ESSENTIAL NUMBER THEORY

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Aaron Landesman

2024

vol. 3 no. 1

msp

Essential Number Theory
Vol. 3, No. 1, 2024
https://doi.org/10.2140/ent.2024.3.1

Invariance of the tame fundamental group under base change between algebraically closed fields

Aaron Landesman

We show that the tame étale fundamental group of a connected normal finite type separated scheme remains invariant upon base change between algebraically closed fields of characteristic $p \geq 0$.

1. Statement of theorem

In a wide range of number theoretic situations, one may want to compare local systems on a variety over one algebraically closed field to local systems on the base change of the variety to a larger algebraically closed field. At least when these local systems are tame, the two notions should be equivalent. Our main result, Theorem 1.1, states that this is indeed true. See Remark 1.6 for some sample uses of this result in number theory.

We now introduce notation to precisely state our main result. Let U be a connected normal finite type separated scheme over an algebraically closed base field k of characteristic p, allowing the possibility $p = 0$. Let $\pi_1(U)$ denote the étale fundamental group of U, where we leave the base point implicit. If $\overline{U}$ is a proper normal scheme containing U as a dense open subscheme, we call $\overline{U}$ a normal compactification of U. If moreover $\overline{U}$ is projective, we call $\overline{U}$ a projective normal compactification of U. Normal compactifications of normal separated finite type schemes always exist, and projective normal compactifications of normal quasiprojective schemes always exist, as described in Remark 1.8.

We next introduce notation to define the numerically tame fundamental group with respect to the above normal compactification $U \to \overline{U}$. We denote this by $\pi_1^{\text{tame}}(U)$, which implicitly depends on the normal compactification $U \subset \overline{U}$. See [Kerz and Schmidt 2010, Appendix, Example 2] for an example demonstrating this dependence on the choice of compactification. Also see Remark 1.9. This numerically tame fundamental group is a quotient of the usual étale fundamental group. Moreover, the prime-to-p étale fundamental group, whose finite quotients correspond to covers of degree relatively prime to p, is a quotient of the tame

MSC2020: 14F35.
Keywords: tameness, fundamental groups, finite étale covers, Abhyankar's lemma, Bertini's theorem.

fundamental group. Here and elsewhere, when $p = 0$, we consider every integer to be relatively prime to p so that the prime-to-p étale fundamental group is the same as the usual étale fundamental group.

First, we introduce the notion of tameness. In order to define tameness, we first recall the definition of the inertia group. Let $E \to U$ be a finite étale Galois G-cover. By convention, we assume Galois covers are connected. Let $s \in \overline{U}$ be a point, let $\overline{E}$ denote the normalization of $\overline{U}$ in the function field of E. Let $t \in \overline{E}$ map to s and define the *decomposition group* of $\overline{E} \to \overline{U}$ at t to be

$$D_{t,\overline{E}/\overline{U}} := \{g \in G : gt = t\}.$$

Then, the *inertia group* of $\overline{E} \to \overline{U}$ at t is $I_{t,\overline{E}/\overline{U}} := \ker(D_{t,\overline{E}/\overline{U}} \to \mathrm{Aut}_s(t))$. Changing our choice of t results in a conjugate inertia group, and so we use $I_{s,\overline{E}/\overline{U}}$ to denote the *inertia group* of $\overline{E} \to \overline{U}$ at s which is the conjugacy class of the subgroup $I_{t,\overline{E}/\overline{U}}$ for any t over s. Note that in the case that the residue fields of s and t agree, the inertia group agrees with the decomposition group. In particular, this automatically holds when the residue field of s is algebraically closed.

We next define tameness. We say $E \to U$ is *tame along s* if the inertia group of $\overline{E} \to \overline{U}$ at s has order prime to p. In the case $E \to U$ is not Galois, we say $E \to U$ is *tame along s* if the Galois closure of $E \to U$ is tame along s. We say $E \to U$ is *tame* if it is tame at every point $s \in \overline{U} - U$.

Finally, we come to the definition of the numerically tame fundamental group. Let $\bar{b} \in U$ denote a geometric point, which we use as a basepoint. For $E \to U$ a finite étale Galois cover, let $\mathrm{Hom}_U(\bar{b}, E)$ denote the set of maps $\bar{b} \to E$ whose composition with $E \to U$ is the given map $\bar{b} \to U$. Following [Kerz and Schmidt 2010, Section 7, page 17] the *numerically tame fundamental group*, $\pi_1^{\mathrm{tame}}(U, \bar{b})$, is by definition the automorphism group of the fiber functor which sends a tame finite étale cover $E \to U$ to $\mathrm{Hom}_U(\bar{b}, E)$. Since every connected finite étale cover is dominated by a Galois finite étale cover, this profinite group is noncanonically in bijection with the profinite set $\lim_{E \to U,\ \text{finite étale tame Galois covers}} \mathrm{Hom}_U(\bar{b}, E)$, and the latter is a torsor under the former, whose trivialization can be obtained by choosing a compatible system of basepoints in each $\mathrm{Hom}_U(\bar{b}, E)$. We remind the reader that $\pi_1^{\mathrm{tame}}(U, \bar{b})$ implicitly depends on the choice of normal compactification $U \to \overline{U}$ because the set of finite étale tame Galois covers implicitly depends on the compactification. In what follows, we will omit the basepoint $\bar{b}$ from the notation, and simply write it as $\pi_1^{\mathrm{tame}}(U)$; see [Schmidt 2002; Kerz and Schmidt 2010] for more background on the numerically tame fundamental group. In particular, when k has characteristic 0, $\pi_1(U) \simeq \pi_1^{\mathrm{tame}}(U)$.

If $X \to Y$ and $Z \to Y$ are morphisms, we denote $X \times_Y Z$ by X_Z. In the case $Z = \mathrm{Spec}\, B$, we also denote $X \times_Y Z$ by X_B.

Our main result is the following theorem.

Theorem 1.1. *Suppose k is an algebraically closed field of characteristic $p \geq 0$ and U is a connected normal separated finite type scheme over k. Let L be any algebraically closed field containing k and $\overline{U}$ any normal compactification of U. Then, the natural map $\pi_1^{\mathrm{tame}}(U_L) \to \pi_1^{\mathrm{tame}}(U)$ is an isomorphism, where tameness for covers of U_L is taken with respect to the normal compactification $U_L \subset (\overline{U})_L$.*

Using the fact that the fundamental group of a scheme is unchanged under inseparable field extensions [Stacks 2005–, Tag 0BQN], we can generalize the above theorem to the case that k and L are only separably closed.

