

# ESSENTIAL NUMBER THEORY

**A note on Tate's conjectures for abelian varieties**

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# A note on Tate's conjectures for abelian varieties

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In this mostly expository note, we explain a proof of Tate's two conjectures for algebraic cycles of arbitrary codimension on certain products of elliptic curves and abelian surfaces over number fields.

## 1. Statement

Let  $X$  be a smooth projective variety over a finitely generated field  $F$ . Let  $\text{Ch}^r(X)$  be the Chow group of codimension  $r$  algebraic cycles of  $X$  defined over  $F$  modulo rational equivalence. Let  $\bar{F}$  be a separable algebraic closure of  $F$  and  $\Gamma_F := \text{Gal}(\bar{F}/F)$ . Tate [1965, Conjecture 1] made the following far-reaching conjecture (often known as *the Tate conjecture*), relating algebraic cycles and  $\Gamma_F$ -invariants of the  $\ell$ -adic cohomology of  $X$ .

**Conjecture 1.1** (Tate I). *For any  $1 \leq r \leq \dim X$  and for any prime  $\ell \neq \text{char}(F)$ , the  $\ell$ -adic cycle class map*

$$\text{Ch}^r(X) \otimes \mathbb{Q}_\ell \rightarrow \text{H}^{2r}(X_{\bar{F}}, \mathbb{Q}_\ell(r))^{\Gamma_F}$$

*is surjective.*

Let  $\text{Ch}_{\text{hom}}^r(X)$  be the quotient group of  $\text{Ch}^r(X)$  modulo  $\ell$ -adic homological equivalence. It is further conjectured (and known when  $\text{char}(F) = 0$ ) that  $\text{Ch}_{\text{hom}}^r(X)$  is independent of  $\ell$ , and the  $\ell$ -adic cycle class map is injective on  $\text{Ch}_{\text{hom}}^r(X) \otimes \mathbb{Q}_\ell$ ; see [Tate 1965, page 97]. In particular, when  $\text{char}(F) = 0$ , Tate I implies an isomorphism  $\text{Ch}_{\text{hom}}^r(X) \otimes \mathbb{Q}_\ell \simeq \text{H}^{2r}(X_{\bar{F}}, \mathbb{Q}_\ell(r))^{\Gamma_F}$  and thus

$$\text{rank } \text{Ch}_{\text{hom}}^r(X) = \dim \text{H}^{2r}(X_{\bar{F}}, \mathbb{Q}_\ell(r))^{\Gamma_F} \quad (1.1.1)$$

for any prime  $\ell$ .

Tate [1965, Conjecture 2] further made a conjecture relating algebraic cycles to poles of zeta functions (often known as *the strong Tate conjecture*). When  $F$  is a number field, we denote by  $L(\text{H}^{2r}(X)(r), s)$  the (incomplete)  $L$ -function associated to the compatible system  $\{\text{H}^{2r}(X_{\bar{F}}, \mathbb{Q}_\ell(r))\}$  of  $\Gamma_F$ -representations, which

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converges absolutely for  $\Re(s) > 1$ . Then Conjecture 2 of [Tate 1965] specializes to the following.

**Conjecture 1.2** (Tate II). *Assume that  $F$  is a number field. Then for any  $1 \leq r \leq \dim X$ ,*

$$\text{rank Ch}_{\text{hom}}^r(X) = -\text{ord}_{s=1} L(H^{2r}(X)(r), s).$$

Tate I for divisors ( $r = 1$ ) is known for various  $X$ , including abelian varieties over any finitely generated fields [Faltings 1983; Zarhin 1975; Tate 1966]. Much less is known when  $r > 1$ . We refer to the surveys [Totaro 2017; Milne 2007; Tate 1994; Ramakrishnan 1989] for a nice summary of known results. The goal of this short note is to present some examples of abelian varieties  $X$  over number fields for which Tate’s conjectures hold for algebraic cycles in *arbitrary* codimension  $r$ .

