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Counting locally supercuspidal newforms

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The trace formula is a versatile tool for computing sums of spectral data across families of automorphic forms. Using specialized test functions, one can treat small families with refined spectral properties. This has proven fruitful in analytic applications. We detail such methodology here, with the aim of counting newforms in certain small families. The result is a general formula for the number of holomorphic newforms of weight k and level N whose local representation type at each $p \mid N$ is a fixed supercuspidal representation σ_p of $\mathrm{GL}_2(\mathbb{Q}_p)$. This is given in terms of local elliptic orbital integrals attached to matrix coefficients of the σ_p. We evaluate the formula explicitly in the case where each σ_p has conductor $\leq p^3$. The technical heart of the paper is the explicit calculation of elliptic orbital integrals attached to such σ_p. We also compute the traces of Hecke operators on the span of these newforms. Some applications are given to biases among root numbers of newforms.

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1. Introduction

1.1. *Overview.* Modular forms are holomorphic functions on the complex upper half-plane $\mathbb{H}$ that obey a type of symmetry under the action of $\mathrm{SL}_2(\mathbb{Z})$ (or a congruence subgroup) on $\mathbb{H}$ by linear fractional transformations. They belong to the realm of analysis, but this symmetry embodies a deep link with number theory and algebra. Indeed, Langlands' famous functoriality conjecture predicts that there is a precise connection between the algebraic structure of the field $\mathbb{Q}$ of rational numbers (as captured by representations of its absolute Galois group) and spectral properties of automorphic forms (the latter being simultaneous eigenfunctions of the Laplace operator and its p-adic analogs, the Hecke operators) [13]. This connection is expressed as an equality of L-functions.

Automorphic forms can be elusive, and for most purposes it is not feasible to study them and their L-functions one at a time. The trace formula is a technique that provides access to averages of spectral data across families of forms, where the family is determined by a choice of test function. For instance, by choosing a test function with certain invariance properties, one obtains a sum of Hecke eigenvalues $\lambda_n(h)$ for all eigenforms h of a given level and weight, i.e., the trace of the Hecke operator T_n on $S_k(N)$ (see, for example, [23]).

The trace formula and its relative cousins have seen widespread use in analytic number theory, with applications to such problems as estimating moments of L-functions with consequent subconvexity bounds for a single L-function, determining the asymptotic distribution of the Hecke eigenvalues of a growing family of cusp forms (vertical Sato–Tate laws), and finding densities of low-lying zeros of families L-functions (Katz–Sarnak philosophy). See [3] for a recent survey of these and other applications.

Our aim in the present article is to train the trace formula microscope more narrowly through the use of specialized test functions, thereby providing access to thinner families in the automorphic spectrum. This is achieved using the "simple trace formula", variants of which have been in use since the 1970s [15, (7.21)]. Our motivation (described in the next section) is to count cusp forms in these thin families. But the explicit and flexible local-to-global techniques detailed here for GL(2) can be used in many other applications.

Counterintuitively, by considering smaller families, in some situations one obtains simpler trace formulas and stronger analytic results. We mention here some examples that illustrate this. First, Hu [18] and Hu, Petrow and Young [19] have recently developed Fourier relative trace formulas for newforms with certain prescribed local representation types. This is used to estimate thin averages of Rankin–Selberg L-functions, leading to improved hybrid subconvexity bounds.

In a different direction, in 2007 Booker and Strömbergsson [4] used the Selberg trace formula to provide evidence for Selberg's conjecture that the first Laplace eigenvalue in the cuspidal spectrum of $\Gamma\backslash\mathbb{H}$ for a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is $\geq \frac{1}{4}$. In verifying the conjecture for $\Gamma = \Gamma_1(N)$ for square-free $N < 857$, they observed that the trace formula simplifies upon sieving out the contribution of oldforms in this case. They were also able to restrict to the even (or odd) part of the spectrum. With Lee in [5], they subsequently extended this work to remove the square-free hypothesis on N. However, in this case removing the oldform contribution introduces further complication. To proceed, they developed a novel method to sieve the spectrum down further to twist-minimal newforms, arriving at a simpler formula. In both papers, working with a thinner family extended the reach of their numerical computations.

A general discussion about the value of isolating small families of automorphic forms is given in [45, Section 1.5]. In the breakthrough papers [44; 45], Petrow and Young established Weyl-type subconvexity bounds for Dirichlet L-functions using a family of Maass forms that is locally principal series at all finite places.