Corollary 1.2. *Suppose k is a separably closed field of characteristic $p \geq 0$ and U is a connected normal separated finite type scheme over k. Let L be any separably closed field containing k and $\overline{U}$ any normal compactification of U. Then, the natural map $\pi_1^{\mathrm{tame}}(U_L) \to \pi_1^{\mathrm{tame}}(U)$ is an isomorphism, where tameness for covers of U_L is taken with respect to the normal compactification $U_L \subset (\overline{U})_L$.*

Remark 1.3. The result Theorem 1.1 for tame fundamental groups described above implies an analogous result for prime-to-p fundamental groups. Namely, let $\pi_1'(U)$ denote the prime-to-p fundamental group, which is the limit of automorphism groups of all Galois finite étale covers of U of degree prime to p. Because prime-to-p covers are all tame, we obtain from Theorem 1.1 that the natural map $\pi_1'(U_L) \to \pi_1'(U)$ is an isomorphism.

Remark 1.4. Theorem 1.1 is surely a folklore theorem. Nevertheless, in its complete form, the author was unable to find it in the literature. The proof written here is primarily a combination of ideas presented to me by Brian Conrad and Jason Starr. In particular, Jason Starr [2016] has written up a separate proof on mathoverflow. The proof in this note is a reorganization of the ideas presented in that post.

Remark 1.5. Many special cases of Theorem 1.1 already exist in the literature. The prime-to-p version of Theorem 1.1 as in Remark 1.3 was previously verified in [Lieblich and Olsson 2010, Corollary A.12] via a proof heavily involving stacks. Separately, this was also shown in [Orgogozo 2003, Corollaire 4.5]. The important special case that U is a curve is also mentioned in [Orgogozo and Vidal 2000, Theorem 6.1], though the proof is omitted there. In characteristic 0, a proof is given in [SGA 1 1971, Exposé XIII, Proposition 4.6] taking $Y = \operatorname{Spec} L$ in the statement there. However, that proof relies on resolution of singularities.

In the case U is proper, this was proven in [Lang and Serre 1957, Théorème 3], [Szamuely 2009, Proposition 5.6.7], [SGA 1 1971, Exposé X, Corollaire 1.8] and also [Stacks 2005–, Tag 0A49].

Remark 1.6. Theorem 1.1 is frequently used in the literature. We provide a few such instances we have come across, but expect that many more examples exist.

In the case U is quasiprojective and k has positive characteristic, the prime-to-p version as in Remark 1.3 is used in [Litt 2021, (4.2.1)] regarding arithmetic representations of fundamental groups.

In the case k has characteristic 0, this result is useful in transferring properties of the fundamental group of a variety over $\mathbb{Q}$ to the corresponding base change to $\mathbb{C}$. For example, this was used in the proof of [Zywina 2010, Lemma 5.2] in order to understand images of Galois representations of abelian varieties. Another sample use is [Landesman 2021, page 701, paragraph 3, proof of Proposition 4.9], where the result was used by the author to estimate average sizes of Selmer groups of elliptic curves over function fields.

As is evident, from the above number theoretic examples, Theorem 1.1 crops up in a variety of situations relevant to number theorists, and so may prove a useful fact in the number theorist's toolkit.

Example 1.7. The tameness hypothesis in the characteristic $p > 0$ case is crucial. If $k \subset L$ are two algebraically closed fields of characteristic $p > 0$, then for U a normal quasiprojective scheme over k, the map $\pi_1(U_L) \to \pi_1(U)$ is not in general an isomorphism. Artin–Schreier covers provide counterexamples in the case $U = \mathbb{A}^1_k$. In more detail, if $\pi_1(\mathbb{A}^1_L) \to \pi_1(\mathbb{A}^1_k)$ were an isomorphism, then the map $H^1(\mathbb{A}^1_k, \mathbb{Z}/(p)) \to H^1(\mathbb{A}^1_L, \mathbb{Z}/(p))$ would also be an isomorphism. The Artin–Schreier exact sequence identifies this with the map

$$k[x]/\{f^p - f : f \in k[x]\} \to L[x]/\{f^p - f : f \in L[x]\},$$

and this map is not surjective because ax^{p-1} for $a \in L - k$ does not lie in the image.

Remark 1.8. Note that the standard definition of the tame fundamental group is more restrictive than our definition in terms of numerical tameness, because the usual definition as in [SGA 1 1971, Expose XIII, 2.1.3] assumes U has a smooth compactification whose boundary is a normal crossings divisor. With this notion from [SGA 1 1971], the tame fundamental group is independent of the choice of compactification.

In contrast, the notion of tame fundamental group we use here makes sense for any normal finite type separated scheme U over k, since we can find a normal compactification of U as follows:

By Nagata compactification, [Stacks 2005–, Tag 0F41] if U is finite type and separated, there exists a quasicompact open immersion $U \to \overline{U}$, where $\overline{U}$ is a proper scheme. One can then replace $\overline{U}$ with its normalization to obtain a proper normal scheme $\overline{U}$, containing U as a dense open.

Moreover, in the case U is quasiprojective, we can also assume $\overline{U}$ is projective by taking any projective scheme $\overline{U}$ containing U as a dense open and then replacing $\overline{U}$ by its normalization.

Remark 1.9. Our notion of the numerically tame fundamental group agrees with the usual notion described in [SGA 1 1971, Expose XIII, 2.1.3] when the compactification of U is smooth with normal crossings boundary by [Schmidt 2002, Proposition 1.14]. This tame fundamental group is not in general independent of the choice of normal compactification; see [Kerz and Schmidt 2010, Appendix, Example 2].

2. Proof of theorem

2.1. ***Idea of proof of Theorem 1.1.*** The proof of Theorem 1.1 is fairly technically involved, but the idea is not too complicated: The key is to verify injectivity of $\pi_1^{\text{tame}}(U_L) \to \pi_1^{\text{tame}}(U)$. As a first step, we reduce from the normal case to the smooth case using that geometrically normal schemes have a dense open smooth subscheme. Then, using Chow's lemma, we reduce to the smooth quasiprojective case. We therefore assume our variety U is smooth and quasiprojective, and prove the theorem by reducing it to the curve case. For this reduction, we fiber U over a variety of one lower dimension, in which case we can apply the curve case to the geometric generic fiber of the fibration.

It remains to deal with the case that U is a quasiprojective smooth curve, which is also the most technically involved part. In this case, we can write U as $\overline{U} - D$, with $\overline{U}$ smooth and projective and D a divisor. To check injectivity, we want to check every finite étale cover of U_L is the base change of some finite étale cover of U. If E is one such cover, we can use spreading out and specialization to obtain an étale cover $U' \to U$ with the same ramification index over each point of D that E has. Then, we construct the cover E' which is the normalization of E in $E \times_{U_L} U'_L$, and verify this is the base change of a cover from k. We do so by applying the projective version of Theorem 1.1, using that E' and U'_L have projective compactifications $\overline{E}'$ and $\overline{U}'_L$ with a finite étale map $\overline{E}' \to \overline{U}'_L$.

We now indicate how we put together the steps described in the above to prove Theorem 1.1. In Section 2.2 (Lemma 2.3), we prove $\pi_1^{\text{tame}}(U_L) \to \pi_1^{\text{tame}}(U)$ is surjective. For injectivity, we first prove the map is injective in the case U is a smooth, connected, and quasiprojective curve in Section 2.4 (Proposition 2.10). We prove in Section 2.14 (Proposition 2.17) that Theorem 1.1 holds for smooth, quasiprojective varieties of all dimensions. We next verify the case that U is smooth, finite type, and separated in Proposition 2.20. Finally, we complete the proof in the case that U is normal, connected, finite type, and separated in Section 2.21.

