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L-values and nonsplit extensions: a simple case

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We explain a construction of explicit extensions — of rational Hodge structures and of p-adic Galois representations — in a simple context: the cohomology of $\mathbb{P}^1 - \{\text{some points}\}$ relative to $\{\text{some other points}\}$. These extensions are naturally related to Dirichlet characters, and we connect the nonsplitting of these extensions to the values at $s = 0$ and $s = 1$ of associated Dirichlet L-functions $L(s, \chi)$. We highlight the close parallels between the proofs of nonsplitting in both the Hodge-theoretic and p-adic cases, emphasizing the use of de Rham theory. We also indicate connections with Euler systems along with variations on these constructions in the setting of modular curves. This paper is intended as an introduction to some of the key ideas in forthcoming constructions of Galois cohomology classes and Euler systems in a range of settings.

1. Introduction

Beginning with Birch and Swinnerton-Dyer's formulation of their celebrated conjecture, if not earlier, number theorists have sought arithmetic explanations for the zeros at special values of s of the L-functions $L(M, s)$ that arise in the context of arithmetic geometry. This encompasses Dirichlet L-series, L-functions of algebraic Hecke characters, the Hasse–Weil L-functions of elliptic curves and other varieties over number fields, etc. For example, conjectures of Beilinson essentially express the order of vanishing at particular s as the dimension of a certain group of extensions

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in a category of mixed Hodge structures, or more ambitiously in a category of mixed motives [Beilinson 1984; Nekovář 1994]. And conjectures of Bloch and Kato essentially express the same orders of vanishing as the dimensions of certain groups of extensions of p-adic Galois representations [Bloch and Kato 1990; Fontaine and Perrin-Riou 1994]. The latter should be the p-adic realizations of the former motivic extensions. These conjectures are only proved for some simple cases, though evidence exists for many interesting L-functions. It is expected that the Galois extensions related to a given $L(M, s)$ and its twists $L(M, \chi, s)$ by Dirichlet characters (or other finite Hecke characters) should form an Euler system, which then yield — via the theory of Euler and Kolyvagin systems — upper bounds on orders of related Selmer groups.

Given an L-function $L(M, s)$ and a special value of s, the expected motivic nature of the related extensions makes it natural to ask: should the expected extensions be concretely realized in cohomology by some general construction? A good rule of thumb here is that if $L(M, s)$ is suitably primitive and indecomposable, then this should be the case if and only if the order of vanishing equals 1: there is generally no good reason to distinguish one line in a space of extensions from another when the space of extensions has dimension greater than 1. Such a guiding principle both explains and predicts the extensions that comprise many of the known examples of Euler systems.[1]

We construct explicit extensions — of rational Hodge structures and of p-adic Galois representations — in a simple context: the cohomology of $\mathbb{P}^1 - \{\text{some points}\}$ relative to {some other points}. These extensions are extensions in the corresponding categories, that is, elements of Ext^1-groups. They are naturally related to Dirichlet characters χ, and for nontrivial χ we demonstrate that they are nonsplit if and only if χ is even and $L(s, \chi)$ vanishes at $s = 0$ to order 1. Our aim in writing this is three-fold: (i) to provide some evidence in a very simple case for the rule of thumb stated above, (ii) to highlight the close parallels between the proofs of nonsplitting in both the Hodge-theoretic and p-adic cases, and (iii) to give a sense, in this very simple case, of the ideas underpinning some recent and forthcoming constructions of new Euler systems (such as [Sangiovanni-Vincentelli and Skinner $\geq$ 2024a; $\geq$ 2024b], but see also [Shang et al. $\geq$ 2024]). We emphasize especially the aim (ii), though we also provide some elaboration on (iii).

In both the Hodge and p-adic cases, the proof of nonsplitting is reduced to an analytic calculation. For the Hodge structures this goes via Hodge theory and the real analytic de Rham isomorphism. For the p-adic Galois representations this goes via the comparison isomorphisms of p-adic Hodge theory as well as a p-adic analytic expression for algebraic de Rham classes. In our simple setting we can appeal to

[1] But like all such 'rules', it should also be taken with a grain of salt.

Monsky–Washnitzer cohomology for the latter, though the final calculation is done in the context of locally analytic functions via Coleman's p-adic integration. In both cases, the crucial input is a simple, explicit description of a cohomology class and its de Rham realization. In fact, another of the key points to carry away from this note is that — at least for the purposes of Euler systems — in many instances such explicit classes can reasonably substitute for motivic constructions of classes (often realized via, say, units or elements of higher Chow groups).

The constructions in our simple case are carried out in Sections 4 and 5. The aim in each section is an explanation of the statements (4.6.b) and (5.6.c), respectively, linking orders of zeros of complex L-functions to the nontriviality of extensions. We also indicate the connection with Euler systems in Sections 5.7 and 5.8, respectively. This is followed in Section 6 with brief sketches of similar constructions and calculations in the cohomology of modular curves, yielding extensions related to L-values of Dirichlet characters (again) and to Hecke characters of imaginary quadratic fields.

We suspect that many of the ideas herein, especially in the simple context in which we work, are known to experts.[2] However, extracting them from the more well-known of the existing literature (such as [Deligne 1989] or [Deligne and Goncharov 2005]) does not seem straightforward, which hopefully lends some usefulness to publishing this note. At the end of Section 3 we give some indication of the relation of this note to other works. Of course, our goal is not so much to prove new results but to explain old results from a perspective that might not be widely known.

2. The setting

Let $X = \mathbb{P}^1_{/\mathbb{Q}} = \operatorname{Proj} \mathbb{Q}[t_0, t_1]$. Let $\infty \in X(\mathbb{Q})$ be the point $\infty = [0 : 1]$. Let $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$, so $\mathbb{A}^1 = \operatorname{Spec} \mathbb{Q}[t]$, $t = t_1/t_0$. Let $Y = \mathbb{A}^1 \setminus \{1\} = \operatorname{Spec} \mathbb{Q}\big[t, \frac{1}{t-1}\big]$. So $Y = X \setminus Z$ for $Z = \{\infty, 1\}$. Let $N \geq 2$ be an integer and let $W = \mu_N^\circ = \operatorname{Spec} \mathbb{Q}[t]/(\Phi_N(t)) \subset \mathbb{A}^1$, for $\Phi_N(t)$ the N-th cyclotomic polynomial. In particular, $X(\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$ is just the Riemann sphere, $Z(\mathbb{C}) = \{\infty, 1\}$, $Y(\mathbb{C}) = X(\mathbb{C}) \setminus Z(\mathbb{C})$ is just the punctured plane $\mathbb{C} \setminus \{1\}$, and $W(\mathbb{C}) = \{\exp(2\pi i a/N) : a \in (\mathbb{Z}/N\mathbb{Z})^\times\}$ is the set of primitive N-th roots of unity. Since $N \geq 2$, $1 \notin W(\mathbb{C})$.

3. The basic idea

To construct and analyze the extensions in this paper we will make use of various cohomology theories for X, Y, Z, and W: the singular and de Rham cohomologies of the manifolds defined by the $\mathbb{C}$-points of these varieties, the étale and algebraic de Rham cohomologies of the varieties, and even crystalline cohomology. Each

[2] Harder's unpublished manuscript [2023], especially §2, provides clear evidence of this.

of these cohomology theories admits relative cohomology for the pairs (X, Y) and (Y, W), yielding exact sequences

$$\cdots \to H^i(X) \to H^i(Y) \to H^{i+1}(X, Y) \to H^{i+1}(X) \to \cdots$$

and

$$\cdots \to H^{i-1}(W) \to H^i(Y, W) \to H^i(Y) \to H^i(W) \to \cdots .$$

Here we have written $H^i(-)$ to denote any of the cohomology theories and have suppressed any reference to coefficients (which may depend on the particular theory). Purity often affords a canonical identification of $H^{i+1}(X, Y)$ (which is sometimes written as $H^{i+1}_Z(X)$) with $H^{i-1}(Z)(-1)$, where the (-1) denotes a twist, whose nature depends on the cohomology theory (e.g., a Tate twist in the case of étale cohomology). We also refer to the first of these sequences as the Gysin sequence for the pair (X, Z).

We will use the first of the above sequences together with purity to define explicit submodules[3] $A \subset H^1(Y)$ and to deduce various properties of A (e.g., the Galois action on A in the case of étale cohomology). We will also define an explicit quotient $H^0(W) \twoheadrightarrow B$ that factors through $H^0(W)/\mathrm{im}(H^0(Y))$. We will then use the second of the above sequences to define an extension via pull-back/push-forward:

$$\begin{array}{ccccc} \frac{H^0(W)}{\mathrm{im}(H^0(Y))} & \hookrightarrow & H^1(Y, W) & \twoheadrightarrow & H^1(Y) \\ \downarrow\!\!\downarrow & & \vdots & & \uparrow \\ B & \hookrightarrow & \mathcal{E} & \twoheadrightarrow & A. \end{array}$$

Here the dashed arrow denotes subquotient. The particular category in which the extension $\mathcal{E}$ belongs depends on the cohomology theory (e.g., the category of Galois modules in the case of étale cohomology). Our aim is to understand when the extension class $\mathcal{E} \in \mathrm{Ext}^1(A, B)$ is nonzero, that is, when the extension $\mathcal{E}$ is nonsplit. This will be achieved by making use of the comparison isomorphisms of the various cohomology theories, which will ultimately reduce the problem to whether a certain formula extracted from de Rham cohomology is nonzero.

A quick glance at a select part of the literature. We very briefly indicate the relation of the construction sketched above to some of the vast body of literature about mixed motives.

1) (Nori motives). Our use of relative cohomology meshes well with Nori's program to construct a general theory of mixed motives using such cohomology groups. A nice exposition of Nori's program is given in [Huber and Müller-Stach 2017]. Not surprisingly there is some overlap of the context we work in with some of the

[3] In the simplest situation considered here, A will be turn out to be all of $H^1(Y)$, but more generally it will just be a submodule (see 5.8).

examples in op. cit., especially [Huber and Müller-Stach 2017, §14.1]. However, the emphasis therein, as in the complementary survey [Huber 2020], is on periods, while the focus herein is on showing that certain explicit extensions of motives, and especially of Galois representations, are nonsplit. Of course, the calculations in Sections 4.6 and 5.6 can be recast in the context of periods and the final results expressed as: certain periods are nonzero if and only if certain extensions are nonsplit (the motivated reader might profit from doing so).

2) ($\mathbb{P}^1$ minus three points) Deligne's influential paper [1989] introduced, among other things, an approach to studying the category of mixed Tate motives (the kind of extensions we construct in this note) via the unipotent fundamental group of $X = \mathbb{P}^1 - \{0, 1, \infty\}$; this was realized more completely in [Deligne and Goncharov 2005]. This essentially realizes extensions of Tate motives in the cohomology of X^n relative to certain normal crossing divisors (which, in the cases considered, can be reinterpreted as being in the cohomology of certain unipotent local systems on X); see especially [Deligne and Goncharov 2005, §3]. This setting is well-adapted for expressing associated periods as iterated integrals (hence the relation to multiple zeta values; see [Brown 2014] for the state of the art). A translation of the construction we explain herein into the setting of [Deligne 1989; Deligne and Goncharov 2005] would surely be interesting, but we content ourselves with noting that the *duals* of the extensions we construct in Sections 4 and 5 can be extracted from the special case of $X = \mathbb{P}^1 - \mu_N^\circ$, $\{a, b\} = \{1, \infty\}$, and $n = 1$ (see [Deligne and Goncharov 2005, Proposition 3.4]).

3) (Harder's Anderson motives). After preparing the first draft of this note we became aware of an unpublished manuscript of Harder [2023] in which he proposes a very similar construction of mixed motives and Hodge structures, which he calls Anderson motives. Indeed, our construction can be viewed as an elucidation of a special case of Harder's construction for curves [2023, §2]. One thing this note includes that is not in op. cit. is an explanation of the nonsplitting of the p-adic Galois representations. Indeed, illustrating how the arguments for Galois representations closely parallel those for the Hodge structures is one of our main points. We also explain — at least in our simple case, but see also [Shang et al. $\geq$ 2024] or [Sangiovanni-Vincentelli and Skinner $\geq$ 2024a] — how these constructions lead to Euler systems, which answers questions raised by Harder.

4) (Beilinson's conjectures) The extensions that we construct — of mixed Hodge structures and of p-adic Galois representations — are shown to be nontrivial precisely when the value of some Dirichlet L-series $L(s, \bar{\chi})$ is nonzero at $s = 1$. By the functional equation, this can be reinterpreted as saying that $\mathrm{ord}_{s=0} L(s, \chi) = 1$. Very generally, Beilinson conjectured that the order of vanishing of an L-function $L(s, M)$ of a motive M at the special value $s = 0$ should (usually) equal the

rank of the group of extensions $\mathrm{Ext}^1_{MM}(\mathbb{Q}(0), M^\vee(1))$, in the category MM of mixed motives, of the trivial Tate motive $\mathbb{Q}(0)$ by the dual motive $M^\vee(1)$. He also conjecture an expression for a certain associated regulator map in terms of the first nonzero coefficient of the Taylor series of $L(s, M)$ at $s = 0$. The expository article [Nekovář 1994] is an excellent introduction to these conjectures. In this paper we essentially construct extensions $\mathbb{Q}_{\bar\chi} \hookrightarrow E \twoheadrightarrow \mathbb{Q}(-1)$ for a motive $\mathbb{Q}_{\bar\chi}$ associated with $\bar\chi$ (what we really construct should be the Hodge and p-adic étale realizations of such motivic extensions). Then $E(1) \in \mathrm{Ext}^1_{MM}(\mathbb{Q}(0), \mathbb{Q}_{\bar\chi}(1))$. Beilinson's conjectures tell as that the right-hand side should be nonzero if and only if $L(0, \chi) = 0$ (as $\mathbb{Q}_{\bar\chi}(1)$ is the dual of $\mathbb{Q}_\chi$ and $L(s, \mathbb{Q}_\chi) = L(s, \chi)$), and we show that if χ is even and $\mathrm{ord}_{s=0} L(s, \chi) = 1$ then $E(1) \neq 0$. Of course, Dirichlet's units theorem already tells us that the rank of $\mathrm{Ext}^1_{MM}(\mathbb{Q}(0), \mathbb{Q}_\chi(1))$ is 1 if χ is even (see [Nekovář 1994, §8, (2)]). Our focus is on showing that a *particular* construction yields a nonsplit extension.

4. Nonsplit extensions of rational Hodge structures

We find nonsplit extensions of rational Hodge structures in the relative cohomology of the pair (Y, W). We check that these extensions are nonsplit essentially by integrating an explicit differential representing a class in the cohomology of Y and recognizing the resulting formulas as expressions for L-values of Dirichlet characters at $s = 1$ (or derivatives at $s = 0$ via the functional equation). The key input here is the explicit de Rham representative of the cohomology class.

Though the idea is simple — and the integration boils down to $\frac{dx}{x} = d \log |x|$! — we have included details of the singular and de Rham cohomology of Y and the pair (Y, W). We have done this partly for the sake of completeness, partly to illustrate the general definitions in this simple case, and partly to more clearly set out a template for other situations (see Section 6). A reader with some familiarity with mixed Hodge structures should be able to grasp the gist quickly upon reading Section 4.2 and fill in details by scanning the subsequent displayed equations. For readers less familiar with Hodge theory, we have included a brief discussion and description of the main players and tried to point to some useful resources, particularly in Sections 4.1, 4.3, and 4.4.

Conventions. In the following, given a variety $\mathcal{V}$ over a subfield of $\mathbb{C}$ and a field F we write $H^i(\mathcal{V}, F)$ for the singular cohomology group $H^i(\mathcal{V}(\mathbb{C}), F)$, and similarly for relative cohomology with respect to a subvariety $\mathcal{U}$ of $\mathcal{V}$. For nonsingular $\mathcal{V}$ and $\mathcal{U}$ these cohomology groups are canonically computed by de Rham cohomology (which gives rise to the Hodge filtrations on the former), and the latter is computed by the real-analytic de Rham complex and by the hypercohomology of both the holomorphic and algebraic de Rham complexes (we abuse notation by not distinguishing our notation for the latter two). We write ι_{dR} for these de Rham-singular isomorphisms. They are functorial in $\mathcal{V}$ and compatible with the long-exact sequences for relative cohomology, Gysin sequences, etc.

Let $\mathbb{Z}(1) = 2\pi i\mathbb{Z}$ and $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes n}$ for any integer n. Note that $\mathbb{Z}(-1)$ is canonically identified with $(2\pi i)^{-1}\mathbb{Z}$. We write $H^i(V, F)(n)$ to mean $H^i(V, F) \otimes \mathbb{Z}(n)$. Keeping track of such 'twists' makes comparisons with de Rham and étale cohomology more clearly functorial. The Hodge filtration on $H^i(V, \mathbb{C})(n)$ comes from that of $H^i(V, \mathbb{C})$ with the index shifted by $+n$, and the weight filtration is also the same but with index shifted by $+2n$, and likewise for relative cohomology. Similarly, our conventions for twists of de Rham cohomology are such that $H^i_{\mathrm{dR}}(V/F)(n)$ is $H^i_{\mathrm{dR}}(V/F)$ with the Hodge filtration shifted by $+n$ and the weight filtration by $+2n$.

4.1. ***Hodge structures and extensions, briefly.*** Recall that a rational mixed Hodge structure is a finite-dimensional $\mathbb{Q}$-space V together with:

- (Hodge filtration) a decreasing filtration $\cdots \supseteq F^p V_{\mathbb{C}} \supseteq F^{p+1} V_{\mathbb{C}} \supseteq \cdots$ of the complex vector space $V_{\mathbb{C}} = V \otimes \mathbb{C}$ such that $F^p V_{\mathbb{C}} = V_{\mathbb{C}}$ if $p \ll 0$ and $F^p V_{\mathbb{C}} = 0$ if $p \gg 0$, and
- (weight filtration) an increasing filtration $\cdots \subseteq W_n V \subseteq W_{n+1} V \subseteq \cdots$ of the rational vector space V such that $W_n V = V$ if $n \gg 0$ and $W_n V = 0$ if $n \ll 0$,

that satisfy:

- (pure graded pieces) the filtration $F^p V_{\mathbb{C}}$ induces a filtration on $\mathrm{gr}_n V_{\mathbb{C}}$ for $\mathrm{gr}_n V = W_n V / W_{n-1} V$,

$$F^p(\mathrm{gr}_n V_{\mathbb{C}}) = (F^p V_{\mathbb{C}} \cap W_n V_{\mathbb{C}} + W_{n-1} V_{\mathbb{C}}) / W_{n-1} V_{\mathbb{C}},$$

and

$$\mathrm{gr}_n V_{\mathbb{C}}^{p,q} := F^p(\mathrm{gr}_n V_{\mathbb{C}}) \cap \overline{F^q(\mathrm{gr}_n V_{\mathbb{C}})}$$

is such that $\mathrm{gr}_n V_{\mathbb{C}}^{p,q} = 0$ if $p + q \neq n$ and

$$\mathrm{gr}_n V_{\mathbb{C}} = \bigoplus_{p+q=n} \mathrm{gr}_n V_{\mathbb{C}}^{p,q}.$$

Here the overline $\overline{(\cdot)}$ denotes the image under the action of complex conjugation on the scalars of $V_{\mathbb{C}} = V \otimes \mathbb{C}$.

