ESSENTIAL
NUMBER THEORY

**On gamma factors for representations
of finite general linear groups**

David Soudry and Elad Zelingher

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We use the Langlands–Shahidi method in order to define the Shahidi gamma
factor for a pair of irreducible generic representations of $GL_n(\mathbb{F}_q)$ and $GL_m(\mathbb{F}_q)$.
We prove that the Shahidi gamma factor is multiplicative and show that it is
related to the Jacquet–Piatetski-Shapiro–Shalika gamma factor. As an application,
we prove a converse theorem based on the absolute value of the Shahidi gamma
factor, and improve the converse theorem of Nien. As another application, we
give explicit formulas for special values of the Bessel function of an irreducible
generic representation of $GL_n(\mathbb{F}_q)$.

1. Introduction

In the representation theory of *p*-adic groups, one method of studying irreducible
representations is by attaching local factors to the representations. These local
factors are complex valued functions of a complex variable. They encode various
properties of the representations in question. These local factors usually arise from
global integrals representing *L*-functions attached to automorphic representations.
Studying these local factors is crucial for understanding the global situation. This
has been done successfully in many cases, including the pioneering works of Jacquet,
Piatetski-Shapiro and Shalika [Jacquet et al. 1983] and Shahidi [1984; 1990].

Let $\mathbb{F}$ be a finite field with cardinality q. A local theory of local factors often
has a finite field analog. It allows one to attach "local constants" to irreducible
representations of the $\mathbb{F}$-points version of the group in consideration. One famous
example is the book by Piatetski-Shapiro [1983], in which he developed the theory
of gamma factors for the tensor product representation of $GL_2 \times GL_1$ over finite
fields. We also mention the works [Roditty-Gershon 2010; Nien 2014; Ye and
Zelingher 2020; Liu and Zhang 2022a; 2022b] as examples. These local constants
usually encode properties analogous to their local factors counterparts. Moreover,
these local constant theories often allow one to consider "toy models" for analogous
local problems. For instance, shortly after Nien's proof [2014] of the analog of

Jacquet's conjecture for finite fields, Chai [2019] proved the conjecture for the
p-adic group case where in his proof he used tools analogous to the ones used by
Nien.

In her master's thesis Roditty-Gershon [2010] defined a finite field analog of the
gamma factor of Jacquet, Piatetski-Shapiro and Shalika [1983]. This gamma factor
represents the tensor product representation, attached to two irreducible generic
representations π and σ of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$, respectively, and is denoted
$\gamma(\pi \times \sigma, \psi)$. Later, Rongqing Ye [2019] showed that $\gamma(\pi \times \sigma, \psi)$ is related to
its local field counterpart through level zero supercuspidal representations. Using
this relation and the local Langlands correspondence, Rongqing Ye and the second
author were able to express $\gamma(\pi \times \sigma, \psi)$ as a product of Gauss sums [Ye and
Zelingher 2021].

The theory of the finite field version of the gamma factor associated to the tensor
product, as it currently appears in the literature, is in some sense not complete. The
first problem is that the gamma factor $\gamma(\pi \times \sigma, \psi)$ is currently not defined for all
irreducible generic representations π and σ. It is only defined when $n \geq m$, and
under the assumption that π is cuspidal (and if $n = m$, σ is also required to be
cuspidal). One can tweak the proofs so they will work for all irreducible generic
representations π and σ, such that π and $\sigma^\vee$ have disjoint cuspidal support, but
that is not enough in order to define $\gamma(\pi \times \sigma, \psi)$ for all pairs π and σ. One can try
to define $\gamma(\pi \times \sigma, \psi)$ naively using the expression involving the Bessel functions
of π and σ (see Section 2B1 for the definition of the Bessel function), but this leads
to the second problem. The second problem is that the current theory lacks the
multiplicativity property of the gamma factor. If one naively extends the definition
$\gamma(\pi \times \sigma, \psi)$ using the approach suggested above, it is not clear that the gamma
factor would be multiplicative. Both of these difficulties need to be resolved for
applications as in [Zelingher 2023].

The Langlands–Shahidi method provides an alternative approach that solves
both of these problems. In this paper, we use this method to define a finite field
version of the Shahidi gamma factor. We briefly describe the construction now. Let
π and σ be representations of Whittaker type of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$, respectively.
In Section 3A, we consider an intertwining operator $U_{\sigma,\pi} : \sigma \circ \pi \to \pi \circ \sigma$, where
$\circ$ denotes parabolic induction. In Section 3B, given Whittaker vectors $v_{\pi,\psi} \in \pi$
and $v_{\sigma,\psi} \in \sigma$, we define Whittaker vectors $v_{\pi,\sigma,\psi} \in \pi \circ \sigma$ and $v_{\sigma,\pi,\psi} \in \sigma \circ \pi$.
By uniqueness of the Whittaker vectors, we have that there exists a constant
$\Gamma(\pi \times \sigma, \psi) \in \mathbb{C}$, such that

$$U_{\sigma,\pi} v_{\sigma,\pi,\psi} = \Gamma(\pi \times \sigma, \psi) \cdot v_{\pi,\sigma,\psi}.$$

We call $\Gamma(\pi \times \sigma, \psi)$ the *Shahidi gamma factor associated to π and σ*. This is a
finite analog of Shahidi's local coefficient [1984].

We prove properties of $\Gamma(\pi \times \sigma, \psi)$, the most important one is that it is multiplicative (Theorem 3.9).

Theorem 1.1. *Let π, σ_1 and σ_2 be representations of Whittaker type of $\mathrm{GL}_n(\mathbb{F})$, $\mathrm{GL}_{m_1}(\mathbb{F})$ and $\mathrm{GL}_{m_2}(\mathbb{F})$, respectively. Then*

$$\Gamma(\pi \times (\sigma_1 \circ \sigma_2), \psi) = \Gamma(\pi \times \sigma_1, \psi) \cdot \Gamma(\pi \times \sigma_2, \psi).$$

We also express $\Gamma(\pi \times \sigma, \psi)$ in terms of the Bessel functions associated with π and σ when both representations are irreducible. We show that if $n \geq m$, then up to some simple factors, $\Gamma(\pi \times \sigma, \psi)$ is given by the naive extension of $\gamma(\pi \times \sigma^\vee, \psi)$ discussed above (Theorem 3.14). We deduce a relation between the Shahidi gamma factor and the Jacquet–Piatetski-Shapiro–Shalika gamma factor (Corollary 3.15).

Theorem 1.2. *Let π and σ be irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$, respectively. Suppose that π is cuspidal and $n \geq m$. If $n = m$, suppose that σ is also cuspidal. Then*

$$\Gamma(\pi \times \sigma, \psi) = q^{m(2n-m-1)/2} \omega_\sigma(-1) \gamma(\pi \times \sigma^\vee, \psi).$$

The relation between both gamma factors allows us to give a representation theoretic interpretation for the absolute value of the Shahidi gamma factor. We show that, in some sense, the absolute value of the Shahidi gamma factor serves as a good substitute for the order of the pole of the local L-factor associated with the tensor product representation. Let us stress that the relation to the Jacquet–Piatetski-Shapiro–Shalika gamma factor is crucial for these results. The following theorem can be seen as an analog of [Jacquet et al. 1983, Section 8.1].

Theorem 1.3. *Let π be an irreducible generic representation of $\mathrm{GL}_n(\mathbb{F})$ and let σ be an irreducible cuspidal representation of $\mathrm{GL}_m(\mathbb{F})$. Then*

$$|q^{-nm/2} \cdot \Gamma(\pi \times \sigma, \psi)| = q^{-d_\pi(\sigma)m/2},$$

where $d_\pi(\sigma)$ is the number of times σ appears in the cuspidal support of π.

This allows us to deduce a converse theorem based on the absolute value of the normalized Shahidi gamma factor. Similar theorems in the local setting were given by Gan and his collaborators in many works (see [Gan and Savin 2012, Lemma 12.3], [Gan and Ichino 2016, Lemma A.6] and [Atobe and Gan 2017, Lemma A.6]), but our proof is done on the "group side" rather than on the "Galois side".

Theorem 1.4. *Let π_1 and π_2 be two irreducible generic representations of $\mathrm{GL}_{n_1}(\mathbb{F})$ and $\mathrm{GL}_{n_2}(\mathbb{F})$, respectively. Assume that for every m and every irreducible cuspidal representation σ of $\mathrm{GL}_m(\mathbb{F})$ we have*

$$|q^{-n_1 m/2} \cdot \Gamma(\pi_1 \times \sigma, \psi)| = |q^{-n_2 m/2} \cdot \Gamma(\pi_2 \times \sigma, \psi)|.$$

Then $n_1 = n_2$ and $\pi_1 \cong \pi_2$.

Our results combined with Nien's converse theorem [2014] allow us to deduce a converse theorem that holds under weaker assumptions. This is similar to [Jiang et al. 2015, Section 2.4].

Theorem 1.5. *Let π_1 and π_2 be irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$ with the same central character. Suppose that for any $1 \leq m \leq \frac{n}{2}$ and any irreducible cuspidal representation σ of $\mathrm{GL}_m(\mathbb{F})$ we have*

$$\Gamma(\pi_1 \times \sigma, \psi) = \Gamma(\pi_2 \times \sigma, \psi).$$

Then $\pi_1 \cong \pi_2$.

As another application of our results, we find explicit formulas for special values of the Bessel function of an irreducible generic representation π. The first formula (Theorem 4.10) expresses $\mathcal{J}_{\pi,\psi}\left(\begin{smallmatrix} & I_{n-1} \\ c & \end{smallmatrix}\right)$ as an exotic Kloosterman sum [Katz 1993, Page 152]. This formula is already known in the literature by the work of Curtis and Shinoda [2004], but our proof is based on multiplicativity of the Shahidi gamma factor, rather than on Deligne–Lusztig theory. The second formula we find (Theorem 4.14) expresses

$$\mathcal{J}_{\pi,\psi}\begin{pmatrix} & & -c' \\ & I_{n-2} & \\ c & & \end{pmatrix}$$

as a twisted convolution of values of the form $\mathcal{J}_{\pi,\psi}\left(\begin{smallmatrix} & I_{n-1} \\ c & \end{smallmatrix}\right)$ and $\mathcal{J}_{\pi,\psi}\left(\begin{smallmatrix} & c' \\ I_{n-1} & \end{smallmatrix}\right)$. Such a formula was given by Chang [1976] for $n = 3$ and then generalized by Shinoda and Tulunay [2005] for $n = 4$. Chang's method is based on the Gelfand–Graev algebra, while our method is based on formulas we found for the Shahidi gamma factor.

This paper is based on a unpublished note by the first author [Soudry 1979].

2. Preliminaries

2A. *Parabolic induction.* Given a sequence of positive integers $n_1, \ldots, n_r$, we denote by $P_{n_1,\ldots,n_r}$ the parabolic subgroup of $\mathrm{GL}_{n_1+\cdots+n_r}(\mathbb{F})$ corresponding to the composition $(n_1, \ldots, n_r)$. That is,

$$P_{n_1,\ldots,n_r} = D_{n_1,\ldots,n_r} \ltimes N_{n_1,\ldots,n_r},$$

where

$$D_{n_1,\ldots,n_r} = \{\mathrm{diag}(g_1, \ldots, g_r) \mid \forall 1 \leq j \leq r, g_j \in \mathrm{GL}_{n_j}(\mathbb{F})\},$$

$$N_{n_1,\ldots,n_r} = \left\{ \begin{pmatrix} I_{n_1} & * & * & * \\ & I_{n_2} & * & * \\ & & \ddots & * \\ & & & I_{n_r} \end{pmatrix} \right\}.$$

Given representations $\pi_1, \ldots, \pi_r$ of $\mathrm{GL}_{n_1}(\mathbb{F}), \ldots, \mathrm{GL}_{n_r}(\mathbb{F})$, respectively, we denote by $\pi_1 \overline{\otimes} \ldots \overline{\otimes} \pi_r$ the inflation of $\pi_1 \otimes \cdots \otimes \pi_r$ to $P_{n_1, \ldots, n_r}$. That is, $\pi_1 \overline{\otimes} \ldots \overline{\otimes} \pi_r$ is a representation of $P_{n_1, \ldots, n_r}$, acting on the space of $\pi_1 \otimes \cdots \otimes \pi_r$, and its action on pure tensors is given by

$$(\pi_1 \overline{\otimes} \ldots \overline{\otimes} \pi_r)(du) v_1 \otimes \cdots \otimes v_r = \pi_1(g_1) v_1 \otimes \cdots \otimes \pi_r(g_r) v_r,$$

where $d = \mathrm{diag}(g_1, \ldots, g_r) \in D_{n_1, \ldots, n_r}$ and $u \in N_{n_1, \ldots, n_r}$, and for every $1 \le j \le r$, $v_j \in \pi_j$.

The parabolic induction $\pi_1 \circ \ldots \circ \pi_r$ is defined as the following representation of $\mathrm{GL}_{n_1 + \cdots + n_r}(\mathbb{F})$:

$$\pi_1 \circ \ldots \circ \pi_r = \mathrm{Ind}_{P_{n_1, \ldots, n_r}}^{\mathrm{GL}_{n_1 + \cdots + n_r}(\mathbb{F})} \pi_1 \overline{\otimes} \ldots \overline{\otimes} \pi_r.$$

A representation π of $\mathrm{GL}_n(\mathbb{F})$ is called *cuspidal* if for every composition $(n_1, \ldots, n_r) \ne (n)$ of n, there does not exist $0 \ne v \in \pi$ such that v is invariant under $N_{n_1, \ldots, n_r}$, i.e., if $v \in \pi$ is such that $\pi(u)v = v$ for every $u \in N_{n_1, \ldots, n_r}$ then $v = 0$. If π is irreducible, π is cuspidal if and only if it is not a subrepresentation of $\pi_1 \circ \ldots \circ \pi_r$ for some $\pi_1, \ldots, \pi_r$ as above, where $r > 1$.

By [Gel'fand 1970, Theorem 2.4], if π is an irreducible representation of $\mathrm{GL}_n(\mathbb{F})$, then there exist $n_1, \ldots, n_r > 0$ with $n_1 + \cdots + n_r = n$ and irreducible cuspidal representations $\pi_1, \ldots, \pi_r$, of $\mathrm{GL}_{n_1}(\mathbb{F}), \ldots, \mathrm{GL}_{n_r}(\mathbb{F})$, such that π is isomorphic to a subrepresentation of the parabolic induction $\pi_1 \circ \ldots \circ \pi_r$. Such $\pi_1, \ldots, \pi_r$ are unique up to ordering. We define the *cuspidal support of π* to be the multiset $\{\pi_1, \ldots, \pi_r\}$.

2B. *Generic representations.* Let $\psi : \mathbb{F} \to \mathbb{C}^*$ be a nontrivial additive character. Let $Z_n \le \mathrm{GL}_n(\mathbb{F})$ be the upper triangular unipotent subgroup. We define a character $\psi : Z_n \to \mathbb{C}^*$ by the formula

$$\psi \begin{pmatrix} 1 & a_1 & * & \ldots & * \\ & 1 & a_2 & \ldots & * \\ & & \ddots & \ddots & \vdots \\ & & & 1 & a_{n-1} \\ & & & & 1 \end{pmatrix} = \psi \left(\sum_{k=1}^{n-1} a_k \right).$$

Let π be a finite dimensional representation of $\mathrm{GL}_n(\mathbb{F})$. π is said to be *generic* if

$$\mathrm{Hom}_{Z_n}(\mathrm{Res}_{Z_n} \pi, \psi) \ne 0.$$

This condition does not depend on the choice of ψ. See Section 3B1. We call a nonzero element in $\mathrm{Hom}_{Z_n}(\mathrm{Res}_{Z_n} \pi, \psi)$ a *ψ-Whittaker functional*. The representation π is generic if and only if there exists $0 \ne v \in \pi$, such that $\pi(u)v = \psi(u)v$ for every $u \in Z_n$. We call such vector a *Whittaker vector with respect to ψ*, or a *ψ-Whittaker vector*. The dimension of the subspace spanned by the ψ-Whittaker vectors of π is $\dim \mathrm{Hom}_{Z_n}(\mathrm{Res}_{Z_n} \pi, \psi)$.

Definition 2.1. We say that π *is of Whittaker type* if π is generic and the subspace spanned by its ψ-Whittaker vectors is one-dimensional.