2.2. ***Surjectivity.*** We first show $\pi_1^{\text{tame}}(U_L) \to \pi_1^{\text{tame}}(U)$ is surjective.

Lemma 2.3. *The map $\pi_1(U_L) \to \pi_1(U)$ is surjective. In particular, $\pi_1^{\text{tame}}(U_L) \to \pi_1^{\text{tame}}(U)$ is surjective.*

Proof. It suffices to verify that the pullback of any connected finite étale cover over U along $U_L \to U$ is connected, see, for example, [Stacks 2005–, Tag 0BN6]. Since L and k are both algebraically closed, the result follows from the fact that connectedness is preserved under base change between algebraically closed fields [EGA IV$_2$ 1965, Proposition 4.5.1]. □

2.4. ***Proof of injectivity in the curve case.*** We next prove injectivity for smooth connected quasiprojective curves U. For this, it suffices to show that any tame Galois finite étale cover E of U_L is the base change of some tame Galois finite étale cover of U. Note that any such cover of U, whose base change is a tame cover E of U_L, is automatically tame, since tameness can be verified after base extension. To prove such an E exists, it suffices to find a connected finite étale cover $F' \to U$ over k so that $F'_L \to U_L$ factors through E.

As a first step, we wish to find a cover U' of U with the same ramification indices as E over points in the normal projective compactification of U.

Notation 2.5. Let $k \to L$ be an inclusion of algebraically closed fields, let U be a smooth curve over k, $\overline{U}$ its regular projective compactification, and $D := \overline{U} - U$. Let $E \to U_L$ be a tame Galois finite étale cover. Let $\bar{E}$ be the normalization of $\overline{U}_L$ inside E.

Lemma 2.6. *With notation as in Notation 2.5, there exists a finite Galois cover $\overline{U}' \to \overline{U}$, étale over U, with the same ramification indices that $\bar{E}$ has over the corresponding points of D_L.*

The idea of this proof is to "spread out and specialize" E. See (2-1) for a diagram.

Proof. To construct $\overline{U}'$, we can find a finitely generated k-subalgebra $A \subset L$ and a finite étale cover $E_A \to U_A$, over A so that $(E_A)_L \simeq E$ and $E_A \to U_A$ is finite étale Galois and tame. Let $\bar{E}_A$ denote the normalization of $\overline{U}_A$ along $E_A \to U_A$. Note that, because of the Galois condition, the ramification index of a point of $\bar{E}_A$ over a point of $\overline{U}_A$ only depends on the image point in $\overline{U}_A$. We may therefore speak of the ramification index over a point of $\overline{U}_A$. Since k is algebraically closed, for any field $K \supset k$, the irreducible components of D_K arise uniquely from the irreducible components of D under scalar extension. We freely use the above observations in what follows.

Let $K(A)$ denote the fraction field of A. Note that the ramification index of $E_{K(A)}$ over each point of $D_{K(A)}$ agrees with that of E over the corresponding point of D_L. Further, we claim that for a general closed point s of $\operatorname{Spec} A$, the ramification index of $s \times_{\operatorname{Spec} A} E_A$ over a point of $s \times_{\operatorname{Spec} A} D_A \simeq D$ agrees with the ramification index of $E_{K(A)}$ over the corresponding generic point of $D_{K(A)}$. To see why this ramification index n is constant over an open set of $\operatorname{Spec} A$, recall that we are assuming the cover $E \to U$ is tame, and so, after possibly shrinking $\operatorname{Spec} A$, we may assume the same of $E_A \to U_A$. By the tameness hypothesis, the ramification index over a point can be identified with one more than the degree of the relative sheaf of differentials at that point; see, for example, [Vakil 2017, page 592]. (The point here is that if the map is locally of the form $t \mapsto us^n$, for t and s uniformizers and u a unit, then the derivative is $dt = d(us^n) = uns^{n-1}ds + s^n du$, which has order precisely $n-1$ if n is not divisible by the characteristic.) So, for $p \in D$ a geometric point, under the identification $p_A \simeq \operatorname{Spec} A$, we see that at any point of $\overline{E}_A \times_{\overline{U}_A} p_A$ over the generic point of $\operatorname{Spec} A$, $\Omega_{\overline{E}_A \times_{\overline{U}_A} p_A / p_A}$ has degree $n-1$. It follows that there is a nonempty open subscheme of $\operatorname{Spec} A$ where $\Omega_{\overline{E}_A \times_{\overline{U}_A} p_A / p_A}$ has degree $n-1$. Hence, the morphism has inertia of order n over some open subscheme of $\operatorname{Spec} A$.

Since k is an algebraically closed field, every closed point of $\operatorname{Spec} A$ has residue field k, so we may choose such a closed point $t \colon \operatorname{Spec} k \to \operatorname{Spec} A$ with the same ramification indices over D as E has over the corresponding points of D_L. Since the locus of geometric points on the base $\operatorname{Spec} A$ where the map $E_A \to U_A$ is a map of connected schemes is constructible [EGA IV$_3$ 1966, Corollaire 9.7.9], we may also assume the fiber of $E_A \to U_A$ over $t : \operatorname{Spec} k \to \operatorname{Spec} A$ is connected. Then, $U' := E_A \times_{\operatorname{Spec} A} \operatorname{Spec} k$ is our desired connected finite étale cover. Finally, we take $\overline{U}'$ to be the normalization of $\overline{U}$ along $U' \to U$. $\square$

Summarizing the situation of Lemma 2.6, we obtain the commutative diagram:

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & E_A \arrow[d] & U' \arrow[l] \arrow[d] \\
U_L \arrow[r] \arrow[d] & U_A \arrow[d] & U \arrow[l] \arrow[d] \\
\operatorname{Spec} L \arrow[r] & \operatorname{Spec} A & \operatorname{Spec} k \arrow[l, "t"']
\end{tikzcd} \tag{2-1}$$

where the four squares are fiber products.

Notation 2.7. Let $\overline{U}' \to \overline{U}$ denote the finite Galois cover of Lemma 2.6. Let $\overline{E}'$ denote the normalization of $\overline{E}$ in $\overline{E} \times_{\overline{U}_L} \overline{U}'_L$ and let $E' := \overline{E}' \times_{\overline{U}'_L} U'_L$, as in the commutative diagram:

$$\begin{tikzcd}
 & E' \arrow[rr] \arrow[dl] \arrow[dd] & & \overline{E}' \arrow[dl] \arrow[dd] & \\
E \arrow[rr] \arrow[dd] & & \overline{E} \arrow[dd] & & \\
 & U'_L \arrow[rr] \arrow[dl] & & \overline{U}'_L \arrow[dl] & \\
U_L \arrow[rr] & & \overline{U}_L & & D_L \arrow[ll]
\end{tikzcd}$$

Remark 2.8. Observe that the finite map $\overline{U}' \to \overline{U}$ of Notation 2.7 restricts to $U' \to U$ over $U \subset \overline{U}$ as U is normal. By Abhyankar's lemma [Freitag and Kiehl 1988, A I.11] (see also [SGA 1 1971, Expose XIII, 5.2]) we obtain that $\overline{U}'$ is regular, hence smooth, as we are working over an algebraically closed field k.

