ESSENTIAL NUMBER THEORY

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2024

vol. 3 no. 1

msp

Essential Number Theory
Vol. 3, No. 1, 2024
https://doi.org/10.2140/ent.2024.3.19

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Following Arthur, we present a trace formula argument proving that discrete automorphic representations on (possibly non-quasisplit) classical groups weakly transfer to general linear groups in the sense that the transfer is compatible with Satake parameters and infinitesimal characters. This result is conditional on the weighted fundamental lemma but no more. We explain how the weak transfer leads to the existence of automorphic Galois representations valued in the C-groups, as formulated by Buzzard and Gee, when the automorphic representations are C-algebraic and satisfy suitable regularity conditions.

1. Introduction

Classical groups are the isometry groups of symmetric, symplectic or (skew-) Hermitian forms. They play vital roles in many areas of mathematics. In number theory they are prominent in the theory of automorphic forms and the Langlands program. One of the key questions is how to transfer automorphic representations on classical groups to general linear groups as predicted by the Langlands functoriality conjecture. There are two main approaches: the converse theorem and the trace formula.

The converse theorem was successfully employed to transfer cuspidal generic automorphic representations on quasisplit classical groups over number fields by Cogdell, Kim, Krishnamurthy, Piatetski-Shapiro, Shahidi, and others; see [Cogdell et al. 2011]. Lomeli [2009] proved the analogous result for split classical groups over global function fields. There is a prospect, arising from the work by Cai,

MSC2020: 11F70, 11F72, 11R39.
Keywords: Langlands functoriality, Langlands correspondence, trace formula, automorphic representations, Galois representations.

Friedberg, Ginzburg and Kaplan [Cai et al. 2019], that the converse theorem method may extend to all classical groups without any genericity condition.

It is perhaps fair to say that the trace formula method requires more groundwork to get started, notably the stabilization of the trace formula and the fundamental lemma as well as their twisted analogues. Since the tools are still developing over global function fields, see [Labesse and Lemaire 2021], we will concentrate on the *number field case* throughout the paper. When it works, the trace formula leads to extra information beyond the existence of transfer to general linear groups, such as parametrization of local and global packets of representations characterized by endoscopic character identities and the Arthur multiplicity formula. This has been carried out for

- quasisplit symplectic and special orthogonal groups by Arthur [2013];
- quasisplit unitary groups by Mok [2015];
- non-quasisplit unitary groups by Kaletha, Minguez, Shin and White [2014], under temperedness and pure-inner-twist hypotheses;
- non-quasisplit odd special orthogonal groups by Ishimoto [2023], under a temperedness hypothesis;
- certain non-quasisplit symplectic and special orthogonal groups under a cohomological hypothesis at infinity by Taïbi [2019].

It is worth mentioning that Clozel and Labesse (see [Labesse 2011]) proved unconditional results on the transfer of cohomological automorphic representations on unitary groups to those on general linear groups (without full endoscopic classifications for them). However the results in the bulleted list are conditional on the proof of the weighted fundamental lemma and some results to be proven. (By "some results", we mean the projected papers in [Arthur 2013], which the author cites as [A25], [A26] and [A27], as well as their analogues for unitary groups, which are also missing at the time of writing this article.) The weighted fundamental lemma is known for split groups by Chaudouard and Laumon [2010; 2012] but it is also needed for nonsplit groups. We also need the "nonstandard weighted fundamental lemma" formulated by Waldspurger [2009] in the stabilization of the twisted trace formula. See the paragraph above Theorem 1.1.2 for further remarks.

Apart from the conditionality mentioned above, the trace formula is believed to yield complete results for all non-quasisplit classical groups as outlined in [Arthur 2013, Chapter 9]. This is a central problem to work out in its own right. It is also pivotal for arithmetic applications involving Shimura varieties since non-quasisplit groups appear naturally in that context. A full solution of the problem would take years to complete.

The first goal of this paper is to explain that Arthur's argument [2013, Chapter 3] is already enough to establish the existence of a weak transfer for *all* classical groups. He states the results for quasisplit symplectic and special orthogonal groups but the argument works generally. Indeed, Arthur himself [2013, Proposition 9.5.2] made this observation; our intention is merely to bring this part of his work to the broader audience.

Here a weak transfer means a transfer of automorphic representations between two reductive groups related via a morphism of their L-groups, such that the Satake parameters at finite places and the infinitesimal characters at infinite places are transported via the L-morphism; see Section 1.1 below. Our argument is relatively simple as long as the stabilization of the twisted (and untwisted) trace formula is accepted. In particular we do not need [A25], [A26] and [A27] from [Arthur 2013], or their analogues mentioned above (nor the main theorems of [Arthur 2013; Mok 2015]). Rather, the weak transfer at hand is conditional only on the weighted fundamental lemma for nonsplit groups and the nonstandard weighted fundamental lemma.

Our approach to the weak transfer is close to Taïbi's [2022], see Remark A.5 therein. The difference is that his argument and theorem are optimized for the intended application. As such, he accepts the main results of [Arthur 2013] and makes a regularity hypothesis to deal with non-quasisplit symplectic and special orthogonal groups. By contrast, we keep a minimal hypothesis as mentioned above and also treat the case of unitary groups in a uniform manner.

As an application and our second goal, we verify Buzzard and Gee's conjecture on the existence of automorphic Galois representations, which amounts to one direction of the global Langlands correspondence, for classical groups. Besides the weak transfer, a crucial ingredient comes from what is known in the construction of automorphic Galois representations for general linear groups. Once this is taken for granted, it is a series of elementary exercises to deduce Buzzard and Gee's conjecture for classical groups (modulo some technical hypotheses discussed below). While we do not claim originality, it may be of interest to see all classical groups treated side by side in the language of C-groups. Previous works usually considered these groups separately; e.g., see [Kret and Shin 2020, Section 6; 2023, Section 2] and the references at the start of Section 3.4 below.

Now we describe the two main goals more precisely in Sections 1.1 and 1.2 below. They correspond to Sections 2 and 3 in the main body of the paper.

1.1. ***Weak transfer.*** Let G and $\tilde{G}$ be connected reductive groups over a number field F, and $\tilde{\xi} : {}^L G \to {}^L \tilde{G}$ be a morphism of L-groups (either the Galois or Weil form, see [Arthur 2005, Section 26]). Assume that $\tilde{G}$ is quasisplit over F. Let S be a finite set of places of F including all infinite places such that G, $\tilde{G}$, and $\tilde{\xi}$ are unramified over F_v for all places $v \notin S$. (For $\tilde{\xi}$, this means that $\tilde{\xi}$ is inflated from an L-morphism with respect to the Galois or Weil group for an extension unramified at v.) At each $v \notin S$, the map $\tilde{\xi}$ induces a map $\tilde{\xi}_*$ from irreducible unramified representations of $G(F_v)$ to those of $\tilde{G}(F_v)$ (on the level of isomorphism classes) by Satake transform, which amounts to the unramified local Langlands correspondence for each of G and $\tilde{G}$.

A weak form of the Langlands functoriality conjecture is the following, see [Langlands 1970, Questions 3 and 5] and the commentary in [Arthur 2021, Section 4] for instance.

Conjecture 1.1.1. *Let $\tilde{\xi} : {}^L G \to {}^L \tilde{G}$ be a morphism of L-groups. For each automorphic representation π of $G(\mathbb{A}_F)$, there exists an automorphic representation Π of $\tilde{G}(\mathbb{A}_F)$ such that, for every $v \notin S$ where π is unramified, Π_v is unramified and isomorphic to $\tilde{\xi}_*(\pi_v)$. Moreover the infinitesimal characters of archimedean components of Π are determined by those of π via $\tilde{\xi}$.*

If Π as above exists, we say that Π is a *weak transfer* (a.k.a. a weak functorial lift) of π. It is said to be weak because the conjecture does not address what happens at the places in S nor what the set of all Π as above looks like. A stronger conjecture can be best formulated in terms of local Arthur packets at *all* places as well as global Arthur packets, as accomplished in the endoscopic classification for classical groups mentioned above. By focusing on the weak version, we bypass the subtlety of Arthur packets at the expense of losing precision.

We are particularly interested in Conjecture 1.1.1 where π appears in the discrete spectrum of the space of L^2-automorphic forms on $G(\mathbb{A}_F)$. Although the beyond endoscopy program was proposed by Langlands to attack this conjecture, the general case is still completely out of reach. Good news is that substantial progress has been made in the (twisted) endoscopic case, namely when $\tilde{\xi}$ realizes G as a (twisted) endoscopic group for $\tilde{G}$. A prominent example is Langlands and Arthur and Clozel's base change [1989] for general linear groups, where $G = \mathrm{GL}_n$ and $\tilde{G} = \mathrm{Res}_{F'/F}\, \mathrm{GL}_n$ (Weil restriction of scalars) for a finite solvable extension F'/F. See [Cogdell 2003, Section 4] for more on the base change and other examples.

This paper is concerned with a weak transfer for classical groups. In this case G is a classical group and $\tilde{G}$ is (the restriction of scalars of) a general linear group; the latter is denoted $\tilde{G}^0(N)$ in the main text. We are divided into Cases S and U:

Case S: G is a special orthogonal or a symplectic group, $\tilde{\xi}$ is the standard embedding.

Case U: G is a unitary group and $\tilde{\xi}$ is the base change embedding (up to a twist).

In these two cases the quasisplit inner form G^* of G may be thought of as a twisted endoscopic group for $\tilde{G}$; see Sections 2.1 and 2.2 for more details. Henceforth we make the following hypothesis as our method crucially relies on the stabilization of the (possibly twisted) trace formula by Arthur and Moeglin and Waldspurger:

(H1) The weighted fundamental lemma (WFL) is true for nonsplit groups. Moreover its nonstandard version is true.

It is worth elaborating on the hypothesis. The stabilization of the twisted trace formula [Mœglin and Waldspurger 2016a; 2016b] requires the twisted weighted fundamental lemma [Mœglin and Waldspurger 2016a, II.4.4], which is reduced by the main result of [Waldspurger 2009] to the WFL for Lie algebras and the nonstandard WFL. The latter two, precisely formulated in Sections 3.6 and 3.7 of [Waldspurger 2009], assert certain identities of weighted orbital integrals on the Lie algebras of two reductive groups which are related by endoscopic data or nonstandard endoscopic data, respectively. As mentioned above, the WFL for Lie algebras remains to be verified for nonsplit groups. The nonstandard WFL is open at this time.

With that said, hypothesis (H1) can be black-boxed since we only need the outcome of the stabilization, namely (2.4.4) and (2.4.6) below. Let us state our first main theorem.

Theorem 1.1.2. *Assuming (H1), Conjecture 1.1.1 is true for Cases S and U above.*

Here is the idea of proof in the essential case when $G = G^*$, i.e., when G is quasisplit; see the proof of Theorem 2.5.1 for complete details. By induction, we may assume that the theorem is known for all classical groups of smaller rank, or finite products thereof. Let π be as in Conjecture 1.1.1. Let c^S and ζ denote the family of Satake parameters of π away from S and the infinitesimal character of π at ∞, respectively. The L-morphism $\tilde{\xi}$ transfers c^S and ζ to a family of Satake parameters $\tilde{c}^S$ and an infinitesimal character $\tilde{\zeta}$ for $\tilde{G}$. We assume that $(\tilde{\zeta}, \tilde{c}^S)$ does not appear in the automorphic spectrum for $\tilde{G}$. The goal is to derive a contradiction.

The main input is the stabilized trace formula relating G and $\tilde{G}$, where the subscript $\tilde{\zeta}, \tilde{c}^S$ indicates the $(\tilde{\zeta}, \tilde{c}^S)$-isotypic part of each trace formula (reviewed in Section 2.4 following [Arthur 2013, Chapter 3]; we recommend [Arthur 2005] for a detailed introduction to the trace formula)

$$I^{\tilde{G}}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}(f) = \sum_{G^{\tilde{\mathfrak{e}}}} \iota(\tilde{\mathfrak{e}}) S^{\tilde{\mathfrak{e}}}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}(f^{\tilde{\mathfrak{e}}}), \tag{1.1.1}$$

where:

- $I^{\tilde{G}}_{\mathrm{disc}}$ is an invariant distribution on $\tilde{G}(\mathbb{A}_F)$, which is the discrete part of the invariant trace formula for the twisted group $\tilde{G}$.
- $G^{\tilde{\mathfrak{e}}}$ stands for the twisted endoscopic group in a twisted elliptic endoscopic datum $\tilde{\mathfrak{e}}$ for $\tilde{G}$ (up to isomorphism); this includes $G^{\tilde{\mathfrak{e}}} = G$.
- $\iota(\tilde{\mathfrak{e}}) \in \mathbb{Q}$ is a positive constant.
- $S^{\tilde{\mathfrak{e}}}_{\mathrm{disc}}$ is a stable distribution on $G^{\tilde{\mathfrak{e}}}(\mathbb{A}_F)$, which is the discrete part of the stable trace formula for the twisted endoscopic group of $\tilde{\mathfrak{e}}$.

- f is a decomposable test function on $\tilde{G}(\mathbb{A}_F)$ whose components away from S belong the unramified Hecke algebras.
- $f^{\tilde{\mathfrak{e}}}$ is a function on $G^{\tilde{\mathfrak{e}}}(\mathbb{A}_F)$ which is a transfer of f.

Although $S^{\tilde{\mathfrak{e}}}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}$ is very complicated in general, the induction hypothesis can be used to show that $S^{\tilde{\mathfrak{e}}}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}$ is equal to the trace on the $(\tilde{\zeta},\tilde{c}^S)$-isotypic part of the L^2-discrete spectrum of $G^{\tilde{\mathfrak{e}}}$. The point is that the "error terms" (the difference between the two quantities in the preceding sentence) all come from classical groups of smaller rank, which have to do with automorphic representations of general linear groups by induction, whereas $(\tilde{\zeta},\tilde{c}^S)$ is unrelated to such representations by hypothesis. In particular, for $\tilde{\mathfrak{e}}$ such that $G^{\tilde{\mathfrak{e}}}=G$, the stable distribution $S^{\tilde{\mathfrak{e}}}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}$ is *not* the zero distribution since π appears in the sum. (Recall that $(\tilde{\zeta},\tilde{c}^S)$ is the image of (ζ,c^S) via $\tilde{\xi}$.)

The left-hand side of (1.1.1) is trivially zero by the assumption that $(\tilde{\zeta},\tilde{c}^S)$ does not appear in the automorphic spectrum of $\tilde{G}$. Hence our preceding observation about $S^{\tilde{\mathfrak{e}}}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}$ tells us that a certain *nonnegative* combination of traces of irreducible representations on different groups on the right-hand side vanishes. We crucially invoke Arthur's vanishing result [2013, Section 3.5], exactly designed for these circumstances and relying on the nonnegativity of coefficients, to show that the right-hand side is term-by-term trivial, i.e., every nonnegative coefficient is zero. This is a contradiction since $S^{\tilde{\mathfrak{e}}}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}$ was seen to be nontrivial.

1.2. ***Automorphic Galois representations.*** For the moment we go back to a general connected reductive group G over a number field F. An automorphic representation π of $G(\mathbb{A}_F)$ is called L-algebraic (resp. C-algebraic) if the infinitesimal character of π at ∞ is algebraic (resp. algebraic after shifting by the half sum of positive roots), see Definition 3.1.1 below. By ${}^CG$ we denote the C-group of G introduced by Buzzard and Gee [2014], which is a certain semiproduct of ${}^LG$ with $\mathbb{G}_m$; see Section 3.1 below. It can also be thought of as the L-group of a central $\mathbb{G}_m$-extension of G.

Fix a prime ℓ. Let S denote the finite set of places of F containing all ℓ-adic and infinite places as well as the finite places v such that either G or π is ramified at v. When $v\notin S$, write

$$\phi_{\pi_v}=\phi^L_{\pi_v}:W_{F_v}\to {}^LG$$

for the unramified Langlands parameter for π_v, with coefficient in $\mathbb{C}$. We also define a C-normalized parameter

$$\phi^C_{\pi_v}:W_{F_v}\to {}^CG$$

by modifying ϕ_{π_v}; see below Lemma 3.1.5 for more details. In this paper, a *Galois representation* $\Gamma_F\to {}^LG(\overline{\mathbb{Q}}_\ell)$ or $\Gamma_F\to {}^CG(\overline{\mathbb{Q}}_\ell)$ always means a continuous semisimple representation which is unramified at all but finitely places and whose restriction to the local Galois group at each place above ℓ is de Rham. When the de Rham condition is satisfied, the Galois representations can be assigned Hodge–Tate cocharacters (Section 3.1).

Buzzard and Gee [2014] formulated the following, see Conjectures 3.1.2 and 3.1.8 below, generalizing from the case of general linear groups in Clozel's work [1990].

Conjecture 1.2.1. *Let $? \in \{L, C\}$, ℓ a prime, and $\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ an isomorphism. For each $?$-algebraic discrete automorphic representation π of $G(\mathbb{A}_F)$, there exists a Galois representation*

$$r = r_{\ell,\iota}(\pi) : \Gamma_F \to {}^{?}G(\overline{\mathbb{Q}}_\ell)$$

such that:

(i) $r|^{\mathrm{ss}}_{W_{F_v}} \cong \iota\phi^{?}_{\pi_v}$ *at finite places $v \notin S$.*

(ii) *The Hodge–Tate cocharacters of r are explicitly determined by the infinitesimal characters of π at ∞.*

Our interest lies in the conjecture when G is a classical group. We will concentrate on the C-algebraic case for two reasons. Firstly, it is more directly related to the geometric Satake equivalence (that is, part (i) of the conjecture is compatible with geometric Satake in the C-algebraic case, see [Zhu 2020b]) and the cohomology of Shimura varieties (e.g., as observed in [Johansson 2013]). Secondly, the C-algebraic case is more general as illustrated by the example of an even unitary group (i.e., of even rank) over a totally real field relative to a CM quadratic extension. Such a group does not possess any L-algebraic automorphic representations whose archimedean components belong to discrete series whereas there are many C-algebraic ones. (In fact, one can go from the C-algebraic case to the L-algebraic case and vice versa after pulling back via a central $\mathbb{G}_m$-extension of G, see [Buzzard and Gee 2014, Section 5], but we do not discuss it further.) With that said, it is worth mentioning that C-algebraicity and L-algebraicity coincide for symplectic, even special orthogonal, and odd unitary groups.

From now, assume that F is a totally real field. In Case U, assume that G is a unitary group with respect to a CM quadratic extension E over F, and write $c \in \mathrm{Gal}(E/F)$ for the nontrivial element. In Case S, set $E := F$ and $c := 1$ (trivial automorphism of F).

We fix π as in Theorem 1.1.2, so the theorem provides us with an automorphic representation Π of $\mathrm{GL}_N(\mathbb{A}_E)$ for a suitable N. Without loss of generality we assume that Π is an isobaric sum of cuspidal automorphic representations of smaller general linear groups: $\Pi = \boxplus_{i=1}^{r} \Pi_i$. (In fact we show that Π can be chosen as such when proving the theorem.) By the strong multiplicity one theorem, such a Π is unique up to isomorphism. (Hence $\Pi_1, \dots, \Pi_r$ are unique up to isomorphism and permutation.) For each i, we write $\Pi_i^\star$ for the contragredient of $\Pi_i \circ c$, where c naturally acts on $\mathrm{GL}_N(\mathbb{A}_E)$. Consider the following hypotheses:

(H2) The infinitesimal character of Π is regular at infinity, see Definition 3.2.1 below.

(H3) Each Π_i is (conjugate) self-dual, i.e., $\Pi_i^\star \cong \Pi_i$ for every i.

Condition (H2) is equivalent to regularity of the infinitesimal character of π at infinity unless G^* is an even special orthogonal group (Lemma 3.2.2). Hypothesis (H3) is implied by a full endoscopic classification theorem, which is a conditional theorem for classical groups as already discussed. Our second main theorem is the following (Theorem 3.2.7).

Theorem 1.2.2. *Assume (H1), (H2), and (H3). Then the C-algebraic version of Conjecture 1.2.1 holds true in Cases S and U above, except that* (*i*) *is true only up to outer automorphism in the even orthogonal case. If we assume only (H1) and (H2) then we have the existence of the Galois representation as in the conjecture satisfying* (*i*) *but possibly not* (*ii*).

Let us outline the steps of the proof:

(Step 1) Prove Conjecture 1.2.1 for cuspidal regular automorphic representations Π_0 of GL_N over totally real or CM fields (see Proposition 3.1.11 below for the precise version).

(Step 2) Combine Step 1 with Theorem 1.1.2 to construct a GL_N-valued Galois representation $R(\pi)$ corresponding to given π on a classical group.

(Step 3) Factor the Galois representation $R(\pi)$ through the L or C-group of G. In Case U, this entails extending the Galois representation along the quadratic extension E/F.