**Theorem 1.3** (Tate I). *Assume that  $F$  is finitely generated with  $\text{char}(F) = 0$ . Then Tate I holds for any abelian variety  $X$  over  $F$  with simple factors all having dimension  $\leq 2$ .*

**Theorem 1.4** (Tate II). *Assume that  $F$  is a number field. Let  $E_1, E_2, E_3, E_4$  be elliptic curves over  $F$ . Let  $A$  be an abelian surface over  $F$ . Then Tate II holds for the following cases:*

- (i)  *$F$  is totally real or imaginary CM and  $X = E_1^{n_1} \times E_2^{n_2}$  for any  $n_1 \geq 1, n_2 \geq 0$ .*
- (ii)  *$F$  is totally real or imaginary CM and  $X = E_1^{n_1} \times E_2^{n_2} \times E_3$  for any  $n_1 \geq 1$  and  $1 \leq n_2 \leq 2$ .*
- (iii)  *$F$  is totally real or imaginary CM and  $X = E_1^{n_1} \times E_2^{n_2} \times E_3 \times E_4$  for any  $1 \leq n_1, n_2 \leq 2$ .*
- (iv)  *$F$  is totally real and  $X = A, X = A^2$ .*

**Remark 1.5.** It is worth mentioning that the special case when  $X = E^n$  is a power of an elliptic curve was considered by Tate himself [Tate 1965, page 106], and played an important role in his formulation of the Sato–Tate conjecture.

**Theorem 1.3** (Tate I) can be deduced from recent theorems on the Hodge conjecture and the Mumford–Tate conjecture [Ramón Marí 2008; Lombardo 2016], as mentioned, e.g., in [Moonen 2017, page 284]. **Theorem 1.4** (Tate II) can be deduced from more recent potential automorphy theorems [Allen et al. 2018; Boxer et al. 2021] and known cases of Langlands functionality, and should also be known to the experts. All these ingredients are available in more generality, but to illustrate the ideas we do not aim for maximal generality in the statement of the theorems.



## 2. Proof of Theorem 1.3 (Tate I)

Choose an embedding  $F \hookrightarrow \mathbb{C}$  and view  $F$  as a subfield of  $\mathbb{C}$ . Since all simple factors of  $X$  have dimension  $\leq 2$ , the Hodge conjecture for  $X_{\mathbb{C}}$  holds (in any codimension  $r$ ) by [Ramón Marí 2008, Theorem 3.15]. In fact in this case all Hodge classes on  $X_{\mathbb{C}}$  are generated by products of divisor classes. Also by [Lombardo 2016, Corollary 1.2], the Mumford–Tate conjecture for  $X$  holds.

Now the desired result follows due to the well-known general fact (see, e.g., [Farfán 2016, Section 6]) that the Mumford–Tate conjecture for the abelian variety  $X$  over  $F$  together with the Hodge conjecture for  $X_{\mathbb{C}}$  (in codimension  $r$ ) implies Tate I (Conjecture 1.1) for  $X$  (in codimension  $r$ ). In particular all Tate classes on  $X$  are also generated by products of divisor classes.

**Remark 2.1.** We refer to [Ramón Marí 2008; Lombardo 2016] for discussions about related previous works on the Hodge and Mumford–Tate conjectures. When  $X$  is a product of elliptic curves, the Hodge conjecture was proved in [Murty 1990] (see also [Gordon 1999, Appendix B, Section 3]) and the same method should also apply to prove Tate I.

## 3. Potential automorphy

Let  $F$  be a number field. Let  $V = \{V_{\ell}\}$  and  $W = \{W_{\ell}\}$  be compatible systems of semisimple  $\ell$ -adic  $\Gamma_F$ -representations (e.g., in the sense of strictly compatible systems of  $\ell$ -adic representations of  $\Gamma_F$  defined over  $\mathbb{Q}$  of [Boxer et al. 2021, Section 2.8]). Recall that  $V$  is *potentially automorphic* if there exists a finite Galois extension  $L/F$  such that the restriction  $V|_{\Gamma_L}$  is automorphic (e.g., in the sense of [Boxer et al. 2021, Definition 9.1.1]). We introduce the following variants of potential automorphy.