By a well known result of Gelfand and Graev, we have that if π is generic and irreducible, then it is of Whittaker type [Gel'fand 1970, Theorem 0.5; Silberger and Zink 2000, Corollary 5.6]. It is well known that irreducible cuspidal representations of $\mathrm{GL}_n(\mathbb{F})$ are generic [Silberger and Zink 2000, Lemma 5.2]. The following result is also well known [loc. cit., Theorem 5.5].

Theorem 2.2. *Let $\pi_1, \ldots, \pi_r$ be representations of Whittaker type of $\mathrm{GL}_{n_1}(\mathbb{F}), \ldots, \mathrm{GL}_{n_r}(\mathbb{F})$, respectively. Then the parabolic induction $\pi_1 \circ \ldots \circ \pi_r$ is a representation of Whittaker type.*

2B1. *Whittaker models and Bessel functions.* Let π be an irreducible generic representation of $\mathrm{GL}_n(\mathbb{F})$. Since π is of Whittaker type, Frobenius reciprocity implies that

$$\dim \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{F})}(\pi, \mathrm{Ind}_{Z_n}^{\mathrm{GL}_n(\mathbb{F})} \psi) = 1.$$

We denote by $\mathcal{W}(\pi, \psi)$ the unique subspace of $\mathrm{Ind}_{Z_n}^{\mathrm{GL}_n(\mathbb{F})} \psi$ that is isomorphic to π. This is *the Whittaker model of π with respect to ψ*.

Recall that for an irreducible representation π of $\mathrm{GL}_n(\mathbb{F})$, we have that its contragredient $\pi^\vee$ is isomorphic to π^ι, where π^ι is the representation acting on the space of π by $\pi^\iota(g) = \pi(g^\iota)$, where for $g \in \mathrm{GL}_n(\mathbb{F})$,

$$g^\iota = {}^t(g^{-1}).$$

(This follows from the fact that for $g \in \mathrm{GL}_n(\mathbb{F})$, the trace characters of π and $\pi^\vee$ are related by $\mathrm{tr}\,\pi^\vee(g) = \mathrm{tr}\,\pi(g^{-1})$, and from the fact that g^{-1} and ${}^t(g^{-1})$ are conjugate.)

Using the isomorphism $\pi^\vee \cong \pi^\iota$, we get an isomorphism of vector spaces $\mathcal{W}(\pi, \psi) \to \mathcal{W}(\pi^\vee, \psi^{-1})$, given by $W \mapsto \tilde{W}$, where

$$\tilde{W}(g) = W(w_n g^\iota),$$

and where $w_n \in \mathrm{GL}_n(\mathbb{F})$ is the long Weyl element

$$w_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}.$$

Under the realization of π by its Whittaker model $\mathcal{W}(\pi, \psi)$, the one-dimensional subspace spanned by the ψ-Whittaker vectors of π is realized as the one-dimensional subspace of $\mathcal{W}(\pi, \psi)$ consisting of functions $W \in \mathcal{W}(\pi, \psi)$, such that $W(gu) = \psi(u)W(g)$, for every $u \in Z_n$ and every $g \in \mathrm{GL}_n(\mathbb{F})$. By [Gel'fand 1970, Proposition 4.5], there exists a (unique) element W in this one-dimensional subspace such that $W(I_n) = 1$. We call this W the *normalized Bessel function of π with respect*

to ψ, and denote it by $\mathcal{J}_{\pi,\psi}$. To summarize, the Bessel function $\mathcal{J}_{\pi,\psi}$ is the unique element in $\mathcal{W}(\pi, \psi)$, such that:

(1) $\mathcal{J}_{\pi,\psi}(I_n) = 1$.

(2) $\mathcal{J}_{\pi,\psi}(gu) = \psi(u)\mathcal{J}_{\pi,\psi}(g)$, for every $g \in \mathrm{GL}_n(\mathbb{F})$ and $u \in Z_n$.

The Bessel function enjoys the following identities that relate it to its complex conjugate and to its contragredient [Nien 2014, Propositions 2.15 and 3.5].

Proposition 2.3. *For any irreducible generic representation π of $\mathrm{GL}_n(\mathbb{F})$ and any $g \in \mathrm{GL}_n(\mathbb{F})$, we have the following identities*:

(1) $\mathcal{J}_{\pi,\psi}(g^{-1}) = \overline{\mathcal{J}_{\pi,\psi}(g)}$.

(2) $\mathcal{J}_{\pi,\psi}(g^{-1}) = \mathcal{J}_{\pi^\vee,\psi^{-1}}(g)$.

Remark 2.4. Let $v_{\pi,\psi}$ be a nonzero ψ-Whittaker vector. If we choose an inner product $(\cdot, \cdot)_\pi$ on π which is invariant under the $\mathrm{GL}_n(\mathbb{F})$-action, we have that the assignment $\ell_{\pi,\psi} : \pi \to \mathbb{C}$ given by $v_\pi \mapsto (v_\pi, v_{\pi,\psi})_\pi$ defines a Whittaker functional. The Whittaker model of π can be described using Frobenius reciprocity as $\mathcal{W}(\pi, \psi) = \{W_{v_\pi} \mid v_\pi \in \pi\}$, where for $g \in \mathrm{GL}_n(\mathbb{F})$ and $v_\pi \in \pi$, we define $W_{v_\pi}(g) = (\pi(g)v_\pi, v_{\pi,\psi})_\pi$. The Bessel function is given by

$$\mathcal{J}_{\pi,\psi}(g) = \frac{(\pi(g)v_{\pi,\psi}, v_{\pi,\psi})_\pi}{(v_{\pi,\psi}, v_{\pi,\psi})_\pi}.$$

All of the properties of the Bessel function listed above now follow immediately from the fact that $(\cdot, \cdot)_\pi$ is an inner product, and that $v_{\pi,\psi}$ is a ψ-Whittaker vector. Moreover, the projection operator to the one-dimensional subspace spanned by the ψ-Whittaker vectors $\mathrm{pr}_{\mathbb{C}v_{\pi,\psi}}$ can be described in two ways. The first way is by using the inner product, in which case for $v_\pi \in \pi$,

$$\mathrm{pr}_{\mathbb{C}v_{\pi,\psi}}(v_\pi) = \frac{(v_\pi, v_{\pi,\psi})_\pi}{(v_{\pi,\psi}, v_{\pi,\psi})_\pi} v_{\pi,\psi}.$$

The second way is by averaging, in which case

$$\mathrm{pr}_{\mathbb{C}v_{\pi,\psi}}(v_\pi) = \frac{1}{|Z_n|} \sum_{u \in Z_n} \psi^{-1}(u)\pi(u)v_\pi.$$

By completing $v_{\pi,\psi}$ to an orthogonal basis of π and using the fact that the subspace spanned by the ψ-Whittaker vectors is one dimensional, we see that

$$\mathrm{tr}(\mathrm{pr}_{\mathbb{C}v_{\pi,\psi}} \circ \pi(g)) = \mathcal{J}_{\pi,\psi}(g).$$

This is [Gel'fand 1970, Proposition 4.5].

2C. *Jacquet–Piatetski-Shapiro–Shalika gamma factors.* Let π and σ be irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$, respectively. For most π and σ, one can define a constant attached to π and σ called the *Jacquet–Piatetski-Shapiro–Shalika gamma factor of π and σ*. It is also known as the *Rankin–Selberg gamma factor of π and σ*. This is a finite field analog of the definition given by Jacquet, Piatetski-Shapiro and Shalika [1983] for p-adic groups. These were explained in Piatetski-Shapiro's lectures in 1976 and studied in an unpublished note from 1979 by the first author [Soudry 1979]. The case $n > m$ was also studied in Roddity-Gershon's master's thesis under the supervision of the first author.

2C1. *The case $n > m$.* In her master's thesis Edva Roditty-Gershon [2010] defined the Jacquet–Piatetski-Shapiro–Shalika gamma factor $\gamma(\pi \times \sigma, \psi)$, under the assumption that π is cuspidal and that $n > m$. Roddity-Gershon's thesis is unpublished, but her main results are presented by Nien [2014]. We briefly review these results now.

The first result is a functional equation that defines the Jacquet–Piatetski-Shapiro–Shalika gamma factor. Suppose that $n > m$ and that π is cuspidal. For any $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\sigma, \psi^{-1})$, and any $0 \le j \le n - m - 1$, we define

$$Z_j(W, W'; \psi) = \sum_{h \in Z_m \backslash \mathrm{GL}_m(\mathbb{F})} \sum_{x \in M_{(n-m-j-1) \times m}(\mathbb{F})} W \begin{pmatrix} h & & \\ x & I_{n-m-j-1} & \\ & & I_{j+1} \end{pmatrix} W'(h).$$

We are now ready to state the functional equation.

Theorem 2.5 [Nien 2014, Theorem 2.10]. *There exists a nonzero constant $\gamma(\pi \times \sigma, \psi) \in \mathbb{C}$, such that for every $0 \le j \le n - m - 1$, every $W \in \mathcal{W}(\pi, \psi)$ and every $W' \in \mathcal{W}(\sigma, \psi^{-1})$, we have*

$$q^{mj} \gamma(\pi \times \sigma, \psi) Z_j(W, W'; \psi) = Z_{n-m-j-1}\left(\pi^\vee \begin{pmatrix} I_m & \\ & w_{n-m} \end{pmatrix} \tilde{W}, \tilde{W}'; \psi^{-1} \right),$$

where $w_{n-m} \in \mathrm{GL}_{n-m}(\mathbb{F})$ is the long Weyl element.

The second result expresses the gamma factor $\gamma(\pi \times \sigma, \psi)$ in terms of the Bessel functions of π and σ.

Proposition 2.6 [Nien 2014, Proposition 2.16]. *Under the assumptions above, we have*

$$\gamma(\pi \times \sigma, \psi) = \sum_{h \in Z_m \backslash \mathrm{GL}_m(\mathbb{F})} \mathcal{J}_{\pi, \psi} \begin{pmatrix} & I_{n-m} \\ h & \end{pmatrix} \mathcal{J}_{\sigma, \psi^{-1}}(h).$$

It follows from Propositions 2.6 and 2.3 that

$$\overline{\gamma(\pi \times \sigma, \psi)} = \gamma(\pi^\vee \times \sigma^\vee, \psi^{-1}).$$

Moreover, applying Theorem 2.5 twice, we get the following corollary regarding the absolute value of $\gamma(\pi \times \sigma, \psi)$.

Corollary 2.7. *We have that*

$$\gamma(\pi \times \sigma, \psi)\gamma(\pi^\vee \times \sigma^\vee, \psi^{-1}) = q^{-m(n-m-1)},$$

and therefore

$$|\gamma(\pi \times \sigma, \psi)| = q^{-m(n-m-1)/2}.$$

2C2. *The case $n = m$.* The case $n = m$ was discussed in Piatetski-Shapiro's lecture and is explained briefly in Rongqing Ye's work [2019].

Let $\mathcal{S}(\mathbb{F}^n)$ be the space of functions $\phi \colon \mathbb{F}^n \to \mathbb{C}$. For a function $\phi \in \mathcal{S}(\mathbb{F}^n)$, we define its Fourier transform $\mathcal{F}_\psi \phi \colon \mathbb{F}^n \to \mathbb{C}$ by the formula

$$\mathcal{F}_\psi \phi(y) = \sum_{x \in \mathbb{F}^n} \phi(x)\psi(\langle x, y \rangle),$$

where if $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{F}^n$, then $\langle x, y \rangle$ is the standard pairing

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i.$$

Let π and σ be irreducible cuspidal representations of GL$_n(\mathbb{F})$. We define for any $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\sigma, \psi^{-1})$ and any $\phi \in \mathcal{S}(\mathbb{F}^n)$

$$Z(W, W', \phi; \psi) = \sum_{g \in Z_n \backslash \mathrm{GL}_n(\mathbb{F})} W(g)W'(g)\phi(e_n g),$$

where $e_n = (0, \ldots, 0, 1) \in \mathbb{F}^n$. We are now ready to introduce the functional equation that defines $\gamma(\pi \times \sigma, \psi)$.

Theorem 2.8 [Ye 2019, Theorem 2.3]. *There exists a nonzero constant $\gamma(\pi \times \sigma, \psi)$, such that for any $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\sigma, \psi^{-1})$, and any $\phi \in \mathcal{S}(\mathbb{F}^n)$ with $\phi(0) = 0$, we have*

$$Z(\tilde{W}, \tilde{W}', \mathcal{F}_\psi \phi; \psi^{-1}) = \gamma(\pi \times \sigma, \psi)Z(W, W', \phi; \psi).$$

Similarly to the case $n > m$, we have an expression of $\gamma(\pi \times \sigma, \psi)$ in terms of the Bessel functions of π and σ.

Proposition 2.9 [Ye 2019, Equation (4.4)]. *Let π and σ be irreducible cuspidal representations of GL$_n(\mathbb{F})$. Then*

$$\gamma(\pi \times \sigma, \psi) = \sum_{g \in Z_n \backslash \mathrm{GL}_n(\mathbb{F})} \mathcal{J}_{\pi,\psi}(g)\mathcal{J}_{\sigma,\psi^{-1}}(g)\psi(\langle e_n g^{-1}, e_1 \rangle),$$

where $e_1 = (1, 0, \ldots, 0) \in \mathbb{F}^n$.

It follows from Propositions 2.9 and 2.3 that

$$\gamma(\pi^\vee \times \sigma^\vee, \psi^{-1}) = \overline{\gamma(\pi \times \sigma, \psi)}.$$

We now move to discuss the absolute value of $\gamma(\pi \times \sigma, \psi)$. In order to do that, we first explain how to extend the functional equation in Theorem 2.8 to all functions in $\mathcal{S}(\mathbb{F}^n)$ for most cases. To begin, we notice that for the indicator function of $0 \in \mathbb{F}^n$, which we denote δ_0, we have that $Z(W, W', \delta_0; \psi) = 0$. We also notice that if π is not isomorphic to $\sigma^\vee$, then $Z(W, W', 1; \psi) = 0$, where 1 represents the constant function. This is because

$$Z(W, W', 1; \psi) = \sum_{g \in Z_n \backslash \mathrm{GL}_n(\mathbb{F})} W(g)W'(g)$$

defines a $\mathrm{GL}_n(\mathbb{F})$-invariant pairing $\mathcal{W}(\pi, \psi) \otimes \mathcal{W}(\sigma, \psi^{-1}) \to \mathbb{C}$, but such nontrivial pairing exists only when π is isomorphic to $\sigma^\vee$. These two observations imply the following extension of the functional equation, in the special case where π is not isomorphic to $\sigma^\vee$.

Proposition 2.10. *Suppose that $\pi \ncong \sigma^\vee$. Then for any $\phi \in \mathcal{S}(\mathbb{F}^n)$ we have*

$$Z(\tilde{W}, \tilde{W}', \mathcal{F}_\psi \phi; \psi^{-1}) = \gamma(\pi \times \sigma, \psi)Z(W, W', \phi; \psi).$$

Proof. Write $\phi = \phi_0 + \phi_1$, where $\phi_0 = \phi - \phi(0)$ and $\phi_1 = \phi(0)$. Then $\phi_0(0) = 0$ and $\mathcal{F}_\psi \phi_1 = q^n \phi(0) \delta_0$. Since Z is linear in ϕ, we have from the discussion above that

$$Z(W, W', \phi; \psi) = Z(W, W', \phi_0; \psi)$$

and that

$$Z(\tilde{W}, \tilde{W}', \mathcal{F}_\psi \phi; \psi^{-1}) = Z(\tilde{W}, \tilde{W}', \mathcal{F}_\psi \phi_0; \psi^{-1}).$$

The statement now follows from Theorem 2.8. $\square$

As a result, we get the following corollary regarding the absolute value of $\gamma(\pi \times \sigma, \psi)$.

Corollary 2.11. *Let π and σ be irreducible cuspidal representations of $\mathrm{GL}_n(\mathbb{F})$ such that $\pi \ncong \sigma^\vee$. Then*

$$\gamma(\pi \times \sigma, \psi)\gamma(\pi^\vee \times \sigma^\vee, \psi^{-1}) = q^n,$$

and therefore

$$|\gamma(\pi \times \sigma, \psi)| = q^{n/2}.$$

Proof. This follows by applying Proposition 2.10 twice, and from the fact that the Fourier transform satisfies

$$\mathcal{F}_{\psi^{-1}} \mathcal{F}_\psi \phi = q^n \phi,$$

for any $\phi \in \mathcal{S}(\mathbb{F}^n)$. $\square$

We are left to deal with the case $\pi \cong \sigma^\vee$. In this case, the gamma factor $\gamma(\pi \times \pi^\vee, \psi)$ can be computed explicitly and it equals -1; see the Appendix.

