ESSENTIAL NUMBER THEORY

Exceptional zeros, sieve parity, Goldbach

John B. Friedlander and Henryk Iwaniec

2022

vol. 1 no. 1

msp

Essential Number Theory
Vol. 1, No. 1, 2022
https://doi.org/10.2140/ent.2022.1.13

Exceptional zeros, sieve parity, Goldbach

John B. Friedlander and Henryk Iwaniec

We survey connections between the possible existence of exceptional real zeros of Dirichlet L-functions and the sieve parity barrier and then show how recent work tying them to the Goldbach problem can be viewed in a considerably generalized framework.

1. Introduction

A fundamental problem in analytic number theory is that of establishing excellent upper and lower bounds in general sieve methods, most especially in the linear sieve. Following a great deal of progress, stretching now over a century, one gradually became aware of a general "parity barrier" which governs the limitations of what one can hope to accomplish, at least in general.

A fundamental problem in analytic number theory is that of establishing zero-free regions for Dirichlet L-functions. In case the corresponding character $\chi \pmod q$ is complex or, alternatively, for all complex zeros $\rho = \beta + i\gamma$ with $\gamma \neq 0$, one has long known how to produce zero-free regions of the type

$$\sigma \geq 1 - c/\log q(|t|+1) \tag{1-1}$$

where $s = \sigma + it$ with a positive constant c. In the remaining situation, where both χ and s are real, much less is known, nothing more recent than a famous "ineffective" estimate of Siegel for the L-function at $s = 1$ which enables a bound like (1-1) but only with the replacement of $\log q$ by q^{ε} with arbitrary $\varepsilon > 0$ and a numerically uncomputable c depending on ε. This exponentially weaker result has been a serious impediment to progress in many basic questions.

It is not unfair to claim that much progress in mathematics proceeds by analogy. The two problems above, in many aspects, ring familiar to each other. The first purpose of this paper is to illustrate ways in which this has been found to be true. Our second purpose is to, in the case of one close recently discovered connection, carry forward this investigation to a new, deeper and more general setting.

Friedlander was supported in part by NSERC grant A5123.
MSC2020: 11M20, 11N05, 11N35, 11P32.
Keywords: primes, sieves, exceptional zeros, Goldbach conjecture.

We recall, that an "exceptional" zero is a real zero β that does lie in the region (1-1) for a constant c. If however there were only a finite number of these we could (since the L-functions do not vanish at $s = 1$) adjust the constant c to exclude them all from the region. Thus the name is really not a very good one for an individual zero since the concept requires an infinite sequence of these. Nevertheless, it is ingrained in the literature; when we use it we are thinking of such a sequence. It is known, essentially due to Landau, that such a sequence of moduli, should one exist, must be very lacunary; the zeros would all be simple, at most one per modulus and indeed with the exceptional moduli q_i satisfying

$$\frac{\log q_{i+1}}{\log q_i} \to \infty.$$

Failing a proof of their nonexistence, it is the lack of any examples of exceptional zeros that leads to the ineffectivity in results such as that of Siegel. Specific real (or nearly real) zeros can and do lead to computationally effective results, even when, as first realized in [Friedlander 1976], they are all the way over at $s = \frac{1}{2}$, a location where the GRH does not prohibit their appearance.

In the absence of a solution to the problem of whether there exist exceptional zeros, there have naturally been attempts to relate the question to other very difficult problems. One class of results of this type deals with showing that the assumption of the existence of exceptional zeros leads to consequences for prime number distribution that are beyond current reach, but are nevertheless expected to be true. There have been in recent years quite a number of such results, several by the current authors; see [Heath-Brown 1983; Friedlander and Iwaniec 2003; 2004; 2005; Merikoski 2021].

These statements, although conditional, can be quite deep and spectacular. For example, in the case of [Friedlander and Iwaniec 2003], we derived asymptotics for the counting of primes $p \le x$ in arithmetic progressions of modulus $q < x^{1/2+\delta}$, so beyond the reach of the generalized Riemann hypothesis. An essential ingredient for this was our asymptotic formula for the divisor function $\tau_3(n)$, $n \le x$ in progressions to modulus $q \le x^{1/2+\delta'}$, which we deduced [Friedlander and Iwaniec 1985] from the expected estimates for exponential sums over relevant varieties, proofs of which were provided for us by Birch and Bombieri, using in turn the Riemann hypothesis for varieties, proved by Deligne. The type of applications of Deligne's work, pioneered in [Friedlander and Iwaniec 1985], has since been extensively developed, for example by Y. Zhang [2014] and, especially, in a whole series of papers by E. Fouvry, E. Kowalski and P. Michel.

Results of this type are not however the primary concern in this paper. On the contrary, we are here highlighting an admittedly smaller class of examples, wherein the assumption of exceptional zeros leads to consequences that are beyond current reach, but are nevertheless expected to be *false*. If one does not believe in the existence of exceptional zeros, then one can dream that this is more promising.

One early example of this class we here consider, by now folklore, shows that the nonexistence of such zeros would follow from improvements, seductively small, in the Brun–Titchmarsh theorem which gives uniform upper bounds for the number of primes in an arithmetic progression. We shall recall this situation in more detail in Section 3.

In more recent years, results have been obtained showing how relatively good bounds for exceptional zeros would follow from assumptions about the less obviously related Goldbach conjecture. The latter famous statement predicts that every even integer exceeding two can be written as the sum of two primes. Hardy and Littlewood [1923] put forth a conjectured asymptotic formula for the number of representations of n as the sum of two primes. Following the normal practice in the subject, we find it simpler to consider a weighted sum over the representations involving the von Mangoldt function, one which leads to an entirely equivalent conjecture. Let

$$G(n) = \sum_{\substack{m_1+m_2=n \\ 2\nmid m_1 m_2}} \Lambda(m_1)\Lambda(m_2). \tag{1-2}$$

The Hardy–Littlewood conjecture predicts that, for n even, we have $G(n) \sim \mathfrak{S}(n)n$ where $\mathfrak{S}(n)$ is a certain positive product over the primes, to be defined in (4-2), and easily large enough to imply Goldbach for all sufficiently large even n.

In Section 4 we recall how even a much weakened form of this conjectured asymptotic completely eliminates the possible existence of any exceptional zeros. Then, in the subsequent sections, we are going to generalize considerably the results of Section 4, for the purpose of showing clearly that the questions are linked to the parity barrier of sieve theory.

But first, in the next section, we give a review of that barrier.

2. Parity problem and the asymptotic sieve

We are interested in counting prime numbers. Beginning from the very earliest works, but especially over the past century, a significant component of this exercise has been the development of sieve methods.

Already from Brun's early successes, a striking achievement was the attainment of upper bounds of the correct order for the number of primes in interesting subsequences of the positive integers.

The attainment of a positive lower bound however seemed always a bit beyond reach. What one could succeed in getting was a lower bound for the number of integers having no more than k prime factors for some value of k, fairly small but invariably greater than one. These results created an interest in the so-called "almost primes".

Gradually, around the middle of the last century, it began to be noticed that the constant factor in the upper bound was never better than twice the expected, though, in the most favorable situations, it could come very close to that.

Analogously, although the lower bound the machinery spewed out for the number of primes was never positive, here too, in the most favorable situations, it could come very close to being so. This has in places been attributed to the incapability of the sieve to distinguish between integers with an odd number of prime factors and those having an even number. The apparent inevitability of this situation has led to the name "parity phenomenon", a name which will seem more clearly appropriate in what follows.

In the same way that, for reasons which are both elementary (think Chebyshev) and analytic (think Riemann), it turns out to be both convenient and elegant to study the primes using the von Mangoldt function, the study of almost primes of order k is facilitated with the introduction of its generalization, given by the Dirichlet convolution

$$\Lambda_k = \mu * \log^k, \tag{2-1}$$

which, as its progenitor ($k = 1$), is supported on integers having at most k distinct prime factors, satisfies (by induction) the recurrence

$$\Lambda_{k+1} = \Lambda_k \cdot \log + \Lambda_k * \Lambda, \tag{2-2}$$

obeys the bounds

$$0 \le \Lambda_k(n) \le (\log n)^k \tag{2-3}$$

and yields the asymptotic formula

$$\sum_{n \le x} \Lambda_k(n) \sim kx(\log x)^{k-1}. \tag{2-4}$$

In case $k = 1$ this last result is of course the prime number theorem and from that and (2-2) one can easily obtain the others. However, it turns out, due to Selberg, that for $k = 2$ and hence for larger k, the formula admits an elementary proof.

