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Cohomology of Fuchsian groups and Fourier interpolation

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We give a new proof of a Fourier interpolation result first proved by Radchenko and Viazovska (2019), deriving it from a vanishing result of the first cohomology of a Fuchsian group with coefficients in the Weil representation.

1. Introduction

Let \mathcal{S} be the space of even Schwartz functions on the real line, and \mathfrak{s} the space of sequences of complex numbers $(a_n)_{n \geq 0}$ such that $|a_n|n^k$ is bounded for all k ; we write $\hat{\phi}(k) = \int_{\mathbb{R}} \phi(x) e^{-2\pi i k x} dx$ for the Fourier transform of $\phi \in \mathcal{S}$. Radchenko and Viazovska [2019] proved the following beautiful “interpolation formula”:

Theorem 1.1. *The map*

$$\Psi : \mathcal{S} \rightarrow \mathfrak{s} \oplus \mathfrak{s}, \quad \phi \mapsto (\phi(\sqrt{n}), \hat{\phi}(\sqrt{n}))_{n \geq 0}$$

is an isomorphism onto the codimension 1 subspace of $\mathfrak{s} \oplus \mathfrak{s}$ cut out by the Poisson summation formula, i.e., the subspace of (x_n, y_n) defined by $\sum_{n \in \mathbb{Z}} x_{n^2} = \sum_{n \in \mathbb{Z}} y_{n^2}$.

This is an abstract interpolation result: The statement implies the existence of a universal formula that computes any value $\phi(x)$ of any even Schwartz function ϕ as a linear combination $\sum a_n(x) \phi(\sqrt{n}) + \sum \hat{a}_n(x) \hat{\phi}(\sqrt{n})$ for some $a_n(x)$, $\hat{a}_n(x)$, but does not specify what those functions are. By contrast, Radchenko and Viazovska first write down this explicit interpolation formula, and then deduce Theorem 1.1 from it. In a sense, what is accomplished in the present paper is to separate the abstract content of this interpolation result from its computational aspect.

The morphism Ψ is in fact a homeomorphism of topological vector spaces with reference to natural topologies. We will give another proof of this theorem. The first step of this proof is to notice that the evaluation points \sqrt{n} occur very naturally in the theory of the *oscillator representation* defined by Segal, Shale and Weil (see [Chan 2012] or [Lion and Vergne 1980] for introductions). Using this observation, the theorem can be reduced to computing the cohomology of a certain Fuchsian group with coefficients in this oscillator representation, and here we prove a more general statement:

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Theorem 1.2. *Let G be $\mathrm{SL}_2(\mathbb{R})$ or a finite cover thereof, Γ a lattice in G , W an irreducible infinite-dimensional (\mathfrak{g}, K) -module, and $W_{-\infty}^*$ the distributional globalization of its dual (see [Section 2.4](#)). Then $H^1(\Gamma, W_{-\infty}^*)$ is always finite-dimensional, and in fact*

$$\dim H^1(\Gamma, W_{-\infty}^*) = \text{multiplicity of } W^{\mathrm{cl}} \text{ in cusp forms on } \Gamma \backslash G, \quad (1)$$

where W^{cl} is the complementary irreducible representation to W (see [Section 2.3](#)).

The theorem can be contrasted with usual Frobenius reciprocity:

$$\dim H^0(\Gamma, W_{-\infty}^*) = \text{multiplicity of } W \text{ in automorphic forms on } \Gamma \backslash G. \quad (2)$$

Note that, in the passage from (1) to (2), “cusp forms” have been replaced by “automorphic forms” and W^{cl} by W . We also emphasize the surprising fact that, in the theorem, the H^1 takes *no account* of the topology on $W_{-\infty}^*$: it is simply the usual cohomology of the discrete group Γ acting on the abstract vector space $W_{-\infty}^*$. The corresponding determination for finite-dimensional W is the subject of automorphic cohomology and is in particular completely understood, going back to [\[Eichler 1957\]](#).

A variant of [Theorem 1.2](#), computing *all* the cohomology groups H^i when W is a spherical principal series representation, was already proved by Bunke and Olbrich in the 1990s. We were unaware of this work when we first proved [Theorem 1.2](#); our original argument has many points in common with [\[Bunke and Olbrich 1998\]](#), most importantly in our usage of surjectivity of the Laplacian both for analytic and algebraic purposes, but also has some substantial differences of setup and emphasis. We will correspondingly give two proofs: the first based on the results of [\[Bunke and Olbrich 1998\]](#), and the second a shortened version of our original argument.

Some other interpolation consequences of [Theorem 1.2](#), where interpolation is understood in the abstract sense as discussed after [Theorem 1.1](#), arise by replacing \mathcal{S} by other spaces of functions carrying natural representations of $\mathrm{SL}_2(\mathbb{R})$ and its finite covers; we discuss this in [Section 6.4](#). For example, Hedenmalm and Montes-Rodríguez [\[2011\]](#) have shown that the functions $e^{i\pi\alpha nt}$, $e^{i\pi\beta n/t}$ are weakly dense in L^∞ if and only if $\alpha\beta = 1$. We will show that an interpolation result holds at the transition point $\alpha\beta = 1$; we thank the referee for bringing [\[Hedenmalm and Montes-Rodríguez 2011\]](#) to our attention.

1.1. Theorem 1.2 implies Theorem 1.1. Here we give an outline of the argument and refer to [Section 6](#) for details.

We pass first to a dual situation. Denote by \mathcal{S}^* the space of tempered distributions, i.e., the continuous dual of \mathcal{S} . For our purposes we regard it as a vector space without topology.

Similarly, we define \mathfrak{s}^* as the continuous dual of \mathfrak{s} , where \mathfrak{s} is topologized by means of the norms $\|(b_n)\|_k := \sup_n b_n(1 + |n|)^k$; thus, \mathfrak{s}^* may be identified with

sequences (a_n) of complex numbers of polynomial growth, where the pairing of $(a_n) \in \mathfrak{s}^*$ and $(b_n) \in \mathfrak{s}$ is given by the rule $\sum a_n b_n$. With this notation, the map

$$\Psi^* : \mathfrak{s}^* \oplus \mathfrak{s}^* \rightarrow \mathcal{S}^*$$

dual to Ψ takes the coordinate functions to the distributions δ_n and $\hat{\delta}_n$:

$$(a_n, b_n)_{n \geq 0} \mapsto \sum a_n \delta_n + b_n \hat{\delta}_n,$$

where

$$\delta_n(\phi) = \phi(\sqrt{n}), \quad \hat{\delta}_n(\phi) = \hat{\phi}(\sqrt{n}).$$

Then [Theorem 1.1](#) is equivalent to the assertion:

(Dual interpolation theorem): Ψ^* is surjective and its kernel consists precisely of the “Poisson summation” relation.

The equivalence of this statement and [Theorem 1.1](#) is not a complete formality because of issues of topology: see [\(52\)](#) for an argument that uses a theorem of Banach.

The next key observation is that the space of distributions spanned by δ_n and by $\hat{\delta}_n$ occur in a natural way in representation theory.

The closure of the span of δ_n (respectively, the closure of the span of $\hat{\delta}_n$) coincide with the e -fixed and f -fixed vectors on the space \mathcal{S}^* of tempered distributions, where

$$e = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad (3)$$

act on \mathcal{S}^* according to the oscillator representation (see [Section 6.1](#) for details), namely e and f multiply ϕ and $\hat{\phi}$, respectively, by $e^{2\pi i x^2}$, see [\(51\)](#).

Let Γ be the group generated by e and f inside $\mathrm{SL}_2(\mathbb{R})$: it is a free group, of index 2 in $\Gamma(2)$, and it lifts to the double cover G of $\mathrm{SL}_2(\mathbb{R})$. As explicated in [Section 6](#), computations of dimensions of modular forms and [Theorem 1.2](#) yield

$$\dim H^0(\Gamma, \mathcal{S}^*) = 1, \quad \dim H^1(\Gamma, \mathcal{S}^*) = 0. \quad (4)$$

The final observation is that:

The kernel and cokernel of $(\mathcal{S}^*)^e \oplus (\mathcal{S}^*)^f \rightarrow \mathcal{S}^*$ compute, respectively, the H^0 and H^1 of Γ acting on \mathcal{S}^* .

This follows from a Mayer–Vietoris-type long exact sequence that computes the cohomology of the free group Γ [[Brown 1982](#), Chapters II and III], namely,

$$\begin{aligned} 0 \rightarrow H^0(\Gamma, \mathcal{S}^*) \rightarrow H^0(\langle e \rangle, \mathcal{S}^*) \oplus H^0(\langle f \rangle, \mathcal{S}^*) \\ \rightarrow H^0(1, \mathcal{S}^*) \rightarrow H^1(\Gamma, \mathcal{S}^*) \rightarrow \dots \end{aligned} \quad (5)$$

Combined with (4), we see that $\mathcal{S}^* = (\mathcal{S}^*)^e + (\mathcal{S}^*)^f$, i.e., the desired surjectivity of Ψ^* , and that the intersection of $(\mathcal{S}^*)^e$ and $(\mathcal{S}^*)^f$ is one-dimensional; this corresponds exactly to the Poisson summation formula.

Another way to look at this is the following. The Poisson summation formula is an obstruction to *surjectivity* in [Theorem 1.1](#) and is closely related to the invariance of the distribution $\sum \delta_n \in \mathcal{S}^*$ by Γ , i.e., the existence of a class in the *zeroth* cohomology of Γ on \mathcal{S}^* . The above discussion shows a less obvious statement: the obstruction to *injectivity* in [Theorem 1.1](#) is precisely the *first* cohomology of Γ on \mathcal{S}^* .

1.2. The proof of [Theorem 1.2](#). The analogue of [Theorem 1.2](#) when W is finite-dimensional and $\Gamma \backslash G$ is compact is (by now) a straightforward exercise; as noted, the ideas go back at least to [\[Eichler 1957\]](#), and the general case is documented in [\[Borel and Wallach 2000\]](#); the noncompact case is less standard but also well known, see, e.g., [\[Casselman 1984\]](#) and [\[Franke 1998\]](#) for a comprehensive treatment.

The main complication of our case is that the coefficients are infinite-dimensional and one might think this renders the question unmanageable. The key point is that W is irreducible as a G -module. This says that, “relative to G ”, it is just as good as a finite-dimensional representation.

We will present two proofs of [Theorem 1.2](#):

- The first proof, in [Section 3](#), relies on the work of Bunke and Olbrich [\[1998\]](#), who computed the cohomology of lattices in $\mathrm{SL}_2(\mathbb{R})$ with coefficients in (the distribution globalization of a) principal series representation. We give a sketch of the argument of [\[Bunke and Olbrich 1998\]](#) for the convenience of the reader, and also because their argument as written does not cover the situation we need. To deduce [Theorem 1.2](#) from these results then requires us to pass from a principal series to a subquotient, which we do in a rather ad hoc way.
- The second proof is our original argument prior to learning of the work of Bunke and Olbrich just mentioned. It generalizes the standard way of computing Γ -cohomology with finite-dimensional coefficients, as given in [\[Borel and Wallach 2000\]](#), to the infinite-dimensional case — at least in cohomological degree 1. Given the content of [\[Bunke and Olbrich 1998\]](#), we have permitted ourselves to abridge some tedious parts of our original argument, and reproduce here in detail the part that is perhaps most distinct from [\[Bunke and Olbrich 1998\]](#) — namely, we express the desired cohomology groups in terms of certain Ext-groups of (\mathfrak{g}, K) -modules and then compute these explicitly.

In both arguments the surjectivity of a Laplacian-type operator plays an essential role. Such results are known since the work of Casselman [\[1984\]](#), and in their work, Bunke and Olbrich prove and utilize such a result both at the level of G and $\Gamma \backslash G$. We include a self-contained proof of such a result for $\Gamma \backslash G$ in [Section 5](#).

1.3. Questions. As we have noted, we prove an *abstract* interpolation result. Can one recover the explicit formula for the interpolating functions, as given in [Radchenko and Viazovska 2019], from this approach? It seems to us that our proof is sufficiently explicit that this is, at least, plausible.

It is very interesting to ask about the situation where Γ is not a lattice. Indeed, if one were to ask about an interpolation formula with evaluation points $0.9\sqrt{n}$, one is immediately led to similar questions for a discrete but *infinite covolume* subgroup of $\mathrm{SL}_2(\mathbb{R})$, whereas considering $1.1\sqrt{n}$ leads to considering a nondiscrete subgroup. Note that Kulikov, Nazarov and Sodin [Kulikov et al. 2025] have recently shown very general results about Fourier uniqueness that imply, in particular, that evaluating f and \hat{f} at $1.1\sqrt{n}$ do not suffice to determine f , but that evaluating them at $0.9\sqrt{n}$ *does*.