**Definition 3.1.** Let  $S$  be a nonempty set of rational primes. Let  $L/F$  be a finite Galois extension.

We say that  $V$  is  *$S$ -strongly automorphic over  $L$* , if for any subextension  $L'/F$  of  $L/F$  with  $L/L'$  solvable, the following conditions are satisfied:

- (i)  $V|_{\Gamma_{L'}}$  is automorphic.
- (ii) Let  $\pi$  be an isobaric automorphic representation on  $\mathrm{GL}_n(\mathbb{A}_{L'})$  associated to  $V|_{\Gamma_{L'}}$  ( $n = \dim V$  and  $\mathbb{A}_{L'}$  is the ring of adèles of  $L'$ ). Write  $\pi = \boxplus_{i=1}^k \pi_i$  as an isobaric direct sum of cuspidal automorphic representations on  $\mathrm{GL}_{n_i}(\mathbb{A}_{L'})$  ( $n = \sum_{i=1}^k n_i$ ). Write  $V|_{\Gamma_{L'}} = \bigoplus_{i=1}^k V_i$  as the corresponding direct sum decomposition into compatible systems of  $\Gamma_{L'}$ -representations. Then the  $\ell$ -adic  $\Gamma_{L'}$ -representation  $V_{i,\ell}$  ( $i = 1, \dots, k$ ) is irreducible for any  $\ell \in S$ . (Notice that the irreducibility of  $V_{i,\ell}$  is conjectured but not known in general).

We say that  $V$  is  $S$ -strongly potentially automorphic, if  $V$  is  $S$ -strongly automorphic over  $L$  for some finite Galois extension  $L/F$ . We say that  $V$  is strongly potentially automorphic, if  $V$  is  $S$ -strongly potentially automorphic for some Dirichlet density one set  $S$ .

We say that  $V$  and  $W$  are jointly  $S$ -strongly potentially automorphic, if  $V$  and  $W$  are both  $S$ -strongly automorphic over  $L$  for some finite Galois extension  $L/F$ . We say that  $V$  and  $W$  are jointly strongly potentially automorphic, if  $V$  and  $W$  are jointly  $S$ -strongly potentially automorphic for some Dirichlet density one set  $S$ .

**Lemma 3.2.** *Let  $V = \{V_\ell\}$  and  $W = \{W_\ell\}$  be compatible systems of semisimple  $\ell$ -adic  $\Gamma_F$ -representations. Let  $S$  be a nonempty set of rational primes:*

- (i) *Assume that  $V$  is  $S$ -strongly potentially automorphic. Then  $L(V, s)$  has meromorphic continuation to all of  $\mathbb{C}$ , and for any  $\ell \in S$ ,*

$$\dim V_\ell^{\Gamma_F} = -\text{ord}_{s=1} L(V, s).$$

- (ii) *Assume that  $V$  and  $W$  are jointly  $S$ -strongly potentially automorphic. Then  $L(V \otimes W, s)$  has meromorphic continuation to all of  $\mathbb{C}$ , and for any  $\ell \in S$ ,*

$$\dim(V_\ell \otimes W_\ell)^{\Gamma_F} = -\text{ord}_{s=1} L(V \otimes W, s).$$

- (iii) *Assume that  $V$  has a finite direct sum decomposition  $V \simeq \bigoplus_{i=1}^k V_i \otimes W_i$  into tensor products of compatible systems of  $\Gamma_F$ -representations. Assume that  $V_i$  and  $W_i$  are jointly  $S$ -strongly potentially automorphic for each  $i$ . Then  $L(V, s)$  has meromorphic continuation to all of  $\mathbb{C}$ , and for any  $\ell \in S$ ,*

$$\dim V_\ell^{\Gamma_F} = -\text{ord}_{s=1} L(V, s).$$

**Remark 3.3.** Lemma 3.2 should be known to the experts and the proof idea, using Brauer's induction theorem and known properties of automorphic  $L$ -functions, is an old one; see, e.g., [Taylor 2002; Harris et al. 2010; Harris 2009]. Notice that (i) also follows as a special case of (iii). We keep (i) to illustrate the ideas.