A (mixed) Hodge structure with graded weight filtration supported on exactly one degree, that is, $\mathrm{gr}_n V = V$ for some (unique) n, is a *pure* Hodge structure of weight n. So the third condition above just says that for a mixed Hodge structure, the induced Hodge structures on the graded pieces of the weight filtration are pure of the corresponding weight. A morphism of mixed Hodge structures is a $\mathbb{Q}$-linear map that is compatible with the Hodge and weight filtrations. Let $\mathbb{Q}-\mathrm{MHS}$ denote the category of mixed rational Hodge structures. Replacing $\mathbb{Q}$ with $\mathbb{R}$ in the above, we get the category $\mathbb{R}-\mathrm{MHS}$ of real mixed Hodge structures. A rational mixed Hodge structure V gives rise to a real mixed Hodge structure $V_{\mathbb{R}} = V \otimes \mathbb{R}$ by extending scalars.

The singular cohomology groups of an algebraic variety (including relative cohomology) are all equipped with canonical rational mixed Hodge structures, and all the maps in the associated long exact sequences (e.g., the Gysin sequence and the sequence for relative cohomology) are morphisms of mixed Hodge structures. This is a consequence of Hodge theory as developed in [Deligne 1971a; 1971b; 1974]; see also [Peters and Steenbrink 2008]. The article [Kedlaya 2008] contains a fairly gentle introduction to Hodge theory for varieties.

The simplest examples of nonzero Hodge structures are the pure Hodge structures $\mathbb{Q}(m) = (2\pi i\mathbb{Q})^{\otimes m} = (2\pi i)^m\mathbb{Q}$, m an integer, with Hodge filtration $F^p\mathbb{Q}(m) = \mathbb{Q}(m)$ if $p \le -m$ and $F^p\mathbb{Q}(m) = 0$ if $p > -m$ and weight filtration $W_n\mathbb{Q}(m) = \mathbb{Q}(m)$ if $n \ge -2m$ and $W_n\mathbb{Q}(m) = 0$ if $n < -2m$. So $\mathbb{Q}(m)$ is pure of weight $-2m$ and $\mathbb{C}(m) = \mathbb{Q}(m) \otimes \mathbb{C} = \mathbb{Q}(m)_{\mathbb{C}} = \mathbb{C}(m)^{-m,-m}$. If $\mathcal{V}$ is a complete, connected variety over $\mathbb{C}$ of dimension d, then $H^{2d}(\mathcal{V}, \mathbb{Q})$ is isomorphic to $\mathbb{Q}(-d)$ as a Hodge structure. Let $\mathbb{R}(m) = \mathbb{Q}(m) \otimes \mathbb{R}$; this is a pure real Hodge structure.

The simplest examples of nontrivial rational mixed Hodge structures are the nonsplit extensions

$$0 \to \mathbb{Q}(n) \to E \to \mathbb{Q}(m) \to 0$$

of $\mathbb{Q}(m)$ by $\mathbb{Q}(n)$, $m < n$, in the category of mixed Hodge structures.[4] Let $\phi_H : \mathbb{C}(m) \to E_{\mathbb{C}}$ be a $\mathbb{C}$-linear splitting compatible with the Hodge filtrations; ϕ_H is unique in this case. Let $\phi_W : \mathbb{Q}(m) \to E$ be a $\mathbb{Q}$-linear splitting respecting the weight filtrations; in this case, ϕ_W is only well-defined up to addition of any element of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(m), \mathbb{Q}(n))$. Let $\phi = \phi_H - \phi_W \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}(m), \mathbb{C}(n))$. Then the image of ϕ in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}(m), \mathbb{C}(n))/\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(m), \mathbb{Q}(n))$ depends only on E and not the choices of ϕ_H or ϕ_W. This yields an identification $\mathrm{Ext}^1_{\mathbb{Q}-\mathrm{MHS}}(\mathbb{Q}(m), \mathbb{Q}(n)) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}(m), \mathbb{C}(n))/\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(m), \mathbb{Q}(n)) \simeq \mathbb{C}/\mathbb{Q}$. Injectivity is a consequence of the observation that E is split if and only if we can choose $\phi_W = \phi_H$, so E is split if and only if $\phi_H - \phi_W \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(m), \mathbb{Q}(n))$ in general. Surjectivity follows by an explicit construction: Let $\phi : \mathbb{C}(m) \to \mathbb{C}(n)$ be a $\mathbb{C}$-linear map (which necessarily preserves the weight filtrations in this case). Consider the vector space $E = \mathbb{Q}(n) \oplus \mathbb{Q}(m)$ with Hodge and weight filtrations

$$F^pE_{\mathbb{C}} = \big\{(a + \phi(b), b) : a \in F^p\mathbb{C}(n), b \in F^p\mathbb{C}(m)\big\},$$

$$W_kE = \begin{cases} E, & k \ge -m, \\ \mathbb{Q}(n), & -n \le k < -m, \\ 0, & k < -n. \end{cases}$$

[4] If $n \le m$ then any such extension is split.

This is a mixed Hodge structure, and the natural inclusion $\mathbb{Q}(n) \hookrightarrow E$ and projection $E \twoheadrightarrow \mathbb{Q}(m)$ are clearly morphisms of Hodge structures: $E \in \mathrm{Ext}^1_{\mathbb{Q}-\mathrm{MHS}}(\mathbb{Q}(m), \mathbb{Q}(n))$. The image of this extension in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}(m), \mathbb{C}(n))/\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(m), \mathbb{Q}(n))$ is just the image of ϕ. Similarly, $\mathrm{Ext}^1_{\mathbb{R}-\mathrm{MHS}}(\mathbb{R}(n), \mathbb{R}(n))$ is identified with the space $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}(m), \mathbb{C}(n))/\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}(n), \mathbb{R}(m)) \simeq \mathbb{C}/\mathbb{R}$ and so is a one-dimensional $\mathbb{R}$-space. For more on extensions of mixed Hodge structure, the interested reader should consult [Carlson 1980] or [Carlson and Hain 1989].

In the rest of Section 4 we will find extensions of the Hodge structures $\mathbb{Q}(-1)$ by $\mathbb{Q}(0)$ as quotients of the relative singular cohomology groups $H^1(Y, W, \mathbb{Q})$ of the pairs (Y, W), and so find extensions in $\mathbb{Q}-\mathrm{MHS}$. To decide whether such an extension E is nontrivial, it suffices to identify a homomorphism $\phi : \mathbb{C}(-1) \to \mathbb{C}(0)$ giving rise to E as above. One way of doing this is as follows: Let $0 \neq \omega \in \mathbb{Q}(-1)$ and find elements $\omega_H \in F^1 E_{\mathbb{C}}$ and $\omega_W \in W_2 E$ that both map to ω. Identifying $\mathbb{Q}(-1)$ with a subspace of E via $\omega \mapsto \omega_W$, the Hodge structure on E is identified with the Hodge structure on $\mathbb{Q}(0) \oplus \mathbb{Q}(-1)$ defined by the ϕ such that $\phi(\omega) = \omega_H - \omega_W$. This extension is split if and only if $\phi \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(m), \mathbb{Q}(n)))$ and so if and only if $\omega_H - \omega_W \in \mathbb{Q}(0)$. If we work instead in $\mathbb{R}-\mathrm{MHS}$, then the criteria to be split becomes $\omega_H - \omega_W \in \mathbb{R}(0)$. In practice (as will be the case below) one can often find an explicit ω_H using Hodge theory, but finding an explicit ω_W — especially one for which the difference $\omega_H - \omega_W$ can be identified — can be difficult. We take a different tack.

To prove nontriviality of the extensions we find, we make use of the fact that all the Hodge structures involved in our constructions have a particular enrichment. This enrichment is the action of an involution ϕ_∞ on the underlying $\mathbb{Q}$-space V of a rational mixed Hodge structure (or $\mathbb{R}$-space of a real mixed Hodge structure) such that the action of $\phi_\infty \otimes \tau$ on $V_{\mathbb{C}} = V \otimes \mathbb{C}$ induces a $\mathbb{C}$-semilinear involution of each $F^p V_{\mathbb{C}}$. Here τ denote the action of complex conjugation on $\mathbb{C}$ and the semilinearity is with respect to τ. We denote by $\mathbb{Q}-\mathrm{MHS}^+$ the category of such enriched rational mixed Hodge structures (morphisms must also respect the action of ϕ_∞). We similarly write $\mathbb{R}-\mathrm{MHS}^+$ for the category of such enriched real mixed Hodge structures. The Hodge structures coming from the singular cohomology of varieties defined over $\mathbb{R}$ or some subfield have a natural enrichment: ϕ_∞ is the involution induced from the action of complex conjugation on the $\mathbb{C}$-points of the variety. The Hodge structures $\mathbb{Q}(m)$ also have natural enrichments: ϕ_∞ acts as multiplication by $(-1)^m$. For a complete, geometrically connected variety $\mathcal{V}$ of dimension d defined over a subfield of $\mathbb{R}$, the enriched Hodge structure on $H^{2d}(\mathcal{V}, \mathbb{Q})$ is isomorphic to that of $\mathbb{Q}(-d)$.

Let $m < n$ be integers. Following the description of $\mathrm{Ext}^1_{\mathbb{Q}-\mathrm{MHS}}(\mathbb{Q}(m), \mathbb{Q}(n))$ above, we find that extensions in $\mathrm{Ext}^1_{\mathbb{Q}-\mathrm{MHS}^+}(\mathbb{Q}(m), \mathbb{Q}(n))$ are those coming from homomorphisms $\phi \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}(m), \mathbb{C}(n))$ such that $\phi((2\pi i)^m) = r i^{m-n} (2\pi i)^n$ for some $r \in \mathbb{R}$. In particular, the group of enriched extensions $\mathrm{Ext}^1_{\mathbb{Q}-\mathrm{MHS}^+}(\mathbb{Q}(m), \mathbb{Q}(n))$ is identified with the image of $i\mathbb{R}$ in $\mathbb{C}/\mathbb{Q}$ if $n-m$ is odd and with $\mathbb{R}/\mathbb{Q}$ otherwise. Similarly, $\mathrm{Ext}^1_{\mathbb{R}-\mathrm{MHS}^+}(\mathbb{R}(m), \mathbb{R}(n))$ is identified with $i\mathbb{R} \xrightarrow{\sim} \mathbb{C}/\mathbb{R}$ if $m-n$ is odd, but $\mathrm{Ext}^1_{\mathbb{R}-\mathrm{MHS}^+}(\mathbb{R}(m), \mathbb{R}(n)) = 0$ if $m-n$ is even.

Our strategy for determining whether an extension E of $\mathbb{Q}(-1)$ by $\mathbb{Q}(0)$ in $\mathbb{Q}-\mathrm{MHS}^+$ is nonzero will be to find an explicit element $0 \neq \omega \in \mathbb{Q}(-1)$ and an explicit lift $\omega_H \in F^1 E_{\mathbb{C}}$. Then E is nonsplit if and only if $\phi_\infty(\omega_H) + \omega_H \neq 0$. This is readily seen by using the description of E as an extension associated with some $\phi \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}(-1), \mathbb{C}(0))$ such that $\phi((2\pi i)^{-1}) = ir \in i\mathbb{R}$ (which is split if and only if $r = 0$). For if E is isomorphic to the enriched mixed Hodge structure on $\mathbb{Q}(0) \oplus \mathbb{Q}(-1)$ for some ϕ such that $\phi((2\pi i)^{-1}) = ir \in i\mathbb{R}$ and if $\omega = (2\pi i)^{-1}a$, then $\omega'_H = (ira, (2\pi i)^{-1}a) \in F^1 E_{\mathbb{C}}$ is a lift of ω and $\phi_\infty(\omega'_H) + \omega'_H = 2ira \in \mathbb{C}(0)$ is nonzero if and only if $r \neq 0$, that is, if and only if $\phi \neq 0$, which – as we have already seen – is the condition for E to be nonsplit in $\mathbb{Q}-\mathrm{MHS}^+$. As ϕ_∞ acts trivially, on $\mathbb{Q}(0)$ and hence on $\mathbb{C}(0)$, we see that $\phi_\infty(\omega_H) + \omega_H = \phi_\infty(\omega'_H) + \omega'_H$. Of course, the 'if' part is even easier to see in this special case: ω_H is the unique lift of ω to $F^1 E_{\mathbb{C}}$ and so the extension being split in $\mathbb{Q}-\mathrm{MHS}^+$ would then imply that $\phi_\infty \omega_H = -\omega_H$. In practice, we will be able to use Hodge theory to explicitly compute $\phi_\infty(\omega_H) + \omega_H$. In this approach we only make use of an explicit lift $\omega_H \in F^1 E_{\mathbb{C}}$ and do not need to also identify a lift $\omega_W \in W_2 E$. As is explained in Section 5, a very similar strategy can be used to show that certain extensions of p-adic Galois representations are nonsplit (see especially Section 5.1).

4.2. ***The extension $\mathcal{E}_{\mathrm{MH}}$.*** Let $F/\mathbb{Q}$ be any extension. The relative singular cohomology $H^i(Y, W, F)$ fits into a long exact sequence

$$\cdots \to H^0(Y, F) \to H^0(W, F) \to H^1(Y, W, F) \to H^1(Y, F) \to H^1(W, F) \to \cdots . \tag{4.2.a}$$

In the case $F = \mathbb{Q}$, each of the cohomology groups in this sequence is endowed with a rational (possibly mixed) Hodge structure, and the maps between groups are morphisms of mixed Hodge structures. In this case, the Hodge structures on $H^0(W, F)$ and $H^1(Y, F)$ are pure of weights 0 and 2, respectively (for more details on the cohomology and Hodge theory of Y and (Y, W) and associated notation, see Sections 4.3 and 4.4 below). In particular, the induced extension

$$0 \to \frac{H^0(W, \mathbb{Q})}{\mathrm{im}(H^0(Y, \mathbb{Q}))} \to H^1(Y, W, \mathbb{Q}) \to H^1(Y, \mathbb{Q}) \to 0 \tag{4.2.b}$$

realizes the mixed Hodge structure on $H^1(Y, W, \mathbb{Q})$ as an extension in the category $\mathbb{Q}-\mathrm{MHS}$ of mixed rational Hodge structures: an extension of a pure Hodge structure of weight 0 by a pure Hodge structure of weight 2. Since each of the varieties X,

Y, Z, and W is defined over $\mathbb{Q}$, the singular cohomology groups considered above all carry the action of an involution, denoted ϕ_∞, induced by the action of complex conjugation on the $\mathbb{C}$-points of the varieties, and the above maps of cohomology groups also respect the action of ϕ_∞. The tensor product $\phi_\infty \otimes \tau$ of ϕ_∞ with the action τ of complex conjugation on the coefficients for $F = \mathbb{C}$ preserves the Hodge filtration (but is only semilinear with respect to τ for the action of $\mathbb{C}$). So all of these Hodge structures, maps, and extensions are actually in the category $\mathbb{Q} - \mathrm{MHS}^+$ of enriched mixed Hodge structures.

Let $V = H^0(W, \mathbb{Q})/\mathrm{im}(H^0(Y, \mathbb{Q}))$. Since $Y(\mathbb{C}) = \mathbb{C} \setminus \{1\}$, $H^1(Y, \mathbb{Q}) \simeq \mathbb{Q}$. As we will see there is a natural $\mathbb{Q}$-basis $c \in H^1(Y, \mathbb{Q})$, which we use to identify $H^1(Y, \mathbb{Q})$ with the 1-dimensional pure Hodge structure $\mathbb{Q}(-1)$ of weight 2 with ϕ_∞-action being multiplication by -1. Then the extension (4.2.b) together with the mixed Hodge structure on $H^1(Y, W, \mathbb{Q})$ and the action of ϕ_∞ defines a class $\mathcal{E}_{\mathrm{MH}} = [H^1(Y, W, \mathbb{Q})] \in \mathrm{Ext}^1_{\mathbb{Q}-\mathrm{MHS}^+}(\mathbb{Q}(-1), V)$. It is natural to ask:

$$\boxed{\text{is } \mathcal{E}_{\mathrm{MH}} \neq 0?}$$

That is, is (4.2.b) a nonsplit extension of enriched mixed Hodge structures?

Let $\lambda : V \twoheadrightarrow \mathbb{Q}(0)$ be any surjective map of (enriched) Hodge structures; since V is pure of weight 0, this is just any surjective linear map from V. Then the push-out of $\mathcal{E}_{\mathrm{MH}}$ by λ yields an extension $\mathcal{E}_{\mathrm{MH},\lambda} \in \mathrm{Ext}^1_{\mathbb{Q}-\mathrm{MHS}^+}(\mathbb{Q}(-1), \mathbb{Q}(0))$. It is clear that $\mathcal{E}_{\mathrm{MH}} \neq 0$ if and only if there exists some λ such that $\mathcal{E}_{\mathrm{MH},\lambda} \neq 0$. So it also natural, and even equivalent, to ask:

$$\boxed{\text{does there exists } \lambda \text{ such that } \mathcal{E}_{\mathrm{MH},\lambda} \neq 0?}$$

The keys to our answer to these question are

- explicit descriptions of some classes in $F^1H^1(Y, \mathbb{C})$ and $F^1H^1(Y, W, \mathbb{C})$ via their corresponding classes in $H^1_{\mathrm{dR}}(Y/\mathbb{C})$ and $H^1_{\mathrm{dR}}((Y, W)/\mathbb{C})$, and
- an analytic calculation with the explicit de Rham classes and their images under ϕ_∞.