Perhaps a more straightforward question is to establish an isomorphism

$$H^i(\Gamma, W_{-\infty}^*) \simeq \mathrm{Ext}_{\mathfrak{g}, K}^i(W, \text{space of automorphic forms for } \Gamma \backslash G), \quad (6)$$

which is valid for general lattices Γ in semisimple Lie groups G and general irreducible (smooth, moderate growth) representations V of G . Bunke and Olbrich have proved this in the cocompact case, and our original argument proceeded by establishing the case $i = 1$ for general lattices in $\mathrm{SL}_2(\mathbb{R})$. Also, Deitmar and Hilgert [2005, Corollary 3.3] prove a result of this type in great generality, but with the space of automorphic forms replaced by the larger space $C^\infty(\Gamma \backslash G)$ without growth constraints.

2. Covering groups of $\mathrm{SL}_2(\mathbb{R})$

Let $q \geq 1$ be a positive integer and let G be the q -fold covering of the group $\mathrm{SL}_2(\mathbb{R})$, i.e., G is a connected Lie group equipped with a continuous homomorphism $G \rightarrow \mathrm{SL}_2(\mathbb{R})$ with kernel of order q . This characterizes G up to unique isomorphism covering the identity of $\mathrm{SL}_2(\mathbb{R})$.

Denote by \mathfrak{g} the shared Lie algebra of G and of $\mathrm{SL}_2(\mathbb{R})$ and $\exp : \mathfrak{g} \rightarrow G$ the exponential map. Also denote by K the preimage of $\mathrm{SO}_2(\mathbb{R})$ inside G ; it is abstractly isomorphic as topological group to $S^1 = \mathbb{R}/\mathbb{Z}$ and we fix such an isomorphism below.

The quotient G/K is identified with the hyperbolic plane \mathbb{H} , on which G acts by isometries. Define the norm of $g \in G$ to be $\|g\| := e^{\mathrm{dist}_{\mathbb{H}}(i, gi)}$. Equivalently, we could use $\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \sqrt{a^2 + b^2 + c^2 + d^2}$ since either of these two norms is bounded by a constant multiple of the other.

2.1. Lie algebra. Let H, X, Y be the standard basis for \mathfrak{g} :

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also use $\kappa = i(X - Y)$, $2p = H - i(X + Y)$, $2m = H + i(X + Y)$, or, in matrix form

$$\kappa = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad 2p = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad 2m = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}. \quad (7)$$

We have $\kappa = ik$, where k generates the Lie algebra of K .

The elements p, m and κ satisfy the commutation relations

$$[p, m] = \kappa, \quad [\kappa, p] = 2p, \quad [\kappa, m] = -2m, \quad (8)$$

which say that p and m (shorthand for *plus* and *minus*) raise and lower κ -weights by 2. The Casimir element \mathcal{C} in the universal enveloping algebra determined by the trace form is given by any of the equivalent formulas:

$$\mathcal{C} = \frac{1}{2}H^2 + XY + YX = \frac{1}{2}\kappa^2 + pm + mp = \frac{1}{2}\kappa^2 + \kappa + 2mp = \frac{1}{2}\kappa^2 - \kappa + 2pm. \quad (9)$$

2.2. Iwasawa decomposition. There is a decomposition

$$G = NAK, \quad (10)$$

where A and N are the connected Lie subgroups of G with Lie algebra $\mathbb{R}.H$ and $\mathbb{R}.X$ respectively. We will parameterize elements of A via

$$a_y := \exp\left(\frac{1}{2}\log(y)H\right),$$

so that a_y projects to the diagonal element of $\mathrm{SL}_2(\mathbb{R})$ with entries $y^{\pm 1/2}$. We will also write $n_x = \exp(xX)$.

2.3. (\mathfrak{g}, K) -modules. Recall that a (\mathfrak{g}, K) -module W means a \mathfrak{g} -module equipped with a compatible continuous action of K . Equivalently, it is described by the following data:

- for each $\zeta \in q^{-1}\mathbb{Z}$, a vector space W_ζ giving the ζ -weight space of K , so that κ acts on W_ζ by ζ ;
- maps $p : W_\zeta \rightarrow W_{\zeta+2}$ and $m : W_\zeta \rightarrow W_{\zeta-2}$ satisfying $[p, m] = \kappa$.

We recall some facts about classification, see [Howe and Tan 1992] for details. Irreducible, infinite-dimensional (\mathfrak{g}, K) -modules belong to one of three classes; in each case, the weight spaces W_ζ have dimension either zero or 1.

- Highest weight modules of weight ζ ; these are determined up to isomorphism by the fact that their nonzero weight spaces occur in weights $\{\zeta, \zeta - 2, \zeta - 4, \dots\}$. W_ζ is killed by p . One computes using (9) that on such modules, the Casimir element \mathcal{C} acts by $\frac{1}{2}\zeta(\zeta + 2)$.
- Lowest weight modules of weight ζ ; these are determined up to isomorphism by the fact that their nonzero weight spaces occur in weights $\{\zeta, \zeta + 2, \zeta + 4, \dots\}$. W_ζ is killed by m . Again, (9) shows that the Casimir element \mathcal{C} acts by $\frac{1}{2}\zeta(\zeta - 2)$.
- Doubly infinite modules, in which the weights are of the form $\zeta + 2\mathbb{Z}$ for $\zeta \in \frac{1}{q}\mathbb{Z}$.

Definition 2.1. For an infinite-dimensional irreducible (\mathfrak{g}, K) -module W we define the complementary irreducible representation W^{cl} to be

| | |
|---|-------------------------------------|
| the irreducible (\mathfrak{g}, K) -module with highest weight $\zeta - 2$ | if W has lowest weight ζ , |
| the irreducible (\mathfrak{g}, K) -module with lowest weight $\zeta + 2$ | if W has highest weight ζ , |
| W | otherwise. |

The representation W^{cl} can be finite-dimensional; this occurs exactly when W is the underlying (\mathfrak{g}, K) -module of a discrete series representation on $\text{SL}_2(\mathbb{R})$.

In [Section 4](#) we use the following key fact about (\mathfrak{g}, K) -modules.

Proposition 2.2. *Let W be an irreducible infinite-dimensional (\mathfrak{g}, K) -module with Casimir eigenvalue λ . Then, for any (\mathfrak{g}, K) -module V :*

- (a) *If $\mathcal{C} - \lambda$ is surjective on V , then $\text{Ext}_{(\mathfrak{g}, K)}^1(W, V) = 0$.*
- (b) *If V is irreducible, $\text{Ext}_{(\mathfrak{g}, K)}^1(W, V)$ is one-dimensional if $V \simeq W^{\text{cl}}$, and is zero otherwise.*

Proof. We will prove these statements in the case where W is a lowest weight module, which is the case of our main application. The same proof works with slight modifications for W a highest weight or doubly infinite module: in every case, one takes an arbitrary lift of a generating vector, and modifies it using the surjectivity of an appropriate operator.

We prove (a). Take W to be generated by a vector v_ζ of lowest weight ζ with $mv_\zeta = 0$. This implies by the classification above that

$$\lambda = \frac{1}{2}\zeta(\zeta - 2). \quad (11)$$

Take an extension $V \rightarrow E \rightarrow W$; to give a splitting we must lift w_ζ to a vector in E of K -type ζ killed by m . Arbitrarily lift w_ζ to $\tilde{w}_\zeta \in E_\zeta$. Then $m\tilde{w}_\zeta \in V_{\zeta-2}$ and it suffices to show that it lies inside the image of $m : V_\zeta \rightarrow V_{\zeta-2}$, for we then modify the choice of \tilde{w}_ζ by any preimage to get the desired splitting. By [\(9\)](#) and [\(11\)](#) we see that $\mathcal{C} - \lambda : V_{\zeta-2} \rightarrow V_{\zeta-2}$ agrees with $2mp$. Since it is surjective, it follows that in particular $m : V_\zeta \rightarrow V_{\zeta-2}$ is surjective.

We pass to (b). Suppose V is irreducible; then $\text{Ext}_{(\mathfrak{g}, K)}^1(W, V)$ vanishes unless V has the same \mathcal{C} -eigenvalue as W . The argument above exhibits an injection of

$$\text{Ext}_{(\mathfrak{g}, K)}^1(W, V) \hookrightarrow \frac{V_{\zeta-2}}{mV_\zeta}$$

and inspection of K -types amongst those irreducibles with the same \mathcal{C} -eigenvalue as V shows that this also vanishes unless $V \simeq W^{\text{cl}}$, in which case it is one-dimensional. It remains only to exhibit a nontrivial extension of W by W^{cl} , which is readily done by explicit computation. \square

2.4. Globalizations. A globalization of a (\mathfrak{g}, K) -module W is any continuous G -representation on a topological vector space \overline{W} such that $(\overline{W})_K = W$. We will consider two instances of this: the *smooth*, or *Casselman–Wallach globalization* W_∞ , and the *distributional globalization* $W_{-\infty}$.

Following [Casselman 1989], the representation W_∞ is the unique globalization of W as a moderate growth Fréchet G -representation. By definition, such a representation is a Fréchet space F (topologized with respect to a family of seminorms) such that for any seminorm $\|\cdot\|_\alpha$, there is an integer N_α and a seminorm $\|\cdot\|_\beta$ for which

$$\|gw\|_\alpha \leq \|g\|^{N_\alpha} \|w\|_\beta.$$

The distributional globalization is a dual notion. Indeed, denote by W^* the K -finite part of the dual of W , equipped with the contragredient (\mathfrak{g}, K) -module structure. Then

$$(W_\infty)^* = (W^*)_{-\infty}, \quad (12)$$

where on the left-hand side, the dual is understood as continuous.

We recall an explicit construction of $W_{-\infty}$, see [Bunke and Olbrich 1998, Sections 2 and 3], although it will not be directly used in the rest of the paper: Given W^* as above, let $V^* \subset W^*$ be a finite-dimensional K -stable subspace that generates W^* as a (\mathfrak{g}, K) -module. Let $(V^*)^* =: V \subset W$ viewed as a K -representation, and consider the space

$$\mathcal{E}_V = \{f \in C^\infty(G, V) \mid f(gk) = k^{-1}f(g), g \in G, k \in K\}.$$

Then the image of W under the map $i : W \rightarrow \mathcal{E}_V$ characterized by

$$\langle i(w)(g), v^* \rangle := \langle w, gv^* \rangle \quad (w \in W, v^* \in V^*)$$

belongs to the space \mathcal{A}_V^G of sections of moderate growth, i.e., of functions $f \in \mathcal{E}_V$ such that for every $X \in U(\mathfrak{g})$, there is $R = R(f, X)$ for which

$$\|f\|_{X,R} = \sup_{g \in G} \frac{|Xf(g)|}{\|g\|^R} < \infty. \quad (13)$$

We note that this differs from the notion of *uniform* moderate growth, where one requires R to be taken independently of X .

The space \mathcal{A}_V^G is topologized as the direct limit of Fréchet spaces with respect to the seminorms $\|\cdot\|_{X,R}$. The map i is injective since V^* generates W^* , and the distributional globalization is defined by

$$W_{-\infty} := \overline{i(W)} \subset \mathcal{A}_V^G.$$

3. First proof of Theorem 1.2: resolutions of principal series

In this section, we derive Theorem 1.2 from the results of Bunke and Olbrich [1998], adapting the arguments of Section 9 therein to nonspherical principal series. The

two essential ingredients of this argument are the following points established by Bunke and Olbrich, which we shall use as “black boxes”:

- acyclicity of Γ acting on spaces of moderate growth functions on G/K , and
- surjectivity of a Laplace-type operator acting on these spaces.

The first point, at least, is intuitively reasonable: it asserts that moderate growth functions on G/K behave like a free Γ -module; this is plausible since the Γ -action on G/K is (at least, virtually) free.

Given these, the idea of the argument for [Theorem 1.2](#) is as follows. We will first show that principal series representations are realized as spaces of moderate growth Laplacian eigenfunctions on G/K ; by the two points mentioned above, this gives a resolution of the principal series by Γ -acyclic modules. This permits us to compute cohomology of principal series representations. Finally, every irreducible representation is realized as a subquotient of such a representation, and we will then prove [Theorem 1.2](#) by a study of the associated long exact sequence in cohomology.