*Proof.* (i) Let  $L/F$  be a finite Galois extension such that  $V$  is  $S$ -strongly automorphic over  $L$ . By Brauer's induction theorem, we may find a virtual decomposition

$$\mathbf{1}_{\Gamma_F} = \sum_{j=1}^k c_j \text{Ind}_{\Gamma_{L_j}}^{\Gamma_F} \psi_j,$$

where  $c_j \in \mathbb{Z}$ ,  $F \subseteq L_j \subseteq L$  with  $L/L_j$  solvable, and  $\psi_j$  is a 1-dimensional representation of  $\text{Gal}(L/L_j)$  ( $j = 1, \dots, k$ ). Since  $V$  is  $S$ -strongly automorphic over  $L$ , we know that for each  $j$  there exists an isobaric direct sum of cuspidal automorphic representations  $\pi_{L_j} = \bigoplus_{i=1}^{m_i} \pi_{L_j, i}$  of  $\text{GL}_n(\mathbb{A}_{L_j})$  and a direct sum decomposition

$V|_{\Gamma_{L_j}} = \bigoplus_{i=1}^{m_j} V_{L_j,i}$  into  $\Gamma_{L_j}$ -representations such that

$$L(V|_{\Gamma_{L_j}}, s) = L(s, \pi_{L_j}) \quad L(V_{L_j,i}, s) = L(s, \pi_{L_j,i}),$$

and each  $\ell$ -adic representation  $V_{L_j,i,\ell}$  is irreducible for any  $\ell \in S$ . Here  $L(s, \pi_{L_j})$  is the (incomplete) standard  $L$ -function as in [Godement and Jacquet 1972] and has meromorphic continuation to all of  $\mathbb{C}$ . Hence

$$L(V \otimes \text{Ind}_{\Gamma_{L_j}}^{\Gamma_F} \psi_j, s) = L(V|_{\Gamma_{L_j}} \otimes \psi_j, s) = \prod_{i=1}^{m_j} L(V_{L_j,i} \otimes \psi_j, s) = \prod_{i=1}^{m_j} L(s, \pi_{L_j,i} \otimes \chi_j),$$

where  $\chi_j$  is the automorphic character on  $\text{GL}_1(\mathbb{A}_{L_j})$  associated to  $\psi_j$ . It follows that

$$L(V, s) = L(V \otimes \mathbf{1}_{\Gamma_F}, s) = \prod_{j=1}^k \prod_{i=1}^{m_j} L(s, \pi_{L_j,i} \otimes \chi_j)^{c_j}$$

and thus  $L(V, s)$  has meromorphic continuation to all of  $\mathbb{C}$ .

Since  $\pi_{L_j,i} \otimes \chi_j$  is cuspidal, by [Jacquet and Shalika 1976] we know that  $L(s, \pi_{L_j,i} \otimes \chi_j)$  has no zero or pole at  $s = 1$ , unless  $\pi_{L_j,i} \otimes \chi_j$  is the trivial representation in which case it has a simple pole at  $s = 1$ . Hence  $-\text{ord}_{s=1} L(V, s)$  equals the number of trivial representations among  $\pi_{L_j,i} \otimes \chi_j$  weighted by  $c_j$ , and so we obtain

$$-\text{ord}_{s=1} L(V, s) = \sum_{j=1}^k \sum_{i=1}^{m_j} c_j \dim \text{Hom}_{\Gamma_{L_j}}(\mathbf{1}_{\Gamma_{L_j}}, V_{L_j,i,\ell} \otimes \psi_j,\ell),$$

for any  $\ell \in S$  by the irreducibility of  $V_{L_j,i,\ell}$ . This evaluates to

$$\sum_{j=1}^k c_j \dim \text{Hom}_{\Gamma_{L_j}}(\mathbf{1}_{\Gamma_{L_j}}, V_{\ell}|_{\Gamma_{L_j}} \otimes \psi_j,\ell),$$

which by the Frobenius reciprocity equals

$$\dim \text{Hom}_{\Gamma_F}(\mathbf{1}_{\Gamma_F}, V_{\ell}) = \dim V_{\ell}^{\Gamma_F}.$$