In retrospect, we can see that this difference in the levels of difficulty between $k = 1$ and larger k is mirrored in the analytic behavior of their generating functions. The Dirichlet series for Λ_k, namely

$$\sum_{n \ge 1} \Lambda_k(n) n^{-s} = (-1)^k \frac{\zeta^{(k)}(s)}{\zeta(s)}, \tag{2-5}$$

has a pole of order k at $s = 1$. As soon as $k \geq 2$ this pole has multiple order and its effect cannot be canceled out by a simple real zero. Still, it does seem strange that zeta in particular feels the need to worry that she might have an exceptional zero.

It is interesting to note that, although for $k = 1$ the contribution to the sum in (2-4) comes entirely from the integers with an odd number of distinct prime factors, on the contrary, for each $k \geq 2$ the contribution comes half from odd and half from even.

The original motivation for Selberg's discovery was that it could then be combined with other arguments (which he implemented, as did Erdös) leading to elementary proofs for the prime number theorem itself. But that is not the issue here (although perhaps some day it could be).

We are concerned with the counting of primes in more general sequences and, with rare exceptions, we are still far from this goal. It was Bombieri [1976] (see also [Friedlander and Iwaniec 1978; 1996; 2010]) who made breakthroughs in enormously generalizing the elementary results for $k \geq 2$ with his asymptotic sieve. To avoid using excessive space and notation we shall give only the flavor of these results.

We consider a sequence (a_n) of nonnegative reals which satisfies certain basic axioms of linear sieve type. Without the possibility of providing an exhaustive list (see [Friedlander and Iwaniec 2010]) we mention the most essential ones.

We consider, for given $d \geq 1$, the congruence sum

$$A_d(x) = \sum_{\substack{n \leq x \\ n \equiv 0 \pmod{d}}} a_n \tag{2-6}$$

and assume it satisfies the approximation

$$A_d(x) = A_1(x)g(d) + r_d(x) \tag{2-7}$$

where the function $g(d)$ in the "main term" is multiplicative and satisfies the linear sieve condition

$$\sum_{p \leq y} g(p) \log p = \log y + c_g + O_A((\log y)^{-A}) \tag{2-8}$$

for arbitrary A and all $y \leq x$.

For the same A and y the "remainder terms" $r_d(y)$ are assumed to satisfy, for every $\varepsilon > 0$, $D = x^{1-\varepsilon}$, the bound

$$\sum_{d \leq D} |r_d(y)| \ll A_1(x)(\log D)^{-A}. \tag{2-9}$$

We remark that, of these conditions, that for the main term, i.e., (2-8), is known to hold for many interesting sequences. On the other hand, the latter assumption (2-9), although expected to hold for many of those natural sequences that are not very sparse (in that they satisfy $A_1(x) \gg x(\log x)^{-B}$ for some B), is in most cases quite difficult to prove.

By weakening the assumption (2-9), requiring it to hold only for some smaller value of D (the“level of distribution”), one can verify it for many sequences and still can get useful results (see [Friedlander and Iwaniec 1978]), but then the connection to the parity principle rapidly falls off.

We now loosely describe the main thrusts of Bombieri's results [1976].

By heuristic arguments, one is led to the conjecture that for a nice sequence (a_n) satisfying (2-8) one might expect, in place of (2-4), the asymptotic formula

$$\sum_{n\le x} a_n \Lambda_k(n) \sim kH \sum_{n\le x} a_n (\log n)^{k-1} \sim kH A_1(x)(\log x)^{k-1}, \tag{2-10}$$

where H is given by the product

$$H = \prod_p (1-g(p))\Big(1-\frac{1}{p}\Big)^{-1}. \tag{2-11}$$

Bombieri shows in particular that, given a sequence (a_n) satisfying (2-8), (2-9) and some quite mild additional conditions, for each $k \ge 2$ the asymptotic formula (2-10) holds. In fact, one gets more precise information which describes, apart from one glaring loophole, a rather precise picture of the contribution to these sums coming from the integers having a specified number of prime factors.

Given our sequence (a_n) having these properties, there exists a function $\delta(x)$, defined up to $o(1)$, such that the following happens. For each integer $r \ge 1$ let $\sum^r$ denote a sum restricted to positive integers with precisely r distinct prime factors. We fix some $k \ge 2$ and some r with $1 \le r \le k$. Then we have

$$\sum_{n\le x}{}^{r} a_n \Lambda_k(n) \sim \delta(x) kH \sum_{n\le x}{}^{r} a_n (\log n)^{k-1}. \tag{2-12}$$

Moreover, the same formula holds with the same value of $\delta(x)$ for every other $r \le k$ having the same parity and with the value $2-\delta(x)$ for every $r \le k$ having the opposite parity.

In particular, we see that $0 \le \delta(x) \le 2$. As it happens, for each such real number, one can give examples of sequences satisfying the axioms which give rise to that particular value. We noted earlier that, for each $k \ge 2$, the contribution to the sum in (2-4) comes half from those integers with an odd number of distinct prime factors and half from those with an even number. We can now say that this happens for the more general sequence (a_n) provided that $\delta(x) = 1$.

Bombieri goes on to show that results of the same type apply to sums over a_n weighted by functions far more general, supported on almost primes. To do this he first studies convolutions of the various Λ_k and finite linear combinations of these. He then shows, using the Weierstrass approximation theorem, that quite general normalized smooth functions f, defined at squarefree $n = p_1 \cdots p_r$ by

$$f_r(n) = F_r\left(\frac{\log p_1}{\log n}, \dots, \frac{\log p_r}{\log n}\right), \tag{2-13}$$

with $F_r(u_1, \dots, u_r)$ continuous and symmetric, one for each value of r, can be closely approximated by these linear combinations. This allows him to deduce statements for the sums

$$\sum_{n\le x}{}^{r}\, a_n F_r\left(\frac{\log p_1}{\log n}, \dots, \frac{\log p_r}{\log n}\right), \tag{2-14}$$

similar to that for the special case (2-12). One needs some growth conditions on the weight function (2-14) which imply that the small prime factors of n do not make an essential contribution. For example, $F_r(u_1, \dots, u_r) \ll u_1 \cdots u_r$ is fine.

3. Primes in arithmetic progressions

That there are relations between the parity barrier and the existence of exceptional zeros becomes particularly evident in connection with the study of the distribution of primes in an arithmetic progression.

Analytic methods have so far succeeded to prove, for example,

$$\psi(x; q, a) = \sum_{\substack{n\le x\\ n\equiv a \pmod q}} \Lambda(n) = \frac{x}{\varphi(q)} - \frac{\chi(a)}{\varphi(q)}\frac{x^\beta}{\beta} + O(x\exp(-c\sqrt{\log x})). \tag{3-1}$$

Here the second term is to be deleted if there is no exceptional zero β. When combined with Siegel's bound, this gives the asymptotic formula, but only with a uniformity in q bounded by an arbitrary fixed power of $\log x$.

For numerous applications it is desirable to have a much wider uniformity so it is of great utility that one has at least an upper bound with that feature, the Brun–Titchmarsh theorem, which is provided by sieve methods.

That upper bound, after years of successive improvement by a constant factor, is

$$\pi(x; q, a) = \sum_{\substack{p\le x\\ p\equiv a \pmod q}} 1 \le \frac{(2+\varepsilon)x}{\varphi(q)\log(x/q)}. \tag{3-2}$$

The Selberg sieve and the beta sieve (see [Friedlander and Iwaniec 2010]) both give this constant 2 and fail to do significantly better. This failure seems inevitable when one considers that the replacement of 2 by $2-\eta$ with a fixed positive η in a range $x > q^{A(\eta)}$, would lead to the banishment of exceptional zeros. The proof of this result (in somewhat weaker form) is found in [Siebert 1983] with a deeper, more precise, statement in [Granville 2020]. The basic idea is to combine (3-1) and (3-2), the latter having been adjusted to a bound for $\psi(x; q, a)$.

Moreover, using more sophisticated ideas, Siebert and then, in definitive form, Granville show this result to be a special case of the following more general statement.

The linear sieve produces specific upper and lower bound functions $F(s)$ and $f(s)$ respectively, first discovered by Jurkat and Richert [1965], (see Section 12.1 of [Friedlander and Iwaniec 2010]), which apply when we are dealing with a sequence (a_n), $n \le x$ satisfying the linear sieve axiom (2-8) and we are sieving by a set of primes $p \le D^{1/s}$. It is known that these functions F, f are optimal in general, although the specific sequences which provide a counterexample do not resemble arithmetic progressions. Siebert, respectively Granville, show that a fixed improvement of the value of either $F(s)$, $f(s)$ *for any value of* s, again in the case of arithmetic progressions and with x larger than a sufficiently large power of q, implies that exceptional zeros do not exist.