3.1. Setup. Fix a Casimir eigenvalue λ , and a lattice $\Gamma \subset G$. Given ζ a one-dimensional representation of K , define the following spaces of smooth functions (compare with [Section 2.4](#), and see [\(13\)](#) in particular for the notion of moderate growth, which is *not* the same as uniform moderate growth):

$$\begin{aligned} \mathcal{A}^G \text{ (resp. } \mathcal{A}) &= \text{moderate growth functions on } G \text{ (resp. on } \Gamma \backslash G), \\ \mathcal{A}_\zeta^G, \mathcal{A}_\zeta &= \text{subspace with right } K\text{-type } \zeta: f(gk) = f(g)\zeta(k), \\ \mathcal{A}_\zeta^G(\lambda), \mathcal{A}_\zeta(\lambda) &= \text{subspace with right } K\text{-type } \zeta \text{ and Casimir eigenvalue } \lambda, \\ \text{Cusp}_\zeta(\lambda) &= \text{subspace of } \mathcal{A}_\zeta(\lambda) \text{ consisting of cuspforms.} \end{aligned} \tag{14}$$

We will first prove a variant of [Theorem 1.2](#) for principal series. Let B be the preimage of the upper triangular matrices inside G , which we recall is the q -fold cover of $\text{SL}_2(\mathbb{R})$; we may write

$$B = MAN,$$

where A and N are as in [\(10\)](#), and $M = Z_K(A) \simeq \mathbb{Z}/2q\mathbb{Z}$. Denote by $\xi \in \mathbb{C}$ the character of A sending $a_y \mapsto y^\xi$. Given a pair of characters (σ, ξ) of K and A respectively, let

$$H = \{f \in C^\infty(G) \mid f(mang) = a^{\xi+1} \sigma^{-1}(m) f(g), f \text{ } K\text{-finite}\} \tag{15}$$

be the Harish-Chandra module of K -finite vectors in the corresponding principal series representation. This depends on σ and ξ , but to simplify the notation we will not include them explicitly. We denote by $H_{-\infty}$ its distributional completion (see [Section 2.4](#)); explicitly, if we identify H as above with functions on K which transform on the left under the character σ^{-1} , then $H_{-\infty}$ is the corresponding space of *distributions* on K .

Let us explicate this in the language of [Section 2.3](#). We will parameterize σ by the value of $d\sigma$ at κ ; this is a class in $q^{-1}\mathbb{Z}$ that we will denote by ζ_0 . A K -basis of H is given by vectors e_ζ with $\zeta \in \zeta_0 + 2\mathbb{Z}$, normalized to take value 1 at the identity of G . The actions of raising and lowering operators are given by

$$pe_\zeta = \frac{1}{2}(\zeta + 1 + \xi)e_{\zeta+2} \quad \text{and} \quad me_\zeta = \frac{1}{2}(-\zeta + 1 + \xi)e_{\zeta-2}, \quad (16)$$

and the action of the Casimir on e_ζ is thereby given by $\frac{1}{2}(\xi^2 - 1)$. From these explicit formulas we readily deduce the following statements:

- (a) If $1 + \xi$ does not belong to $\pm\zeta_0 + 2\mathbb{Z}$, then H is irreducible.
- (b) If $1 + \xi$ belongs to either $\zeta_0 + 2\mathbb{Z}$ or $-\zeta_0 + 2\mathbb{Z}$ but not both, then H has the structure

$$0 \rightarrow \bar{V} \rightarrow H \rightarrow V \rightarrow 0, \quad (17)$$

where \bar{V} , V are irreducible (\mathfrak{g}, K) -modules; \bar{V} is the module of highest (lowest) weight ζ according to whether $-\xi - 1$ or $1 + \xi$ belongs to $\zeta_0 + 2\mathbb{Z}$, and $V = \bar{V}^{\text{cl}}$.

- (c) If $1 + \xi$ belongs to both¹ $\zeta_0 + 2\mathbb{Z}$ and $-\zeta_0 + 2\mathbb{Z}$, and $\xi \geq 1$, then H has the structure of an extension

$$V^+ \oplus V^- \rightarrow H \rightarrow F,$$

where V^- is the highest weight representation of weight $-\xi - 1$, and V^+ the lowest weight representation of weight $\xi + 1$, whereas F is the finite-dimensional representation of dimension ξ with weights $-\xi + 1, -\xi + 3, \dots, \xi - 1$. A similar dual description is valid when $\xi \leq 0$, where the finite-dimensional representation now occurs as a subrepresentation.

In the following proposition, we will assume that we are in either cases (a) or (b) of the above classification, that is, H is either irreducible, or decomposes as

$$0 \rightarrow \bar{V} \rightarrow H \rightarrow V \rightarrow 0, \quad (18)$$

where both the subrepresentation and quotient are irreducible (\mathfrak{g}, K) -modules.

Proposition 3.1. *Let G be the degree q connected cover of $\text{SL}_2(\mathbb{R})$. Denote by λ the eigenvalue by which \mathcal{C} acts on H_K ; then there are natural isomorphisms*

$$H^0(\Gamma, H_{-\infty}) \simeq \mathcal{A}_\zeta(\lambda), \quad H^1(\Gamma, H_{-\infty}) \simeq \text{Cusp}_\zeta(\lambda),$$

$$H^i(\Gamma, H_{-\infty}) = 0 \quad \text{for } i \geq 2,$$

where ζ is any K -weight generating the dual (\mathfrak{g}, K) -module H^* .

The condition on ζ is automatic when H is irreducible, and in the case when H is reducible is equivalent to asking that ζ belongs to the K -weights of \bar{V}^* .

¹This happens only when $\zeta_0 \in \mathbb{Z}$, and in particular the representation descends to a representation of $\text{SL}_2(\mathbb{R})$.

Proof. In Section 9 of [Bunke and Olbrich 1998] this result is proven in the case of $q = 1$ and the trivial K -type. We will outline the argument to make clear that it remains valid in the situation where we now work, i.e., permitting a covering of $\mathrm{SL}_2(\mathbb{R})$ and an arbitrary K -type.

Fix $v^* \in H^*$ of K -type ζ , normalized as in the discussion preceding (16). Then the rule sending $\mathcal{D} \in H_{-\infty}$ to the function $\mathcal{D}(gv^*)$ on G induces an isomorphism

$$H_{-\infty} \simeq \mathcal{A}_{\zeta}^G(\lambda). \quad (19)$$

We will outline a direct proof of this isomorphism. Injectivity, at least, follows readily: if \mathcal{D} lies in the kernel, it would annihilate the (\mathfrak{g}, K) -module generated by v^* , which is all of H^* , and by continuity \mathcal{D} is then zero.

For surjectivity, one first checks that K -finite functions lie in the image of the map—that is to say, a function f of fixed right and left K -types, and with a specified Casimir eigenvalue, occurs in the image of the map above. Such an f is uniquely specified up to constants: using the decomposition $G = KAK$, the Casimir eigenvalue amounts to a second-order differential equation for the function $y \mapsto f(a_y)$ for $y \in (1, \infty)$, and of the two-dimensional space of solutions only a one-dimensional subspace extends smoothly over $y = 1$; see [Kitaev 2017, pp. 12 and 13] for an explicit description both of the differential equation and a hypergeometric basis for the solutions.² It follows from this uniqueness that f must agree with $\mathcal{D}(gv^*)$ where \mathcal{D} and v^* match the left and right K -types of f . To pass from surjectivity onto K -finite vectors to surjectivity, we take arbitrary $f \in \mathcal{A}_{\zeta}^G(\lambda)$ and expand it as a sum $\sum_{\xi} f_{\xi}$ of left K -type. Each f_{ξ} has a preimage v_{ξ} according to the previous argument; so one must verify that $\sum_{\xi} v_{\xi}$ converges inside $H_{-\infty}$, and for this it is enough to show that $\|v_{\xi}\|$ grows polynomially with respect to $|\xi|$ (here we compute $\|v_{\xi}\|$ as the L^2 -norm restricted to K in (15)). For this we “effectivize” the previous argument: The moderate growth property of f implies a bound of the form $|f_{\xi}(g)| \leq c\|g\|^N$, uniform in ξ . On the other hand, $f_{\xi} = v_{\xi}(gv^*)$, and such a matrix coefficient always is not too small:

$$|v_{\xi}(gv^*)| \geq (1 + |\xi|)^{-M} \|v_{\xi}\| \quad \text{for some choice of } \|g\| \leq (1 + |\xi|)^M. \quad (20)$$

Such lower estimates on matrix coefficients can be obtained by keeping track of error bounds in asymptotic expressions. They are developed in greater generality in the Casselman–Wallach theory, see, e.g., Corollary 12.4 of [Bernstein and Krötz 2014] for a closely related result. Combining (20) with the upper bound on f_{ξ} shows that $\|v_{\xi}\| \leq c(1 + |\xi|)^{MN+M}$ as desired.

²There are other references in the mathematical literature but Kitaev explicitly considers the universal cover.

This concludes our sketch of proof of (19), that is to say, $H_{-\infty}$ is the kernel of

$$\mathcal{A}_{\zeta}^G \xrightarrow{\mathcal{C}-\lambda} \mathcal{A}_{\zeta}^G \quad (21)$$

in the notation of (14). We now invoke surjectivity of a Laplace operator: the morphism $\mathcal{C} - \lambda$ of (21) is surjective, by [Bunke and Olbrich 1998, Theorem 2.1]; and consequently (21) is in fact a resolution of $H_{-\infty}$. Moreover, [Bunke and Olbrich 1998, Theorem 5.6] asserts that the higher cohomology of Γ acting on \mathcal{A}_{ζ}^G vanishes; it is for this argument that Bunke and Olbrich use “moderate growth” rather than “uniform moderate growth”. Consequently, the Γ -cohomology of $H_{-\infty}$ can be computed by taking Γ -invariants on the complex (21):

$$(\mathcal{A}_{\zeta}^G)^{\Gamma} \xrightarrow{\mathcal{C}-\lambda} (\mathcal{A}_{\zeta}^G)^{\Gamma}.$$

Clearly, the H^0 here coincides with $\mathcal{A}_{\zeta}(\lambda)$. On the other hand, the image of $\mathcal{C} - \lambda$ contains the orthogonal complement of cusp forms (see [Bunke and Olbrich 1998, Theorem 6.3]; compare Proposition 4.1), and so the H^1 coincides with the cokernel of $\mathcal{C} - \lambda$ acting on cusp forms; there we can pass to the orthogonal complement and identify $H^1 \simeq \text{Cusp}_{\zeta}(\lambda)$ as desired.³ \square

Lemmas 3.2 and 3.3 below will be useful in the sequel. We omit the proof of the first one.

Lemma 3.2. *Let ζ be, as in Proposition 3.1, a K -weight on H^* which generates the latter as (\mathfrak{g}, K) -module; fix $v_{\zeta} \in H^*$ nonzero of weight ζ . For any (\mathfrak{g}, K) -module V , there is an isomorphism*

$$\text{Hom}_{(\mathfrak{g}, K)}(H^*, V) \rightarrow V_{\zeta}(\lambda), \quad f \mapsto f(v_{\zeta}), \quad (22)$$

where $V_{\zeta}(\lambda)$ is the subspace of V_{ζ} killed by $\mathcal{C} - \lambda$.

The second is a precise statement of Frobenius reciprocity, stated in a less formal way in (2).

Lemma 3.3. *Let V be a finite length (\mathfrak{g}, K) -module. Then there is an isomorphism*

$$H^0(\Gamma, V_{-\infty}^*) \simeq \text{Hom}_{(\mathfrak{g}, K)}(V, \mathcal{A}_K),$$

where $V_{-\infty}^*$ is the distributional globalization of V^* .

One of the earliest versions of such a statement can be found in [Gelfand et al. 1969, Chapter 1, Section 4]. For completeness we outline the proof, in our language, in Remark 4.3.

For reducible principal series as in (18), we prove:

³In fact, $\mathcal{C} - \lambda$ is adjoint to $\mathcal{C} - \bar{\lambda}$, but the kernel of the latter of either is only nonzero if λ is real, so we do not keep track of the complex conjugate.

Proposition 3.4. *Let $H_{-\infty}$ with Casimir eigenvalue λ decompose as in (18). Then the quotient map $H \rightarrow V$ induces an isomorphism, after passing to distribution globalizations and Γ -cohomology,*

$$H^1(\Gamma, V_{-\infty}) \simeq H^1(\Gamma, H_{-\infty}) (\simeq \text{Cusp}_{\zeta}(\lambda), \text{ by Proposition 3.1}).$$

Proof. The discussion around (17) shows that inverting both ξ and σ gives rise to another principal series \bar{H} which fits into the exact sequence

$$0 \rightarrow V \rightarrow \bar{H} \rightarrow \bar{V} \rightarrow 0, \quad (23)$$

i.e., for which the roles of subrepresentation and quotient are swapped between \bar{V} and V . We will deduce the result by playing off Proposition 3.1 applied to (the distribution globalization of) H , and the same Proposition applied to \bar{H} .