(ii) Let  $L/F$  be a finite Galois extension such that both  $V$  and  $W$  are  $S$ -strongly automorphic over  $L$ . By the same notation and argument in the proof of (i), we know that for each  $j$  there exists an isobaric direct sum of cuspidal representations  $\pi_{L_j} = \boxplus_{i=1}^{m_j} \pi_{L_j,i}$  (resp.  $\Pi_{L_j} = \boxplus_{i'=1}^{m'_j} \Pi_{L_j,i'}$ ), together with a corresponding decomposition into  $\Gamma_{L_j}$ -representations  $V|_{\Gamma_{L_j}} \simeq \bigoplus_{i=1}^{m_j} V_{L_j,i}$  (resp.  $W|_{\Gamma_{L_j}} \simeq \bigoplus_{i'=1}^{m'_j} W_{L_j,i'}$ ) such that each  $\ell$ -adic representation  $V_{L_j,i,\ell}$  (resp.  $W_{L_j,i',\ell}$ ) is irreducible for any  $\ell \in S$ .

It follows that

$$L(V \otimes W, s) = \prod_{j=1}^k L(V \otimes W \otimes \mathbf{1}_{\Gamma_F}, s) = \prod_{j=1}^k \prod_{i=1}^{m_j} \prod_{i'=1}^{m'_j} L(s, \pi_{L_j, i} \times (\Pi_{L_j, i'} \otimes \chi_j))^{c_j},$$

where  $L(s, \pi_{L_j, i} \times (\Pi_{L_j, i'} \otimes \chi_j))$  is the (incomplete) Rankin–Selberg  $L$ -function as in [Jacquet et al. 1983], and thus  $L(V \otimes W, s)$  has meromorphic continuation to all of  $\mathbb{C}$ .

Since  $\pi_{L_j, i}$  and  $\Pi_{L_j, i'} \otimes \chi_j$  are cuspidal, we know that  $L(s, \pi_{L_j, i} \times (\Pi_{L_j, i'} \otimes \chi_j))$  has no zero at  $s = 1$  by [Shahidi 1980]; see also [Moreno 1985, Lemma 3.1; Sarnak 2004, page 721]. Also by [Jacquet and Shalika 1981, (4.6) and (4.11)] (see also [Mœglin and Waldspurger 1989, Appendice; Cogdell and Piatetski-Shapiro 2004, Theorem 2.4]), it has no pole at  $s = 1$ , unless  $\pi_{L_j, i} \simeq (\Pi_{L_j, i'} \otimes \chi_j)^\vee$  in which case it has a simple pole at  $s = 1$ . The latter happens if and only if  $V_{L_j, i} \simeq (W_{L_j, i'} \otimes \psi_j)^\vee$ . Hence

$$-\text{ord}_{s=1} L(V, s) = \sum_{j=1}^k \sum_{i=1}^{m_j} \sum_{i'=1}^{m'_j} c_j \dim \text{Hom}_{\Gamma_{L_j}}(\mathbf{1}_{\Gamma_{L_j}}, V_{L_j, i, \ell} \otimes W_{L_j, i', \ell} \otimes \psi_{j, \ell})$$

for any  $\ell \in S$  by the irreducibility of  $V_{L_j, i, \ell}$  and  $W_{L_j, i', \ell}$ . This evaluates to

$$\sum_{j=1}^k c_j \dim \text{Hom}_{\Gamma_{L_j}}(\mathbf{1}_{\Gamma_{L_j}}, (V_\ell \otimes W_\ell)|_{\Gamma_{L_j}} \otimes \psi_{j, \ell}),$$

which by the Frobenius reciprocity equals

$$\dim \text{Hom}_{\Gamma_F}(\mathbf{1}_{\Gamma_F}, V_\ell \otimes W_\ell) = \dim(V_\ell \otimes W_\ell)^{\Gamma_F}.$$

(iii) It follows directly from (ii) and the factorization  $L(V, s) = \prod_{i=1}^k L(V_i \otimes W_i, s)$ .  $\square$