Step 1 follows by combining the work of many authors as recalled in the proof of Proposition 3.1.11, if Π_0 is moreover (conjugate) self-dual up to a character. Without hypothesis (H3), we need to appeal to more recent work by Harris, Lan, Taylor and Thorne [Harris et al. 2016] and Scholze [2015]. In this case we lose control of the Hodge–Tate cocharacter. (See the last paragraph in the proof of Proposition 3.1.11.) This is why part (ii) of Conjecture 1.1.1 is not verified when (H3) is not assumed. Other than this, the argument is the same whether (H3) is assumed or not.

In Step 2 we start from a weak transfer $\pi \mapsto \Pi = \boxplus_{i=1}^r \Pi_i$ and apply Step 1 to construct Galois representations R_i from Π_i. The desired Galois representation is essentially $\bigoplus_{i=1}^r R_i$ but this is not literally true. We need to keep a careful track of L and C-normalizations.

In Step 3 the main input is Bellaïche and Chenevier's [2011] result on the sign of Galois representations. Thanks to this, the argument is relatively simple in Case S. More work is needed in Case U, but knowing the sign again allows us to factor the extended Galois representation through the C-group.

Remark 1.2.3. When F is a global function field of characteristic $p > 0$, if $\ell \neq p$ then Conjecture 1.2.1 can be stated in terms of the L-group of G, without imposing condition (ii) or algebraicity. (Every automorphic representation is considered algebraic.) Then Conjecture 1.2.1 is true for every G and every cuspidal π by V. Lafforgue [2018].

1.3. *Complements.* We comment on the prospect of removing hypotheses (H1), (H2), and (H3). The author is cautiously optimistic that the removal of (H1) would be attainable within the next few years. It may be possible to weaken the regularity condition (H2) in Theorem 1.2.2 to weak regularity of Π at infinity in the sense of [Fakhruddin and Pilloni 2019, Section 9.1]; the weak regularity (and oddness) of Π is always satisfied if π has regular infinitesimal character at infinity, even when G is an even special orthogonal group. A crucial input is [Boxer and Pilloni 2021, Theorem 6.11.2], which relaxes the regularity assumption on π in Proposition 3.1.11 to weak regularity. The proof of Proposition 3.2.4, except the assertions on signs, goes through with the weakening of (H2) as long as both (H1) and (H3) are assumed. The only missing ingredient is the analogue of the main results of [Bellaïche and Chenevier 2011] when Π is weakly regular (and odd) but not regular. To remove (H3), the main problem is to compute the Hodge–Tate weights of the automorphic Galois representations in [Harris et al. 2016; Scholze 2015] as mentioned above. Partial results are available in [A'Campo 2024, Theorem 1.0.6; Hevesi 2023, Theorem 1.1].

There are other ways to strengthen Theorems 1.1.2 and 1.2.2. Theorem 1.1.2 is going to be eventually superseded by a full endoscopic classification; the point of our theorem lies in the simplicity and uniformity of the argument. Theorem 1.2.2 can be upgraded by listing more properties satisfied by the Galois representation r. For instance, we can ask for a description of the image of complex conjugation at real places of F, see Remark 3.2.8. Another question is to prove local-global compatibility at *all* finite places v, namely that the Weil–Deligne representation associated with r at v corresponds to the v-component of the automorphic representation via the local Langlands correspondence. This is known in the setting of Proposition 3.1.11 for GL_N. (If π is not conjugate self-dual up to a character then the compatibility is known away from places above ℓ.) From this, our existing arguments should justify the local-global compatibility for G at all finite places (avoiding places above ℓ if (H3) is not assumed), at least if G is quasisplit. In fact, such a reasoning already appears in the proof of [Kret and Shin 2023, Theorem 2.4 (i), (iv)] and [Kret and Shin 2020, Theorem 6.4(SO-i)] in some special cases. If G is not quasisplit then the same should work once the local Langlands correspondence for G becomes available in a way that is compatible with the local Langlands for its quasisplit inner form.

Finally one can try to characterize those Galois representations which correspond to automorphic representations in Conjecture 1.2.1. In fact it is fruitful to view the Galois representations as global L-parameters and extend the Galois representations to some sort of global A-parameters as in [Johansson and Thorne 2020, Section 4]. Then a natural problem is to formulate local and global A-packet classifications for algebraic automorphic representations by means of such Galois-theoretic A-parameters. We hope to address this elsewhere.

1.4. ***Notation and conventions.*** Let k be a perfect field. Denote by $\bar{k}$ an algebraic closure of k. Write $\Gamma_{k'/k} := \mathrm{Gal}(k'/k)$ for any Galois extension k'/k and put $\Gamma_k := \Gamma_{\bar{k}/k}$. When T is a torus over k, write $X^*(T) := \mathrm{Hom}_{\bar{k}}(T, \mathbb{G}_m)$ and $X_*(T) := \mathrm{Hom}_{\bar{k}}(\mathbb{G}_m, T)$. Put $X^*(T)_R := X^*(T) \otimes_{\mathbb{Z}} R$ for $\mathbb{Z}$-algebras R, which is an $R[\Gamma_k]$-module. Define $X_*(T)_R$ likewise. Let $\widehat{T}$ denote the dual torus of T over $\mathbb{C}$ equipped an action of Γ_k.

From now on, let F be a number field. Write $\mathbb{A}_F$ for the ring of adèles and $\mathbb{A}_F^S$ for the ring of adèles away from S, where S is a finite set of places of F. For each place v of F, write W_{F_v} for the local Weil group. We fix the embeddings $\iota_v : \overline{F} \hookrightarrow \overline{F}_v$ at each v, which induce the injections $\Gamma_{F_v} \hookrightarrow \Gamma_F$. If v is a complex place, then there are two $\mathbb{R}$-isomorphisms $\iota_1, \iota_2 : \overline{F}_v \cong \mathbb{C}$. For each complex embedding $\tau : F \hookrightarrow \mathbb{C}$ inducing the place v, we write $\iota_\tau : \overline{F} \hookrightarrow \mathbb{C}$ for either $\iota_1\iota_v$ or $\iota_2\iota_v$, whichever induces τ via the inclusion $F \subset \overline{F}$. If τ is a real embedding inducing v then set $\iota_\tau := \iota_v$. Thus we have $\iota_\tau : \overline{F} \hookrightarrow \mathbb{C}$ extending every embedding $\tau : F \hookrightarrow \mathbb{C}$.

Let F_0 be a subfield of F (allowing $F_0 = F$), and S a finite set of places of F_0 containing all infinite places. Then $\Gamma_{F,S}$ denotes the Galois group $\mathrm{Gal}(F_S/F)$, where $F_S \subset \overline{F}$ is the maximal extension of F which is unramified at every place of F which lies above some place of F_0 in S.

Let G^* be a connected quasisplit reductive group over F, with an F-pinning $(B^*, T^*, \{X_\alpha^*\})$. Let $\widehat{G}^*$ denote the Langlands dual group over $\mathbb{C}$ equipped with a Γ_F-action on $\widehat{G}^*$ (called an L-action), a Γ_F-pinning $(\widehat{B}^*, \widehat{T}^*, \{\widehat{X}^*_{\alpha^\vee}\})$, and a Γ_F-equivariant bijection between the based root datum of $\widehat{G}^*$ and the dual based root datum of G^*. This allows us to define the Galois form of the L-group

$$^LG^* := \widehat{G}^* \rtimes \Gamma_F.$$

It is also convenient to use $\Gamma_{F'/F}$ in place of Γ_F, where F' is a finite extension of F over which G^* splits. Only in Section 2 we will occasionally consider the Weil form of the L-group, with the Weil group of F in place of Γ_F. We will often fix an isomorphism $\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ and also view $\widehat{G}^*$ and $^LG^*$ over $\overline{\mathbb{Q}}_\ell$. Write $S_{\mathrm{bad}}(G^*)$ for the set of places v of F which are either infinite or such that $G^*_{F_v}$ is ramified.

At $v \notin S_{\mathrm{bad}}(G^*)$, the pinning determines a hyperspecial subgroup $K_v^* \subset G^*(F_v)$. Unramified representations of $G^*(F_v)$ at $v \notin S_{\mathrm{bad}}(G^*)$ are always meant to be relative to this K_v^*.

Let G be a connected reductive group over F with an isomorphism $\mathfrak{i} : G^*_{\overline{F}} \simeq G_{\overline{F}}$ such that $\mathfrak{i}^{-1}\sigma(\mathfrak{i})$ is an inner automorphism of $G^*_{\overline{F}}$ for every $\sigma \in \Gamma_F$. Such a pair $(G, \mathfrak{i})$ is called an *inner twist* of G^* over F, and classified up to isomorphism by the Galois cohomology valued in the adjoint group $H^1(F, G^{*,\mathrm{ad}})$, whose image in $H^1(F_v, G^{*,\mathrm{ad}})$ is trivial for v not contained a finite set of places S. Then $H^1(F, G^{*,\mathrm{ad}}(\mathbb{A}_F^S)) = \bigoplus_{v \notin S} H^1(F_v, G^{*,\mathrm{ad}})$ is trivial, so $\mathfrak{i}$ is defined over $\mathbb{A}_F^S$ after conjugation by an element of $G^{*,\mathrm{ad}}(\overline{\mathbb{A}_F^S})$. Thereby we obtain an isomorphism $G^*(\mathbb{A}_F^S) \cong G(\mathbb{A}_F^S)$, canonical up to $G^*(\mathbb{A}_F^S)$-conjugacy. Put $S_{\mathrm{bad}}(G) := S_{\mathrm{bad}}(G^*) \cup S$. At each $v \notin S_{\mathrm{bad}}(G)$, we transport hyperspecial subgroups K_v^* to $K_v \subset G(F_v)$ via the isomorphism and use them for the notion of unramified representations. We transfer the F-pinning for G^* to a pinning for G via $\mathfrak{i}$ so that the based root data for G^* and G are Γ_F-equivariantly identified. Thereby we may and will identify the L-group ${}^LG$ with ${}^LG^*$, and transfer $(\widehat{B}^*, \widehat{T}^*, \{\widehat{X}^*_{\alpha^\vee}\})$ for $\widehat{G}^*$ to $(\widehat{B}, \widehat{T}, \{\widehat{X}_{\alpha^\vee}\})$ for $\widehat{G}$.

For a place v of G, we often write G_v to mean $G \times_F F_v$. Write $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{v \mid \infty} F_v$, and $G_\infty := (\mathrm{Res}_{F/\mathbb{Q}} G) \times_{\mathbb{Q}} \mathbb{R} = \prod_{v \mid \infty} G_v$. We fix a maximal compact subgroup $K_\infty = \prod_{v \mid \infty} K_v \subset G_\infty(\mathbb{R}) = \prod_{v \mid \infty} G(F_v)$.

By $\mathcal{H}(G)$ we denote the space of smooth compactly supported functions on $G(\mathbb{A}_F)$ which are bi-K-finite under some compact subgroup $K = \prod_v K_v \subset G(\mathbb{A}_F)$, where K_v is the fixed hyperspecial subgroup (resp. maximal compact subgroup) at all but finitely many v (resp. all infinite places v). Let $\mathcal{H}(G_\infty)$ denote the space of smooth compactly supported bi-K_∞-finite functions on $G_\infty(\mathbb{R})$. Let S be a finite set of finite places of F containing $S_{\mathrm{bad}}(G)$. At $v \notin S$, let $\mathcal{H}_{\mathrm{ur}}(G_v)$ denote the unramified Hecke algebra of bi-K_v-invariant functions on $G(F_v)$. Take $\mathcal{H}^S_{\mathrm{ur}}(G)$ to be the unramified Hecke algebra of compactly supported bi-K^S-invariant functions on $G(\mathbb{A}_F^S)$, where $K^S = \prod_{v \notin S} K_v$ is the product of fixed hyperspecial subgroups. The analogous definition of $\mathcal{H}(G)$, possibly with decorations, makes sense when G is a nontrivial coset in a twisted group, e.g., $G = G(N)$ as in Section 2.2 below.

Write A_G for the maximal $\mathbb{Q}$-split torus in the center of $\mathrm{Res}_{F/\mathbb{Q}} G$. (We have $A_G = \{1\}$ for the classical groups to be considered.) Put

$$[G] := G(F)\backslash G(\mathbb{A}_F)/A_G(\mathbb{R})^0.$$

Let $L^2_{\mathrm{disc}}([G])$ denote the discrete part of the L^2-space of functions on $[G]$, viewed as a $G(\mathbb{A}_F)$-module by right translation. Every irreducible $G(\mathbb{A}_F)$-subrepresentation is referred to as a discrete automorphic representation. Denote by $L^2_{\mathrm{disc}}([G])^{S-\mathrm{ur}}$ the subspace generated by discrete automorphic representations which are unramified away from S. Write $\mathcal{C}^S(G)$ for the set in which each member is a family of semisimple $\widehat{G}$-conjugacy classes $c_v \subset {}^L G_v$ over finite places $v \notin S$ such that c_v maps to the geometric Frobenius element under the projection from ${}^L G_v$ to the unramified Galois group over F_v. By the Satake isomorphism, each c_v corresponds to a $\mathbb{C}$-algebra morphisms $\mathcal{H}_{\mathrm{ur}}(G_v) \to \mathbb{C}$ at $v \notin S$. Thereby $\mathcal{C}^S(G)$ is identified with the set of $\mathbb{C}$-algebra morphisms $\mathcal{H}^S_{\mathrm{ur}}(G) \to \mathbb{C}$.

Write $G_{\infty,\mathbb{C}} := (\mathrm{Res}_{F/\mathbb{Q}}\, G) \times_{\mathbb{Q}} \mathbb{C} = \prod_{\tau: F \hookrightarrow \mathbb{C}} G_\tau$, where $G_\tau := G \times_{F,\tau} \mathbb{C}$. Let $T_{\infty,\mathbb{C}} = \prod_\tau T_\tau$ be a maximal torus in $G_{\infty,\mathbb{C}}$. The Lie algebra of $T_{\infty,\mathbb{C}}$ is denoted by $\mathfrak{t}_{\infty,\mathbb{C}}$. Write $\Omega_\infty = \prod_\tau \Omega_\tau$ for the Weyl group of $T_{\infty,\mathbb{C}}$ in $G_{\infty,\mathbb{C}}$. We often write Ω for Ω_τ for simplicity.

We use $\mathfrak{Z}(G_\infty)$ to denote the center of the universal enveloping algebra of the Lie algebra of $G_{\infty,\mathbb{C}}$. By the Harish-Chandra isomorphism, we may identify $\mathfrak{Z}(G_\infty) = \mathbb{C}[\mathfrak{t}_{\infty,\mathbb{C}}]^\Omega$. Write $\mathcal{C}_\infty(G)$ for the set of $\mathbb{C}$-algebra morphisms $\mathfrak{Z}(G_\infty) \to \mathbb{C}$, or equivalently

$$\mathcal{C}_\infty(G) = \mathfrak{t}^*_{\infty,\mathbb{C}}/\Omega = X^*(T_\infty)_{\mathbb{C}}/\Omega_\infty = X_*(\widehat{T}_\infty)_{\mathbb{C}}/\Omega_\infty = \prod_\tau X_*(\widehat{T}_\tau)_{\mathbb{C}}/\Omega. \quad (1.4.1)$$

Let $\pi = \otimes'_v \pi_v$ be an irreducible admissible representation of $G(\mathbb{A}_F)$ such that π is unramified outside S. At each $v \notin S$, each π_v corresponds to a semisimple $\widehat{G}$-conjugacy class $c(\pi_v) \subset {}^L G_v$ known as the Satake parameter of π_v, and vice versa. By assigning to π the infinitesimal character at ∞ and the Satake parameters away from S, we obtain a map

$$\pi \mapsto (\zeta_{\pi_\infty}, (c(\pi_v))_{v \notin S}) \in \mathcal{C}_\infty(G) \times \mathcal{C}^S(G).$$

According to the decomposition (1.4.1), we write

$$\zeta_{\pi_\infty} = (\zeta_{\pi_\infty,\tau})_{\tau: F \hookrightarrow \mathbb{C}}.$$

For π as above, we have an unramified L-parameter $\phi_{\pi_v} : W_{F_v} \to {}^L G_v$ at each $v \notin S$ and an archimedean L-parameter $\phi_{\pi_v} : W_{F_v} \to {}^L G_v$ at $v \mid \infty$. The relation to the above map is as follows. For $v \notin S$, ϕ_{π_v} sends lifts of the geometric Frobenius element into $c(\pi_v)$. For $v \mid \infty$ and each $\tau : F \hookrightarrow \mathbb{C}$ inducing v, if we identify $\overline{F}_v = \mathbb{C}$ via τ thus $W_{\overline{F}_v} = \mathbb{C}^\times \subset W_{F_v}$, then $\phi_{\pi_v}|_{\mathbb{C}^\times}$ is $\widehat{G}$-conjugate to a map of the form

$$z \in \mathbb{C}^\times \mapsto \lambda(z)\lambda'(\bar{z}) \in \widehat{T}_\tau \subset \widehat{G}_\tau = \widehat{G}_v$$

such that $\lambda = \zeta_{\pi_\infty,\tau}$.

When v is a place of F, we denote by $|\cdot|_v$ the usual norm character on $F_v^\times$ or W_{F_v} valued in positive real numbers, satisfying the product formula. Our normalization at finite places v is that a uniformizer in $F_v^\times$ and a lift of the geometric Frobenius in W_{F_v} both map to the inverse of the residue field cardinality. By $\det_N : \mathrm{GL}_N \to \mathbb{G}_m$ we mean the determinant map, and $|\det_N|_v : \mathrm{GL}_N(F_v) \to \mathbb{R}_{>0}$ the map $x \mapsto |\det_N(x)|_v$. We often omit N and v and simply write $|\cdot|$, det, and $|\det|$.

Given a finite dimensional representation r (typically of a local Weil group), r^{ss} stands for its semisimplification. By an (ℓ-adic) *Galois representation* of Γ_F, where F is a number field, we mean a continuous semisimple representation of Γ_F on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space which is unramified at almost all places of F and de Rham at ℓ. More generally, when G is as above, an ${}^L G$ or ${}^C G$-valued *Galois representation* is a continuous representation

$$\Gamma_F \to {}^L G(\overline{\mathbb{Q}}_\ell) \quad \text{or} \quad R : \Gamma_F \to {}^C G(\overline{\mathbb{Q}}_\ell)$$

which:

- Is unramified at almost all places of F.
- Commutes with the projections from Γ_F and the L or C-groups onto the Galois group $\Gamma_{F'/F}$, where F'/F is a Galois extension with respect to which ${}^L G$ or ${}^C G$ is formed.
- $i \circ R$ is semisimple and de Rham at ℓ for i a faithful algebraic representation (see [Borel 1979, Section 2.6]) of the L-group or C-group.

For G over F as above, write $\mathcal{E}_{\mathrm{ell}}(G)$ for a set of representatives for isomorphism classes of (standard) elliptic endoscopic data $(H, \mathcal{H}, s, \xi)$ as in [Kottwitz and Shelstad 1999, Section 2.1]; see [Langlands and Shelstad 1987, Section 1.2]. We refer to H as an elliptic endoscopic group for G. We will always be in the case when $\mathcal{H}$ can be taken to be the L-group of H. Our notation for such a datum is usually $\mathfrak{e} = (G^{\mathfrak{e}}, {}^L G^{\mathfrak{e}}, s^{\mathfrak{e}}, \xi^{\mathfrak{e}})$. The set $\mathcal{E}_{\mathrm{ell}}(G)$ always contains a unique element $\mathfrak{e}_0$ whose endoscopic group is a quasisplit inner form of G. Write $\mathcal{E}^{<}_{\mathrm{ell}}(G)$ for the complement $\mathcal{E}_{\mathrm{ell}}(G)\backslash\{\mathfrak{e}_0\}$. Every endoscopic group in $\mathcal{E}^{<}_{\mathrm{ell}}(G)$ has strictly lower semisimple rank than G.

The cyclotomic character has Hodge–Tate weight -1 in our convention.

2. Weak transfer

2.1. ***Classical groups.*** Let $m, n \in \mathbb{Z}_{>0}$. We introduce the quasisplit classical groups Sp_{2n}, SO_{2n+1}, SO^{η}_{2n}, and U_n, naturally sitting inside (the restriction of scalars of) general linear group GL_m. (Compare with [Arthur 2013, Chapters 1 and 9] and [Waldspurger 2010, Section 1].) For unitary groups, we write N instead of m in anticipation of Section 2.2.