1.2. ***Description of main results.*** Given an integer

$$N = \prod_{p \mid N} p^{N_p} > 1,$$

let $H_k(N)$ be the set of cuspidal Hecke newforms of level N and weight k. Each $h \in H_k(N)$ corresponds to a cuspidal automorphic representation π_h of $G(\mathbb{A}) = G(\mathbb{R}) \prod'_p G(\mathbb{Q}_p)$ where $G = \mathrm{PGL}_2$. The representation π_h factors as a restricted tensor product

$$\pi_h \cong \bigotimes_{p\le\infty}{}' \pi_{h,p}$$

of infinite-dimensional irreducible admissible representations of the local groups. We know that $\pi_{h,\infty} = \pi_k$ is the weight k discrete series representation, that for each prime $p \nmid N$, $\pi_{h,p}$ is an unramified principal series representation with Satake parameters determined by the p-th Hecke eigenvalue of h, and that for each $p\,|\,N$, $\pi_{h,p}$ is ramified of conductor p^{N_p} (see, for example, [12]).

There is an algorithm, due to Loeffler and Weinstein [35], to determine the isomorphism class of each ramified $\pi_{h,p}$ given h. Here we consider the opposite problem, namely to understand the cusp forms h with prescribed local ramification behavior. To this end, we define the following spaces of newforms. For each $p\,|\,N$, fix an irreducible admissible representation σ_p of $\mathrm{PGL}_2(\mathbb{Q}_p)$ of conductor p^{N_p}, and let $\hat\sigma = (\sigma_p)_{p\,|\,N}$. We then let $H_k(\hat\sigma)$ be the set of weight k newforms of level N having the local representation type σ_p at each $p\,|\,N$:

$$H_k(\hat\sigma) = \{h \in H_k(N) \mid \pi_{h,p} \cong \sigma_p \text{ for all } p\,|\,N\}.$$

Defining

$$S_k(\hat\sigma) = \mathrm{Span}\, H_k(\hat\sigma), \qquad S_k^{\mathrm{new}}(N) = \mathrm{Span}\, H_k(N),$$

we have

$$S_k^{\mathrm{new}}(N) = \bigoplus_{\hat\sigma} S_k(\hat\sigma), \tag{1-1}$$

where $\hat\sigma$ ranges over all tuples as above.

The dimensions of the spaces $S_k^{\mathrm{new}}(N)$ have been computed by Greg Martin in [37], by sieving the well-known dimension formulas for the full spaces $S_k(N)$. It is an open problem to refine these dimension formulas by computing $\dim S_k(\hat\sigma) = |H_k(\hat\sigma)|$ for each tuple $\hat\sigma$. More generally one can ask for the traces of Hecke operators on $S_k(\hat\sigma)$. A complete solution to this problem seems well out of reach, but even special cases are of great interest. For example, such information would enable investigations into the effect of the underlying representation type on various statistical properties of cusp forms.

In some special cases, asymptotic results about $|H_k(\hat\sigma)|$ are known. When p is a finite prime, the representation σ_p of $G(\mathbb{Q}_p)$ is either principal series, special, or supercuspidal [7, Section 9.11]. Only the latter two types are square-integrable (assuming unitary central character), and these are amenable to study via the trace formula. Kim, Shin and Templier [21] gave asymptotics for automorphic representations with prescribed supercuspidal local behavior in a quite general setting. In the case of $\mathrm{PGL}_2(\mathbb{Q})$, their work shows that if each σ_p is supercuspidal,

$$|H_k(\hat\sigma)| \sim \tfrac{1}{12}(k-1) \prod_{p\,|\,N} d_{\sigma_p} \tag{1-2}$$

as $k, N \to \infty$, where d_{σ_p} is the formal degree of σ_p, suitably normalized. They use the trace formula, and the main technical input is a bound for the elliptic orbital integrals attached to supercuspidal matrix coefficients. In a related earlier work, Weinstein [56] gave asymptotics for cusp forms with prescribed local inertial types, concluding that the set of types lacking global realization is finite. Fixing inertial type is weaker than fixing the local representation, but this result includes types which are not square-integrable. This is discussed further in a recent paper of Dieulefait, Pacetti and Tsaknias [10].

We remark that in Corollary 7.2 we will show that the asymptotic (1-2) is in fact an equality when $k \geq 3$ is odd (so in particular the nebentypus is nontrivial) and N has a prime factor $p > 3$ with $\operatorname{ord}_p(N)$ odd.