Although the normalization $\bar{E} \to \overline{U}_L$ of $\overline{U}_L$ in $E \to U_L$ is not necessarily étale, we now show the finite surjection $\bar{E}' \to \overline{U}'_L$ is étale.

Lemma 2.9. *With notation as in Notations 2.5 and 2.7, $\bar{E}' \to \overline{U}'_L$ is étale.*

Proof. Since $E' \to U'_L$ is étale by construction, it is enough to check $\bar{E}' \to \overline{U}'_L$ is étale over all points of $\overline{U}'_L$ lying above a point of D_L. Indeed, this is where we crucially use the assumption that $E \to U$ is tame. Since being étale can be checked in the local ring at each such point, étaleness of $\bar{E}' \to \overline{U}'_L$ follows from a version of Abhyankar's lemma, using that the ramification orders of $\overline{U}'_L \to \overline{U}_L$ and $\bar{E} \to \overline{U}_L$ agree over each point of D_L, by Lemma 2.6. For a precise form of Abhyankar's lemma applicable in this setting; see, for example, [Stacks 2005–, Tag 0EYH]. □

We are now prepared to complete the curve case of Theorem 1.1.

Proposition 2.10. *Theorem 1.1 holds in the case that U is a smooth connected curve.*

Proof. Let U be a smooth connected curve. We use notation from Notation 2.5 and Notation 2.7. By Lemma 2.3, we only need to check injectivity of $\pi_1^{\text{tame}}(U_L) \to \pi_1^{\text{tame}}(U)$. Since $E' \to U_L$ is a finite étale cover of U_L dominating $E \to U_L$, to complete the proof in the case that U is a smooth curve, it suffices to show $E' \to U_L$ is the base change of some tame finite étale cover $F' \to U$ over k. Note here that tameness of $F' \to U$ is automatic once we show it base changes to $E' \to U_L$, as tameness can be verified after base extension. We showed in Lemma 2.9 that $\bar{E}' \to \overline{U}'_L$ is a finite étale cover. Since $\overline{U}'$ is projective and normal, by [Lang and Serre 1957, Théorème 3], we obtain that there is some finite étale cover $\bar{F}' \to \overline{U}'$ over k with $\bar{E}' \simeq (\bar{F}')_L$. (Alternatively, see [Szamuely 2009, Proposition 5.6.7], [SGA 1 1971, Exposé X, Corollaire 1.8], and [Stacks 2005–, Tag 0A49].) We then find $F' := \bar{F}' \times_{\overline{U}'} U'$ is a finite étale cover of U satisfying $(F')_L \simeq E'$, and so this is the desired cover. □

2.11. ***Dominating compactifications.*** In order to complete the reduction from the higher dimensional case to the curve case, we will want to know that Theorem 1.1 holds for one compactification $U \to Y$ whenever it holds for another compactification $U \to X$ with a compatible map $X \to Y$. The next couple lemmas are devoted to verifying this.

Lemma 2.12. *Suppose W is a connected smooth separated finite type scheme over a field k and $\beta : W \subset X$ and $\alpha : W \subset Y$ are two normal compactifications with a map $f : X \to Y$ so that $\alpha = f \circ \beta$. If a finite étale Galois cover $E \to W$ is tame with respect to α, it is also tame with respect to β.*

Proof. Tameness can be checked after field extension, so we will assume k is algebraically closed. Fix a point $s \in Y$ and a preimage $t \in X$ with $f(t) = s$. Let F_Y denote the normalization of Y in the function field of E and let F_X denote the normalization of X in the function field of E. We assume F_Y is tame over s and wish to show F_X is tame over t.

We next claim that there is a map $F_X \to F_Y$. Let F denote the normalization of $F_Y \times_Y X$. It is enough to show the natural map $F \to F_X$ induced by the universal property of normalization is an isomorphism, as we then obtain a map $F_X \simeq F \to F_Y \times_Y X \to F_Y$. Because the normalization map is finite by [Stacks 2005–, Tag 03GR and Tag 035B], both F and F_X are finite over X. Therefore, the map $F \to F_X$ is a birational map which is finite (because it is quasifinite and proper) between normal schemes over k. It follows from a version of Zariski's main theorem that $F \to F_X$ is an isomorphism [Stacks 2005–, Tag 0AB1].

We now conclude the proof. Let v be a point of F_X over t and $u \in F_Y$ be the image of v under the map $F_X \to F_Y$. Since v maps to u, we have an inclusion of decomposition groups $D_{v,F_X/X} \subset D_{u,F_Y/Y}$. Since we are assuming k is algebraically closed, this is identified with an inclusion of inertia groups $I_{v,F_X/X} \subset I_{u,F_Y/Y}$. Hence, up to conjugacy, the inertia group at t is a subgroup of the inertia group at s and so tameness at s implies tameness at t. □

Lemma 2.13. *With the same notation as in Lemma 2.12, if Theorem 1.1 holds with respect to the normal compactification $W \subset X$, Theorem 1.1 also holds with respect to the compactification $W \subset Y$.*

Proof. By Lemma 2.3, it suffices to verify injectivity for the map $\pi_1^{\text{tame}}(W_L) \to \pi_1^{\text{tame}}(W)$ with respect to the compactification $W \subset Y$. Using [Szamuely 2009, Corollary 5.5.8], we can rephrase this as showing that if $k \subset L$ is an extension of algebraically closed fields and $E \to W_L$ is any tame (with respect to $W \to Y$) finite étale Galois cover, then E arises as the base change of a cover $F \to W$ over k. By Lemma 2.12, this cover is also tame with respect to the normal compactification $W \to X$. By assumption Theorem 1.1 holds for the compactification $W \to X$, and so $E \to W_L$ is the base change of a cover $F \to W$ over k, as we wished to show. □

2.14. ***Proof of injectivity in the smooth and quasiprojective case.*** In this section, specifically in Proposition 2.17, we prove Theorem 1.1 in the case that U is a smooth connected quasiprojective variety. To start, we use Bertini's theorem to obtain a fibration away from a codimension 2 subset of U. This fibration will allow us to run an induction on the dimension.