We first consider the long exact sequence associated to (the distribution globalization of) (18), namely

$$\begin{aligned} 0 \rightarrow H^0(\Gamma, \bar{V}_{-\infty}) \rightarrow H^0(\Gamma, H_{-\infty}) \xrightarrow{\Omega} H^0(\Gamma, V_{-\infty}) \\ \rightarrow H^1(\Gamma, \bar{V}_{-\infty}) \rightarrow H^1(\Gamma, H_{-\infty}) \xrightarrow{\Pi} H^1(\Gamma, V_{-\infty}) \rightarrow 0. \end{aligned} \quad (24)$$

We have used here that the next group $H^2(\Gamma, \bar{V}_{-\infty})$ of the sequence vanishes: it is isomorphic to $H^3(\Gamma, V_{-\infty})$ by the long exact sequence associated to (23) and Proposition 3.1, and that H^3 vanishes always. Indeed, let $\bar{\Gamma}$ be the image of $\Gamma \rightarrow \text{PSL}_2(\mathbb{R})$, and $\mu \leq \Gamma$ the kernel of $\Gamma \rightarrow \bar{\Gamma}$; if V is a $\mathbb{C}[\Gamma]$ -module then $H^i(\Gamma, V) = H^i(\bar{\Gamma}, V^{\mu})$, and being a lattice in $\text{PSL}_2(\mathbb{R})$, the virtual cohomological dimension of $\bar{\Gamma}$ is at most 2.

We must show that the penultimate map Π of (24) is an isomorphism. For this it is enough to show that

$$\dim \text{cokernel } \Omega \geq \dim H^1(\Gamma, \bar{V}_{-\infty}).$$

By applying Proposition 3.1 to \bar{H} , we find that $H^1(\Gamma, \bar{V}_{-\infty})$ is a quotient of $\text{Cusp}_{\chi}(\lambda)$, for χ a weight in V^* . It therefore suffices to show that

$$\dim \text{cokernel } \Omega \geq \dim \text{Cusp}_{\chi}(\lambda). \quad (25)$$

We will prove this by exhibiting a subspace

$$H^0(\Gamma, V_{-\infty})^{\text{cusp}} \subset H^0(\Gamma, V_{-\infty}) \quad (26)$$

of the codomain of Ω , which does not meet the image of Ω , and whose dimension equals that of $\text{Cusp}_{\chi}(\lambda)$.

The space $H^0(\Gamma, V_{-\infty})$ is identified, by means of Frobenius reciprocity (see Lemma 3.3) with the space of homomorphisms from the dual (\mathfrak{g}, K) -module V^* to the K -finite vectors \mathcal{A}_K in the space of automorphic forms. Define $H^0(\Gamma, V_{-\infty})^{\text{cusp}}$

to be the subspace corresponding to homomorphisms $V^* \rightarrow \mathcal{A}_K$ that are actually valued in cusp forms. We now show the two properties of this subspace $H^0(\Gamma, V_{-\infty})^{\text{cusp}}$ asserted after (26):

- Its dimension equals that of $\text{Cusp}_\chi(\lambda)$. To see this, apply Lemma 3.2 to \bar{H} , with $\zeta = \chi$ and V the K -finite vectors of the space of cusp forms; it yields an isomorphism

$$\text{Hom}_{(\mathfrak{g}, K)}(H^*, \text{Cusp}_K) \simeq \text{Cusp}_\chi(\lambda).$$

But homomorphisms from H^* to Cusp_K factor through V^* by semisimplicity of the space of cusp forms (which in turn follows by unitarity). This shows that the space $\text{Hom}_{(\mathfrak{g}, K)}(V^*, \text{Cusp}_K)$ has the same dimension as $\text{Cusp}_\chi(\lambda)$, as required.

- It intersects trivially the image of Ω . This amounts to the statement that no homomorphism from V^* to Cusp_K can be extended to a homomorphism from H^* to \mathcal{A}_K . Suppose, then, that $f : H^* \rightarrow \mathcal{A}_K$ is a (\mathfrak{g}, K) -module homomorphism whose restriction to V^* is nonzero and has cuspidal image. We now make use of the orthogonal projection map from all automorphic forms to cusp forms, which exists because one can sensibly take the inner product of a cusp form with any function of moderate growth. Post-composing f with this projection gives a morphism from H^* to the semisimple (\mathfrak{g}, K) -module Cusp_K ; since H^* is a nontrivial extension of \bar{V}^* by V^* , this morphism is necessarily trivial on the subrepresentation V^* , a contradiction. \square

Now let us deduce Theorem 1.2. We divide into three cases according to how the representation W of the theorem can be fit into a principal series. Our division corresponds to the division (a), (b), (c) enunciated after (16), and the statements below about the structure of W can all be deduced from the statements given there.

- W is an irreducible principal series, equivalently, W is doubly infinite. In this case, $W^{\text{cl}} = W$, and combining Proposition 3.1 and Lemma 3.2 gives the statement of Theorem 1.2.
- W is an irreducible subquotient of a principal series H with exactly two composition factors. In this case we can suppose that $W = V^*$ with notation as in (18). In that notation we have $W^* = V$ and $W^{\text{cl}} = \bar{V}^*$. Proposition 3.4 gives $H^1(\Gamma, V_{-\infty}) \simeq \text{Cusp}_\zeta(\lambda)$, and Lemma 3.2 shows that $\text{Cusp}_\zeta(\lambda)$ is identified with the space of (\mathfrak{g}, K) -homomorphisms from H^* to the space of cusp forms; by semisimplicity of the target such a homomorphism factors through the irreducible quotient $\bar{V}^* = W^{\text{cl}}$. This proves Theorem 1.2 in this case.
- W is an irreducible subquotient of a principal series with more than two composition factors. In this case, W is necessarily a highest- or lowest-weight module factoring through $\text{SL}_2(\mathbb{R})$, and there is an exact sequence

$$F \rightarrow H \rightarrow \mathcal{D}, \tag{27}$$

where F is finite-dimensional and \mathcal{D} is the sum of W^* and another highest- or lowest-weight module. Here, $W^{\text{cl}} = F^* \simeq F$ and [Theorem 1.2](#) is equivalent to the vanishing of $H^1(\Gamma, W_{-\infty}^*)$. In the case of a discrete series that factors through $\text{PSL}_2(\mathbb{R})$, this vanishing follows from [\[Bunke and Olbrich 1998, Proposition 8.2\]](#), and the remaining case of an “odd” discrete series is handled by the same argument. Namely, use the long exact sequence associated to (27); the argument of [Proposition 3.1](#) shows that $H^1(\Gamma, H_{-\infty}) = 0$, and also $H^2(\Gamma, F) = 0$ by Poincaré duality because F is nontrivial. Thus also $H^1(\Gamma, \mathcal{D}_{-\infty}) = 0$ and so its summand $H^1(\Gamma, W_{-\infty}^*)$ also vanishes.

4. Second proof of [Theorem 1.2](#): extensions of (\mathfrak{g}, K) -modules

Our original proof of [Theorem 1.2](#) proceeds by a reduction to a computation in the category of (\mathfrak{g}, K) -modules. The two essential ingredients of this argument are:

- (a) The Casselman–Wallach theory [\[Casselman 1989; Wallach 1992\]](#) which gives a canonical equivalence between suitable categories of *topological* G -representations and *algebraic* (\mathfrak{g}, K) -modules.
- (b) Surjectivity of a Laplace-type operator acting, now, on spaces of moderate growth functions on $\Gamma \backslash G$.

We will not prove (a), although we will briefly sketch an elementary proof of what we use from it. We will prove (b) in the next section.

Let λ be the eigenvalue by which the Casimir $C \in Z(\mathfrak{g})$ of (9) acts on W (the irreducible (\mathfrak{g}, K) -module from the statement of [Theorem 1.2](#)). We will use the notation \mathcal{A} from (14) for the space of smooth, *uniform* moderate growth functions f on $\Gamma \backslash G$, i.e., for which there exists R such that for all $X \in \mathfrak{U}$,

$$\|f\|_{X,R} = \sup_{g \in G} \frac{|Xf(g)|}{\|g\|^R} < \infty \quad (28)$$

(compare with (13), and beware that we are using the same notation as in [Section 3](#), but for a slightly different space). We use *uniform* moderate growth because it interfaces more readily with the Casselman–Wallach theory; by contrast, [Section 3](#) used moderate growth because this is used in the acyclicity result mentioned after (21).

Also consider the following subspaces of \mathcal{A} :

$\mathcal{A}_{\lambda-\text{nil}}$ = K -finite functions on which $C - \lambda$ acts nilpotently,

$\text{Cusp}(\lambda)$ = subspace of $\mathcal{A}_{\lambda-\text{nil}}$ consisting of cusp forms.

The precise form of (b) we will use is this:

Proposition 4.1. *The image of the map $C - \lambda : \mathcal{A}_K \rightarrow \mathcal{A}_K$ is precisely the orthogonal complement to $\text{Cusp}(\lambda)$ inside \mathcal{A}_K .*

This is almost [Bunke and Olbrich 1998, Theorem 6.3], except there the argument is for moderate growth functions rather than uniform moderate growth; they state on page 73 that the same proof remains valid in the uniform moderate growth setting. Also, Cassleman [1984, Theorem 4.4] proves, for the trivial K -type, that \mathcal{C} is surjective on spaces of Eisenstein distributions, from which a similar result can be extracted. Since the above statement is in a sense the crux of the argument, and neither reference gives it in precisely this form, we have given a self-contained proof in Section 5. Our proof follows a slightly different strategy and is perhaps of independent interest.

4.1. Proof of Theorem 1.2: reduction to (\mathfrak{g}, K) extensions. We begin the proof of Theorem 1.2 assuming Proposition 4.1. This will proceed in three steps:

- (i) First, using a topological version of Shapiro's lemma, we make the identification $H^1(\Gamma, W_{-\infty}^*) \simeq \text{Ext}_G^1(W_{\infty}, \mathcal{A})$.
- (ii) Next, we pass from the category of G -modules to that of (\mathfrak{g}, K) -modules and produce an isomorphism $\text{Ext}_G^1(W_{\infty}, \mathcal{A}) \simeq \text{Ext}_{(\mathfrak{g}, K)}^1(W, \mathcal{A}_{\lambda-\text{nil}})$.
- (iii) Finally, we compute that $\text{Ext}_{(\mathfrak{g}, K)}^1(W, \mathcal{A}_{\lambda-\text{nil}})$ is isomorphic to the promised space of cuspforms, using the explicit computations from Section 2.3.

In practice, for technical reasons, we carry out (iii) first and then show that the map of (ii) is an isomorphism.

We begin by constructing an isomorphism

$$H^1(\Gamma, W_{-\infty}^*) \simeq \text{Ext}_G^1(W_{\infty}, \mathcal{A}), \quad (29)$$

where W_{∞} is the smooth globalization of W .

On the left, we have the ordinary group cohomology of the discrete group Γ acting on the vector space $W_{-\infty}^*$, without reference to topology. On the right here we use a *topological* version of Ext defined as follows: present \mathcal{A} as a directed union $\varinjlim \mathcal{A}(R)$ of moderate growth Fréchet G -representations (see Section 2.4) obtained by imposing a specific exponent of growth R in (28). The right-hand side is then defined to be the direct limit $\varinjlim \text{Ext}_G^1(W_{\infty}, \mathcal{A}(R))$, where the elements of each Ext group are represented by isomorphism classes of short exact sequences⁴ $\mathcal{A}(R) \rightarrow ? \rightarrow W_{\infty}$, with $?$ a moderate growth Fréchet G -representation and the maps are required to be continuous.

The statement (29) is then a version of Shapiro's lemma in group cohomology. Let us spell out the relationship: for $G_1 \leq G_2$ of finite index, and W a finite-dimensional

⁴Here, the notion of exact sequence is the usual one, with no reference to topology: the first map is injective, and its image is the kernel of the second, surjective map.

G_1 -representation, Shapiro's lemma supplies an isomorphism

$$\begin{aligned} H^1(G_1, W^*) &\stackrel{(i)}{\simeq} H^1(G_2, I_{G_1}^{G_2} W^*) \\ &\stackrel{(ii)}{\simeq} H^1(G_2, (I_{G_1}^{G_2} \mathbb{C}) \otimes W^*) \stackrel{(iii)}{\simeq} \text{Ext}_{G_2}^1(W, I_{G_1}^{G_2} \mathbb{C}). \end{aligned} \quad (30)$$

Here $I_{G_1}^{G_2}$ is the induction from G_1 to G_2 , and we used in (i) Shapiro's lemma in its standard form [Brown 1982, Chapter 3, Sections 5 and 6]; at step (ii) the projection formula $I_{G_1}^{G_2} W^* \simeq I_{G_1}^{G_2} \mathbb{C} \otimes W^*$, and at step (iii) the relationship between group cohomology and Ext-groups which results by deriving the relationship $\text{Hom}_{G_2}(W, V) = (V \otimes W^*)^{G_2}$.

Our statement (29) is precisely analogous to the isomorphism of (30) with Γ playing the role of G_1 , G playing the role of G_2 , and with topology inserted. It can be proven simply by writing down the explicit maps from far left to far right in (30) and checking that they respect topology and are inverse to one another. There is only one point that is not formal: to prove that there is a well-defined map from left to right, one needs to check that the extension of G -representations arising in (iii) by “inflating” a cocycle $j : \Gamma \rightarrow W_{-\infty}^*$ indeed has moderate growth. This requires growth bounds on j , and these follow simply by writing out $j(\gamma)$, for arbitrary $\gamma \in \Gamma$, in terms of the values of j on a generating set using the cocycle relation. We observe that some “automatic continuity” argument of this nature is needed, because, in the statement of (29), the topology of W figures on the right-hand side but not on the left.

