q_1, \dots, q_m)$

and $s := \gcd(p_1, \dots, p_l)$. Since the S^1 -action is effective, $\gcd(r, s) = 1$. Equation (3-2) implies the weight identity

$$\sum_{\alpha=1}^l p_\alpha - \sum_{\beta=1}^m q_\beta = -z_j. \quad (3-19)$$

If the S^1 -action is quasifree, i.e., the isotropy group of any point is either trivial or the whole S^1 , then the p_α and q_β are all equal to 1. This equation implies that $z_j \leq n$, so Q is compact.

In general, the situation is more complicated. For simplicity of discussion, without loss of generality, we assume there is a unique connected component of the critical set Q on which $u = z_j$ and, by an abuse of notation, we will still denote this component by Q . Fix $\epsilon > 0$ small such that z_j is the only critical value in $(z_j - \epsilon, z_j + \epsilon)$. We denote by (X, D, L, ω_z) and (X', D', L', ω'_z) the Kähler reduction space for $z \in (z_j - \epsilon, z_j)$ and $z \in (z_j, z_j + \epsilon)$, respectively.

- (Type I, $m = l = 1$) X and X' are naturally isomorphic and Q sits inside both as a divisor. We have $p_1 - q_1 = -z_j$. To compare the log canonical divisors $K_X + D$ and $K_{X'} + D'$, we first investigate the linear model case.

Consider the action of $\mathbb{C}^*$ on $\mathbb{C}_{w_1, w_2}^2$ with weight $(\lambda^{-q_1}, \lambda^{p_1})$. The situation is very similar to the discussion of weighted projective spaces (see for example [Ross and Thomas 2011]), except that we allow weights of mixed sign (so there is a variation of GIT). For the two quotients here, we have global charts $X_- := \{w_1 = 1\}/\mathbb{Z}_{q_1}$ and $X_+ := \{w_2 = 1\}/\mathbb{Z}_{p_1}$. So the underlying variety of both is identified with $\mathbb{C}$ and the boundary divisors are given by $D_- = (1 - 1/q_1)[0]$ and $D_+ = (1 - 1/p_1)[0]$. We claim that

$$L_- - L_+ = \frac{1}{p_1 q_1} [0]. \quad (3-20)$$

To see this, we choose integers a, b such that $q_1 a - p_1 b = 1$. Since w_1 has weight $-q_1$ and w_2 has weight p_1 on the fibers of $L_\pm^{-1}$, the rational function $w_1^{-a} w_2^{-b}$ has weight 1, so it defines a section of both L_- and L_+ . On X_- it vanishes at $w_2 = 0$ with order $-b$ on the chart upstairs, so it contributes to $-b/q_1[0]$. Similarly, on X_+ it contributes to $-a/p_1[0]$. Adding together yields (3-20).

In our global setting, the above discussion implies that

$$D - \left(1 - \frac{1}{q_1}\right)Q = D' - \left(1 - \frac{1}{p_1}\right)Q \quad (3-21)$$

and

$$L - L' = \frac{1}{p_1 q_1} Q. \quad (3-22)$$

In particular,

$$K_X + D = K_{X'} + D' + \frac{z_j}{p_1 q_1} Q. \quad (3-23)$$

- (Type II, $m > 1, l > 1$) In this case, we have a flipping birational map

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X' \\ & \searrow & \swarrow \\ & Y & \end{array}$$

Here Y is the (singular) Kähler quotient at the critical value z_j . Since π and π^{-1} are both defined away from Zariski closed sets of codimension at least 2, we have

$$\pi_* D = D', \pi_* K_X = K_{X'}. \quad (3-24)$$

- (Type III, $m > 1, l = 1$) In this case, there is a contraction morphism $\pi : X \rightarrow X'$. The exceptional divisor E is a weighted projective space bundle over Q . Each fiber is isomorphic to $\mathbb{P}(q_1/r, \dots, q_m/r)$. To compare the log canonical divisors, we again refer to the model situation first.

Consider the action of $\mathbb{C}^*$ on $\mathbb{C}_{q_1, \dots, q_m, p_1}^{m+1}$ with weights $(-q_1, \dots, -q_m, p_1)$. We have two GIT quotients $X_- = \{(w_1, \dots, w_m) \neq 0\}/\mathbb{C}^*$ and $X_+ = \{w_{m+1} = 1\}/\mathbb{Z}_{p_1}$. Since $\gcd(p_1, r) = 1$, the only possible codimension 1 orbifold locus is given by the reduced divisor E set-theoretically cut out by $\{w_{m+1} = 0\}$ in X_- . Notice that E is isomorphic to the weighted projective space $\mathbb{P}(q_1/r, \dots, q_m/r)$. It is also the exceptional set of the natural contraction map $X_- \rightarrow X_+$. The boundary divisor is $D_- = (1 - 1/r)E$ and $D_+ = 0$. We claim that