We should mention as well that Granville considers also, and in considerable detail, the corresponding problem in which one sieves by small primes, the integers in a short interval.

Before we leave the topic of arithmetic progressions, we draw attention to an interesting feature of Bombieri's sieve in this case. Naturally enough, the results of the last section are applicable in particular to this most basic sequence $\{n \le x; n \equiv a \pmod q\}$. Moreover, for this particular sequence, the level of distribution axiom (2-9) holds uniformly in the modulus q in a much wider range than $q \ll (\log x)^A$, which was our limit for $k = 1$. Hence, we have the following result.

For each integer $k \ge 2$ and $(a, q) = 1$ there holds the asymptotic formula

$$\sum_{\substack{n \le x \\ n \equiv a \pmod q}} \Lambda_k(n) \sim k \frac{x}{\varphi(q)} (\log x)^{k-1}, \tag{3-3}$$

now valid for q in the much larger range

$$\log q = o(\log x).$$

The proof of this is to be found in [Friedlander 1981] for $k = 2$ and extends easily to larger k. As was the situation with $\zeta(s)$, for $k \ge 2$ the principal L-function has a pole of multiple order, whereas any potential exceptional zero must be simple. This fact offers an analytic explanation for the resulting extra level of uniformity as compared to that for $k = 1$.

4. The Goldbach problem

In relation to this problem Hardy and Littlewood [1923] conjectured the following asymptotic formula for the sum (1-2).

$$G(n) = \sum_{\substack{m_1+m_2=n \\ 2\nmid m_1m_2}} \Lambda(m_1)\Lambda(m_2) \sim \mathfrak{S}(n)n, \tag{4-1}$$

for n even, where

$$\mathfrak{S}(n) = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p\mid n \\ p>2}}\left(1+\frac{1}{p-2}\right). \tag{4-2}$$

A rather weakened (though still seemingly far from reach) form of the Hardy–Littlewood conjecture which features in our work is as follows.

Weak Hardy–Littlewood–Goldbach conjecture. *For all sufficiently large even n, we have*

$$\delta\mathfrak{S}(n)n < G(n) < (2-\delta)\mathfrak{S}(n)n, \tag{4-3}$$

for some fixed $0 < \delta < 1$.

In [Friedlander and Iwaniec 2021; Friedlander et al. 2022] the following result is proved.

Theorem. *Assume that the Weak Hardy–Littlewood–Goldbach conjecture holds for all sufficiently large even n. Then, there are no zeros of any Dirichlet L-function in the region* (1-1) *with a positive constant c which is now allowed to depend on δ.*

Earlier results in this direction had been given in [Fei 2016; Bhowmik et al. 2019; Bhowmik and Halupczok 2021; Jia 2022; Goldston and Suriajaya 2021]. Those works had narrowed the escape window for the exceptional zeros but did not close it tightly.

In the following sections we are going to consider the arguments that lead to this theorem but in considerably more general form.

5. A generalized Goldbach problem

We let $a(\ell)$, $b(m)$ be given sequences of real numbers having some interesting arithmetical structure and, for every $n \geq 2$ we consider

$$F(n) = \sum_{\ell+m=n} a(\ell)b(m). \tag{5-1}$$

For example, if $a(\ell) = \Lambda(\ell)$, $b(m) = \Lambda(m)$ for $2\nmid \ell m$ then $F(n)$ reduces to the sum $G(n)$ in (1-2). We shall, in any case, be interested in the representations $\ell + m = n$

with ℓ, m being almost primes, hence having a number of prime factors bounded by a fixed quantity, say $r \geq 1$. From now on, some of the constants implied in our estimates may depend on r.

In the appendix we employ heuristic arguments to predict an asymptotic formula

$$F(n) \sim \mathfrak{S}(n)\Phi(n), \tag{5-2}$$

as $n \to \infty$, n even and where $\Phi(n)$ will be defined in (12-2). Then, in Section 14, we mention somewhat weaker estimates

$$\delta\mathfrak{S}(n)\Phi(n) < F(n) < (2-\delta)\mathfrak{S}(n)\Phi(n), \tag{5-3}$$

with a fixed $0 < \delta < 1$ for all even n sufficiently large. The punchline of this heuristic thinking, as it was in [Friedlander et al. 2022], is the following.

Conclusion. The region $s = \sigma + it$ with

$$\sigma \geq 1 - c/\log q(|t|+1) \tag{5-4}$$

is free of zeros of $L(s, \chi)$ for all characters $\chi \pmod q$ and all $q \geq 3$, where $c = c(\delta)$ is a positive constant computable in terms of δ.

Remarks. Although our results are more general than those in [Friedlander et al. 2022] we shall appeal to some of the statements there without change. In particular, the Bombieri version of zero density estimates is a key input to both works; see (4.3) in [Friedlander et al. 2022].

Our generalization from $G(n)$ to $F(n)$ lets us see the parity issue of sieve methods in a more transparent, picturesque context. The arguments we provide are amenable to still further generalization than we have given in this work. However, this would have made the paper more complicated and the extra results would have drifted the topic away from this very connection.

Incidentally, one should not lose hope of proving the original Goldbach conjecture before killing off the exceptional characters because, to this end, when one is not worried about quantitative bounds, one can skip counting many inconvenient representations. Ironically, the existence of exceptional characters might conceivably help to solve the original Goldbach problem, as it does for the twin prime problem and for other questions about prime numbers. In this connection, see as we have mentioned earlier, [Heath-Brown 1983; Friedlander and Iwaniec 2003; 2004; 2005; Merikoski 2021].

6. A series of $F(n)$

Let $N \geq q \geq 3$. We are going to consider the series

$$S(N,q) = \sum_{n \equiv 0 \pmod q} F(n) e^{-n/N} \tag{6-1}$$

by means of L-functions, similarly to [Goldston and Suriajaya 2021; Friedlander and Iwaniec 2021; Friedlander et al. 2022]. First, we detect the congruence $n = \ell + m \equiv 0 \pmod q$ by characters $\chi \pmod q$, getting

$$S(N,q) = \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \chi(-1) A(N,\chi) B(N,\bar\chi) + E(N,q) \tag{6-2}$$

where

$$A(N,\chi) = \sum_{\ell} \chi(\ell) a(\ell) e^{-\ell/N}, \quad B(N,\bar\chi) = \sum_{m} \bar\chi(m) b(m) e^{-m/N} \tag{6-3}$$

and $E(N,q)$ is the contribution from the terms ℓ, m with $(\ell m, q) \neq 1$, that is

$$E(N,q) = \sum\sum_{\substack{\ell + m \equiv 0 \pmod q \\ (\ell m, q) \neq 1}} a(\ell) b(m) e^{-(\ell+m)/N}. \tag{6-4}$$

Remark. Naturally, one may think that the main part of (6-2) comes from the principal character χ_0, but the exceptional character χ_1 cannot be dismissed. All the other characters will be shown to yield a negligible contribution. The last term (6-4) will also turn out to be negligible due to the properties of $a(\ell)$.

7. Properties of $a(\ell)$

We assume throughout that $a(\ell)$ is supported on squarefree almost primes and that $a(\ell)$ is quite small if ℓ has a small prime factor. We express this latter property in the following fashion:

$$a(\ell) \ll \log p, \quad \text{for all } p \mid \ell. \tag{7-1}$$

We assume that $a(1) \ll 1$. As for $\ell > 1$, the examples

$$a(\ell) = \Lambda(\ell), \quad a(\ell) = \Lambda_r(\ell)(\log \ell)^{1-r}$$

and the r-fold convolution

$$a(\ell) = (\Lambda * \cdots * \Lambda)(\ell)(\log \ell)^{1-r},$$

all satisfy (7-1); see (2-3). Our assumption means that $a(\ell)$ is majorized by

$$\mathcal{C}(\ell)=\sum_{\substack{p_1\cdots p_r=\ell\\ p_1<\cdots<p_r}}\log p_1,\quad \text{if } \omega(\ell)=r\geq 1, \tag{7-2}$$

where $\omega(\ell)$ as usual denotes the number of distinct prime factors of ℓ. For $\ell=1$ we set $\mathcal{C}(1)=1$. Note that the subsequence $a(d\ell)$ also satisfies (7-1).

Remark. We do not assume that $a(\ell)$ is positive nor that it is equidistributed over reduced residue classes except for the heuristic arguments in the Appendix. The arguments in that section are loose and lacking in mathematical rigor. They serve in this presentation as a motivation to expect the asymptotic formula (12-1), (12-2) (a generalization of the Hardy–Littlewood formula for $G(n)$), which we use in Section 13 to build a reliable model $R(N,q)$ for $S(N,q)$ and then to compare the two in the discussions of Section 14.