As the next step towards Theorem 1.2, observe that there is a natural map

$$\text{Ext}_G^1(W_\infty, \mathcal{A}) \rightarrow \text{Ext}_{(\mathfrak{g}, K)}^1(W, \mathcal{A}_{\lambda-\text{nil}}), \quad (31)$$

where the right-hand side is taken in the category of (\mathfrak{g}, K) -modules.

This “natural map” associates to an extension $\mathcal{A} \rightarrow E \rightarrow W_\infty$ the underlying sequence of K -finite vectors in each of \mathcal{A} , E , W_∞ which are annihilated by some power of $C - \lambda$ (in the case of W_∞ , this space is exactly W , on which $C - \lambda$ acts trivially). That the resulting sequence remains exact follows from surjectivity of $C - \lambda$ in the form of Proposition 4.1. We explicate this: one must verify that each element $w \in W$ has a preimage in E_K killed by some power of $(C - \lambda)$. First, take an arbitrary preimage of w in E and average it over K to produce a preimage $e \in E_K$. Then $(C - \lambda)e$ belongs to the image of \mathcal{A}_K , and can be written as $f_1 + f_2$ with $f_1 \in \text{Cusp}(\lambda) \subset \ker(C - \lambda)$ and $f_2 \in \text{Cusp}(\lambda)^\perp$. Choose, by Proposition 4.1, a class $e' \in \mathcal{A}_K$ with $(C - \lambda)e' = f_2$; then $e - e'$ still lifts w and is now killed by $(C - \lambda)$.

We will show in Section 4.2 that the right-hand side of (31) has dimension

$$m = \text{the multiplicity of } W^{\text{cl}} \text{ in } \text{Cusp}(\lambda),$$

and in Section 4.3 that (31) is actually an isomorphism. This will conclude the proof, remembering that the left-hand side is identified, by means of (29), with $H^1(\Gamma, W_{-\infty}^*)$.

4.2. Evaluation of the (\mathfrak{g}, K) -ext. We compute the (\mathfrak{g}, K) -extension on the right-hand side of (33). The space $\mathrm{Cusp}(\lambda)$ decomposes as a finite direct sum of irreducible (\mathfrak{g}, K) -modules; this follows from the similar L^2 statement, see [Borel 1997, Section 8]. Since each of these irreducible summands has infinitesimal character λ , their underlying (\mathfrak{g}, K) -modules can belong to at most three isomorphism classes, as described in Section 2.3; among these is W^{cl} , the “complementary (\mathfrak{g}, K) -module to W ” from Definition 2.1. Accordingly we decompose

$$\mathcal{A}_{\lambda-\mathrm{nil}} = \mathrm{Cusp}(\lambda)^\perp \oplus (W^{\mathrm{cl}})^m \oplus \bigoplus_{\substack{V \subset \mathrm{Cusp}(\lambda) \\ V \not\cong W^{\mathrm{cl}}}} V, \quad (32)$$

where $\mathrm{Cusp}(\lambda)^\perp$ is the orthogonal complement of $\mathrm{Cusp}(\lambda)$ within $\mathcal{A}_{\lambda-\mathrm{nil}}$, and m is the multiplicity of W^{cl} in $\mathrm{Cusp}(\lambda)$.

The splitting (32) induces a similar direct sum splitting of $\mathrm{Ext}_{(\mathfrak{g}, K)}^1(W, \mathcal{A}_{\lambda-\mathrm{nil}})$. But Proposition 4.1 implies that $\mathcal{C} - \lambda$ defines a surjection from $\mathrm{Cusp}(\lambda)^\perp$ to itself, and so, applying Proposition 2.2,

$$\mathrm{Ext}_{(\mathfrak{g}, K)}^1(W, \mathrm{Cusp}(\lambda)^\perp) = 0.$$

The remaining two summands evaluate via the second part of Proposition 2.2 to \mathbb{C}^m and 0 respectively. This yields

$$\mathrm{Ext}_{(\mathfrak{g}, K)}^1(W, \mathcal{A}_{\lambda-\mathrm{nil}}) \simeq \mathbb{C}^m.$$

This concludes the proof that the right-hand side of (31) has dimension m .

4.3. Comparison of topology and (\mathfrak{g}, K) extensions. To conclude, we must show that the map of (31) is in fact an *isomorphism*.

Injectivity of the resulting map on Ext-groups follows using the Casselman–Wallach theory of canonical globalization; the result is formulated in exactly the form we need in [Bernstein and Krötz 2014, Proposition 11.2], namely, a splitting at the level of (\mathfrak{g}, K) -modules automatically gives rise to a continuous splitting.⁵

For surjectivity, one cannot directly apply the Casselman–Wallach theory because \mathcal{A} is “too big”. However, we saw in Section 4.2 that the right-hand side of (31) actually is generated by the image of $\mathrm{Ext}_{(\mathfrak{g}, K)}^1(W, \mathrm{Cusp}(\lambda))$. The space $\mathrm{Cusp}(\lambda)$ has finite length, and then the results of [Casselman 1989] (in the form of the equivalence of categories, see [Wallach 1992, Corollary, Section 11.6.8]) implies that each such extension of (\mathfrak{g}, K) -modules arises from an extension of smooth globalizations, which readily implies the desired surjectivity.

⁵We sketch the idea of the argument to emphasize that what we use is relatively elementary: Given an abstract (\mathfrak{g}, K) -module splitting $\varphi : W \rightarrow \mathcal{A}$ we must show that it does not distort norms too far. Fixing a generating set w_1, \dots, w_r for W , one shows using bounds similar to (20) that any vector $w \in W$ can be written as $\sum h_i \star w_i$ where h_i are bi- K -finite functions on G and the norms of the h_i are not too large in terms of the norms of w . This permits one to bound the size of $\varphi(w) = \sum h_i \star \varphi(w_i)$.

Remark 4.2. Together, the isomorphisms (29) and (31) give an isomorphism

$$H^1(\Gamma, W_{-\infty}^*) \simeq \text{Ext}_{(\mathfrak{g}, K)}^1(W, \mathcal{A}_{\lambda-\text{nil}}). \quad (33)$$

The analogous statement in *all* cohomological degrees has been proved for cocompact Γ in [Bunke and Olbrich 1997, Theorem 1.4]. However, our argument does not generalize to this case, at least in any routine way: it is not immediately clear to us how to generalize the cocycle growth argument to H^i for $i > 1$.

Remark 4.3. For completeness, because we made use of it earlier, we outline the argument for the much easier degree 0 version of (33), i.e., “Frobenius reciprocity”:

$$H^0(\Gamma, W_{-\infty}^*) \simeq \text{Hom}_{(\mathfrak{g}, K)}(W, \mathcal{A}_{\lambda-\text{nil}}), \quad (34)$$

where we now allow W to be any finite length (\mathfrak{g}, K) -module.

The standard construction of Frobenius reciprocity identifies $H^0(\Gamma, W_{-\infty}^*)$ with continuous G -homomorphisms from W_{∞} to \mathcal{A} ; then, restriction to K -finite vectors defines a class in $\text{Hom}_{(\mathfrak{g}, K)}(W_K, \mathcal{A}) \simeq \text{Hom}_{(\mathfrak{g}, K)}(W_K, \mathcal{A}_{\lambda-\text{nil}})$. This restriction map is an isomorphism by the Casselman–Wallach theory [Wallach 1992, Theorem, Section 11.6.7], taking the target space to be the subspace of \mathcal{A} comprising functions which are (i) by killed by a fixed large power of $(\mathcal{C} - \lambda)$ and (ii) have finite norm (28) for all X and for some fixed large R . This proves (34).

Now (34) implies Lemma 3.3: W is annihilated by an ideal of finite codimension in $Z(\mathfrak{g})$; as such, the image of any (\mathfrak{g}, K) -homomorphism from W to moderate growth functions automatically has image inside functions of uniform moderate growth [Borel 1997, 5.6], and therefore has image in $\mathcal{A}_{\lambda-\text{nil}}$.

5. Surjectivity of Casimir on the space of automorphic forms

The primary analytic ingredient in both proofs is the surjectivity of a Laplacian-type operator; in the first proof this is used on spaces of functions both on G and on $\Gamma \backslash G$, and in the second proof it is used only on $\Gamma \backslash G$. We will now give a self-contained proof of the second version, Proposition 4.1. As noted after that proposition, this statement is essentially in the literature, but given its importance it seemed appropriate to give a self-contained proof.

We follow here the notation of Section 4; in particular, \mathcal{A} is defined using the notion of *uniform* moderate growth. It is enough to show that every function orthogonal to $\text{Cusp}(\lambda)$ occurs in the image of $\mathcal{C} - \lambda : \mathcal{A}_K \rightarrow \mathcal{A}_K$. The basic strategy is as follows:

- (i) In Section 5.4, we decompose elements of \mathcal{A}_K into functions “near the cusp” and functions of rapid decay.
- (ii) In Section 5.5, we construct preimages under $\mathcal{C} - \lambda$ for functions in each subspace. Doing this “near the cusp” amounts to solving an ODE; the construction of preimages for functions of rapid decay is carried out via L^2 -spectral theory.

Since $C - \lambda$ commutes with K , it suffices to prove [Proposition 4.1](#) with \mathcal{A}_K replaced by its subspace \mathcal{A}_ζ with K -type ζ . In what follows, we will regard ζ as fixed.

5.1. Cusps. It is convenient to fix once and for all a fundamental domain for $\Gamma \backslash G$: we take

$$\mathfrak{F} = \{z \in \mathbb{H} : d(z, i) \leq d(\gamma z, i) \text{ for all } \gamma \in \Gamma - \{e\}\}, \quad (35)$$

which describes a convex hyperbolic polygon which is (up to boundary) a fundamental domain for Γ acting on \mathbb{H} ; its pullback to G via $g \mapsto g \cdot i$ is a fundamental domain for $\Gamma \backslash G$, which will often be denoted by the same letter. In particular, \mathfrak{F} can be decomposed in the following way, where the sets intersect only along their boundary:

$$\mathfrak{F} = \mathfrak{F}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \dots \cup \mathcal{C}_h, \quad (36)$$

with \mathfrak{F}_0 compact and each \mathcal{C}_i a *cusp*, that is to say, a G -translate of a region of the form $\{x + iy : a \leq x \leq b, y \geq Y_0\}$. In the Iwasawa coordinates $G = NAK$ of [\(10\)](#), the pullback of \mathcal{C}_i to G therefore has the form

$$\tilde{\mathcal{C}}_i = g_i \cdot \{n_x a_y k : a \leq x \leq b, y \geq Y_0, k \in K\}. \quad (37)$$

The map $\tilde{\mathcal{C}}_i \rightarrow \Gamma \backslash G$ is injective on the interior of $\tilde{\mathcal{C}}_i$. We will often identify $\tilde{\mathcal{C}}_i$ with its image in $\Gamma \backslash G$.

5.2. The constant term and moderate growth functions in the cusp. Let $f \in \mathcal{A}_\zeta$. Fix a cusp i ; we write Γ_i^N for $\Gamma \cap g_i N g_i^{-1}$. The constant term $f_i^N : g_i N g_i^{-1} \backslash G \rightarrow \mathbb{C}$ is defined by the rule

$$f_i^N : x \mapsto \text{average value of } f(g_i n_t g_i^{-1} x) \text{ for } t \in \mathbb{R}. \quad (38)$$

The function $f(g_i n_t g_i^{-1} x)$ is periodic in t and therefore the notion of its average value makes sense. Moreover, the above map is right G -equivariant. A basic (and elementary) fact is that f_i^N is asymptotic to f inside $\tilde{\mathcal{C}}_i$; indeed the function $f - f_i^N$ has rapid decay in $\tilde{\mathcal{C}}_i$, as proved in [\[Borel 1997, 7.5\]](#). Here, we say that a function $J : \tilde{\mathcal{C}}_i \rightarrow \mathbb{C}$ has *rapid decay* if, for any $X_1, \dots, X_r \in \mathfrak{g}$ and any positive integer N we have

$$\sup_{\tilde{\mathcal{C}}_i} \|g\|^N |X_1 \dots X_r J(g)| < \infty. \quad (39)$$

Let us consider more generally functions f on G that are left N -invariant and have fixed right K -type ζ . Such a function may be identified, by means of pullback by $y \mapsto a_y$, with a function f on \mathbb{R}_+ . The condition of the original N -invariant function on G having finite norm under $\|\cdot\|_{X,R}$ for all X , with notation as in [\(28\)](#), is equivalent to asking that

$$\left| \left(y \frac{d}{dy} \right)^j f \right| < C_j \cdot (|y|^{-1} + |y|)^R \quad \text{for all } j. \quad (40)$$

That this condition is necessary is seen by applying (28) to X a product of elements in $\text{Lie}(A)$. To see that it is sufficient, we fix \mathfrak{U} belonging to the universal enveloping algebra of \mathfrak{g} ; now, for any $k \in K$, we may write \mathfrak{U} as a sum of terms $\sum c_i(k)(\text{Ad}(k^{-1})\mathfrak{U}_{N,i})(\text{Ad}(k^{-1})\mathfrak{U}_{A,i})\mathfrak{U}_{K,i}$ where the terms belong to fixed bases for the universal enveloping algebra of N , A and K respectively, and the coefficients $c_i(k)$ are bounded independently of k . This permits us to bound $\mathfrak{U}f(nak)$ and we see that the bound (40) suffices.