When N is square-free, each σ_p is necessarily special. Going beyond asymptotics, Kimball Martin [38] computed $|H_k(\hat{\sigma})|$ explicitly in this case, by applying Yamauchi's trace formula for Atkin–Lehner operators. As an interesting consequence, he discovered that there is a bias among newforms of square-free level, favoring root number $+1$: letting $S_k^{\pm}(N)$ denote the span of the newforms of root number ± 1, we have

$$\dim S_k^{+}(N) - \dim S_k^{-}(N) \geq 0,$$

when N is square-free, with the inequality being strict with finitely many explicit exceptions. For example, if $N > 3$ and $k > 2$,

$$\dim S_k^{+}(N) - \dim S_k^{-}(N) = c_N h(-N), \tag{1-3}$$

where $c_N \in \left\{\frac{1}{2}, 1, 2\right\}$ is a constant depending on the equivalence class of N modulo 8, and $h(-N)$ is the class number of $\mathbb{Q}(\sqrt{-N})$.

In the present paper, we further investigate the case where each σ_p is supercuspidal. Our first main result is Theorem 4.2 giving, for such tuples $\hat{\sigma}$, a general formula for the trace of a Hecke operator $T_{\mathtt{n}}$ on $S_k(\hat{\sigma})$ as a main term plus a finite sum of elliptic orbital integrals $\Phi(\gamma, f)$. This theorem is obtained from the adelic GL_2 trace formula using a test function f built using supercuspidal matrix coefficients at the ramified places. In Section 3.3 we show how each global elliptic orbital integral can be factorized into a product of local ones, multiplied by a global measure term that is computed in Theorem 4.16. This global measure is the source of the class numbers of quadratic number fields that appear in classical trace formulas. The local orbital integrals at primes not dividing the level are evaluated explicitly over an arbitrary local field of characteristic 0 in Sections 4.4 and 4.5. We have kept these calculations as general as possible in order that they may find use in other applications of the trace formula.

Theorem 4.2 thereby reduces explicit evaluation of $\operatorname{tr}(T_{\mathtt{n}} \mid S_k(\hat{\sigma}))$ to the calculation of certain local elliptic orbital integrals at the places dividing the level. We demonstrate proof of concept in Sections 5 and 6 by carrying out the latter in the special case where each σ_p has conductor $\le p^3$. As recalled in Section 3.1, the supercuspidals come in two series: the unramified supercuspidals, of conductor p^{2r}, and the ramified supercuspidals, of conductor p^{2r+1}. We thus treat the first ($r=1$) family in each series. The result is the following explicit formula for $\operatorname{tr}(T_{\mathtt{n}}\,|\,S_k(\hat\sigma))$ under this restriction. We allow nontrivial nebentypus, which requires the tuple $\hat\sigma$ to satisfy a global central character constraint described in Section 5.3. Of course, when $\dim S_k(\hat\sigma)=1$ as sometimes happens when k and N are small, it provides a direct way to compute the Fourier coefficients of the associated newform.

Theorem 1.1. *Let $N=S^2T^3>1$ for S, T relatively prime and square-free, and let ω' be a Dirichlet character of level N and conductor dividing ST. Let $\hat\sigma=(\sigma_p)_{p\,|\,N}$ be a tuple of supercuspidal representations, with σ_p of conductor p^2 (resp. p^3) if $p\,|\,S$ (resp. $p\,|\,T$), chosen compatibly with ω' as in Section 5.3. For $k>2$ satisfying $\omega'(-1)=(-1)^k$, let $S_k(\hat\sigma)\subseteq S_k^{\mathrm{new}}(N,\omega')$ be the associated space of newforms. Then for $(\mathtt{n},N)=1$ and $T_{\mathtt{n}}$ the usual Hecke operator defined in Section 4.1,*

$$\begin{aligned}\operatorname{tr}(T_{\mathtt{n}}\,|\,S_k(\hat\sigma))\\ =\mathtt{n}^{(k/2)-1}\Bigg[&\overline{\omega'(\sqrt{\mathtt{n}})}\tfrac{1}{12}(k-1)\prod_{p\,|\,S}(p-1)\prod_{p\,|\,T}\tfrac12(p^2-1)\\ &+\frac12\sum_{M\,|\,T}\Phi\left(\begin{pmatrix} & -\mathtt{n}M\\ 1 & \end{pmatrix}\right)+\sum_{M\,|\,T}\ \sum_{1\le r<\sqrt{4\mathtt{n}/M}}\Phi\left(\begin{pmatrix}0 & -\mathtt{n}M\\ 1 & rM\end{pmatrix}\right)\Bigg],\end{aligned}$$