Define antidiagonal matrices J_m, $J_m^* \in \mathrm{GL}_m(\mathbb{Z})$ and $J'_{2n} \in \mathrm{GL}_{2n}(\mathbb{Z})$ as follows:

$$J_m = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}, \quad J_m^* = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{m-1} & & & \end{pmatrix}, \quad J'_{2n} = \begin{pmatrix} & -J_n \\ J_n & \end{pmatrix}.$$

When $m = 2n$, let $\eta : \Gamma_{F_\eta/F} \to \{\pm 1\}$ be a faithful character. (So F_η/F is a quadratic extension if $\eta \neq 1$, and $F_\eta = F$ if $\eta = 1$.) If $\eta = 1$ then set $J_m^\eta := J_m$. If $\eta \neq 1$, choose $\alpha \in \mathcal{O}_F^\times$ whose square roots generate F_η over F. Then define J_{2n}^η from J_{2n} by replacing the 2×2-matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in the middle with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -\alpha \end{smallmatrix}\right)$.

Case S. We define the $\mathcal{O}_F$-group schemes

$$G \in \{\mathrm{Sp}_m, \mathrm{O}_m^\eta, \mathrm{O}_m\},$$

with $m = 2n$ in the first two cases, and $m = 2n+1$ in the last case, by the following formula

$$G := \{g \in \mathrm{GL}_m : {}^t g J g = J\}, \quad J \in \{J'_m, J_m^\eta\}, \text{ respectively},$$

on $\mathcal{O}_F$-algebra valued points. The connected component of the identity in O_m^η (resp. O_{2n+1}) is denoted by SO_{2n}^η (resp. SO_{2n+1}). By abuse of notation, we still write Sp_{2n}, SO_{2n}^η, and SO_{2n+1} for the F-group schemes obtained by base change. We often omit η in case $\eta = 1$. Each group contains a Borel subgroup B over F: if G is SO_m or Sp_{2n} then B consists of upper triangular matrices in G; if $G = \mathrm{SO}_{2n}^\eta$ with $\eta \neq 1$ then B consists of matrices (g_{ij}) such that $g_{ij} = 0$ if $i > j$ and $(i, j) \neq (n+1, n)$. In the following examples, we make an explicit choice of a maximal torus T in B and describe the character group of T as well as the half sum of positive roots ρ. When A_i are square matrices for $1 \le i \le r$, let $\mathrm{diag}(A_1, \dots, A_r)$ denote the block diagonal matrix.

$G = \mathrm{Sp}_{2n}$. We take $T = \{\mathrm{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) : t_1, \dots, t_n \in \mathbb{G}_m\}$ and use the coordinates to identify $X^*(T) = \mathbb{Z}^n$ with trivial Γ_F-action. We have the Weyl group $\Omega = \{\pm 1\}^n \rtimes S_n$, where $(\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$ acts on $(a_i) \in X^*(T)$ by sending each a_i to $a_i^{\epsilon_i}$, and S_n acts by permuting $a_1, \dots, a_n$. By computation $\rho = (n, n-1, \dots, 2, 1)$.

$G = \mathrm{SO}_{2n}^\eta$ *(allowing* $\eta = 1$*)*. Take $T = \{\mathrm{diag}(t_1, \dots, t_{n-1}, s, t_{n-1}^{-1}, \dots, t_1^{-1}) : t_1, \dots, t_{n-1} \in \mathbb{G}_m, s \in \mathrm{SO}_2^\eta\}$. Using b as the last coordinate we identify $X^*(T) = \mathbb{Z}^n$, with Γ_F acting through η on the last coordinate as $\{\pm 1\}$. The Weyl group Ω is the index two subgroup of $\{\pm 1\}^n \rtimes S_n$ consisting of $(\epsilon_1, \dots, \epsilon_n, \sigma)$ such that $\prod_{i=1}^n \epsilon_i = 1$. Each element of Ω acts on $\mathbb{Z}^n$ in the same way as in the Sp_{2n}-case. We have $\rho = (n-1, n-2, \dots, 1, 0)$.

$G = \mathrm{SO}_{2n+1}$. Here $T = \{\mathrm{diag}(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}) : t_1, \dots, t_n \in \mathbb{G}_m\}$ and $X^*(T) = \mathbb{Z}^n$ with trivial Γ_F-action. The Weyl group $\Omega = \{\pm 1\}^n \rtimes S_n$ acts on $X^*(T)$ in the same way as above, and $\rho = \frac{1}{2}(2n-1, 2n-3, \dots, 3, 1)$.

For each G the choice of (B, T) as above extends to an F-pinning (a.k.a. F-splitting, see [Kottwitz and Shelstad 1999, Section 1.2]). The Langlands dual groups $\widehat{G}$, as reductive groups over $\mathbb{C}$, are described as $\widehat{\mathrm{Sp}_{2n}} = \mathrm{SO}_{2n+1}$, $\widehat{\mathrm{SO}^{\eta}_{2n}} = \mathrm{SO}_{2n}$, and $\widehat{\mathrm{SO}_{2n+1}} = \mathrm{Sp}_{2n}$, equipped with pinnings for $\widehat{G}$ chosen in the same way as for G. The L-action of Γ_F on $\widehat{G}$ is trivial when G is the split group Sp_{2n}, SO_{2n}, or SO_{2n+1}, whereas the action for $G = \mathrm{SO}^{\eta}_{2n}$ with $\eta \neq 1$ factors through $\mathrm{Gal}(F_\eta/F)$ with the nontrivial element acts as the outer automorphism $\hat{\theta}^\circ : g \mapsto \vartheta g \vartheta^{-1}$ on SO_{2n}, where

$$\vartheta = \mathrm{diag}\left(I_{n-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I_{n-1} \right) \in \mathrm{SO}_{2n}(\mathbb{C}).$$

Set $F' := F$ unless $G = \mathrm{SO}^{\eta}_{2n}$, in which case $F' := F_\eta$, so that the Γ_F-action factors through $\Gamma_{F'/F}$. Then the F'/F-form of the L-group ${}^L G_{F'/F} = \widehat{G} \rtimes \Gamma_{F'/F}$ is given as follows; we will often omit the subscript F'/F:

$${}^L\mathrm{Sp}_{2n} = \mathrm{SO}_{2n+1}, \quad {}^L\mathrm{SO}^{\eta}_{2n} = \begin{cases} \mathrm{O}_{2n}, & \eta \neq 1, \\ \mathrm{SO}_{2n}, & \eta = 1, \end{cases} \quad {}^L\mathrm{SO}_{2n+1} = \mathrm{Sp}_{2n},$$

where ${}^L\mathrm{SO}^{\eta}_{2n} = \mathrm{O}_{2n}$ when $\eta \neq 1$ by sending the nontrivial element of $\mathrm{Gal}(F_\eta/F)$ to ϑ.

Endoscopic groups $G^{\mathfrak{e}}$ in $\mathcal{E}_{\mathrm{ell}}(G)$ have the following forms, where $0 \le n' \le n$ and $\eta', \eta'_1, \eta'_2 : \Gamma_F \to \{\pm 1\}$ are continuous characters, understanding that $\eta \neq 1$ (resp. $\eta = 1$) in any factor of the form SO^{η}_2 (resp. SO^{η}_0) in the list:

- $G = \mathrm{Sp}_{2n}$: $G^{\mathfrak{e}} = \mathrm{SO}^{\eta'}_{2n'} \times \mathrm{Sp}_{2n-2n'}$.
- $G = \mathrm{SO}^{\eta}_{2n}$: $G^{\mathfrak{e}} = \mathrm{SO}^{\eta'_1}_{2n'} \times \mathrm{SO}^{\eta'_2}_{2n-2n'}$, $\eta'_1\eta'_2 = \eta$.
- $G = \mathrm{SO}_{2n+1}$: $G^{\mathfrak{e}} = \mathrm{SO}_{2n'+1} \times \mathrm{SO}_{2n+1-2n'}$.

There is redundancy in the second and third items, which can be removed by imposing $n' \le \lfloor n/2 \rfloor$; see [Arthur 2013, Section 1.2] or [Waldspurger 2010, Section 1.8] for a description of full endoscopic data.

Case U. In this case, let E be a quadratic extension of F. Write c for the nontrivial element in $\mathrm{Gal}(E/F)$. Define U_N as an $\mathcal{O}_F$-group scheme by

$$\mathrm{U}_N := \{g \in \mathrm{Res}_{\mathcal{O}_E/\mathcal{O}_F} \mathrm{GL}_N : {}^t g J^*_N c(g) = J^*_N\}$$

on $\mathcal{O}_F$-algebra valued points. Again we still write U_N for $\mathrm{U}_N \times_{\mathcal{O}_F} F$. This group contains a Borel subgroup B (resp. a maximal torus T) over F consisting of upper triangular (resp. diagonal) matrices in U_N so that

$$T = \{(t_1, \dots, t_N) : t_i \in \mathrm{Res}_{E/F} \mathbb{G}_m, t_i \cdot c(t_{N+1-i}) = 1, i = 1, \dots, N\}.$$

By fixing an F-algebra embedding $\tau_0 : E \hookrightarrow \overline{F}$, we obtain a projection $(\mathrm{Res}_{E/F}\mathbb{G}_m)_{\overline{F}} \to \mathbb{G}_{m,\overline{F}}$ induced by $E \otimes_F \overline{F} \to \overline{F}$, $a \otimes b \mapsto \tau(a)b$, thereby $T_{\overline{F}} \cong \mathbb{G}^N_{m,\overline{F}}$. This leads to an identification

$$X^*(T) = X_*(\hat{T}) = \mathbb{Z}^N \quad \text{via} \quad \tau_0,$$

with the Γ_F-action factoring through $\Gamma_{E/F}$, and $c \in \Gamma_{E/F}$ acts as $(a_i) \mapsto (-a_{N+1-i})$. (If $\tau_0 c$ was used instead of τ_0, then the identification changes by $(a_i) \mapsto (-a_{N+1-i})$.) We compute $\rho = \left(\frac{1}{2}(N-1), \frac{1}{2}(N-3), \dots, \frac{1}{2}(1-N)\right)$. The above choice of (B, T) extends to an F-pinning.

The map τ_0 induces a projection $(\mathrm{Res}_{E/F}\mathrm{GL}_N)_{\overline{F}} \to \mathrm{GL}_{N,\overline{F}}$ inducing $\mathrm{U}_{N,\overline{F}} \cong \mathrm{GL}_{N,\overline{F}}$ and also $\widehat{\mathrm{U}}_N \cong \mathrm{GL}_N$ as a complex reductive group. The standard pinning for GL_N is carried over to a pinning for $\widehat{\mathrm{U}}_N$. The L-action of Γ_F, factoring through $\Gamma_{E/F}$, is given by $c \in \Gamma_{E/F}$ acting as $\hat{\theta}(g) := J_N^{*}\,{}^t g^{-1} (J_N^*)^{-1}$ for $g \in \hat{\mathrm{U}}_N \cong \mathrm{GL}_N$. This determines the structure of the L-group:

$$\tilde{\xi}_0 : {}^L\mathrm{U}_N \cong \mathrm{GL}_N \rtimes \Gamma_F, \quad {}^L(\mathrm{U}_N)_{E/F} \cong \mathrm{GL}_N \rtimes \Gamma_{E/F} \quad \text{via } \tau_0.$$

We also let $\tilde{\xi}_0$ denote either map or the common restriction to the dual group: $\widehat{\mathrm{U}}_N \cong \mathrm{GL}_N$. If τ_0 is replaced with a conjugate embedding $\tau_0 c$, then the above isomorphism is composed with $g \rtimes \gamma \mapsto \hat{\theta}(g) \rtimes \gamma$. Let v be a finite place of F. Recall that $\iota_v : \overline{F} \hookrightarrow \overline{F}_v$ is fixed (Section 1.4), which gives rise to

$$\tau_{0,v} : E \xhookrightarrow{\tau_0} \overline{F} \xhookrightarrow{\iota_v} \overline{F}_v.$$

Write u for the place of E indued by $\overline{F}_v$ via $\tau_{0,v}$. As we did for $\tilde{\xi}_0$, we obtain an isomorphism

$$\tilde{\xi}_u : {}^L(\mathrm{U}_N)_{F_v} = \mathrm{GL}_N \rtimes \Gamma_{F_v} \quad \text{via } \tau_{0,v}.$$

The maps $\tilde{\xi}_0$ and $\tilde{\xi}_u$ fit in a commutative square with the natural embeddings ${}^L(\mathrm{U}_N)_{F_v} \hookrightarrow {}^L\mathrm{U}_N$ and $\mathrm{GL}_N \rtimes \Gamma_{F_v} \hookrightarrow \mathrm{GL}_N \rtimes \Gamma_F$. Similarly, let $\sigma : F \hookrightarrow \mathbb{C}$ be an embedding. Write v for the infinite place of F induced by σ. We have chosen $\iota_\sigma : \overline{F} \hookrightarrow \mathbb{C}$ to extend σ in Section 1.4. Write $\tau_{0,\sigma} := \iota_\sigma \tau_0$. Then we obtain

$$\tilde{\xi}_{\tau_{0,\sigma}} : {}^L(\mathrm{U}_N)_{F_v} = \mathrm{GL}_N \rtimes \Gamma_{F_v} \quad \text{via } \tau_{0,\sigma}.$$

For the embedding $\tau_{0,\sigma}c$ conjugate to $\tau_{0,\sigma}$, we define $\tilde{\xi}_{\tau_{0,\sigma}c}$ to be $\tilde{\xi}_{\tau_{0,\sigma}}$ followed by $g \rtimes \gamma \mapsto \hat{\theta}(g) \rtimes \gamma$. Similarly, if a finite place v splits in E as u and u' then $\tilde{\xi}_{u'}$ is set to be $\tilde{\xi}_u$ composed with $g \rtimes \gamma \mapsto \hat{\theta}(g) \rtimes \gamma$. To sum up, we defined

$$\tilde{\xi}_\tau \text{ for all embeddings } E \hookrightarrow \mathbb{C} \quad \text{and} \quad \tilde{\xi}_u \text{ for all finite places } u \text{ of } E.$$

When v is an infinite place, we also fix an isomorphism $\overline{F}_v \cong \mathbb{C}$ and still write $\tau_{0,v}$ for the composite map $E \hookrightarrow \overline{F}_v \cong \mathbb{C}$. This map induces $\hat{T}_{\tau_{0,v}} \cong \mathbb{G}_m^N$ over $\mathbb{C}$, thus $X_*(\hat{T}_{\tau_{0,v}}) = \mathbb{Z}^N$.

Endoscopic groups in $\mathcal{E}_{\mathrm{ell}}(\mathrm{U}_N)$ have the form $\mathrm{U}_{N_1} \times \mathrm{U}_{N_2}$ for integers $N_1 \geq N_2 \geq 0$ and $N_1 + N_2 = N$. See [Rogawski 1990, Section 4.6]; compare [Waldspurger 2010, Section 1.8] or [Mok 2015, Section 2.4]) for more details on full endoscopic data. We note that the Weil form (rather than the Galois form) of the L-group is needed to describe the L-morphisms in the endoscopic data.

2.2. ***Twisted general linear groups.*** Consider Cases S and U together. Keep the same E and c as above in Case U; set $E = F$ and $c = 1 \in \mathrm{Gal}(E/F)$ in Case S for uniformity. For $N \in \mathbb{Z}_{\geq 1}$ we introduce the groups

$$\tilde{G}^0(N) := \mathrm{Res}_{E/F}\,\mathrm{GL}_N \quad \text{and} \quad \tilde{G}(N) := \tilde{G}^0(N) \rtimes \langle\theta\rangle,$$

where $\langle\theta\rangle$ is an order 2 group with θ acting on $\tilde{G}^0(N)$ as $\theta(g): g \mapsto J_N^{*\,t} c(g)^{-1} (J_N^*)^{-1}$. Fix a standard pinning $(B_N, T_N, \{X_N\})$ of $\tilde{G}^0(N)$, which is stabilized by θ. In particular, T_N is the diagonal maximal torus of $\tilde{G}^0(N)$. Write $G(N) := \tilde{G}^0(N) \rtimes \theta$ for the θ-coset in $\tilde{G}(N)$. We also let $G(N)$ stand for the datum $(\tilde{G}(N), \theta)$ as in [Arthur 2013, page 125]. For simplicity of notation we will often write ${}^L G(N)$ and $\widehat{G(N)}$ for ${}^L\tilde{G}^0(N)$ and $\widehat{\tilde{G}^0(N)}$.

Denote by $\tilde{\mathcal{E}}_{\mathrm{ell}}(N)$ a set of representatives for isomorphism classes of twisted endoscopic data for $(\tilde{G}(N), \theta)$. Each element of $\tilde{\mathcal{E}}_{\mathrm{ell}}(N)$ is represented by a quadruple $\tilde{\mathfrak{e}} = (G^{\tilde{\mathfrak{e}}}, {}^L G^{\tilde{\mathfrak{e}}}, s^{\tilde{\mathfrak{e}}}, \xi^{\tilde{\mathfrak{e}}})$; see [Kottwitz and Shelstad 1999]. By $\tilde{\mathcal{E}}_{\mathrm{sim}}(N)$ we mean the subset of simple twisted endoscopic data in $\tilde{\mathcal{E}}_{\mathrm{ell}}(N)$, i.e., the data where $G^{\tilde{\mathfrak{e}}}$ attains maximal semisimple rank.

We give an explicit parametrization of $\tilde{\mathcal{E}}_{\mathrm{ell}}(N)$ by means of the twisted endoscopic group $G^{\tilde{\mathfrak{e}}}$ following [Arthur 2013, Section 1.2] and [Rogawski 1990, Section 4.7]. For simple endoscopic data we will write G and $\tilde{\xi}$ for $G^{\tilde{\mathfrak{e}}}$ and $\xi^{\tilde{\mathfrak{e}}}$, and describe $\tilde{\xi}$ explicitly.

Case S. The twisted endoscopic groups are parametrized by triples

$$(N_O, N_S, \eta), \quad N_O, N_S \in \mathbb{Z}_{\geq 0}, N_O + N_S = N, N_S \text{ is even}, \eta : \Gamma_F \to \{\pm 1\},$$

where the continuous character η is trivial if $N_O = 0$, nontrivial if $N_O = 2$, and arbitrary if $N_O > 2$. The corresponding $G^{\tilde{\mathfrak{e}}}$ is $\mathrm{SO}^{\eta}_{N_O} \times \mathrm{SO}_{N_S+1}$ if N is even, and $\mathrm{Sp}_{N_O-1} \times \mathrm{SO}_{N_S+1}$ if N is odd. In each case, $\xi^{\tilde{\mathfrak{e}}}$ can be described as in [Arthur 2013, page 11]. (If N is odd then η only affects $\xi^{\tilde{\mathfrak{e}}}$, not $G^{\tilde{\mathfrak{e}}}$.)

The triple corresponds to an element of $\tilde{\mathcal{E}}_{\mathrm{sim}}(N)$ precisely when $N_O = 0$ or $N_S = 0$. If $N = 2n$, then we have $(0, N, 1)$ and $(N, 0, \eta)$. In the first case, $G = \mathrm{SO}_{2n+1}$ and

$$\tilde{\xi} : {}^L G = \mathrm{Sp}_{2n} \hookrightarrow \mathrm{GL}_{2n}$$

is the standard embedding, inducing the map on cocharacter groups

$$X_*(\widehat{T}) = \mathbb{Z}^n \to X_*(\widehat{T}_{2n}) = \mathbb{Z}^{2n}, \quad (a_i)_{i=1}^n \mapsto (a_1, \ldots, a_n, -a_n, \ldots, -a_1).$$

The triple $(N, 0, \eta)$ corresponds to $G = \mathrm{SO}^{\eta}_{2n}$ and

$$\tilde{\xi} : {}^L G = \mathrm{O}_{2n} \hookrightarrow \mathrm{GL}_{2n}$$

is again the standard embedding, inducing the map on cocharacter groups

$$X_*(\widehat{T}) = X^*(T) = \mathbb{Z}^n \to X_*(\widehat{T}_{2n}) = \mathbb{Z}^{2n}, \quad (a_i)_{i=1}^n \mapsto (a_1, \ldots, a_n, -a_n, \ldots, -a_1).$$

Strictly speaking the codomain of $\tilde{\xi}$ is $\mathrm{GL}_{2n} \times \Gamma_{F_\eta/F}$, but the image of $\tilde{\xi}$ in the Galois factor is dictated by the fact that $\tilde{\xi}$ is an L-morphism, so we often omit it from the formula. The same will apply to $\tilde{\xi}$ below when N is odd.