We summarize all cases in the following proposition.

Proposition 2.12. *Let π and σ be irreducible cuspidal representations of* GL$_n(\mathbb{F})$:

- *If $\pi \not\cong \sigma^\vee$ then $|\gamma(\pi \times \sigma, \psi)| = q^{n/2}$.*
- *If $\pi \cong \sigma^\vee$ then $|\gamma(\pi \times \sigma, \psi)| = 1$.*

3. Shahidi gamma factors (local coefficients)

In this section, we use the Langlands–Shahidi method in order to define a gamma factor for two representations of Whittaker type of finite general linear groups. This is the finite field analog of Shahidi's local coefficient, which uses an intertwining operator. The treatment in Sections 3A–3C is a finite field analog of Shahidi's work on local coefficients over local fields [Shahidi 1984]. Unlike the Jacquet–Piatetski-Shapiro–Shalika gamma factors discussed in Section 2C, the Shahidi gamma factor can be defined uniformly for all irreducible generic representations of GL$_n(\mathbb{F})$ and GL$_m(\mathbb{F})$, regardless of $n > m$ or whether the representations are cuspidal. We prove properties of the Shahidi gamma factor, where the most important one is the multiplicativity property, which explains how this gamma factor behaves under parabolic induction. We end this section by expressing the Shahidi gamma factor in terms of the Bessel functions associated with the representations, and showing its relation to the Jacquet–Piatetski-Shapiro–Shalika gamma factor.

3A. *The intertwining operator.* Let n and m be positive integers and let π and σ be representations of GL$_n(\mathbb{F})$ and GL$_m(\mathbb{F})$, respectively. We define a linear map $\sigma \otimes \pi \to \pi \otimes \sigma$ acting on pure tensors by component swap:

$$\mathrm{sw}_{\sigma,\pi}(v_\sigma \otimes v_\pi) = v_\pi \otimes v_\sigma.$$

For a function $f\colon \mathrm{GL}_{n+m}(\mathbb{F}) \to \sigma \otimes \pi$, we denote by $\tilde{f}\colon \mathrm{GL}_{n+m}(\mathbb{F}) \to \pi \otimes \sigma$ the function $\tilde{f}(g) = \mathrm{sw}_{\sigma,\pi}(f(g))$.

We consider the following intertwining operator $T_{\sigma,\pi}\colon \sigma \circ \pi \to \pi \circ \sigma$, defined for $f \in \sigma \circ \pi$ and $g \in \mathrm{GL}_{n+m}(\mathbb{F})$ by the formula

$$T_{\sigma,\pi} f(g) = \sum_{p \in P_{n,m}} (\pi \overline{\otimes} \sigma)(p^{-1}) \tilde{f}(\hat{w}_{n,m} pg),$$

where $\hat{w}_{n,m}$ is the following Weyl element

$$\hat{w}_{n,m} = \begin{pmatrix} & I_m \\ I_n & \end{pmatrix}.$$

Using the decomposition $P_{n,m} = D_{n,m} \ltimes N_{n,m}$, we may write every $p \in P_{n,m}$ in a unique way $p = du$, where $u \in N_{n,m}$ and $d = \operatorname{diag}(g_1, g_2) \in D_{n,m}$, with $g_1 \in \mathrm{GL}_n(\mathbb{F})$ and $g_2 \in \mathrm{GL}_m(\mathbb{F})$. It follows from the identity

$$\operatorname{diag}(g_2, g_1)\hat{w}_{n,m} = \hat{w}_{n,m} \operatorname{diag}(g_1, g_2),$$

and from the left $D_{m,n}$-equivariance property of f that

$$T_{\sigma,\pi} f(g) = |D_{n,m}| \cdot U_{\sigma,\pi} f(g),$$

where

$$U_{\sigma,\pi} f(g) = \sum_{u \in N_{n,m}} \tilde{f}(\hat{w}_{n,m} u g).$$

By construction, we have that $T_{\sigma,\pi}$ and $U_{\sigma,\pi}$ are nonzero elements of the space

$$\operatorname{Hom}_{\mathrm{GL}_{n+m}(\mathbb{F})}(\sigma \circ \pi, \pi \circ \sigma).$$

3B. *The Shahidi gamma factor.* Suppose now that π and σ are representations of Whittaker type of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$, respectively. By Theorem 2.2 we have that the parabolically induced representations $\sigma \circ \pi$ and $\pi \circ \sigma$ are also of Whittaker type. Let $v_{\sigma,\psi} \in \sigma$ and $v_{\pi,\psi} \in \pi$ be nonzero ψ-Whittaker vectors for σ and π, respectively. We may define a nonzero ψ-Whittaker vector $f_{v_{\sigma,\psi}, v_{\pi,\psi}}$ for $\sigma \circ \pi$ by the formula

$$f_{v_{\sigma,\psi}, v_{\pi,\psi}}(g) = \begin{cases} \psi(u)(\sigma \overline{\otimes} \pi)(p) v_{\sigma,\psi} \otimes v_{\pi,\psi} & g = p\hat{w}_{n,m} u, \ p \in P_{m,n}, \ u \in Z_{n+m}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we may define $f_{v_{\pi,\psi}, v_{\sigma,\psi}} \in \pi \circ \sigma$.

Since $U_{\sigma,\pi}$ is an intertwining operator, we have that $U_{\sigma,\pi} f_{v_{\sigma,\psi}, v_{\pi,\psi}}$ is a ψ-Whittaker vector of $\pi \circ \sigma$. Since $f_{v_{\pi,\psi}, v_{\sigma,\psi}}$ is the unique nonzero ψ-Whittaker vector of $\pi \circ \sigma$ up to scalar, we must have that

$$U_{\sigma,\pi} f_{v_{\sigma,\psi}, v_{\pi,\psi}} = \gamma \cdot f_{v_{\pi,\psi}, v_{\sigma,\psi}},$$

where $\gamma \in \mathbb{C}$. It is easy to check that this number γ does not depend on the choice of ψ-Whittaker vectors $v_{\sigma,\psi}$ and $v_{\pi,\psi}$.

In order to ease the notation, we denote $v_{\sigma,\pi,\psi} = f_{v_{\sigma,\psi}, v_{\pi,\psi}}$, where we suppress $v_{\sigma,\psi}$ and $v_{\pi,\psi}$ from the notation. Similarly, we denote $v_{\pi,\sigma,\psi} = f_{v_{\pi,\psi}, v_{\sigma,\psi}}$.

Definition 3.1. The Shahidi gamma factor of π and σ with respect to ψ is the unique number $\Gamma(\pi \times \sigma, \psi) \in \mathbb{C}$, such that

$$U_{\sigma,\pi} v_{\sigma,\pi,\psi} = \Gamma(\pi \times \sigma, \psi) \cdot v_{\pi,\sigma,\psi}.$$

Remark 3.2. If $\pi \circ \sigma$ is irreducible, then so is $\sigma \circ \pi$, and since $U_{\sigma,\pi}$ is a nonzero intertwining operator, it is an isomorphism and $\Gamma(\pi \times \sigma, \psi)$ must be nonzero.

However, in the general case it is not obvious at this point that $\Gamma(\pi \times \sigma, \psi)$ is nonzero. We will show this later.

Remark 3.3. As in Remark 2.4, we may choose invariant inner products $(\cdot, \cdot)_\pi$ and $(\cdot, \cdot)_\sigma$ on π and σ, respectively. We then have a natural inner product $(\cdot, \cdot)_{\sigma \otimes \pi}$ on $\sigma \otimes \pi$, which defines an inner product on $\sigma \circ \pi$ by the formula

$$(f_1, f_2)_{\sigma \circ \pi} = \sum_{g \in P_{m,n} \backslash \mathrm{GL}_{n+m}(\mathbb{F})} (f_1(g), f_2(g))_{\sigma \otimes \pi}.$$

Using this inner product, the Whittaker functional $\ell_{\sigma \circ \pi, \psi}(f) = (f, v_{\sigma, \pi, \psi})_{\sigma \circ \pi}$ is related to the Whittaker functionals $\ell_{\sigma, \psi}(v_\sigma) = (v_\sigma, v_{\sigma, \psi})_\sigma$ and $\ell_{\pi, \psi}(v_\pi) = (v_\pi, v_{\pi, \psi})_\pi$ by the formula

$$\ell_{\sigma \circ \pi, \psi}(f) = \sum_{u \in N_{n,m}} \ell_{\sigma, \psi} \otimes \ell_{\pi, \psi}(f(\hat{w}_{n,m} u)) \psi^{-1}(u).$$

Similarly, by exchanging the roles of π and σ, we have that Whittaker functional $\ell_{\pi \circ \sigma, \psi}$ is given by a similar formula. Using the definitions of the inner products, and the fact that elements in $\pi \circ \sigma$ are left invariant under $N_{n,m}$, we see that $U_{\pi, \sigma}$ is the adjoint of $U_{\sigma, \pi}$, with respect to our choice of inner products. Using the relation between $v_{\sigma, \pi, \psi}$ and $v_{\pi, \sigma, \psi}$, we obtain the following relation

$$\ell_{\sigma \circ \pi, \psi} \circ U_{\pi, \sigma} = \Gamma(\pi \times \sigma, \psi) \cdot \ell_{\pi \circ \sigma, \psi}.$$

This is how the Shahidi gamma factor is usually defined in the literature.

3B1. *Dependence on ψ.* For any $a \in \mathbb{F}^*$, let $\psi_a : \mathbb{F} \to \mathbb{C}^*$ be the additive character

$$\psi_a(x) = \psi(ax).$$

It is well known that all nontrivial additive characters of $\mathbb{F}$ are of the form ψ_a for some $a \in \mathbb{F}^*$. In this section, we give a relation between $\Gamma(\pi \times \sigma, \psi)$ and $\Gamma(\pi \times \sigma, \psi_a)$.

Let $a \in \mathbb{F}^*$. Suppose that τ is a generic representation of $\mathrm{GL}_k(\mathbb{F})$ with a nonzero ψ-Whittaker vector $v_{\tau, \psi}$. Let

$$d_k = \mathrm{diag}(1, a, a^2, \ldots, a^{k-1}).$$

Then we have that $\tau(d_k)v_{\tau, \psi}$ is a nonzero ψ_a-Whittaker vector of τ. The map $v \mapsto \tau(d_k)v$ is a linear isomorphism from the subspace spanned by the ψ-Whittaker vectors of τ to the subspace spanned by the ψ_a-Whittaker vectors of τ. In particular, if $v_{\tau, \psi}$ is the unique (up to scalar multiplication) ψ-Whittaker vector of τ, then $\tau(d_k)v_{\tau, \psi}$ is the unique (up to scalar multiplication) ψ_a-Whittaker vector of τ.

Let π and σ be representations of Whittaker type of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$, respectively. Let $v_{\pi, \psi} \in \pi$ and $v_{\sigma, \psi} \in \sigma$ be nonzero ψ-Whittaker vectors. Assume

that π and σ have central characters, and denote them by ω_π and ω_σ, respectively. Let

$$v_{\sigma,\pi,\psi_a} = f_{\sigma(d_m)v_{\sigma,\psi},\pi(d_n)v_{\pi,\psi}}.$$

Similarly, we define v_{π,σ,ψ_a}.

We first express v_{σ,π,ψ_a} in terms of $v_{\sigma,\pi,\psi}$. We will use this relation later to show a relation between the gamma factors $\Gamma(\pi \times \sigma, \psi_a)$ and $\Gamma(\pi \times \sigma, \psi)$.

Proposition 3.4. *We have*

$$v_{\sigma,\pi,\psi_a} = \omega_\sigma(a)^{-n} \rho(d_{n+m}) v_{\sigma,\pi,\psi},$$

where $\rho(d_{n+m})$ denotes right translation by d_{n+m}.

Proof. Let $f = \omega_\sigma(a)^{-n} \rho(d_{n+m}) v_{\sigma,\pi,\psi} \in \sigma \circ \pi$. By the discussion above, f is a ψ_a-Whittaker vector of $\sigma \circ \pi$.

We have that

$$f(\hat{w}_{n,m}) = \omega_\sigma(a)^{-n} v_{\sigma,\pi,\psi}(\hat{w}_{n,m} d_{n+m}).$$

Writing $d_{n+m} = \mathrm{diag}(d_n, a^n d_m)$, we have $\hat{w}_{n,m} d_{n+m} = \mathrm{diag}(a^n d_m, d_n) \hat{w}_{n,m}$, and hence

$$f(\hat{w}_{n,m}) = (\sigma(d_m) \otimes \pi(d_n)) v_{\sigma,\pi,\psi}(\hat{w}_{n,m}) = \sigma(d_m) v_{\sigma,\psi} \otimes \pi(d_n) v_{\pi,\psi}.$$

This shows that $f = v_{\sigma,\pi,\psi_a}$, as both are ψ_a-Whittaker vectors in $\sigma \circ \pi$, and both agree at the point $\hat{w}_{n,m}$. $\qquad\square$

Theorem 3.5. *We have*

$$\Gamma(\pi \times \sigma, \psi_a) = \omega_\pi(a)^m \cdot \omega_\sigma(a)^{-n} \cdot \Gamma(\pi \times \sigma, \psi).$$

Proof. By definition, we have that

$$\Gamma(\pi \times \sigma, \psi_a) v_{\pi,\sigma,\psi_a} = U_{\sigma,\pi} v_{\sigma,\pi,\psi_a}.$$

By Proposition 3.4,

$$\Gamma(\pi \times \sigma, \psi_a) \omega_\pi(a)^{-m} \rho(d_{n+m}) v_{\pi,\sigma,\psi} = \omega_\sigma(a)^{-n} U_{\sigma,\pi} \rho(d_{n+m}) v_{\sigma,\pi,\psi}.$$

Therefore, we get that

$$\Gamma(\pi \times \sigma, \psi_a) \omega_\sigma(a)^n \omega_\pi(a)^{-m} v_{\pi,\sigma,\psi} = U_{\sigma,\pi} v_{\sigma,\pi,\psi},$$

which implies that

$$\Gamma(\pi \times \sigma, \psi_a) \omega_\sigma(a)^n \omega_\pi(a)^{-m} = \Gamma(\pi \times \sigma, \psi),$$

as required. $\qquad\square$

3B2. *Relation between* $\Gamma(\pi \times \sigma, \psi)$ *and* $\Gamma(\sigma^\vee \times \pi^\vee, \psi^{-1})$. In this section, we analyze the relation between $\Gamma(\pi \times \sigma, \psi)$ and $\Gamma(\sigma^\vee \times \pi^\vee, \psi)$.

Recall that for a finite dimensional representation τ of GL$_k(\mathbb{F})$, we have that $\tau^\vee \cong \tau^\iota$. See Section 2B1. If $v_{\tau,\psi}$ is a nonzero ψ-Whittaker vector for τ, then $\tau(w_k)v_{\tau,\psi}$ is a nonzero ψ^{-1}-Whittaker vector for τ^ι.

We have that

$$\pi^\iota \circ \sigma^\iota \cong (\sigma \circ \pi)^\iota$$

by the isomorphism $S_{\sigma,\pi} : (\sigma \circ \pi)^\iota \cong \pi^\iota \circ \sigma^\iota$ that sends $f \in (\sigma \circ \pi)^\iota$ to the function

$$(S_{\sigma,\pi} f)(g) = \tilde{f}(\hat{w}_{n,m} g^\iota).$$

Let

$$v_{\pi^\iota,\sigma^\iota,\psi^{-1}} = f_{\pi(w_n)v_{\pi,\psi},\sigma(w_m)v_{\sigma,\psi}} \in \pi^\iota \circ \sigma^\iota.$$

Then $v_{\pi^\iota,\sigma^\iota,\psi^{-1}}$ is a nonzero ψ^{-1}-Whittaker vector of $\pi^\iota \circ \sigma^\iota$. On the other hand, by the discussion above, a nonzero ψ^{-1}-Whittaker vector of $(\sigma \circ \pi)^\iota$ is given by $\rho(w_{m+n})v_{\sigma,\pi,\psi}$, where $\rho(w_{m+n})$ represents right translation by w_{m+n}. Therefore $S_{\sigma,\pi}\rho(w_{m+n})v_{\sigma,\pi,\psi}$ is another nonzero ψ^{-1}-Whittaker vector of $\pi^\iota \circ \sigma^\iota$.