$$L_- = \frac{1}{p_1 r} E. \quad (3-25)$$

To see this, we consider the function w_{m+1} , which has weight p_1 on the fibers of L_-^{-1} , so it defines a section of $L_-^{p_1}$. At the same time, working in a local chart, say $\{w_1 = 1\}/\mathbb{Z}_{q_1}$, of X_- , one sees that upstairs w_{m+1} has vanishing order 1, but the zero set $\{w_{m+1} = 0\}$ has an isotropy group $\mathbb{Z}_r$. So it really defines the divisor $(1/r)E$. Then (3-25) follows.

In our global case, we then have

$$\pi_* K_X = K_{X'}, \quad D = \pi_*^{-1} D' + \left(1 - \frac{1}{r}\right) E \quad (3-26)$$

and

$$L = \pi^* L' + \frac{1}{p_1 r} E. \quad (3-27)$$

Notice also that $-(K_X + D) + z_j L$ is trivial along the weighted projective fibers of E , so

$$K_X + D = \pi^*(K_{X'} + D') + aE, \quad (3-28)$$

where

$$a = \frac{z_j}{p_1 r} = \frac{\sum_{\beta} q_{\beta} - p_1}{p_1 r} > 0.$$

- (Type IV, $m = 1, l > 1$) In this case, there is a contraction morphism $\pi : X' \rightarrow X$. The exceptional divisor E is a weighted projective space bundle over Q . Each fiber is isomorphic to $\mathbb{P}(p_1/s, \dots, p_l/s)$.

Similar to the Type III situation, we find that

$$D' = \pi_*^{-1} D + \left(1 - \frac{1}{s}\right) E \quad (3-29)$$

and

$$L' = \pi^* L - \frac{1}{q_1 s} E. \quad (3-30)$$

Also one can compute

$$K_{X'} + D' = \pi^*(K_X + D) + aE, \quad (3-31)$$

where

$$a = -\frac{z_j}{q_1 s} = \frac{\sum_{\alpha} p_{\alpha} - q_1}{q_1 s} < 0.$$

Remark 3.4. If we only have Type II and Type III critical values, then, as z increases, we are essentially running a minimal model program along the $\mathbb{Q}$ -divisor L .

Lemma 3.5. *There exists a $\delta > 0$ such that, for every regular value $z > 0$, one can find an effective $\mathbb{Q}$ -divisor Δ_z on X_z such that the following hold:*

- *The pair $(X_z, D_z + \Delta_z)$ is δ -lc, i.e., the log discrepancy of any divisor over X_z is at least δ .*
- *$K_{X_z} + D_z + \Delta_z \sim_{\mathbb{Q}} 0$.*
- *Δ_z is a big divisor.*

Proof. We start with a positive regular value z sufficiently close to 0. From (3-16), we see that $-(K_{X_z} + D_z)$ is the limit of ample divisors, so it is itself a nef divisor. Since $\min_X u < 0$, the volume of (X_z, ω_z) as $z \rightarrow 0$ does not go to zero. So $-(K_{X_z} + D_z)$ is big. Since (X_z, D_z) is klt, by Bertini's theorem we may find an effective $\mathbb{Q}$ -divisor Δ_z on X_z such that the statements hold for some $\delta > 0$ small.

Next, we prove the properties continue to hold with the same δ when we cross a critical value z_j . There are four cases:

- (Type I) We simply let $\Delta' = \Delta + (z_j/p_1 q_1)Q$, so $D + \Delta = D' + \Delta'$. Then $(X', D' + \Delta')$ is naturally isomorphic to $(X, D + \Delta)$ and Δ' is big.
- (Type II) We let $\Delta' = \pi_* \Delta$, which is big since Δ is big. Then, by (3-24), $K_{X'} + D' + \Delta' \sim_{\mathbb{Q}} 0$. By [Kollár and Mori 1998, Lemma 3.38], the log discrepancy of $(X', D' + \Delta')$ agrees with that of $(X, D + \Delta)$.
- (Type III) We let $\Delta' = \pi_* \Delta$, which is big since Δ is big. Then, by (3-26), $K_{X'} + D' + \Delta' \sim_{\mathbb{Q}} 0$. By [Kollár and Mori 1998, Lemma 2.30], the log discrepancy of $(X', D' + \Delta')$ agrees with that of $(X, D + \Delta)$.
- (Type IV) We let $\Delta' = \pi^* \Delta - aE$. It is crucial here that $a < 0$, so that Δ' is again big and effective. Then, by (3-31), $K_{X'} + D' + \Delta' \sim_{\mathbb{Q}} 0$. By [loc. cit.], the log discrepancy of $(X', D' + \Delta')$ agrees with that of $(X, D + \Delta)$. $\square$

An immediate consequence is that the coefficients of D_z are uniformly bounded above by $1 - \delta$. So, at each codimension 1 orbifold point, the order is uniformly bounded by $1/\delta$. In particular, the Type I critical values are uniformly bounded.