If $N = 2n+1$, simple data correspond to $(N, 0, \eta)$, thus $G = \mathrm{Sp}_{2n}$ and

$$\tilde{\xi} : {}^L G_{F_\eta/F} = \mathrm{SO}_{2n+1} \times \Gamma_{F_\eta/F} \hookrightarrow \mathrm{GL}_{2n+1}$$

given by the standard embedding on SO_{2n+1} and $\eta : \Gamma_{F_\eta/F} \hookrightarrow \{\pm 1\} \subset \mathrm{GL}_{2n+1}$ on the Galois group. The induced map on cocharacters is

$$X_*(\widehat{T}) = \mathbb{Z}^n \to X_*(\widehat{T}_{2n+1}) = \mathbb{Z}^{2n+1}, \quad (a_i)_{i=1}^n \mapsto (a_1, \ldots, a_n, 0, -a_n, \ldots, -a_1).$$

Case U. The twisted endoscopic groups in $\tilde{\mathcal{E}}_{\mathrm{ell}}(N)$ are parametrized by quadruples

$$(N_1, N_2, \kappa_1, \kappa_2), \quad N_1, N_2 \in \mathbb{Z}_{\geq 0}, N_1 + N_2 = N, \kappa_1, \kappa_2 \in \{\pm 1\},$$

with (κ_1, κ_2) either $(1, -1)$ or $(-1, 1)$ if N is even, and $(1, 1)$ or $(-1, -1)$ if N is odd, modulo the equivalence $(N_1, N_2, \kappa_1, \kappa_2) \sim (N_2, N_1, \kappa_2, \kappa_1)$. (Compare with [Mok 2015, Section 2.4], but beware of a small inaccuracy that the equivalence between endoscopic data is incorrect there.) For each quadruple we have a twisted endoscopic group $G^{\mathfrak{e}} = \mathrm{U}_{N_1} \times \mathrm{U}_{N_2}$, with respect to the same E/F, which is part of a twisted endoscopic datum. We refer to *loc. cit.* for a formula for the L-morphism $\xi^{\tilde{\mathfrak{e}}}$, which depends on κ_1, κ_2.

The subset $\tilde{\mathcal{E}}_{\mathrm{sim}}(N)$ corresponds to quadruples $(N, 0, \kappa_1, \kappa_2)$. Set $\kappa := \kappa_1 \in \{\pm 1\}$. We need not keep track of κ_2 as it is determined by N and κ_1. In both cases the twisted endoscopic group is $G = \mathrm{U}_N$; let $\tilde{\xi}_+, \tilde{\xi}_- : {}^L\mathrm{U}_N \to {}^L\tilde{G}^0(N)$ denote the L-morphisms corresponding to $\kappa = 1, -1$, respectively. Let $\tau_0 : E \hookrightarrow \overline{F}$ be the embedding fixed in Section 2.1. Then $\widehat{G(N)} = \mathrm{GL}_N \times \mathrm{GL}_N$, where the copies of GL_N are indexed by τ_0 and $\tau_0 c$ in the order, and $\Gamma_{E/F}$ acts by permuting the two factors. The "base change" morphism $\tilde{\xi}_+$ is easy to describe

$$\begin{aligned} \tilde{\xi}_+ : {}^L(\mathrm{U}_N)_{E/F} \overset{\tau_0}{=} \mathrm{GL}_N \rtimes \Gamma_{E/F} &\to {}^L\tilde{G}^0(N) = (\mathrm{GL}_N \times \mathrm{GL}_N) \rtimes \Gamma_{E/F}, \\ g \rtimes \gamma &\mapsto (g, \hat{\theta}(g)) \rtimes \gamma = (g, J_N^{*\,t} g^{-1} (J_N^*)^{-1}) \rtimes \gamma. \end{aligned} \tag{2.2.1}$$

This map is independent of the choice of τ_0: if τ is replaced with τc, then the first identification is twisted by $g \rtimes \gamma \mapsto \hat{\theta}(g) \rtimes \gamma$ while the second map becomes $g \rtimes \gamma \mapsto (g, \hat{\theta}(g))$ (if the first component is still labeled by τ) so the changes are canceled out, while the last identification is unchanged.

The map $\tilde{\xi}_+$ induces a map on the cocharacter groups

$$X_*(\widehat{T}) \overset{\tau}{=} \mathbb{Z}^N \to X_*(\widehat{T}_N) = \mathbb{Z}^N \oplus \mathbb{Z}^N, \quad (a_i) \mapsto ((a_i), (-a_{N+1-i}))$$

in accordance with (2.2.1). Similarly we can describe the map induced by $\tilde{\xi}_+$:

$$X_*(\widehat{T}_\infty) = \bigoplus_\sigma X_*(\widehat{T}) \to X_*(\widehat{T}_{N,\infty}) = \bigoplus_\tau X_*(\widehat{T}_N),$$

where the first sum is over embeddings $\sigma : F \hookrightarrow \mathbb{C}$ and the second over $\tau : E \hookrightarrow \mathbb{C}$. Namely if $(a_i) \in X_*(\hat{T})$ denotes the σ-component, then the image is supported on the $\tau_{0,\sigma}$ and $\tau_{0,\sigma}c$ components on the right, and the map is $(a_i) \mapsto ((a_i), (-a_{N+1-i}))$.

We refer to [Mok 2015, Section 2.4] for a description of $\tilde{\xi}_-$, which will be needed only in a minor way, and leaves it as an exercise to describe the induced map on cocharacter groups. We just remark that $\tilde{\xi}_-$ is not defined on L-groups relative to a Galois extension; we need the Weil form of the L-groups.

2.3. ***Global parameters.*** Keep the notation from the preceding subsection. We introduce (conjugate) self-dual parameters for general linear groups, which will serve as parameters for automorphic representations of classical groups. We are following [Arthur 2013, Section 1.4] in spirit, but our situation is simpler in that we do not need the seed theorems of Arthur (namely [Arthur 2013, Theorems 1.4.1 and 1.4.2]) as we will prove only weak transfers.

For $m \in \mathbb{Z}_{\geq 1}$, let $\Psi_{\text{sim}}(m)$ denote the set of (isomorphism classes of) unitary cuspidal automorphic representations of $G(m, \mathbb{A}_F) = \mathrm{GL}_m(\mathbb{A}_E)$. Write $\Psi(N)$ for the set of formal global parameters

$$\psi = \boxplus_{i \in I} \mu_i \boxtimes \nu_{n_i}, \quad \mu_i \in \Psi_{\text{sim}}(m_i), m_i, n_i \in \mathbb{Z}_{\geq 1}, \tag{2.3.1}$$

where I is a finite index set, ν_{n_i} is an irreducible n_i-dimensional algebraic representation of $\mathrm{SL}_2(\mathbb{C})$, and $\sum_{i \in I} m_i n_i = N$. Given ψ is considered equal to another parameter $\psi' = \boxplus_{i' \in I'} \mu_{i'} \boxtimes \nu_{n'_i}$ if there exists a bijection $f : I \to I'$ such that $\mu_i = \mu_{f(i)}$ and $n_i = n_{f(i)}$ for all $i \in I$.

Given $\mu \in \Psi_{\text{sim}}(m)$, let $\mu^\star := \mu^\vee \circ c \in \Psi_{\text{sim}}(m)$ denote its conjugate-dual. This definition extends to $\Psi(N)$ by setting $\psi^\star := \boxplus_{i \in I} \mu_i^\star \boxtimes \nu_{n_i}$. Put

$$\tilde{\Psi}(N) := \{\psi \in \Psi(N) : \psi^\star = \psi\}.$$

Let S be a finite set of places of F containing all the places of F ramified in E. Write $\Psi^S(N)$ for the subset of $\psi \in \Psi(N)$ which are unramified outside S; the latter means that μ_i are all unramified outside S in (2.3.1). Put $\tilde{\Psi}^S(N) := \tilde{\Psi}(N) \cap \Psi^S(N)$. We define $\mathcal{C}_\infty(N)$ and $\mathcal{C}^S(N)$ to be the sets of $\mathbb{C}$-algebra characters of $\mathfrak{Z}(\tilde{G}^0(N)_\infty)$ and $\mathcal{H}^S_{\text{ur}}(\tilde{G}^0(N))$, respectively. We have a map

$$\psi \in \Psi^S(N) \mapsto (\zeta_{\psi,\infty}, c^S(\psi)) \in \mathcal{C}_\infty(N) \times \mathcal{C}^S(N)$$

defined as follows. Given ψ as in (2.3.1), we have $(\zeta_{\mu_{i,\infty}}, c^S(\mu_i)) \in \mathcal{C}_\infty(m_i) \times \mathcal{C}^S(m_i)$. The block diagonal embedding $\prod_{i\in I}\prod_{j=1}^{n_i} \mathrm{GL}_{m_i} \to \mathrm{GL}_{m_i n_i}$ induces a map

$$\prod_{i\in I}\prod_{j=1}^{n_i}(\mathcal{C}_\infty(m_i)\times\mathcal{C}^S(m_i)) \to \mathcal{C}_\infty(N)\times\mathcal{C}^S(N).$$

We define $(\zeta_{\psi_\infty}, c^S(\psi))$ to be the image of

$$\left(\zeta_{\mu_{i,\infty}} + \frac{n_i+1-2j}{2},\, q_v^{(n_i+1-2j)/2} c^S(\mu_i)\right)_{i\in I, 1\le j\le n_i},$$

where the sum $\zeta_{\mu_{i,\infty}} + a$ with $a \in \mathbb{Q}$ means that the sum is taken in $X_*(\widehat{T}_{m_i})_{\mathbb{C}}/\Omega_{m_i}$, and $a \in \mathbb{Q} = X_*(\mathbb{G}_m)_{\mathbb{Q}}$ embeds into $X_*(\widehat{T}_{m_i})_{\mathbb{Q}}$ via the inclusion of $\mathbb{G}_m = Z(\widehat{\tilde{G}^0(m_i)})^{\Gamma_F}$ in $\widehat{T}_{m_i}$; the product $q_v^b c^S(\psi)$ with $b \in \mathbb{Q}$ is taken in $\widehat{\tilde{G}^0(m_i)}$, where $q_v^b \in \mathbb{G}_m(\mathbb{C})$ is viewed as a central element of the dual group of $\tilde{G}^0(m_i)$. Our definition of $(\zeta_{\psi,\infty}, c^S(\psi))$ is given explicitly such that it is consistent with the local A-parameters at ∞ and finite places away from S obtained from localizing ψ.

2.4. ***Stabilized trace formulas.*** Let G be an inner form of a quasisplit classical group as in Section 2.1. (In fact the discussion below in the untwisted case works for general reductive groups as in the relevant parts of [Arthur 2013, Chapter 3].)

Let us begin by introducing the notion of Hecke types following [Arthur 2013, page 129]. We freely use the notation and the choices made from Section 1.4. Let S be a finite set of places of F containing $S_{\mathrm{bad}}(G)$. Let κ_S^∞ be an open compact subgroup of $\prod_v G(F_v)$, where v runs over finite places in S. Write K^S for the product of hyperspecial subgroups K_v^0 over finite places $v \notin S$, so $\kappa_S^\infty K^S$ is an open compact subgroup of $G(\mathbb{A}_F^\infty)$. Fix a finite set τ_∞ consisting of irreducible representations of a fixed maximal compact subgroup K_∞ of $G_\infty(\mathbb{R}) = \prod_{v\mid\infty} G(F_v)$. The pair $\kappa = (\tau_\infty, \kappa_S^\infty K^S)$ arising this way is called a Hecke type. Write $\mathcal{H}(G)_\kappa$ for the subspace generated by $f = f^\infty f_\infty \in \mathcal{H}(G)$ such that f^∞ is biinvariant under $\kappa_S^\infty K^S$ and such that f_∞ transforms under left and right translations under K_∞ according to representations in τ_∞.

Let $h \in \mathcal{H}_{\mathrm{ur}}^S(G)$ and $z \in \mathfrak{Z}(G_\infty)$. By evaluating $c^S \in \mathcal{C}^S(G)$ and $\zeta \in \mathcal{C}_\infty(G)$ at h and z respectively (see Section 1.4), we obtain the numbers to be denoted by $\widehat{h}(c^S) \in \mathbb{C}$ and $\zeta(z) \in \mathbb{C}$. Moreover h and z act on $\mathcal{H}_{\mathrm{ur}}^S(G)$ and $\mathcal{H}(G_\infty)$, written as $f^S \mapsto h * f^S$ and $f_\infty \mapsto z * f_\infty$, such that for irreducible admissible representations π^S of $G(\mathbb{A}_F^S)$ and π_∞ of $G_\infty(\mathbb{R})$,

$$\pi^S(h * f^S) = \widehat{h}(c^S(\pi^S))\pi^S(f^S), \quad \pi_\infty(z * f_\infty) = \zeta_{\pi_\infty}(z)\pi_\infty(f_\infty). \tag{2.4.1}$$

In particular we have identities by taking the traces of both sides in (2.4.1). The commuting action of (h, z) on $\mathcal{H}^S_{\mathrm{ur}}(G) \times \mathcal{H}(G_\infty)$, again denoted by $*$, obviously extends to $\mathcal{H}(G(\mathbb{A}_F), K^S)$.

Let $t \in \mathbb{R}_{\geq 0}$. Write $I^G_{\mathrm{disc},t}$ for the discrete part of the trace formula, which is an invariant linear form on $\mathcal{H}(G)$. The restriction of $I^G_{\mathrm{disc},t}$ to $\mathcal{H}(G)_\kappa$ decomposes as a finite sum of eigen-linear forms of $\mathcal{H}^S_{\mathrm{ur}}(G)$. Moreover, we can further decompose as a finite sum of eigen-linear forms for the action of $\mathfrak{Z}(G_\infty)$ on $\mathcal{H}(G_\infty)$. Thus we can write

$$I^G_{\mathrm{disc},t}(f) = \sum_{(\zeta, c^S) \in \mathcal{C}_\infty(G) \times \mathcal{C}^S(G)} I^G_{\mathrm{disc},\zeta,c^S}(f), \quad f \in \mathcal{H}(G(\mathbb{A}_F), K^S), \tag{2.4.2}$$

where $I^G_{\mathrm{disc},\zeta,c^S}$ are (ζ, c^S)-eigen-linear forms:

$$I^G_{\mathrm{disc},\zeta,c^S}((h, z) * f) = \widehat{h}(c^S)\zeta(z) I^G_{\mathrm{disc},\zeta,c^S}(f), \quad h \in \mathcal{H}^S_{\mathrm{ur}}(G), z \in \mathfrak{Z}(G_\infty). \tag{2.4.3}$$

The ζ and c^S appearing in (2.4.2) should be thought of as the infinitesimal characters at ∞ and the away-from-S Satake parameters for the automorphic representations contributing to $I_{\mathrm{disc},t}$. For a fixed Hecke type κ, the sum (2.4.2) runs over a finite set depending only on κ and not on $f \in \mathcal{H}(G)_\kappa$ by Harish-Chandra's finiteness theorem.

Note that t is determined by ζ to be the norm of the imaginary part of ζ; see [Arthur 2013, page 123]. That is, for a fixed ζ and c^S, the linear form $I^G_{\mathrm{disc},\zeta,c^S}$ in (2.4.2) is nontrivial for at most one t. Hence the meaning of $I^G_{\mathrm{disc},\zeta,c^S}$ is unambiguous even if we do not include t in the notation.

Write $R^G_{\mathrm{disc},t}$ for the regular representation of $G(\mathbb{A}_F)$ on $L^2_{\mathrm{disc}}([G])$; see Section 1.4. Just like $I^G_{\mathrm{disc},t}$ the invariant distribution $\operatorname{tr} R^G_{\mathrm{disc},t}$ decomposes as

$$\operatorname{tr} R^G_{\mathrm{disc},t}(f) = \sum_{(\zeta, c^s) \in \mathcal{C}_\infty(G) \times \mathcal{C}^S(G)} \operatorname{tr} R^G_{\mathrm{disc},\zeta,c^S}(f), \quad f \in \mathcal{H}(G(\mathbb{A}_F), K^S).$$

To discuss stable distributions, we will only consider G with the following property: for every finite sequence $\mathfrak{e}_i = (G^{\mathfrak{e}}_i, \mathcal{G}^{\mathfrak{e}}_i, s^{\mathfrak{e}}_i, \xi^{\mathfrak{e}}_i)$ indexed by $i = 1, \ldots, r$, where $\mathfrak{e}_i$ is an elliptic endoscopic datum for $G^{\mathfrak{e}}_{i-1}$ over F for $2 \leq i \leq r$, we can take $\mathcal{G}^{\mathfrak{e}}_i = {}^L G^{\mathfrak{e}}_i$ for all $1 \leq i \leq r$. (That is, $\mathfrak{e}_i$ is isomorphic to an endoscopic datum whose second entry is given by the L-group of the first entry.) The purpose of the simplifying hypothesis is to dispense with any discussion of z-extensions. This suffices for our needs as the classical groups in Section 2.1 satisfy the condition.

Now we consider elliptic endoscopic data $\mathfrak{e} = (G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\mathfrak{e}}, \xi^{\mathfrak{e}})$ for G over F. Denote by $f^{\mathfrak{e}} \in \mathcal{H}(G^{\mathfrak{e}}(\mathbb{A}_F))$ a Langlands–Shelstad transfer of f. Arthur inductively defined stable linear forms $S^{\mathfrak{e}}_{\mathrm{disc},t} = S^{G^{\mathfrak{e}}}_{\mathrm{disc},t} : \mathcal{H}(G^{\mathfrak{e}}) \to \mathbb{C}$ for each $\mathfrak{e}$ satisfying the fundamental identity

$$I^G_{\mathrm{disc},t}(f) = \sum_{\mathfrak{e}\in\mathcal{E}_{\mathrm{ell}}(G)} \iota(\mathfrak{e}) S^{\mathfrak{e}}_{\mathrm{disc},t}(f^{\mathfrak{e}}), \tag{2.4.4}$$

where $\iota(\mathfrak{e}) \in \mathbb{Q}_{>0}$ is an explicit constant. For quasisplit $G = G^{\mathfrak{e}_0}$, the equality should be viewed as an inductive definition of $S^G_{\mathrm{disc},t}$; the inductive procedure is based on the fact that the semisimple rank of $G^{\mathfrak{e}}$ is less than that of G for $\mathfrak{e} \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$. The role of the stabilization of the trace formula is to tell us that the inductive definition of $S^G_{\mathrm{disc},t}$ indeed yields a stable linear form. If G is not quasisplit then both sides of (2.4.4) are a priori defined, and the content of the stabilization is that the equality holds in (2.4.4). See the explanation between (3.2.3) and (3.2.4) in [Arthur 2013] for more details.

The transfer $f^{\mathfrak{e}}$ has trivial stable orbital integrals unless $S \supset S_{\mathrm{bad}}(G^{\mathfrak{e}})$, which we assume from now. In particular if $f \in \mathcal{H}(G(\mathbb{A}_F), K^S)$ then $f^{\mathfrak{e}} \in \mathcal{H}(G^{\mathfrak{e}}(\mathbb{A}_F), K^{\mathfrak{e},S})$, where $K^{\mathfrak{e},S}$ is the product of fixed hyperspecial subgroups of $G^{\mathfrak{e}}(F_v)$ over $v \notin S$. Based on (2.4.2) and (2.4.4), we can adapt the argument from [Arthur 2013, Lemma 3.3.1] to decompose $S^{\mathfrak{e}}_{\mathrm{disc},t}$ into stable linear forms

$$S^{\mathfrak{e}}_{\mathrm{disc},t}(f^{\mathfrak{e}}) = \sum_{(\zeta',c'^{,S})\in\mathcal{C}_\infty(G^{\mathfrak{e}})\times\mathcal{C}^S(G^{\mathfrak{e}})} S^{\mathfrak{e}}_{\mathrm{disc},\zeta',c'^{,S}}(f^{\mathfrak{e}}), \quad f \in \mathcal{H}(G^{\mathfrak{e}}(\mathbb{A}_F), K^{\mathfrak{e},S}),$$

such that each $S^{\mathfrak{e}}_{\zeta',c'^{,S}}$ satisfies the analogue of (2.4.3). If G is quasisplit, then this applies in particular to $G^{\mathfrak{e}} = G$, that is, we have a stable linear form $S^G_{\mathrm{disc},\zeta,c^S} : \mathcal{H}(G(\mathbb{A}_F), K^S) \to \mathbb{C}$ for (ζ, S) as before. Given $(\zeta, c^S) \in \mathcal{C}_\infty(G) \times \mathcal{C}^S(G)$, define

$$S^{\mathfrak{e}}_{\mathrm{disc},\zeta,c^S} := \begin{cases} \sum_{(\zeta',c'^{,S})\mapsto(\zeta,c^S)} S^{\mathfrak{e}}_{\mathrm{disc},\zeta',c'^{,S}} & \text{if } S \supset S_{\mathrm{bad}}(G^{\mathfrak{e}}), \\ 0 & \text{otherwise.} \end{cases}$$

where the sum is taken over the pairs such that $\zeta' \mapsto \zeta$ and $c'^{,S} \mapsto c^S$ under the natural maps $\mathcal{C}_\infty(G^{\mathfrak{e}}) \to \mathcal{C}_\infty(G)$ and $\mathcal{C}^S(G^{\mathfrak{e}}) \to \mathcal{C}^S(G)$ induced by $\xi^{\mathfrak{e}}$. Then we have a refinement of (2.4.4) as in [Arthur 2013, Lemma 3.3.1]:

$$I^G_{\mathrm{disc},\zeta,c^S}(f) = \sum_{\mathfrak{e}\in\mathcal{E}_{\mathrm{ell}}(G)} \iota(\mathfrak{e}) S^{\mathfrak{e}}_{\mathrm{disc},\zeta,c^S}(f^{\mathfrak{e}}). \tag{2.4.5}$$

More precisely, the refinement by c^S is done in [loc. cit.] but not by infinitesimal characters. The argument of [loc. cit.] based on multipliers works in the same way to give refinement by ζ as long as the archimedean transfer is compatible with infinitesimal characters; such compatibility is stated and proved in either of [Mezo 2013, Lemma 24] and [Mœglin and Waldspurger 2016a, I.2.8. Corollary], including the twisted case. This point is also explained in [Taïbi 2019, page 867].