Lemma 7.1. *For $x\geq 2$ we have*

$$\sum_{\ell\leq x}|a(\ell)|\ell^{-1}\ll \log x. \tag{7-3}$$

Proof. For the sum over ℓ prime we have the bound

$$\sum_{p\leq x}\frac{\log p}{p}\ll \log x. \tag{7-4}$$

For the sum over ℓ having $r\geq 2$ prime factors we use the bound

$$\sum_{\substack{p_1\cdots p_r\leq x\\ p_1<\cdots<p_r}}(p_1\cdots p_r)^{-1}\log p_1\ll \log x, \tag{7-5}$$

which follows by repeated application of (7-4). □

Actually, we can derive from (7-1) the following bound.

Lemma 7.2. *We have*

$$\sum_{x<\ell\leq qx}|a(\ell)|\ell^{-1}\ll \log q. \tag{7-6}$$

Proof. If $x\leq q^r$ the result follows from (7-3). If ℓ is prime the result follows from

$$\sum_{x<\ell\leq qx}\frac{\log p}{p}=\log q+O(1).$$

Now, let $\ell=p\ell'$, $x<\ell\leq qx$ where ℓ' has all of its $r-1$ prime factors smaller than p. Then, for $x>q^r$ we have $\ell'\leq (qx)^{1-1/r}\leq x^{1-1/r^2}$. Hence, the contribution to the sum (7-6) is bounded by

$$\sum_{\ell'\le x^{1-1/r^2}} \frac{\mathcal{C}(\ell')}{\ell'} \sum_{x/\ell'<p\le qx/\ell'} \frac{1}{p} \ll \frac{\log q}{\log x} \sum_{\ell'\le x} \frac{\mathcal{C}(\ell')}{\ell'} \ll \log q;$$

see (7-3) for the function(7-2) . □

Lemma 7.3. *For $x \ge 2$ we have*

$$\sum_{\ell\le x} |a(\ell)| \ll x. \tag{7-7}$$

Proof. For the sum over ℓ prime we have the bound $O(x)$. For the sum over ℓ having $r \ge 2$ prime factors, $\sqrt{x} < \ell \le x$, we estimate as follows:

$$\sum_{\substack{\sqrt{x}<p_1\cdots p_r\le x\\ p_1<\cdots<p_r}} \log p_1 \ll \sum_{\substack{p_1\cdots p_{r-1}\le x^{1-1/(2r)}\\ p_1<\cdots<p_{r-1}}} \frac{\log p_1}{p_1\cdots p_{r-1}} \frac{rx}{\log x} \ll x.$$

The contribution of $\ell \le \sqrt{x}$ is negligible. □

By similar arguments one shows that (use the Brun–Titchmarsh theorem) that

$$\sum_{\substack{\ell\le x\\ \ell\equiv\alpha \pmod q}} |a(\ell)| \ll \frac{x}{\varphi(q)} \quad \text{if } (\alpha, q) = 1 \text{ and } x \ge q^{r+1}. \tag{7-8}$$

Lemma 7.4. *For $x \ge 2$ and p prime*, *we have*

$$\sum_{\substack{p\ne\ell\le x\\ \ell\equiv 0 \pmod p}} |a(\ell)| \ll \frac{x}{p}. \tag{7-9}$$

Proof. The contribution of those ℓ having all prime factors $\ge p$ is bounded by (apply the sieve over the range $P(p)$: the product of all primes less than p)

$$\sum_{\substack{p<\ell\le x/p\\ (\ell,P(p))=1}} \log p \ll \frac{x}{p}\frac{\log p}{\log p} = \frac{x}{p}.$$

If p is not the smallest prime divisor of ℓ then $a(\ell p)$ with $1 \le \ell \le x/p$ satisfies (7-1) so, as in the proof of (7-7), we get a contribution $\ll x/p$. □

Lemma 7.5. *Let $r \ge 1$. For $x \ge 2$ and p prime we have*

$$\sum_{\substack{\ell\le x\\ \ell\equiv 0 \pmod p\\ \omega(\ell)=r+1}} |a(\ell)| \ll \frac{x}{p}\frac{\log p}{\log x}\left(\log\left(1+\frac{\log x}{\log p}\right)\right)^{r-1}, \tag{7-10}$$

where, *we recall that $\omega(\ell)$ denotes the number of distinct prime factors of ℓ.*

Proof. If $p > \sqrt{x}$, then (7-10) follows from (7-9). If $p \le \sqrt{x}$, and $r = 1$ then (7-10) is obvious. If $p \le \sqrt{x}$, and $r \ge 2$, then, using (7-5), we see that the sum is bounded by

$$\begin{aligned}\sum_{\substack{p_1\cdots p_r \le x/p\\ p_1<\cdots<p_r}} \min(\log p, \log p_1) &\ll \frac{x}{p\log x} \sum_{p_1<\cdots<p_{r-1<x}} \frac{\min(\log p, \log p_1)}{p_1\cdots p_{r-1}}\\ &\ll \frac{x}{p}\frac{\log p}{\log x} \sum_{0\le j<r}\Big(\sum_{p<p'<x}\frac{1}{p'}\Big)^j\\ &\ll \frac{x}{p}\frac{\log p}{\log x}\Big(\log\frac{\log x}{\log p}\Big)^{r-1}.\end{aligned}$$

□

Corollary 7.6. *Suppose $a(\ell)$ is supported on squarefree numbers having at most r prime factors and that* (7-1) *holds. Then, for $x \ge 2$ and $z \ge 2$ we have*

$$\sum_{p\le z}\ \sum_{\substack{\ell\le x\\ \ell\equiv 0 \pmod p}} |a(\ell)| \ll x\frac{\log z}{\log x}\Big(\log\Big(1+\frac{\log x}{\log z}\Big)\Big)^{r-1}. \tag{7-11}$$

Actually, for $r \ge 2$ we can take the stronger exponent $r-2$ rather than $r-1$.

Lemma 7.7. *For $x \ge 2$ and $q \ge 2$ we have*

$$\sum_{\substack{\ell\le x\\ (\ell,q)\ne 1}} |a(\ell)| \ll \frac{x}{\log x}(\log\log 2x)^{r-1}\log\log 2q. \tag{7-12}$$

Here, $r \ge 1$ is the bound for the number of prime divisors of ℓ.

Proof. This follows from (7-10) and the easy bound

$$\sum_{p\,|\,q}\frac{\log p}{p} \ll \log\log 2q.$$

□

Lemma 7.8. *Let $d = (\alpha, q) \ne 1$. For $x \ge q^{2r+2}$ we have*

$$\sum_{\substack{\ell\le x\\ \ell\equiv\alpha \pmod q}} |a(\ell)| \ll \frac{x\log p(d)}{\varphi(q)\log x}\Big(\log\frac{\log x}{\log p(d)}\Big)^{r-1}, \tag{7-13}$$

where $p(d)$ denotes the smallest prime divisor of d and the implied constant depends only on r.

Proof. The contribution of $\ell \le q^{2r}$ is negligible by (7-7). Let $q^{2r} \le \ell \le x$. We write $\ell = p\ell'$ with ℓ' having at most $r-1$ prime divisors, each of them smaller than p. Therefore $p > q^2$, $\ell' < x^{1-1/r}$ and $a(\ell) \ll \mathcal{C}(\ell')$, $d \mid \ell'$, where we recall the definition (7-2). Since $d \ne 1$, $r \ge 2$. The contribution of these terms to (7-13) is estimated as follows:

$$\sum_{\substack{\ell' < x^{1-1/r} \\ \ell' \equiv 0 \pmod d}} \mathcal{C}(\ell') \sum_{\substack{q < p \le x/\ell' \\ p\ell' \equiv \alpha \pmod q}} 1 \ll \frac{x}{\varphi(q/d)\log x} \sum_{\substack{\ell' \le x \\ \ell' \equiv 0 \pmod d}} \frac{\mathcal{C}(\ell')}{\ell'}$$

by the Brun–Titchmarsh theorem for primes $p \equiv \beta \pmod{q/d}$ where $\beta\ell' \equiv \alpha \pmod q$. Note that $(\beta, q/d) = 1$ because $p \nmid q$. The above sum of $\mathcal{C}(\ell')/\ell'$ is estimated using the arrangements as in the proof of (7-10). Let $p(d)$ denote the least prime divisor of d. Then, the sum of $\mathcal{C}(\ell')/\ell'$ over $\ell' \equiv 0 \pmod d$, $\ell' \le x$ is estimated by

$$\frac{1}{d} \sum_{\substack{\ell \le x \\ \omega(\ell) \le r-2}} \frac{\mathcal{C}(d\ell)}{\ell} \le \frac{1}{d} \sum_{0 \le s \le r-2} \Bigg(\sum_{p_1 < \cdots < p_s \le p(d)} \frac{\log p_1}{p_1 \cdots p_s} \Bigg) \Bigg(\sum_{p(d) < p \le x} \frac{1}{p} \Bigg)^{r-2-s}$$
$$\ll \frac{\log p(d)}{d} \left(\log \frac{\log x}{\log p(d)} \right)^{r-2}.$$

Here, if $s = 0$ the sum over $p_1 < \cdots < p_s$ is taken to have the value 1.