Proposition 3.6. *We have*

$$v_{\pi^\iota,\sigma^\iota,\psi^{-1}} = S_{\sigma,\pi}\rho(w_{m+n})v_{\sigma,\pi,\psi}. \tag{3-1}$$

Proof. We have that

$$S_{\sigma,\pi}\rho(w_{m+n})v_{\sigma,\pi,\psi}(\hat{w}_{m,n}) = \mathrm{sw}_{\sigma,\pi}\, v_{\sigma,\pi,\psi}(w_{m+n}) = \pi(w_n)v_{\pi,\psi} \otimes \sigma(w_m)v_{\sigma,\psi},$$

where in the last step we used the fact that $\mathrm{diag}(w_m, w_n)\hat{w}_{n,m} = w_{n+m}$.

Since $S_{\sigma,\pi}\rho(w_{m+n})v_{\sigma,\pi,\psi}$ and $v_{\pi^\iota,\sigma^\iota,\psi^{-1}}$ are both ψ^{-1}-Whittaker vectors for the representation of Whittaker type $\pi^\iota \circ \sigma^\iota$, and they both agree at the point $\hat{w}_{m,n}$, they are equal. $\square$

Similarly, let

$$v_{\sigma^\iota,\pi^\iota,\psi^{-1}} = f_{\sigma(w_m)v_{\sigma,\psi},\pi(w_n)v_{\pi,\psi}} \in \sigma^\iota \circ \pi^\iota.$$

Using Proposition 3.6 with roles of the representations π and σ exchanged, we have

$$v_{\sigma^\iota,\pi^\iota,\psi^{-1}} = S_{\pi,\sigma}\rho(w_{m+n})v_{\pi,\sigma,\psi}. \tag{3-2}$$

Theorem 3.7. *Let π and σ be representations of Whittaker type of* GL$_n(\mathbb{F})$ *and* GL$_m(\mathbb{F})$, *respectively. Then*

$$\Gamma(\pi \times \sigma, \psi) = \Gamma(\sigma^\vee \times \pi^\vee, \psi^{-1}).$$

Proof. By definition,

$$U_{\pi^\iota,\sigma^\iota} v_{\pi^\iota,\sigma^\iota,\psi^{-1}} = \Gamma(\sigma^\iota \times \pi^\iota, \psi^{-1}) \cdot v_{\sigma^\iota,\pi^\iota,\psi^{-1}}. \tag{3-3}$$

Substituting (3-1) and (3-2) in (3-3), we get

$$U_{\pi^\iota,\sigma^\iota} S_{\sigma,\pi} \rho(w_{m+n}) v_{\sigma,\pi,\psi} = \Gamma(\sigma^\iota \times \pi^\iota, \psi^{-1}) \cdot S_{\pi,\sigma} \rho(w_{m+n}) v_{\pi,\sigma,\psi}.$$

A simple computation shows that

$$U_{\pi^\iota,\sigma^\iota} \circ S_{\sigma,\pi} = S_{\pi,\sigma} \circ U_{\sigma,\pi}.$$

Hence, we get that

$$S_{\pi,\sigma} \rho(w_{m+n}) U_{\sigma,\pi} v_{\sigma,\pi,\psi} = \Gamma(\sigma^\iota \times \pi^\iota, \psi^{-1}) \cdot S_{\pi,\sigma} \rho(w_{m+n}) v_{\pi,\sigma,\psi},$$

which implies that

$$U_{\sigma,\pi} v_{\sigma,\pi,\psi} = \Gamma(\sigma^\iota \times \pi^\iota, \psi^{-1}) \cdot v_{\pi,\sigma,\psi}.$$

Therefore, we must have

$$\Gamma(\sigma^\iota \times \pi^\iota, \psi^{-1}) = \Gamma(\pi \times \sigma, \psi),$$

and the statement in the theorem follows, since $\sigma^\iota \cong \sigma^\vee$ and $\pi^\iota \cong \pi^\vee$. $\qquad \square$

Combining Theorem 3.7 with Theorem 3.5, we get the following corollary.

Corollary 3.8. *Let π and σ be representations of Whittaker type of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$, respectively. Assume that both π and σ have central characters, and denote them by ω_π and ω_σ, respectively. Then*

$$\Gamma(\pi \times \sigma, \psi) = \Gamma(\sigma^\vee \times \pi^\vee, \psi) \cdot \omega_\pi(-1)^m \omega_\sigma(-1)^n.$$

3C. *Multiplicativity of Gamma factors.* In this section, we show that $\Gamma(\pi \times \sigma, \psi)$ is multiplicative.

Let π be a representation of Whittaker type of $\mathrm{GL}_n(\mathbb{F})$. Let $m = m_1 + m_2$, and let σ_1 and σ_2 be representations of Whittaker type of $\mathrm{GL}_{m_1}(\mathbb{F})$ and $\mathrm{GL}_{m_2}(\mathbb{F})$, respectively. By Theorem 2.2, the parabolic induction $\sigma_1 \circ \sigma_2$ is also a representation of Whittaker type. Hence, the gamma factor $\Gamma(\pi \times (\sigma_1 \circ \sigma_2), \psi)$ is well defined. We will show the following theorem.

Theorem 3.9. $\qquad \Gamma(\pi \times (\sigma_1 \circ \sigma_2), \psi) = \Gamma(\pi_1 \times \sigma_1, \psi) \cdot \Gamma(\pi_2 \times \sigma_2, \psi).$

The proof of this theorem will occupy the remaining subsections of this section.

Remark 3.10. Let $\sigma \subset \sigma_1 \circ \sigma_2$ be the unique irreducible generic subrepresentation of $\sigma_1 \circ \sigma_2$. We have that ψ-Whittaker vectors of σ are the same as ψ-Whittaker vectors of $\sigma_1 \circ \sigma_2$. Hence, Theorem 3.9 implies that

$$\Gamma(\pi \times \sigma, \psi) = \Gamma(\pi \times \sigma_1, \psi) \cdot \Gamma(\pi \times \sigma_2, \psi).$$

Before proving the theorem, we mention two other multiplicative properties that follow immediately from the theorem.

The first property is that the gamma factor is also multiplicative in the first variable. This follows from Theorem 3.9 combined with Theorem 3.7.

Corollary 3.11. *Let π_1 and π_2 be representations of Whittaker type of $\mathrm{GL}_{n_1}(\mathbb{F})$ and $\mathrm{GL}_{n_2}(\mathbb{F})$, respectively, and let σ be a representation of Whittaker type of $\mathrm{GL}_m(\mathbb{F})$. Then*

$$\Gamma((\pi_1 \circ \pi_2) \times \sigma, \psi) = \Gamma(\pi_1 \times \sigma, \psi) \cdot \Gamma(\pi_2 \times \sigma, \psi).$$

The second corollary allows us to express the gamma factor of two parabolically induced representations as the product of the gamma factors of the components of the parabolic induction. It follows by repeatedly using multiplicativity in both variables.

Corollary 3.12. *Let $\pi_1, \ldots, \pi_r$ and $\sigma_1, \ldots, \sigma_t$ be irreducible generic representations of $\mathrm{GL}_{n_1}(\mathbb{F}), \ldots, \mathrm{GL}_{n_r}(\mathbb{F})$ and $\mathrm{GL}_{m_1}(\mathbb{F}), \ldots, \mathrm{GL}_{m_t}(\mathbb{F})$, respectively. Then*:

(1) *We have*

$$\Gamma((\pi_1 \circ \ldots \circ \pi_r) \times (\sigma_1 \circ \ldots \circ \sigma_t), \psi) = \prod_{i=1}^{r} \prod_{j=1}^{t} \Gamma(\pi_i \times \sigma_j, \psi).$$

(2) *If π is the unique irreducible generic subrepresentation of $\pi_1 \circ \ldots \circ \pi_r$ and σ is the unique irreducible generic subrepresentation of $\sigma_1 \circ \ldots \circ \sigma_t$, then*

$$\Gamma(\pi \times \sigma, \psi) = \prod_{i=1}^{r} \prod_{j=1}^{t} \Gamma(\pi_i \times \sigma_j, \psi).$$

In the next subsections we make preparations for the proof of Theorem 3.9.

3C1. *Transitivity of parabolic induction.* Let τ_1, τ_2 and τ_3 be finite dimensional representations of $\mathrm{GL}_{n_1}(\mathbb{F})$, $\mathrm{GL}_{n_2}(\mathbb{F})$ and $\mathrm{GL}_{n_3}(\mathbb{F})$, respectively.

We realize elements in $(\tau_1 \circ \tau_2) \otimes \tau_3$ as functions $\mathrm{GL}_{n_1+n_2}(\mathbb{F}) \to \tau_1 \otimes \tau_2 \otimes \tau_3$ in the obvious way. Similarly, we realize elements in $\tau_1 \otimes (\tau_2 \circ \tau_3)$ as functions $\mathrm{GL}_{n_2+n_3} \to \tau_1 \otimes \tau_2 \otimes \tau_3$ in the obvious way.

Consider the space $(\tau_1 \circ \tau_2) \circ \tau_3$. We will regard elements of this space as functions

$$f: \ \mathrm{GL}_{n_1+n_2+n_3}(\mathbb{F}) \times \mathrm{GL}_{n_1+n_2}(\mathbb{F}) \to \tau_1 \otimes \tau_2 \otimes \tau_3,$$

where $f(g; h)$ means evaluating f at $g \in \mathrm{GL}_{n_1+n_2+n_3}(\mathbb{F})$ and then evaluating the resulting function at $h \in \mathrm{GL}_{n_1+n_2}(\mathbb{F})$. We will similarly regard elements of $\tau_1 \circ (\tau_2 \circ \tau_3)$ as functions

$$f: \ \mathrm{GL}_{n_1+n_2+n_3}(\mathbb{F}) \times \mathrm{GL}_{n_2+n_3}(\mathbb{F}) \to \tau_1 \otimes \tau_2 \otimes \tau_3.$$

We have an isomorphism of representations

$$L_{\tau_1,\tau_2;\tau_3} \colon (\tau_1 \circ \tau_2) \circ \tau_3 \to \tau_1 \circ \tau_2 \circ \tau_3,$$

given by mapping a function $f \in (\tau_1 \circ \tau_2) \circ \tau_3$ to

$$L_{\tau_1,\tau_2;\tau_3} f(g) = f(g; I_{n_1+n_2}),$$

where $g \in \mathrm{GL}_{n_1+n_2+n_3}(\mathbb{F})$.

Similarly, we have an isomorphism of representations

$$L_{\tau_1;\tau_2,\tau_3} \colon \tau_1 \circ (\tau_2 \circ \tau_3) \to \tau_1 \circ \tau_2 \circ \tau_3,$$

given by mapping a function $f \in \tau_1 \circ (\tau_2 \circ \tau_3)$ to

$$L_{\tau_1;\tau_2,\tau_3} f(g) = f(g; I_{n_2+n_3}),$$

where again $g \in \mathrm{GL}_{n_1+n_2+n_3}(\mathbb{F})$.

Assume now that τ_1, τ_2 and τ_3 have nonzero ψ-Whittaker vectors, $v_{\tau_1,\psi}$, $v_{\tau_2,\psi}$ and $v_{\tau_3,\psi}$, respectively, and assume that up to scalar multiplication, these Whittaker vectors are unique. We denote, as before, the following nonzero ψ-Whittaker vectors $v_{\tau_1,\tau_2,\psi} = f_{v_{\tau_1,\psi},v_{\tau_2,\psi}} \in \tau_1 \circ \tau_2$ and $v_{\tau_2,\tau_3,\psi} = f_{v_{\tau_2,\psi},v_{\tau_3,\psi}} \in \tau_2 \circ \tau_3$. We also define the following nonzero ψ-Whittaker vectors $v_{\tau_1,\tau_2\circ\tau_3,\psi} = f_{v_{\tau_1,\psi},v_{\tau_2,\tau_3,\psi}} \in \tau_1 \circ (\tau_2 \circ \tau_3)$ and $v_{\tau_1\circ\tau_2,\tau_3,\psi} = f_{v_{\tau_1,\tau_2,\psi},v_{\tau_3,\psi}} \in (\tau_1 \circ \tau_2) \circ \tau_3$. Finally, we define

$$v_{\tau_1,\tau_2,\tau_3,\psi} = L_{\tau_1;\tau_2,\tau_3} v_{\tau_1,\tau_2\circ\tau_3,\psi} = L_{\tau_1,\tau_2;\tau_3} v_{\tau_1\circ\tau_2,\tau_3,\psi} \in \tau_1 \circ \tau_2 \circ \tau_3.$$

Then $v_{\tau_1,\tau_2,\tau_3,\psi}$ is the ψ-Whittaker vector in $\tau_1 \circ \tau_2 \circ \tau_3$ supported on the double coset $P_{n_1,n_2,n_3} \hat{w}_{n_3,n_2,n_1} Z_{n_1+n_2+n_3}$, with $v_{\tau_1,\tau_2,\tau_3,\psi}(\hat{w}_{n_3,n_2,n_1}) = v_{\tau_1,\psi} \otimes v_{\tau_2,\psi} \otimes v_{\tau_3,\psi}$, where

$$\hat{w}_{n_3,n_2,n_1} = \begin{pmatrix} & & I_{n_1} \\ & I_{n_2} & \\ I_{n_3} & & \end{pmatrix}.$$

3C2. *Intertwining operators.* We return to the notations of the beginning of this section. Let π be a representation of Whittaker type of $\mathrm{GL}_n(\mathbb{F})$. Let $m = m_1 + m_2$, and let σ_1 and σ_2 be representations of Whittaker type of $\mathrm{GL}_{m_1}(\mathbb{F})$ and $\mathrm{GL}_{m_2}(\mathbb{F})$, respectively.

Using the isomorphisms from the previous section, we obtain maps such that the following diagrams are commutative:

$$
\begin{CD}
\sigma_1 \circ (\sigma_2 \circ \pi) @>{\mathrm{id}_{\sigma_1} \otimes U_{\sigma_2,\pi}}>> \sigma_1 \circ (\pi \circ \sigma_2) \\
@V{L_{\sigma_1;\sigma_2,\pi}}VV @VV{L_{\sigma_1;\pi,\sigma_2}}V \\
\sigma_1 \circ \sigma_2 \circ \pi @>{\tilde{U}_{\sigma_2,\pi}}>> \sigma_1 \circ \pi \circ \sigma_2
\end{CD}
\tag{3-4}
$$

$$
\begin{CD}
(\sigma_1 \circ \pi) \circ \sigma_2 @>{U_{\sigma_1,\pi} \otimes \mathrm{id}_{\sigma_2}}>> (\pi \circ \sigma_1) \circ \sigma_2 \\
@V{L_{\sigma_1,\pi;\sigma_2}}VV @VV{L_{\pi,\sigma_1;\sigma_2}}V \\
\sigma_1 \circ \pi \circ \sigma_2 @>{\tilde{U}_{\sigma_1,\pi}}>> \pi \circ \sigma_1 \circ \sigma_2
\end{CD}
\tag{3-5}
$$

$$
\begin{CD}
(\sigma_1 \circ \sigma_2) \circ \pi @>{U_{\sigma_1 \circ \sigma_2,\pi}}>> \pi \circ (\sigma_1 \circ \sigma_2) \\
@V{L_{\sigma_1,\sigma_2;\pi}}VV @VV{L_{\pi;\sigma_1,\sigma_2}}V \\
\sigma_1 \circ \sigma_2 \circ \pi @>{\tilde{U}_{\sigma_1 \circ \sigma_2,\pi}}>> \pi \circ \sigma_1 \circ \sigma_2
\end{CD}
\tag{3-6}
$$

Let us explain these diagrams. We begin with explaining (3-4). The map $\mathrm{id}_{\sigma_1} \otimes U_{\sigma_2,\pi} \colon \sigma_1 \otimes (\sigma_2 \circ \pi) \to \sigma_1 \otimes (\pi \circ \sigma_2)$ is a homomorphism of representations. It defines a homomorphism $\sigma_1 \circ (\sigma_2 \circ \pi) \to \sigma_1 \circ (\pi \circ \sigma_2)$, which we keep denoting by $\mathrm{id}_{\sigma_1} \otimes U_{\sigma_2,\pi}$. By unwrapping the definitions, we see that the map $\tilde{U}_{\sigma_2,\pi} \colon \sigma_1 \circ \sigma_2 \circ \pi \to \sigma_1 \circ \pi \circ \sigma_2$ is given by the formula

$$
\tilde{U}_{\sigma_2,\pi}(f)(g) = \sum_{u_{n,m_2} \in N_{n,m_2}} \mathrm{sw}_{\sigma_2,\pi} \, f\left(\begin{pmatrix} I_{m_1} & & \\ & I_{m_2} & \\ & & I_n \end{pmatrix} \begin{pmatrix} I_{m_1} & & \\ & & \\ & u_{n,m_2} \end{pmatrix} g \right).
\tag{3-7}
$$

The commutative diagram (3-5) is similar. We get by unwrapping the definitions that the map $\tilde{U}_{\sigma_1,\pi} \colon \sigma_1 \circ \pi \circ \sigma_2 \to \pi \circ \sigma_1 \circ \sigma_2$ is given by

$$
\tilde{U}_{\sigma_1,\pi}(f)(g) = \sum_{u_{n,m_1} \in N_{n,m_1}} \mathrm{sw}_{\sigma_1,\pi} \, f\left(\begin{pmatrix} & I_{m_1} & \\ I_n & & \\ & & I_{m_2} \end{pmatrix} \begin{pmatrix} u_{n,m_1} & & \\ & & I_{m_2} \end{pmatrix} g \right).
\tag{3-8}
$$

Finally, in the diagram (3-6), we have that $\tilde{U}_{\sigma_1 \circ \sigma_2,\pi} \colon \sigma_1 \circ \sigma_2 \circ \pi \to \pi \circ \sigma_1 \circ \sigma_2$ is given by

$$
\tilde{U}_{\sigma_1 \circ \sigma_2,\pi}(f)(g) = \sum_{u_{n,m} \in N_{n,m}} \tilde{f}\left(\begin{pmatrix} & I_{m_1} & \\ & & I_{m_2} \\ I_n & & \end{pmatrix} u_{n,m} g \right),
\tag{3-9}
$$

where for $g \in \mathrm{GL}_{n+m}(\mathbb{F})$, we mean $\tilde{f}(g) = \mathrm{sw}_{\sigma_1,\pi} \, \mathrm{sw}_{\sigma_2,\pi} \, f(g)$.