These come together as follows: Let $0 \neq \omega \in H^0(\Omega^1_{X/\mathbb{C}}(\log Z))$. Via the de Rham isomorphism ι_{dR}, the differential ω determines classes $c = \iota_{\mathrm{dR}}([\omega]) \in F^1H^1(Y, \mathbb{C})$ and $c_H = \iota_{\mathrm{dR}}([\omega]) \in F^1H^1(Y, W, \mathbb{C})$ in the Hodge filtrations. Just as explained in the final paragraph of Section 4.1, $\mathcal{E}_{\mathrm{MH},\lambda} \neq 0$ if and only if the image of $(1+\phi_\infty)c_H \in V_{\mathbb{C}} = H^0(W, \mathbb{C})/\mathrm{im}(H^0(Y, \mathbb{C}))$ is nonzero under λ. In particular, $\mathcal{E}_{\mathrm{MH}} \neq 0$ if and only if $(1+\phi_\infty)c_H \neq 0$ in $V_{\mathbb{C}} = H^0(W, \mathbb{C})/\mathrm{im}(H^0(Y, \mathbb{C}))$. In some instances ω can be chosen so that $(1+\phi_\infty)\omega = d\eta$ for some explicit real analytic function η on Y. Then $(1+\phi_\infty)c_H$ is just the image of $\eta|_W \in H^0(W, \mathbb{C})$. In particular, $\mathcal{E} \neq 0$ if and only if $\lambda(\eta|_W) \neq 0$ for some homomorphism $\lambda : H^0(W, \mathbb{C}) \to \mathbb{C}$ (not necessarily $\mathbb{Q}$-valued) that is trivial on the image of $H^0(Y, \mathbb{C})$. In particular, to show that $\mathcal{E} \neq 0$

it will be enough to write down a sufficiently explicit ω so that η can be determined and seen to satisfy $\lambda(\eta|_W) \neq 0$ for some such λ. Note that $\lambda(\eta|_W)$ is just a linear combination of the values of η on the points in W.

Arguing this way we will show that $\mathcal{E}_{\mathrm{MH}} \neq 0$ if, for example, there exists a nontrivial even primitive Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. For a more precise result, see (4.6.a) below.

4.3. ***The cohomology of Y.*** As $Y(\mathbb{C})$ is just the Riemann sphere $X(\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$ minus the two points ∞ and 1, the singular cohomology group $H^1(Y, \mathbb{Q})$ is isomorphic to $\mathbb{Q}$. A somewhat explicit isomorphism is given as follows.

Recall the long exact sequence for relative cohomology for the (open) inclusion $Y(\mathbb{C}) \subset X(\mathbb{C})$:

$$\cdots \to H^1(X, F) \to H^1(Y, F) \xrightarrow{\partial} H^2(X, Y, F) \to H^2(X, F) \to H^2(Y, F) \to \cdots. \quad (4.3.a)$$

The group $H^2(X, Y, F)$ is naturally identified with the space $H^0(Z, F)(-1) = \bigoplus_{z\in Z(\mathbb{C})} (2\pi i)^{-1} F$. Under this identification, the induced map $H^0(Z, F) \to H^2(X, F)(1)$ is just the cycle class map.[5] In particular, Poincaré duality canonically identifies $H^2(X, F)(1)$ with the F-dual of $H^0(X, F)$ and the map $H^0(Z, F) \to H^2(X, F)(1)$ with the dual of the natural map $H^0(X, F) \to H^0(Z, F)$, $H^0(Z, F)$ being, of course, self-dual in the obvious way. As $H^1(X, F) = H^1(\mathbb{P}^1, F) = 0$, it follows that

$$\partial : H^1(Y, F) \xrightarrow{\sim} \left\{ ((2\pi i)^{-1} a_z)_{z\in Z(\mathbb{C})} : a_z \in F, \sum_{z\in Z(\mathbb{C})} a_z = 0 \right\} \subset H^0(Z, F)(-1). \quad (4.3.b)$$

In particular, as $Z(\mathbb{C}) = \{\infty, 1\}$,

$$\partial : H^1(Y, F) \xrightarrow{\sim} \left\{ ((2\pi i)^{-1} a, -(2\pi i)^{-1} a) : a \in F \right\} \simeq F.$$

As the Hodge structure on $H^0(Z, \mathbb{Q})$ is pure of weight 0, $H^0(Z, \mathbb{Q})(-1)$ is pure of weight 2, and so the isomorphism (4.3.b) implies that the Hodge structure on $H^1(Y, \mathbb{Q})$ is pure of weight 2. This can also be seen as follows. The Hodge filtration $F^\bullet H^1(Y, \mathbb{C})$ on $H^1(Y, \mathbb{C}) = H^1(Y, \mathbb{Q}) \otimes \mathbb{C}$ is defined via the de Rham isomorphism $\iota_{\mathrm{dR}} : H^1_{\mathrm{dR}}(Y/\mathbb{C}) \xrightarrow{\sim} H^1(Y, \mathbb{C})$ and the Hodge filtration on $H^1_{\mathrm{dR}}(Y/\mathbb{C})$. The de Rham cohomology $H^*_{\mathrm{dR}}(Y/\mathbb{C})$ is computed by the hypercohomology of both the de Rham complex $DR_Y = [\mathcal{O}_Y \xrightarrow{d} \Omega^1_Y]$ and the log de Rham complex $DR_X(\log Z) = [\mathcal{O}_X \xrightarrow{d} \Omega^1_X(\log Z)]$; the natural map $DR_X(\log Z) \to DR_Y$ is a quasi-isomorphism. Here '$(\log Z)$' denotes the complex with log poles along Z

[5] This is essentially the definition of the cycle class map.

(for more on the log de Rham complex and its cohomology, see [Kedlaya 2008, §1.9], [Esnault and Viehweg 1992, §2], or [Peters and Steenbrink 2008, §4]). Let $A = \mathbb{Q}\left[t, \frac{1}{t-1}\right]$. Since $Y = \operatorname{Spec} A$ is affine, the hypercohomology of the de Rham complex for Y is the cohomology of the complex itself. The Hodge filtration $F^\bullet H^1_{\mathrm{dR}}(Y/\mathbb{C})$ is just the image of the hypercohomology of the usual filtration on $DR_X(\log Z)$. In particular,

$$\begin{aligned} F^0 H^1_{\mathrm{dR}}(Y/\mathbb{C}) &= H^1_{\mathrm{dR}}(Y/\mathbb{C}) = \Omega^1_{A\otimes\mathbb{C}}/d(A\otimes\mathbb{C}) = \mathbb{C}\frac{dt}{1-t}, \\ F^1 H^1_{\mathrm{dR}}(Y/\mathbb{C}) &= \operatorname{im}(H^0(\Omega^1_X(\log Z))) = H^0(\Omega^1_X(\log Z)) = \mathbb{C}\frac{dt}{1-t}, \\ F^2 H^1_{\mathrm{dR}}(Y/\mathbb{C}) &= 0. \end{aligned}$$

The weight filtration $W_\bullet H^1(Y, \mathbb{Q})$ on $H^1(Y, \mathbb{Q})$ is given by $0 = W_0 H^1(Y, \mathbb{Q}) = W_1 H^1(Y, \mathbb{Q}) = \operatorname{im}(H^1(X, \mathbb{Q})) \subset W_2 H^1(Y, \mathbb{Q}) = H^1(Y, \mathbb{Q})$. Indeed, in this case the n-th part $W_n H^1(Y, \mathbb{C})$ of the weight filtration is the image of the hypercohomology of $W_{n-1} DR_X(\log Z)$, where $W_n DR_X(\log Z) = [0]$ $(n < 0)$, $W_0 DR_X(\log Z) = DR_X$, and $W_m DR_X(\log Z) = DR_X(\log Z)$ $(m \geq 1)$; see [Peters and Steenbrink 2008, Theorem 4.2]. It is a fundamental result of Hodge theory that this weight filtration on $H^1(Y, \mathbb{C})$ is actually rational.

Note that the compatibility of the de Rham isomorphisms with the long exact sequence (4.3.a) shows that the class $\iota_{\mathrm{dR}}([\omega_a]) \in H^1(Y, \mathbb{C})$ of the differential $\omega_a = (2\pi i)^{-1} a \frac{dt}{1-t} \in H^0(\Omega^1_{X/\mathbb{C}}(\log Z))$ satisfies

$$\partial(\iota_{\mathrm{dR}}([\omega]) = ((2\pi i)^{-1} a_\infty, (2\pi i)^{-1} a_1), \quad a_\infty = -a_1 = a. \tag{4.3.c}$$

This is because the corresponding boundary map for de Rham cohomology just takes the class of ω to $(\operatorname{Res}_z(\omega))_{z\in Z}$. In particular, the de Rham isomorphism induces an identification

$$H^1_{\mathrm{dR}}(Y/\mathbb{C}) \supset (2\pi i)^{-1} F \frac{dt}{1-t} \xrightarrow[\sim]{\iota_{\mathrm{dR}}} H^1(Y, F) \tag{4.3.d}$$

for any subfield $F \subset \mathbb{C}$.

Let

$$\omega = \frac{dt}{1-t} \in H^0(\Omega^1_X(\log Z)) \quad \text{and} \quad \omega^{\mathrm{an}} = (2\pi i)^{-1}\omega.$$

Let $[\omega^{\mathrm{an}}] \in F^1 H^1_{\mathrm{dR}}(Y/\mathbb{C})$ be the corresponding class. Then

$$H^1_{\mathrm{dR}}(Y/\mathbb{C}) = F^1 H^1_{\mathrm{dR}}(Y/\mathbb{C}) = \mathbb{C}[\omega^{\mathrm{an}}].$$

Let

$$c = \iota_{\mathrm{dR}}([\omega^{\mathrm{an}}]) \in F^1 H^1(Y, \mathbb{C}).$$

Then $H^1(Y, \mathbb{C}) = F^1 H^1(Y, \mathbb{C}) = \mathbb{C}c$. It follows from (4.3.d) that $c \in H^1(Y, \mathbb{Q})$.

4.4. ***The cohomology of*** (Y, W)**.** We can compute the relative singular cohomology groups $H^1(Y, W, F)$ as the cohomology of the mapping cone $\mathrm{Cone}(C^\bullet(Y, F) \to C^\bullet(W, F))[-1]$ for $C^\bullet(Y, F)$, $C^\bullet(W, F)$ the singular cochain complexes with F coefficients and the map being that induced by the inclusion $W(\mathbb{C}) \hookrightarrow Y(\mathbb{C})$. Concretely, this mapping cone is $C^\bullet(Y, F) \oplus C^{\bullet-1}(W, F))$ with differential $d(a, b) = (d^\bullet a, -d^{\bullet-1}b - a|_W)$; the map to $C^\bullet(Y)$ is just projection onto the first summand.

The de Rham cohomology of the pair is similarly computed but with $C^\bullet(Y, F)$ and $C^\bullet(W, F)$ replaced by the de Rham complexes DR_Y and DR_W, respectively, or even $DR_X(\log Z)$ and DR_W. From the definition of the mapping cone, it is easy to see that $\mathrm{Cone}(DR_Y \to DR_W)[-1]$ (resp. $\mathrm{Cone}(DR_X(\log Z) \to DR_W)[-1]$) can be replaced with the quasiisomorphic subcomplex $DR_Y(-W) = [\mathcal{O}_Y(-W) \xrightarrow{d} \Omega^1_Y]$ (resp. $DR_X(\log Z)(-W) = [\mathcal{O}_X(-W) \xrightarrow{d} \Omega^1_X(\log Z)]$).

The Hodge filtration on $H^1(Y, W, \mathbb{C})$ is again defined via the de Rham isomorphism. In particular, it is given by the images of the hypercohomology of the usual filtration on $DR_X(\log Z)(-W)$. Recall that $A = \mathbb{Q}\big[t, \frac{1}{t-1}\big]$. Much as for $H^1_{\mathrm{dR}}(Y, \mathbb{C})$, we have

$$F^0 H^1_{\mathrm{dR}}((Y, W)/\mathbb{C}) = H^1_{\mathrm{dR}}((Y, W)/\mathbb{C}) = \Omega^1_{A\otimes\mathbb{C}}/d(\Phi_N(t)A \otimes \mathbb{C}),$$
$$F^1 H^1_{\mathrm{dR}}((Y, W)/\mathbb{C}) = \mathrm{im}(H^0(\Omega^1_X(\log Z))) = H^0(\Omega^1_X(\log Z)) = \mathbb{C}\frac{dt}{t-1},$$
$$F^2 H^1_{\mathrm{dR}}((Y, W)/\mathbb{C}) = 0.$$

The weight filtration on $H^1(Y, W, \mathbb{Q})$ is $W_\bullet H^1(Y, W, \mathbb{Q})$ with $W_{-1} H^1(Y, W, \mathbb{Q}) = 0$, $W_0 H^1(Y, W, \mathbb{Q}) = W_1 H^1(Y, W, \mathbb{Q}) = \mathrm{im}(H^0(W, \mathbb{Q}))$, and $W_2 H^1(Y, W, \mathbb{Q}) = H^1(Y, W, \mathbb{Q})$. Note that $W_0 H^1(Y, W, \mathbb{Q})/W_{-1} H^1(Y, W, \mathbb{Q}) = \mathrm{im}(H^0(W, \mathbb{Q}))$ and the induced Hodge filtration is indeed the unique Hodge structure pure of weight 0. (For more on the Hodge structures on relative cohomology see [Peters and Steenbrink 2008, §5.5].) Note also that $W_2 H^1(Y, W, \mathbb{Q})/W_1 H^1(Y, W, \mathbb{Q}) \xrightarrow{\sim} H^1(Y, \mathbb{Q})$ and the induced Hodge filtration is just the one on $H^1(Y, \mathbb{Q})$ described above. This just makes explicit in this setting the general fact that the extension (4.2.b) realizes the mixed Hodge structure on $H^1(Y, W, \mathbb{Q})$ as an extension in the category of mixed Hodge structures.

For ω^{an} as before, let $[\omega^{\mathrm{an}}]_W \in F^1 H^1_{\mathrm{dR}}((Y, W)/\mathbb{C})$. Then $F^1 H^1(Y, W, \mathbb{C}) = \mathbb{C}\iota_{\mathrm{dR}}([\omega^{\mathrm{an}}]_W)$, and the isomorphism $F^1 H^1(Y, W, \mathbb{C}) \xrightarrow{\sim} F^1 H^1(Y, \mathbb{C})$ maps $c_H = \iota_{\mathrm{dR}}([\omega^{\mathrm{an}}]_W)$ to $c = \iota_{\mathrm{dR}}([\omega^{\mathrm{an}}])$.

4.5. ***The involution*** ϕ_∞**.** Since each of the varieties X, Y, Z, and W is defined over $\mathbb{Q}$, the cohomology groups considered above — singular and de Rham — all carry the action of an involution, denoted ϕ_∞, induced by the action of complex conjugation on the $\mathbb{C}$-points of the varieties. The maps in (4.3.a), (4.3.b), and (4.2.a) are all compatible with the actions of ϕ_∞ as are the de Rham isomorphisms ι_{dR}.

Moreover, ϕ_∞ interacts well with the Hodge filtrations: $\phi_\infty(F^p(-)) = \overline{F^p(-)}$. That is, $\phi_\infty \circ \tau$, for τ the action of complex conjugation on the coefficients, preserves the Hodge filtrations. Since $\overline{F^1H^1(Y,\mathbb{C})} = F^1H^1(Y,\mathbb{C})$ and $F^1H^1(Y,W,\mathbb{C}) \xrightarrow{\sim} F^1H^1(Y,\mathbb{C})$, the extension (4.2.b) is a split extension of Hodge structures enriched with the involutions ϕ_∞ if and only if $\overline{F^1H^1(Y,W,\mathbb{C})} = F^1H^1(Y,W,\mathbb{C})$. But this is often not the case, as we show below.

4.6. ***The calculation.*** The class $\phi_\infty(\iota_{\mathrm{dR}}([\omega^{\mathrm{an}}]_W)) = \iota_{\mathrm{dR}}(\phi_\infty([\omega^{\mathrm{an}}]_W))$ is represented by the real analytic differential $\phi_\infty^*\omega^{\mathrm{an}} = (2\pi i)^{-1}\frac{d\bar{t}}{1-\bar{t}}$ via the real-analytic de Rham isomorphism. Then $(1+\phi_\infty)\iota_{\mathrm{dR}}([\omega^{\mathrm{an}}]_W)$ is represented by the real analytic differential

$$(2\pi i)^{-1}\left(\frac{dt}{1-t} + \frac{d\bar{t}}{1-\bar{t}}\right) = -(2\pi i)^{-1} d\log|t-1|^2 = d(-(2\pi i)^{-1}\log|t-1|^2).$$

Let $\eta = -(2\pi i)^{-1}\log|t-1|^2$. This is a real-analytic function on Y. It follows that $(1+\phi_\infty)\iota_{\mathrm{dR}}([\omega^{\mathrm{an}}]_W)$ is the image of the class $\eta|_W \in H^0_{\mathrm{dR}}(W/\mathbb{C}) = H^0(W,\mathbb{C})$, which is just $(\eta(\zeta))_{\zeta\in W(\mathbb{C})} = \bigoplus_{\zeta\in W(\mathbb{C})}\mathbb{C} = H^0(W,\mathbb{C})$.

Let $\zeta_N = \exp(2\pi i/N) \in \mu_N^\circ(\mathbb{C}) = W(\mathbb{C})$. Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a nontrivial character and let

$$\lambda_\chi : H^0(W,\mathbb{C}) \to \mathbb{C}, \quad \lambda_\chi((x_\zeta)_{\zeta\in W(\mathbb{C})}) = \sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) x_{\zeta_N^a}.$$

As χ is nontrivial, λ_χ is 0 on the image of $H^0(Y,\mathbb{C})$, which is just the image of the diagonal embedding $\mathbb{C} \hookrightarrow \bigoplus_{\zeta\in W(\mathbb{C})}\mathbb{C}$. Then

$$\lambda_\chi(\eta|_W) = -2(2\pi i)^{-1} \sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} \chi(a)\log|\zeta_N^a - 1|.$$

If χ is odd (so $\chi(-1) = -1$) then the sum is 0 as $|\zeta_N^a - 1| = |\zeta_N^{-a} - 1|$. But if χ is even (so $\chi(a) = \chi(-a)$), then the sum equals

$$2(2\pi i)^{-1}\frac{N_0}{\tau(\bar{\chi}_0)} L(1,\bar{\chi}_0) \prod_{\substack{\ell \text{ prime} \\ \ell \mid N \\ \ell \nmid N_0}} (1-\chi_0(\ell))$$

by a well-known formula for the value of the Dirichlet series $L(s,\bar{\chi}_0)$ at the point $s = 1$ (see [Washington 1997, Theorem 4.9]). Here χ_0 is the primitive character associated with χ, N_0 is its conductor, $\bar{\chi}_0 = \chi_0^{-1}$, and $\tau(\bar{\chi}_0)$ is the usual Gauss sum. By the functional equation for $L(s,\chi_0)$, the last displayed expression equals

$$-(2\pi i)^{-1} 4L'(0,\chi_0) \prod_{\substack{\ell \text{ prime} \\ \ell \mid N \\ \ell \nmid N_0}} (1-\chi_0(\ell)) = -(2\pi i)^{-1} 4L'(0,\chi).$$

As noted before, $\mathcal{E}_{\mathrm{MH}}$ is a nonsplit extension of enriched Hodge structures if and only if $\lambda(\eta|_W) \neq 0$ for some $\lambda : H^0(W, \mathbb{C}) \to \mathbb{C}$ that vanishes on the image of $H^0(Y, \mathbb{C})$. Such λ are exactly the linear combinations of the λ_χ for χ running over the nontrivial characters of $(\mathbb{Z}/N\mathbb{Z})^\times$. So as a consequence of the calculation above we have:

$$\boxed{\begin{array}{c}\text{there is a nontrivial even character } \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \\ \text{such that } \mathrm{ord}_{s=0} L(s, \chi) = 1\end{array} \iff \mathcal{E}_{\mathrm{MH}} \neq 0.} \tag{4.6.a}$$

The left-hand side is satisfied, of course, if there is a primitive even character modulo N.