This completes the proof of (7-13), using $d\varphi(q/d) \ge \varphi(q)$. □

8. Properties of $b(m)$

We could work with $b(m)$ as with $a(\ell)$ but for simplicity (in order to apply (3.3) of [Friedlander et al. 2022] without modification) we shall assume that

$$b(m) = \sum_{hk=m} \lambda(h)\Lambda(k), \tag{8-1}$$

where $\lambda(h)$ is supported on squarefree almost primes and

$$\lambda(h) \ll \log p \quad \text{for all } p \mid h. \tag{8-2}$$

We take $\lambda(1) = 1$. If $h > 1$, for example, $\lambda(h) = \Lambda_r(h)(\log h)^{1-r}$ is good. Note that $\lambda(h)$ satisfies (7-3)–(7-13). Moreover, we have

$$\sum_{m \le x} |b(m)| \ll x \log x \tag{8-3}$$

for every $x \ge 2$, because

$$\sum_{h \le x} |\lambda(h)| h^{-1} \ll \log x, \tag{8-4}$$

by (7-3) for the lambda function. Actually, we have the stronger result

$$\sum_{x<h\leq qx} |\lambda(h)|h^{-1} \ll \log q, \tag{8-5}$$

for every $x \geq 2$; see (7-6) for the lambda function.

Remark. Many interesting functions supported on almost primes can be well-approximated by sums of functions like $\lambda * \Lambda$. For example, we can take

$$b(m) = F_r\left(\frac{\log p_1}{\log m}, \dots, \frac{\log p_r}{\log m}\right)(\log m)^2$$

if $m = p_1 \cdots p_r$, where we recall F_r in (2-13) is as in Bombieri's asymptotic sieve; see Chapters 3 and 16 of [Friedlander and Iwaniec 2010].

In the case $b(m) = \Lambda(m)$ we have $\lambda(h) = 0$ except for $\lambda(1) = 1$. Therefore, in this special case some of our estimates can be improved by a log factor from those displayed. In particular, in (8-3) the factor $\log x$ can be removed and in (8-4) the "sum" is bounded. In the arguments of the following sections, this special case is therefore much easier, yet the need for these slightly stronger bounds would complicate the exposition. Since the results for this particular example are anyway just those already given in [Friedlander et al. 2022], we omit them from this presentation.

9. Evaluation of $S(N, q)$, first steps

Let $\chi_1 \pmod q$ be a real primitive character of modulus q such that $L(s, \chi_1)$ has a simple real zero β_1 close to $s = 1$. We single out the contributions of χ_0 and χ_1 to (6-2) and estimate the remaining parts as follows:

$$Q(N, q) = \sum_{\chi \neq \chi_0, \chi_1} \chi(-1)A(N, \chi)B(N, \bar{\chi}) = S(H, N, q) + T(H, N, q), \tag{9-1}$$

say, where $S(H, N, q)$ is the partial sum restricted to $h \leq H$ and $T(H, N, q)$ is the complementary partial sum. The first one is bounded by

$$\left(\sum_{\ell} |a(\ell)|e^{-\ell/N}\right) \sum_{h\leq H} |\lambda(h)| \sum_{\chi \neq \chi_0, \chi_1} \left|\sum_k \chi(k)\Lambda(k)e^{-hk/N}\right|. \tag{9-2}$$

The sum over ℓ in (9-2) is bounded by $O(N)$; see (7-7). The sum over k is (3.3) from [Friedlander et al. 2022], so it satisfies

$$\sum_{\chi \neq \chi_0, \chi_1} \left|\sum_k \chi(k)\Lambda(k)e^{-hk/N}\right| \ll \frac{N}{h}(1-\beta_1)\log q, \tag{9-3}$$

provided that $Nh^{-1} \geq q^b$ for a suitably large b; see (5.1) and (3.5) of [Friedlander et al. 2022]. This condition is satisfied for $N \geq Hq^b$. Hence, we get

$$S(H, N, q) \ll N^2(1-\beta_1)(\log q)(\log H). \tag{9-4}$$

Recall that (9-3) exploits the Bombieri zero density theorem with the repulsion effect of the exceptional zero β_1. We do not apply this effect, nor do we need it, for the estimation of $T(H, N, q)$. We write

$$T(H, N, q) = \sum_{\chi \neq \chi_0, \chi_1} \chi(-1)\Bigl(\sum_{\ell} a(\ell)\chi(\ell)e^{-\ell/N}\Bigr)\sum_{h>H}\sum_{k}\bar{\chi}(hk)\lambda(h)\Lambda(k)e^{-hk/N}.$$

Hence, inserting the corresponding sum for the missing two characters and using orthogonality, we find that

$$\begin{aligned}T&(H, N, q)\\ &= \varphi(q)\mathop{\sum\sum\sum}_{\substack{\ell+hk\equiv 0 \pmod q\\ (\ell,q)=1,\, h>H}} a(\ell)\lambda(h)\Lambda(k)e^{-(\ell+hk)/N} + O\Bigl(N^2\sum_{h>H}|\lambda(h)|h^{-1}e^{-h/2N}\Bigr),\end{aligned}$$

on using the trivial bound for the contribution of the two additional characters. Using (7-8), we see that the above main term is also bounded by the above error term. Moreover, this error term is $\ll N^2\log(N/H)$, as seen by applying (8-5) for $x = H, qH, q^2H, \ldots$. Choosing $H = Nq^{-b}$ we conclude that

$$T(H, N, q) \ll N^2 \log q. \tag{9-5}$$

On adding these estimates (9-4) and(9-5), we see that the sum in (9-1) satisfies $Q(N, q) \leq \varepsilon(N, q)N^2 \log N$ where

$$\varepsilon(N, q) \ll (1-\beta_1)\log q + \frac{\log q}{\log N}. \tag{9-6}$$

We still need to estimate $E(N, q)$ in (6-2) which is given by (6-4). This term is negligible and is actually smaller than the main term by a saving factor $\log N$. Nevertheless, we give simpler arguments producing an estimate somewhat weaker, yet still sufficient for our applications; see (9-7) and (9-8). Recall that $a(\ell)$, $\lambda(h)$ are supported on squarefree numbers having at most r prime divisors. By (7-13) we obtain

$$\begin{aligned}|E(N, q)| &\leq \sum_{\substack{d\mid q\\ d\neq 1}}\mathop{\sum\sum}_{\substack{\ell+m\equiv 0 \pmod q\\ (m,q)=d}} |a(\ell)b(m)|e^{-(\ell+m)/N}\\ &\ll \frac{N}{\varphi(q)}\frac{(\log\log N)^{r-1}}{\log N}\sum_{\substack{d\mid q\\ d\neq 1}}(\log p(d))W(N, d)\end{aligned}$$

where

$$W(N,d)=\sum_{m\equiv 0 \ (\mathrm{mod}\ d)}|b(m)|e^{-m/N}\le \sum_{uv=d}\ \sum\sum_{\substack{h\equiv 0 \ (\mathrm{mod}\ u)\\ k\equiv 0 \ (\mathrm{mod}\ v)}}\lambda(h)\Lambda(k)e^{-hk/N}.$$

Since k is prime we have $v=1$ or $v=k$. The sum with $v=1$ contributes

$$\begin{aligned}W_1(N,d)&=\sum_{h\equiv 0 \ (\mathrm{mod}\ d)}\lambda(h)\sum_k\Lambda(k)e^{-hk/N}\\&\ll\sum_{h\equiv 0 \ (\mathrm{mod}\ d)}|\lambda(h)|h^{-1}e^{-h/2N}\\&\ll\frac{\log d}{d}N(\log\log N)^{r-1},\end{aligned}$$

by the trivial bound $\lambda(h)\ll\log d$. Then we need, here and later, the easy bound

$$\sum_{d\mid q}(\log d)^2d^{-1}\ll(\log\log q)^3.$$

Next, the sum with $v=k$ contributes

$$W_2(N,d)=\sum_{uv=d}\Lambda(v)\sum_{h\equiv 0 \ (\mathrm{mod}\ u)}|\lambda(h)|e^{-hv/N}.$$