where $\omega'(\sqrt{\mathtt{n}})$ is taken to be 0 if $\mathtt{n}$ is not a perfect square. Each orbital integral $\Phi(\gamma)$ as above may be evaluated explicitly using

$$\Phi(\gamma)=\frac{2h(E)}{w_E\,2^{\omega(d_E)}}\Phi_\infty(\gamma)\prod_{p\,|\,N}\Phi_p(\gamma)\prod_{\ell\,|\,\Delta_\gamma,\,\ell\nmid N}\Phi_\ell(\gamma). \tag{1-4}$$

Here, ℓ and p denote prime numbers, Δ_γ is the discriminant of the characteristic polynomial of γ, $E=\mathbb{Q}[\gamma]$ is an imaginary quadratic field with class number $h(E)$, discriminant d_E (with $\omega(d_E)$ distinct prime factors) and w_E roots of unity. Given

$$\gamma=\begin{pmatrix}0 & -\mathtt{n}M\\ 1 & rM\end{pmatrix}$$

for $0\le r<\sqrt{4\mathtt{n}/M}$, the factors in (1-4) are given explicitly as follows.

Taking $\theta_\gamma=\arctan(\sqrt{|\Delta_\gamma|}/rM)$ (interpreted as $\frac{\pi}{2}$ if $r=0$),

$$\Phi_\infty(\gamma)=-\frac{\sin((k-1)\theta_\gamma)}{\sin(\theta_\gamma)}$$

(as in Proposition 4.3).

Suppose $\ell \mid \Delta_\gamma$ and $\ell \nmid N$. Then if γ is hyperbolic in $G(\mathbb{Q}_\ell)$,

$$\Phi_\ell(\gamma) = |\Delta_\gamma|_\ell^{-1/2}$$

(as in Proposition 4.4). *If γ is elliptic in $G(\mathbb{Q}_\ell)$, then* (as in Proposition 4.8 and (4-20), (4-21))

$$\Phi_\ell(\gamma) = e_\gamma(\ell) \sum_{j=0}^{\mathrm{ord}_\ell(b)} \ell^j \left(1 + \frac{2 - e_\gamma(\ell)}{\ell} \delta_{j>0}\right),$$

where $\delta_{j>0}$ is an indicator function, $e_\gamma(\ell) \in \{1, 2\}$ is 2 if and only if ℓ ramifies in E, and b is defined by $\Delta_\gamma = b^2 d_E$ for d_E the discriminant of E.

Suppose $p \mid N$. If γ is hyperbolic in $G(\mathbb{Q}_p)$, then $\Phi_p(\gamma) = 0$. So we will assume that γ is elliptic in $G(\mathbb{Q}_p)$. We consider the three cases $p \mid M$, $p \mid (T/M)$, and $p \mid S$ separately. If $p \mid M$, then $\Phi_p(\gamma) = 0$ unless there exists y such that $y^2 \equiv -pt_p/\mathtt{n}M$ mod p, *where t_p is the parameter of the fixed supercuspidal representation $\sigma_p = \sigma_{t_p}^{\zeta_p}$ of conductor p^3* (see Section 5.2). *In this case,*

$$\Phi_p\left(\begin{pmatrix} 0 & -\mathtt{n}M \\ 1 & rM \end{pmatrix}\right) = \overline{\zeta_p}\left[e\left(-\frac{yrM}{p^2}\right)\omega_p(y) + \delta(p \neq 2)\, e\left(\frac{yrM}{p^2}\right)\omega_p(-y)\right]$$

(as in Proposition 6.4), *where ζ_p and ω_p are the root number and central character of σ_p respectively, $e(x) = e^{2\pi i x}$ and δ is an indicator function.*

If $p \mid (T/M)$, then $\Phi_p(\gamma) = 0$ unless the characteristic polynomial P_γ of γ has a nonzero double root modulo p, say

$$P_\gamma(X) \equiv (X - z)^2 \bmod p \quad \textit{for some } z \in (\mathbb{Z}/p\mathbb{Z})^*. \tag{1-5}$$