Proposition 3.13. *We have*

$$\tilde{U}_{\sigma_1 \circ \sigma_2, \pi} = \tilde{U}_{\sigma_1, \pi} \circ \tilde{U}_{\sigma_2, \pi}.$$

Proof. Let $f \in \sigma_1 \circ \sigma_2 \circ \pi$ and $g \in \mathrm{GL}_{n+m}(\mathbb{F})$. Then by (3-7) and (3-8),

$$(\tilde{U}_{\sigma_1,\pi} \circ \tilde{U}_{\sigma_2,\pi} f)(g) = \sum_{X \in M_{n \times m_1}(\mathbb{F})} \sum_{Y \in M_{n \times m_2}(\mathbb{F})} \tilde{f} \left(\begin{pmatrix} I_{m_1} & & \\ & I_{m_2} & \\ & & I_n \end{pmatrix} \begin{pmatrix} I_{m_1} & & \\ & I_n & Y \\ & & I_{m_2} \end{pmatrix} \right.$$
$$\left. \times \begin{pmatrix} & I_{m_1} & \\ I_n & & \\ & & I_{m_2} \end{pmatrix} \begin{pmatrix} I_n & X & \\ & I_{m_1} & \\ & & I_{m_2} \end{pmatrix} g \right).$$

A simple computation shows that

$$\begin{pmatrix} I_{m_1} & & \\ & I_{m_2} & \\ & & I_n \end{pmatrix} \begin{pmatrix} I_{m_1} & & \\ & I_n & Y \\ & & I_{m_2} \end{pmatrix} \begin{pmatrix} & I_{m_1} & \\ I_n & & \\ & & I_{m_2} \end{pmatrix} \begin{pmatrix} I_n & X & \\ & I_{m_1} & \\ & & I_{m_2} \end{pmatrix}$$
$$= \begin{pmatrix} & I_{m_1} & \\ & & I_{m_2} \\ I_n & & \end{pmatrix} \begin{pmatrix} I_n & X & Y \\ & I_{m_1} & \\ & & I_{m_2} \end{pmatrix}.$$

Hence, we get

$$(\tilde{U}_{\sigma_1,\pi} \circ \tilde{U}_{\sigma_2,\pi} f)(g) = \sum_{X \in M_{n \times m_1}(\mathbb{F})} \sum_{Y \in M_{n \times m_2}(\mathbb{F})} \tilde{f} \left(\begin{pmatrix} & I_{m_1} & \\ & & I_{m_2} \\ I_n & & \end{pmatrix} \begin{pmatrix} I_n & X & Y \\ & I_{m_1} & \\ & & I_{m_2} \end{pmatrix} g \right),$$

and the last sum is $\tilde{U}_{\sigma_1 \circ \sigma_2, \pi}(f)(g)$ by (3-9). □

3C3. *Proof of Theorem 3.9.* Let $v_{\pi,\psi}$, $v_{\sigma_1,\psi}$ and $v_{\sigma_2,\psi}$ be nonzero ψ-Whittaker vectors of π, σ_1 and σ_2, respectively. We keep the notations from the previous section. We are now ready to prove Theorem 3.9.

Proof. By definition, we have

$$(\mathrm{id}_{\sigma_1} \otimes U_{\sigma_2,\pi})(v_{\sigma_1,\sigma_2 \circ \pi,\psi}) = \Gamma(\pi \times \sigma_2, \psi) v_{\sigma_1, \pi \circ \sigma_2, \psi}.$$

Since $L_{\sigma_1;\sigma_2,\pi} v_{\sigma_1,\sigma_2 \circ \pi,\psi} = v_{\sigma_1,\sigma_2,\pi,\psi}$ and $L_{\sigma_1;\pi,\sigma_2} v_{\sigma_1,\pi \circ \sigma_2,\psi} = v_{\sigma_1,\pi,\sigma_2,\psi}$, we get from the commutative diagram (3-4) that

$$\tilde{U}_{\sigma_2,\pi} v_{\sigma_1,\sigma_2,\pi,\psi} = \Gamma(\pi \times \sigma_2, \psi) v_{\sigma_1,\pi,\sigma_2,\psi}.$$

Similarly, we have that

$$(U_{\sigma_1,\pi} \otimes \mathrm{id}_{\sigma_2})(v_{\sigma_1 \circ \pi, \sigma_2, \psi}) = \Gamma(\pi \times \sigma_1, \psi) v_{\pi \circ \sigma_1, \sigma_2, \psi},$$

and we get from the commutative diagram (3-5) that

$$\tilde{U}_{\sigma_1,\pi} v_{\sigma_1,\pi,\sigma_2,\psi} = \Gamma(\pi \times \sigma_1, \psi) v_{\pi,\sigma_1,\sigma_2,\psi}.$$

Finally, we have

$$U_{\sigma_1 \circ \sigma_2,\pi} v_{\sigma_1 \circ \sigma_2,\pi,\psi} = \Gamma(\pi \times (\sigma_1 \circ \sigma_2), \psi) v_{\pi,\sigma_1 \circ \sigma_2,\psi}.$$

Since $L_{\sigma_1,\sigma_2;\pi} v_{\sigma_1 \circ \sigma_2,\pi,\psi} = v_{\sigma_1,\sigma_2,\pi,\psi}$ and $L_{\pi;\sigma_1,\sigma_2} v_{\pi,\sigma_1 \circ \sigma_2,\psi} = v_{\pi,\sigma_1,\sigma_2,\psi}$, we get from the commutative diagram (3-6)

$$\tilde{U}_{\sigma_1 \circ \sigma_2,\pi} v_{\sigma_1,\sigma_2,\pi,\psi} = \Gamma(\pi \times (\sigma_1 \circ \sigma_2), \psi) v_{\pi,\sigma_1,\sigma_2,\psi}.$$

Since $\tilde{U}_{\sigma_1 \circ \sigma_2,\pi} = \tilde{U}_{\sigma_1,\pi} \circ \tilde{U}_{\sigma_2,\pi}$, we get that

$$\Gamma(\pi \times (\sigma_1 \circ \sigma_2), \psi) v_{\pi,\sigma_1,\sigma_2,\psi} = \Gamma(\pi \times \sigma_1, \psi) \Gamma(\pi \times \sigma_2, \psi) v_{\pi,\sigma_1,\sigma_2,\psi},$$

and the theorem follows. $\qquad\square$

3D. *Expression in terms of Bessel functions.* In this section, we express the Shahidi gamma factor of two irreducible generic representations in terms of their Bessel functions.

Let π and σ be irreducible generic representations of $GL_n(\mathbb{F})$ and $GL_m(\mathbb{F})$, respectively. We assume that π and σ are realized by their Whittaker models $\mathcal{W}(\pi, \psi)$ and $\mathcal{W}(\sigma, \psi)$, respectively. We choose the Whittaker vectors of π and σ to be their corresponding Bessel functions, i.e., we choose $v_{\pi,\psi} = \mathcal{J}_{\pi,\psi}$ and $v_{\sigma,\psi} = \mathcal{J}_{\sigma,\psi}$. We denote $\mathcal{J}_{\sigma,\pi,\psi} = v_{\sigma,\pi,\psi} = f_{\mathcal{J}_{\sigma,\psi},\mathcal{J}_{\pi,\psi}}$ and similarly $\mathcal{J}_{\pi,\sigma,\psi} = v_{\pi,\sigma,\psi} = f_{\mathcal{J}_{\pi,\psi},\mathcal{J}_{\sigma,\psi}}$.

Assume that $n \geq m$. By definition, we have that for any $g \in GL_{n+m}(\mathbb{F})$,

$$(U_{\sigma,\pi} \mathcal{J}_{\pi,\sigma,\psi})(g) = \Gamma(\pi \times \sigma, \psi) \mathcal{J}_{\pi,\sigma,\psi}(g).$$

Substituting $g = \hat{w}_{m,n}$, we get

$$\Gamma(\pi \times \sigma, \psi) \mathcal{J}_{\pi,\sigma,\psi}(\hat{w}_{m,n}) = \sum_{u \in N_{n,m}} sw_{\sigma,\pi} \mathcal{J}_{\pi,\sigma,\psi}(\hat{w}_{n,m} u \hat{w}_{m,n}),$$

and therefore

$$\Gamma(\pi \times \sigma, \psi) \mathcal{J}_{\pi,\psi} \otimes \mathcal{J}_{\sigma,\psi} = \sum_{A \in M_{n \times m}(\mathbb{F})} sw_{\sigma,\pi} \mathcal{J}_{\sigma,\pi,\psi} \begin{pmatrix} I_m & \\ A & I_n \end{pmatrix}. \qquad (3\text{-}10)$$

In order for $\mathcal{J}_{\sigma,\pi,\psi} \begin{pmatrix} I_m & \\ A & I_n \end{pmatrix}$ not to vanish, we must have $\begin{pmatrix} I_m & \\ A & I_n \end{pmatrix} \in P_{m,n} \hat{w}_{n,m} Z_{n+m}$, so there must exist $\begin{pmatrix} p_1 & x \\ & p_2 \end{pmatrix} \in P_{m,n}$ and $\begin{pmatrix} u_1 & y \\ & u_2 \end{pmatrix} \in Z_{n+m}$, where $u_1 \in Z_n$ and $u_2 \in Z_m$, such that

$$\begin{pmatrix} p_1 & x \\ & p_2 \end{pmatrix} \begin{pmatrix} I_m & \\ A & I_n \end{pmatrix} = \begin{pmatrix} & I_m \\ I_n & \end{pmatrix} \begin{pmatrix} u_1 & y \\ & u_2 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} p_1 + xA & x \\ p_2 A & p_2 \end{pmatrix} = \begin{pmatrix} & u_2 \\ u_1 & y \end{pmatrix}.$$

Therefore, we have $p_1 + xA = 0$ and $x = (0_{m \times (n-m)}, u_2)$.

In order to proceed, we will separate two cases, the case where $n > m$ and the case where $n = m$.

3D1. *The case $n > m$.* In this case, we write $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, where $A_1 \in M_{(n-m) \times m}(\mathbb{F})$ and $A_2 \in M_{m \times m}(\mathbb{F})$ and $x = (0_{m \times (n-m)}, u_2)$. Then $p_1 + xA = 0$ implies $p_1 + u_2 A_2 = 0$, and therefore A_2 is invertible.

Write

$$\begin{pmatrix} I_n & \\ A & I_m \end{pmatrix} = \begin{pmatrix} I_m & & \\ A_1 & I_{n-m} & \\ A_2 & & I_m \end{pmatrix}$$

$$= \begin{pmatrix} I_m & & -A_2^{-1} \\ A_1 & I_{n-m} & \\ A_2 & & \end{pmatrix} \begin{pmatrix} I_m & & A_2^{-1} \\ & I_{n-m} & -A_1 A_2^{-1} \\ & & I_m \end{pmatrix}$$

$$= \begin{pmatrix} -A_2^{-1} & I_m & \\ & A_1 & I_{n-m} \\ & A_2 & \end{pmatrix} \hat{w}_{n,m} \begin{pmatrix} I_m & & A_2^{-1} \\ & I_{n-m} & -A_1 A_2^{-1} \\ & & I_m \end{pmatrix}.$$

Therefore, we have

$$\mathcal{J}_{\sigma,\pi,\psi} \begin{pmatrix} I_m & \\ A & I_n \end{pmatrix}$$

$$= \psi \begin{pmatrix} I_{n-m} & -A_1 A_2^{-1} \\ & I_m \end{pmatrix} \sigma(-A_2^{-1}) \otimes \pi \begin{pmatrix} A_1 & I_{n-m} \\ A_2 & \end{pmatrix} \mathcal{J}_{\sigma,\psi} \otimes \mathcal{J}_{\pi,\psi}. \quad (3\text{-}11)$$

Substituting (3-11) back in (3-10), we get

$$\Gamma(\pi \times \sigma, \psi) \mathcal{J}_{\pi,\psi} \otimes \mathcal{J}_{\sigma,\psi}$$

$$= \sum_{\substack{A_1 \in M_{(n-m) \times m}(\mathbb{F}) \\ A_2 \in \mathrm{GL}_m(\mathbb{F})}} \psi \begin{pmatrix} I_{n-m} & -A_1 A_2^{-1} \\ & I_m \end{pmatrix} \pi \begin{pmatrix} A_1 & I_{n-m} \\ A_2 & \end{pmatrix} \otimes \sigma(-A_2^{-1}) \mathcal{J}_{\pi,\psi} \otimes \mathcal{J}_{\sigma,\psi}. \quad (3\text{-}12)$$

We evaluate both sides of (3-12) at (I_n, I_m) to get

$$\Gamma(\pi \times \sigma, \psi) = \sum_{\substack{A_1 \in M_{(n-m) \times m}(\mathbb{F}) \\ A_2 \in \mathrm{GL}_m(\mathbb{F})}} \psi \begin{pmatrix} I_{n-m} & -A_1 A_2^{-1} \\ & I_m \end{pmatrix} \mathcal{J}_{\pi,\psi} \begin{pmatrix} A_1 & I_{n-m} \\ A_2 & \end{pmatrix} \mathcal{J}_{\sigma,\psi}(-A_2^{-1}).$$

Writing

$$
\begin{pmatrix} A_1 & I_{n-m} \\ A_2 & \end{pmatrix} = \begin{pmatrix} I_{n-m} & A_1 A_2^{-1} \\ & I_m \end{pmatrix} \begin{pmatrix} & I_{n-m} \\ A_2 & \end{pmatrix},
$$

we get

$$
\mathcal{J}_{\pi,\psi} \begin{pmatrix} A_1 & I_{n-m} \\ A_2 & \end{pmatrix} = \psi \begin{pmatrix} I_{n-m} & A_1 A_2^{-1} \\ & I_m \end{pmatrix} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & I_{n-m} \\ A_2 & \end{pmatrix},
$$

and therefore

$$
\Gamma(\pi \times \sigma, \psi) = \sum_{\substack{A_1 \in M_{(n-m)\times m}(\mathbb{F}) \\ A_2 \in \mathrm{GL}_m(\mathbb{F})}} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & I_{n-m} \\ A_2 & \end{pmatrix} \mathcal{J}_{\sigma,\psi}(-A_2^{-1}).
$$

The summand is independent of A_1. Using the equivariance properties of the Bessel function, we get that the summand is invariant under Z_m left translations of A_2. Finally, using the properties of the Bessel function discussed in Section 2B1, we get

$$
\Gamma(\pi \times \sigma, \psi) = q^{m(2n-m-1)/2} \omega_\sigma(-1) \sum_{x \in Z_m \backslash \mathrm{GL}_m(\mathbb{F})} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & I_{n-m} \\ x & \end{pmatrix} \mathcal{J}_{\sigma^\vee,\psi^{-1}}(x),
$$

where $q^{m(2n-m-1)/2} = |M_{(n-m)\times m}(\mathbb{F})| \cdot |Z_m|$.