The partial sum of $W_2(N,d)$ with $u=1$ is

$$W_{21}(N,d)=\Lambda(d)\sum_h|\lambda(h)|e^{-dh/N}\ll\frac{\Lambda(d)}{d}N;$$

see (7-7) for the λ function. The remaining part of $W_2(N,d)$ is

$$W_{22}(N,d)=\sum_{\substack{uv=d\\ u\ne 1}}\Lambda(v)\sum_{h\equiv 0 \ (\mathrm{mod}\ u)}|\lambda(h)|e^{-hv/N}.$$

Hence,

$$\begin{aligned}\sum_{d\mid q}(\log p(d))W_{22}(N,d)&\le\sum_{uv\mid q}\Lambda^2(v)\sum_{\substack{u\mid h\\ (h,q)\ne 1}}|\lambda(h)|e^{-hv/N}\\&\ll\sum_{v\mid q}\Lambda^2(v)\sum_{(h,q)\ne 1}|\lambda(h)|e^{-hv/N}\end{aligned}$$

because $\tau(h)\ll_r 1$. Applying (7-12), we find this is bounded by

$$\Big(\sum_{v\mid q}\frac{\Lambda^2(v)}{v}\Big)\frac{N}{\log N}(\log\log N)^r\ll\frac{N}{\log N}(\log\log N)^{r+3}.$$

Gathering the above estimates, we obtain

$$E(N,q) \ll \frac{N^2}{\varphi(q)}\frac{(\log\log N)^{2r+2}}{\log N}. \tag{9-7}$$

This is stronger than we needed, namely

$$E(N,q) \ll N^2\frac{\log q}{\varphi(q)}. \tag{9-8}$$

Now, (6-2) becomes

$$\varphi(q)S(N,q)$$
$$= A(N,\chi_0)B(N,\chi_0) + \chi_1(-1)A(N,\chi_1)B(N,\chi_1) + (\varepsilon(N,q)N^2\log N). \tag{9-9}$$

The coprimality of ℓ, m with q in the main term $A(N,\chi_0)B(N,\chi_0)$ can be dropped within the existing error term, specifically

$$A(N,\chi_0) = A(N,1) + O\left(N\frac{(\log\log N)^r}{\log N}\right) \tag{9-10}$$

and

$$B(N,\chi_0) = B(N,1) + O(N(\log\log N)^r), \tag{9-11}$$

by direct applications of (7-12) for $a(\ell)$ and $b(m)/\log N$ respectively. Note that $(\log\log N)^r \ll (\log(\log N/\log q))^r \log q$.

10. Evaluation of $A(N,\chi_1)$ and $B(N,\chi_1)$

The exceptional character pretends to be the Möbius function on squarefree numbers, so we are able to replace

$$A(N,\chi_1) = \sum_\ell \chi_1(\ell)a(\ell)e^{-\ell/N} \tag{10-1}$$

by

$$A(N,\mu) = \sum_\ell \mu(\ell)a(\ell)e^{-\ell/N}. \tag{10-2}$$

In this section we use the Linnik zero repulsion phenomenon (see [Bombieri 1987]) to estimate the error caused in making this replacement. For ℓ squarefree we have

$$|\chi_1(\ell) - \mu(\ell)| \le \sum_{p\,|\,\ell}(1+\chi_1(p)). \tag{10-3}$$

Hence

$$|A(N,\chi_1) - A(N,\mu)| \le \sum_p (1+\chi_1(p)) \sum_{\ell\equiv 0 \ (\mathrm{mod}\ p)} |a(\ell)|e^{-\ell/N}. \tag{10-4}$$

The contribution of $\ell = p$ is bounded by $\Psi(N)$, where

$$\Psi(y) = \sum_p (1 + \chi_1(p))(\log p)e^{-p/y}. \tag{10-5}$$

For $y \geq z = q^b$ we have (apply (5.3) of [Friedlander et al. 2022]):

$$\Psi(y) \ll (1-\beta_1) y \log y + y(\log y)^{-1}. \tag{10-6}$$

For $p > z$ and $N > z$ we write

$$e^{-\ell/N} \leq e^{-\ell/2N} e^{-p/2N} \leq 6e^{-\ell/2N}(e^{-p/2N} - e^{-p/z}).$$

Hence, the terms ℓ, p with $\ell \neq p > z$ contribute to (10-4) at most

$$\frac{N}{\log N}\left(\log\left(1 + \frac{\log N}{\log z}\right)\right)^{r-1} \sum_p (1 + \chi_1(p)) \frac{\log p}{p}(e^{-p/2N} - e^{-p/z}) \tag{10-7}$$

by (7-10). The sum over all p above is equal to

$$\int_z^{2N} \Psi(y) y^{-2} dy \ll (1-\beta_1)(\log N)^2 + \log\left(\frac{\log 2N}{\log z}\right)$$

by (10-6). Hence (10-7) is bounded by

$$N\left(\log\left(1 + \frac{\log N}{\log z}\right)\right)^r \left((1-\beta_1)\log N + \frac{1}{\log N}\right). \tag{10-8}$$

For $p \leq z$ we use (7-11) obtaining a contribution to (10-4) at most

$$N\frac{\log z}{\log N}\left(\log\left(1 + \frac{\log N}{\log z}\right)\right)^r. \tag{10-9}$$

Combining estimates (10-6), (10-7), (10-9), we conclude that, if $N \geq q^b$, then

$$|A(N, \chi_1) - A(N, \mu)| \ll \eta(N, q)N \tag{10-10}$$

where

$$\eta(N, q) \ll \left((1-\beta_1)\log N + \frac{\log q}{\log N}\right)\left(\log \frac{\log N}{\log q}\right)^r. \tag{10-11}$$

Similarly, we can replace $B(N, \chi_1)$ by $B(N, \mu)$. Since the function $b(m)/\log N$ satisfies, for $m \leq N^{2021}$, the same conditions as $a(\ell)$, hence the same arguments as those between (10-3) and (10-11) yield

$$|B(N, \chi_1) - B(N, \mu)| \ll \eta(N, q)N \log N, \tag{10-12}$$

where $\eta(N, q)$ satisfies (10-11), the contribution of $m > N^{2021}$ being microscopic.

11. Evaluation of $S(N, q)$, conclusion

Collecting the results of the last two sections we formulate our basic result:

Proposition 11.1. *Let $a(\ell)$ and $b(m)$ be supported on squarefree numbers having at most r prime factors. Suppose* (7-1), (8-1), (8-2) *hold. Then, for $N \geq q^b$ with a suitable constant b, we have*

$$\varphi(q)S(N, q) = A(N, 1)B(N, 1) + \chi_1(-1)A(N, \mu)B(N, \mu) + \eta(N, q)N^2 \log N \tag{11-1}$$

with

$$\eta(N, q) \ll \left((1-\beta_1)\log N + \frac{\log q}{\log N}\right)\left(\log \frac{\log N}{\log q}\right)^r, \tag{11-2}$$

the implied constant depending on r and where $\chi_1 \pmod q$ is the exceptional character and β_1 is the zero of $L(s, \chi_1)$ in the segment

$$1 - c(\log q)^{-1} < \beta_1 < 1 \tag{11-3}$$

with c a small positive constant.

Proof. In (9-9) use (9-10), (9-11) to replace χ_0 by 1, use (10-10) and (10-12) to replace χ_1 by μ. □

In case $b(m) = \Lambda(m)$ we have $\lambda(h) = 0$ except for $\lambda(1) = 1$ and, as mentioned in Section 8, in this special, much easier case some of our estimates can be improved by a log factor from those displayed. As an upshot, our final formula (11-1) holds with the error term $\eta(N, q)N^2$.

For $N = q^A$ with A a large exponent we have

$$\eta(N, q) \ll (A(1-\beta_1)\log q + A^{-1})(\log A)^r. \tag{11-4}$$

Given $\delta > 0$ we can make

$$|\eta(N, q)| < \delta \tag{11-5}$$

if the exceptional zero satisfies (11-3) with c sufficiently small:

$$A \asymp \frac{1}{\delta}\left(\log \frac{1}{\delta}\right)^r, \quad c \leq A^{-2}. \tag{11-6}$$

We can write (11-1) in the form

$$\varphi(q)S(N, q) = \sum_\ell \sum_m (1+\chi_1(-1)\mu(\ell m))a(\ell)b(m)e^{-(\ell+m)/N} + \eta(N, q)N^2 \log N, \tag{11-7}$$

where $\eta(N, q)$ satisfies (11-2).