This motivates the following definition: Fix $Y_0 > 0$ and denote by $\mathcal{P}_{\geq Y_0}$ the space of smooth functions on \mathbb{R} supported in $y > Y_0$ satisfying (40) for some R . Because of the restriction that $y > Y_0$, this is equivalent to ask that all derivatives are “uniformly” polynomially bounded, i.e., there is R such that, for all j , there exists a constant C_j with

$$|d^j f / dy^j| < C_j(2 + |y|)^{R-j}. \quad (41)$$

5.3. The subspace Eis_λ of Eisenstein series with eigenvalue λ . To each cusp \mathcal{C}_j , we attach an Eisenstein series $E^j(s)$, which is an \mathcal{A}_ζ -valued meromorphic function of the complex variable s , characterized by the fact that for $\text{Re}(s) \gg 1$ it equals

$$E^j(s, g) = \sum_{\gamma \in \Gamma_N^j \backslash \Gamma} H(g_i^{-1} \gamma g)^s,$$

where H is the unique function on G with right K -type ζ , invariant on the left by N , and on A given by $a_y \mapsto y$.

The resulting vector-valued function is holomorphic when $\text{Re}(s) = \frac{1}{2}$ and we denote its value at $s = \frac{1}{2} + it$ by E_t^j . In words, E_t^j is the unitary Eisenstein series of K -type ζ with parameter $t \in \mathbb{R}$ attached to the j -th cusp of $\Gamma \backslash G$. Finally, denoting by λ_t the eigenvalue of \mathcal{C} on E_t^j , let

$$\text{Eis}(\lambda) := \bigoplus_j \{\text{span of all Eisenstein series } E_t^j, \text{ with } t \in \mathbb{R}, \text{ such that } \lambda_t = \lambda\},$$

so that $\text{Eis}(\lambda)$ is a finite-dimensional subspace of \mathcal{A}_ζ annihilated by $\mathcal{C} - \lambda$. However, if the quadratic function $t \mapsto \lambda_t - \lambda$ happens to have a double zero, we include in the above space the derivative $\frac{d}{dt} E_t^j$, for this is also annihilated by $\mathcal{C} - \lambda$. The Casimir eigenvalue of $E^j(s, g)$ is quadratic in s and therefore the dimension of $\text{Eis}(\lambda)$ is at most twice the number of cusps.

5.4. Decomposition of \mathcal{A}_ζ . Consider the subspace of \mathcal{A}_ζ consisting of L^2 -eigenfunctions of the Casimir with eigenvalue λ ; call this $\text{Discrete}(\lambda)$.

Lemma 5.1. *Let $\tilde{\mathcal{C}}_i$ be the cusps for a fundamental domain for the action of Γ on G as in (36). Then every $f \in \mathcal{A}_\zeta$, perpendicular to $\text{Cusp}(\lambda)$, can be written as the sum*

$$f = f_s + \sum_i f_{c_i}, \quad (42)$$

where:

- (i) The function f_s is smooth, has rapid decay at all the cusps, and is perpendicular to $\text{Eis}(\lambda) \oplus \text{Discrete}(\lambda)$.
- (ii) Each f_{c_i} is supported in the cusp $\tilde{\mathcal{C}}_i$ and, with reference to the identification (37):

$$\tilde{\mathcal{C}}_i = g_i \cdot \{n_x a_y k : a \leq x \leq b, y \geq Y_0, k \in K\}$$

has the form

$$n_x a_y k \mapsto P(y) \zeta(k), \quad (43)$$

where P belongs to the space $\mathcal{P}_{\geq Y_0}$ described after (41).

Observe that, although f is only assumed orthogonal to cusp forms, we arrange that f_s is orthogonal also to $\text{Eis}(\lambda)$ and all of $\text{Discrete}(\lambda)$. This is possible because there is a lot of freedom in the decomposition (42). It will be very convenient later.

Proof. This is a straightforward cut-off process; the only delicacy is to ensure that f_s is in fact perpendicular to $\text{Eis}(\lambda)$ and $\text{Discrete}(\lambda)$. We start from f_i^N , the constant term along the i -th cusp as defined in (38). Take φ_i, ψ_i smooth functions on \mathbb{R}_+ where:

- $\varphi_i = 0$ for $y < Y_0$ and $\varphi_i = 1$ for $y > 2Y_0$.
- ψ_i is supported in $(Y_0, 2Y_0)$.

We consider φ_i and ψ_i as functions on $\tilde{\mathcal{C}}_i$ described by the rules $g_i n_x a_y k \mapsto \varphi_i(y)$ and $g_i n_x a_y k \mapsto \psi_i(y) \zeta(k)$ respectively. Now put $f_s = f - \sum_i (\varphi_i f_i^N + \psi_i)$ so that

$$f = f_s + \sum \underbrace{(\varphi_i f_i^N + \psi_i)}_{f_{c_i}}. \quad (44)$$

We will show that, for suitable choice of ψ_i , (44) is the desired splitting of f . All the properties except perpendicularity to $\text{Discrete}(\lambda) \oplus \text{Eis}(\lambda)$ follow from general properties of the constant term discussed in Section 5.2. In particular, the uniform bound on the functions P associated — as in (43) — to the various f_{c_i} follow from the condition that f has uniform moderate growth.

Observe that $\varphi_i f_i^N$ and ψ_i are both perpendicular to all cuspidal functions and in particular to $\text{Cusp}(\lambda)$, because they both arise from functions on $g_i N g_i^{-1} \cap \Gamma \backslash G$ which are left invariant by $g_i N g_i^{-1}$. Therefore f_s is also perpendicular to $\text{Cusp}(\lambda)$.

It remains to choose ψ_i in such a way that f_s is indeed perpendicular to the orthogonal complement of $\text{Cusp}(\lambda)$ inside $\text{Discrete}(\lambda) \oplus \text{Eis}(\lambda)$; call this space $\widetilde{\text{Eis}}(\lambda)$, as it is (potentially) a finite-dimensional enlargement of $\text{Eis}(\lambda)$. To do this, for each $\mathcal{E} \in \widetilde{\text{Eis}}(\lambda)$ we should have

$$\left\langle \sum_i f - \varphi_i f_i^N, \mathcal{E} \right\rangle = \sum_i \langle \psi_i, \mathcal{E}_i^N \rangle_{\tilde{\mathcal{C}}_i}.$$

The right-hand side can be considered as a linear mapping from the vector space of possible ψ_i to the finite-dimensional dual $\widetilde{\text{Eis}}(\lambda)^*$ of the vector space $\widetilde{\text{Eis}}(\lambda)$. It is enough to show this mapping is surjective, and for this it is enough to show that its dual is injective. But the dual map is identified with the constant term

$$\widetilde{\text{Eis}}(\lambda) \rightarrow \bigoplus_i C^\infty(T_i, 2T_i), \quad \mathcal{E} \mapsto (\mathcal{E}_i^N)(g_i a_y)$$

and this is injective: if \mathcal{E}^N vanished in $(T_i, 2T_i)$ then it — being real-analytic — vanishes identically; if this is so for all i , then \mathcal{E} would be a cusp form, contradicting the definition of $\widetilde{\text{Eis}}(\lambda)$. \square

5.5. Surjectivity of $\mathcal{C} - \lambda$. We now show surjectivity of $\mathcal{C} - \lambda$ on each of the two pieces of \mathcal{A}_ζ corresponding to the decomposition of [Lemma 5.1](#).

5.5.1. Surjectivity on the cusp.

Lemma 5.2. *The operator $\mathcal{C} - \lambda$ is surjective on the space of functions on G which*

- *are left N -invariant and have fixed right K -type ζ , and*
- *lie in the space $\mathcal{P}_{\geq Y_0}$ described before [\(41\)](#) when pulled back to \mathbb{R}_+ by means of $y \mapsto a_y$.*

Proof. Let $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ be extended to a function $F : G \rightarrow \mathbb{C}$ by left N -invariance and with fixed κ -weight equal to ζ , so that F has the form

$$F(na_y \exp(\theta k)) = f(y) e^{i\zeta\theta}.$$

Observe that for arbitrary $X_1 \in \mathfrak{n} = \text{Lie}(N)$ and $X_2, \dots, X_k \in \mathfrak{g}$ we have

$$(X_1 \dots X_k F) \text{ is identically zero on } NA.$$

Indeed, the left-hand side is the partial derivative $\partial_{t_1} \dots \partial_{t_k}$ of $F(nae^{t_1 X_1} \dots e^{t_k X_k})$ evaluated at $t_i = 0$, which vanishes since F is independent of t_1 . From this observation, it follows that the action of the operator $\mathcal{C} = \frac{1}{2}H^2 - H + 2XY$ on f agrees with the action of $\frac{1}{2}H^2 - H$ on $f(y)$. Since H acts on f via $2y \frac{d}{dy}$, we get that $\mathcal{C} - \lambda$ acts as the differential operator

$$2y^2 \frac{d^2}{dy^2} - \lambda.$$

We show that $\mathcal{C} - \lambda$ is surjective on $\mathcal{P}_{\geq Y_0}$ explicitly, by constructing a g with $(\mathcal{C} - \lambda)g = f$ via the method of variation of parameters.

The homogeneous solutions to the equation $(2y^2 \frac{d^2}{dy^2} - \lambda)g = 0$ are given by y^{p_1}, y^{p_2} , where the p_i are roots of $2p(1-p) + \lambda = 0$. We assume that $p_1 \neq p_2$, the $p_1 = p_2$ case is similar. A solution to $(\mathcal{C} - \lambda)g = f$ can then be found by taking

$$g = b_1(y)y^{p_1} + b_2(y)y^{p_2},$$

where the b_i satisfy

$$\frac{db_i}{dy} = (-1)^i \frac{1/2}{p_1 - p_2} f(y) y^{-p_i-1}.$$

Taking f as in (41), we take $b_i = \pm \frac{1}{2}(p_i - p_2) \int_{Y_0}^y f(y) y^{-p_i-1}$ for $y > Y_0$ and $b_i(y) = 0$ for $y \leq Y_0$. By construction, if f belongs to $\mathcal{P}_{\geq Y_0}$ then so does b_i and so also g . \square

5.5.2. Surjectivity on functions of rapid decay.

Proposition 5.3. *The image of the map $\mathcal{C} - \lambda : \mathcal{A}_\zeta \rightarrow \mathcal{A}_\zeta$ contains all functions of rapid decay that are orthogonal to $\text{Eis}(\lambda)$ and $\text{Discrete}(\lambda)$.*

Proof. Let f be such a function. We fix an orthonormal basis $\{\varphi_i\}$ for the discrete spectrum of $\mathcal{C} - \lambda$ on $L^2(\Gamma \backslash G)_\zeta$, where the subscript means that we restrict to K -type ζ . For constants μ_j depending only on the width of the various cusps, we have, following, e.g., [Borel 1997, Section 13],

$$f = \sum_i \langle f, \varphi_i \rangle \varphi_i + \mu_j \sum_j \int_{t \geq 0} \langle f, E_t^j \rangle E_t^j dt. \quad (45)$$

A priori this is an equality inside L^2 . Let λ_i and λ_t be, respectively, the eigenvalues of $\mathcal{C} - \lambda$ on φ_i and E_t ; by the assumption on f , these are nonvanishing except when $\langle f, \varphi_i \rangle = 0$ or when $\langle f, E_t \rangle = 0$.

Define $\tilde{f} \in L^2$ by the rule

$$\tilde{f} = \sum_{\lambda_i \neq 0} \frac{\langle f, \varphi_i \rangle}{\lambda_i} \varphi_i + \sum_j \int_{t \in \mathbb{R}} \frac{\langle f, E_t^j \rangle}{\lambda_t} E_t^j dt. \quad (46)$$

It is not hard to see that the right-hand side defines an L^2 -function: The function $\langle f, E_t^j \rangle$ is holomorphic in a neighborhood of $t \in i\mathbb{R}$, as follows from holomorphicity of $t \mapsto E_t^j$ and absolute convergence of the integral defining $\langle f, E_t^j \rangle$. Moreover, by assumption, this holomorphic function vanishes when $\lambda_t = 0$. In particular the function $\langle f, E_t^j \rangle / \lambda_t$ is holomorphic, too; this follows from what we just said if the quadratic function $t \mapsto \lambda_t$ has distinct zeroes, and in the case when it has a double zero t_0 we recall that the derivatives $(dE_t^j/dt)|_{t=t_0}$ also belong to $\text{Eis}(\lambda)$. Therefore, the integrand in (46) is locally integrable in t , and then its global integrability follows from (45).