Proposition 2.15. *Let U be a smooth connected quasiprojective variety of dimension $d > 1$. Choose a projective normal compactification $U \subset \overline{U}$. There is a closed subscheme $Z \subset U$ of codimension at least 2 and a projective normal compactification $U - Z \to X$ satisfying the following three properties:*

(1) *The closed subscheme Z lies in the smooth locus of $\overline{U}$.*

(2) *There is a map $X \to \overline{U}$ so that the composition $U - Z \to X \to \overline{U}$ agrees with the composition $U - Z \to U \to \overline{U}$.*

(3) *There is a dominant generically smooth map $\alpha \colon X \to \mathbb{P}^{d-1}_k$ with geometrically irreducible generic fiber.*

Proof. Let $U \subset \overline{U}$ be the given projective normal compactification. Choose an embedding $U \subset \overline{U} \subset \mathbb{P}^n_k$. Replacing $\mathbb{P}^n_k$ by the span of $\overline{U}$ in $\mathbb{P}^n_k$, we may also assume $\overline{U}$ is nondegenerate. Choose a general codimension d plane $H \subset \mathbb{P}^n_k$ such that $H \cap \overline{U}$ is smooth of dimension 0, $H \cap (\overline{U} - U) = \varnothing$, and so that, if $J' \subset \mathbb{P}^n_k$ is a general codimension $d-1$ plane containing H, we have $J' \cap \overline{U}$ is smooth and geometrically irreducible of dimension 1. This is possible because $\overline{U}$ is normal, hence smooth away from codimension 2, and by Bertini's theorem, as in [Jouanolou 1983, Theoreme 6.10(2) and (3)].

Define $Z := H \cap U = H \cap \overline{U}$ for H as in the previous paragraph. For H general as above, the following three conditions are satisfied: $Z \subset U$ has codimension at least 2, Z does not meet $\overline{U} - U$, and, for a general plane J' containing H, the intersection $J' \cap \overline{U}$ is smooth and geometrically irreducible. The second property verifies condition (1) in the statement because it shows $Z \subset U \subset \overline{U}$ and U is contained in the smooth locus of $\overline{U}$. Take $X \to \overline{U}$ to be the blow up of $\overline{U}$ along $Z \subset \overline{U}$. This verifies condition (2) in the statement.

To conclude, we will show condition (3) in the statement holds. Namely, we will show there is a dominant map $X \to \mathbb{P}^{d-1}_k$ whose generic fiber is smooth and geometrically irreducible. Geometrically, this map is induced by projection of $\overline{U}$ away from the plane H, and sends a point $x \in U - Z$ to $\operatorname{Span}(x, H)$, where we view $\operatorname{Span}(x, H)$ as a point of $\mathbb{P}^{d-1}_k$ parametrizing codimension $d-1$ planes $J' \subset \mathbb{P}^n_k$ containing H. The above-described map $U - Z \to \mathbb{P}^{d-1}_k$ extends to a map on the blow up $X = \operatorname{Bl}_{\overline{U} \cap H} \overline{U} \to \mathbb{P}^{d-1}_k$, where the fiber over a point $[J'] \in \mathbb{P}^{d-1}_k$ (parametrizing codimension $d-1$ planes $J' \subset \mathbb{P}^n_k$ containing H) is $J' \cap \overline{U}$. By construction of H so that $J' \cap \overline{U}$ is smooth and geometrically irreducible for a general codimension $d-1$ plane $J' \subset \mathbb{P}^{d-1}_k$ containing H, the generic fiber of the map $X \to \mathbb{P}^{d-1}_k$ is smooth and geometrically irreducible. $\square$

Assuming we have a fibration as in Proposition 2.15, we next show that the fiber of a tame Galois finite étale cover $E \to U_L$, when restricted to the generic point of $\mathbb{P}^{d-1}_L$, is the base change of a Galois finite étale cover over the generic point of $\mathbb{P}^{d-1}_k$.

Proposition 2.16. *Assume U is a smooth connected k-variety of dimension $d \geq 1$ with a normal projective compactification $U \to \overline{U}$ and a dominant generically smooth map $\alpha \colon \overline{U} \to \mathbb{P}^{d-1}_k$ with geometrically irreducible generic fiber. Let η_k*

denote the generic point of $\mathbb{P}^{d-1}_k$ *and* η_L *denote the geometric generic point of* $\mathbb{P}^{d-1}_L$. *Any given tame finite étale Galois cover* $E \to U_L$ *restricts to a Galois finite étale cover* $E_{\eta_L} \to U_{\eta_L}$ *(with respect to the compactification* $U_{\eta_L} \subset \overline{U}_{\eta_L}$*) which is the base change of some Galois finite étale cover* $F_{\eta_k} \to U_{\eta_k}$.

Proof. Let $\bar\eta_k$ and $\bar\eta_L$ denote compatible algebraic geometric generic points of $\mathbb{P}^{d-1}_k$ and $\mathbb{P}^{d-1}_L$, with corresponding generic points η_k and η_L. By this, we mean that $\bar\eta_k$ has residue field which is the algebraic closure of $\kappa(\eta_k)$ and similarly for L. Moreover, they are compatible in the sense that we specify an embedding $\kappa(\bar\eta_k) \to \kappa(\bar\eta_L)$ restricting to the inclusion $\kappa(\eta_k) \to \kappa(\eta_L)$. Let $E_{\bar\eta_L} := E \times_{\mathbb{P}^{d-1}_L} \bar\eta_L$, which we note is smooth and of dimension 1. Because $E \to U_L$ is tame with respect to $U_L \to \overline{U}_L$, we obtain that $E_{\bar\eta_L} \to U_{\bar\eta_L}$ is tame with respect to $U_{\bar\eta_L} \to \overline{U}_{\bar\eta_L}$. By the curve case of Theorem 1.1, shown in Proposition 2.10, $E_{\bar\eta_L}$ arises as the base change of some cover $F_{\bar\eta_k} \to U_{\bar\eta_k}$. That is, $(F_{\bar\eta_k})_{\bar\eta_L} \simeq E_{\bar\eta_L}$.

To conclude the proof, we only need realize $F_{\bar\eta_k} \to U_{\bar\eta_k}$ as the base change of a map over η_k so that the above isomorphism $(F_{\bar\eta_k})_{\bar\eta_L} \simeq E_{\bar\eta_L}$ is the base change of an isomorphism over η_L. For K a field, we use K^s to denote its separable closure. We can realize $\bar\eta_k \to \eta_k$ as the composition of a purely inseparable morphism $\bar\eta_k \to \eta^s_k$ and a separable morphism $\eta^s_k \to \eta_k$ by taking $\eta^s_k := \operatorname{Spec} \kappa(\eta_k)^s$. Since $\bar\eta_k \to \eta^s_k$ is a universal homeomorphism, the same is true of $U_{\bar\eta_k} \to U_{\eta^s_k}$, and so the map induces an isomorphism of étale fundamental groups $\pi_1(U_{\bar\eta_k}) \to \pi_1(U_{\eta^s_k})$ [Stacks 2005–, Tag 0BQN]. It follows that $F_{\bar\eta_k} \to U_{\bar\eta_k}$ is the base change of a morphism $F_{\eta^s_k} \to U_{\eta^s_k}$ over η^s_k. Moreover, by spreading out, there is a finite Galois extension $\eta'_k \to \eta_k$ so that $F_{\eta^s_k} \to U_{\eta^s_k}$ is the base change of a morphism $F_{\eta'_k} \to U_{\eta'_k}$ over η'_k. We next want to verify this is the base change of a map over η_k, which we will do by producing descent data along the extension $\eta'_k \to \eta_k$.