3D2. *The case $n = m$.* In this case, we have $-p_1 = xA$, and therefore A is invertible. We write

$$
\begin{pmatrix} I_n & \\ A & I_n \end{pmatrix} = \begin{pmatrix} I_n & -A^{-1} \\ & A \end{pmatrix} \begin{pmatrix} I_n & A^{-1} \\ & I_n \end{pmatrix} = \begin{pmatrix} -A^{-1} & I_n \\ & A \end{pmatrix} \hat{w}_{n,n} \begin{pmatrix} I_n & A^{-1} \\ & I_n \end{pmatrix}.
$$

Therefore, we have

$$
\mathcal{J}_{\sigma,\pi,\psi} \begin{pmatrix} I_m & \\ A & I_n \end{pmatrix} = \psi \begin{pmatrix} I_n & A^{-1} \\ & I_n \end{pmatrix} \sigma(-A^{-1}) \otimes \pi(A) \mathcal{J}_{\sigma,\psi} \otimes \mathcal{J}_{\pi,\psi}. \tag{3-13}
$$

Substituting (3-13) in (3-10), we get

$$
\Gamma(\pi \times \sigma, \psi) \mathcal{J}_{\pi,\psi} \otimes \mathcal{J}_{\sigma,\psi}
$$
$$
= \sum_{A \in \mathrm{GL}_n(\mathbb{F})} \psi \begin{pmatrix} I_n & A^{-1} \\ & I_n \end{pmatrix} \pi(A) \otimes \sigma(-A^{-1}) \mathcal{J}_{\pi,\psi} \otimes \mathcal{J}_{\sigma,\psi}. \tag{3-14}
$$

Evaluating both sides of (3-14) at (I_n, I_n), we get

$$
\Gamma(\pi \times \sigma, \psi) = \sum_{A \in \mathrm{GL}_n(\mathbb{F})} \psi \begin{pmatrix} I_n & A^{-1} \\ & I_n \end{pmatrix} \mathcal{J}_{\pi,\psi}(A) \mathcal{J}_{\sigma,\psi}(-A^{-1}).
$$

The summand is invariant under Z_n left translations. Using the properties of the Bessel function discussed in Section 2B1, we get

$$\Gamma(\pi \times \sigma, \psi) = q^{n(n-1)/2} \omega_\sigma(-1) \sum_{x \in Z_n \backslash \mathrm{GL}_n(\mathbb{F})} \psi \begin{pmatrix} I_n & x^{-1} \\ & I_n \end{pmatrix} \mathcal{J}_{\pi,\psi}(x) \mathcal{J}_{\sigma^\vee, \psi^{-1}}(x).$$

3D3. *Summary of cases.* We conclude this section by writing down formulas for the Shahidi gamma factor for a pair of irreducible generic representations, in terms of their Bessel functions for all cases. In order to do that, we use Theorem 3.7 and the formulas from Sections 3D1 and 3D2.

Theorem 3.14. *Let π and σ be irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$, respectively:*

(1) *If $n > m$, then*

$$\Gamma(\pi \times \sigma, \psi) = q^{m(2n-m-1)/2} \omega_\sigma(-1) \sum_{x \in Z_m \backslash \mathrm{GL}_m(\mathbb{F})} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & I_{n-m} \\ x & \end{pmatrix} \mathcal{J}_{\sigma^\vee, \psi^{-1}}(x).$$

(2) *If $n = m$, then*

$$\Gamma(\pi \times \sigma, \psi) = q^{n(n-1)/2} \omega_\sigma(-1) \sum_{x \in Z_n \backslash \mathrm{GL}_n(\mathbb{F})} \psi \begin{pmatrix} I_n & x^{-1} \\ & I_n \end{pmatrix} \mathcal{J}_{\pi,\psi}(x) \mathcal{J}_{\sigma^\vee, \psi^{-1}}(x).$$

(3) *If $n < m$, then*

$$\Gamma(\pi \times \sigma, \psi) = q^{n(2m-n-1)/2} \omega_\pi(-1) \sum_{x \in Z_n \backslash \mathrm{GL}_n(\mathbb{F})} \mathcal{J}_{\pi,\psi}(x) \mathcal{J}_{\sigma^\vee, \psi^{-1}} \begin{pmatrix} & I_{m-n} \\ x & \end{pmatrix}.$$

Theorem 3.14 allows us to give a relation between the Jacquet–Piatetski-Shapiro–Shalika gamma factors defined in Section 2C and the Shahidi gamma factor. By Propositions 2.6 and 2.9, we get the following corollary.

Corollary 3.15. *Let π be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$ and let σ be an irreducible generic representation of $\mathrm{GL}_m(\mathbb{F})$. Then we have the equality*

$$\Gamma(\pi \times \sigma, \psi) = q^{m(2n-m-1)/2} \omega_\sigma(-1) \gamma(\pi \times \sigma^\vee, \psi)$$

in either of the following cases:

(1) *$n > m$.*

(2) *$n = m$ and σ is cuspidal.*

4. Applications

4A. *Quantitative interpretation of gamma factors.* In this section, we give a representation theoretic interpretation of the absolute value of the Shahidi gamma factor. Our results relate the absolute value of a normalized version of the Shahidi gamma factor with the cuspidal support of the representations.

For irreducible generic representations π and σ of GL$_n(\mathbb{F})$ and GL$_m(\mathbb{F})$, respectively, we define the normalized Shahidi gamma factor by

$$\Gamma^*(\pi \times \sigma, \psi) = q^{-nm/2}\Gamma(\pi \times \sigma, \psi).$$

It follows from Corollary 3.12 that under this normalization, the gamma factor is still multiplicative, i.e., the following proposition holds.

Proposition 4.1. *Let $\pi_1, \ldots, \pi_r$ and $\sigma_1, \ldots, \sigma_t$ be irreducible generic representations of GL$_{n_1}(\mathbb{F}), \ldots, $ GL$_{n_r}(\mathbb{F})$ and GL$_{m_1}(\mathbb{F}), \ldots, $ GL$_{m_t}(\mathbb{F})$. Suppose that π is the unique irreducible generic subrepresentation of $\pi_1 \circ \ldots \circ \pi_r$ and that σ is the unique irreducible generic subrepresentation of $\sigma_1 \circ \ldots \circ \sigma_t$. Then*

$$\Gamma^*(\pi \times \sigma, \psi) = \prod_{i=1}^{r}\prod_{j=1}^{t} \Gamma^*(\pi_i \times \sigma_j, \psi).$$

By Corollaries 3.15 and 2.7 and Proposition 2.12, we have the following proposition, which allows us to express the size of the absolute value of $\Gamma(\pi \times \sigma, \psi)$ where π and σ are cuspidal.

Proposition 4.2. *Let π and σ be irreducible cuspidal representations of GL$_n(\mathbb{F})$ and GL$_m(\mathbb{F})$, respectively. Then*

$$|\Gamma^*(\pi \times \sigma, \psi)| = \begin{cases} q^{-n/2} & n = m \text{ and } \pi \cong \sigma, \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 4.2 tells us that the size of the normalized Shahidi gamma factor serves as a "Kronecker delta function" for cuspidal representations. It could be thought of an analog of [Jacquet et al. 1983, Section 8.1]. Combining this with the multiplicativity property, we get the following theorem, that allows us to recover the cuspidal support of an irreducible generic representation π by computing $|\Gamma^*(\pi \times \sigma, \psi)|$ for any irreducible cuspidal σ.

Theorem 4.3. *Let π be an irreducible generic representation of GL$_n(\mathbb{F})$ and let σ be an irreducible cuspidal representation of GL$_m(\mathbb{F})$. Then*

$$|\Gamma^*(\pi \times \sigma, \psi)| = q^{-d_\pi(\sigma)m/2},$$

where $d_\pi(\sigma)$ is the number of times that σ appears in the cuspidal support of π.

Proof. Suppose that the cuspidal support of π is $\{\pi_1, \ldots, \pi_r\}$. Then π is the unique irreducible generic subrepresentation of $\pi_1 \circ \ldots \circ \pi_r$. The result now follows immediately from Propositions 4.1 and 4.2. □

As a corollary, we get the following converse theorem, which allows us to determine whether generic representations of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$ are isomorphic based on the absolute value of their normalized gamma factors. It is an analog of [Atobe and Gan 2017, Lemma A.6], but our proof is on the "group side" rather than on the "Galois side".

Theorem 4.4. *Let π_1 and π_2 be irreducible generic representations of $\mathrm{GL}_{n_1}(\mathbb{F})$ and $\mathrm{GL}_{n_2}(\mathbb{F})$, respectively. Suppose that for every $m > 0$ and every irreducible cuspidal representation σ of $\mathrm{GL}_m(\mathbb{F})$ we have*

$$|\Gamma^*(\pi_1 \times \sigma, \psi)| = |\Gamma^*(\pi_2 \times \sigma, \psi)|.$$

Then $n_1 = n_2$ and $\pi_1 \cong \pi_2$.

Proof. By Theorem 4.3, π_1 and π_2 have the same cuspidal support. By Theorem 2.2, there exists a unique irreducible generic representation with a given cuspidal support. □

As another corollary, we explain that the functional equations in Theorem 2.5 and Proposition 2.10 fail for π and σ, whenever the cuspidal support of π has a nonempty intersection with the cuspidal support of $\sigma^\vee$.

Corollary 4.5. *Suppose that π and σ are irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{GL}_m(\mathbb{F})$, respectively, and that $n > m$ (respectively, $n = m$). Suppose that the cuspidal support of π has a nonempty intersection with the cuspidal support of $\sigma^\vee$. Then the functional equation in Theorem 2.5 (respectively, Theorem 2.8) does not hold for π and σ.*

Proof. If the functional equation holds for π and σ, then it also holds for $\pi^\vee$ and $\sigma^\vee$. This can be seen by applying complex conjugation to the functional equation, which sends the ψ-Whittaker functions to ψ^{-1}-Whittaker functions of the contragredient. As in Corollary 2.7 (respectively, Corollary 2.11), we get that $|\gamma(\pi \times \sigma, \psi)| = q^{-m(n-m-1)/2}$. Whenever $\gamma(\pi \times \sigma, \psi)$ is defined, it is given by the formula in Proposition 2.6 (respectively, Proposition 2.9), and therefore the formula in Corollary 3.15 holds. This implies that $|\Gamma^*(\pi \times \sigma^\vee, \psi)| = 1$.

On the other hand, because π and $\sigma^\vee$ have common elements in their cuspidal support, we have that $|\Gamma^*(\pi \times \sigma^\vee, \psi)| < 1$. □

Remark 4.6. In his unpublished manuscript [Soudry 1979], the first author showed that whenever the cuspidal support of π does not intersect the cuspidal support of $\sigma^\vee$, the relevant functional equation holds. Due to length considerations, we do not include the proofs here.

4B. *Consequences for the converse theorem.* Our results from Section 4A allow us to improve Nien's results regarding the converse theorem for irreducible generic representations of finite general linear groups.

Nien [2014] showed the following theorem.

Theorem 4.7. *Let π_1 and π_2 be two irreducible cuspidal representations of $GL_n(\mathbb{F})$ with the same central character. Suppose that for every $1 \le m \le \frac{n}{2}$, and every irreducible generic representation σ of $GL_m(\mathbb{F})$ we have*

$$\gamma(\pi_1 \times \sigma, \psi) = \gamma(\pi_2 \times \sigma, \psi). \tag{4-1}$$

Then $\pi_1 \cong \pi_2$.

Using our results and Theorem 4.7, we are able to deduce the following converse theorem, where π_1 and π_2 can be arbitrary generic representations (rather than just cuspidal representations), and (4-1) needs to be verified only for cuspidal representations σ (rather than for all generic representations). This is similar to [Jiang et al. 2015, Section 2.4].

Theorem 4.8. *Let π_1 and π_2 be two irreducible generic representations of $GL_n(\mathbb{F})$ with the same central character. Suppose that for every $1 \le m \le \frac{n}{2}$, and every irreducible cuspidal representation σ of $GL_m(\mathbb{F})$ we have*

$$\Gamma^*(\pi_1 \times \sigma, \psi) = \Gamma^*(\pi_2 \times \sigma, \psi). \tag{4-2}$$

Then $\pi_1 \cong \pi_2$.

Proof. Our proof is by induction on the cardinality of the cuspidal support of π_1.

We first notice that by Proposition 4.1, we have that for any $1 \le m \le \frac{n}{2}$ and any irreducible generic representation σ of $GL_m(\mathbb{F})$,

$$\Gamma^*(\pi_1 \times \sigma, \psi) = \Gamma^*(\pi_2 \times \sigma, \psi).$$

Suppose that π_1 is cuspidal, then its cuspidal support is of cardinality 1. If π_2 is not cuspidal, then its cuspidal support contains an irreducible cuspidal representation τ of $GL_k(\mathbb{F})$, where $k \le \frac{n}{2}$. Since $k < n$, we have by Theorem 4.3 that $|\Gamma^*(\pi_1 \times \sigma, \psi)| = 1$. We also have by Theorem 4.3 that $|\Gamma^*(\pi_2 \times \sigma, \psi)| < 1$, which is a contraction. Therefore, π_2 is also cuspidal, and by Corollary 3.15 and Theorem 4.7, we have that π_1 and π_2 are isomorphic.

Suppose now that π_1 is not cuspidal. Let $\{\tau_1, \ldots, \tau_r\}$ the cuspidal support of π_1 and let $\{\tau'_1, \ldots, \tau'_{r'}\}$ be the cuspidal support of π_2. Without loss of generality, we have that τ_1 is an irreducible cuspidal representation of $GL_{n_1}(\mathbb{F})$, where $n_1 \le \frac{n}{2}$. Then by Theorem 4.3 we have that $|\Gamma^*(\pi_1 \times \tau_1, \psi)| < 1$. Since $n_1 \le \frac{n}{2}$, we have that $\Gamma^*(\pi_1 \times \tau_1, \psi) = \Gamma^*(\pi_2 \times \tau_1, \psi)$, and therefore $|\Gamma^*(\pi_2 \times \tau_1, \psi)| < 1$. By Theorem 4.3, this implies that τ_1 is in the cuspidal support of π_2. Without loss of

generality, we may assume that $\tau_1' = \tau_1$. By Proposition 4.1, we deduce that for any irreducible generic representation σ of $\mathrm{GL}_m(\mathbb{F})$ where $m \leq \frac{n}{2}$,

$$\prod_{j=2}^{r} \Gamma^*(\tau_j \times \sigma, \psi) = \prod_{j=2}^{r'} \Gamma^*(\tau_j' \times \sigma, \psi). \tag{4-3}$$

Let π_1' be the unique irreducible generic representation of $\mathrm{GL}_{n-n_1}(\mathbb{F})$ with cuspidal support $\{\tau_2, \ldots, \tau_r\}$, and let π_2' be the unique irreducible generic representation of $\mathrm{GL}_{n-n_1}(\mathbb{F})$ with cuspidal support $\{\tau_2', \ldots, \tau_{r'}'\}$. For $i = 1, 2$, the central characters of π_i and π_i' are related by $\omega_{\pi_i} = \omega_{\pi_i'} \cdot \omega_{\tau_1}$. Therefore, we have that π_1' and π_2' also have the same central character. By Proposition 4.1, we have that (4-3) implies that for every $m \leq \frac{n}{2}$ and every irreducible generic representation σ of $\mathrm{GL}_m(\mathbb{F})$,

$$\Gamma^*(\pi_1' \times \sigma, \psi) = \Gamma^*(\pi_2' \times \sigma, \psi).$$

By induction $\pi_1' \cong \pi_2'$, and therefore $\{\tau_2, \ldots, \tau_r\} = \{\tau_2', \ldots, \tau_{r'}'\}$. Hence, $\pi_1 \cong \pi_2$, as required. $\square$

4C. *Special values of the Bessel function.* In this section, we use our results regarding multiplicativity of the Shahidi gamma factor, and its relation to the Jacquet–Piatetski-Shapiro–Shalika gamma factor in order to find an explicit formula for special values of the Bessel function of irreducible generic representations of $\mathrm{GL}_n(\mathbb{F})$. For two blocks, such a formula was given by Curtis and Shinoda [2004, Lemma 3.5]. However, their proof uses Deligne–Lusztig theory, while our proof only uses Green's character values for irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$; see [Gel'fand 1970, Section 6] and [Nien 2017, Section 3.1]. We also provide a formula for a simple value consisting of three blocks. This generalizes a formula of Chang [1976] for irreducible generic representations of $\mathrm{GL}_3(\mathbb{F})$.