Under this condition, we have (as in Proposition 6.5 and its remarks)

$$\Phi_p(\gamma) = \frac{\overline{\omega_p(z)}}{p} \sum_{n=1}^{\mathrm{ord}_p(\Delta_\gamma)-1} \sum_{c \bmod p} \mathcal{N}_\gamma(c, n) \sum_{y=1}^{p-1} e\left(\frac{yc}{zp}\right) e\left(-\frac{t_p}{yzp}\right)^{\delta(n=1)},$$

where $t_p \in (\mathbb{Z}/p\mathbb{Z})^*$ is the parameter of $\sigma_p = \sigma_{t_p}^{\zeta_p}$, ω_p is its central character, $e(x) = e^{2\pi i x}$, and

$$\mathcal{N}_\gamma(c, n) = \#\{b \bmod p^{n+1} \mid P_\gamma(b) \equiv cp^n \bmod p^{n+1}\}.$$

Finally, suppose $p \mid S$. If (1-5) *is satisfied, then* (as in Proposition 6.8),

$$\Phi_p(\gamma) = -\overline{\omega_p(z)} + \frac{\overline{\omega_p(z)}}{p} \sum_{n=1}^{\mathrm{ord}_p(\Delta_\gamma)-1} \left[(p-1)\,\mathcal{N}_\gamma(0, n) - \sum_{c=1}^{p-1} \mathcal{N}_\gamma(c, n)\right]$$

for $\mathcal{N}_\gamma(c, n)$ as above. On the other hand, if P_γ is irreducible modulo p, then

$$\Phi_p(\gamma) = -\overline{\nu(\gamma)} - \overline{\nu^p(\gamma)},$$

where ν is the primitive character of $\mathbb{F}_{p^2}^*$ attached to the fixed supercuspidal σ_p of conductor p^2 (see Section 5.1), $\omega_p = \nu|_{\mathbb{F}_p^*}$, *and we interpret the above to mean $-\overline{\nu(x)} - \overline{\nu^p(x)}$ if $x \in \mathbb{F}_{p^2}^*$ has the same minimum polynomial over $\mathbb{F}_p$ as the reduction of γ mod p.*

Remarks. (1) What we call $S_k(N, \omega')$ is usually called $S_k(N, \omega'^{-1})$. See the beginning of Section 4 for explanation. The reason we assume that the conductor of ω' divides ST is that this is necessary for the existence of tuples $\hat{\sigma}$ given the conductor hypotheses, by [55, Proposition 3.4].

(2) The theorem contains various simple conditions under which an orbital integral as in (1-4) vanishes. These are summarized and established in Proposition 5.6.

(3) Analytic applications often require uniform bounds for the orbital integrals appearing on the geometric side. Such bounds were established in a much more general context by Kim, Shin and Templier [21, (1.5), (1.6), (1.8)]. Using these, they proved a vertical (fixed p) equidistribution result for p-th Hecke eigenvalues in $S_k(\hat{\sigma})$ as $N \to \infty$, refining the result of Serre [51]. Their paper includes several helpful examples to explain their results in the setting of PGL(2). The explicit formulas for local orbital integrals developed in the present paper illustrate their bounds; see the remarks after Proposition 6.5, for example.

(4) Although we describe some interesting consequences of Theorem 1.1 below in Section 1.3, perhaps the main utility of this article is the methodology leading to the theorem, rather than this particular trace formula. Indeed, there are any number of variants that one could pursue simply by doing some additional local computations and updating the set of relevant global γ's on the geometric side:

- One could capture newforms with prescribed representation type at some places, and, less restrictively, prescribed local conductor at some other places. For the latter places, the local elliptic orbital integral calculation is carried out in [28].

- We have excluded the case where $\mathrm{ord}_p(N) = 1$ at a prime p for the same reason that we impose $k > 2$: the matrix coefficients of the local representations in such cases are square-integrable but not integrable [23, Proposition 14.3; 53]. So these functions cannot be used directly in the trace formula. One could incorporate these representation types either by using pseudocoefficients [21, Example 6.6; 30; 43], or, via the Jacquet–Langlands correspondence, by computing the corresponding local orbital integrals on a quaternion algebra [22].