**Lemma 3.4.** *Assume that  $F$  is a number field. Let  $E_1, E_2, E_3, E_4$  be elliptic curves over  $F$ . Let  $A$  be an abelian surface over  $F$ :*

- (i) *If  $F$  is totally real or imaginary CM, then  $\{\text{Sym}^{k_1} \text{H}^1(E_{1, \bar{F}}, \bar{\mathbb{Q}}_\ell)\}$  and  $\{\text{Sym}^{k_2} \text{H}^1(E_{2, \bar{F}}, \bar{\mathbb{Q}}_\ell)\}$  are jointly strongly potentially automorphic for any  $k_1, k_2 \geq 0$ .*
- (ii) *If  $F$  is totally real or imaginary CM, then  $\{\text{Sym}^{k_1} \text{H}^1(E_{1, \bar{F}}, \bar{\mathbb{Q}}_\ell)\}$  and  $\{\text{Sym}^{k_2} \text{H}^1(E_{2, \bar{F}}, \bar{\mathbb{Q}}_\ell) \otimes \text{Sym}^{k_3} \text{H}^1(E_{3, \bar{F}}, \bar{\mathbb{Q}}_\ell)\}$  are jointly strongly potentially automorphic for any  $k_1 \geq 0, 0 \leq k_2 \leq 2$ , and  $0 \leq k_3 \leq 1$ .*
- (iii) *If  $F$  is totally real or imaginary CM, then*

$$\{\text{Sym}^{k_1} \text{H}^1(E_{1, \bar{F}}, \bar{\mathbb{Q}}_\ell) \otimes \text{Sym}^{k_3} \text{H}^1(E_{3, \bar{F}}, \bar{\mathbb{Q}}_\ell)\}$$

and

$$\{\mathrm{Sym}^{k_2} \mathrm{H}^1(E_{2,\bar{F}}, \bar{\mathbb{Q}}_\ell) \otimes \mathrm{Sym}^{k_4} \mathrm{H}^1(E_{4,\bar{F}}, \bar{\mathbb{Q}}_\ell)\}$$

are jointly strongly potentially automorphic for any  $0 \leq k_1, k_2 \leq 2$  and  $0 \leq k_3, k_4 \leq 1$ .

(iv) If  $F$  is totally real, then  $\{\mathrm{H}^{k_1}(A_{\bar{F}}, \bar{\mathbb{Q}}_\ell)\}$  and  $\{\mathrm{H}^{k_2}(A_{\bar{F}}, \bar{\mathbb{Q}}_\ell)\}$  are jointly strongly potentially automorphic for any  $0 \leq k_1, k_2 \leq 4$ .

*Proof.* (i) If one of  $E_1$  or  $E_2$  has CM, say  $E_1$  has CM, then  $\{\mathrm{Sym}^{k_1} \mathrm{H}^1(E_{1,\bar{F}}, \bar{\mathbb{Q}}_\ell)\}$  is automorphic, as an isobaric direct sum of automorphic characters on  $\mathrm{GL}_1(\mathbb{A}_F)$ , and possibly automorphic inductions of automorphic characters on  $\mathrm{GL}_1(\mathbb{A}_K)$  for a quadratic extension  $K/F$ . In particular, we know that  $\{\mathrm{Sym}^{k_1} \mathrm{H}^1(E_{1,\bar{F}}, \bar{\mathbb{Q}}_\ell)|_{\Gamma_L}\}$  is  $S$ -strongly automorphic over any finite Galois extension  $L/F$  and any nonempty set  $S$  of primes. The result follows if  $E_2$  also has CM. If  $E_2$  has no CM, then  $\{\mathrm{H}^1(E_{2,\bar{F}}, \bar{\mathbb{Q}}_\ell)\}$  is strongly irreducible in the sense defined before [Allen et al. 2018, Lemma 7.1.1] (i.e., for any finite extension  $F'/F$ , the representation  $\mathrm{H}^1(E_{2,\bar{F}}, \bar{\mathbb{Q}}_\ell)|_{\Gamma_{F'}}$  is irreducible for  $\ell$  in a Dirichlet density one set of primes), and we can apply [loc. cit., Corollary 7.1.11] to  $\{\mathrm{Sym}^{k_2} \mathrm{H}^1(E_{2,\bar{F}}, \bar{\mathbb{Q}}_\ell)\}$  together with [loc. cit., Proposition 6.5.13] to obtain the desired joint  $S$ -strong potential automorphy for a Dirichlet density one set  $S$  of primes. If neither of  $E_1$  and  $E_2$  has CM, then the desired result follows from the more general [loc. cit., Theorem 7.1.10] together with [loc. cit., Proposition 6.5.13]. (In the case  $F = \mathbb{Q}$ , we may also directly apply [Newton and Thorne 2021, Theorem A (non-CM case) and Theorem A.1 (CM case)]).