4C1. *Special value formula for two blocks.* Fix an algebraic closure $\bar{\mathbb{F}}$ of $\mathbb{F}$. For every positive integer n, let $\mathbb{F}_n$ be the unique extension of degree n in $\bar{\mathbb{F}}$. Let $\mathrm{N}_{\mathbb{F}_n/\mathbb{F}} \colon \mathbb{F}_n^* \to \mathbb{F}^*$ and $\mathrm{Tr}_{\mathbb{F}_n/\mathbb{F}} \colon \mathbb{F}_n \to \mathbb{F}$ be the norm and the trace maps, respectively. Let $\widehat{\mathbb{F}_n^*}$ be the character group consisting of all multiplicative characters $\alpha \colon \mathbb{F}_n^* \to \mathbb{C}^*$.

It is known that irreducible cuspidal representations of $\mathrm{GL}_n(\mathbb{F})$ are in a bijection with Frobenius orbits of size n of $\widehat{\mathbb{F}_n^*}$, that is, every irreducible cuspidal representation π of $\mathrm{GL}_n(\mathbb{F})$ corresponds to a set of size n of the form $\{\alpha, \alpha^q, \ldots, \alpha^{q^{n-1}}\}$, where $\alpha \in \widehat{\mathbb{F}_n^*}$.

We first recall Nien's result regarding the computation of the Jacquet–Piatetski-Shapiro–Shalika gamma factor $\gamma(\pi \times \chi, \psi)$ where π is an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$ and χ is a representation of $\mathrm{GL}_1(\mathbb{F})$, that is, $\chi \colon \mathbb{F}^* \to \mathbb{C}^*$ is a multiplicative character. Nien's result expresses $\gamma(\pi \times \chi, \psi)$ as a Gauss sum.

Proposition 4.9 [Nien 2014, Theorem 1.1]. *Let π be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$ associated with the Frobenius orbit $\{\alpha, \alpha^q, \ldots, \alpha^{q^{n-1}}\}$, where $\alpha \in \widehat{\mathbb{F}_n^*}$. Let $\chi : \mathbb{F}^* \to \mathbb{C}^*$ be a multiplicative character. Then*

$$\gamma(\pi \times \chi, \psi) = (-1)^{n+1}\chi(-1)^{n+1}q^{-n+1}\sum_{\xi \in \mathbb{F}_n^*}\alpha^{-1}(\xi)\chi^{-1}(\mathrm{N}_{\mathbb{F}_n/\mathbb{F}}(\xi))\psi(\mathrm{Tr}_{\mathbb{F}_n/\mathbb{F}}(\xi)).$$

Nien's proof only uses Green's character formula for irreducible cuspidal representations, and does not use Deligne–Lusztig theory.

We are ready to state our result regarding special two blocks values of the Bessel function.

Theorem 4.10. *Let $n > 1$, and let π be an irreducible generic representation of $\mathrm{GL}_n(\mathbb{F})$ with cuspidal support $\{\pi_1, \ldots, \pi_r\}$, where for every $1 \le j \le r$, π_j is an irreducible cuspidal representation of $\mathrm{GL}_{n_j}(\mathbb{F})$ corresponding to the Frobenius orbit $\{\alpha_j, \alpha_j^q, \ldots, \alpha_j^{q^{n_j-1}}\}$, where $\alpha_j \in \widehat{\mathbb{F}_{n_j}^*}$ is a multiplicative character. Then for any $c \in \mathbb{F}^*$,*

$$\mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-1} \\ c & \end{pmatrix} = (-1)^{n+r}q^{-n+1}\sum_{\substack{\xi_1 \in \mathbb{F}_{n_1}^*, \ldots, \xi_r \in \mathbb{F}_{n_r}^* \\ \prod_{j=1}^r \mathrm{N}_{\mathbb{F}_{n_j}/\mathbb{F}}(\xi_j) = (-1)^{n-1}c^{-1}}}\prod_{j=1}^r\left(\alpha_j^{-1}(\xi_j)\psi(\mathrm{Tr}_{\mathbb{F}_{n_j}/\mathbb{F}}(\xi_j))\right).$$

Proof. By Theorem 3.14, we have that

$$\Gamma(\pi \times \chi, \psi) = q^{n-1}\sum_{x \in \mathbb{F}^*}\mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-1} \\ x & \end{pmatrix}\chi^{-1}(-x).$$

Multiplying by $\chi(-c)$ and averaging over all $\chi \in \widehat{\mathbb{F}^*}$, and using the fact that a sum of a nontrivial character on a group is zero, we get

$$\frac{1}{|\mathbb{F}^*|}\sum_{\chi \in \widehat{\mathbb{F}^*}}\Gamma(\pi \times \chi, \psi)\chi(-c) = q^{n-1}\mathcal{J}_{\pi,\psi}\begin{pmatrix} & I_{n-1} \\ c & \end{pmatrix}. \tag{4-4}$$

By Corollary 3.12, we have that

$$\Gamma(\pi \times \chi, \psi) = \prod_{j=1}^r\Gamma(\pi_j \times \chi, \psi).$$

By Corollary 3.15 and Proposition 4.9, we have that

$$\Gamma(\pi_j \times \chi, \psi) = (-1)^{n_j+1}\chi(-1)^{n_j}\sum_{\xi \in \mathbb{F}_{n_j}^*}\alpha_j^{-1}(\xi)\chi(\mathrm{N}_{\mathbb{F}_{n_j}/\mathbb{F}}(\xi))\psi(\mathrm{Tr}_{\mathbb{F}_{n_j}/\mathbb{F}}(\xi)).$$

Therefore, we get that $\Gamma(\pi \times \chi, \psi)$ is given by

$$(-1)^{n+r} \sum_{\xi_1 \in \mathbb{F}_{n_1}^*, \dots, \xi_r \in \mathbb{F}_{n_r}^*} \left(\prod_{j=1}^r \alpha_j^{-1}(\xi_j) \psi(\mathrm{Tr}_{\mathbb{F}_{n_j}/\mathbb{F}}(\xi_j)) \right) \chi\left((-1)^n \prod_{j=1}^r \mathrm{N}_{\mathbb{F}_{n_j}/\mathbb{F}}(\xi_j) \right).$$

(4-5)

Substituting the expression (4-5) for $\Gamma(\pi \times \chi, \psi)$ in (4-4), and using the fact that a sum of a nontrivial of character over a group is zero, we get the desired result. $\square$

Remark 4.11. The expression for $\Gamma(\pi \times \chi, \psi)$ in (4-5) is originally due to Kondo [1963]. He computed it for the Godement–Jacquet gamma factor. One can show directly that the Godement–Jacquet gamma factor coincides with the Shahidi gamma factor for representations for which both factors are defined. Our proof, which is based on Nien's result and on multiplicativity of gamma factors, is different than the one given by Kondo. See also another proof in [Macdonald 1998, Chapter IV, Section 6, Example 4].

Remark 4.12. In [Zelingher 2023], a vast generalization of the method in the proof of Theorem 4.10 is used in order to find formulas for

$$\mathcal{J}_{\pi,\psi} \begin{pmatrix} & I_{n-m} \\ c I_m & \end{pmatrix}.$$

However, [Zelingher 2023] relies on the results of [Ye and Zelingher 2021], which in turn rely on the local Langlands correspondence. The proof given here does not rely on such results.

4C2. *Special value formula for three blocks.* In this subsection, we use our results to prove a formula for special values of the Bessel function, for a simple value consisting of three blocks. This generalizes a formula given by Chang [1976] for $\mathrm{GL}_3(\mathbb{F})$, generalized later by Shinoda and Tulunay [2005] to $\mathrm{GL}_4(\mathbb{F})$. Our proof is different from Chang's proof, which is based on the Gelfand–Graev algebra.

We start with the following proposition.

Proposition 4.13. *Let π be an irreducible generic representation of $\mathrm{GL}_n(\mathbb{F})$. Then for any $c \in \mathbb{F}^*$, and any $g \in \mathrm{GL}_n(\mathbb{F})$, we have*

$$\mathcal{J}_{\pi,\psi}(g) \mathcal{J}_{\pi,\psi} \begin{pmatrix} I_{n-1} \\ c \end{pmatrix}$$

$$= q^{-(n-1)} \sum_{{}^t x = (x_1, \dots, x_{n-1}) \in \mathbb{F}^{n-1}} \psi(-x_{n-1}) \mathcal{J}_{\pi,\psi} \left(g \begin{pmatrix} I_{n-1} & x \\ & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} \\ c & \end{pmatrix} \right).$$

Proof. Let $m = 1$ and let $\sigma = \chi : \mathbb{F}^* \to \mathbb{C}^*$ be a multiplicative character. By (3-12), we have

$$\Gamma(\pi \times \chi, \psi) \mathcal{J}_{\pi,\psi} = \sum_{\substack{{}^t x \in \mathbb{F}^{n-1} \\ a \in \mathbb{F}^*}} \chi(-a^{-1}) \psi \begin{pmatrix} I_{n-1} & -a^{-1}x \\ & 1 \end{pmatrix} \pi \begin{pmatrix} x & I_{n-1} \\ a & \end{pmatrix} \mathcal{J}_{\pi,\psi}.$$

We multiply by $\chi(-c)$ and average over $\chi \in \widehat{\mathbb{F}^*}$. Using the fact that a sum of a nontrivial character over a group is zero, and using (4-4), we get

$$q^{n-1} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & I_{n-1} \\ c & \end{pmatrix} \mathcal{J}_{\pi,\psi} = \sum_{{}^t x = (x_1,\ldots,x_{n-1}) \in \mathbb{F}^{n-1}} \psi(-c^{-1}x_{n-1}) \pi \begin{pmatrix} x & I_{n-1} \\ c & \end{pmatrix} \mathcal{J}_{\pi,\psi}.$$

Using the decomposition

$$\begin{pmatrix} x & I_{n-1} \\ c & \end{pmatrix} = \begin{pmatrix} I_{n-1} & c^{-1}x \\ & 1 \end{pmatrix} \begin{pmatrix} & I_{n-1} \\ c & \end{pmatrix},$$

and changing the summation variable x to $c \cdot x$, we get the desired result. $\qquad\square$

Theorem 4.14. *Suppose $n \geq 3$. Then for any irreducible generic representation π of $\mathrm{GL}_n(\mathbb{F})$ and any $c, c' \in \mathbb{F}^*$, we have*

$$\mathcal{J}_{\pi,\psi} \begin{pmatrix} & & -c' \\ & I_{n-2} & \\ c & & \end{pmatrix}$$

$$= \sum_{s \in \mathbb{F}^*} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & I_{n-1} \\ s^{-1}c & \end{pmatrix} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & sc' \\ I_{n-1} & \end{pmatrix} (\psi(s) - 1) + \frac{\delta_{cc',1}}{q^{n-2}},$$

where

$$\delta_{cc',1} = \begin{cases} 1 & cc' = 1, \\ 0 & otherwise. \end{cases}$$

Proof. We substitute $g = \begin{pmatrix} & c' \\ I_{n-1} & \end{pmatrix}$ in Proposition 4.13 to get

$$\mathcal{J}_{\pi,\psi} \begin{pmatrix} & I_{n-1} \\ c & \end{pmatrix} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & c' \\ I_{n-1} & \end{pmatrix}$$

$$= q^{-(n-1)} \sum_{{}^t x = (x_1,\ldots,x_{n-1}) \in \mathbb{F}^{n-1}} \psi(-x_{n-1}) \mathcal{J}_{\pi,\psi} \begin{pmatrix} & cc' \\ cx & I_{n-1} \end{pmatrix}.$$

If $x_{n-1} = 0$, then $\begin{pmatrix} & cc' \\ cx & I_{n-1} \end{pmatrix}$ lies in the mirabolic subgroup. By [Nien 2014, Lemma 2.14], we have that the Bessel function is zero for elements in the mirabolic subgroup that do not lie in the upper unipotent subgroup Z_n. Therefore, we get that

if $x_{n-1} = 0$, then $x = 0$ and

$$\psi(x_{n-1}) \mathcal{J}_{\pi,\psi} \begin{pmatrix} cc' \\ cx & I_{n-1} \end{pmatrix} = \delta_{cc',1}.$$

Suppose now that $x_{n-1} = t \neq 0$. Denote

$${}^t x' = (x_1, \ldots, x_{n-2}) \in \mathbb{F}^{n-2}.$$

Then we have

$$\begin{pmatrix} cc' \\ cx & I_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & t^{-1}c' \\ & I_{n-2} & t^{-1}x' \\ & & 1 \end{pmatrix} \begin{pmatrix} & -t^{-1}c' \\ & I_{n-2} \\ tc & \end{pmatrix} \begin{pmatrix} 1 & 0 & (tc)^{-1} \\ & I_{n-2} & -t^{-1}x' \\ & & 1 \end{pmatrix}.$$

Since we have q^{n-2} elements in $\mathbb{F}^{n-1}$ with $x_{n-1} = t$, we get that

$$\mathcal{J}_{\pi,\psi} \begin{pmatrix} I_{n-1} \\ c \end{pmatrix} \mathcal{J}_{\pi,\psi} \begin{pmatrix} c' \\ I_{n-1} \end{pmatrix} = \frac{\delta_{cc',1}}{q^{n-1}} + q^{-1} \sum_{t \in \mathbb{F}^*} \psi(-t) \mathcal{J}_{\pi,\psi} \begin{pmatrix} & -t^{-1}c' \\ & I_{n-2} \\ tc & \end{pmatrix}.$$

We proceed as in [Chang 1976, Page 379; Shinoda and Tulunay 2005, Lemma 4.2]. We replace c with $s^{-1}c$ and c' with sc', where $s \in \mathbb{F}^*$, to get

$$\mathcal{J}_{\pi,\psi} \begin{pmatrix} I_{n-1} \\ s^{-1}c \end{pmatrix} \mathcal{J}_{\pi,\psi} \begin{pmatrix} sc' \\ I_{n-1} \end{pmatrix}$$
$$= \frac{\delta_{cc',1}}{q^{n-1}} + q^{-1} \sum_{t \in \mathbb{F}^*} \psi(-st) \mathcal{J}_{\pi,\psi} \begin{pmatrix} & -t^{-1}c' \\ & I_{n-2} \\ tc & \end{pmatrix}. \quad (4\text{-}6)$$

Summing (4-6) over $s \in \mathbb{F}^*$, we get

$$\sum_{s \in \mathbb{F}^*} \mathcal{J}_{\pi,\psi} \begin{pmatrix} I_{n-1} \\ s^{-1}c \end{pmatrix} \mathcal{J}_{\pi,\psi} \begin{pmatrix} sc' \\ I_{n-1} \end{pmatrix}$$
$$= \frac{q-1}{q^{n-1}} \delta_{cc',1} - q^{-1} \sum_{t \in \mathbb{F}^*} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & -t^{-1}c' \\ & I_{n-2} \\ tc & \end{pmatrix}. \quad (4\text{-}7)$$

Multiplying (4-6) by $\psi(s)$ and summing over $s \in \mathbb{F}^*$, we get

$$\sum_{s \in \mathbb{F}^*} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & I_{n-1} \\ s^{-1}c & \end{pmatrix} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & sc' \\ I_{n-1} & \end{pmatrix} \psi(s)$$

$$= -\frac{\delta_{cc',1}}{q^{n-1}} + \frac{q-1}{q} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & & -c' \\ & I_{n-2} & \\ c & & \end{pmatrix}$$

$$- q^{-1} \sum_{1 \neq t \in \mathbb{F}^*} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & & -t^{-1}c' \\ & I_{n-2} & \\ tc & & \end{pmatrix}. \quad (4\text{-}8)$$

Subtracting (4-7) from (4-8), we get the desired result. $\qquad \square$

Remark 4.15. Using the formulas in Theorem 4.10 and its proof, one can show that if the cuspidal support of π does not contain any irreducible representation of GL$_1(\mathbb{F})$, then we have a simpler formula:

$$\mathcal{J}_{\pi,\psi} \begin{pmatrix} & & -c' \\ & I_{n-2} & \\ c & & \end{pmatrix} = \sum_{s \in \mathbb{F}^*} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & I_{n-1} \\ s^{-1}c & \end{pmatrix} \mathcal{J}_{\pi,\psi} \begin{pmatrix} & sc' \\ I_{n-1} & \end{pmatrix} \psi(s). \quad (4\text{-}9)$$

However, if the cuspidal support of π contains irreducible representations of GL$_1(\mathbb{F})$, this simpler formula does not hold.