Now we apply a deep result in [Birkar 2021, Corollary 1.4], which implies that the set of all X_z belongs to a bounded family of normal projective varieties. We claim that the order of the orbifold group at every point in the underlying variety X_z is uniformly bounded above. There might be many ways of seeing this; for example, a quick way is to apply the lower semicontinuity result of [Blum and Liu 2021] (together with [de Fernex et al. 2011]) to get a uniform lower bound on the normalized volume at points of X_z .

Now Proposition 3.3 follows from:

Lemma 3.6. *Given a critical value z_j of Type II, III or IV, there is a quotient singularity in the underlying variety X_z , for a regular value z close to z_j , with the orbifold order at least $\delta^{n-1}z_j/(n-1)$. In particular, its normalized volume is at most $(n-1)^n/\delta^{n-1}z_j$.*

Proof. Suppose at z_j the weights are given by $-q_1, \dots, -q_m, p_1, \dots, p_l$. Then $\sum_{\beta} q_{\beta} \geq z_j$. In particular, there is a β with $q_{\beta} \geq z_j/(n-1)$. Notice that we need to deal with codimension 1 orbifold loci. Since $\gcd(q_1, \dots, q_m, p_1, \dots, p_l) = 1$ and, by the above discussion, the gcd of any $m+l-1$ weights is at most $1/\delta$, it is easy to see that, on X_z , for $z \in (z_j - \epsilon, z_j)$ there is at least one orbifold singularity with order $\delta^{n-1}q_{\beta} \geq \delta^{n-1}z_j/(n-1)$. $\square$

The idea of using Kähler reduction to study Kähler–Ricci shrinkers, while quite natural, appears to be novel. As another application, we show that every smooth Kähler–Ricci shrinker is simply connected. This is also proved independently in [Esparza 2025a] using a version of the Hirzebruch–Riemann–Roch formula on noncompact manifolds.

Proposition 3.7. *A Kähler–Ricci shrinker X is simply connected.*

Proof. If X is compact, it is a standard result that a Fano manifold is simply connected. If X is noncompact, it is known that X has finite fundamental group [Wylie 2008]. Let $\phi : \tilde{X} \rightarrow X$ denote the universal cover, where $\tilde{X}$ inherits the Kähler–Ricci shrinker structure from X . As before, we can perturb the soliton vector field ξ to obtain a holomorphic vector field η , which generates an effective S^1 -action with a proper Hamiltonian potential u . This S^1 -action naturally lifts to $\tilde{X}$ with Hamiltonian potential given by ϕ^*u .

We perform Kähler reduction for regular values ϵ of u near 0. As mentioned earlier, since u is proper, each of its level sets is connected [Hilgert et al. 1994, Theorem 4.1]. We claim that, as long as $\phi : \tilde{X} \rightarrow X$ is a nontrivial covering, the quotient $(\phi^*u)^{-1}(\epsilon)/S^1$ is a nontrivial covering of $u^{-1}(\epsilon)/S^1$. To prove this, it is sufficient to show that the lift of each S^1 -orbit on X remains a loop in $\tilde{X}$. Notice that the S^1 -orbit on X is generated by the flow of $J\nabla u$. We can use the negative gradient flow line of u to contract this S^1 -orbit to a point, showing that this orbit is homotopically trivial in X . By the properties of covering spaces, the lift of each S^1 -orbit on X to $\tilde{X}$ must also remain a loop.

Let $X_\epsilon = u^{-1}(\epsilon)/S^1$. By the proof of Lemma 3.5, for sufficiently small ϵ , the pair (X_ϵ, D_ϵ) is klt and $-(K_{X_\epsilon} + D_\epsilon)$ is big and nef. Then, by [Hacon and McKernan 2007, Corollary 1.14] (see also [Zhang 2006, Corollary 1]), we conclude that X_ϵ is simply connected. Therefore, the covering map $(\phi^*u)^{-1}(\epsilon)/S^1 \rightarrow u^{-1}(\epsilon)/S^1$ must be trivial. Consequently, $\tilde{X} \rightarrow X$ must also be a trivial covering, implying that X itself is simply connected. $\square$

Finally, we give an intrinsic description of the algebraic structure of X in terms of the vector field ξ . The ring R_X of regular functions on X is given by

$$R_X = \bigoplus_{\alpha \in \text{Lie}(\mathbb{T})^*} R_{X,\alpha}, \quad (3-32)$$

where $R_{X,\alpha}$ consists of homogeneous functions with weight α . Notice that $\langle \alpha, \xi \rangle > 0$ for any nonzero α with $R_{X,\alpha} \neq 0$. The polarized affine cone Y is isomorphic to $\text{Spec}(R_X)$.