(ii) By the same argument in (i), there are a finite Galois extension  $L/F$  and a Dirichlet density one set  $S$  of primes such that  $\{\mathrm{Sym}^{k_i} \mathrm{H}^1(E_{i,\bar{F}}, \bar{\mathbb{Q}}_\ell)\}$  is  $S$ -strongly automorphic over  $L$  for any  $1 \leq i \leq 3$ . Hence by the functorial products for  $\mathrm{GL}(2) \times \mathrm{GL}(2) \rightarrow \mathrm{GL}(4)$  [Ramakrishnan 2000, Theorem M] and  $\mathrm{GL}(2) \times \mathrm{GL}(3) \rightarrow \mathrm{GL}(6)$  [Kim and Shahidi 2002, Theorem A], we know that  $\{\mathrm{Sym}^{k_2} \mathrm{H}^1(E_{2,\bar{F}}, \bar{\mathbb{Q}}_\ell) \otimes \mathrm{Sym}^{k_3} \mathrm{H}^1(E_{3,\bar{F}}, \bar{\mathbb{Q}}_\ell)\}$  is also  $S$ -strongly automorphic over  $L$  for any  $0 \leq k_2 \leq 2$  and  $0 \leq k_3 \leq 1$ . The result then follows.

(iii) By the same argument in (ii), there are a finite Galois extension  $L/F$  and a Dirichlet density one set  $S$  of primes such that  $\{\mathrm{Sym}^{k_i} \mathrm{H}^1(E_{i,\bar{F}}, \bar{\mathbb{Q}}_\ell) \otimes \mathrm{Sym}^{k_j} \mathrm{H}^1(E_{j,\bar{F}}, \bar{\mathbb{Q}}_\ell)\}$  is  $S$ -strongly automorphic over  $L$  for any  $0 \leq k_i \leq 2$  and  $0 \leq k_j \leq 1$ , which gives the result.

(iv) The result follows from [Boxer et al. 2021, Theorem 9.3.1] and its proof.  $\square$

**Remark 3.5.** For each item of Lemma 3.4, the proof supplies a Dirichlet density one set  $S$  of primes such that the joint  $S$ -strong potential automorphy holds. Since compatible systems in Lemma 3.4 come from elliptic curves and abelian surfaces, one should also be able to prove directly that the irreducible conditions required in Definition 3.1(ii) hold for all primes  $\ell$ , and hence the joint  $S$ -strong potential



automorphy holds for the set  $S$  of all primes. For the purpose of the proof of [Theorem 1.4](#) (Tate II) below, any nonempty  $S$  suffices.

#### 4. Proof of [Theorem 1.4](#) (Tate II)

Let  $1 \leq r \leq \dim X$ . Let  $V = \{H^{2r}(X_{\bar{F}}, \bar{\mathbb{Q}}_\ell(r))\}$ . By [Theorem 1.3](#) (Tate I), we know from [\(1.1.1\)](#) that  $\text{rank Ch}_{\text{hom}}^r(X) = \dim V_\ell^{\Gamma_F}$  for any prime  $\ell$ . Thus it remains to show that  $\dim V_\ell^{\Gamma_F} = -\text{ord}_{s=1} L(V, s)$  for some prime  $\ell$ :