The discussion so far can be adapted to the twisted case, as this case is covered in [Arthur 2013, Sections 3.1–3.3]. For the twisted group $\tilde{G}(N)$ introduced in

Section 2.1, denote by $I^{G(N)}_{\mathrm{disc},t}$ the twisted invariant trace formula and by $\tilde{\mathcal{E}}_{\mathrm{ell}}(N)$ a set of representatives for isomorphism classes of twisted endoscopic data. Each $\tilde{\mathfrak{e}} \in \tilde{\mathcal{E}}_{\mathrm{ell}}(N)$ is again represented by a quadruple $(G^{\tilde{\mathfrak{e}}}, {}^L G^{\tilde{\mathfrak{e}}}, s^{\tilde{\mathfrak{e}}}, \xi^{\tilde{\mathfrak{e}}})$, where $G^{\tilde{\mathfrak{e}}}$ is a product of one or two classical groups as listed in Section 2.2.

Recall that we defined $\mathcal{C}_\infty(N)$ and $\mathcal{C}^S(N)$ in Section 2.3. Put $K(N)^S \subset \tilde{G}^0(N)(\mathbb{A}^S_F)$ for the product of hyperspecial subgroups coming from the obvious integral model of $\tilde{G}^0(N)$ over $\mathcal{O}_F$. We have $h \in \mathcal{H}^S_{\mathrm{ur}}(\tilde{G}^0(N))$ and $z \in \mathfrak{Z}(\tilde{G}^0(N)_\infty)$ act on $\mathcal{H}(G(N, \mathbb{A}^S_F), K(N)^S)$ and $\mathcal{H}(G(N)_\infty)$, respectively, such that the analogue of (2.4.1) holds for representations of $\tilde{G}(N, \mathbb{A}^S_F)$ and $\tilde{G}(N)_\infty$. The decomposition (2.4.2) admits a twisted analogue

$$I^{G(N)}_{\mathrm{disc},t}(f) = \sum_{(\tilde{\zeta}, \tilde{c}^S) \in \mathcal{C}_\infty(N) \times \mathcal{C}^S(N)} I^{G(N)}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}(f), \quad f \in \mathcal{H}(G(N, \mathbb{A}_F), K(N)^S),$$

where each $I^{G(N)}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}$ is an invariant linear form on $\mathcal{H}(G(N))$ satisfying the eigenproperty analogous to (2.4.3). As before, $I^{G(N)}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}$ is nontrivial for at most one t, so there is no danger if t is omitted in the subscript.

Provided that $S \supset S_{\mathrm{bad}}(G^{\tilde{\mathfrak{e}}})$, the L-morphism $\xi^{\tilde{\mathfrak{e}}} : {}^L G^{\tilde{\mathfrak{e}}} \to {}^L \tilde{G}^0(N)$ induces maps $\mathcal{C}_\infty(G^{\tilde{\mathfrak{e}}}) \to \mathcal{C}_\infty(N)$ and $\mathcal{C}^S(G^{\tilde{\mathfrak{e}}}) \to \mathcal{C}^S(N)$. Thereby we put, for each $(\tilde{\zeta}, \tilde{c}^S) \in \mathcal{C}_\infty(N) \times \mathcal{C}^S(N)$,

$$S^{\tilde{\mathfrak{e}}}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S} := \sum_{(\zeta, c^S) \mapsto (\tilde{\zeta}, \tilde{c}^S)} S^{\tilde{\mathfrak{e}}}_{\mathrm{disc},\zeta,c^S},$$

as a stable linear form on $\mathcal{H}(G^{\tilde{\mathfrak{e}}})$. If $S \not\supset S_{\mathrm{bad}}(G^{\tilde{\mathfrak{e}}})$ then set $S^{\tilde{\mathfrak{e}}}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S} := 0$.

The stabilization of the twisted trace formula due to Moeglin and Waldspurger [2016b, X.8.1] shows that, if $f^{\tilde{\mathfrak{e}}}$ denotes a Langlands–Shelstad–Kottwitz transfer of $f \in \mathcal{H}(G(N))$ then the twisted analogue of (2.4.4) holds:

$$I^{G(N)}_{\mathrm{disc},t}(f) = \sum_{\tilde{\mathfrak{e}} \in \tilde{\mathcal{E}}_{\mathrm{ell}}(N)} \iota(\tilde{\mathfrak{e}}) S^{\tilde{\mathfrak{e}}}_{\mathrm{disc},t}(f^{\tilde{\mathfrak{e}}}), \tag{2.4.6}$$

where $\iota(\tilde{\mathfrak{e}}) \in \mathbb{Q}_{>0}$ is an explicit constant. For $(\tilde{\zeta}, \tilde{c}^S)$ as above, we refine the preceding formula again by [Arthur 2013, Lemma 3.3.1] (see the paragraph below (2.4.5)):

$$I^{G(N)}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}(f) = \sum_{\tilde{\mathfrak{e}} \in \tilde{\mathcal{E}}_{\mathrm{ell}}(N)} \iota(\tilde{\mathfrak{e}}) S^{\tilde{\mathfrak{e}}}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}(f^{\tilde{\mathfrak{e}}}). \tag{2.4.7}$$

2.5. ***Weak transfer for classical groups.*** Let G^* be a quasisplit classical group as in Case S or U of Section 2.1. Let $\tilde{\xi} : {}^L G^* \to {}^L \tilde{G}^0(N)$ be the L-morphism such that G^* and $\tilde{\xi}$ constitute a simple twisted endoscopic group for $(\tilde{G}(N), \theta)$ as in Section 2.2. Let $(G, \mathfrak{i})$ be an inner twist of G^* over F (Section 1.4).

Theorem 2.5.1 (quasisplit case). *Assume (H1) in Section 1.1 and let $G = G^*$. Fix a finite set $S \supset S_{\mathrm{bad}}(G)$:*

(1) *For $(\zeta, c^S) \in \mathcal{C}_\infty(G) \times \mathcal{C}^S(G)$ write $(\tilde{\zeta}, \tilde{c}^S) \in \mathcal{C}_\infty(N) \times \mathcal{C}^S(N)$ for the image of (ζ, c^S) under $\tilde{\xi}$. Unless $(\tilde{\zeta}, \tilde{c}^S) = (\zeta_{\psi,\infty}, c^S(\psi))$ for some $\psi \in \tilde{\Psi}^S(N)$,*

$$\operatorname{tr} R^G_{\mathrm{disc},\zeta,c^S}(f) = I^G_{\mathrm{disc},\zeta,c^S}(f) = S^G_{\mathrm{disc},\zeta,c^S}(f) = 0, \quad f \in \mathcal{H}(G(\mathbb{A}_F), K^S).$$

(2) *We have a $G(\mathbb{A}_F)$-equivariant decomposition*

$$L^2_{\mathrm{disc}}([G])^{S-\mathrm{ur}} = \bigoplus_{\psi} \bigoplus_{(\zeta,c^S)\mapsto(\zeta_{\psi,\infty},c^S(\psi))} L^2_{\mathrm{disc},\zeta,c^S}([G])$$

where the first sum runs over $\psi \in \tilde{\Psi}^S(N)$, and the second over $(\zeta, c^S) \in \mathcal{C}_\infty(G) \times \mathcal{C}^S(G)$ which map to $(\zeta_{\psi,\infty}, c^S(\psi))$ under $\tilde{\xi}$. (See Section 2.3 for the notation.)

This theorem corresponds to [Arthur 2013, Proposition 3.4.1, Corollary 3.4.3]. Arthur's main global theorems (Section 1.5 therein) show that only a proper subset of $\tilde{\Psi}^S(N)$ contributes in (i) and (ii), consisting of the ones coming from square-integrable parameters of G. The soft argument here does not narrow down the set of ψ as much. Theorem 2.5.1 is proven essentially in the same way as [Arthur 2013, Proposition 3.4.1, Corollary 3.4.3]. We give some details for the convenience of the reader, taking for granted the key input [Arthur 2013, Proposition 3.5.1] on vanishing.

Proof. Assume that $(\tilde{\zeta}, \tilde{c}^S) \neq (\zeta_{\psi,\infty}, c^S(\psi))$ for any $\psi \in \tilde{\Psi}^S(N)$. Let us show (i) and (ii) by induction on N.

Let us check (i) and (ii) when G is a torus; this serves as the base case. Concretely $G = \mathrm{SO}_2^\eta$ (allowing $\eta = 1$) in Case S, and $G = \mathrm{U}_1$ in Case U. Since the two cases are similar, we only consider the latter case. Then $L^2_{\mathrm{disc}}([G])^{S-\mathrm{ur}} = \bigoplus_\chi \chi$, where $\chi : \mathrm{U}_1(\mathbb{A}_F) \backslash \mathrm{U}_1(\mathbb{A}_F) \to \mathbb{C}^\times$ is an automorphic character unramified outside S. This matches the decomposition on the right-hand side of (ii) since each χ determines a unique conjugate self-dual Hecke character $\psi : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ by $\psi(x) = \chi(x/x^c)$ and a unique pair (ζ, c^S) recording the infinitesimal character and the Satake parameter of χ. Turning to the displayed formula of (i), we see that the first equality holds because a torus has no proper parabolic subgroup, and that the second equality holds because a torus permits no elliptic endoscopic data other than the tautological one. Now the vanishing of the quantities in (i) follows from the decomposition of (ii).

Now we proceed with the induction hypothesis- suppose that (i) and (ii) are known for all quasisplit classical groups which are simple twisted endoscopic groups of $G(N')$ for all $N' < N$ and that G is a simple twisted endoscopic group for $G(N)$. (Here $N > 1$.)

Recall that $I^G_{\mathrm{disc},t} - \mathrm{tr}\, R^G_{\mathrm{disc},t}$ is by definition a linear combination of traces of induced representations from discrete automorphic representations π_M on proper Levi subgroups M of G. So the same is true for $I^G_{\mathrm{disc},\zeta,c^S} - \mathrm{tr}\, R^G_{\mathrm{disc},\zeta,c^S}$. Hence, if the latter were nonzero, then there exists a proper Levi M of G such that (ζ, c) is the image of $\mathfrak{c} = (\zeta_M, c^S_M) \in \mathcal{C}_\infty(M) \times \mathcal{C}^S(M)$ associated with some discrete automorphic representation π_M of $M(\mathbb{A}_F)$. We can write $M = M_h \times M_l$ with M_h a classical group, where M_h is realized as a twisted endoscopic group for $G(N - 2N')$, and $M_l = G(N')$ with $N' < N$. According to $M = M_h \times M_l$, we decompose $\mathfrak{c} = (\mathfrak{c}_h, \mathfrak{c}_l)$. By induction hypothesis for M_h, we have $\mathfrak{c}_h$ map to $(\zeta_{\psi_h,\infty}, c^S(\psi_h))$ for some $\psi_h \in \tilde{\Psi}(N - 2N')$. On the other hand, since the L^2-discrete spectrum of M_l is completely accounted for by $\Psi(N')$ thanks to [Mœglin and Waldspurger 1989] (see [Arthur 2013, pages 23–25] for explanation), we have $\mathfrak{c}_l = (\zeta_{\psi_l,\infty}, c^S(\psi_l))$ for some $\psi_l \in \Psi(N')$. Since (ζ, c) is the image of $(\mathfrak{c}_h, \mathfrak{c}_l)$ under parabolic induction, we see that $(\tilde{\zeta}, \tilde{c}^S) = (\zeta_{\psi,\infty}, c^S(\psi))$ for $\psi = \psi_h \boxplus \psi_l \boxplus \psi_l^\star \in \tilde{\Psi}(N)$. This is a contradiction. We conclude that

$$I^G_{\mathrm{disc},\zeta,c^S}(f) = \mathrm{tr}\, R^G_{\mathrm{disc},\zeta,c^S}(f), \quad f \in \mathcal{H}(G(\mathbb{A}_F), K^S). \tag{2.5.1}$$

Now $I^G_{\mathrm{disc},\zeta,c^S} - S^G_{\mathrm{disc},\zeta,c^S}$ is a linear combination of $S^{\mathfrak{e}}_{\mathrm{disc},\zeta,c^S}$ over $\mathfrak{e} \in \mathcal{E}^<_{\mathrm{ell}}(G)$. If the difference were nonzero, then for some $\mathfrak{e}$,

$$S^{\mathfrak{e}}_{\mathrm{disc},\zeta,c^S} = \sum_{(\zeta',c'^{,S}) \mapsto (\zeta,c^S)} S^{\mathfrak{e}}_{\mathrm{disc},\zeta',c'^{,S}}$$

is nontrivial. Since $G^{\mathfrak{e}}$ is a product of quasisplit classical groups G_1 and G_2 of lower rank (see Section 2.1), by arguing as in the preceding paragraph based on the induction hypothesis for G_1 and G_2, we reach a similar contradiction. (The difference is that there is no general linear factor in G and that the role of parabolic induction is played by the endoscopic transfer via $\xi^{\mathfrak{e}}$.) Hence

$$I^G_{\mathrm{disc},\zeta,c^S}(f) = S^G_{\mathrm{disc},\zeta,c^S}(f), \quad f \in \mathcal{H}(G(\mathbb{A}_F), K^S). \tag{2.5.2}$$

By the initial hypothesis, $I^{G(N)}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S} = 0$. Applying (2.4.7), (2.5.1) and (2.5.2), we obtain

$$0 = I^{G(N)}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}(\tilde{f}) = \sum_{\tilde{\mathfrak{e}} \in \tilde{\mathcal{E}}_{\mathrm{ell}}(N)} \iota(\tilde{\mathfrak{e}})\, \mathrm{tr}\, R^{G^{\tilde{\mathfrak{e}}}}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}(\tilde{f}^{\tilde{\mathfrak{e}}}). \tag{2.5.3}$$

The sum runs over the set of $\tilde{\mathfrak{e}}$ such that $G^{\tilde{\mathfrak{e}}}$ is unramified outside S; thus it is a finite sum. Each $\mathrm{tr}\, R^{G^{\tilde{\mathfrak{e}}}}_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}$ is a positive linear combination of traces of finitely many discrete automorphic representations $\pi^{\tilde{\mathfrak{e}}}$ of $G^{\tilde{\mathfrak{e}}}(\mathbb{A}_F)$. If $\tilde{f}$ is chosen from the Hecke algebra on $G(N)$ of a fixed Hecke type κ then each $\tilde{f}^{\tilde{\mathfrak{e}}}$ belongs to the Hecke algebra on $G^{\tilde{\mathfrak{e}}}$ of a Hecke type $\kappa^{\tilde{\mathfrak{e}}}$ determined by κ. Thus the set of contributing $\pi^{\tilde{\mathfrak{e}}}$ is contained in a finite set depending only on κ, by the condition that $\pi^{\tilde{\mathfrak{e}}}$ should be unramified outside S and that the components of $\pi^{\tilde{\mathfrak{e}}}$ at S should have finitely many types dictated by $\kappa^{\tilde{\mathfrak{e}}}$. (The discussion of this paragraph is based on the explanation between (3.4.11) and (3.4.13) of [Arthur 2013]. The two key facts are that a compatible family therein arises exactly from an element of the Hecke algebra on $G(N)$ and that a compatible family always has a Hecke type.)

The preceding paragraph tells us that Arthur's vanishing result [2013, Proposition 3.5.1] applies to (2.5.3). As a result, every summand in (2.5.3) is identically zero. In particular this is true for $G^{\tilde{\mathfrak{e}}} = G$, namely $\operatorname{tr} R^G_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}$ is an empty linear combination. That is, $\operatorname{tr} R^G_{\mathrm{disc},\tilde{\zeta},\tilde{c}^S}(f) = 0$ for all f. This completes the proof of (i) in light of (2.5.1) and (2.5.2).

Part (ii) follows immediately from (i) since $\operatorname{tr} R^G_{\mathrm{disc},\zeta,c^S} = 0$, which implies $L^2_{\mathrm{disc},\zeta,c^S}([G]) = 0$, unless (ζ, c^S) maps to $(\zeta_{\psi,\infty}, c^S(\psi))$ for some $\psi \in \tilde{\Psi}^S(N)$. □

Theorem 2.5.2 (general case). *Assume (H1). Let $(G, \mathfrak{i})$ be an inner twist of G^* over F. For each $\zeta \in \mathcal{C}_\infty(G)$ and $c^S \in \mathcal{C}^S(G)$,*

$$\operatorname{tr} R^G_{\mathrm{disc},\zeta,c^S}(f) = I^G_{\mathrm{disc},\zeta,c^S}(f) = 0, \qquad f \in \mathcal{H}(G(\mathbb{A}_F), K^S),$$

unless ξ sends (ζ, c^S) to $(\zeta_{\psi,\infty}, c^S(\psi))$ for some $\psi \in \tilde{\Psi}(N)$. There is a $G(\mathbb{A}_F)$-equivariant decomposition

$$L^2_{\mathrm{disc}}([G])^{S-\mathrm{ur}} = \bigoplus_{\psi} \bigoplus_{(\zeta,c^S)\mapsto(\zeta_{\psi,\infty},c^S(\psi))} L^2_{\mathrm{disc},\zeta,c^S}([G]),$$

where the sums run over $\psi \in \tilde{\Psi}^S(N)$ and $(\zeta, c^S) \in \mathcal{C}_\infty(G) \times \mathcal{C}^S(G)$ such that $\xi((\zeta, c^S)) = (\zeta_{\psi,\infty}, c^S(\psi))$.

Proof. We induct on N as in the proof of Theorem 2.5.1. The argument there carries over to show that

$$I^G_{\mathrm{disc},\zeta,c^S}(f) = \operatorname{tr} R^G_{\mathrm{disc},\zeta,c^S}(f), \qquad f \in \mathcal{H}(G(\mathbb{A}_F), K^S),$$

using the fact that a proper Levi subgroup of G is a product of $G'(N)$ with $N' < N$ and a non-quasisplit classical group of lower rank than G; the induction hypothesis is applied to the latter.

Now we consider (2.4.5). Since the stable distributions on the right-hand vanish by Theorem 2.5.1 (if $\mathfrak{e} \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$, we can also argue as in the proof of that theorem), we deduce that $I^G_{\mathrm{disc},\zeta,c^S}(f) = 0$. Hence $\operatorname{tr} R^G_{\mathrm{disc},\zeta,c^S}$ vanishes as well, and the assertion about $L^2_{\mathrm{disc}}([G])$ follows. □

Theorem 2.5.2 can be rephrased as the existence of a weak endoscopic lift for G as a twisted endoscopic group of $(\tilde{G}(N), \theta)$ in the next corollary. Let us introduce a notion that will be used here and in the next section. Let π_i be a cuspidal automorphic representation of $\mathrm{GL}_{N_i}(\mathbb{A}_F)$ for $i = 1, \ldots, r$. Following [Clozel 1990, Definition 1.2], the isobaric sum of $\pi_1, \dots, \pi_r$, denoted by $\boxplus_{i=1}^r \pi_i$, is defined to be an automorphic representation Π of $\mathrm{GL}_{\sum_i N_i}(\mathbb{A}_F)$ such that Π_v is isomorphic to the Langlands subquotient of the normalized parabolic induction from $\bigotimes_{i=1}^r \pi_{i,v}$ at every place v of F. As remarked in [loc. cit.] an automorphic representation of $\mathrm{GL}_N(\mathbb{A}_F)$ is written as an isobaric sum in a unique way (up to permutation) by a result of Jacquet and Shalika.