Similarly, the space of algebraic sections of $-mK_X$ is given by

$$\mathfrak{R}_m := \bigoplus_{\alpha \in \text{Lie}(\mathbb{T})^*} \mathfrak{R}_{m,\alpha},$$

where $\mathfrak{R}_{m,\alpha}$ consists of homogeneous sections with weight α . There exists a positive constant C such that $\mathfrak{R}_{m,\alpha} \neq 0$ only when $\langle \alpha, \xi \rangle \geq -Cm$. Now the direct sum

$$\mathfrak{R} := \bigoplus_{m \geq 0} \mathfrak{R}_m$$

is a finitely generated graded algebra over R_X , and X is isomorphic to $\text{Proj}_{\text{Spec}(R_X)} \mathfrak{R}$.

More recently, it has been proved by Esparza [2025b, Section 5] that the algebraic structure of X is indeed uniquely determined by its complex-analytic structure.

4. Discussion

Theorem 1.1 proves the algebraicity of a Kähler–Ricci shrinker (X, J, ω, f) . It remains an open question to understand the metric geometry of a Kähler–Ricci shrinker. For example, one expects suitable conical behavior at infinity.

Conjecture 4.1 (asymptotic conicity). *A Kähler–Ricci shrinker (X, J, ω, f) is weakly asymptotically conical, in the sense that it has a unique asymptotic cone at infinity, which is a Kähler cone metric on the base Y of the associated polarized Fano fibration.*

Under the assumption of quadratic curvature decay, Conjecture 4.1 holds by [Munteanu and Wang 2017; Conlon et al. 2024]. Notice that some explicit examples of Kähler–Ricci shrinkers have been constructed in [Koiso 1990; Cao 1996; Feldman et al. 2003; Dancer and Wang 2011; Futaki and Wang 2011; Li 2010; Cifarelli 2024]. They are all weakly asymptotically conical.

We also expect that a Kähler–Ricci shrinker is a *canonical* differential geometric object associated to a polarized Fano fibration. In particular, we have:

Conjecture 4.2 (KRS uniqueness). *A polarized Fano fibration $(\pi : X \rightarrow Y, \xi)$ with X smooth admits at most one Kähler–Ricci shrinker, up to the action of $\text{Aut}(X, \xi)$ (the group of automorphisms of X preserving ξ).*

When Y is a point, we know X is a Fano manifold, and then the uniqueness is known due to [Tian and Zhu 2002; 2000]. The interest of Conjecture 4.2 lies in the analysis in the noncompact setting, *without* a priori prescribing the asymptotics at infinity. This is in the spirit of a free boundary problem. Conjecture 4.2 has been proved in [Cifarelli 2022] in the toric setting and in [Esparza 2025b] in the case of quadratic curvature decay.

The existence of a Kähler–Ricci shrinker on a polarized Fano fibration is expected to be related to algebrogeometric K-stability. The notion of K-polystability in this setting is defined in [Sun and Zhang 2024]. The following conjecture can be viewed as a natural extension of the theorem of Chen, Donaldson and Sun [Chen et al. 2015] to the noncompact and relative setting.

Conjecture 4.3 (KRS existence). *A polarized Fano fibration $(\pi : X \rightarrow Y, \xi)$ with X smooth admits a Kähler–Ricci shrinker if and only if it is K-polystable.*

In general, it is natural to study singular Kähler–Ricci shrinkers. We expect that, with more effort, Theorem 1.1 can be generalized to the case of singular Kähler–Ricci shrinkers. One can also formulate more general versions of Conjectures 4.2 and 4.3 for general polarized Fano fibrations (without assuming that X is smooth).

The result of this paper motivates new perspectives on the algebrogeometric study of Fano fibrations. In [Sun and Zhang 2024], the notion of *weighted volume* is introduced for a polarized Fano fibration and a local-to-global comparison is proved, which serves as the algebrogeometric analogue of the Bishop–Gromov volume comparison. Furthermore, generalizing [Donaldson and Sun 2017; Chen et al. 2018], a 2-step degeneration theory is proposed to degenerate a Fano fibration germ into a K-polystable polarized Fano fibration. This point of view provides new insights into the minimal model program in birational geometry, for example the question of classifying polarized Fano fibrations, the boundedness question for Fano fibrations and the question on termination of flips. It also bridges the study of singularities of Kähler–Ricci flow on projective manifolds and algebraic geometry. For more details, see [Sun and Zhang 2024].

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