- One could capture Maass newforms with prescribed local behavior by taking the archimedean component f_∞ of the test function to be bi-SO(2)-invariant, as described, for example, in [26, Chapters 3 and 4]. In this case, the inclusion of a supercuspidal matrix coefficient at some place p will annihilate the continuous and residual spectra, but at least two such places would be needed in order to annihilate the hyperbolic and unipotent terms on the geometric side of the trace formula, as explained in Theorem 3.3 below. Further, in this case γ need no longer be elliptic in $G(\mathbb{R})$ in order to contribute nontrivially, so there are more relevant γ's that would have to be considered.

• The nonarchimedean local calculations in the present paper are all carried out over arbitrary p-adic fields, so with some additional global considerations one could work over a number field.

The technical heart of the paper is Section 6, in which we calculate local elliptic orbital integrals attached to the supercuspidal representations of conductor $\le p^3$. Character values of supercuspidal representations on various groups appear in many places, but the orbital integral calculations in Section 6 are new. Some related calculations were made by Palm in his doctoral thesis [43, Section 9.11]. Although there are some errors in that work, the methods have been adapted for our computations.

In Section 7 we illustrate Theorem 1.1 by computing dimension formulas and some examples of $\operatorname{tr}(T_{\mathtt{n}} \mid S_k(\hat{\sigma}))$ for $\mathtt{n} > 1$.

1.3. ***Dimension formulas and root number bias.*** Upon taking $\mathtt{n} = 1$ in Theorem 4.2, we obtain a general formula for $\dim S_k(\hat{\sigma})$, given in Theorem 7.1. As shown there, the list of relevant γ can be narrowed considerably when $\mathtt{n} = 1$; only $M = T, \frac{T}{2}$ contribute to the formula when $T > 3$. We will state some special cases below, but first we provide some additional motivation.

Simple supercuspidals are the representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ with conductor p^3. Assuming trivial central character, they can be parametrized by the pairs (t, ζ) where $t \in (\mathbb{Z}/p\mathbb{Z})^*$ and $\zeta \in \{\pm 1\}$. There are thus $2(p-1)$ such representations, denoted σ_t^ζ, and each is constructed in the same way via compact induction from a character $\chi_{t,\zeta}$ of a certain open compact-mod-center subgroup H_t' of $\mathrm{GL}_2(\mathbb{Q}_p)$.

An interesting question is whether each member of such a local family has the same global multiplicity, in the following sense. For $T > 1$ square-free, consider $N = T^3$ in (1-1), with $\hat{\sigma}$ running over all tuples $(\sigma_{t_p}^{\zeta_p})_{p \mid T}$. (We assume trivial central character for the moment, though the general case is considered in the main body of this paper.) In this case we have the dimension formula

$$\dim S_k^{\mathrm{new}}(T^3) = \tfrac{1}{12}(k-1) \prod_{p \mid T} (p^2-1)(p-1) \tag{1-6}$$

as in [37]. Since there is no immediately apparent reason for nature favoring one simple supercuspidal over another, one might surmise that the subspaces $S_k(\hat{\sigma})$ all have the same dimension, i.e., that the asymptotic (1-2), which in the present situation becomes

$$\dim S_k(\hat{\sigma}) \sim \tfrac{1}{12}(k-1) \prod_{p \mid T} \tfrac{1}{2}(p^2-1), \tag{1-7}$$

is an equality. (Note that the right-hand side of (1-7) results from dividing (1-6) by the number $2(p-1)$ of simple supercuspidals at each place $p\,|\,T$.) This would be consistent with a 2011 calculation of Gross [17, p. 1255], who fixed the tuple of parameters $(t_p)_{p|N}$ and allowed the ζ_p parameters to vary. Using the trace formula he showed

$$\sum_{(\zeta_p)_{p|T}} \dim S_k((\sigma_{t_p}^{\zeta_p})_{p|T}) = \tfrac{1}{12}(k-1)\prod_{p|T}(p^2-1), \tag{1-8}$$

which is what one would expect, upon dividing (1-6) by the number of tuples $(t_p)_{p|T}$.