Suppose that χ is quadratic as well as even. Then $\mathcal{E}_{\mathrm{MH},\chi} = H^1(Y, W, \mathbb{Q})/\ker(\lambda_\chi)$ is an extension of enriched Hodge structures that fits into a commutative diagram:

$$\begin{array}{ccccc} \frac{H^0(W,\mathbb{Q})}{\mathrm{im}(H^0(Y,\mathbb{Q}))} & \hookrightarrow & H^1(Y, W, \mathbb{Q}) & \twoheadrightarrow & H^1(Y, \mathbb{Q}) \\ \downarrow{\scriptstyle \lambda_\chi} & & \downarrow{\scriptstyle /\ker(\lambda_\chi)} & & \| \\ \mathbb{Q} & \hookrightarrow & \mathcal{E}_{\mathrm{MH},\chi} & \twoheadrightarrow & \mathbb{Q}c. \end{array}$$

As $\ker(\lambda_\chi)$ is clearly stable under ϕ_∞,

$$\mathcal{E}_{\mathrm{MH},\chi} \in \mathrm{Ext}^1_{\mathbb{Q}-\mathrm{MHS}^+}(\mathbb{Q}c, \mathbb{Q}) = \mathrm{Ext}^1_{\mathbb{Q}-\mathrm{MHS}^+}(\mathbb{Q}(-1), \mathbb{Q}(0)).$$

Note that $\mathcal{E}_{\mathrm{MH},\chi}$ is just the image of $\mathcal{E}_{\mathrm{MH}}$ under the map induced by λ_χ. The calculation above shows

$$\boxed{\chi \text{ even and nontrivial, } \mathrm{ord}_{s=0} L(s, \chi) = 1 \iff \mathcal{E}_{\mathrm{MH},\chi} \neq 0.} \tag{4.6.b}$$

4.6.1. Remark. The fact that $\mathcal{E}_{\mathrm{MH},\chi}$ is split when χ is odd is consistent with the fact that $L(0, \chi) \neq 0$ for χ odd and primitive, and so we do not expect extensions.

5. Nonsplit extensions of p-adic Galois representations

We explain how arguments analogous to those in Section 4 yield statements analogous to (4.6.a) and (4.6.b) for certain extensions of p-adic Galois representations that occur in the relative étale cohomology of the pair (Y, W). Just as in the case of the extensions of mixed Hodge structures, we check that these Galois extensions are nonsplit by integrating an explicit differential representing a class in the cohomology of Y and recognizing the resulting formulas as expressions for L-values of Dirichlet characters. Only in this case the integration takes place in the context of p-adic rigid analysis, and the passage from étale cohomology to rigid geometry goes via the comparison theorems of p-adic Hodge theory. We explain how this calculation essentially computes the Bloch–Kato logarithm of these Galois extensions. We also explain how these extensions are the first layer of an Euler system. A reader with some familiarity with p-adic Hodge theory should be able to grasp the gist quickly upon reading Section 5.2 and fill in details by scanning the subsequent displayed equations. For readers less familiar with p-adic Hodge theory, we have included — much as we did in Section 4 — a brief introduction and some useful resources, particularly in Sections 5.1, 5.3, and 5.4.

Conventions. Let $\overline{\mathbb{Q}}$ be a fixed separable algebraic closure of $\mathbb{Q}$. We fix an embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, which we use to identify $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}$. For each prime ℓ we also fix a separable algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ and an embedding $\iota_\ell : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. The latter identifies $G_{\mathbb{Q}_\ell} = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ with a decomposition group of $G_\mathbb{Q}$ for the prime ℓ. We let $I_\ell \subset G_{\mathbb{Q}_\ell}$ be the inertia subgroup and $\mathrm{frob}_\ell \in G_{\mathbb{Q}_\ell}/I_\ell$ the arithmetic Frobenius element. In particular, we identify $\overline{\mathbb{Q}}$ with a subfield of $\overline{\mathbb{Q}}_p$ via ι_p. Let $\mathbb{Q}_p^{\mathrm{ur}} \subset \overline{\mathbb{Q}}_p$ be the maximal unramified extension of $\mathbb{Q}_p$.

Let $\epsilon : G_\mathbb{Q} \to \mathbb{Z}_p^\times$ be the p-adic Galois character giving the action of $G_\mathbb{Q}$ on all p-th-power roots of unity and so on $\mathbb{Z}_p(1) = \varprojlim_r \mu_{p^r}$. The exponential map $\exp : \mathbb{Z}(1) \to \mathbb{C}^\times$ identifies $\varprojlim_r (\mathbb{Z}(1) \otimes \mathbb{Z}/p^r\mathbb{Z})$ with $\varprojlim_r \mu_{p^r} = \mathbb{Z}_p(1)$. We let $\underline{\zeta} \in \mathbb{Z}_p(1)$ be the $\mathbb{Z}_p$-basis that is the image of $2\pi i \in \mathbb{Z}(1)$.

Given a variety $\mathcal{V}$ defined over $\mathbb{Q}$ we let $\overline{\mathcal{V}}$ denote its base change $\mathcal{V}_{/\overline{\mathbb{Q}}}$ over $\overline{\mathbb{Q}}$. The role of the de Rham-singular isomorphisms in the preceding section will here be played by the de Rham-étale comparison isomorphisms of p-adic Hodge theory. This essentially allows us to compare $H^1_{\text{ét}}(\overline{\mathcal{V}}, \mathbb{Q}_p)$ with $H^1_{\mathrm{dR}}(\mathcal{V}/\mathbb{Q}_p)$ (for good $\mathcal{V}$), with the additional complication that the comparison is not direct but passes through the D_{dR}-functor: for a finite-dimensional continuous $\mathbb{Q}_p$-linear $G_{\mathbb{Q}_p}$-representation M, $D_{\mathrm{dR}}(M) = (M \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_{\mathbb{Q}_p}}$, where B_{dR} is the usual de Rham period ring. The de Rham-étale comparison isomorphism is a canonical functorial isomorphism $\iota_{\mathrm{dR},p}$ of $H^1_{\mathrm{dR}}(\mathcal{V}/\mathbb{Q}_p)$ with $D_{\mathrm{dR}}(H^1_{\text{ét}}(\overline{\mathcal{V}}, \mathbb{Q}_p))$, and similarly for relative cohomology with respect to a subvariety $\mathcal{U} \subset \mathcal{V}$ (at least for $\mathcal{U}$ a normal crossings divisor or a complement of such). These isomorphisms are functorial in $\mathcal{V}$ and compatible with the long-exact sequences for relative cohomology, Gysin sequences, etc.

5.1. ***p-adic Galois representations and their period rings.*** Let $L/\mathbb{Q}_p$ be a finite extension of $\mathbb{Q}_p$ and let V be a finite-dimensional L-space equipped with a continuous L-linear action of $G_{\mathbb{Q}_p}$. Simple examples of such are the one-dimensional $\mathbb{Q}_p$-spaces $\mathbb{Q}_p(n) = (\mathbb{Z}_p(1)^{\otimes n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, on which $G_{\mathbb{Q}_p}$ acts[6] via the n-th-power ϵ^n of the p-adic cyclotomic character ϵ. For any V we write $V(n)$ for $V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$ (the n-th Tate twist of V) with $G_{\mathbb{Q}_p}$ acting on both factors. If $\chi : G_{\mathbb{Q}_p} \to L^\times$ is any continuous character, then we let $L(\chi)$ be the one-dimensional L-space with $\sigma \in G_{\mathbb{Q}_p}$ acting as multiplication by $\chi(\sigma)$. Unlike for $\mathbb{Q}_p(n)$, the representation

[6]Of course, the Galois group $G_\mathbb{Q}$ also acts on $\mathbb{Q}_p(n)$. In fact, in subsequent sections we will largely be interested in $G_{\mathbb{Q}_p}$-actions that are the restrictions of $G_\mathbb{Q}$-actions.

$L(\chi)$ has an implicit L-basis; hence, identifying $L(\chi)(n)$ with $L(\chi\epsilon^n)$ requires choosing a basis of $\mathbb{Q}_p(n)$. Other important examples of V's arise in arithmetic geometry. For a complete, geometrically connected variety $\mathcal{V}$ of dimension d defined over a subfield of $\mathbb{Q}_p$, $H^{2d}_{\text{ét}}(\mathcal{V}_{/\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is isomorphic to $\mathbb{Q}_p(-d)$. More generally, if $\mathcal{V}$ is any variety over $\mathbb{Q}_p$, then the étale cohomology groups $H^{*}_{\text{ét}}(\mathcal{V}_{/\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ (as well as relative cohomology groups) are finite-dimensional $\mathbb{Q}_p$-spaces with $\mathbb{Q}_p$-linear continuous actions of $G_{\mathbb{Q}_p}$, and all the maps in the associated exact sequences (e.g., the Gysin sequences and the sequences for relative cohomology) are maps of such representations.

There are subclasses of p-adic Galois representations that figure prominently in arithmetic geometry:

$$\left\{\begin{matrix}\text{crystalline}\\ \text{reps.}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{semistable}\\ \text{reps.}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{potentially}\\ \text{semistable}\\ \text{reps.}\end{matrix}\right\} = \left\{\begin{matrix}\text{de Rham}\\ \text{reps.}\end{matrix}\right\}.$$

Each class is characterized by a period ring $B_?$, $? =$ crys, st, or dR, respectively. These period rings are topological $\mathbb{Q}_p^{\text{ur}}$-algebras (even domains), and even a $\overline{\mathbb{Q}}_p$-algebra in the case of B_{dR}. Each is equipped with a continuous action of $G_{\mathbb{Q}_p}$ compatible with the action on $\mathbb{Q}_p^{\text{ur}}$ (on $\overline{\mathbb{Q}}_p$ in the case of B_{dR}) and such that $B_?^{G_{\mathbb{Q}_p}} = \mathbb{Q}_p$. It is always true that the $\mathbb{Q}_p$-dimension of $D_?(V) := (V \otimes_{\mathbb{Q}_p} B_?)^{G_{\mathbb{Q}_p}}$ is at most that of V, and by definition V belongs to the corresponding class for ? if and only if the $\mathbb{Q}_p$-dimension of $D_?(V)$ equals the $\mathbb{Q}_p$-dimension of V. (If $G_{\mathbb{Q}_p}$ were replaced by G_K for a general finite extension $K/\mathbb{Q}_p$, the picture would be slightly different.) If V is an L-space, then so is $D_?(V)$ and one can also check whether V belongs to the category ? by comparing dimensions over L. The ring B_{crys} is a subring of both B_{dR} and B_{st}, and if we fix a branch of the p-adic logarithm, then B_{st} can viewed as a sub-B_{crys}-algebra of B_{dR}. There is a canonical inclusion $\mathbb{Z}_p(1) \hookrightarrow B_{\text{crys}}$ and we let $\underline{t} \in B_{\text{crys}}$ be the image of $\underline{\zeta}$. The element $\underline{t}$ is invertible in B_{crys} (hence also in the other rings) and

$$D_?(\mathbb{Q}_p(n)) = \mathbb{Q}_p(\underline{\zeta}^{\otimes n} \otimes \underline{t}^{-n}).$$

So the representations $\mathbb{Q}_p(n)$ are all crystalline. Clearly then, V is crystalline (or semistable or de Rham) if and only if $V(n)$ is for some integer n. More generally, each of these classes of representations is stable under direct sums, duals, tensor products, taking subrepresentations or quotients (and hence subquotients). However, they are not closed under extensions. For more on p-adic Galois representations and these period rings the interested reader should consult [Berger 2004; 2013] or [Conrad and Brinon 2009].

Suppose $\mathcal{V}$ is a variety over $\mathbb{Q}_p$. The étale cohomology groups $H^{*}_{\text{ét}}(\mathcal{V}_{/\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ are all de Rham (equivalently, potentially semistable), as are the relative cohomology groups. If $\mathcal{V}$ has a smooth complete model over $\mathbb{Z}_p$, then $H^*_{\text{ét}}(\mathcal{V}_{/\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is crystalline. Not at all surprisingly, if $\mathcal{V}$ has a semistable model over $\mathbb{Z}_p$, then $H^*_{\text{ét}}(\mathcal{V}_{/\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is semistable.

The ring B_{dR} of de Rham periods has a natural decreasing filtration: $F^i B_{\text{dR}} = \underline{t}^i B^+_{\text{dR}}$ for a certain subring B^+_{dR} containing $\underline{t}$. This induces a finite exhaustive filtration on $D_{\text{dR}}(V)$ for any V: $F^i D_{\text{dR}}(V) = (V \otimes_{\mathbb{Q}_p} \underline{t}^i B^+_{\text{dR}})^{G_{\mathbb{Q}_p}}$. The rings B_{crys} and B_{st} are equipped with a semilinear Frobenius ϕ_p. That is, ϕ_p is a continuous endomorphism that acts semilinearly with respect to the usual (arithmetic) Frobenius frob_p on the maximal unramified extension $\mathbb{Q}_p^{\text{ur}}$ of $\mathbb{Q}_p$ (so $\phi(ax) = \text{frob}_p(a)\phi(x)$ for $a \in \mathbb{Q}_p^{\text{ur}}$ and $x \in B_{\text{crys}}, B_{\text{st}}$). The ring B_{st} also has a nilpotent endomorphism N (sometimes called a monodromy operator) such satisfying $N\phi_p = p\phi_p N$. The Frobenius ϕ_p acts on $\underline{t}$ as multiplication by p. In particular, in the case of $\mathbb{Q}_p(n)$ we have $D(n) := D_{\text{dR}}(\mathbb{Q}_p(n)) = D_{\text{crys}}(\mathbb{Q}_p(n)) = \mathbb{Q}_p(\underline{\zeta}^{\otimes n} \otimes \underline{t}^{-n})$ is a free $\mathbb{Q}_p$-space of rank one with ϕ_p acting as multiplication by p^{-n}. The filtration on $D(n)$ is such that $F^i D(n) = D(n)$ if $i \leq -n$ and $F^i D = 0$ otherwise.

Analogously to Section 4, in the rest of Section 5 we will construct an extension of $\mathbb{Q}_p(-1)$ by some $\mathbb{Q}_p(\chi)$, for χ a finite character, in the category of crystalline representations of $G_{\mathbb{Q}_p}$. We will investigate when this extension is nonsplit. Suppose then that we have a crystalline extension

$$0 \to V \to E \to \mathbb{Q}_p(m) \to 0.$$

Applying the $D_{\text{crys}}(-)$ functor we obtain an extension

$$0 \to D(V) \to D(E) \to D(m) \to 0$$

of filtered $\mathbb{Q}_p$-spaces with a $\mathbb{Q}_p$-linear action of ϕ_p. The extension E is split if and only if the extension $D(E)$ is.[7] Suppose that $F^{-m} D(V) = 0$. Let $0 \neq \omega \in D(m)$ and $\omega_H \in F^{-m} D(E)$ that maps to 0. As $F^{-m} D(E) \cap D(V) = 0$, the $\mathbb{Q}_p$-map $\phi : D(m) \to D(E)$ that takes ω to ω_H is the unique splitting of E as filtered $\mathbb{Q}_p$-spaces. It follows that the extension $D(E)$ is split if and only if $\phi(\phi_p \omega) = \phi_p \phi(\omega) = \phi_p \omega_H$. As $\phi_p \omega = p^m \omega$, this holds if and only if $\phi_p \omega_H = p^m \omega_H$, or, equivalently, $(1 - p^{-m}\phi_p)\omega_H = 0$.

In practice, we will be able to use Hodge theory to find ω_H and to compute $(1 - p^{-m}\phi_p)\omega_H$.

5.2. ***The extension $\mathcal{E}_{\mathbb{Q}_p,\text{ét}}$.*** Let $F/\mathbb{Q}_p$ be any finite extension. From the long exact sequence

$$\cdots \to H^0_{\text{ét}}(\overline{Y}, F) \to H^0_{\text{ét}}(\overline{W}, F) \to H^1_{\text{ét}}(\overline{Y}, \overline{W}, F) \to H^1_{\text{ét}}(\overline{Y}, F) \to H^1_{\text{ét}}(\overline{W}, F) \to \cdots \quad (5.2.a)$$

[7]The ‘only if’ direction is clear. The ‘if’ direction is a consequence of the equivalence of crystalline representations and *admissible* filtered ϕ_p-modules.

of étale cohomology groups we obtain an extension of $F[G_{\mathbb{Q}}]$-modules

$$0 \to \frac{H^0_{\text{ét}}(\overline{W}, F)}{\operatorname{im}(H^0_{\text{ét}}(\overline{Y}, F))} \to H^1_{\text{ét}}(\overline{Y}, \overline{W}, F) \to H^1_{\text{ét}}(\overline{Y}, F) \to 0. \tag{5.2.b}$$

Let $V_F = H^0_{\text{ét}}(\overline{W}, F)/\operatorname{im}(H^0_{\text{ét}}(\overline{Y}, F))$. As we will see, $H^1_{\text{ét}}(\overline{Y}, F) \simeq F(-1)$ as $F[G_{\mathbb{Q}}]$-modules, and there is a natural F-basis $c \in H^1_{\text{ét}}(\overline{Y}, F)$, which we will use to identify $H^1_{\text{ét}}(\overline{Y}, F)$ with $F(-1)$. Then the extension (5.2.b) yields a class $\mathcal{E}_{F,\text{ét}} = [H^1_{\text{ét}}(\overline{Y}, \overline{W}, F)] \in \operatorname{Ext}^1_{F[G_{\mathbb{Q}}]}(V_F, F(-1))$. This is just the p-adic étale analog of the extension class $\mathcal{E}$ of rational Hodge structures considered in the preceding section. As in that case, it is natural to ask:

$$\text{is } \mathcal{E}_{F,\text{ét}} \neq 0?$$

And much as before, the keys to our answer to this question are

- explicit descriptions of some classes in $H^1_{\text{ét}}(\overline{Y}, F)$ and the action of $G_{\mathbb{Q}}$ on these classes,
- the action of a p-adic Frobenius ϕ_p on the de Rham versions of the cohomology groups in (4.2.b) and its action on the de Rham realizations of the explicit classes, and
- the reduction, via p-adic Hodge theory, to a p-adic analytic calculation with the de Rham realizations of the explicit classes and their images under ϕ_p.