Remark 4.16. Using the expression in Theorem 4.10, we have that the expression on the right hand side of (4-9) is an exponential sum that generalizes the Friedlander–Iwaniec character sum; see [Kowalski 2015, Theorem 7.3, formula (27) and Remark 7.4]. The Friedlander–Iwaniec character sum played a role in Zhang's work on the twin prime conjecture [2014].

Appendix: Computation of $\gamma(\pi \times \pi^\vee, \psi)$ when π is cuspidal

In this appendix, we compute the Jacquet–Piatetski-Shapiro-Shalika gamma factor $\gamma(\pi \times \sigma, \psi)$ in the special case where π and σ are irreducible cuspidal representations of GL$_n(\mathbb{F})$ and $\pi \cong \sigma^\vee$. We will prove the following theorem.

Theorem A.1. *Let π be an irreducible cuspidal representation of* GL$_n(\mathbb{F})$. *Then*

$$\gamma(\pi \times \pi^\vee, \psi) = -1.$$

This was done in [Ye 2019, Corollary 4.3]. We provide another proof, since the proof in [loc. cit.] relies on results of representations of p-adic groups.

For future purposes, we will prove the following general lemma. We will show that Theorem A.1 follows from it.

We denote by $P_n \leq \text{GL}_n(\mathbb{F})$ the mirabolic subgroup.

Lemma A.2. *Let G be a finite group and let $H \leq G$ be a subgroup. Suppose that H is a semidirect product of the form $H = N \rtimes \mathrm{GL}_n(\mathbb{F})$. Let $\Psi : H \to \mathbb{C}^*$ be a character which is trivial on $\mathrm{GL}_n(\mathbb{F})$. Let τ be an irreducible representation of G, such that:*

(1) $\dim \mathrm{Hom}_H(\mathrm{Res}_H \tau, \Psi) = 1$.

(2) $\dim \mathrm{Hom}_{N \rtimes P_n}(\mathrm{Res}_{N \rtimes P_n} \tau, \mathrm{Res}_{N \rtimes P_n} \Psi) = 1$.

(3) *There exists a functional $\ell \in \mathrm{Hom}_{Z_n}(\mathrm{Res}_{Z_n} \tau, \mathbb{C})$ and a vector $v_0 \in \tau$, such that*

$$\sum_{p \in Z_n \backslash P_n} \sum_{n \in N} \ell(\tau(np)v_0)\Psi^{-1}(n) = 1.$$

Then

$$\sum_{g \in Z_n \backslash \mathrm{GL}_n(\mathbb{F})} \sum_{n \in N} \ell(\tau(ng)v_0)\Psi^{-1}(n)\psi(\langle e_n g, e_1 \rangle) = -1.$$

Remark A.3. *If $\mathbb{F}^* \leq H$ lies in the center of G, then (1) implies that the restriction of the central character of τ to $\mathbb{F}^* \leq H$ is trivial.*

Proof. Notice that we have a containment

$$\mathrm{Hom}_H(\mathrm{Res}_H \tau, \Psi) \subset \mathrm{Hom}_{N \rtimes P_n}(\mathrm{Res}_{N \rtimes P_n} \tau, \mathrm{Res}_{N \rtimes P_n} \Psi).$$

Since both spaces are one dimensional, we have that they are equal. Denote for $v \in \tau$,

$$L(v) = \sum_{p \in Z_n \backslash P_n} \sum_{n \in N} \ell(\tau(np)v)\Psi^{-1}(n).$$

Then $L \in \mathrm{Hom}_{N \rtimes P_n}(\mathrm{Res}_{N \rtimes P_n} \tau, \mathrm{Res}_{N \rtimes P_n} \Psi)$ and $L \neq 0$ because $L(v_0) = 1$. Therefore, $L \in \mathrm{Hom}_H(\mathrm{Res}_H \tau, \Psi)$, which implies that $L(\tau(g)v) = L(v)$ for any $v \in \tau$, and any $g \in \mathrm{GL}_n(\mathbb{F})$.

Denote

$$S = \sum_{g \in Z_n \backslash \mathrm{GL}_n(\mathbb{F})} \sum_{n \in N} \ell(\tau(ng)v_0)\Psi^{-1}(n)\psi(\langle e_n g, e_1 \rangle).$$

We have

$$S = \sum_{g \in P_n \backslash \mathrm{GL}_n(\mathbb{F})} L(\tau(g)v_0)\psi(\langle e_n g, e_1 \rangle) = \sum_{g \in P_n \backslash \mathrm{GL}_n(\mathbb{F})} \psi(\langle e_n g, e_1 \rangle).$$

We decompose this sum through the center of $\mathrm{GL}_n(\mathbb{F})$

$$S = \sum_{g \in (\mathbb{F}^* \cdot P_n) \backslash \mathrm{GL}_n(\mathbb{F})} \sum_{a \in \mathbb{F}^*} \psi(\langle e_n a g, e_1 \rangle).$$

We have that for $t \in \mathbb{F}$,

$$\sum_{a \in \mathbb{F}^*} \psi(at) = \begin{cases} -1 & t \neq 0, \\ q - 1 & t = 0. \end{cases}$$

Therefore, we get

$$S = (q - 1) \sum_{\substack{g \in (\mathbb{F}^* \cdot P_n) \backslash \mathrm{GL}_n(\mathbb{F}) \\ \langle e_n g, e_1 \rangle = 0}} 1 - \sum_{\substack{g \in (\mathbb{F}^* \cdot P_n) \backslash \mathrm{GL}_n(\mathbb{F}) \\ \langle e_n g, e_1 \rangle \neq 0}} 1,$$

which we rewrite as

$$S = \sum_{\substack{g \in P_n \backslash \mathrm{GL}_n(\mathbb{F}) \\ \langle e_n g, e_1 \rangle = 0}} 1 - \frac{1}{|\mathbb{F}^*|} \sum_{\substack{g \in P_n \backslash \mathrm{GL}_n(\mathbb{F}) \\ \langle e_n g, e_1 \rangle \neq 0}} 1.$$

Consider the right action of $\mathrm{GL}_n(\mathbb{F})$ on $\mathbb{F}^n \setminus \{0\}$. This action is transitive. The stabilizer of e_n is the mirabolic subgroup P_n. Therefore, for $x = (x_1, \ldots, x_n) \in \mathbb{F}^n \setminus \{0\}$, we have that

$$S_x = \sum_{\substack{g \in P_n \backslash \mathrm{GL}_n(\mathbb{F}) \\ e_n g = x}} 1 = 1.$$

This implies that

$$S = \sum_{\substack{x \in \mathbb{F}^n \setminus \{0\} \\ x_1 = 0}} S_x - \frac{1}{|\mathbb{F}^*|} \sum_{\substack{x \in \mathbb{F}^n \setminus \{0\} \\ x_1 \neq 0}} S_x = (q^{n-1} - 1) - q^{n-1} = -1,$$

as required. $\qquad\qquad\square$

We move to prove Theorem A.1.

Proof. We will use Lemma A.2 in the following setup. Let $G = \mathrm{GL}_n(\mathbb{F}) \times \mathrm{GL}_n(\mathbb{F})$, and let $H = \mathrm{GL}_n(\mathbb{F})$ embedded diagonally. Let $N = \{I_n\}$ and $\Psi = 1$.

Let π be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$, then by Schur's lemma, the space

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{F})}(\pi \otimes \pi^\vee, \mathbb{C})$$

is one-dimensional. Since π is cuspidal, By [Gel'fand 1970, Theorem 2.2], the restriction of π to the mirabolic subgroup P_n is irreducible. Therefore, by Schur's lemma the space

$$\mathrm{Hom}_{P_n}(\mathrm{Res}_{P_n} \pi \otimes \mathrm{Res}_{P_n} \pi^\vee, \mathbb{C})$$

is also one-dimensional.

We take $\tau = \mathcal{W}(\pi, \psi) \otimes \mathcal{W}(\pi^\vee, \psi^{-1})$, and $\ell \colon \mathcal{W}(\pi, \psi) \otimes \mathcal{W}(\pi^\vee, \psi^{-1}) \to \mathbb{C}$ to be the functional defined on pure tensors by

$$\ell(W \otimes W') = W(I_n) \cdot W'(I_n).$$

We have that $\ell \in \mathrm{Hom}_{Z_n}(\mathrm{Res}_{Z_n} \tau, 1)$. Let $v_0 = \mathcal{J}_{\pi, \psi} \otimes \mathcal{J}_{\pi^\vee, \psi^{-1}}$.

Consider

$$\sum_{p \in Z_n \backslash P_n} \sum_{n \in N} \ell(\tau(np)v_0) \Psi^{-1}(n) = \sum_{p \in Z_n \backslash P_n} \mathcal{J}_{\pi, \psi}(p) \mathcal{J}_{\pi^\vee, \psi^{-1}}(p).$$

By [Nien 2014, Lemma 2.14], we have that if $\mathcal{J}_{\pi, \psi}(p) \neq 0$ for $p \in P_n$, then $p \in Z_n$. Therefore,

$$\sum_{p \in Z_n \backslash P_n} \mathcal{J}_{\pi, \psi}(p) \mathcal{J}_{\pi^\vee, \psi^{-1}}(p) = \sum_{p \in Z_n \backslash Z_n} \mathcal{J}_{\pi, \psi}(p) \mathcal{J}_{\pi^\vee, \psi^{-1}}(p) = 1.$$

Thus, we showed that the required properties for Lemma A.2 are satisfied.

Using Proposition 2.9, we have

$$\gamma(\pi \times \pi^\vee, \psi) = \sum_{g \in Z_n \backslash \mathrm{GL}_n(\mathbb{F})} \mathcal{J}_{\pi, \psi}(g) \mathcal{J}_{\pi^\vee, \psi^{-1}}(g) \psi(\langle e_n g^{-1}, e_1 \rangle).$$

Replacing g with g^{-1} and using Proposition 2.3, we have

$$\gamma(\pi \times \pi^\vee, \psi) = \sum_{g \in Z_n \backslash \mathrm{GL}_n(\mathbb{F})} \mathcal{J}_{\pi, \psi}(g) \mathcal{J}_{\pi^\vee, \psi^{-1}}(g) \psi(\langle e_n g, e_1 \rangle),$$

and therefore by Lemma A.2

$$\gamma(\pi \times \pi^\vee, \psi) = \sum_{g \in Z_n \backslash \mathrm{GL}_n(\mathbb{F})} \sum_{n \in N} \ell(\tau(ng)v) \Psi^{-1}(n) \psi(\langle e_n g, e_1 \rangle) = -1,$$

as required.   □

References

[Atobe and Gan 2017] H. Atobe and W. T. Gan, "Local theta correspondence of tempered representations and Langlands parameters", *Invent. Math.* **210**:2 (2017), 341–415. MR Zbl

[Chai 2019] J. Chai, "Bessel functions and local converse conjecture of Jacquet", *J. Eur. Math. Soc. (JEMS)* **21**:6 (2019), 1703–1728. MR Zbl

[Chang 1976] B. Chang, "Decomposition of Gelfand–Graev characters of $\mathrm{GL}_3(q)$", *Comm. Algebra* **4**:4 (1976), 375–401. MR Zbl

[Curtis and Shinoda 2004] C. W. Curtis and K.-i. Shinoda, "Zeta functions and functional equations associated with the components of the Gelfand–Graev representations of a finite reductive group", pp. 121–139 in *Representation theory of algebraic groups and quantum groups*, edited by T. Shoji et al., Adv. Stud. Pure Math. **40**, Math. Soc. Japan, Tokyo, 2004. MR Zbl

[Gan and Ichino 2016] W. T. Gan and A. Ichino, "The Gross–Prasad conjecture and local theta correspondence", *Invent. Math.* **206**:3 (2016), 705–799. MR Zbl

[Gan and Savin 2012] W. T. Gan and G. Savin, "Representations of metaplectic groups I: epsilon dichotomy and local Langlands correspondence", *Compos. Math.* **148**:6 (2012), 1655–1694. MR Zbl

[Gel'fand 1970] S. I. Gel'fand, "Representations of the full linear group over a finite field", *Mat. Sb. (N.S.)* **83(125)** (1970), 15–41. MR

[Jacquet et al. 1983] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, "Rankin–Selberg convolutions", *Amer. J. Math.* **105**:2 (1983), 367–464. MR Zbl

[Jiang et al. 2015] D. Jiang, C. Nien, and S. Stevens, "Towards the Jacquet conjecture on the local converse problem for p-adic GL$_n$", *J. Eur. Math. Soc. (JEMS)* **17**:4 (2015), 991–1007. MR Zbl

[Katz 1993] N. M. Katz, "Estimates for Soto–Andrade sums", *J. Reine Angew. Math.* **438** (1993), 143–161. MR Zbl

[Kondo 1963] T. Kondo, "On Gaussian sums attached to the general linear groups over finite fields", *J. Math. Soc. Japan* **15** (1963), 244–255. MR Zbl

[Kowalski 2015] E. Kowalski, "Gaps between prime numbers and primes in arithmetic progressions [after Y. Zhang and J. Maynard]", pp. 327–366 in *Journées de Géométrie Algébrique d'Orsay*, Astérisque **367-368**, Société Mathématique de France, Paris, 2015. MR Zbl

[Liu and Zhang 2022a] B. Liu and Q. Zhang, "Gamma factors and converse theorems for classical groups over finite fields", *J. Number Theory* **234** (2022), 285–332. MR Zbl

[Liu and Zhang 2022b] B. Liu and Q. Zhang, "On a converse theorem for G$_2$ over finite fields", *Math. Ann.* **383**:3-4 (2022), 1217–1283. MR Zbl

[Macdonald 1998] I. G. Macdonald, *Symmetric functions and orthogonal polynomials*, University Lecture Series **12**, American Mathematical Society, Providence, RI, 1998. MR Zbl

[Nien 2014] C. Nien, "A proof of the finite field analogue of Jacquet's conjecture", *Amer. J. Math.* **136**:3 (2014), 653–674. MR Zbl

[Nien 2017] C. Nien, "$n \times 1$ local gamma factors and Gauss sums", *Finite Fields Appl.* **46** (2017), 255–270. MR Zbl

[Piatetski-Shapiro 1983] I. Piatetski-Shapiro, *Complex representations of* GL(2, K) *for finite fields* K, Contemporary Mathematics **16**, American Mathematical Society, Providence, RI, 1983. MR Zbl

[Roditty-Gershon 2010] E.-A. Roditty-Gershon, *On gamma factors and bessel functions for representations of general linear groups over finite fields*, master's thesis, Tel Aviv University, 2010.

[Shahidi 1984] F. Shahidi, "Fourier transforms of intertwining operators and Plancherel measures for GL(n)", *Amer. J. Math.* **106**:1 (1984), 67–111. MR Zbl

[Shahidi 1990] F. Shahidi, "A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups", *Ann. of Math. (2)* **132**:2 (1990), 273–330. MR Zbl

[Shinoda and Tulunay 2005] K. Shinoda and I. Tulunay, "Representations of the Hecke algebra for GL$_4$(q)", *J. Algebra Appl.* **4**:6 (2005), 631–644. MR Zbl

[Silberger and Zink 2000] A. J. Silberger and E.-W. Zink, "The characters of the generalized Steinberg representations of finite general linear groups on the regular elliptic set", *Trans. Amer. Math. Soc.* **352**:7 (2000), 3339–3356. MR Zbl

[Soudry 1979] D. Soudry, "On gamma functions of pairs over finite fields", unpublished note, 1979.

[Ye 2019] R. Ye, "Rankin–Selberg gamma factors of level zero representations of GL$_n$", *Forum Math.* **31**:2 (2019), 503–516. MR Zbl

[Ye and Zelingher 2020] R. Ye and E. Zelingher, "Exterior square gamma factors for cuspidal representations of GL$_n$: finite field analogs and level-zero representations", *Israel J. Math.* **240**:2 (2020), 889–934. MR Zbl

[Ye and Zelingher 2021] R. Ye and E. Zelingher, "Epsilon factors of representations of finite general linear groups", *J. Number Theory* **221** (2021), 122–142. MR Zbl

[Zelingher 2023] E. Zelingher, "On values of the Bessel function for generic representations of finite general linear groups", *Adv. Math.* **434** (2023), art. id. 109314. MR Zbl

[Zhang 2014] Y. Zhang, "Bounded gaps between primes", *Ann. of Math.* (2) **179**:3 (2014), 1121–1174. MR Zbl

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