However, equation (1-7) is *not* in fact an equality in general, for the simple reason that, as we spell out at (5-22), the right-hand side of (1-7) fails to be an integer for infinitely many values of T. This is manifested in recent work of Pi and Qi [46], who considered a sum different from that treated by Gross, namely, varying the t_p and ζ_p parameters subject to the constraint $(-1)^{k/2}\prod_{p|T}\zeta_p = \epsilon$ for fixed $\epsilon \in \{\pm 1\}$. This amounts to counting the newforms with root number ϵ. They found, for $k \geq 4$ even and square-free $T > 3$, that

$$\dim S_k^{\mathrm{new}}(T^3)^{\pm} = \tfrac{1}{24}(k-1)\prod_{p|T}(p^2-1)(p-1) \pm \tfrac{1}{2}c_T\,\varphi(T)\,h(-T), \tag{1-9}$$

where c_T and h are as in (1-3) and φ is Euler's φ-function. This shows that, just as in the case of square-free level, there is a bias in favor of positive root number. Instead of the Arthur–Selberg trace formula, they used a Petersson formula obtained using the simple supercuspidal new vector matrix coefficient from [27].

By evaluating the $S = \mathtt{n} = 1$ case of Theorem 1.1, in Section 7.4 we obtain an explicit formula for $\dim S_k(\hat{\sigma})$ that refines each of the above results. For example, we have the following.

Theorem 1.2. *Let $N = T^3$ for $T > 3$ odd and square-free, let $k > 2$ be even, and let $\hat{\sigma} = (\sigma_{t_p}^{\zeta_p})_{p|N}$ be a tuple of simple supercuspidal representations with trivial central characters. Then*

$$\dim S_k(\hat{\sigma}) = \tfrac{1}{12}(k-1)\prod_{p|N}\tfrac{1}{2}(p^2-1) + \Delta(\hat{t})\,\epsilon(k,\hat{\zeta})\,b_T\,h(-T), \tag{1-10}$$

where $\Delta(\hat{t}) \in \{0,1\}$ is nonzero if and only if $-pt_p/T$ is a square modulo p for each $p\,|\,T$, $\epsilon(k,\hat{\zeta}) = (-1)^{k/2}\prod_{p|N}\zeta_p$ is the common global root number of the newforms comprising $H_k(\hat{\sigma})$, b_T is a certain power of 2 depending on T mod 8, and $h(-T)$ is the class number of $\mathbb{Q}[\sqrt{-T}]$.

This is a special case of Theorem 7.17, which also allows for T even. The presence of $\Delta(\hat{t})$ demonstrates that the dimension is not simply a function of the weight, level and root number (even when the right-hand side of (1-7) is an integer). Indeed, as described in [6] for example, each σ_p has attached a ramified quadratic extension of $\mathbb{Q}_p$, namely $E_{\sigma_p} = \mathbb{Q}_p(\sqrt{t_p p})$, which depends only on the Legendre symbol (t_p/p). So $\dim S_k(\hat{\sigma})$ depends only on T, k, the fields E_{σ_p}, and the global root number. (If T is even, the dimension also depends on the local root number ζ_2.)

The second term in (1-10) comes from an elliptic orbital integral. These do not appear in (1-8), but combine to form the error term in (1-9). Indeed, the local root number already appears as a coefficient in our local test function at the places dividing T, so the global root number naturally appears in the elliptic orbital integral that yields the error term in (1-10). This helps explain the positive bias of the root number in this situation.

At the end of Section 7.4, we indicate how our results recover the dimension formula (1-6) and the root number bias (1-9). In Theorem 7.16 we find that the root numbers of newforms of level 27 have a strict bias toward -1 (among the possibilities $\pm 1, \pm i$) when $k \equiv 5 \bmod 6$ and the nebentypus has conductor 3.

In a more recent paper, K. Martin [39] addressed the question of root number bias in $S_k^{\mathrm{new}}(N)$ for arbitrary levels. He showed that there is a bias towards root number $+1$ with one exception: when $N = S^2$ for a square-free number S and $(-1)^{k/2} = -\prod_{p\mid S}(-1)$, then the root number has a strict negative bias when k is sufficiently large. In discussing why the exceptions arise, he noted that the picture is obscured by the existence of newforms of level S^2 which are twists of forms of lower level. (No such forms exist in the $N = T^3$ case discussed above.)

Theorem 1.1 allows us to investigate this further, since the subspace $S_k^{\min}(S^2) \subseteq S_k^{\mathrm{new}}(S^2)$ spanned by the newforms which are not twists of newforms of lower level is the direct sum

$$S_k^{\min}(S^2) = \bigoplus_{\hat{\sigma}} S_k(\hat{\sigma}),$$

ranging over all $\hat{\sigma} = (\sigma_p)_{p\mid S}$ with each σ_p a supercuspidal representation of conductor p^2 (a "depth zero" supercuspidal) and trivial central character.