(i) By the Künneth formula and the decomposition of  $H^1(E_{i,\bar{F}}, \bar{\mathbb{Q}}_\ell)^{\otimes k_i}$  into symmetric powers of  $H^1(E_{i,\bar{F}}, \bar{\mathbb{Q}}_\ell)$  ( $i = 1, 2$ ), we have an isomorphism of semisimple  $\Gamma_F$ -representations

$$\begin{aligned} H^{2r}(X_{\bar{F}}, \bar{\mathbb{Q}}_\ell(r)) \\ \simeq \bigoplus_{\substack{0 \leq k_i \leq n_i \\ i=1,2}} m_{k_1, k_2} (\text{Sym}^{k_1} H^1(E_{1,\bar{F}}, \bar{\mathbb{Q}}_\ell) \otimes \text{Sym}^{k_2} H^1(E_{2,\bar{F}}, \bar{\mathbb{Q}}_\ell))^{\frac{1}{2}(k_1 + k_2)}, \end{aligned}$$

where  $m_{k_1, k_2} \geq 0$  are certain multiplicities (nonzero only if  $k_1 + k_2 \leq 2r$  is even). The result then follows from [Lemma 3.2 \(iii\)](#) and [Lemma 3.4 \(i\)](#).

(ii) Similarly, if we set  $n_3 = 1$  then we have an isomorphism of semisimple  $\Gamma_F$ -representations

$$H^{2r}(X_{\bar{F}}, \bar{\mathbb{Q}}_\ell(r)) \simeq \bigoplus_{\substack{0 \leq k_i \leq n_i \\ 1 \leq i \leq 3}} m_{k_1, k_2, k_3} (\otimes_{1 \leq i \leq 3} \text{Sym}^{k_i} H^1(E_{i,\bar{F}}, \bar{\mathbb{Q}}_\ell))^{\frac{1}{2}(k_1 + k_2 + k_3)},$$

where  $m_{k_1, k_2, k_3} \geq 0$  are certain multiplicities (nonzero only if  $k_1 + k_2 + k_3 \leq 2r$  is even). The result then follows from [Lemma 3.2\(iii\)](#) and [Lemma 3.4\(ii\)](#).

(iii) Similarly, the result follows from [Lemma 3.2\(iii\)](#) and [Lemma 3.4\(ii\)](#).

(iv) For  $X = A$ , the result follows from [Lemma 3.2\(i\)](#) and [Lemma 3.4\(iv\)](#). For  $X = A^2$ , by the Künneth formula, we have an isomorphism of semisimple  $\Gamma_F$ -representations

$$H^{2r}(X_{\bar{F}}, \bar{\mathbb{Q}}_\ell(r)) \simeq \bigoplus_{\substack{k_1 + k_2 = 2r \\ 0 \leq k_1, k_2 \leq 4}} (H^{k_1}(A_{\bar{F}}, \bar{\mathbb{Q}}_\ell) \otimes H^{k_2}(A_{\bar{F}}, \bar{\mathbb{Q}}_\ell))(r).$$

The result then follows from [Lemma 3.2\(iii\)](#) and [Lemma 3.4\(iv\)](#).

**Remark 4.1.** When  $X$  is an abelian surface of the type  $\text{Res}_{K/F} E$ , where  $F$  is totally real,  $K/F$  is a quadratic CM extension and  $E$  is an elliptic curve over  $K$ , Tate II was proved in [[Virdol 2015](#)] using a similar argument. We also refer to [[Johansson 2017](#); [Taylor 2020](#)] for more detailed analysis for  $L$ -functions of abelian surfaces.

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# ESSENTIAL NUMBER THEORY

2022 vol. 1 no. 1

The cubic case of Vinogradov's mean value theorem	1
D. R. HEATH-BROWN	
Exceptional zeros, sieve parity, Goldbach	13
JOHN B. FRIEDLANDER and HENRYK IWANIEC	
A note on Tate's conjectures for abelian varieties	41
CHAO LI and WEI ZHANG	
A Diophantine problem about Kummer surfaces	51
WILLIAM DUKE	
Quartic index form equations and monogenizations of quartic orders	57
SHABNAM AKHTARI	
Modularity lifting theorems	73
TOBY GEE	