These combine to provide an answer to the question about the nonvanishing of $\mathcal{E}_{F,\text{ét}}$ much in the same way that their real and complex analogs answered the question about the nonvanishing of $\mathcal{E}_{\text{MH}}$.

5.3. ***The étale cohomology of $\overline{Y}$.*** In the long exact sequence for the relative étale cohomology for the (open) inclusion $Y \subset X$,

$$\cdots \to H^1_{\text{ét}}(\overline{X}, F) \to H^1_{\text{ét}}(\overline{Y}, F) \xrightarrow{\partial_{\text{ét}}} H^2_{\text{ét}}(\overline{X}, \overline{Y}, F) \\ \to H^2_{\text{ét}}(\overline{X}, F) \to H^2_{\text{ét}}(Y, F) \to \cdots, \tag{5.3.a}$$

the group $H^2_{\text{ét}}(\overline{X}, \overline{Y}, F)$ is naturally identified with the space $H^0_{\text{ét}}(\bar{Z}, F(-1)) = \bigoplus_{z \in \underline{Z}(\overline{\mathbb{Q}})} F \otimes \underline{\zeta}^\vee$. This identification is such that the induced map $H^0_{\text{ét}}(\bar{Z}, F) \to H^2_{\text{ét}}(\overline{X}, F(1)) = F$ is just the cycle class map. It follows that

$$\partial_{\text{ét}} : H^1_{\text{ét}}(\overline{Y}, F) \xrightarrow{\sim} \left\{ (a_z \otimes \underline{\zeta}^\vee)_{z \in Z(\overline{\mathbb{Q}})} : a_z \in F, \sum_{z \in Z(\overline{\mathbb{Q}})} a_z = 0 \right\} \\ \subset H^0_{\text{ét}}(\bar{Z}, F(-1)). \tag{5.3.b}$$

In particular, as $Z(\overline{\mathbb{Q}}) = \{\infty, 1\}$, $\partial_{\text{ét}} : H^1_{\text{ét}}(\overline{Y}, F) \xrightarrow{\sim} \{(a \otimes \underline{\zeta}^\vee, -a \otimes \underline{\zeta}^\vee) : a \in F\} \simeq F$. The action of $G_{\mathbb{Q}}$ on $H^1_{\text{ét}}(\overline{Y}, F)$ is easily read off from this: The Galois action on

$H^0_{\text{ét}}(\bar{Z}, F(-1))$ is just given by $\sigma(a)_z = \epsilon(\sigma)^{-1} a_{\sigma^{-1}(z)} \otimes \underline{\zeta}^\vee$ for $a = (a_z \otimes \underline{\zeta}^\vee)_{z \in Z(\overline{\mathbb{Q}})}$ and $\sigma \in G_{\mathbb{Q}}$. Since the points of Z are defined over $\mathbb{Q}$, this shows that $H^1_{\text{ét}}(\overline{Y}, F) \simeq F(-1)$ as an $F[G_{\mathbb{Q}}]$-module.

Let $c_{\text{ét}} \in H^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_p)$ be the class corresponding under $\partial_{\text{ét}}$ to

$$(c_\infty, c_1) = (1 \otimes \underline{\zeta}^\vee, -1 \otimes \underline{\zeta}^\vee).$$

Then

$$\sigma c_{\text{ét}} = \epsilon^{-1}(\sigma) c_{\text{ét}}, \quad \sigma \in G_{\mathbb{Q}}, \tag{5.3.c}$$

and $H^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_p) = \mathbb{Q}_p c_{\text{ét}} \simeq \mathbb{Q}_p(-1)$.

The singular-étale comparison isomorphisms $\iota_{\text{ét}}$ identify the sequence (4.2.a) with (5.2.a) and the isomorphism (4.3.b) with (5.3.b) (with $(2\pi i)^{-1}$ being identified with $1 \otimes \underline{\zeta}^\vee$). It follows that

$$\iota_{\text{ét}}(c) = c_{\text{ét}}.$$

However, this is not needed in the following.

5.4. *D_{dR} and the Frobenius ϕ_p.* The étale cohomology groups in (5.2.a) are all de Rham representations of $G_{\mathbb{Q}_p}$. In particular, applying the D_{dR}-functor yields a commutative diagram

$$\begin{array}{ccccc}
\dfrac{H^0_{\mathrm{dR}}(W/\mathbb{Q}_p)}{\mathrm{im}(H^0_{\mathrm{dR}}(Y/\mathbb{Q}_p))} & \hookrightarrow & H^1_{\mathrm{dR}}((Y, W)/\mathbb{Q}_p) & \xrightarrow{\alpha_{\mathrm{dR}}} & H^1_{\mathrm{dR}}(Y/\mathbb{Q}_p) \\
\Big\| \iota_{\mathrm{dR},p} & & \Big\| \iota_{\mathrm{dR},p} & & \Big\| \iota_{\mathrm{dR},p} \\
\dfrac{D_{\mathrm{dR}}(H^0_{\text{ét}}(\overline{W}, \mathbb{Q}_p))}{\mathrm{im}(D_{\mathrm{dR}}(H^0_{\text{ét}}(\overline{Y}, \mathbb{Q}_p))} & \hookrightarrow & D_{\mathrm{dR}}(H^1_{\text{ét}}(\overline{Y}, \overline{W}, \mathbb{Q}_p)) & \xrightarrow{\alpha_{\text{ét}}} & D_{\mathrm{dR}}(H^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_p)),
\end{array} \tag{5.4.a}$$

where the vertical arrows are the de Rham comparison isomorphisms of p-adic Hodge theory. These spaces are all filtered $\mathbb{Q}_p$-spaces: the Hodge filtration on the top line is identified with the filtration induced from the filtration $\underline{t}^i B^+_{\mathrm{dR}}$ on B_{dR} on the bottom line. All the maps are morphisms of filtered $\mathbb{Q}_p$-spaces.

A splitting of the extension (5.2.b) would give a splitting of the bottom line of this diagram, and hence a splitting of the top, as filtered $\mathbb{Q}_p$-spaces. We will show that this does not happen in general, at least if we also take into account the action of an additional operator on these spaces — a Frobenius operator ϕ_p (which replaces ϕ_∞ in this p-adic context). The splittings would also be splittings for the action of ϕ_p, and we will show that such splittings do not exist when certain values of p-adic L-functions are nonzero.

To explain what ϕ_p is and illustrate its role, we make the simplifying hypothesis that

$$p \nmid N. \tag{5.4.b}$$

The varieties X, Y, and Z have smooth models over $\mathbb{Z}$—just replace $\mathbb{Q}$ with $\mathbb{Z}$—and W has a smooth model over $\mathbb{Z}[\frac{1}{N}]$—just replace $\mathbb{Q}$ with $\mathbb{Z}[\frac{1}{N}]$. Hence they also all have smooth models $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$, and $\mathscr{W}$ over $\mathbb{Z}_p$ under (5.4.b). The inclusions $W \hookrightarrow Y \hookrightarrow X$ and $Z \hookrightarrow X$ extend to these models. This implies that the cohomology groups in (5.2.b) are all crystalline representations of $G_{\mathbb{Q}_p}$, and so D_{dR} can be replaced with the crystalline functor D_{crys} in the bottom line of (5.4.a). The modules $D_{\mathrm{crys}}(-) = (- \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_{\mathbb{Q}_p}}$ inherit a $\mathbb{Q}_p$-linear action of the crystalline Frobenius ϕ_p from B_{crys}. In particular, if $\mathcal{E}_{\mathbb{Q}_p,\text{ét}}$ were 0 after restriction to $G_{\mathbb{Q}_p}$ then the bottom line in (5.4.a) would be simultaneously split as an extension of filtered $\mathbb{Q}_p$-spaces and as an extension of $\mathbb{Q}_p[\phi_p]$-modules.

From the Galois action (5.3.c) we see that

$$c_{\mathrm{crys}} = c_{\text{ét}} \otimes \underline{t} \in D_{\mathrm{crys}}(H^1(\overline{Y}, \mathbb{Q}_p)) = D_{\mathrm{dR}}(H^1(\overline{Y}, \mathbb{Q}_p))$$

is a $\mathbb{Q}_p$-basis of $D_{\mathrm{crys}}(H^1(\overline{Y}, \mathbb{Q}_p)) = D_{\mathrm{crys}}(\mathbb{Q}_p c)$. As ϕ_p acts on $\underline{t}$ as multiplication by p, it follows that

$$\phi_p c_{\mathrm{crys}} = p c_{\mathrm{crys}}. \tag{5.4.c}$$

Noting that $\omega = \frac{dt}{1-t} \in H^0(\Omega^1_{X/\mathbb{Q}_p}(\log Z))$, we let

$$c_{\mathrm{dR}} = [\omega] \in F^1 H^1_{\mathrm{dR}}(Y/\mathbb{Q}_p) \ \text{ and } \ c_{\mathrm{dR},H} = [\omega]_W \in F^1 H^1_{\mathrm{dR}}((Y, W)/\mathbb{Q}_p).$$

The de Rham comparison isomorphisms of p-adic Hodge theory are compatible with the boundary map in the sequence (5.3.a), in the sense that

$$H^1_{\mathrm{dR}}(Y/\mathbb{Q}_p) \overset{\iota_{\mathrm{dR},p}}{=} D_{\mathrm{dR}}(H^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_p)) \xrightarrow{\partial_{\text{ét}} \otimes \mathrm{id}} D_{\mathrm{dR}}(H^0_{\text{ét}}(\overline{Z}, \mathbb{Q}_p(-1))) \overset{\iota_{\mathrm{dR},p}}{=} H^0_{\mathrm{dR}}(Z/\mathbb{Q}_p)(-1)$$

is just the boundary map (the residue map) in the corresponding sequence for de Rham cohomology. As $\underline{\zeta}^\vee \otimes \underline{t}$ is identified with 1 by $\iota_{\mathrm{dR},p}$, it follows that

$$\iota_{\mathrm{dR},p}(c_{\mathrm{dR}}) = c_{\mathrm{crys}},$$

and (5.4.c) shows[8] that the induced action of ϕ_p on c_{dR} is just

$$\phi_p c_{\mathrm{dR}} = p c_{\mathrm{dR}}. \tag{5.4.d}$$

This implies that $(1 - p^{-1}\phi_p)c_{\mathrm{dR},H} \in H^1_{\mathrm{dR}}((Y, W)/\mathbb{Q}_p)$ is the image of something in $H^0_{\mathrm{dR}}(W/\mathbb{Q}_p)$. As $c_{\mathrm{dR},H} \in F^1 H^1_{\mathrm{dR}}((Y, W)/\mathbb{Q}_p)$ and $c_{\mathrm{dR}} \in F^1 H^1_{\mathrm{dR}}(Y/\mathbb{Q}_p)$ and since $\alpha_{\mathrm{dR}} : F^1 H^1_{\mathrm{dR}}((Y, W)/\mathbb{Q}_p) \xrightarrow{\sim} F^1 H^1_{\mathrm{dR}}(Y/\mathbb{Q}_p)$, this 'something' is nonzero modulo the image of $H^0_{\mathrm{dR}}(Y/\mathbb{Q}_p)$ if and only if the bottom of (5.4.a) is a nonsplit extension of filtered $\mathbb{Q}_p$-spaces equipped with a $\mathbb{Q}_p[\phi_p]$-module structure.

[8]This also follows as the spaces being compared are one-dimensional, but this argument works in more general settings.

Ideally there would be $\omega' \in H^0(\Omega^1_{X/\mathbb{Q}_p}(\log Z))$ such that $(1 - p^{-1}\phi_p)c_{\mathrm{dR},H} = [\omega']_W$. As $0 = [\omega'] \in H^1_{\mathrm{dR}}(Y/\mathbb{Q}_p)$, it would have to be that $\omega' = d\eta$ for some $\eta \in H^0(Y, \mathcal{O}_{Y/\mathbb{Q}_p})$ and hence that $(1 - p^{-1}\phi_p)c_{\mathrm{dR},H}$ is the image of $\eta|_W$. The nonvanishing of this image (and so of the class $\mathcal{E}_{F,\text{ét}}|_{G_{\mathbb{Q}_p}}$) would be equivalent to $\lambda(\eta|_W) \neq 0$ for some $\mathbb{Q}_p$-homomorphism $\lambda : H^0_{\mathrm{dR}}(W/\mathbb{Q}_p) \to \overline{\mathbb{Q}}_p$ that vanishes on the image of $H^0_{\mathrm{dR}}(Y/\mathbb{Q}_p)$. Unfortunately, this ideal situation does not hold in general. However, we can essentially realize it by passing from algebraic de Rham cohomology to another cohomology theory, one where the whole of the cohomology group $H^1_{\mathrm{dR}}((Y, W)/\mathbb{Q}_p)$ can be represented by differentials, much as $H^1_{\mathrm{dR}}((Y, W)/\mathbb{C})$ can be represented by real analytic differentials.

5.5. ***Monsky–Washnitzer cohomology.*** Let W, X, Y, etc., be the special fibers of $\mathscr{W}$, $\mathscr{X}$, $\mathscr{Y}$, etc. The de Rham cohomology groups on the top line of (5.4.a) are naturally identified with the corresponding Monsky–Washnitzer (MW) cohomology of the corresponding special fibers, yielding a commutative diagram

$$\begin{array}{ccccc} \dfrac{H^0_{\mathrm{dR}}(W/\mathbb{Q}_p)}{\mathrm{im}(H^0_{\mathrm{dR}}(Y/\mathbb{Q}_p))} & \hookrightarrow & H^1_{\mathrm{dR}}((Y, W)/\mathbb{Q}_p) & \xrightarrow{\alpha_{\mathrm{dR}}} & H^1_{\mathrm{dR}}(Y/\mathbb{Q}_p) \\ \Vert & & \Vert & & \Vert \\ \dfrac{H^0_{\mathrm{MW}}(\mathsf{W},\mathbb{Q}_p)}{\mathrm{im}(H^0_{\mathrm{MW}}(\mathsf{Y},\mathbb{Q}_p))} & \hookrightarrow & H^1_{\mathrm{MW}}(\mathsf{Y}, \mathsf{W}, \mathbb{Q}_p) & \xrightarrow{\alpha_{\mathrm{MW}}} & H^1_{\mathrm{MW}}(\mathsf{Y}, \mathbb{Q}_p). \end{array} \tag{5.5.a}$$

The MW cohomology groups are defined as follows. Let

$$A_0^\dagger = \mathbb{Z}_p\langle t, x\rangle^\dagger/((t-1)x-1)$$

be the weak completion of $A_0 = \mathbb{Z}_p\big[t, \frac{1}{t-1}\big]$ and let

$$\Omega^1_{A_0^\dagger} = (A_0^\dagger dt + A_0^\dagger dx)/A_0^\dagger(x\,dt + (t-1)dx)$$

be the module of continuous differentials. Here $\mathbb{Z}_p\langle t, x\rangle^\dagger$ consists of the power series $\sum_{n,m=0}^{\infty} a_{n,m}t^n x^m$, $a_{n,m} \in \mathbb{Z}_p$, for which there exists a constant $C > 0$ and a real number $0 < \rho < 1$ such that $|a_{n,m}|_p \le C\rho^{n+m}$ for all n, m. Let $A^\dagger = A_0^\dagger \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and

$$\Omega^1_{A^\dagger} = \Omega^1_{A_0^\dagger} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Then the cohomology group $H^1_{\mathrm{MW}}(\mathsf{Y}, \mathbb{Q}_p)$ is canonically computed by the cohomology of the complex $DR_Y^\dagger = [A^\dagger \xrightarrow{d} \Omega^1_{A^\dagger}]$. Similarly, $H^1_{\mathrm{MW}}(\mathsf{Y}, \mathsf{W}, \mathbb{Q}_p)$ is computed by the cohomology of the complex $DR^\dagger_{\mathsf{Y}}(-\mathsf{W}) = [\Phi_N(t)A^\dagger \to \Omega^1_{A^\dagger}]$, and so

$$H^1_{\mathrm{MW}}(\mathsf{Y}, \mathbb{Q}_p) = \Omega^1_{A^\dagger}/dA^\dagger \quad \text{and} \quad H^1_{\mathrm{MW}}(\mathsf{Y}, \mathsf{W}, \mathbb{Q}_p) = \Omega^1_{A^\dagger}/d(\Phi_N(t)A^\dagger).$$

The maps between the top and bottom rows of (5.5.a) are induced by the obvious maps of complexes $DR_Y \to DR^\dagger_{\mathsf{Y}}$ and $DR_Y(-W) \to DR^\dagger_{\mathsf{Y}}(-\mathsf{W})$. In particular, the map from $F^1H^1_{\mathrm{dR}}(Y/\mathbb{Q}_p) = \Omega^1_A/dA$ to $H^1_{\mathrm{MW}}(\mathrm{Y}, \mathbb{Q}_p)$ is just the obvious one, and similarly for $F^1H^1_{\mathrm{dR}}((Y, W)/\mathbb{Q}_p) = \Omega^1_A/d(\Phi_N(t)A)$.

The Monsky–Washnitzer cohomology groups are also equipped with a canonical Frobenius action induced by any homomorphism $F_p : A_0^\dagger \to A_0^\dagger$ that reduces mod p to the usual Frobenius map on $A_0/pA_0 = A_0^\dagger/pA_0^\dagger$. In this case there is a unique such F_p that sends t to t^p. The canonical Frobenius action on the Monsky–Washnitzer cohomology groups agrees with the Frobenius action ϕ_p on the de Rham cohomology groups. For more on Monsky–Washnitzer cohomology and the various objects introduced above, the interested reader should see [van der Put 1986].

Our problem now becomes one of finding an explicit $\eta \in A^\dagger$ such that $d\eta = (1 - p^{-1}F_p^*)\omega$. To do this we enlarge the class of functions we are working with.