We claim that \tilde{f} has uniform moderate growth and

$$(\mathcal{C} - \lambda) \tilde{f} = f$$

as desired.

In fact, the summation and integrals in both (45) and (46) are absolutely convergent, uniformly on compact sets, and they define functions of uniform moderate growth; moreover, any derivative $X \tilde{f}$ coincides with the corresponding summation

inserting X inside the sums and integrals. The proof of these claims follow from nontrivial, but relatively standard, estimates. We summarize these estimates, with references. A convenient general reference for all the analysis required is that of Iwaniec [1995]; he works only with the trivial K -type, but analytical issues are exactly the same if we work with a general K -type.

We examine the first summand of (46) first. Let λ_i be the $(\mathcal{C} - \lambda)$ -eigenvalue of φ_i . Then the easy upper bound in Weyl's law (compare [Iwaniec 1995, (7.11), Corollary 11.2] for the sharp Weyl law in the spherical case; the same proof applies with K -type) gives

$$\#\{i : |\lambda_i| \leq T\} \leq \text{const} \cdot T^2. \quad (47)$$

For any $r \geq 0$ we have an estimate

$$|\langle f, \varphi_i \rangle| \leq c_r (1 + |\lambda_i|)^{-r}, \quad (48)$$

arising from integration by parts and Cauchy–Schwarz (using $\|\varphi_i\|_{L^2} = 1$). Finally, there is a constant N with the following property: for any invariant differential operator $X \in \mathfrak{U}$ of degree d , we have a bound

$$|X\varphi_i(g)| \leq (1 + |\lambda_i|)^{d+N} \|g\|^N. \quad (49)$$

This is derivable from a Sobolev estimate, again using the normalization $\|\varphi_i\|_{L^2} = 1$; see, e.g., [Bernstein and Reznikov 2002, (3.7)]. These estimates suffice to treat the cuspidal summand of (46).

Now we discuss the integral summand of (46). To examine absolute convergence of the integral, one reasons exactly as for cusp forms, but rather than pointwise estimates in t one only looks at averages over $T \leq t \leq T + 1$. In place of the L^2 -normalization of φ_i we have the estimate

$$\int_T^{T+1} \int_{\text{ht} \leq Y} |E_t^j(g)|^2 \ll T^2 + \log(Y),$$

where $\text{ht} \leq Y$ means that we integrate over the complement of the set $y \geq Y$ in each cusp. This bound is derived from the Maass–Selberg relations (compare [Iwaniec 1995, Proposition 6.8 and (6.35) and (10.9)]) and average bounds on the scattering matrix (equation (10.13) of the same reference). From this, one obtains in the same way as the cuspidal case bounds on $\int_T^{T+1} |\langle f, E_t^j \rangle|^2$ and $\int_T^{T+1} |XE_t^j|^2$ that are of the same quality as (48) and (49) and the same analysis as for the cuspidal spectrum goes through. \square

5.6. Proof of the proposition. We now prove Proposition 4.1, that is to say, that the image of $\mathcal{C} - \lambda$ is the orthogonal complement of cusp forms. Take $f \in \mathcal{A}_\zeta$ and write $f = f_s + \sum f_{c_i}$ as in Lemma 5.1. By Lemma 5.2 and Proposition 5.3 there are functions $g_i, g \in \mathcal{A}_\zeta$ with

$$(\mathcal{C} - \lambda)g_i = f_{c_i}, \quad (\mathcal{C} - \lambda)g = f_s,$$

where, in the case of g_i , we use [Lemma 5.2](#) to produce a function on $\tilde{\mathcal{C}}_i$, and then extend it by zero to get an element of \mathcal{A}_ζ . Then $g + \sum_i g_i$ is the desired preimage of f under $\mathcal{C} - \lambda$. \square

6. Interpolation and cohomology

We will recall background on the Segal–Shale–Weil representation (see [\[Lion and Vergne 1980\]](#) for details) necessary to explain why the foregoing results imply the interpolation formula of Radchenko and Viazovska [\[2019\]](#). We have already outlined the argument in [Section 1.1](#) and what remains is to explain in detail where the actual numbers in [\(4\)](#) come from.

6.1. The Weil representation. Let $L^2(\mathbb{R})_+$ be the Hilbert space of even square integrable functions on \mathbb{R} , and let \mathcal{S} be the subspace of even Schwartz functions, i.e., even smooth functions f such that

$$\sup_{x \in \mathbb{R}} \left| x^n \frac{d^m}{dx^m} f(x) \right| < \infty \quad (50)$$

for any pair (m, n) of nonnegative integers. Let G be the degree 2 cover of $\mathrm{SL}_2(\mathbb{R})$. There is a unique unitary representation of G on $L^2(\mathbb{R})_+$, the Weil (or oscillator) representation, for which \mathcal{S} is precisely the subspace of smooth vectors and such that the action of \mathfrak{g} on \mathcal{S} is given by

$$X \cdot \phi(x) = -i\pi x^2 \phi(x), \quad Y \cdot \phi(x) = \frac{-i}{4\pi} \frac{\partial^2}{\partial x^2} \phi(x), \quad H \cdot \phi(x) = \left(x \frac{d}{dx} + \frac{1}{2} \right) \phi(x).$$

It then follows that $\kappa = i(X - Y)$ acts by

$$\kappa \cdot \phi(x) = \left(\pi x^2 - \frac{1}{4\pi} \frac{\partial^2}{\partial x^2} \right) \phi(x).$$

The normalization ensures that the action of G is unitary and that the relation $\sigma X \sigma^{-1} = Y$ is preserved, where $\sigma : \mathcal{S} \rightarrow \mathcal{S}$ is the Fourier transform

$$\sigma(\phi)(\xi) = \hat{\phi}(\xi) := \int_{\mathbb{R}} \phi(x) e^{-2\pi i x \xi} dx.$$

Moreover, with respect to the seminorms of [\(50\)](#), the topological vector space \mathcal{S} has the structure of a moderate growth Fréchet representation of G .

The vector $v_{1/2} := e^{-\pi x^2}$ has κ -weight $\frac{1}{2}$ and Casimir eigenvalue $-\frac{3}{8}$. The other K -finite vectors in \mathcal{S} are spanned by its Lie algebra translates; they have the form $q(x) e^{-\pi x^2}$ for q an even polynomial, and have κ -weights $\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \dots$

6.2. The lattice Γ . If $X \in \mathfrak{g}$ is nilpotent, the projection map identifies $\exp(\mathbb{R}X) \subset G$ with the corresponding 1-parameter subgroup of $\mathrm{SL}_2(\mathbb{R})$. In particular, the map $G \rightarrow \mathrm{SL}_2(\mathbb{R})$ splits over any one-parameter unipotent subgroup; thus the groups of upper and lower-triangular matrices have distinguished lifts in G .

In particular, the elements $e = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $f = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ defined in (3) have distinguished lifts \tilde{e}, \tilde{f} to G . They act in the Weil representation by

$$\tilde{e} \cdot \phi(x) = e^{-2\pi i x^2} \phi(x), \quad \tilde{f} \cdot \phi(x) = \sigma \tilde{e} \sigma^{-1} \phi(x). \quad (51)$$

Let $\Gamma \in \mathrm{SL}_2(\mathbb{Z})$ be the subgroup freely generated by e and f . It is the subgroup of $\Gamma(2)$ whose diagonal entries are congruent to 1 mod 4, and is conjugate to $\Gamma_1(4)$.

Lemma 6.1. *There is a splitting $\Gamma \rightarrow G$ which extends the splitting over the two subgroups $\langle e \rangle$ and $\langle f \rangle$. The image of Γ in this splitting are precisely the elements of its preimage leaving fixed the distribution $\mathcal{Q} := \sum_{n \in \mathbb{Z}} \delta_{n^2}$; see Section 1.1 for the definition of δ_n .*

Proof. The lift \tilde{e} of e to G fixes \mathcal{Q} . By Poisson summation, so does the lift \tilde{f} of f . The group $\tilde{\Gamma}$ generated by \tilde{e} and \tilde{f} surjects onto Γ with kernel of size at most two. But $\tilde{\Gamma}$ fixes \mathcal{Q} , and the two lifts of any $g \in \mathrm{SL}_2(\mathbb{R})$ to G act on \mathcal{S} by different signs, so the map $\tilde{\Gamma} \rightarrow \Gamma$ is injective. \square

6.3. Conclusion of the proof. We now fill in the deduction, already sketched in the introduction, of the interpolation Theorem 1.1 from Theorem 1.2.

We first handle a detail of topology from the discussion of Section 1.1, namely, the equivalence between the interpolation statement and its “dual” form. For a Fréchet space F we denote its continuous dual by F^* ; we regard it as an abstract vector space without topology. Then, for $\eta : E \rightarrow F$ a continuous map of Fréchet spaces,

$$\text{if } \eta^* : F^* \rightarrow E^* \text{ is bijective, then } \eta \text{ is a homeomorphism.} \quad (52)$$

Indeed, following [Trèves 1967, Theorem 37.2], a continuous homomorphism $\eta : E \rightarrow F$ of Fréchet spaces is surjective if η^* is injective and its image is weakly closed. Applying this in the situation of (52), we see at least that η is surjective. It is injective because the image of η^* is orthogonal to the kernel of η , and then we apply the open mapping theorem to see that it is a homeomorphism.

To verify the equivalence, asserted in Section 1.1, between Theorem 1.1 and its dual version, we apply (52) to the map Ψ of Theorem 1.1, with codomain the closed subspace of $\mathfrak{s} \oplus \mathfrak{s}$ defined by $\sum_{n \in \mathbb{Z}} \phi(n) = \sum_{n \in \mathbb{Z}} \hat{\phi}(n)$.

The other point that was not proved in Section 1.1 was (4), the actual evaluation of H^0 and H^1 for the dual of the oscillator representation, namely

$$\dim H^0(\Gamma, \mathcal{S}^*) = 1, \quad \dim H^1(\Gamma, \mathcal{S}^*) = 0. \quad (53)$$

Now, \mathcal{S}^* is precisely the distribution globalization of the dual of \mathcal{S}_K , i.e., it is the $W_{-\infty}^*$ of the statement of Theorem 1.2 if we take W to be \mathcal{S}_K . Therefore Theorem 1.2 reduces us to showing that the multiplicity of \mathcal{S}_K (resp. $\mathcal{S}_K^{\mathrm{cl}}$) in the space of automorphic forms (resp. cusp forms) for Γ equals 1 (resp. 0).

From [Section 6.1](#), the K -finite vectors S_K are a realization of the (\mathfrak{g}, K) -module of lowest weight $\frac{1}{2}$, whose complementary representation $(S_K)^{\text{cl}}$ is the (\mathfrak{g}, K) -module of highest weight $-\frac{3}{2}$. In general, a homomorphism from a lowest weight (\mathfrak{g}, K) -module to any (\mathfrak{g}, K) -module W is uniquely specified by the image of the lowest weight vector, which can be an arbitrary element of W killed by m ; and the dual statement about highest weight modules is also valid.

It follows that (\mathfrak{g}, K) -homomorphisms from S_K (respectively $(S_K)^{\text{cl}}$) to the space \mathcal{A} of automorphic forms correspond exactly to holomorphic forms of weight $\frac{1}{2}$ (respectively, antiholomorphic forms of weight $-\frac{3}{2}$); the conditions of being killed by m or p precisely translate to being holomorphic or antiholomorphic. The desired conclusion [\(53\)](#) now follows from:

Lemma 6.2. (a) *The space of holomorphic forms for Γ of weight $\frac{1}{2}$ is one-dimensional, and the space of cuspidal holomorphic forms of this weight is trivial.*

(b) *The space of **cuspidal** holomorphic forms for Γ of weight $\frac{3}{2}$ is trivial; therefore, the space of cuspidal antiholomorphic forms for Γ of weight $-\frac{3}{2}$ is also trivial.*

Proof. For (a), the group Γ is conjugate to $\Gamma_1(4)$, for which the space of modular forms of weight $\frac{1}{2}$ is spanned by the theta series $\theta_{1/2}(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i z n^2}$ [[Serre and Stark 1977](#)].