We next set up notation for descent data. Observe that $\eta_k \simeq \operatorname{Spec} k(x_1, \dots, x_n)$ and $\eta_L \simeq \operatorname{Spec} L(x_1, \dots, x_n)$. Let $M := \Gamma(\eta'_k, \mathcal{O}_{\eta'_k})$ so that $\eta'_k = \operatorname{Spec} M$. It follows that the two maps of schemes $\eta'_k \to \eta_k$ and $\eta_L \to \eta_k$ correspond to the extensions of fields $k(x_1, \dots, x_n) \to M$ and $k(x_1, \dots, x_n) \to L(x_1, \dots, x_n)$. It is a standard fact that these are linearly disjoint, see Lemma A.3. Let $M_L := M \otimes_k L$. Since M and $L(x_1, \dots, x_n)$ are linearly disjoint, base extension defines a bijective map

$$\operatorname{Gal}(M/k(x_1, \dots, x_n)) \simeq \operatorname{Gal}(M_L/L(x_1, \dots, x_n)).$$

We denote the above Galois group by G. As described in [Bosch et al. 1990, Section 6.2, Example B], specifying descent data for $F_{\eta'_k} \to U_{\eta'_k}$ along $\eta'_k \to \eta_k$, is equivalent to specifying an isomorphism $\phi_{F,k,\sigma} : F_{\eta'_k} \to F_{\eta'_k}$ for each $\sigma \in G$, defining an action of G on $F_{\eta'_k}$. (We warn the reader that the action is only defined over η_k and not over η'_k.) Since $U_{\eta'_k}$ is the base change of U_{η_k}, we do have descent data $\phi_{U,k,\sigma} : U_{\eta'_k} \to U_{\eta'_k}$. The descent data $\phi_{F,k,\sigma}$ we wish to produce should live over the descent data for $\phi_{U,k,\sigma}$, in the sense the diagram

$$\begin{array}{ccc} F_{\eta'_k} & \xrightarrow{\phi_{F,k,\sigma}} & F_{\eta'_k} \\ \downarrow & & \downarrow \\ U_{\eta'_k} & \xrightarrow{\phi_{U,k,\sigma}} & U_{\eta'_k} \end{array} \tag{2-2}$$

should commute. Let $\eta'_L := \eta'_k \times_{\eta_k} \eta_L$. Since we do have descent data for $F_{\eta'_L} \to U_{L'}$ along $\eta'_L \to \eta_L$, we have $\phi_{F,L,\sigma}$ and $\phi_{U,L,\sigma}$ so that

$$\begin{array}{ccc} F_{\eta'_L} & \xrightarrow{\phi_{F,L,\sigma}} & F_{\eta'_L} \\ \downarrow & & \downarrow \\ U_{\eta'_L} & \xrightarrow{\phi_{U,L,\sigma}} & U_{\eta'_L} \end{array} \tag{2-3}$$

commutes.

We wish to show that $\phi_{F,L,\sigma}$ is the base change of a unique map $\phi_{F,k,\sigma}$ along $\operatorname{Spec} L \to \operatorname{Spec} k$. Indeed, consider the η_k scheme $\operatorname{Aut}_{\phi_{U,k,\sigma}}(F_{\eta'_k})$ of automorphisms of $F_{\eta'_k}$ over the specified automorphism $\phi_{U,k,\sigma}$ of $U_{\eta'_k}$. Note that $\operatorname{Aut}_{\phi_{U,k,\sigma}}(F_{\eta'_k}) \times_{\eta_k} \eta_L \simeq \operatorname{Aut}_{\phi_{U,L,\sigma}}(F_{\eta'_L})$. Moreover, for $N \in \{k, L\}$, since the automorphisms of $F_{\eta'_N}$ over $\phi_{U,N,\sigma}$ are given by composing any given automorphism over $\phi_{U,N,\sigma}$ with an automorphisms of $F_{\eta'_N}$ over $U_{\eta'_N}$, $\operatorname{Aut}_{\phi_{U,k,\sigma}}(F_{\eta'_k})$ and $\operatorname{Aut}_{\phi_{U,L,\sigma}}(F_{\eta'_L})$ are both G torsors. Since the residue field of each point of $\operatorname{Aut}_{\phi_{U,k,\sigma}}(F_{\eta'_k})$ over η_k is linearly disjoint from the field extension $\kappa(\eta_k) \to \kappa(\eta_L)$ by Lemma A.3, there is a bijection between the points of $\operatorname{Aut}_{\phi_{U,k,\sigma}}(F_{\eta'_k})$ and $\operatorname{Aut}_{\phi_{U,L,\sigma}}(F_{\eta'_L})$. Since the latter is the trivial G torsor, we also obtain $\operatorname{Aut}_{\phi_{U,k,\sigma}}(F_{\eta'_k})$ is the trivial G torsor. In other words there is a unique map $\phi_{F,k,\sigma}$ over $\phi_{U,k,\sigma}$ whose base change to η_L is $\phi_{F,L,\sigma}$. Choosing these $\phi_{F,k,\sigma}$ whose base change is $\phi_{F,L,\sigma}$, we find that the $\phi_{F,k,\sigma}$ define descent data (because the $\phi_{F,L,\sigma}$ do). Hence, $F_{\eta'_k} \to U_{\eta'_k}$ is the base change of a map $F_{\eta_k} \to U_{\eta_k}$, as desired. □

We now complete the proof of Theorem 1.1 in the case U is smooth quasiprojective with a projective normal compactification. Since U is quasiprojective, recall that such a projective normal compactification exists by Remark 1.8.

Proposition 2.17. *Theorem 1.1 holds when $U \to \overline{U}$ is a projective normal compactification and U is smooth and quasiprojective.*

Proof. The case $d = 1$ holds by Proposition 2.10, and $d = 0$ is trivial, so we now assume $d > 1$.

By Proposition 2.15, there is a $Z \subset U \subset \overline{U}$ and a projective normal compactification $U - Z \to X$ satisfying the properties given there. Then, since Z as in Proposition 2.15 has codimension at least 2, $\pi_1^{\text{tame}}(U - Z) \simeq \pi_1^{\text{tame}}(U)$ because the tame fundamental group of a smooth variety is unchanged by removing any set of codimension at least 2, as shown in Lemma A.2. Above, the tameness conditions for both schemes $U - Z$ and U are taken with respect to the projective normal compactification $\overline{U}$.

Observe that Z is in the smooth locus of $\overline{U}$ by Proposition 2.15(1) and $U \to X$ is a normal compactification of U. Using Proposition 2.15(2) to verify the hypotheses of Lemma 2.13, it suffices to prove Theorem 1.1 for the compactification $U - Z \to X$ in place of $U \to \overline{U}$.