5.6. ***Coleman integration and the calculation.*** The ring $A^\dagger$ is a subring of the rigid analytic functions on the affinoid $Y_{\mathrm{an}} = \mathrm{spm}(\mathcal{A})$ for $\mathcal{A} = \mathbb{Q}_p\langle t, x\rangle/((t-1)x - 1))$, where $\mathbb{Q}_p\langle t, x\rangle$ is the standard Tate algebra. The geometric points of Y_{an} comprise the set

$$Y_{\mathrm{an}}(\overline{\mathbb{Q}}_p) = \{t \in \mathcal{O}_{\overline{\mathbb{Q}}_p} : |t-1|_p = 1\},$$

for $\mathcal{O}_{\overline{\mathbb{Q}}_p}$ the ring of integers of $\overline{\mathbb{Q}}_p$. That is, $Y_{\mathrm{an}}(\overline{\mathbb{Q}}_p)$ is $\mathcal{O}_{\overline{\mathbb{Q}}_p}$ with the open disc of radius 1 around 1 removed. The above identification just sends a $\mathbb{Q}_p$-homomorphism $\mathcal{A} \twoheadrightarrow \mathcal{A}/\mathfrak{m} \hookrightarrow \overline{\mathbb{Q}}_p$, $\mathfrak{m} \in \mathrm{spm}(\mathcal{A})$, to the image of t under this homomorphism. The ring $A^\dagger$ is then identified with a subring of the locally analytic functions $\mathcal{A}_{\mathrm{loc}}$ on Y_{an} over $\overline{\mathbb{Q}}_p$, where $\mathcal{A}_{\mathrm{loc}}$ is the ring of $\overline{\mathbb{Q}}_p$-valued functions $f(t)$ on the set $\{t \in \mathcal{O}_{\overline{\mathbb{Q}}_p} : |t-1|_p = 1\}$ such that (i) $\sigma(f(t)) = f(\sigma(t))$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ for some finite extension $L/\mathbb{Q}_p$ and (ii) on some open disc $\{t \in \mathcal{O}_{\overline{\mathbb{Q}}_p} : |t - t_0|_p < \epsilon\}$ around each point t_0, $f(t)$ is equal to a convergent power series in $t - t_0$. There is an obvious notion of locally analytic differentials on $Y_{\mathrm{an}}(\overline{\mathbb{Q}}_p)$ over $\overline{\mathbb{Q}}_p$ and we denote the $\mathcal{A}_{\mathrm{loc}}$-module of such by $\Omega^1_{\mathcal{A}_{\mathrm{loc}}}$. There is also an induced embedding $\Omega^1_{A^\dagger} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \hookrightarrow \Omega^1_{\mathcal{A}_{\mathrm{loc}}}$, which is compatible with the differentials

$$d : A^\dagger \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \to \Omega^1_{A^\dagger} \quad \text{and} \quad d : \mathcal{A}_{\mathrm{loc}} \to \Omega^1_{\mathcal{A}_{\mathrm{loc}}}.$$

We will make use of Coleman integration (see [Besser 2012]), which is a $\overline{\mathbb{Q}}_p$-linear map $\int : \Omega^1_{A^\dagger} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \to \mathcal{A}_{\mathrm{loc}}/\overline{\mathbb{Q}}_p$, to determine η:

$$\eta = \int (1 - p^{-1}F_p^*)\omega \in A_{\mathrm{loc}}/\overline{\mathbb{Q}}_p.$$

Note that η is only well-defined up to the addition of a constant, an ambiguity that does not affect the value $\lambda(\eta|_W)$.

The Frobenius F_p on $\mathcal{A}^\dagger$ is the restriction of $F_{p,\mathrm{loc}} : Y_{\mathrm{an}}(\overline{\mathbb{Q}}_p) \to Y_{\mathrm{an}}(\overline{\mathbb{Q}}_p)$, $t \mapsto t^p$, in the sense that $(F_p f)(t) = f(t^p)$ for $f \in A^\dagger$. The theory of Coleman integration provides a unique $\overline{\mathbb{Q}}_p$-linear map $\int : \Omega^1_{A^\dagger} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \to \mathcal{A}_{\mathrm{loc}}/\overline{\mathbb{Q}}_p$ such that (a) $d \circ \int : \Omega^1_{A^\dagger} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \hookrightarrow \Omega^1_{\mathcal{A}_{\mathrm{loc}}}$ and $\int \circ d : A^\dagger \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \to \mathcal{A}_{\mathrm{loc}}/\overline{\mathbb{Q}}_p$ are the canonical maps, and (b) $F_p^* \circ \int = \int \circ F_p^*$. The condition (b) actually holds for *any* lift F_p of the Frobenius map. Not surprisingly, it is relatively straightforward to show that $\int \frac{dt}{t-1} = \log_p(1-t)$, where $\log_p$ is the usual Iwasawa branch of the p-adic logarithm (so $\log_p(p) = 0$); see [Besser 2012, §1.2]. It then follows from (b) that

$$\eta = -\log_p(1-t) + p^{-1} \log_p(1-t^p) \in \mathcal{A}_{\mathrm{loc}}/\overline{\mathbb{Q}}_p.$$

Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$ be any nontrivial character. Then

$$\lambda_{\chi,\mathrm{dR}} : H^0_{\mathrm{dR}}(W/\overline{\mathbb{Q}}_p) \to \mathbb{Q}_p, \quad \lambda_{\chi,\mathrm{dR}}((x_\zeta)_{\zeta \in W(\overline{\mathbb{Q}}_p)}) = \frac{1}{\tau(\chi_0)} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) x_{\zeta_N^a},$$

is 0 on the image of $H^0_{\mathrm{dR}}(Y/\overline{\mathbb{Q}}_p)$, which is the image of the diagonal embedding $\overline{\mathbb{Q}}_p \hookrightarrow \bigoplus_{\zeta \in W(\overline{\mathbb{Q}}_p)} \overline{\mathbb{Q}}_p$. Here χ_0 is the primitive Dirichlet character associated to χ and $\tau(\chi_0) = \sum_{a \in (\mathbb{Z}/N_0\mathbb{Z})^\times} \chi_0(a) \zeta_{N_0}^a$ is its usual Gauss sum. Then

$$\begin{aligned} \lambda_{\chi,\mathrm{dR}}(\eta|_W) &= -\frac{1}{\tau(\chi_0)} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) \eta(\zeta_N^a) \\ &= -\frac{1}{\tau(\chi_0)} (1 - \bar{\chi}(p) p^{-1}) \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) \log_p(1 - \zeta_N^a). \end{aligned}$$

If χ is odd (so $\chi(-a) = -\chi(a)$), then the last sum vanishes, as $\log_p(1 - \zeta_N^{-a}) = \log_p(-\zeta_N^{-a}(1 - \zeta_N^a)) = \log_p(1 - \zeta_N^a)$. But if χ is even (so $\chi(a) = \chi(-a)$), then the sum equals

$$L_p(1, \bar{\chi}_0) \prod_{\substack{\ell \text{ prime} \\ \ell \mid N \\ \ell \nmid N_0}} (1 - \chi_0(\ell))$$

by a well-known formula for the value of the p-adic Dirichlet L-function $L_p(s, \bar{\chi}_0)$ at the point $s = 1$ (see [Washington 1997, Theorem 5.18]). Here, as before, N_0 is the conductor of χ_0. As $L_p(1, \bar{\chi}_0) \neq 0$ (see [Washington 1997, Corollary 5.30]) we see — just as in the complex case — that $\lambda_{\chi,\mathrm{dR}}(\eta|_W)$ is nonzero if and only if $\chi_0(\ell) \neq 1$ for all $\ell \mid N$, $\ell \nmid N_0$. And, also as before, this is equivalent to $\mathrm{ord}_{s=0} L(s, \chi) = 1$. Hence the nonvanishing of $\lambda_{\chi,\mathrm{dR}}(\eta|_W)$ also agrees with $\mathrm{ord}_{s=0} L(s, \chi) = 1$.

As noted before, $\mathcal{E}_{\mathbb{Q}_p,\text{ét}}|_{G_{\mathbb{Q}_p}}$ is a nonsplit extension of p-adic Galois representations if and only if $\lambda(\eta|_W) \neq 0$ for some nonzero $\lambda : H^0_{\mathrm{dR}}(W/\mathbb{Q}_p) \to \overline{\mathbb{Q}}_p$ that vanishes on the image of $H^0_{\mathrm{dR}}(Y/\mathbb{Q}_p)$. Such λ are exactly the nonzero linear combinations of the $\lambda_{\mathrm{dR},\chi}$ for χ running over the nontrivial characters of $(\mathbb{Z}/N\mathbb{Z})^\times$.

So as a consequence of the calculation above we have:

$$\boxed{\begin{array}{c}\text{there exists a nontrivial even character}\\ \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times\\ \text{such that } \operatorname{ord}_{s=0} L(s,\chi) = 1\end{array} \iff \mathcal{E}_{\mathbb{Q}_p,\text{ét}}|_{G_{\mathbb{Q}_p}} \neq 0.} \tag{5.6.a}$$

The left-hand side is, of course, satisfied if there is a primitive even character modulo N.

Just as in the case of extensions of Hodge structures, this can be refined. Suppose χ is $\mathbb{Q}_p^\times$-valued (which holds, for example, if $\phi(N) \mid (p-1)$). Then

$$\lambda_{\chi,\text{ét}} : H^0_{\text{ét}}(\overline{W}, \mathbb{Q}_p) \to \mathbb{Q}_p(\chi), \quad \lambda_{\chi,\text{ét}}((x_\zeta)_{\zeta\in W(\overline{\mathbb{Q}})}) = \sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) x_{\zeta_N^a},$$

is a $\mathbb{Q}_p[G_\mathbb{Q}]$-module homomorphism. Here we view $\mathbb{Q}_p(\chi)$ as $\mathbb{Q}_p$ but with $G_\mathbb{Q}$ action via the Galois character χ. So $1 \in \mathbb{Q}_p(\chi)$ is a $\mathbb{Q}_p$-basis and $\sigma\cdot 1 = \chi(\sigma)\cdot 1 = \chi(\sigma)$. It follows that $\mathcal{E}_{\chi,\text{ét}} = H^1_{\text{ét}}(\overline{Y}, \overline{W}, \mathbb{Q}_p)/\ker(\lambda_\chi)$ is an extension of $\mathbb{Q}_p[G_\mathbb{Q}]$-modules that fits into a commutative diagram:

$$\begin{array}{ccccc}
\frac{H^0_{\text{ét}}(\overline{W},\mathbb{Q}_p)}{\operatorname{im}(H^0_{\text{ét}}(\overline{Y},\mathbb{Q}_p))} & \hookrightarrow & H^1_{\text{ét}}(\overline{Y}, \overline{W}, \mathbb{Q}_p) & \twoheadrightarrow & H^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_p)\\
\downarrow \lambda_{\chi,\text{ét}} & & \downarrow /\ker(\lambda_{\chi,\text{ét}}) & & \|\\
\mathbb{Q}_p(\chi) & \hookrightarrow & \mathcal{E}_{\chi,\text{ét}} & \twoheadrightarrow & \mathbb{Q}_p c_{\text{ét}} = \mathbb{Q}_p(-1).
\end{array} \tag{5.6.b}$$

In particular, $\mathcal{E}_{\chi,\text{ét}} \in \operatorname{Ext}^1_{\mathbb{Q}_p[G_\mathbb{Q}]}(\mathbb{Q}(\chi), \mathbb{Q}c_{\text{ét}}) = \operatorname{Ext}^1_{\mathbb{Q}_p[G_\mathbb{Q}]}(\mathbb{Q}_p(\chi), \mathbb{Q}_p(-1))$. The calculation above shows that

$$\boxed{\chi \text{ even and nontrivial, } \operatorname{ord}_{s=0} L(s,\chi) = 1 \iff \mathcal{E}_{\chi,\text{ét}}|_{G_{\mathbb{Q}_p}} \neq 0.} \tag{5.6.c}$$

5.6.1. Remark. The fact that $\mathcal{E}_{\chi,\text{ét}}|_{G_{\mathbb{Q}_p}} = 0$ if χ is odd is consistent with the fact that $L(0,\chi) \neq 0$ for χ odd and primitive, and so we do not expect extensions.

5.6.2. Remark. A careful reader may have noted that the definitions of $\lambda_{\chi,\text{dR}}$ and $\lambda_{\chi,\text{ét}}$ differ by a factor of $\tau(\chi_0)$. This difference is partly explained by the commutativity of

$$\begin{array}{ccc}
H^0_{\text{dR}}(W/\mathbb{Q}_p) & \xrightarrow{\iota_{\text{dR},p}} & D_{\text{dR}}(H^0_{\text{ét}}(\overline{W}, \mathbb{Q}_p))\\
\downarrow \lambda_{\chi,\text{dR}} & & \downarrow \lambda_{\chi,\text{ét}}\otimes\text{id}\\
\mathbb{Q}_p & \xrightarrow{a\mapsto a(1\otimes\tau(\chi_0))} & D_{\text{dR}}(\mathbb{Q}_p(\chi)).
\end{array}$$

Note that $D_{\text{dR}}(\mathbb{Q}_p(\chi)) = \mathbb{Q}_p(1\otimes\tau(\chi_0)) \subset \mathbb{Q}_p(\chi)\otimes_{\mathbb{Q}_p} B_{\text{dR}}$. This relation figures into the derivation of the expression for the Bloch–Kato logarithm given in the supplement below.

5.6.3. Remark. The careful reader may also have noted that we have not fully succeeded in avoiding special units: the formula for $L_p(1, \bar{\chi}_0)$ involves p-adic logs of what are essentially cyclotomic units (and similarly for $L(1, \bar{\chi}_0)$ in the Hodge case). But even this can be avoided by working with modular curves in place of the projective line, as explained in the example from Section 6.1 below.

5.7. ***Vista: the Bloch–Kato logarithm.*** The extension $\mathcal{E}_{\chi,\text{ét}}$ determines a class $z_\chi \in H^1(\mathbb{Q}, \mathbb{Q}_p(\chi\epsilon))$ as follows: Take the 1-Tate twist of the extension $\mathbb{Q}_p(\chi) \hookrightarrow \mathcal{E}_{\chi,\text{ét}} \twoheadrightarrow \mathbb{Q}_p(-1)\ (= \mathbb{Q}_p c_{\text{ét}})$. This gives an extension

$$\mathbb{Q}_p(\chi\epsilon) \hookrightarrow \mathcal{E}_{\chi,\text{ét}}(1) \twoheadrightarrow \mathbb{Q}_p\ (= \mathbb{Q}_p(c_{\text{ét}} \otimes \underline{\zeta})).$$

Here we have identified $\mathbb{Q}_p(\chi)(1)$ with $\mathbb{Q}_p(\chi\epsilon)$ using the basis $1 \otimes \underline{\zeta} \in \mathbb{Q}_p(\chi)(1)$. Let $\tilde{c} \in \mathcal{E}_{\chi,\text{ét}}(1)$ be any element mapping to $c_{\text{ét}} \otimes \underline{\zeta}$. Then z_χ is just the class of the 1-cocycle $\sigma \mapsto \sigma\tilde{c} - \tilde{c}$. The class z_χ is just the image of $c_{\text{ét}} \otimes \underline{\zeta}$ under the boundary map $\mathbb{Q}_p(c_{\text{ét}} \otimes \underline{\zeta}) \to H^1(\mathbb{Q}, \mathbb{Q}_p(\chi\epsilon))$ of the long exact cohomology sequence associated with the short exact sequence displayed above.

Assuming (5.4.b), we showed that the restriction of $\mathcal{E}_{\chi,\text{ét}}$ to $G_{\mathbb{Q}_p}$ is nontrivial, provided some value of a p-adic L-function is nonzero. This nontriviality is equivalent to $\text{loc}_p(z_\chi) \in H^1(\mathbb{Q}_p, \mathbb{Q}_p(\chi\epsilon))$ being nonzero. As the extension $\mathcal{E}_\chi|_{G_{\mathbb{Q}_p}}$ is a crystalline extension, so is its 1-Tate twist. Hence $\text{loc}_p(z_\chi)$ belongs to the Bloch–Kato subspace

$$H^1_f(\mathbb{Q}_p, \mathbb{Q}_p(\chi\epsilon)) = \ker\bigl\{H^1(\mathbb{Q}_p, \mathbb{Q}_p(\chi\epsilon)) \to H^1(\mathbb{Q}_p, \mathbb{Q}_p(\chi\epsilon) \otimes_{\mathbb{Q}_p} B_{\text{crys}})\bigr\}.$$

This group is computed by the extended Bloch–Kato exponential

$$\widetilde{\exp}_{\text{BK}} : \frac{D_{\text{crys}}(\mathbb{Q}_p(\epsilon\chi)) \oplus \bigl(D_{\text{dR}}(\mathbb{Q}_p(\chi\epsilon))/F^0 D_{\text{dR}}(\mathbb{Q}_p(\chi\epsilon))\bigr)}{\bigl\{\bigl((1-\phi_p)x, x \bmod F^0 D_{\text{dR}}(\mathbb{Q}_p(\chi\epsilon))\bigr) : x \in D_{\text{crys}}(\mathbb{Q}_p(\chi\epsilon))\bigr\}} \xrightarrow{\sim} H^1_f(\mathbb{Q}_p, \mathbb{Q}_p(\chi\epsilon)),$$

which is a boundary map in the long-exact sequence of $G_{\mathbb{Q}_p}$-cohomology for the tensor product over $\mathbb{Q}_p$ of $\mathbb{Q}_p(\chi\epsilon)$ with the short exact sequence $\mathbb{Q}_p \hookrightarrow B_{\text{crys}} \twoheadrightarrow B_{\text{crys}} \oplus (B_{\text{dR}}/B^+_{\text{dR}})$, the last arrow being $x \mapsto ((1-\phi_p x), x \bmod B^+_{\text{dR}})$. The inverse of this is the Bloch–Kato logarithm. As $\mathbb{Q}_p(\chi\epsilon)$ is a crystalline representation of $G_{\mathbb{Q}_p}$ (assuming (5.4.b)), $D_{\text{crys}}(\mathbb{Q}_p(\chi\epsilon)) = D_{\text{dR}}(\mathbb{Q}_p(\chi\epsilon))$, so the restriction of $\widetilde{\exp}_{\text{BK}}$ to the $D_{\text{crys}}(\mathbb{Q}_p(\epsilon\chi))$ summand induces an isomorphism

$$\widetilde{\exp}_{\text{BK}} : \frac{D_{\text{crys}}(\mathbb{Q}_p(\epsilon\chi))}{\bigl\{(1-\phi_p)x : x \in F^0 D_{\text{crys}}(\mathbb{Q}_p(\chi\epsilon))\bigr\}} \xrightarrow{\sim} H^1_f(\mathbb{Q}_p, \mathbb{Q}_p(\chi\epsilon)).$$

In this particular case, $F^0 D_{\text{crys}}(\mathbb{Q}_p(\chi\epsilon)) = F^0 D_{\text{dR}}(\mathbb{Q}_p(\chi\epsilon)) = 0$, so we have

$$\widetilde{\exp}_{\text{BK}} : D_{\text{crys}}(\mathbb{Q}_p(\chi\epsilon)) \xrightarrow{\sim} H^1_f(\mathbb{Q}_p, \mathbb{Q}_p(\chi\epsilon)).$$

Let $\widetilde{\log}_{\mathrm{BK}} : H^1_f(\mathbb{Q}_p, \mathbb{Q}_p(\chi\epsilon)) \xrightarrow{\sim} D_{\mathrm{crys}}(\mathbb{Q}_p(\chi\epsilon))$ be the inverse of $\widetilde{\exp}_{\mathrm{BK}}$. It is natural to ask whether we can identify the element $\lambda_{\mathrm{crys}} \in D_{\mathrm{crys}}(\mathbb{Q}_p(\epsilon\chi))$ such that $\widetilde{\log}_{\mathrm{BK}}(\mathrm{loc}_p(z_\chi)) = \lambda_{\mathrm{crys}}$. As it turns out, we have already computed this:

$$D_{\mathrm{crys}}(\mathbb{Q}_p(\chi\epsilon)) = D_{\mathrm{dR}}(\mathbb{Q}_p(\chi\epsilon)) = D_{\mathrm{dR}}(\mathbb{Q}_p(\chi)(1)) = \mathbb{Q}_p(1 \otimes \underline{\zeta} \otimes \underline{t}^{-1})$$

and

$$\boxed{\widetilde{\log}_{\mathrm{BK}}(\mathrm{loc}_p(z_\chi)) = L_p(1, \bar{\chi}_0) \prod_{\substack{\ell \text{ prime} \\ \ell \mid N \\ \ell \nmid N_0}} (1 - \chi_0(\ell)) \cdot (1 \otimes \underline{\zeta} \otimes \underline{t}^{-1}).} \tag{5.7.a}$$

So the Bloch–Kato logarithm of $\mathrm{loc}_p(z_\chi)$ is naturally identified with the value of a p-adic L-function.