For (b), we use the fact that multiplication by θ injects the space of weight $\frac{3}{2}$ forms into the space of weight 2 forms. The space of weight 2 cusp forms for $\Gamma_1(4)$ is, however, trivial; indeed, the compactified modular curve $X_1(4)$ has genus zero. The final assertion follows by complex conjugation. \square

6.4. Variants: odd Schwartz functions, higher dimensions, Heisenberg uniqueness. We now show how the same ideas give several other interpolation theorems without changing the group $\Gamma = \langle e, f \rangle$; it may also be of interest to consider (∞, p, q) -triangle groups.

6.4.1. Odd Schwartz functions. The discussion of [Section 6.1](#) on the even Weil representation \mathcal{S} carries verbatim to its odd counterpart \mathcal{T} , whose (\mathfrak{g}, K) -module of K -finite vectors is spanned by the translates of the lowest weight vector $v_{3/2} = x e^{-\pi x^2}$. As above, we compute using [Theorem 1.2](#), to get

$$H^0(\Gamma, \mathcal{T}^*) = \mathbb{C}, \quad H^1(\Gamma, \mathcal{T}^*) = 0.$$

Indeed, the zeroth cohomology $H^0(\Gamma, \mathcal{T}^*)$ is identified with the space of modular forms of weight $\frac{3}{2}$, a one-dimensional space spanned by θ^3 , as can be deduced from [[Cohen and Oesterlé 1977](#)]. As for $H^1(\Gamma, \mathcal{T}^*)$, its dimension is equal to the multiplicity of \mathcal{T}^{cl} in the space of cusp forms on Γ . The representation \mathcal{T}^{cl} has highest weight $-\frac{1}{2}$, and the vanishing of H^1 results from the absence of holomorphic

cuspidal forms of weight $\frac{1}{2}$ on Γ as in [Lemma 6.2](#). We then deduce an interpolation theorem as in [Section 1](#), noting that in addition to the δ_n the distributions $\phi \mapsto \phi'(0)$ (resp. $\phi \mapsto \hat{\phi}'(0)$) are also e - (resp. f -)invariant. Arguing as in [Section 1.1](#) recovers a nonexplicit version of the interpolation theorem of Radchenko and Viazovska for odd Schwartz functions, see [\[Radchenko and Viazovska 2019, Theorem 7\]](#).

6.4.2. Radial Schwartz functions on \mathbb{R}^d . We may, similarly, consider instead the representation \mathcal{S}_d of $\mathrm{SL}_2(\mathbb{R})$ on radial Schwartz functions on \mathbb{R}^d . This is, for reasons very similar to that enunciated in [Section 6.1](#), a lowest weight representation of the double cover of $\mathrm{SL}_2(\mathbb{R})$, but now of lowest weight $\frac{d}{2}$ generated by $e^{-\pi(x_1^2 + \dots + x_d^2)}$. We claim that in all cases the corresponding H^1 continues to vanish. Indeed, for d even the complementary representation W^{cl} is finite-dimensional and does not occur in cusp forms; for d odd, occurrences of W^{cl} in cusp forms correspond just as before to holomorphic cusp forms of weight $\frac{1}{2}(4-d)$ for $\Gamma(2)$, and these do not exist for any odd d . Therefore we find that the values of f and \hat{f} at radii \sqrt{n} determine f , subject only to a finite-dimensional space of constraints (the dimension is equal to that of weight $\frac{d}{2}$ holomorphic forms for $\Gamma(2)$).

6.4.3. Heisenberg uniqueness. A result of Hedenmalm and Montes-Rodríguez [\[2011\]](#) asserts that the map

$$L^1(\mathbb{R}) \rightarrow \text{sequences}, \quad h \mapsto \int h(t) e^{\pi i \alpha n t} dt, \int h(t) e^{\pi i \beta n/t} dt \quad (54)$$

is injective if and only if $\alpha\beta \leq 1$. In their terminology, this yields an example of a “Heisenberg uniqueness pair”. We thank the referee for bringing this result to our attention. Using our techniques, we show that an abstract interpolation formula—admittedly, on a eccentric function space—holds at the transition point $\alpha\beta = 1$.

Theorem 6.3. *Let \mathcal{H} be the space of smooth functions on \mathbb{R} with the property that $x^{-2}h(x^{-1})$ extends from $\mathbb{R} - \{0\}$ to a smooth function on \mathbb{R} . Fix α, β with $\alpha\beta = 1$ and for $n \in \mathbb{Z}$ write $a_n = \int h(t) e^{\pi i \alpha n t} dt$ and $b_n = \int h(t) e^{\pi i \beta n/t} dt$. Then the map*

$$h \mapsto \left((a_n), (b_n), h(0), \lim_{x \rightarrow \infty} x^2 h(x) \right)$$

defines a linear isomorphism of \mathcal{H} with a codimension 3 subspace S of $\mathfrak{s}^2 \oplus \mathbb{C}^2$.

In this form, this neither implies nor is implied by the results of [\[Hedenmalm and Montes-Rodríguez 2011\]](#), but it would be interesting to see if our methods can give results closer to theirs, e.g., by considering different completions of the underlying representation.

We obtain [Theorem 6.3](#) in a similar way to [Theorem 1.1](#)—namely, by applying [Theorem 1.2](#) for the same Γ , but with a different coefficient system. Note that we

⁶Note that integration by parts shows that a_n, b_n indeed belong to the space \mathfrak{s} of sequences with rapid decay, introduced in [Section 1](#).

can and will assume that $\alpha = \beta = 1$ by rescaling. We now consider the space W of smooth 1-forms on $\mathbb{P}_{\mathbb{R}}^1 = \mathbb{R} \cup \infty$, which we may think of equivalently as smooth functions $\Phi(x, y)$ on $\mathbb{R}^2 - \{0\}$ satisfying

$$\Phi(\lambda x, \lambda y) = \lambda^{-2} \Phi(x, y).$$

The 1-form on $\mathbb{P}_{\mathbb{R}}^1$ associated to Φ is characterized by the fact that, when pulled back to $\mathbb{R}^2 - \{0\}$, it gives the 1-form $\Phi(x, y)(x dy - y dx)$. Write

$$a_n(\Phi) = \int \Phi(x, 1) e^{\pi i n x} dx, \quad b_n(\Phi) = \int \Phi(1, y) e^{\pi i n y} dy. \quad (55)$$

Write $h(x) = \Phi(x, 1)$. We note that $x^{-2}h(1/x) = \Phi(1, x)$, and so extends over 0. The map $\Phi \mapsto h(x) = \Phi(x, 1)$ thus identifies W with the space \mathcal{H} described in the theorem. We are reduced then to proving:

Claim. *The rule*

$$\Phi \mapsto (a_n, b_n, \Phi(0, 1), \Phi(1, 0)) \quad (56)$$

defines an isomorphism of W with a codimension 3 subspace of $\mathfrak{s}^2 \oplus \mathbb{C}^2$.

Proof of Claim. We apply [Theorem 1.2](#) to the (\mathfrak{g}, K) -module W_K ; the distribution globalization “ $W_{-\infty}^*$ ” that appears in [Theorem 1.2](#) is simply the topological dual W^* to W .

To analyze the e -invariants on W^* , take an arbitrary e -invariant distribution \mathcal{D} on W . The identification $\Phi \mapsto h$ between the space of -2 -homogeneous Φ and $h \in \mathcal{H}$ contains $C_c^\infty(\mathbb{R})$ in its image; thus, we can consider \mathcal{D} as a distribution on the real line, i.e., given any $h \in C_c^\infty(\mathbb{R})$, we form the corresponding Φ and evaluate \mathcal{D} on it. The result is a periodic distribution under $x \mapsto x + 2$ which must be in the closed subspace spanned by the a_n for $n \in \mathbb{Z}$ — write this distribution $\sum c(n) a_n$. Then the difference $\mathcal{D} - \sum c(n) a_n$ vanishes on $C_c^\infty(\mathbb{R})$, and is therefore a linear combination of the Taylor coefficients of $\Phi(1, y)$ at $y = 0$; the only such distribution that is invariant under e is $\Phi \mapsto \Phi(1, 0)$. It follows that $(W^*)^e$ is spanned topologically by the a_n and evaluation at $(1, 0)$. Similarly, $(W^*)^f$ is spanned topologically by the b_n and evaluation at $(0, 1)$.

We will now compute the cohomology of Γ on W^* .

The space W is identified with a reducible principal series of $\mathrm{SL}_2(\mathbb{R})$ which is an extension $D_2^+ \oplus D_2^- \rightarrow W \rightarrow \mathbb{C}$, where D_2^\pm are the holomorphic and antiholomorphic discrete series of weight 2; the map $W \rightarrow \mathbb{C}$ is the integration over $\mathbb{P}_{\mathbb{R}}^1$. Now [Theorem 1.2](#) implies that $H^1(\Gamma, (D_2^\pm)^*)$ vanishes, whereas $H^0(\Gamma, (D_2^\pm)^*)$ has dimension 2 in both the $+$ and $-$ cases. There is therefore an exact sequence

$$0 \rightarrow \mathbb{C} \rightarrow H^0(\Gamma, W^*) \rightarrow \mathbb{C}^4 \rightarrow H^1(\Gamma, \mathbb{C}) \rightarrow H^1(\Gamma, W^*) \rightarrow 0.$$

The map $\mathbb{C}^4 \rightarrow H^1(\Gamma, \mathbb{C})$ is surjective with two-dimensional kernel, for it amounts to the map from the four-dimensional space of (holomorphic and antiholomorphic)

Eisenstein series for Γ to the two-dimensional cohomology.⁷ This proves that $H^1(\Gamma, W^*)$ vanishes, whereas $H^0(\Gamma, W^*)$ is three-dimensional.

We now apply the Mayer–Vietoris sequence (5). In our current context, it implies that $(W^*)^e$ and $(W^*)^f$ span all of W^* , and their intersection is precisely three-dimensional. This concludes the proof of the claim. \square

It may be of interest to describe the three linear constraints that define this codimension 3 subspace. We follow the notations above. The invariants of Γ on W^* have, as basis A, I, J where

$$\begin{aligned} A(\Phi) &= \int_{\mathbb{P}^1} \Phi, \quad I(\Phi) = \sum_{(m,n) \neq (0,0)} \Phi(m, n) - 2 \sum_{2|n} \Phi(m, n), \\ J(\Phi) &= \sum_{(m,n) \neq (0,0)} \Phi(m, n) - 2 \sum_{2|m} \Phi(m, n), \end{aligned}$$

where in both cases the sum is conditionally convergent (e.g., one can sum over large discs of increasing radii). Then A corresponds to the relation $a_0 = b_0$, whereas both I and J give rise to a relation by expanding the stated intertwiner in two different ways. For example, we compute $I(\Phi)$ in two ways, firstly by summing first over n ,

$$\begin{aligned} \sum_{m \neq 0} m^{-2} \sum_n \left(\Phi\left(1, \frac{n}{m}\right) - 2\Phi\left(1, \frac{2n}{m}\right) \right) + 2 \left(\sum n^{-2} - 2 \sum (2n)^{-2} \right) \Phi(0, 1) \\ \stackrel{\text{P.S.}}{=} - \sum_{m \neq 0, t \in \mathbb{Z}} |m|^{-1} b_{m(2t+1)} + \frac{\pi^2}{6} \Phi(0, 1), \end{aligned}$$

where P.S. stands for Poisson summation, and secondly by summing first over m ,

$$\begin{aligned} \sum_{n \neq 0} \left(n^{-2} \sum_m \Phi\left(\frac{m}{n}, 1\right) - \sum_m 2(2n)^{-2} \Phi\left(\frac{m}{2n}, 1\right) \right) 2 \left(\sum m^{-2} - 2 \sum (m)^{-2} \right) \Phi(1, 0) \\ \stackrel{\text{P.S.}}{=} \sum |n|^{-1} a_{n(4t+2)} - \frac{\pi^2}{3} \Phi(1, 0). \end{aligned}$$

Thus we find that the image of W is cut out by the three relations $a_0 = b_0$,

$$\frac{\pi^2}{3} \Phi(1, 0) + \frac{\pi^2}{6} \Phi(0, 1) = \sum_{m \neq 0, t \in \mathbb{Z}} |m|^{-1} b_{m(2t+1)} + \sum_{n \neq 0, t \in \mathbb{Z}} |n|^{-1} a_{n(4t+2)},$$

and dually

$$\frac{\pi^2}{6} \Phi(1, 0) + \frac{\pi^2}{3} \Phi(0, 1) = \sum_{m \neq 0, t \in \mathbb{Z}} |m|^{-1} a_{m(2t+1)} + \sum_{n \neq 0, t \in \mathbb{Z}} |n|^{-1} b_{n(4t+2)}.$$

⁷Indeed, this map records the obstruction to extending an embedding of $D_2^+ \oplus D_2^-$ into the space of automorphic forms, to the larger space W . An embedding of $D_2^+ \oplus D_2^-$ into the space of automorphic forms corresponds to a pair (f, g) of a holomorphic and antiholomorphic 1-form, and it extends to W when $f dz + g \bar{dz}$ is an exact differential.

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