Remarks. Our conditions on $a(\ell)$ and $\lambda(h)$ imply that $A(N,1) \ll N$ and $B(N,1) \ll N \log N$. However, the formula (11-1) is meaningful if

$$A(N,1) \asymp N, \quad B(N,1) \asymp N \log N, \quad \text{for } N \geq q^b. \tag{11-8}$$

As we have already mentioned, the factor $\log N$ in the error term of (11-1) can be deleted if $b(m) = \Lambda(m)$. This is the case of $\lambda(h) = \Lambda_0(h)$, which function vanishes except for $\lambda(1) = \Lambda_0(1) = 1$.

12. Asymptotic formula for $F(n)$: prediction

Recall that $F(n)$ is given by (5-1) with $a(\ell)$ satisfying (7-1) and $b(m)$ given by (8-1) with $\lambda(h)$ satisfying (8-2). As such, $F(n)$ is a generalization of the Goldbach sum so it is too much to be expected to evaluate it unconditionally. Nevertheless, in the Appendix we show heuristic arguments which permit us to predict the following generalization of the Hardy–Littlewood conjecture (4-1).

Corollary. *Under the above-mentioned* (*in Sections 7 and 8*) *conditions on the sequences* $a(\ell)$, $b(m) = (\lambda * \Lambda)(m)$, *we have*

$$F(n) = \sum_{\ell+m=n} a(\ell)b(m) \sim \mathfrak{S}(n)\Phi(n) \tag{12-1}$$

as $n \to \infty$, *n even*, *where* $\mathfrak{S}(n)$ *is given by* (4-2) *and*

$$\Phi(n) = \sum\sum_{\ell+h<n} a(\ell)\lambda(h)h^{-1}. \tag{12-2}$$

Examples. If $b(m) = \Lambda(m)$, we have $\lambda(1) = 1$, $\lambda(h) = 0$ for $h > 1$. Hence

$$\Phi(n) = \sum_{\ell<n-1} a(\ell).$$

Moreover, if $a(\ell) = \Lambda(\ell)$ then we have $\Phi(n) \sim n$ and

$$F(n) = G(n) = \sum_{\ell+m=n} \Lambda(\ell)\Lambda(m) \sim \mathfrak{S}(n)n, \tag{12-3}$$

recovering (4-1). More generally, keeping $b(m) = \Lambda(m)$ but choosing $a(\ell) = \Lambda_k(\ell)/(\log n)^{k-1}$, we have

$$F(n) = \sum_{\ell+m=n} a(\ell)\Lambda(m) \sim k\mathfrak{S}(n)n. \tag{12-4}$$

13. Evaluation of $R(N, q)$

Injecting the asymptotic formula (12-1) into the series (6-1) we obtain the following model for $S(N, q)$:

$$\begin{aligned} R(N,q) &= \sum_{n\equiv 0 \pmod q} \mathfrak{S}(n)\Phi(n)e^{-n/N} \\ &= \sum_{\ell}\sum_{h} a(\ell)\lambda(\ell)h^{-1} \sum_{\substack{n\equiv 0 \pmod q\\ \ell+h<n,\, n \text{ even}}} \mathfrak{S}(n)e^{-n/N}. \end{aligned} \tag{13-1}$$

Using (6.5) of [Friedlander et al. 2022] one can derive the asymptotic formula

$$\sum_{\substack{n\equiv 0 \pmod q\\ n\le x,\, n \text{ even}}} \mathfrak{S}(n) \sim \frac{x}{\varphi(q)}.$$

Hence, the last sum over n in (13-1) is asymptotic to

$$\sim \frac{1}{\varphi(q)}\int_{\ell+h}^{\infty} e^{-x/N}dx = \frac{N}{\varphi(q)}e^{-(\ell+h)/N}$$

and

$$\varphi(q)R(N,q) \sim N\biggl(\sum_{\ell} a(l)e^{-\ell/N}\biggr)\biggl(\sum_{h}\lambda(h)h^{-1}e^{-h/N}\biggr). \tag{13-2}$$

On the other hand, we have

$$B(N,1) = \sum_{m} b(m)e^{-m/N} = \sum_{h}\lambda(h)\sum_{k}\Lambda(k)e^{-hk/N} \sim N\sum_{h}\lambda(h)h^{-1}e^{-h/N}.$$

Hence, (13-2) becomes

$$\varphi(q)R(N,q) \sim A(N,1)B(N,1). \tag{13-3}$$

This should be compared with (11-1) subject to the conditions (11-8).

14. Exceptional zero effects

It is instructive to observe what happens if we compare the legitimate formula (11-1) with the heuristic (13-3) in the range $N = q^A$. Take A sufficiently large and assume, as we may, that the exceptional constant $c \le A^{-2}$ so that $\eta(N, q)$ is negligible. It follows that $A(N, \mu)B(N, \mu)$ is significantly smaller than $A(N, 1)B(N, 1)$. This observation is attractive if the coefficients $a(\ell)$, $b(m)$ are each supported on almost primes having a fixed parity in the number of their prime divisors, because the

Möbius function is then constant and

$$A(N,\mu)=\mu_A A(N,1)\quad\text{where } \mu_A=\pm 1,$$
$$B(N,\mu)=\mu_B B(N,1)\quad\text{where } \mu_B=\pm 1.$$

Hence

$$|A(N,\mu)B(N,\mu)|=|A(N,1)B(N,1)|$$

and

$$\varphi(q)S(N,q)=\nu A(N,1)B(N,1)+o(N^2\log N)$$

with $\nu=0$ or 2. This inconsistency with (13-3) implies that the exceptional character does not exist! Indeed, it means that one may, a fortiori, kill the exceptional character by assuming the weaker conjecture (5-3) with any $0<\delta<1$, under suitable conditions on the coefficients $a(\ell)$, $b(m)$, as has been done in [Friedlander and Iwaniec 2021] and [Friedlander et al. 2022] for $a(\ell)=\Lambda(\ell)$, $b(m)=\Lambda(m)$.

If, on the other hand, we choose instead $a(\ell)=\Lambda_a(\ell)(\log\ell)^{1-a}$ and $\lambda(h)=\Lambda_b(h)(\log h)^{1-b}$ with numbers $a+b>2$, that is not both 1, then the effect on the exceptional zero no longer shows itself in our arguments. The point is that the series

$$\sum_{\ell}\Lambda_a(\ell)\ell^{-s}=(-1)^a\frac{\zeta(s)^{(a)}}{\zeta(s)}$$

has a pole at $s=1$ of order a, while the series

$$\sum_{\ell}\mu(\ell)\Lambda_a(\ell)\ell^{-s}=\zeta(s)\sum_{n}\frac{\mu(n)}{n^s}(\log n)^a\prod_{p\,|\,n}\Bigl(1-\frac{1}{p^s}\Bigr)$$

has only a simple pole at $s=1$ for any $a\geq 1$. Hence $A(N,\mu)$ is smaller than $A(N,1)$ by a factor $(\log N)^{a-1}$, so it yields a negligible contribution if $a\geq 2$. Similarly for $B(N,\mu)$ if $b\geq 2$. This is the same feature which, in the case of arithmetic progressions, led to the wider range of uniformity in Selberg's formula (2-4) and, more generally, in (3-3).

In view of the above, our formula (11-1) is relevant to the issue of exceptional characters only if its coefficients $a(\ell)$, $b(m)$, can be approximated, via the Weierstrass theorem, by linear combinations of scaled down $\Lambda_a(\ell)$, $\Lambda_b(m)$, in which $a=b=1$ appears (cannot be canceled out). The components with $a+b>2$ can be dismissed in (the highest order term of) $A(N,\mu)B(N,\mu)$.

We encourage the reader to learn the Bombieri approximations by the von Mangoldt functions from the original paper [Bombieri 1976] and to look at Chapters 3 and 16 of [Friedlander and Iwaniec 2010]; see also [Friedlander and Iwaniec 1985], especially Section 20.