For the remainder of the proof, we now rename $U - Z$ as U and X as $\overline{U}$. In particular, by Proposition 2.15(3), we may now assume there is a generically smooth dominant map $\overline{U} \to \mathbb{P}^{d-1}_k$.

With notation as in Proposition 2.16, any tame Galois finite étale cover $E_L \to U_L$ restricts to a cover $E_{\eta_L} \to U_{\eta_L}$ which is the base change of a tame Galois finite étale cover $F_{\eta_k} \to U_{\eta_k}$.

Define F to be the normalization of U in the function field of F_{η_k}. We claim that $F_L \simeq E_L$ as covers of U_L. This will complete the proof, as it implies $F \to U$ is tame finite étale and connected, since the same is true of $F_L \to U_L$.

To see $F_L \simeq E_L$ as covers of U_L, we know E_L is the normalization of U_L in $K(E_L) = K(E_{\eta_L})$. Further, since L/k has a separating transcendence basis (since k is algebraically closed, hence perfect), it follows that F_L is normal and has function field $K(E_L)$. Moreover, the universal property of normalization induces a birational map $F_L \to E$. Since both F_L and E are finite over U_L, the map $F_L \to E$ is finite. It then follows from a version of Zariski's main theorem that $F_L \to E$ is an isomorphism [Stacks 2005–, Tag 0AB1]. □

2.18. ***Proof of injectivity in the smooth case.*** Having verified the smooth quasiprojective case, we next verify the smooth finite type and separated case. The general idea is to use Chow's lemma to reduce to the projective case, but there are a number of technical details. We start by explaining the geometric consequence that Chow's lemma gives us.

Lemma 2.19. *Suppose that U is a smooth separated scheme of finite type over an algebraically closed field k with a normal compactification $\alpha : U \to \overline{U}$. There is a closed subscheme $Z \subset U$ of codimension at least 2 and a normal projective compactification $\beta : U - Z \to X$ with a projective map $f : X \to \overline{U}$ so that $\alpha|_{U-Z} = f \circ \beta$.*

Proof. Using Chow's lemma, we can find a projective scheme X with a birational projective map $f : X \to \overline{U}$; see [Stacks 2005–, Tag 0200 and Tag 0201].

We next construct a subscheme $Z \subset U$ of codimension at least 2 and a birational map $\beta : U - Z \to X$. Since f is birational, there is a dense open $W \subset U$ over which f is an isomorphism, so we obtain a map $g : W \to X$ which is an isomorphism onto its image. Because U is regular in codimension 1 and X is proper, there is a scheme $Z \subset U$ of codimension at least 2 so that $g : W \to X$ extends to a birational map $\beta : U - Z \to X$. Now, restricting f, we get a map $f' : f^{-1}(\alpha(U - Z)) \to U - Z$.

We claim β factors through $f^{-1}(\alpha(U - Z))$ and thus defines a section to f'. Indeed, consider the composition $f \circ \beta : U - Z \to X \to \overline{U}$. This agrees with α over the dense open W, and hence agrees with the given open immersion $U - Z \to U \xrightarrow{\alpha} \overline{U}$ on W. Because $U - Z$ is separated, $f \circ \beta$ must agree with the above open immersion on all of $U - Z$. This implies that β sends $U - Z$ to $f^{-1}(\alpha(U - Z))$.

Let $\beta' : U - Z \to f^{-1}(\alpha(U - Z))$ denote the map whose composition with $f^{-1}(\alpha(U - Z)) \to X$ is β. We will show next that β' is a closed immersion. We have seen above that β' is a section to f'. Therefore, β' is a monomorphism. Moreover since f' is projective, hence proper, β' is also proper, as any section to a proper map is proper via the cancellation theorem [Vakil 2017, 10.1.19] applied to the composition $f' \circ \beta'$. Since β' is a proper monomorphism, it is a closed immersion [Stacks 2005–, Tag 04XV], hence projective.

We now conclude the proof. By the above, the composition $U - Z \to f^{-1}(\alpha(U - Z)) \to X$ is the composition of a closed immersion and an open immersion into a projective scheme. This implies $U - Z$ is quasiprojective, and $U - Z \to X$ is a normal projective compactification, as desired. By construction, $\alpha|_{U-Z} = f \circ \beta$. $\square$

We are now ready to reduce the proof of Theorem 1.1 to the general smooth case over an algebraically closed field, which follows without much difficulty by applying the above lemma.

Proposition 2.20. *Theorem 1.1 holds when U is smooth.*

Proof. Recall that U is now smooth, finite type, and separated over $k = \bar{k}$ but not necessarily quasiprojective. Using Nagata compactification [Stacks 2005–, Tag 0F41] as described in Remark 1.8, we can find a normal compactification $\alpha : U \to \overline{U}$. By Lemma 2.19, there is a closed subscheme $Z \subset U$ of codimension at least 2 and a projective normal compactification $\beta : U - Z \to X$ with a projective map $f : X \to \overline{U}$ so that $\alpha|_{U-Z} = f \circ \beta$.

For $Z \subset U$ of codimension at least 2 as in Lemma 2.19, we have $\pi_1^{\text{tame}}(U) \simeq \pi_1^{\text{tame}}(U - Z)$ by Lemma A.2. Therefore, it is enough to prove the theorem for the compactification $U - Z \to \overline{U}$. By Lemma 2.13, it is enough to prove the theorem for the compactification $U - Z \to X$ in place of $U - Z \to \overline{U}$. Finally, the theorem holds for the projective compactification $U - Z \to X$ by Proposition 2.17. $\square$

2.21. ***Proof of injectivity in the general case.*** We now complete the proof of the theorem for normal connected quasiprojective schemes, using that we have proven it for smooth U.

Proof of Theorem 1.1. By Lemma 2.3, the map $\pi_1^{\text{tame}}(U_L) \to \pi_1^{\text{tame}}(U)$ is surjective. To complete the proof, we wish to show it is injective. To verify the map $\pi_1^{\text{tame}}(U_L) \to \pi_1^{\text{tame}}(U)$ is injective, by [Szamuely 2009, Corollary 5.5.8], it is enough to show that if $E \to U_L$ is any connected finite étale cover, then E is isomorphic to $\widetilde{F}_L$ for $\widetilde{F} \to U$ some connected finite étale cover. To see this, start with some $E \to U_L$. Let $W \subset U$ denote the maximal dense smooth open subscheme of U. Since we have already shown the map $\pi_1^{\text{tame}}(W_L) \to \pi_1^{\text{tame}}(W)$ is an isomorphism in Proposition 2.20, we know that $E \times_{U_L} W_L$ is isomorphic to the base change of some finite étale cover $F \to W$ along $\operatorname{Spec} L \to \operatorname{Spec} k$. Let $\widetilde{F}$ denote the normalization of U in F. Since U is normal, $\widetilde{F} \to U$ is a finite morphism. The setup this far is summarized by the commutative diagrams:

$$\begin{array}{ccc} E \times_{U_L} W_L & \longrightarrow & E \\ \downarrow & & \downarrow \\ W_L & \longrightarrow & U_L \end{array} \qquad\qquad \begin{array}{ccc} F & \longrightarrow & \widetilde{F} \\ \downarrow & & \downarrow \\ W & \longrightarrow & U \end{array}$$