The equality in (5.7.a) can be seen as follows. For crystalline $\mathbb{Q}_p$-representations V of $G_{\mathbb{Q}_p}$, the groups $H^0(\mathbb{Q}_p, V)$ and $H^1_f(\mathbb{Q}_p, V)$ are functorially computed by the complex $C_{\mathrm{crys}}(V) = [D_{\mathrm{crys}}(V) \to D_{\mathrm{crys}}(V) \oplus D_{\mathrm{dR}}(V)/F^0 D_{\mathrm{dR}}(V)]$, where the arrow is the map $x \mapsto ((1-\phi_p)x, x \bmod F^0 D_{\mathrm{dR}}(V))$. Applying this to the two sequences in the 1-Tate twist of the commutative diagram (5.6.b), employing the snake lemma to compute the boundary map

$$\begin{aligned} H^0(C_{\mathrm{crys}}(\mathbb{Q}_p)) &= H^0(C_{\mathrm{crys}}(\mathbb{Q}_p(c_{\text{ét}} \otimes \underline{\zeta}))) \\ &\to H^1(C_{\mathrm{crys}}(\lambda_{\chi,\text{ét}}(H^0_{\text{ét}}(\overline{W})(1))) = H^1(C_{\mathrm{crys}}(\mathbb{Q}_p(\chi\epsilon))), \end{aligned}$$

and appealing to the relation in Remark 5.6.2 yields the displayed formula for $\widetilde{\log}_{\mathrm{BK}}(\mathrm{loc}_p(z_\chi))$, which is just the image of $c_{\text{ét}} \otimes \underline{\zeta} \otimes \underline{t}^{-1} \in H^0(C_{\mathrm{crys}}(\mathbb{Q}_p(c_{\text{ét}} \otimes \underline{\zeta})))$ under the above boundary map.

5.8. *Vista: Euler systems.* A variation on the definition of the classes z_χ yields an Euler system. For a reader with some familiarity with Euler systems this should not be surprising in light of the relation (5.7.a). Recall that we are assuming that χ is nontrivial and $\mathbb{Q}_p$-valued and that $p \nmid N$ (all for simplicity).

First we note that we can replace $H^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_p)$ with $H^1_{\text{ét}}(\overline{Y}, \mathbb{Z}_p)$ in the definition of $c_{\text{ét}}$. So in particular, $z_\chi \in H^1(\mathbb{Q}, \mathbb{Z}_p(\chi\epsilon))$ with $\mathbb{Z}_p(\chi\epsilon)$ the free $\mathbb{Z}_p$-module of rank one with $\sigma \in G_{\mathbb{Q}}$ acting via multiplication by $\chi\epsilon(\sigma)$. The other classes of our Euler system come from slightly modifying the definition of $\mathcal{E}_{\chi,\text{ét}}$. For each integer M such that $(N, M) = 1$ we let $Z_M = \mu_M \cup \{\infty\} \subset X$ and $Y_M = X \setminus Z_M$. Note that $W \subset Y_M$. Note also that we recover Y by taking $M = 1$. Then just as in Section 5.3 we have $H^1_{\text{ét}}(\overline{Y}_M, \mathbb{Z}_p) \hookrightarrow H^0(\overline{Z}_M, \mathbb{Z}_p(-1)) = \bigoplus_{z \in Z_M(\overline{\mathbb{Q}})} \mathbb{Z}_p(-1)$ with image equal to $\left\{(a_z \otimes \underline{\zeta}^\vee)_{z \in Z_M(\overline{\mathbb{Q}})} : \sum_z a_z = 0\right\}$. For $\zeta \in \mu_M$ we let $c_{\text{ét},\zeta} \in H^1_{\text{ét}}(\overline{Y}_M, \mathbb{Z}_p)$ be the class corresponding to $a_\infty = 1$, $a_\zeta = -1$, and $a_z = 0$ otherwise. The Galois group $G_{\mathbb{Q}}$ acts on $c_{\text{ét},\zeta}$ as $\sigma c_{\text{ét},\zeta} = \epsilon(\sigma)^{-1} c_{\text{ét},\sigma(\zeta)}$. In particular, $G_{\mathbb{Q}[\mu_M]}$

acts on $c_{\text{ét},\zeta}$ as multiplication by ϵ^{-1}. That is, $\mathbb{Z}_p c_{\text{ét},\zeta} \simeq \mathbb{Z}_p(-1)$ as a $\mathbb{Z}_p[G_{\mathbb{Q}[\mu_M]}]$-module. Pulling back to $H^1_{\text{ét}}(\overline{Y}_M, \overline{W}, \mathbb{Z}_p)$ and then pushing out by λ_χ as before yields an extension $\mathcal{E}_{\chi,\zeta} \in \text{Ext}^1_{\mathbb{Z}_p[G_{\mathbb{Q}[\mu_M]}]}(\mathbb{Z}_p(-1), \mathbb{Z}_p(\chi))$ and hence a class $z_{\chi,\zeta} \in H^1(\mathbb{Q}[\mu_M], \mathbb{Z}_p(\chi\epsilon))$. Note that if $\zeta \in \mu_{M'}$ for some $M' \mid M$, then these are just the restrictions to $G_{\mathbb{Q}[\mu_M]}$ of the extension and class defined with M' in place of M. Furthermore, it follows from the action of $G_{\mathbb{Q}}$ on the $c_{\text{ét},\chi}$ and the action of $G_{\mathbb{Q}}$ on $H^1(\mathbb{Q}[\mu_M], \mathbb{Z}_p(\chi\epsilon))$ (particularly in terms of cycle representatives) that

$$\sigma z_{\chi,\zeta} = z_{\chi,\sigma(\zeta)}. \tag{5.8.a}$$

As before, let $\zeta_M = e^{2\pi i/M} \in \mu_M$. We now set

$$z_{\chi,M} = \bar{\chi}(M) z_{\chi,\zeta_M} \in H^1(\mathbb{Q}[\mu_M], \mathbb{Z}_p(\chi\epsilon)).$$

It should not be surprising that

$$\boxed{\left\{z_{\chi,M} \in H^1(\mathbb{Q}[\mu_M], \mathbb{Z}_p(\chi\epsilon)) : (M, N) = 1\right\} \text{ is an Euler system.}}$$

Here, by an Euler system we mean a collection of cohomology classes as in [Rubin 2000]. In particular, the $z_{\chi,M}$ satisfy the norm relations

$$\boxed{\text{cor}_{\mathbb{Q}[\mu_{M\ell}]/\mathbb{Q}[\mu_M]} z_{\chi,M\ell} = \begin{cases} (1 - \bar{\chi}(\ell)\text{frob}_\ell^{-1}) z_{\chi,M}, & \ell \nmid NMp, \\ z_{\chi,M}, & \ell \mid M. \end{cases}} \tag{5.8.b}$$

We quickly explain how to see these relations.

Since the restriction map $H^1(\mathbb{Q}[\mu_M], \mathbb{Z}_p(\chi\epsilon)) \hookrightarrow H^1(\mathbb{Q}[\mu_{M\ell}], \mathbb{Z}_p(\chi\epsilon))$ is an injection, it is enough to check that the equality of the norm relation holds in $H^1(\mathbb{Q}[\mu_{M\ell}], \mathbb{Z}_p(\chi\epsilon))$. From (5.8.a) we see that

$$\begin{aligned} &\text{cor}_{\mathbb{Q}[\mu_{M\ell}]/\mathbb{Q}[\mu_M]} z_{\chi,M\ell} \\ &\qquad = \bar{\chi}(M\ell) \sum_{\sigma \in \text{Gal}(\mathbb{Q}[\mu_{M\ell}]/\mathbb{Q}[\mu_M])} z_{\chi,\sigma(\zeta_{M\ell})} \in H^1(\mathbb{Q}[\mu_{M\ell}], \mathbb{Z}_p(\chi\epsilon)). \end{aligned} \tag{5.8.c}$$

We consider the map $f : Y_{M\ell} \to Y_M$, $f(t) = t^\ell$. This induces a commutative diagram

$$\begin{array}{ccccc} & & & & \mathbb{Z}_p c_{\chi,\zeta_M} \\ & & & & \downarrow \\ \dfrac{H^0_{\text{ét}}(\overline{W},\mathbb{Z}_p)}{\text{im}(H^0_{\text{ét}}(\overline{Y}_M,\mathbb{Z}_p))} & \hookrightarrow & H^1_{\text{ét}}(\overline{Y}_M, \overline{W}, \mathbb{Z}_p) & \twoheadrightarrow & H^1_{\text{ét}}(\overline{Y}_M, \mathbb{Z}_p) \\ \downarrow f^* & & \downarrow f^* & & \downarrow f^* \\ \dfrac{H^0_{\text{ét}}(\overline{W},\mathbb{Z}_p)}{\text{im}(H^0_{\text{ét}}(\overline{Y}_{M\ell},\mathbb{Z}_p))} & \hookrightarrow & H^1_{\text{ét}}(\overline{Y}_{M\ell}, \overline{W}, \mathbb{Z}_p) & \twoheadrightarrow & H^1_{\text{ét}}(\overline{Y}_{M\ell}, \mathbb{Z}_p) \\ \downarrow \lambda_{\chi,\text{ét}} & & & & \\ \mathbb{Z}_p(\chi) & & & & \end{array}$$

It follows that the extension obtained by pulling back c_{χ,ζ_M} and pushing out by $\lambda_\chi \circ f^*$ is the same as that obtained by pulling back f^*c_{χ,ζ_M} and pushing out by λ_χ. As we have

$$\lambda_\chi \circ f^* = \chi(\ell)\lambda_\chi \quad \text{and} \quad f^*c_{\chi,\zeta_M} = \sum_{\zeta^\ell=\zeta_M} c_{\chi,\zeta},$$

it follows that

$$\chi(\ell)z_{\chi,M} = \sum_{\zeta^\ell=\zeta_M} z_{\chi,\zeta} = \begin{cases} \sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}[\mu_{M\ell}]/\mathbb{Q}[\mu_M])} z_{\chi,\sigma(\zeta_{M\ell})} + z_{\chi,\zeta_M^{\bar\ell}}, & \ell\nmid M, \\ \sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}[\mu_{M\ell}]/\mathbb{Q}[\mu_M])} z_{\chi,\sigma(\zeta_{M\ell})}, & \ell \mid M. \end{cases} \tag{5.8.d}$$

Here we have used that $\zeta_{M\ell} = \zeta_M^{\bar\ell}\zeta_\ell^{\bar M}$, where $\bar\ell\ell \equiv 1 \bmod M$ and $\bar M M \equiv 1 \bmod \ell$, and so $\sigma(\zeta_{M\ell}) = \zeta_M^{\bar\ell}\sigma(\zeta_\ell^{\bar M})$. Comparing (5.8.d) with (5.8.c) yields

$$\mathrm{cor}_{\mathbb{Q}[\mu_{M\ell}]/\mathbb{Q}[\mu_M]} z_{\chi,M\ell} = \begin{cases} \bar\chi(M)z_{\chi,\zeta_M} - \bar\chi(M\ell)z_{\chi,\zeta_M^{\bar\ell}}, & \ell\nmid M, \\ \bar\chi(M)z_{\chi,\zeta_M}, & \ell \mid M. \end{cases}$$

If $\ell\nmid MNp$, then z_{χ,ζ_M} is unramified at ℓ and $\mathrm{frob}_\ell^{-1} z_{\chi,\zeta_M} = z_{\chi,\mathrm{frob}_\ell^{-1}(\zeta_M)} = z_{\chi,\zeta_M^{\bar\ell}}$. The norm relations (5.8.b) follow.

5.8.1. Remark. There is nothing in this section that requires χ to be $\mathbb{Q}_p$-valued or N to be prime to p. One can replace $\mathbb{Z}_p$ with the ring of integers $\mathcal{O}$ for any finite extension of $\mathbb{Q}_p$ and take χ to be any nontrivial $\mathcal{O}$-valued Dirichlet character. The arguments carry over immediately. The trivial character can also be handled, albeit with some additional modification (to ensure that the chosen functional λ is still trivial on the image of $H^0(Y_M, \mathcal{O})$).

5.8.2. Remark. The proof of the norm relations we have given here — which may seem much more involved than that for cyclotomic units (see [Rubin 2000, III.2]) — provides a template for an approach that carries over to many other settings, such as in [Shang et al. ≥ 2024] and [Sangiovanni-Vincentelli and Skinner ≥ 2024a].

5.8.3. Remark. To obtain special value formulas from this (or any) Euler system one also needs to relate the restrictions to $G_{\mathbb{Q}_p}$ of the Euler system classes to values of a p-adic L-function, that is, prove a so-called explicit reciprocity law. This is essentially the point of the calculation in Section 5.7. The general case can be handled similarly. The only real obstacle to overcome is that if $p \mid M$ (or N) then the naive integral models $\mathscr{Y}_M$ and $\mathscr{X}$ of Y_M and X are not such that $\mathscr{Y}_M$ is the complement of a smooth (or even normal crossings) divisor in $\mathscr{X}$. But it is not hard to establish the existence of such models over $\mathbb{Z}_p[\mu_{p^r}]$ for $p^r \,||\, M$. With this in hand, the arguments presented previously carry over with only slight modification.

6. Some variations, very briefly

The constructions in Section 5 can be viewed as a very special case of a general set-up. Indications of this are provided by the variations on the construction and analysis of $\mathcal{E}_{\chi,\text{ét}}$ described briefly in this section. These additional special cases can be used to recover the Euler system for Dirichlet characters and for Hecke characters of imaginary quadratic fields along with their connection with p-adic L-functions (see [Shang et al. ≥ 2024]). Though we do not include a discussion here, a simple variation on these constructions involving products of modular curves can be used to recover Kato's Euler system for an eigenform. Examples of new Euler systems (also with connections to p-adic L-functions) obtained using the same template are given in [Sangiovanni-Vincentelli and Skinner ≥ 2024a; ≥ 2024b].

6.1. ***Dirichlet characters (again).*** Let $N \geq 4$. Let $Y_1(N) \subset X_1(N)$ be the usual modular curves for the congruence subgroup $\Gamma_1(N)$, and let $C_1(N) = X \setminus Y$ be the cusps. These have models as smooth varieties over $\mathbb{Q}$. The cusps $C_1(N) = \Gamma_1(N) \setminus \mathbb{P}^1(\mathbb{Q})$ of $X_1(N) = \Gamma_1(N) \setminus [\mathbb{H} \sqcup \mathbb{P}^1(\mathbb{Q})]$ are in bijection with the set $\{(\bar{a}, \bar{c}) \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} : (a, c, N) = 1\}/\sim$ where $(\bar{a}_1, \bar{c}_1) \sim (\bar{a}_2, \bar{c}_2) \iff (\bar{a}_2, \bar{c}_2) = \pm(\bar{a}_1 + m\bar{c}_1, \bar{c}_1)$ for some $m \in \mathbb{Z}$. The bijection is given by $\mathbb{P}^1(\mathbb{Q}) \ni \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right] \mapsto (\bar{a}, \bar{c})$, $a, c \in \mathbb{Z}$, $(a, c) = 1$. When we write $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ for some element in $\mathbb{P}^1(\mathbb{Q})$ or the cusp it represents, we will always mean $a, c \in \mathbb{Z}$ and $(a, c) = 1$. Let $C_0 \subset C_1(N)$ be the set of cusps represented by some $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ with $(c, N) = 1$; there are $\frac{\phi(N)}{2}$ of them. Similarly, let $C_\infty \subset C_1(N)$ be the set of cusps represented by some $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ with $N \mid c$; there are also $\frac{\phi(N)}{2}$ of them. We take the models of $X_1(N)$ and $Y_1(N)$ over $\mathbb{Q}$ such that each cusp in C_∞ is defined over $\mathbb{Q}$ and each cusp $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ in C_0 is defined over $\mathbb{Q}[\mu_N]^+$: The action of $G_\mathbb{Q}$ on the cusps is such that if $\sigma \in G_\mathbb{Q}$ maps to $m \in (\mathbb{Z}/N\mathbb{Z})^\times = \mathrm{Gal}(\mathbb{Q}[\mu_N]/\mathbb{Q})$, then $\sigma \cdot \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ is represented by $\left[\begin{smallmatrix} a \\ c' \end{smallmatrix}\right]$ with $c \equiv mc' \bmod N$. Note that C_0 and C_∞ are $\mathbb{Q}$-subvarieties of $X = X_1(N)$.

Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a nontrivial, primitive, even Dirichlet character. There exists an Eisenstein series G_χ of weight 2 and level N with q-expansion

$$G_\chi(\tau) = \sum_{n=1}^{\infty} \left(\sum_{d \mid n} \bar{\chi}\left(\frac{n}{d}\right) d \right) q^n, \quad q = e^{2\pi i \tau}.$$

The constant term $c_P(G_\chi)$ of G_χ is 0 at any cusp $P \notin C_0$ and at $P = \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right] \in C_0$ it is $c_P(G_\chi) = \bar{\chi}(c)\tau(\bar{\chi})L(-1, \chi)/2N^2$. Let $\omega_\chi = G_\chi(\tau)d\tau$. This defines a holomorphic differential on $Y = X \setminus C_0$ with log poles along C_0. Let $\omega_\chi^{\text{an}} = \tau(\chi)\omega_\chi$. By considering the residues of the differential ω_χ^{an} at the cusps in C_0 (which are essentially the constant terms) and using that the Hecke eigenvalues of G_χ distinguish it from cuspforms, one can see that $c_\chi = \iota_{\text{dR}}([\omega_\chi^{\text{an}}]) \in H^1(Y, \mathbb{Q}(\chi))$, where $\mathbb{Q}(\chi)$ is the finite extension of $\mathbb{Q}$ obtained by adjoining the values of χ.