Corollary 2.5.3. *Assume (H1). For every discrete automorphic representation π of $G(\mathbb{A}_F)$ unramified away from S, there exists an automorphic representation Π of $G^0(N, \mathbb{A}_F)$, which is an isobaric sum of cuspidal representations, such that $\Pi^\vee \cong \Pi \circ c$ and $(\zeta_{\pi_\infty}, c^S(\pi))$ maps to $(\zeta_{\Pi_\infty}, c^S(\Pi))$ via $\tilde{\xi}$.*

Proof. Since π appears in $L^2_{\mathrm{disc}}([G])^{S-\mathrm{ur}}$, it appears in $L^2_{\mathrm{disc},\zeta,c^S}([G])$ for some (ζ, c^S) mapping to $(\zeta_{\psi,\infty}, c^S(\psi))$ as in Theorem 2.5.2. In particular $(\zeta, c^S) = (\zeta_{\pi_\infty}, c^S(\pi))$. Writing ψ in the form (2.3.1), we can take Π to be the isobaric sum

$$\boxplus_{i\in I}(\mu_i|\det|^{(n_i-1)/2} \boxplus \mu_i|\det|^{(n_i-3)/2} \boxplus \cdots \boxplus \mu_i|\det|^{(1-n_i)/2}).$$

By construction $(\zeta_{\psi,\infty}, c^S(\psi)) = (\zeta_{\Pi_\infty}, c^S(\Pi))$. Since $\psi^\star = \psi$, it follows that $\Pi^\vee \cong \Pi \circ c$. ☐

3. Automorphic Galois representations

3.1. ***The Buzzard–Gee conjecture.*** Throughout this subsection, let G be a connected reductive group over a number field F (which need not be a classical group). Let ℓ be a prime number and $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ an isomorphism. We work with fixed ℓ and ι at a time, but note that the conjectures below predict the existence of weakly compatible systems of Galois representations in a suitable sense as ℓ and ι vary.

Let $G_{\infty,\mathbb{C}} = \prod_\tau G_\tau$ and $T_{\infty,\mathbb{C}} = \prod_\tau T_\tau$ be as in Section 1.4. Fix a Borel subgroup $B_{\infty,\mathbb{C}} = B_\tau$ containing $T_{\infty,\mathbb{C}}$. The half sum of positive roots is denoted by $\rho_\infty = (\rho_\tau)_\tau \in X^*(T_{\infty,\mathbb{C}})_\mathbb{Q}$. We also view ρ_∞ as the half sum of positive coroots of $\widehat{T}_{\infty,\mathbb{C}}$ relative to $\widehat{B}_{\infty,\mathbb{C}}$, thus an element of $X_*(\widehat{T}_{\infty,\mathbb{C}})_\mathbb{Q}$. We also have $\rho \in X^*(T) = X_*(\widehat{T})$ as the half sum of positive roots for T and B as in Section 1.4. The pairs (B, T) and (B_τ, T_τ) determine isomorphisms $X^*(T) \cong X^*(T_\tau)$ and $X_*(\widehat{T}) \cong X_*(\widehat{T}_\tau)$, under which ρ maps to ρ_τ.

Let $\pi = \otimes'_v \pi_v$ be a discrete automorphic representation of $G(\mathbb{A}_F)$. We assigned the infinitesimal character $\zeta_{\pi_\infty} = (\zeta_{\pi,\tau}) \in X^*(T_{\infty,\mathbb{C}})_\mathbb{C}/\Omega_\infty = \bigoplus_\tau X_*(\widehat{T})_\mathbb{C}/\Omega$ in Section 1.4. We introduce two notions of algebraicity for π in terms of ζ_{π_∞}.

Definition 3.1.1. We say that π is *L-algebraic* if $\zeta_{\pi_\infty} \in X^*(T_{\infty,\mathbb{C}})/\Omega$. If ζ_{π_∞} belongs to the image of $X^*(T_{\infty,\mathbb{C}}) + \rho_\infty$ in $X^*(T_{\infty,\mathbb{C}})_\mathbb{C}/\Omega$ then π is said to be *C-algebraic*. The representation π is *regular* if ζ_{π_∞} is regular as an Ω-orbit in $X^*(T_{\infty,\mathbb{C}})_\mathbb{C}$, i.e., each element of the orbit has the trivial stabilizer in Ω.

The L and C-algebraicity conditions are independent of the choice of $T_{\infty,\mathbb{C}}$ and $B_{\infty,\mathbb{C}}$; see [Buzzard and Gee 2014, Section 2.3]. An equivalent definition can be given by imposing similar conditions on $\zeta_{\pi_{\infty,\tau}}$, T_τ, and ρ_τ for every $\tau : F \hookrightarrow \mathbb{C}$.

Write $S_{\mathrm{ram}}(\pi)$ for the set of places v of F such that either $v \in S_{\mathrm{bad}}(G)$ or π_v is ramified. Let $S(\ell)$ denote the set of places of F above ℓ. At a finite place $v \notin S_{\mathrm{ram}}(\pi)$ of F, let $\phi_{\pi_v} : W_{F_v} \to {}^L G(\mathbb{C})$ denote the unramified L-parameter for π_v (Section 1.4). Changing coefficients by ι, we obtain

$$\iota\phi_{\pi_v} : W_{F_v} \to {}^L G(\overline{\mathbb{Q}}_\ell).$$

Given a Galois representation $r : \Gamma_F \to {}^L G(\overline{\mathbb{Q}}_\ell)$ which is de Rham at ℓ and an embedding $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_\ell$, we follow [Buzzard and Gee 2014, Section 2.4] to assign a Hodge–Tate cocharacter $\mu_{\mathrm{HT}}(r, \sigma) : \mathbb{G}_m \to \iota\widehat{G}$ over $\mathbb{C}_\ell$, whose $\widehat{G}(\mathbb{C}_\ell)$-conjugacy class is defined over $\overline{\mathbb{Q}}_\ell$; here $\iota\widehat{G}$ stands for the base change of $\widehat{G}$ from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$ via ι or its further base extension to $\mathbb{C}_\ell$. (Such a base change is implicit in the notation ${}^L G(\overline{\mathbb{Q}}_\ell)$.) Thereby we obtain a conjugacy class of cocharacters $\mathbb{G}_m \to \iota\widehat{G}$ over $\overline{\mathbb{Q}}_\ell$, which in turn gives an element of $X_*(\iota\widehat{T})/\Omega$. We denote the resulting element by

$$\mu_{r,\sigma} \in X_*(\iota\widehat{T})/\Omega.$$

Conjecture 3.1.2. *Suppose that π is L-algebraic. There exists a Galois representation*

$$r = r_{\ell,\iota}(\pi) : \Gamma_F \to {}^L G(\overline{\mathbb{Q}}_\ell)$$

such that:

(1) $r|^{\mathrm{ss}}_{W_{F_v}} \cong \iota\phi_{\pi_v}$ *at finite places* $v \notin S_{\mathrm{ram}}(\pi) \cup S(\ell)$.

(2) $\mu_{r,\iota\tau} = -\iota\zeta_{\pi,\tau}$ *for every embedding* $\tau : F \hookrightarrow \mathbb{C}$.

Remark 3.1.3. The negative sign in (ii), which does not appear in [Buzzard and Gee 2014, Section 3.2], is due to the different sign convention. (The cyclotomic character has Hodge–Tate weight 1 there; see [loc. cit., Section 2.4].) In this conjecture and the next conjecture, we omit the statement on the image of complex conjugation as we fell short of proving it in the case of interest, see Remark 3.2.8 below.

Remark 3.1.4. When $G = \mathrm{GL}_N$, choosing T to be the diagonal maximal torus, we can identify each member of $X^*(T_{\iota^{-1}\sigma})/\Omega_\tau$ with ordered n integers $(a_i)_{i=1}^n$ with $a_1 \geq a_2 \geq \cdots \geq a_n$. Similarly, each member of $X^*(T_{\infty,\mathbb{C}})_{\mathbb{Q}}/\Omega$ can be regarded as ordered rational numbers $(a_i)_{i=1}^n$ such that $a_1 \geq a_2 \geq \cdots \geq a_n$. In particular, if π is L-algebraic or C-algebraic, then we can write $-\zeta_{\pi,\tau} = (a_i)_{i=1}^n$ for a suitable set of a_i as such. So condition (ii) above may be understood as an equality of multisets for $G = \mathrm{GL}_N$.

Following [Zhu 2020b] (which gives a different but equivalent definition of C-groups as in [Buzzard and Gee 2014]) the C-group of G is defined by taking the semidirect product

$$^{C}G := {}^{L}G \rtimes \mathbb{G}_m, \quad (1 \rtimes t)(g \rtimes 1)(1 \rtimes t)^{-1} = \mathrm{Ad}(\rho(t))g \rtimes 1, \quad g \in {}^{L}G, t \in \mathbb{G}_m.$$

This is well defined because $\mathrm{Ad}(\rho)$ is an algebraic action of $\mathbb{G}_m$ on ${}^{L}G$ (although ρ need not be an algebraic cocharacter into $\hat{G}$). We can also write ${}^{C}G = \widehat{G} \rtimes (\mathbb{G}_m \times \Gamma_F)$ with $\mathbb{G}_m$ and Γ_F acting on $\widehat{G}$ via the $\mathrm{Ad}(\rho)$-action and the L-action respectively, since the Galois action and the $\mathbb{G}_m$-action on $\widehat{G}$ commute. It is convenient to fix a finite Galois extension F'/F over which G splits, and use the finite Galois forms of the L-group ${}^{L}G_{F'/F} = {}^{L}G \rtimes \Gamma_{F'/F}$ and similarly for the C-group ${}^{C}G_{F'/F} = {}^{L}G_{F'/F} \rtimes \mathbb{G}_m$. From now on, we use the finite Galois form and drop F'/F from the subscript unless specified otherwise. We will use the natural $\widehat{G}$-conjugation on ${}^{C}G$, with coefficients in $\overline{\mathbb{Q}}_\ell$ or $\mathbb{C}$, to define the notion of isomorphism for local parameters and global Galois representations valued in ${}^{C}G$. (It does not make any difference if we use the conjugation by $\widehat{G} \rtimes \mathbb{G}_m$ instead.) For the purpose of this section $\mathbb{G}_m$, ${}^{L}G$, ${}^{C}G$, etc. will mean the topological groups of $\overline{\mathbb{Q}}_\ell$ or $\mathbb{C}$-valued points (though they can also be viewed as groups over $\overline{\mathbb{Q}}_\ell$ or $\mathbb{C}$); the coefficient field is suppressed if there is no danger of confusion.

Write $\widehat{T}^{\mathrm{ad}}$ for the image of $\widehat{T}$ in the adjoint group of $\widehat{G}$.

Lemma 3.1.5. *If there exists $\tilde{\rho} \in X_*(\widehat{T})$ which is Γ_F-invariant and has the same image in $X_*(\widehat{T}^{\mathrm{ad}})$ as ρ, then ${}^{C}G \cong {}^{L}G \times \mathbb{G}_m$ via $g \rtimes t \mapsto (g\tilde{\rho}(t), t)$ with the inverse map $(g, t) \mapsto g\tilde{\rho}(t)^{-1} \rtimes t$. These maps are $\widehat{G}$-equivariant: the image of $h(g \rtimes t)h^{-1}$ equals $(hg\tilde{\rho}(t)h^{-1}, t)$ for $h \in \hat{G}$.*

Proof. This is a straightforward verification. □

Let v be a finite place of F not in $S_{\mathrm{ram}}(\pi)$. We introduce a C-normalization of the unramified L-parameter for π_v (with $\mathbb{C}$-coefficient), which is natural from the viewpoint of the geometric Satake equivalence, see [Zhu 2020b, Section 1.4]:

$$\phi^C_{\pi_v} : W_{F_v} \to {}^{C}G = {}^{L}G \rtimes \mathbb{G}_m, \quad x \mapsto \phi_{\pi_v}(x)2\rho(|x|^{1/2}) \rtimes |x|^{-1}. \tag{3.1.1}$$

It is elementary to check that $\phi^C_{\pi_v}$ is well defined up to $\widehat{G}$-conjugacy. Indeed, if ϕ_{π_v} is conjugated by an element of $\widehat{G}$ then the resulting $\phi^C_{\pi_v}$ is conjugated by the same element. When $\tilde{\rho}$ as in Lemma 3.1.5 exists, the isomorphism therein gives an alternative description of $\phi^C_{\pi_v}$:

$$\phi^C_{\pi_v} : W_{F_v} \to {}^{L}G \times \mathbb{G}_m, \quad x \mapsto (\phi_{\pi_v}(x)2(\rho - \tilde{\rho})(|x|^{1/2}), |x|^{-1}). \tag{3.1.2}$$

Example 3.1.6. When G is Sp_{2n} or SO^{η}_{2n}, we take $\tilde{\rho} = \rho$. In this case $F' = F$ except for the case of SO^{η}_{2n} with $\eta \neq 1$; then take $F' = E$. For $G = \mathrm{SO}_{2n+1}$, we take $F' = F$. In this case no $\tilde{\rho}$ as in the lemma exists. For GL_N, we can take $\tilde{\rho} = (N-1, N-2, \ldots, 1, 0)$ with $F' = F$. So when $G = \mathrm{GL}_N$, (3.1.2) reads

$$\phi^C_{\pi_v}(x) = (\phi_{\pi_v}(x)|x|^{(1-N)/2}, |x|^{-1}). \tag{3.1.3}$$

For $G = \mathrm{U}_N$, we take $F' = E$. For odd N we can take $\tilde{\rho} = \rho$, but there does not exist $\tilde{\rho}$ as in Lemma 3.1.5 if N is even. (For instance, $(N-1, N-2, \ldots, 0)$ is not Γ_F-invariant.)

Example 3.1.7. For SO_{2n+1} (with $F' = F$), we have two maps

$$\begin{aligned} &\mathrm{Sp}_{2n} \times \mathbb{G}_m \to \mathrm{GSp}_{2n}, && (g, t) \mapsto gt, \\ &\mathrm{Sp}_{2n} \times \mathbb{G}_m \to {}^C\,\mathrm{SO}_{2n+1} = \mathrm{Sp}_{2n} \rtimes \mathbb{G}_m, && (g, t) \mapsto g2\rho(t)^{-1} \rtimes t^2. \end{aligned}$$

whose kernels are both generated by $(-1, -1)$. This induces an isomorphism

$${}^C\,\mathrm{SO}_{2n+1} \cong \mathrm{GSp}_{2n}\,.$$

Under this isomorphism, (3.1.1) reads

$$\phi^C_{\pi_v} : W_{F_v} \to \mathrm{GSp}_{2n}, \quad x \mapsto \phi_{\pi_v}(x)|x|^{-1/2}.$$

We return to a general discussion. Let $\tau : F \hookrightarrow \overline{\mathbb{Q}}_\ell$ be an embedding. To a Galois representation $r^C : \Gamma_F \to {}^C G(\overline{\mathbb{Q}}_\ell)$ which is de Rham at ℓ, we assign a Hodge–Tate cocharacter $\mu_{\mathrm{HT}}(r^C, \tau) : \mathbb{G}_m \to \widehat{G} \rtimes \mathbb{G}_m$ over $\mathbb{C}_\ell$, which gives rise to an element

$$\mu_{r^C,\tau} \in X_*(\iota\widehat{T} \times \mathbb{G}_m)/\Omega,$$

as in the case of L-group valued representations. Indeed, ${}^C G$ is the L-group of a $\mathbb{G}_m$-extension of G, see [Buzzard and Gee 2014] and [Zhu 2020b], and $\widehat{T} \times \mathbb{G}_m$ is a maximal torus of $\widehat{G} \rtimes \mathbb{G}_m$ whose Weyl group is naturally isomorphic to Ω, the Weyl group for $\widehat{T}$ in $\widehat{G}$. The action of $\omega \in \Omega$ on $X_*(\widehat{T} \times \mathbb{G}_m) = X_*(\widehat{T}) \oplus X_*(\mathbb{G}_m) = X_*(\widehat{T}) \oplus \mathbb{Z}$, induced by the $\widehat{G}$-conjugation on $\widehat{G} \rtimes \mathbb{G}_m$, is that $\omega(a, b) = (\omega a + b(\omega\rho - \rho), b)$, where ωa and $\omega\rho$ are computed using the natural ω-action on $X_*(\widehat{T})$. Define $\zeta^C_{\pi,\tau}$ by

$$-\zeta^C_{\pi,\tau} = (-\zeta_{\pi,\tau} - \rho, 1) \in X_*(\widehat{T} \times \mathbb{G}_m)_{\mathbb{Q}}/\Omega. \tag{3.1.4}$$

This is well defined since if $\zeta_{\pi,\tau} \in X_*(\widehat{T})_{\mathbb{Q}}$ denotes any representative in its Ω-orbit (still denoted $\zeta_{\pi,\tau}$) then $\omega(-\zeta_{\pi,\tau} - \rho, 1) = (-\omega\zeta_{\pi,\tau} - \rho, 1)$ by the preceding formula. When $\tilde{\rho}$ as in Lemma 3.1.5 exists, composition with the isomorphism ${}^C G \cong {}^L G \times \mathbb{G}_m$ gives an alternative description

$$-\zeta^C_{\pi,\tau} = (-\zeta_{\pi,\tau} - \rho + \tilde{\rho}, 1) \in X_*(\widehat{T} \times \mathbb{G}_m)_{\mathbb{Q}}/\Omega. \tag{3.1.5}$$

The reader is cautioned that even though $\widehat{T} \times \mathbb{G}_m$ serves as a maximal torus in both ${}^C G$ and ${}^L G \times \mathbb{G}_m$ via the natural inclusions, the isomorphism ${}^C G \cong {}^L G \times \mathbb{G}_m$

does not induce the identity map on $\widehat{T} \times \mathbb{G}_m$. Rather the induced map "shifts" by $\tilde{\rho}$, which explains the difference between (3.1.4) and (3.1.5). While (3.1.4) is for general ${}^C G$-valued representations, (3.1.5) is for ${}^L G \times \mathbb{G}_m$-valued representations and requires the existence of $\tilde{\rho}$.

The C-algebraic version of Buzzard and Gee's conjecture is adapted to our setting as follows.

Conjecture 3.1.8. *Suppose that π is C-algebraic. There exists a Galois representation*

$$r^C = r^C_{\ell,\iota}(\pi) : \Gamma_F \to {}^C G(\overline{\mathbb{Q}}_\ell)$$

such that:

(1) $r^C|^{\mathrm{ss}}_{W_{F_v}} \cong \iota\phi^C_{\pi_v}$ *at finite places $v \notin S_{\mathrm{ram}}(\pi) \cup S(\ell)$.*

(2) $\mu_{r^C,\iota\tau} = -\iota\zeta^C_{\pi,\tau}$ *for every embedding $\tau : F \hookrightarrow \mathbb{C}$.*

Remark 3.1.9. Condition (i) implies that the composition of r^C with the projection ${}^C G(\overline{\mathbb{Q}}_\ell) \to \mathbb{G}_m(\overline{\mathbb{Q}}_\ell)$ is ω_ℓ^{-1}, the inverse cyclotomic character, in view of (3.1.1). This convention is consistent with [Zhu 2020a] but opposite to that of [Buzzard and Gee 2014, Section 5.3, Conjecture 5.40], where the composition is ω_ℓ.

Remark 3.1.10. When $\rho \in X_*(\widehat{T})$ (not just $\rho \in X_*(\widehat{T})_{\mathbb{Q}}$), Conjectures 3.1.2 and 3.1.8 are equivalent via the isomorphism ${}^C G \cong {}^L G \times \mathbb{G}_m$ of Lemma 3.1.5 given by $\tilde{\rho} = \rho$. Indeed, L-algebraicity coincides with C-algebraicity in that case. Further, r as in the former conjecture gives rise to r^C in the latter conjecture by $r^C(\gamma) := (r(\gamma), \omega_\ell(\gamma)^{-1})$ via the isomorphism. Conversely r can be recovered from r^C by projection.

Conjectures 3.1.2 and 3.1.8 are known for general linear groups under certain hypotheses as we now recall. The case of classical groups will be eventually derived from this result.

Proposition 3.1.11. *Let F, E be as in Section 2.2 and $\star$ as in Section 2.3. Conjectures 3.1.2 and 3.1.8 are true for every discrete automorphic representation π of* $\mathrm{GL}_N(\mathbb{A}_E)$ (*in particular E serves as the field F in the conjectures*) *if the following hold*:

- π *is regular* (*and L or C-algebraic as assumed in the conjectures*).
- $\pi^\star \cong \pi \otimes (\chi \circ N_{E/F})$ *for a Hecke character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$.*

If π is regular but does not satisfy the second condition, then Conjectures 3.1.2 and 3.1.8 are true except for the assertions on Hodge–Tate cocharacters.