To complete the proof, we only need to show $\widetilde{F} \to U$ is tame finite étale and there is an isomorphism $\widetilde{F}_L \simeq E$ over U_L. Indeed, since $\widetilde{F}$ is normal and finite over U, the base change $\widetilde{F}_L$ is also normal and finite over U_L. It follows that $\widetilde{F}_L$ is the normalization of U_L in $F_L \simeq E \times_{U_L} W_L$. But, since E is also the normalization of U_L in $E \times_{U_L} W_L$, we obtain that $E \simeq \widetilde{F}_L$. Since $\widetilde{F}_L \simeq E \to U_L$ is tame finite étale, it follows that $\widetilde{F} \to U$ is also tame finite étale, completing the proof. □

Appendix: Collected lemmas

In this appendix, we collect several lemmas used in the course of the above proof. These are all quite standard, and we only include them for completeness. We include them in this appendix and not in the body so as not to distract from the flow of the proof.

We begin with two standard results on how the tame fundamental group behaves upon passing to open subschemes. These follow from the usual well-known versions for the full étale fundamental group, but we spell out the usual proof for the reader's convenience.

Lemma A.1. *Let Y be a normal quasiprojective connected scheme and $W \subset Y$ be a nonempty open. Then the natural map $\pi_1(W) \to \pi_1(Y)$ is surjective. In particular, $\pi_1^{\text{tame}}(W) \to \pi_1^{\text{tame}}(Y)$ is surjective, where tameness for Y is taken with respect to a projective normal compactification $Y \to \overline{Y}$ and tameness for W is taken with respect to $W \to Y \to \overline{Y}$.*

Proof. Assuming surjectivity of $\pi_1(W) \to \pi_1(Y)$, surjectivity of $\pi_1^{\mathrm{tame}}(W) \to \pi_1^{\mathrm{tame}}(Y)$ follows from commutativity of the square

$$\begin{array}{ccc} \pi_1(W) & \longrightarrow & \pi_1(Y) \\ \downarrow & & \downarrow \\ \pi_1^{\mathrm{tame}}(W) & \longrightarrow & \pi_1^{\mathrm{tame}}(Y) \end{array} \tag{A-1}$$

and the fact that the vertical maps are surjective.

It remains to verify $\pi_1(W) \to \pi_1(Y)$ is surjective. We need to check any connected finite étale cover $E \to Y$ has pullback $E \times_Y W$ which is also connected. First, we claim E is normal. Indeed, since normality is equivalent to being R1 and S2, E is normal because the properties of being R1 and S2 are preserved under étale morphisms. Therefore, E is normal and connected, hence integral. Then, $E \times_Y W$ is a nonempty open subscheme of the integral scheme E, hence connected. □

For a proof of the next lemma in the case of fundamental groups, instead of tame fundamental groups; see [Szamuely 2009, Corollary 5.2.14].

Lemma A.2. *Let U be a connected smooth k-scheme and $V \subset U$ a closed subscheme of codimension at least 2. Then the natural map $\pi_1^{\mathrm{tame}}(U - V) \to \pi_1^{\mathrm{tame}}(U)$ is an isomorphism, where tameness for U is taken with respect to a projective normal compactification $U \to \overline{U}$, and tameness for $U - V$ is taken with respect to $U - V \to U \to \overline{U}$.*

Proof. The map is surjective by Lemma A.1, so it suffices to verify injectivity. For this, we have to show that any tame finite étale cover $E \to U - V$ extends uniquely to a tame finite étale cover E' of U. If $E \to U - V$ is tame, it follows from the definition of tameness and our compatible choices of compactifications that any extension will automatically also be tame. Hence, it suffices to show there is a unique extension. Uniqueness is immediate because E' is necessarily normal, and hence must be the normalization of U in E. So it suffices to check that the normalization E' of U in E is a finite étale cover of U, restricting to E over $U - V$. That E' restricts to E over $U - V$ is clear and $E' \to U$ is finite by finiteness of normalization. Finally, $E' \to U$ is étale by Zariski–Nagata purity as in [SGA 1 1971, Exposé X, Théorème 3.1] because it is étale over all codimension 1 points and U is smooth. □

Finally, we record a field-theory result on linear disjointness of certain extensions.

Lemma A.3. *Suppose $k \to L$ are algebraically closed fields. Let $k(x_1, \ldots, x_n) \to F$ by any finite separable extension. Then $k(x_1, \ldots, x_n) \to F$ and $k(x_1, \ldots, x_n) \to L(x_1, \ldots, x_n)$ are linearly disjoint extensions.*

Proof. We want to show the only finite separable extension of $k(x_1, \dots, x_n)$ in $L(x_1, \dots, x_n)$ is $k(x_1, \dots, x_n)$. To this end, let F be some finite separable extension of $k(x_1, \dots, x_n)$ in $L(x_1, \dots, x_n)$. So, to see F is equal to $k(x_1, \dots, x_n)$, it suffices to show $F \otimes_{k(x_1,\dots,x_n)} F$ is a domain. We have a containment

$$F \otimes_{k(x_1,\dots,x_n)} F \subset L(x_1, \dots, x_n) \otimes_{k(x_1,\dots,x_n)} L(x_1, \dots, x_n),$$

so it suffices to show

$$L(x_1, \dots, x_n) \otimes_{k(x_1,\dots,x_n)} L(x_1, \dots, x_n)$$

is a domain. Indeed, this is a localization of

$$L[x_1, \dots, x_n] \otimes_{k[x_1,\dots,x_n]} L[x_1, \dots, x_n] \simeq (L \otimes_k L)[x_1, \dots, x_n],$$

so it suffices to show $L \otimes_k L$ is a domain. This then holds because L is a domain, and a domain over an algebraically closed field is still a domain upon base change to any larger algebraically closed field, i.e., the property of being geometrically integral is preserved under base change between algebraically closed fields. □

Acknowledgements

I would like to thank Brian Conrad, Jason Starr, and Sean Cotner for key ideas in the proof. I also thank several anonymous referees for numerous incredibly thorough readings, many extremely helpful comments, and multitudes of thoughtful suggestions. Additionally, I thank Sean Cotner for a detailed reading, and thorough comments. I thank Peter Haine, Daniel Litt, Martin Olsson, Tamás Szamuely and the referees for helpful comments. This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE-1656518.

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Received 15 Jul 2022. Revised 19 Dec 2023.

AARON LANDESMAN:

landesman@math.harvard.edu
Department of Mathematics FAS, Harvard University, Cambridge, MA, United States

ESSENTIAL NUMBER THEORY

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Essential Number Theory (ISSN 2834-4634 electronic, 2834-4626 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

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ESSENTIAL NUMBER THEORY

2024 vol. 3 no. 1

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