Let $L \supset \mathbb{Q}(\chi)$ be any finite extension of $\mathbb{Q}_p$. Then similar considerations show that $G_{\mathbb{Q}}$ acts on the corresponding class $c_{\chi,\text{ét}} = \iota_{\text{ét}}(c_\chi) \in H^1_{\text{ét}}(\overline{Y}, L)$ via $\bar{\chi}\epsilon^{-1}$, that is, $Lc_{\chi,\text{ét}} \simeq L(\bar{\chi}\epsilon^{-1})$ as $L[G_{\mathbb{Q}}]$-modules. We then obtain an extension $\mathcal{E}^{\text{mod}}_{\chi,\text{ét}}$ as a subquotient of the relative cohomology group $H^1_{\text{ét}}(\overline{Y}, \overline{W}, L)$ analogously to $\mathcal{E}_{\chi,\text{ét}}$, where now $W = C_\infty$. Let $\lambda^{\text{mod}}_{\chi,\text{ét}} : H^0_{\text{ét}}(\overline{W}, L) \to L$ be the $L[G_{\mathbb{Q}}]$-homomorphism such that

$$\lambda^{\text{mod}}_{\chi,\text{ét}}((c_P)_{P\in C_\infty}) = c_{\left[\begin{smallmatrix}1\\N\end{smallmatrix}\right]} - c_{\left[\begin{smallmatrix}a\\N\end{smallmatrix}\right]}$$

for some fixed a with $(a, N) = 1$, $a \not\equiv \pm 1 \bmod N$. Note that $\lambda^{\text{mod}}_{\chi,\text{ét}}$ is trivial on the image of $H^0_{\text{ét}}(\overline{Y}, L)$. The extension $\mathcal{E}^{\text{mod}}_{\chi,\text{ét}}$ is the pullback/pushout

$$\begin{array}{ccccc}
\dfrac{H^0_{\text{ét}}(\overline{W}, L)}{\operatorname{im} H^0_{\text{ét}}(\overline{Y}, L)} & \hookrightarrow & H^1_{\text{ét}}(\overline{Y}, \overline{W}, L) & \twoheadrightarrow & H^1_{\text{ét}}(\overline{Y}, L) \\
\Big\downarrow \lambda^{\text{mod}}_{\chi,\text{ét}} & & \vdots & & \Big\uparrow \\
L & \hookrightarrow & \mathcal{E}^{\text{mod}}_{\chi,\text{ét}} & \twoheadrightarrow & Lc_{\chi,\text{ét}},
\end{array}$$

with the dashed arrow denoting a subquotient.

We analyze the extension

$$\mathcal{E}^{\text{mod}}_{\chi,\text{ét}} \in \operatorname{Ext}^1_{L[G_{\mathbb{Q}}]}(L, Lc_{\chi,\text{ét}}) = \operatorname{Ext}^1_{L[G_{\mathbb{Q}}]}(L, L(\bar{\chi}\epsilon^{-1}))$$

just as we did $\mathcal{E}_{\chi,\text{ét}}$ in Section 5. Suppose — again for simplicity — that χ is valued in $\mathbb{Q}_p$ (so we may take $L = \mathbb{Q}_p$) and $p \nmid N$. Then $D_{\text{crys}}(\mathbb{Q}_p c_{\chi,\text{ét}}) = D_{\text{dR}}(\mathbb{Q}_p c_{\chi,\text{ét}}) = \mathbb{Q}_p(c_{\chi,\text{ét}} \otimes \tau(\bar{\chi})\underline{t})$, and it is easy to see — by comparing residues at cusps — that $\iota_{\text{dR},p}([\omega^{\text{alg}}_\chi]) = c_{\chi,\text{ét}} \otimes \frac{1}{\tau(\chi)}\underline{t}$, where $\omega^{\text{alg}}_\chi = 2\pi i \omega_\chi \in H^0(\Omega^1_{X/\mathbb{Q}_p}(\log C_0))$. Specifically, ϕ_p acts on $[\omega^{\text{alg}}_\chi]$ as multiplication by $\chi(p)p$ and we seek to understand whether $(1 - \bar{\chi}(p)p^{-1}\phi_p)[\omega^{\text{alg}}_\chi]_W \in H^1_{\text{dR}}((Y, W)/\mathbb{Q}_p)$ is the image of something nontrivial in $H^0_{\text{dR}}(W/\mathbb{Q}_p)$ that is nonzero under $\lambda^{\text{mod}}_{\chi,\text{ét}}$. We now replace the passage to Monsky–Washnitzer cohomology with restriction to the rigid cohomology of the ordinary locus of X (the rigid analytic subspace of points corresponding to elliptic curves with ordinary reduction at p) and also with partial compact support in W. This moves the calculation into the realm of overconvergent p-adic modular forms, just as passage to MW cohomology moved the calculation to the realm of overconvergent functions on the affinoid Y_{an} in Sections 5.5 and 5.6. The action of ϕ_p on a p-adic modular form $f(q) \in \mathbb{Q}_p[\![q]\!]$ of weight 2 is just $f(q) \mapsto pf(q^p)$, and the differential on p-adic modular functions is just the p-adic Maass–Shimura operator $\theta = q\frac{d}{dq}$. In particular, we want to find an overconvergent p-adic modular function $\eta(q)$ (a form of weight 0) such that $\theta\eta = G_\chi(q) - \bar{\chi}(p)G_\chi(q^p)$. Then $(1 - \bar{\chi}(p)p^{-1}\phi_p)[\omega^{\text{alg}}_\chi]_W$ is the image of $\eta|_W \in H^0_{\text{dR}}(W/\mathbb{Q}_p)$, and so we want to know whether $\lambda^{\text{mod}}_{\chi,\text{ét}}(\eta|_W) \neq 0$. It is easy to identify η from the q-expansion of $G_\chi(q) - \bar{\chi}(p)G_\chi(q^p)$: $\eta = E^{\text{ord}}_{\bar{\chi},0}$, the p-ordinary weight-0 Eisenstein series with

q-expansion

$$E^{\mathrm{ord}}_{\bar\chi,0}(q) = \tfrac{1}{2}L_p(1,\bar\chi) + \sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d \bmod n \\ p\nmid d}} \bar\chi(d)d^{-1}\Bigg)q^n.$$

The existence of such an Eisenstein series is an easy consequence of Katz's Eisenstein measure (which also provides a proof of the existence of the p-adic L-function $L_p(s,\bar\chi)$ as a p-adic measure — see [Serre 1973] and [Katz 1975]). It follows that

$$\lambda^{\mathrm{mod}}_{\chi,\text{ét}}(\eta|W) = c_{\left[\frac{1}{N}\right]}(E^{\mathrm{ord}}_{\chi,0}) - c_{\left[\frac{a}{N}\right]}(E^{\mathrm{ord}}_{\chi,0}) = \tfrac{1}{2}(1-\chi(a))L_p(1,\bar\chi).$$

For a satisfying $\chi(a)\neq 1$, this shows that $\mathcal{E}^{\mathrm{mod}}_{\chi,\text{ét}}|_{G_{\mathbb{Q}_p}}$ is nonsplit if $L_p(1,\bar\chi)\neq 0$. One can associate with $\mathcal{E}^{\mathrm{mod}}_{\chi,\text{ét}}$ a cohomology class $z^{\mathrm{mod}}_\chi \in H^1_f(\mathbb{Q},\mathbb{Q}_p(\chi\epsilon))$ by tensoring the extension over $\mathbb{Q}_p$ with $\mathbb{Q}_p(\chi\epsilon)$, just as we associated z_χ with $\mathcal{E}_{\chi,\text{ét}}$. Then unwinding the preceding analysis as in Section 5.7 yields

$$\widetilde{\log}_{\mathrm{BK}}(\mathrm{loc}_p(z^{\mathrm{mod}}_\chi)) = \tfrac{1}{2}(1-\chi(a))L_p(1,\bar\chi)\cdot(1\otimes\underline{\zeta}\otimes\underline{t}^{-1}) \in D_{\mathrm{crys}}(\mathbb{Q}_p(\chi\epsilon)).$$

6.1.1. Remark. We conclude with a few remarks:

(1) Unlike for z_χ, which was constructed from the cohomology of $\mathbb{P}^1\setminus\{\infty,1\}$, this computation of the Bloch–Kato logarithm of z^{mod}_χ does not rely on a formula for the special value $L_p(1,\bar\chi)$ in terms of p-adic logs of cyclotomic units, but instead comes naturally via the value of a constant term of a p-adic Eisenstein series, and it is via the latter that Serre and Katz (*re-*)*constructed* the p-adic L-function [Serre 1973; Katz 1975]. The construction of z^{mod}_χ (via $\mathcal{E}^{\mathrm{mod}}_{\chi,\text{ét}}$) can be viewed as a cohomological expression of the Serre–Katz construction. Our next construction of cohomology classes — for Hecke characters of imaginary quadratic fields — lends itself to a similar interpretation.

(2) The class z^{mod}_χ can be extended to an Euler system

$$\big\{z^{\mathrm{mod}}_{\chi,M} \in H^1(\mathbb{Q}[\mu_M],\mathbb{Z}_p(\chi\epsilon)) : (M,N)=1\big\}.$$

The classes $z^{\mathrm{mod}}_{\chi,M}$ are just the cohomology classes associated with extensions constructed via pullback/pushforward from simple, natural variations on the Eisenstein classes $\omega_\chi = G_\chi(\tau)d\tau$. However, unlike for z_χ (and $z_{\chi,M}$), the construction described above does not immediately imply that the class z^{mod}_χ (or $z^{\mathrm{mod}}_{\chi,M}$) belongs to $H^1(\mathbb{Q},\mathbb{Z}_p(\chi\epsilon))$. This can be shown, though, via a more careful use of the comparison isomorphisms of p-adic Hodge theory: Assuming $p\nmid N$, we can work with smooth integral models of X, Y, $Z=X\setminus Y$, and $W=C_\infty$ over $\mathbb{Z}_p$. Then $\omega_\chi\in H^0(\Omega^1_{X/\mathbb{Z}_p}(\log Z))$ and $\iota_{\mathrm{dR},p} : H^1_{\mathrm{dR}}(Y/\mathbb{Z}_p) \xrightarrow{\sim} (H^1_{\text{ét}}(\overline{Y},\mathbb{Z}_p)\otimes_{\mathbb{Z}_p} A_{\mathrm{crys}})^{G_{\mathbb{Q}_p}}$, where $A_{\mathrm{crys}}\subset B_{\mathrm{crys}}$ is the usual integral crystalline ring. As $\underline{t}$ is not divisible by a nonunit of $\mathbb{Z}_p^{\mathrm{ur}}$ in A_{crys}, the relation $\iota_{\mathrm{dR}}([\omega^{\mathrm{alg}}_\chi]) = c_{\chi,\text{ét}}\otimes\frac{1}{\tau(\chi)}\underline{t}$ then implies that

$c_{\chi,\text{ét}} \in H^1_{\text{ét}}(\overline{Y}, \mathbb{Z}_p)$. A variation of this argument, similar to [Faltings 2005, §10], can be used to handle the case when $p \mid N$ (and also $z^{\text{mod}}_{\chi,M}$ when $p \mid M$).

(3) As mention in Section 3, a very similar construction of extensions can be found in Harder's unpublished work [2023]. Essentially the same construction can also be found in unpublished work of Romyar Sharifi and Preston Wake. However, neither detect nonsplitting without reference to a comparison with the extension classes defined by modular units.

6.2. ***Hecke characters.*** Let ℓ be a prime, and let $X = X_0(\ell)$ and $Y_0(\ell)$ be the usual modular curves for the congruence subgroup $\Gamma_0(\ell)$, which we view as smooth curves over $\mathbb{Q}$ via the usual canonical models. The cusps $C = X \setminus Y$ consists of two points, usually denoted ∞ and 0 and both defined over $\mathbb{Q}$. The unique holomorphic Eisenstein series E of weight 2, level ℓ, and trivial character, which has q-expansion

$$E(\tau) = \frac{(1-\ell)\zeta(-1)}{2} + \sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d \mid n \\ \ell \nmid d}} d\Bigg) q^n, \quad q = e^{2\pi i\tau},$$

defines a class $\omega_E = E(\tau)d\tau \in H^0(\Omega^1_{X/\mathbb{C}}(\log C))$ and $c_E = \iota_{\text{dR}}[\omega_E] \in H^1(Y, \mathbb{C})$ actually belongs to $H^1(Y, \mathbb{Q})$. So $c_{E,\text{ét}} = \iota_{\text{ét}}(c_E) \in H^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_p)$. The action of $G_{\mathbb{Q}}$ on $c_{E,\text{ét}}$ is via ϵ^{-1}. That is, $\mathbb{Q}_p c_{E,\text{ét}} \simeq \mathbb{Q}_p(-1)$ as a $\mathbb{Q}_p[G_{\mathbb{Q}}]$-module.

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}$. Let

$$R = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : \ell \mid c\right\}$$

be the usual Eichler order of level ℓ (so $(R \otimes \widehat{\mathbb{Z}}) \cap \mathrm{GL}_2(\mathbb{Q})^+ = \Gamma_0(\ell)$). Fix an embedding $K \hookrightarrow M_2(\mathbb{Q})$ such that $R \cap K = \mathcal{O}$. Let $\tau_0 \in \mathbb{H}$ be such that its stabilizer in $\mathrm{GL}_2(\mathbb{Q})^+$ is $K^\times$. Then

$$W = \left\{[\tau_0, x] \in Y(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})^+ \setminus [\mathbb{H} \times \mathrm{GL}_2(\mathbb{A}_f)/(R \otimes \widehat{\mathbb{Z}})^\times] : x \in (K \otimes \mathbb{A}_f)^\times\right\}$$

is a collection of CM points on Y. It is in bijection with the class group of K. The set W is defined over K and each point in W is defined over the Hilbert class field H of K. The action of G_K on W is described via CM theory: Let $\mathrm{Art}_K : K^\times \setminus (K \otimes \mathbb{A}_f)^\times \twoheadrightarrow G_K^{\text{ab}}$ be Artin map of class field theory, with geometric normalizations. If $\sigma \in G_K$ is such that the image of $\sigma \in G_K^{\text{ab}}$ is $\mathrm{Art}_K(z)$ then $\sigma \cdot [\tau_0, x] = [\tau, zx]$.

We view W as a K-subvariety of Y. Let $\psi : K^\times \setminus (K \otimes \mathbb{A}_f)^\times/(\mathcal{O} \otimes \widehat{\mathbb{Z}})^\times \to \mathbb{C}^\times$ be a character of the class group of K. We also view this as a character of G_K via the projection to $\mathrm{Gal}(H/K)$ and the Artin map. Suppose — for simplicity — that ψ takes values in $\mathbb{Q}_p$. Then $\lambda^K_{\psi,\text{ét}} : H^0_{\text{ét}}(\overline{W}, \mathbb{Q}_p) \twoheadrightarrow \mathbb{Q}_p(\psi)$, $\lambda^K_{\psi,\text{ét}}((c_w)_{w\in W}) = \sum_{w=[\tau_0,x]\in W} \psi(x)c_w$, is a G_K-equivariant map. Here $\mathbb{Q}_p(\psi)$ is just $\mathbb{Q}_p$ with

$\sigma \in G_K$ acting via multiplication by $\psi(\sigma)$. The usual pull-back/push-forward construction then yields an extension $\mathcal{E}^K_{\psi,\text{ét}}$:

$$\begin{array}{ccccc}
\frac{H^0_{\text{ét}}(\overline{W},\mathbb{Q}_p)}{\operatorname{im} H^0_{\text{ét}}(\overline{Y},\mathbb{Q}_p)} & \hookrightarrow & H^1_{\text{ét}}(\overline{Y},\overline{W},\mathbb{Q}_p) & \twoheadrightarrow & H^1_{\text{ét}}(\overline{Y},\mathbb{Q}_p) \\
\big\downarrow \lambda^K_{\psi,\text{ét}} & & \big\downarrow & & \big\uparrow \\
\mathbb{Q}_p(\psi) & \hookrightarrow & \mathcal{E}^K_{\psi,\text{ét}} & \twoheadrightarrow & \mathbb{Q}_p c_{E,\text{ét}},
\end{array}$$

which defines a class in $\operatorname{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p(\psi), \mathbb{Q}_p c_{E,\text{ét}}) = \operatorname{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p(\psi), \mathbb{Q}_p(-1))$. And associated with this is a class $z^K_\psi \in H^1(K, \mathbb{Q}_p(\psi\epsilon))$.

Suppose p splits in K: $p = v\bar{v}$. The Bloch–Kato logarithm of $\operatorname{loc}_v(z_\psi) \in H^1_f(K_v, \mathbb{Q}_p(\psi\epsilon))$ can be computed following the same method employed for $\operatorname{loc}_p(z^{\text{mod}}_\chi)$. The upshot is that $\widetilde{\log}_{\text{BK}}(\operatorname{loc}_v(z^K_\psi))$ is a multiple of a natural basis of $D_{\text{crys}}(\mathbb{Q}_p(\psi\epsilon))$, with that multiple being expressed as

$$\sum_{w=[\tau_0,x]\in W} \psi(x) E^{\text{ord}}_0(w), \tag{6.2.a}$$

where E^{ord}_0 is the p-ordinary weight-0 Eisenstein series with q-expansion

$$E^{\text{ord}}_0(q) = (1-\ell^{-1})\zeta_p(1) + \sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d\,|\,n \\ p\nmid d \\ \ell\nmid n/d}} d^{-1}\Bigg)q^n.$$

Note that $\theta E^{\text{ord}}_0 = E(q) - E(q^p)$ which is identified with $(1 - p^{-1}\phi_p)[\omega_E]$ in the rigid cohomology of the ordinary locus of Y, so the expression (6.2.a) is just $\lambda_{\psi,\text{dR}}(E^{\text{ord}}_0|W)$. Via Katz's construction of the p-adic L-function of $\bar{\psi}$ relative to the choice of v [1975], the expression (6.2.a) can be seen to be a simple multiple of the value at $s = 1$ of the p-adic L-function. That is, the Bloch–Kato logarithm of $\operatorname{loc}_v(z^K_\psi)$ is naturally expressed as a value of a p-adic L-function for $\bar{\psi}$.

6.2.1. Remark. Just as for z^{mod}_χ, the class z^K_ψ can be extended to an Euler system for $\mathbb{Z}_p(\psi\epsilon)$ over K in the sense of Rubin [2000]. This involves varying W over CM points defined over ring (and even ray) class extensions as well as varying the Eisenstein class. In this way, one can recover/reconstruct the Euler system for ψ over K previously defined by Rubin [1991] using elliptic units along with its connection with Katz's two-variable p-adic L-function.

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