Proof. The last assertion will be addressed at the end of proof. Until then we assume that π satisfies both conditions. We begin with the case when π is cuspidal and C-algebraic. Let us represent $\zeta_{\pi,\tau}$ by $(a_1, \dots, a_n) - \left(\frac{1}{2}(n-1), \dots, \frac{1}{2}(n-1)\right)$ with

$(a_i)_{i=1}^n \in \mathbb{Z}^n$. By [Barnet-Lamb et al. 2014, Theorem 2.1.1] (which summarizes a theorem due to many people; the sign condition in that theorem was shown to be superfluous by [Patrikis 2015]), there exists a semisimple Galois representation $R = R_{\ell,\iota}(\pi) : \Gamma_E \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$ such that

$$R|_{W_{E_v}}^{\mathrm{ss}} \cong \iota\phi_{\pi_v}|\cdot|_v^{(1-N)/2}, \quad v \notin S_{\mathrm{ram}}(\pi) \cup S(\ell), \tag{3.1.6}$$

$$\mu_{R,\iota\tau} = (a_1, \ldots, a_n) = -\zeta_{\pi,\tau} + \left(\tfrac{1}{2}(n-1), \ldots, \tfrac{1}{2}(n-1)\right). \tag{3.1.7}$$

After choosing $\tilde{\rho}$ as in Example 3.1.6, we identify ${}^C\mathrm{GL}_N \cong \mathrm{GL}_N \times \mathbb{G}_m$ as in Lemma 3.1.5. Then we define an $\mathrm{GL}_N \times \mathbb{G}_m$-valued representation

$$r^C : \Gamma_E \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell) \times \mathbb{G}_m(\overline{\mathbb{Q}}_\ell), \quad \gamma \mapsto (R(\gamma), \omega_\ell^{-1}(\gamma)).$$

Comparing (3.1.6) with (3.1.3), we verify part (i) of Conjecture 3.1.8. The cocharacter $\zeta_{\pi,\tau}^C$ in part (ii) of the conjecture becomes a $\mathrm{GL}_N \times \mathbb{G}_m$-valued cocharacter in view of (3.1.5):

$$t \mapsto ((-\zeta_{\pi,\tau} - \rho + \tilde{\rho})(t), t) = \left(\left(-\zeta_{\pi,\tau} + \left(\tfrac{1}{2}(n-1), \ldots, \tfrac{1}{2}(n-1)\right)\right)(t), t\right).$$

This coincides with $\mu_{r^C,\iota\tau}$ in view of (3.1.7) and the fact that the Hodge–Tate cocharacter of ω_ℓ^{-1} is the tautological map $t \mapsto t$ on $\mathbb{G}_m$.

We turn to the case of cuspidal L-algebraic π. Then $\pi' := \pi|\det|^{(N-1)/2}$ is cuspidal, regular, and C-algebraic. So there exists $R(\pi')$ such that (3.1.6) and (3.1.7) hold with π' in place of π. We take $r = r_{\ell,\iota}(\pi) := R(\pi')$. Then $r|_{W_{E_v}}^{\mathrm{ss}} \cong \iota\phi_{\pi'_v}|\cdot|_v^{(1-N)/2} \cong \iota\phi_{\pi_v}$ at $v \notin S_{\mathrm{ram}}(\pi) \cup S(\ell)$, so (i) of Conjecture 3.1.2 is satisfied. Similarly (ii) follows from (3.1.7) for $r = R(\pi')$.

From now, let π be a noncuspidal discrete automorphic representation. By [Mœglin and Waldspurger 1989]

$$\pi = \boxplus_{j=1}^r \pi_0|\det|^{(r+1-2j)/2}$$

as an isobaric sum, for some $N_0, r \in \mathbb{Z}_{\geq 1}$ and π_0 a cuspidal automorphic representation of $\mathrm{GL}_{N_0}(\mathbb{A}_E)$, where $N = N_0 r$. If π is regular L-algebraic then $\pi_j := \pi_0|\det|^{(r+1-2j)/2}$ is regular, L-algebraic, and unramified outside $S_{\mathrm{ram}}(\pi)$. By the preceding argument, we have $r_{\ell,\iota}(\pi_j)$ corresponding to π_j satisfying Conjecture 3.1.2. Then $r := \bigoplus_j r_{\ell,\iota}(\pi_j)$ is the Galois representation corresponding to π predicted by the conjecture. We leave to the reader to verify Conjecture 3.1.8 when π is regular C-algebraic and noncuspidal as no new idea is needed.

Finally, if the second condition on π is not assumed, we can run the same argument as above except that we apply the theorems of Harris, Lan, Taylor and Thorne [2016] and Scholze [2015] instead of [Barnet-Lamb et al. 2014, Theorem 2.1.1] to obtain Galois representations. The only difference in the outcome is that the Hodge–Tate weights have not been identified for the Galois representations in [Harris et al. 2016; Scholze 2015], so we are unable to verify (ii) in Conjectures 3.1.2 and 3.1.8. ☐

3.2. *Existence of Galois representations for classical groups.* From here until the end of the paper, we use the same notation as in Section 2.5, including G^*, N, and $\tilde{\xi} : {}^L G \hookrightarrow {}^L \tilde{G}^0(N)$. In Case U, take $\tilde{\xi}$ to be the standard base change morphism $\tilde{\xi}_+$ (rather than $\tilde{\xi}_-$). In Case S, we recall that $\tilde{\xi}|_{\widehat{G}^*}$ is the standard embedding of $\widehat{G}^*$ into GL_N.

Let ℓ be a prime and choose an isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. Let S be a finite set of places of F which contains all places above ℓ and ∞ such that G is unramified at places outside S.

Definition 3.2.1. A discrete automorphic representation π of $G(\mathbb{A}_F)$ is said to be *std-regular* if $\tilde{\xi}(\zeta_{\pi_\infty}) \in \mathcal{C}_\infty(N)$ is regular.

Lemma 3.2.2. *If π is std-regular then it is regular. The two conditions are equivalent unless G is an inner form of SO^η_{2n}.*

Proof. As we explicated the map $X_*(\widehat{T}) \to X_*(\widehat{T}_N)$ induced by $\tilde{\xi}$ in Section 2.2, the lemma follows from the definition. ☐

Example 3.2.3. When $G = \mathrm{SO}^\eta_{2n}$, a Weyl group orbit in $X_*(\widehat{T}) = \mathbb{Z}^n$ is uniquely represented by (a_i) such that $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n|$. If ζ_{π_∞} corresponds to such a tuple (a_i) then π is regular if strict inequalities hold everywhere, and std-regular if furthermore $a_n \neq 0$.

Let $r : \Gamma_E \to \mathrm{GL}_m(\overline{\mathbb{Q}}_\ell)$ be a Galois representation. Define another representation $r^\perp$ by

$$r^\perp(\gamma) := {}^t r(c\gamma c^{-1})^{-1},$$

which is isomorphic to the dual representation $r^\vee$ in Case S. Let $\chi : \Gamma_E \to \overline{\mathbb{Q}}_\ell^\times$ be a Galois character such that $\chi(c\gamma c^{-1}) = \chi(\gamma)$ for all $\gamma \in \Gamma_E$ (which is automatic in Case S). From now assume that r is irreducible. Provided that $r^\perp \cong r\chi$, we recall how to define a sign

$$\mathrm{sgn}(r, \chi) \in \{\pm 1\}$$

following [Bellaïche and Chenevier 2011, Section 1.1]. In Case S, we obtain a nonzero Γ_F-equivariant pairing $r \otimes r \to \chi^{-1}$ up to a nonzero scalar. According to whether the pairing is orthogonal or symplectic (it cannot be both since r is irreducible), we assign 1 or -1 as the value of $\mathrm{sgn}(r, \chi)$. When χ is trivial, we just write $\mathrm{sgn}(r)$ and refer to it as the sign of r. Of course if m is odd then always $\mathrm{sgn}(r, \chi) = 1$. In Case U, by assumption there exists $h \in \mathrm{GL}_m(\overline{\mathbb{Q}}_\ell)$, unique up to nonzero scalars, such that $r^\perp = hrh^{-1}\chi$. Then it is elementary to check that ${}^t h = \mathrm{sgn}(r, \chi) h$ for $\mathrm{sgn}(r, \chi) \in \{\pm 1\}$, which does not depend on the choice of h.

Henceforth we restrict E and F as follows in order to access Proposition 3.1.11:

(Case S) $E = F$ is a totally real field.

(Case U) F is a totally real field, and E is a CM quadratic extension of F.

Consider the following hypotheses — see the paragraph above Theorem 1.2.2. The two versions of (H2) are equivalent to each other since $\zeta_{\Pi_\infty} = \tilde{\xi}(\zeta_{\pi_\infty})$.

(H2) π is std-regular.

(H3) In Corollary 2.5.3, if Π is written as an isobaric sum $\Pi = \boxplus_{i=1}^{r} \Pi_i$ then Π_i is (conjugate) self-dual for every i, i.e., $\Pi_i^\star = \Pi_i$.

Proposition 3.2.4. *Let E and F be as above. Assume (H1). Let π be a discrete automorphic representation of $G(\mathbb{A}_F)$ which is unramified outside S, C-algebraic, and satisfying (H2) and (H3). Then there exists a continuous semisimple Galois representation*

$$R = R_{\ell,\iota}(\pi) : \Gamma_{E,S} \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$$

with the following property. If $G^* = \mathrm{Sp}_{2n}$ or SO_{2n}^{η} (Case S), we have:

(i) $R|_{W_{F_v}}^{\mathrm{ss}} \cong \iota\tilde{\xi}\phi_{\pi_v}$ *for every place v of F not above S.*

(ii) $\mu_{R,\iota\sigma} = -\iota\xi(\zeta_{\pi,\sigma})$ *for embeddings $\sigma : F \hookrightarrow \mathbb{C}$.*

(iii) $R^\vee \cong R$. *When $G^* = \mathrm{SO}_{2n}^{\eta}$, every self-dual irreducible constituent of R has sign* 1.

(iv) $\det R = \mathbf{1}$ *if $G^* = \mathrm{Sp}_{2n}$ and* $\det R = \eta$ *if $G^* = \mathrm{SO}_{2n}^{\eta}$.*

If $G^* = \mathrm{SO}_{2n+1}$ (Case S) then:

(i') $R|_{W_{F_v}}^{\mathrm{ss}} \cong \iota(\tilde{\xi}\phi_{\pi_v}|\cdot|^{(1-N)/2})$ *for every place v of F not above S.*

(ii') $\mu_{R,\iota\sigma} = -\iota\tilde{\xi}(\zeta_{\pi,\sigma}) + \big(\frac{1}{2}(N-1), \dots, \frac{1}{2}(N-1)\big)$ *for embeddings $\sigma : F \hookrightarrow \mathbb{C}$.*

(iii') $R^\perp \cong R \otimes \omega_\ell^{N-1}$. *For every irreducible constituent r of R such that $r^\perp \cong r \otimes \omega_\ell^{N-1}$, we have* $\mathrm{sgn}(r, \omega_\ell^{N-1}) = -1$.

If $G^* = \mathrm{U}_N$ (Case U) then with $\tilde{\xi}_u, \tilde{\xi}_\tau$ as in Section 2.1:

(i'') $R|_{W_{E_u}}^{\mathrm{ss}} \cong \iota(\tilde{\xi}_u\phi_{\pi_v}|\cdot|_u^{(1-N)/2})$ *for every place u of E not above S, where v is the place of F restricted from u.*

(ii'') $\mu_{R,\iota\tau} = -\iota\tilde{\xi}_\tau(\zeta_{\pi,\tau|_F}) + (\frac{N-1}{2}, \dots, \frac{N-1}{2})$ *for embeddings $\tau : E \hookrightarrow \mathbb{C}$.*

(iii'') $R^\perp \cong R \otimes \omega_\ell^{N-1}$. *For every irreducible constituent r of R such that $r^\perp \cong r \otimes \omega_\ell^{N-1}$, we have* $\mathrm{sgn}(r, \omega_\ell^{N-1}) = 1$.

If (H1) and (H2) are assumed but not (H3), then the above is true except (ii), (ii'), and (ii'').

Remark 3.2.5. In fact the proof below shows that every irreducible constituent of R in (iii) (resp. (iii') and (iii")) is self-dual (resp. self-dual up to ω_ℓ^{N-1}) thanks to (H3).

Remark 3.2.6. We could have stated the U_N-case uniformly with the SO_{2n+1}-case if we rewrite R as a Galois representation $\Gamma_{F,S} \to {}^L G(N)(\overline{\mathbb{Q}}_\ell)$ via a variant of Shapiro's lemma. Then (i") and (ii") can be merged into (i') and (ii'). E.g., both (i') and (i") assert $R|^{\mathrm{ss}}_{W_{F_v}} \cong \iota\tilde{\xi}\phi_{\pi_v}|\cdot|_v^{(1-N)/2}$ in this formulation. However the current formulation for unitary groups is convenient in Section 3.4.

Proof. Let $\Pi = \boxplus_{i=1}^r \Pi_i$ be the automorphic representation of $G(N,\mathbb{A}_F) = \mathrm{GL}_N(\mathbb{A}_E)$ which is a functorial lift of π as in Corollary 2.5.3. We are going to apply Proposition 3.1.11 to each Π_i. The proof will be presented only when (H1), (H2), and (H3) are assumed. If (H3) is dropped then we lose track of Hodge–Tate cocharacters according to Proposition 3.1.11 but the argument is identical other than that. This explains the last assertion of Proposition 3.2.4.

According to (H3), each Π_i is a cuspidal automorphic representation of $\mathrm{GL}_{m_i}(\mathbb{A}_E)$ such that $\Pi_i^\star \cong \Pi_i$ and $\sum_i m_i = N$. Since $(\zeta_{\Pi_\infty}, c^S(\Pi)) = \tilde{\xi}(\zeta_{\pi_\infty}, c^S(\pi))$, the std-regularity of π implies that Π is regular. Moreover the description of ρ and $\tilde{\xi}$ in Sections 2.1 and 2.2 tells us that:

- If $G^* = \mathrm{Sp}_{2n}$ then π is also L-algebraic; Π is both L and C-algebraic.
- If $G^* = \mathrm{SO}_{2n}^\eta$ then π is also L-algebraic; Π is L-algebraic but not C-algebraic.
- If $G^* = \mathrm{SO}_{2n+1}$ then Π is C-algebraic but not L-algebraic.
- If $G^* = \mathrm{U}_N$ then Π is C-algebraic; it is not L-algebraic if N is even.

Suppose $G^* = \mathrm{SO}_{2n+1}$. Since Π is regular C-algebraic, we see that $\Pi|\det|^{(1-N)/2}$ is regular L-algebraic, so $\Pi_i' := \Pi_i|\det|^{(1-N)/2}$ is regular L-algebraic as well. Moreover $(\Pi_i')^\star \cong \Pi_i'|\det|^{N-1}$, so Proposition 3.1.11 yields a Galois representation $r_i' := r_{\ell,\iota}(\Pi_i')$. Then $R := \bigoplus_{i=1}^r r_i'$ satisfies (i') and (ii') in light of properties (i) and (ii) of Conjecture 3.1.2 for r_i'. Indeed, (i') is checked as follows:

$$R|^{\mathrm{ss}}_{W_{F_v}} \cong \iota\phi_{\Pi_v'} \cong \iota\phi_{\Pi_v}|\cdot|^{(1-N)/2} \cong \iota\tilde{\xi}\phi_{\pi_v}|\cdot|^{(1-N)/2}, \quad v \notin S.$$

As for (ii'), since $\mu_{r_i',\iota\sigma} = \iota\zeta_{\Pi_i',\sigma}$ for every i, we have

$$\mu_{R,\iota\sigma} = -\iota\zeta_{\Pi|\det|^{(1-N)/2},\sigma} = -\iota\tilde{\xi}(\zeta_{\pi,\sigma}) + \left(\tfrac{1}{2}(N-1), \ldots, \tfrac{1}{2}(N-1)\right).$$

Moreover, we have $\phi_{\Pi_v}^\vee \cong \phi_{\Pi_v}$ since $\Pi^\vee \cong \Pi$, so the displayed formula implies that $R^\perp \cong R \otimes \omega_\ell^{N-1}$. The rest of (iii') is verified by [Bellaïche and Chenevier 2011, Corollary 1.3] (their n is our N, which is even; their η_λ is trivial). This finishes the proof when G^* is SO_{2n+1}.

The case $G^* = \mathrm{U}_N$ can be treated as in the SO_{2n+1}-case, by defining Π'_i, r'_i, and R in the same way. There is only a minor difference in showing (i”):

$$R|^{\mathrm{ss}}_{W_{E_u}} \cong \iota\phi_{\Pi'_u} \cong \iota\phi_{\Pi_u}|\cdot|^{(1-N)/2} \cong \iota\tilde{\xi}_u\phi_{\pi_v}|\cdot|^{(1-N)/2}, \quad v \notin S.$$

The justification of (ii’) also goes through for (ii”) with a similar change. The proof of (iii”) is identical to that of (iii’) except that we use the conjugate duality and invoke [Bellaïche and Chenevier 2011, Theorem 1.2] rather than Corollary 1.3 therein.

Now consider $G^* = \mathrm{Sp}_{2n}$ or SO^η_{2n}. Then Π is regular L-algebraic so each Π_i is regular L-algebraic, cuspidal, and $\Pi_i^\vee \cong \Pi_i$. By Proposition 3.1.11, there is a corresponding Galois representation $r_i := r_{\ell,\iota}(\Pi_i)$. Taking $R := \bigoplus_{i=1}^r r_i$, we deduce (i) and (ii) for R from the properties of r_i as in the preceding paragraph. It follows from (i) that R is self-dual. When $G^* = \mathrm{SO}^\eta_{2n}$, [Bellaïche and Chenevier 2011, Corollary 1.3] (their n is our N, which is even; their η_λ equals our ω_ℓ^{1-N} in the case at hand, so $\eta_\lambda(c) = -1$) tells us that the irreducible self-dual constituents of R are orthogonal, so the proof of (iii) is complete. Finally, to show (iv), it suffices to check that $\det R|_{W_{F_v}}$ equals $\mathbf{1}$ if $G^* = \mathrm{Sp}_{2n}$ and η_v if $G^* = \mathrm{SO}^\eta_{2n}$ for $v \notin S$. This follows from part (i). Indeed, this is obvious if $G^* = \mathrm{Sp}_{2n}$ since the image of $\tilde{\xi}$ is contained in SO_{2n+1}; if $G^* = \mathrm{SO}^\eta_{2n}$, it is enough to note that the composite map $\det \circ \tilde{\xi} : {}^L\mathrm{SO}^\eta_{2n} \to \mathrm{GL}_{2n} \to \mathbb{G}_m$ is given by the projection ${}^L\mathrm{SO}^\eta_{2n} \twoheadrightarrow \mathrm{Gal}(F_\eta/F)$ followed by η. □

When $\phi_1, \phi_2 : W_{F_v} \to {}^C G(\overline{\mathbb{Q}}_\ell)$ are two parameters, we write $\phi_1 \overset{\circ}{\cong} \phi_2$ to mean

- $\phi_1 \cong \phi_2$ if $G^* \ncong \mathrm{SO}^\eta_{2n}$, and
- $\phi_1 \cong \phi_2$ or $\hat{\theta}^\circ(\phi_1) \cong \phi_2$ if $G^* \cong \mathrm{SO}^\eta_{2n}$.

Similarly if $\mu_1, \mu_2 \in X_*(\widehat{T})_{\mathbb{Q}}/\Omega$ then $\mu_1 \overset{\circ}{=} \mu_2$ means $\mu_1 = \mu_2$ if $G^* \ncong \mathrm{SO}^\eta_{2n}$, and $\mu_1 = \mu_2$ or $\hat{\theta}^\circ(\mu_1) = \mu_2$ if $G^* \ncong \mathrm{SO}^\eta_{2n}$.

Theorem 3.2.7. *Let E and F be as above and assume (H1). Let π be as in Proposition 3.2.4 satisfying (H2) and (H3). Then Conjecture 3.1.8 holds true if $G^* \ncong \mathrm{SO}^\eta_{2n}$, and it holds up to outer automorphism if $G^* \cong \mathrm{SO}^\eta_{2n}$. More precisely, there exists a continuous semisimple Galois representation*

$$r^C = r^C_{\ell,\iota}(\pi) : \Gamma_{F,S} \to {}^C G(\overline{\mathbb{Q}}_\ell)$$

such that:

(1) $r^C|^{\mathrm{ss}}_{W_{F_v}} \overset{\circ}{\cong} \iota\phi^C_{\pi_v}$ *for every place v of F not above S.*

(2) $\mu_{r^C,\iota\sigma} \overset{\circ}{=} -\iota\zeta^C_{\pi,\sigma}$ *for every $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_\ell$.*

If we drop (H3), then the theorem still holds true except for part (ii).