Appendix: Heuristic arguments

Again we recall that $F(n)$ is given by (5-1) with $a(\ell)$ satisfying (7-1) and $b(m)$ given by (8-1) with $\lambda(\ell)$ satisfying (8-2). The coefficients $a(\ell)$, $b(m)$ are small if ℓ, m have small prime divisors so we can assume that ℓ, m are odd and that $n = \ell + m$ is even. We write

$$\Lambda(k) = -\sum_{d \mid k} \mu(d) \log d$$

and replace $b = \lambda * \Lambda$ by

$$-\sum_{\substack{dh \mid m \\ d<y}} \lambda(h)\mu(d) \log d,$$

where y is neither too small nor too large. Next, we interpret the equation $\ell + m = n$ by the congruence $\ell \equiv n \pmod{dh}$ with $(\ell, n) = 1$, $\ell < n$ and $(dh, n) = 1$. Arguing by the randomness of $\mu(n)$, we replace $F(n)$ by

$$-\sum_{\substack{d<y \\ (d,n)=1}} \mu(d) \log d \sum_{\substack{h<n/d \\ (h,n)=1}} \lambda(h) \sum_{\substack{\ell<n-dh,\, (\ell,n)=1 \\ \ell\equiv n \pmod{dh}}} a(\ell).$$

Next, assuming the equidistribution of $a(\ell)$ over reduced residue classes, we replace the sum over ℓ by

$$\frac{1}{\varphi(dh)} \sum_{\substack{\ell<n-dh, \\ (\ell,n)=1}} a(\ell).$$

We may think of $\ell < n$ as being not very close to n because otherwise $m = n - \ell$ would be very small, hence so would $b(m)$. Similarly, $h < n - \ell$ should not be close to $n - \ell$ because otherwise $k = (n - \ell)/h$ would be very small. Therefore, the sum over d,

$$-\sum_{\substack{d<y,\, dh<n-\ell \\ (d,n)=1}} \frac{\mu(d)}{\varphi(d)} \log d,$$

is not short, so it is reasonable to replace it by the infinite series

$$-\sum_{(d,n)=1} \frac{\mu(d)}{\varphi(d)} \log d = \mathfrak{S}(n);$$

see for example Lemma 19.3 of [Iwaniec and Kowalski 2004]. Now, we can drop the restriction $(\ell h, n) = 1$ because $a(\ell)$, $\lambda(h)$ are supported on almost primes and are relatively small if ℓ, h have any small prime divisors. For the same reason, we have already replaced $\varphi(dh)$ by $\varphi(d)h$.

The above lines show how we are led to the conjecture (12-1). The arguments of Hardy and Littlewood are rather different. They approach the issue by way of the circle method rather than using the randomness of the Möbius function.

Acknowledgement

We thank D.A. Goldston and A.I. Suriajaya whose preprint [Goldston and Suriajaya 2021] sparked our interest in the relevence to the Goldbach problem of the question of exceptional zeros, leading to the joint paper [Friedlander et al. 2022] and to the current work.

We thank Lillian Pierce for having invited us to submit this work on the occasion of the inauguration of her exciting new journal.

We thank the referees for many interesting suggestions.

References

[Bhowmik and Halupczok 2021] G. Bhowmik and K. Halupczok, "Conditional bounds on Siegel zeros", pp. 25–39 in *Combinatorial and additive number theory IV*, edited by M. B. Nathanson, Springer Proc. Math. Stat. **347**, Springer, 2021. MR

[Bhowmik et al. 2019] G. Bhowmik, K. Halupczok, K. Matsumoto, and Y. Suzuki, "Goldbach representations in arithmetic progressions and zeros of Dirichlet L-functions", *Mathematika* **65**:1 (2019), 57–97. MR Zbl

[Bombieri 1976] E. Bombieri, "The asymptotic sieve", *Rend. Accad. Naz. XL* (5) **1(2)** (1976), 243–269. MR Zbl

[Bombieri 1987] E. Bombieri, "Le grand crible dans la théorie analytique des nombres", pp. i+87 Astérisque **18**, Soc. Mat. de France, Paris, 1987. 2ieme ed. MR Zbl

[Fei 2016] J. Fei, "An application of the Hardy–Littlewood conjecture", *J. Number Theory* **168** (2016), 39–44. MR Zbl

[Friedlander 1976] J. B. Friedlander, "On the class numbers of certain quadratic extensions", *Acta Arith.* **28**:4 (1976), 391–393. MR

[Friedlander 1981] J. B. Friedlander, "Selberg's formula and Siegel's zero", pp. 15–23 in *Recent progress in analytic number theory* (Durham, 1979), vol. 1, edited by H. Halberstam and C. Hooley, Academic Press, London, 1981. MR

[Friedlander and Iwaniec 1978] J. Friedlander and H. Iwaniec, "On Bombieri's asymptotic sieve", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **5**:4 (1978), 719–756. MR

[Friedlander and Iwaniec 1985] J. B. Friedlander and H. Iwaniec, "Incomplete Kloosterman sums and a divisor problem", *Ann. of Math.* (2) **121**:2 (1985), 319–350. With an appendix by Bryan J. Birch and Enrico Bombieri. MR

[Friedlander and Iwaniec 1996] J. Friedlander and H. Iwaniec, "Bombieri's sieve", pp. 411–430 in *Analytic number theory* (Allerton Park, IL, 1995), vol. 1, edited by B. C. Berndt et al., Progr. Math. **138**, Birkhäuser, Boston, 1996. MR Zbl

[Friedlander and Iwaniec 2003] J. B. Friedlander and H. Iwaniec, "Exceptional characters and prime numbers in arithmetic progressions", *Int. Math. Res. Not.* **2003**:37 (2003), 2033–2050. MR

[Friedlander and Iwaniec 2004] J. B. Friedlander and H. Iwaniec, "Exceptional characters and prime numbers in short intervals", *Selecta Math.* (*N.S.*) **10**:1 (2004), 61–69. MR

[Friedlander and Iwaniec 2005] J. B. Friedlander and H. Iwaniec, "The illusory sieve", *Int. J. Number Theory* **1**:4 (2005), 459–494. MR

[Friedlander and Iwaniec 2010] J. Friedlander and H. Iwaniec, *Opera de cribro*, American Mathematical Society Colloquium Publications **57**, American Mathematical Society, Providence, RI, 2010. MR Zbl

[Friedlander and Iwaniec 2021] J. Friedlander and H. Iwaniec, "Note on a note of Goldston and Suriajaya", preprint, 2021. arXiv 2105.09038

[Friedlander et al. 2022] J. B. Friedlander, D. A. Goldston, H. Iwaniec, and A. I. Suriajaya, "Exceptional zeros and the Goldbach problem", *J. Number Theory* **233** (2022), 78–86. MR

[Goldston and Suriajaya 2021] D. A. Goldston and A. I. Suriajaya, "Note on the Goldbach conjecture and Landau–Siegel zeros", preprint, 2021. arXiv 2104.09407v1

[Granville 2020] A. Granville, "Sieving intervals and Siegel zeros", preprint, 2020. arXiv 2010.01211

[Hardy and Littlewood 1923] G. H. Hardy and J. E. Littlewood, "Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes", *Acta Math.* **44**:1 (1923), 1–70. MR Zbl

[Heath-Brown 1983] D. R. Heath-Brown, "Prime twins and Siegel zeros", *Proc. London Math. Soc.* (3) **47**:2 (1983), 193–224. MR Zbl

[Iwaniec and Kowalski 2004] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications **53**, American Mathematical Society, Providence, RI, 2004. MR Zbl

[Jia 2022] C. H. Jia, "On the conditional bounds for Siegel zeros", *Acta Math. Sin.* (*Engl. Ser.*) **38**:5 (2022), 869–876. MR Zbl

[Jurkat and Richert 1965] W. B. Jurkat and H.-E. Richert, "An improvement of Selberg's sieve method, I", *Acta Arith.* **11** (1965), 217–240. MR Zbl

[Merikoski 2021] J. Merikoski, "Exceptional characters and prime numbers in sparse sets", preprint, 2021. arXiv 2108.01355

[Siebert 1983] H. Siebert, "Sieve methods and Siegel's zeros", pp. 659–668 in *Studies in pure mathematics*, edited by P. Erdős, Birkhäuser, Basel, 1983. MR Zbl

[Zhang 2014] Y. Zhang, "Bounded gaps between primes", *Ann. of Math.* (2) **179**:3 (2014), 1121–1174. MR Zbl

Received 24 Aug 2021. Revised 22 Dec 2021.

JOHN B. FRIEDLANDER:

frdlndr@math.toronto.edu
Department of Mathematics, University of Toronto, Toronto, ON, Canada

HENRYK IWANIEC:

iwaniec@comcast.net
Department of Mathematics, Rutgers University, Piscataway, NJ, United States

ESSENTIAL NUMBER THEORY

msp.org/ent

See inside back cover or msp.org/ent for submission instructions.

Essential Number Theory (ISSN 2834-4634 electronic, 2834-4626 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ENT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
https://msp.org/

ESSENTIAL NUMBER THEORY

2022 vol. 1 no. 1