The proof is the same whether we assume (H3) or not. Without (H3), we lose property (ii) of the theorem only because we do not know (ii), (ii'), and (ii'') in Proposition 3.2.4. With this understanding, we will present the proof in Section 3.3 and Section 3.4 below in the case that all of (H1), (H2), and (H3) are assumed.

Remark 3.2.8. Buzzard and Gee also makes a prediction on the image of complex conjugation at each real place but we do not see how to prove it completely beyond some partial results. For instance, in the proof of Proposition 3.2.4 in Case S, every r'_i is totally odd by [Taylor 2012; Taïbi 2016; Caraiani and Le Hung 2016], but this alone does not determine the image of complex conjugation (up to conjugacy) under R. Thus the information is insufficient to pin down the image of complex conjugation under r^C in Theorem 3.2.7. The image is sometimes identified under additional hypotheses; see [Kret and Shin 2020, Theorem 6.5; 2023, Theorem 2.4].

3.3. ***Proof of Theorem 3.2.7: Case S.*** Write $R = R_{\ell,\iota}(\pi) : \Gamma_F \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$ for the Galois representation as in Proposition 3.2.4. (We are in the $E = F$ case.) We will divide into three cases according to G^*. When G^* is either Sp_{2n} or SO^η_{2n}, we will prove Conjecture 3.1.2 as this is equivalent to Theorem 3.2.7 but notationally simpler; see Remark 3.1.10.

If $G^* = \mathrm{Sp}_{2n}$ then $R^\vee \cong R$ and every self-dual irreducible constituent is orthogonal by (iii) of Proposition 3.2.4. Hence, possibly after a GL_{2n+1}-conjugation, R factors as

$$\Gamma_{E,S} \longrightarrow \mathrm{O}_{2n+1}(\overline{\mathbb{Q}}_\ell) \xrightarrow{\tilde{\xi}} \mathrm{GL}_{2n+1}(\overline{\mathbb{Q}}_\ell).$$

Take $r^C_{\ell,\iota}(\pi) : \Gamma_{E,S} \to \mathrm{O}_{2n+1}(\overline{\mathbb{Q}}_\ell)$ to be the first map. By Proposition 3.2.4(iv), the image of $r^C_{\ell,\iota}(\pi)$ is contained in $\mathrm{SO}_{2n+1}(\overline{\mathbb{Q}}_\ell)$. Since the natural map $\widehat{T}/\Omega \to \widehat{T}_{2n+1}/\Omega_{2n+1}$ is injective, one deduces (i) and (ii) of Conjecture 3.1.2 from (i) and (ii) of Proposition 3.2.4.

Next consider $G^* = \mathrm{SO}^\eta_{2n}$. As in the Sp_{2n}-case, again from Proposition 3.2.4(iii), we obtain

$$r^C_{\ell,\iota}(\pi) : \Gamma_{F,S} \to \mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$$

such that $\iota(\eta) \circ r^C_{\ell,\iota}(\pi) \cong R_{\ell,\iota}(\pi)$. The difference is that $\widehat{T}/\Omega \to \widehat{T}_{2n}/\Omega_{2n}$ is not a bijection but induces a bijection on the set of $\hat{\theta}^\circ$-orbits on $\widehat{T}/\Omega \to \widehat{T}_{2n}$ onto $\widehat{T}_{2n}/\Omega_{2n}$. With this observation, (i) and (ii) of Conjecture 3.1.2 are implied by (i) and (ii) of Proposition 3.2.4.

In the remaining case $G^* = \mathrm{SO}_{2n+1}$, we identify ${}^C\mathrm{SO}_{2n+1} = \mathrm{GSp}_{2n}$ as in Example 3.1.7. Let $R = R_{\ell,\iota}(\pi) : \Gamma_F \to \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_\ell)$ be the Galois representation corresponding to π by Proposition 3.2.4. By (iii') of the proposition, there is a symplectic pairing $(R \otimes \omega_\ell^{n-1}) \otimes (R \otimes \omega_\ell^{n-1}) \to \omega_\ell^{-1}$. After conjugation, $R \otimes \omega_\ell^{n-1}$ factors through the standard embedding $\tilde{\eta}^C : \mathrm{GSp}_{2n} \to \mathrm{GL}_{2n}$. Denote the resulting representation by

$$r^C = r^C_{\ell,\iota}(\pi) : \Gamma_{F,S} \to \mathrm{GSp}_{2n}(\overline{\mathbb{Q}}_\ell).$$

Write $\lambda : \mathrm{GSp}_{2n} \to \mathbb{G}_m$ for the similitude character. Since the symplectic pairing is valued in ω_ℓ^{-1}, we have

$$\lambda r^C = \omega_\ell^{-1}.$$

By construction, the properties of R in Proposition 3.2.4 tell us that

$$\tilde{\eta}^C(r^C|^{\mathrm{ss}}_{W_{F_v}}) \cong \iota(\tilde{\eta}\phi_{\pi_v} \cdot |\cdot|^{-1/2}) = \tilde{\eta}^C(\iota\phi_{\pi_v} \cdot |\cdot|^{-1/2}),$$
$$\tilde{\eta}^C(\mu_{r^C,\iota\sigma}) = \mu_{\tilde{\eta}^C r^C,\iota\sigma} = -\iota\tilde{\eta}(\zeta_{\pi,\sigma}) + \left(\tfrac{1}{2}, \dots, \tfrac{1}{2}\right) = \tilde{\eta}^C\left(-\iota\zeta_{\pi,\sigma} + \left(\tfrac{1}{2}, \dots, \tfrac{1}{2}\right)\right).$$

On the other hand, we have

$$\lambda(\iota(\tilde{\eta}\phi_{\pi_v} \cdot |\cdot|^{-1/2})) = |\cdot|^{-1} = \lambda r^C|_{W_{F_v}} = \lambda(r^C|^{\mathrm{ss}}_{W_{F_v}}),$$
$$\lambda\left(-\iota\zeta_{\pi,\sigma} + \left(\tfrac{1}{2}, \dots, \tfrac{1}{2}\right)\right) = 1 = \mu_{\omega_\ell^{-1},\iota\sigma} = \mu_{\lambda r^C,\iota\sigma} = \lambda(\mu_{r^C,\iota\sigma}).$$

To deduce the theorem, we need to show that the above relations hold without taking $\tilde{\eta}^C$ and λ at both ends. This is implied by the following facts. Firstly, if semisimple elements $g_1, g_2 \in \mathrm{GSp}_{2n}(\overline{\mathbb{Q}}_\ell)$ are such that $\tilde{\eta}^C(g_1)$, $\tilde{\eta}^C(g_2)$ are conjugate and $\lambda(g_1) = \lambda(g_2)$ then g_1, g_2 are conjugate in $\mathrm{GSp}_{2n}(\overline{\mathbb{Q}}_\ell)$; see [Kret and Shin 2023, Lemmas 1.1, 1.3]. Secondly, the analogous injectivity is also true on the level of conjugacy classes of cocharacters via the isomorphism $X_*(T_{\mathrm{GSp}}) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell^\times \cong T_{\mathrm{GSp}}(\overline{\mathbb{Q}}_\ell)$, which is equivariant for the Weyl group action, where T_{GSp} is a maximal torus of GSp_{2n} over $\overline{\mathbb{Q}}_\ell$. The proof in the SO_{2n+1}-case is complete.

3.4. *Proof of Theorem 3.2.7: Case U.* Recall that E is a CM quadratic extension of a totally real field F in this case. Throughout this section we set

$$\tilde{\rho}(t) := \mathrm{diag}(t^{N-1}, t^{N-2}, \dots, t, 1) \in \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell) \cong \hat{\mathrm{U}}_N(\overline{\mathbb{Q}}_\ell), \quad t \in \mathbb{G}_m,$$

where the isomorphism is fixed as in Section 2.1. (The same $\tilde{\rho}$ appeared in Example 3.1.6 for odd unitary groups. Here $\tilde{\rho}$ is also considered for even unitary groups as Lemma 3.1.5 is irrelevant here.) A key point in the proof is to extend a GL_N-valued representation of $\Gamma_{E,S}$ to a ${}^C\mathrm{U}$-valued representation of $\Gamma_{F,S}$. We begin with two lemmas to help address this problem. Similar problems were considered in related settings; see [Clozel et al. 2008, Section 2.1; Bellaïche and Chenevier 2009, Appendix A.11; Barnet-Lamb et al. 2014, Section 1] (see [Buzzard and Gee 2014, Section 8.3] for a comparison with C-groups), and [Kret and Shin 2020, Appendix A] for instance.

Lemma 3.4.1. *Let $R : \Gamma_{E,S} \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$ be a Galois representation. If there exists $h \in \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$ such that*

$$ {}^t h = h \quad \text{and} \quad R^\perp(\gamma) = hR(\gamma)h^{-1} \cdot \omega_\ell(\gamma)^{N-1}, \quad \gamma \in \Gamma_{E,S}, \tag{3.4.1}$$

then there exists a Galois representation

$$\tilde{R} : \Gamma_{F,S} \to {}^C\,\mathrm{U}_N(\overline{\mathbb{Q}}_\ell) = \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell) \rtimes (\mathbb{G}_m \times \{1, c\})$$

uniquely determined by:

- $\tilde{R}(\gamma) = R(\gamma)\tilde{\rho}(\omega_\ell(\gamma)) \rtimes (\omega_\ell^{-1}(\gamma), 1)$ *for all* $\gamma \in \Gamma_{E,S}$.
- $\tilde{R}(c) = h^{-1} J_N \rtimes (-1, c)$.

Proof. The uniqueness is clear. The main point is to check that the two conditions on $\tilde{R}$ define a group homomorphism. This amounts to checking that $\tilde{R}(c)^2 = 1$ and $\tilde{R}(c)\tilde{R}(\gamma)\tilde{R}(c)^{-1} = \tilde{R}(c\gamma c^{-1})$ for $\gamma \in \Gamma_{E,S}$. Set $h_0 := h^{-1}J_N = h^{-1}J_N^{-1}$ and let $\tilde{\rho}$ be as in Example 3.1.6. We compute

$$\begin{aligned}\tilde{R}(c)^2 &= (h_0 \rtimes (-1, c))(h_0 \rtimes (-1, c)) = (h_0 \rtimes (-1, 1))(J_N^{*}\,{}^t h_0^{-1} J_N^{*,-1} \rtimes (-1, 1))\\ &= h_0 \tilde{\rho}(-1) J_N^{*}\,{}^t h_0^{-1} J_N^{*,-1} \tilde{\rho}(-1)^{-1} = h_0 J_N\,{}^t h_0^{-1} J_N^{-1} = h^{-1}\,{}^t h = 1.\end{aligned}$$

$$\begin{aligned}&\tilde{R}(c)\tilde{R}(\gamma)\tilde{R}(c)^{-1}\\ &\quad= (h_0 \rtimes (-1, c))(R(\gamma)\tilde{\rho}(\omega_\ell(\gamma)) \rtimes (\omega_\ell^{-1}(\gamma), 1))(h_0 \rtimes (-1, c))^{-1}\\ &\quad= (h_0 \rtimes (-1, 1))(J_N^{*}\,{}^t R(\gamma)^{-1} \tilde{\rho}(\omega_\ell(\gamma))^{-1} J_N^{*,-1}\\ &\qquad\qquad \rtimes (\omega_\ell^{-1}(\gamma), 1))(h_0 \rtimes (-1, 1))^{-1}\\ &\quad= h_0 J_N ({}^t R(\gamma)^{-1} \tilde{\rho}(\omega_\ell(\gamma))^{-1} J_N^{-1} \rtimes (\omega_\ell^{-1}(\gamma), 1)) h_0^{-1}\\ &\quad= h^{-1}\,{}^t R(\gamma)^{-1} \tilde{\rho}(\omega_\ell(\gamma))^{-1} J_N^{-1} \tilde{\rho}(\omega_\ell(\gamma))^{-1} h_0^{-1} \tilde{\rho}(\omega_\ell(\gamma)) \rtimes (\omega_\ell^{-1}(\gamma), 1).\end{aligned}$$

By an explicit computation with $\tilde{\rho}$ and J_N, we verify that

$$J_N^{-1} \tilde{\rho}(\omega_\ell(\gamma))^{-1} = \tilde{\rho}(\omega_\ell(\gamma)) J_N^{-1} \omega_\ell(\gamma)^{1-N}.$$

Substituting in the above formula and using $h = J_N^{-1} h_0^{-1}$, we obtain

$$\tilde{R}(c)\tilde{R}(\gamma)\tilde{R}(c)^{-1} = h^{-1} \cdot {}^t R(\gamma)^{-1} h \tilde{\rho}(\omega(\gamma)) \cdot \omega_\ell(\gamma)^{1-N} \rtimes (\omega_\ell(\gamma)^{-1}, 1).$$

On the other hand, we see from (3.4.1) that

$$R(c\gamma c^{-1}) = {}^t R^\perp(\gamma)^{-1} = h^{-1}\,{}^t R(\gamma)^{-1} h \cdot \omega_\ell(\gamma)^{1-N}$$

so $\tilde{R}(c\gamma c^{-1}) = h^{-1}\,{}^t R(\gamma)^{-1} h \cdot \omega_\ell(\gamma)^{1-N} \tilde{\rho}(\omega_\ell(\gamma)) \rtimes (\omega_\ell^{-1}(\gamma), 1)$. We conclude that $\tilde{R}(c)\tilde{R}(\gamma)\tilde{R}(c)^{-1} = \tilde{R}(c\gamma c^{-1})$, recalling that $\omega_\ell(\gamma)$ lies in the center of $\mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$. □

Lemma 3.4.2. *Let $R : \Gamma_{E,S} \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$ be a semisimple Galois representation such that*:

- $R^\star \cong R \otimes \omega_\ell^{N-1}$.
- *Every irreducible subrepresentation $R_0 \subset R$ such that $R_0^\star \cong R_0 \otimes \omega_\ell^{N-1}$ has* $\mathrm{sgn}(R_0, \omega_\ell^{N-1}) = 1$.

Then there exists a Galois representation

$$\tilde{R} : \Gamma_{F,S} \to {}^C\,\mathrm{U}_N(\overline{\mathbb{Q}}_\ell) = \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell) \rtimes (\mathbb{G}_m \times \{1, c\})$$

such that:

- $\tilde{R}(\gamma) = R(\gamma)\tilde{\rho}(\omega_\ell(\gamma)) \rtimes (\omega_\ell^{-1}(\gamma), 1)$ *for all* $\gamma \in \Gamma_{E,S}$.
- $\tilde{R}(c) = h^{-1} J_N \rtimes (-1, c)$ *for a symmetric matrix* $h \in \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$.

Proof. Since $R^\star \cong R \otimes \omega_\ell^{N-1}$, we can decompose R into irreducibles

$$R \cong \left(\bigoplus_{i=1}^{r} R_i\right) \oplus \left(\bigoplus_{j=1}^{s} (R_j \oplus (R_j^\perp \otimes \omega_\ell^{1-N}))\right)$$

such that $R_i^\star \cong R_i \otimes \omega_\ell^{N-1}$ and $R_j^\star \not\cong R_j \otimes \omega_\ell^{N-1}$ for every i, j. (Recall that $R_j^\star \cong R_j^\perp$.) Write $d_i := \dim R_i$ and $d_j := \dim R_j$. For each i, since $\mathrm{sgn}(R_i, \omega_\ell^{N-1}) = 1$, there exists $h_i \in \mathrm{GL}_{d_i}(\overline{\mathbb{Q}}_\ell)$ satisfying (3.4.1) for h_i and R_i in place of h and R. For $1 \le j \le s$, take

$$h_j := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \in \mathrm{GL}_{2d_j}(\overline{\mathbb{Q}}_\ell),$$

where 0 and I stand for the zero and identity $d_j \times d_j$ matrices. Then it satisfies (3.4.1) for h_j and $R_j^\perp \otimes \omega_\ell^{1-N}$ in place of h and R by construction. Hence if we form $h \in \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$ as a block diagonal matrix according to the decomposition of R by putting together h_i and h_j, then (3.4.1) holds true for h and R. By Lemma 3.4.1 we obtain the desired $\tilde{R}$. □

Now we put ourselves in the setting of Theorem 3.2.7 for $G^* = \mathrm{U}_N$ and let $R : \Gamma_{E,S} \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$ be the representation coming from Proposition 3.2.4. Since R satisfies the condition of Lemma 3.4.2, we obtain

$$r^C : \Gamma_{F,S} \to {}^C\,\mathrm{U}_N(\overline{\mathbb{Q}}_\ell) \overset{\tilde{\xi}_0}{=} \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell) \rtimes (\{1, c\} \times \mathbb{G}_m)$$

as in the lemma. (We renamed $\tilde{R}$ as r^C.) By construction the following composition is equal to the representation (R, ω_ℓ^{-1}):

$$\Gamma_{E,S} \xrightarrow{r^C} \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell) \rtimes \mathbb{G}_m \xrightarrow{\varsigma} \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell) \times \mathbb{G}_m,$$

where $\varsigma : g \rtimes t \mapsto g\tilde{\rho}(t)$ is the isomorphism from Lemma 3.1.5.

Our goal is to verify (i) and (ii) of Theorem 3.2.7 for r^C. Since the codomain of r^C is identified with $\mathrm{GL}_N(\overline{\mathbb{Q}}_\ell) \rtimes (\{1, c\} \times \mathbb{G}_m)$ via $\tilde{\xi}_0$ above, we want to do the same with $\phi^C_{\pi_v} : W_{F_v} \to {}^C U_{F_v}$ via ${}^C U_{F_v} \cong \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell) \rtimes (\{1, c\} \times \mathbb{G}_m)$ given by $\tilde{\xi}_u : {}^L U_{F_v} \cong \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell) \rtimes \{1, c\}$ (and the identity map on the $\mathbb{G}_m$-factor of the C-group), which is consistent with $\tilde{\xi}_0$. For each $\sigma : F \hookrightarrow \mathbb{C}$, similarly $\zeta_{\pi,\sigma} \in X_*(\hat{T}_\sigma)_{\mathbb{Q}}$ is viewed as an element of $X_*(\mathbb{G}_m^N)_{\mathbb{Q}}$ via $\tilde{\xi}_{\tau_{0,\sigma}}$; see Case U of Section 2.1 for the discussions on $\tilde{\xi}_0$, $\tilde{\xi}_u$, and $\tilde{\xi}_{\tau_{0,\sigma}}$. Therefore (i) and (ii) are equivalent to the following assertions; see Section 2.1 for $\tau_{0,v}$ and $\tau_{0,\sigma}$:

(a) $\varsigma r^C|^{\mathrm{ss}}_{W_{F_v}} \cong \iota\tilde{\xi}_u(\phi^C_{\pi_v})$, for each finite place v of F not contained in S, and the place u of E induced by $\tau_{0,v} : E \hookrightarrow \overline{F}_v$.

(b) $\mu_{\varsigma r^C, \iota\sigma} = (-\iota\tilde{\xi}_{\tau_{0,\sigma}}(\zeta^C_{\pi,\sigma}), 1)$ for every embedding $\sigma : F \hookrightarrow \mathbb{C}$.

We observed that $\varsigma r^C = (R, \omega_\ell^{-1})$. Hence (a) holds after restriction to W_{E_u} by Proposition 3.2.4 (i”). Assertion (a) follows from this because the isomorphism class on each side is determined by its restriction to W_{E_u}; this is a special case of [Gan et al. 2012, Theorem 8.1(ii)]. As for (b), let $\tau_{0,\sigma} : E \hookrightarrow \mathbb{C}$ be as in Section 2.1, which extends σ. The Hodge–Tate cocharacters can be computed after taking a finite base extension, so

$$\mu_{\varsigma r^C, \iota\sigma} = \mu_{\varsigma r^C|_{\Gamma_E}, \iota\tau_{0,\sigma}} = \mu_{(R, \omega_\ell^{-1}), \iota\tau_{0,\sigma}}.$$

Hence (b) is a consequence of Proposition 3.2.4(ii”) as well as the fact that ω_ℓ has Hodge–Tate weight -1. □

Acknowledgments. The idea to write this paper was conceived during a conversation with Tasho Kaletha, Peter Scholze, and others at the IHES summer school in July 2022. I am grateful to the organizers (Pierre-Henri Chaudouard, Wee Teck Gan, Tasho Kaletha, and Yiannis Sakellaridis) for creating the opportunity. I thank Alexander Bertoloni Meli, Toby Gee, and Alex Youcis for their comments on an earlier draft. Finally I am thankful to the referees for their valuable comments and suggestions.

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Received 13 Feb 2023. Revised 28 Mar 2024.

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Essential Number Theory (ISSN 2834-4634 electronic, 2834-4626 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ENT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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ESSENTIAL NUMBER THEORY

2024 vol. 3 no. 1