In fact, even without using a specialized trace formula, we can infer the existence of negative bias for the root numbers in $S_k^{\min}(S^2)$ for many pairs (S, k) by the following heuristic coming from finite fields (see Section 5.1 for more detail and a summary of the construction of depth zero supercuspidals). Given an odd prime p, there are $p-1$ primitive characters of $\mathbb{F}_{p^2}^*$ with trivial restriction to $\mathbb{F}_p^*$. It follows that the number of σ_p as above is $\frac{1}{2}(p-1)$. If $p \equiv 3 \bmod 4$, this number is odd, so the set of such σ_p contains a preponderance either of local root number $\epsilon_p = +1$ or $\epsilon_p = -1$. So if S is a product of such primes, then for some integer $c \geq 1$ there are c more tuples $\hat{\sigma}$ with one nonarchimedean sign $\epsilon_{\mathrm{fin}} = \prod_{p\mid S} \epsilon_p$ than the other. By (1-2), the spaces $S_k(\hat{\sigma})$ all have roughly the same dimension $\frac{1}{12}(k-1)\prod_{p\mid S}(p-1)$, up to variations of lower magnitude when $k+S$ is sufficiently large. Then with $k/2$ of the appropriate parity, there is a bias towards root number $\epsilon = (-1)^{k/2}\epsilon_{\mathrm{fin}} = -1$, with roughly $c\frac{1}{12}(k-1)\prod_{p\mid S}(p-1)$ more forms of global root number -1 than $+1$. (We will show that in fact $c = 1$; see Proposition 7.6.)

To make a precise statement, we first apply Theorem 1.1 with $\mathtt{n} = T = 1$ to obtain the following.

Theorem 1.3. *Let $N = S^2$ for $S > 1$ square-free, let $k > 2$ be even, and let $\hat{\sigma} = (\sigma_{\nu_p})_{p\,|\,N}$ be a tuple of depth zero supercuspidal representations with trivial central characters, with ν_p the primitive character of $\mathbb{F}_{p^2}^*$ associated to σ_p. Then*

$$\dim S_k(\hat{\sigma}) = \tfrac{1}{12}(k-1)\prod_{p\,|\,S}(p-1) + D_4(S)\,\tfrac{1}{4}\epsilon(k,\hat{\sigma}) \prod_{\text{odd }p\,|\,S} 2 + D_3(S)\,b(k)\,\tfrac{1}{3}(-1)^{\delta_{3|S}} \prod_{p\,|\,S,\,p\neq 3} B(\nu_p), \tag{1-11}$$

where $\epsilon(k,\hat{\sigma})$ is the common global root number of the newforms in $S_k(\hat{\sigma})$, $D_4(S) \in \{0,1\}$ is 0 if and only if S is divisible by a prime $p \equiv 1 \bmod 4$, $D_3(S) \in \{0,1\}$ is 0 if and only if S is divisible by a prime $p \equiv 1 \bmod 3$, δ is the indicator function defined in (2-1),

$$b(k) = \begin{cases} 1 & \text{if } 6\,|\,k, \\ -1 & \text{if } k \equiv 2 \bmod 6, \\ 0 & \text{otherwise}, \end{cases} \tag{1-12}$$

and, for $p \equiv 2 \bmod 3$,

$$B(\nu_p) = \begin{cases} -2 & \text{if the order of } \nu_p \text{ (in the character group of } \mathbb{F}_{p^2}^*) \text{ divides } \tfrac{1}{3}(p+1), \\ 1 & \text{otherwise.} \end{cases}$$

The above is a special case of Theorem 7.3, which allows for nontrivial nebentypus and k odd. We will use Theorem 1.3 to derive an explicit formula for the bias

$$\Delta(S^2,k)^{\min} = \dim S_k^{\min}(S^2)^+ - \dim S_k^{\min}(S^2)^-$$

for $k > 2$ even and $S > 1$ square-free. This is given in Proposition 7.6. For the time being, we just state the following consequence, which is somewhat different from the behavior observed for the larger spaces of newforms of level S^2 appearing in [39, Theorem 1.1(3) and Proposition 1.3].

Proposition 1.4. *Assume $k \geq 4$ is even. With notation as above, $\Delta(S^2,k)^{\min} = 0$ in each of the following situations*: (i) $D_4(S) = D_3(S) = 0$, (ii) *S is divisible by some prime $p \equiv 5 \bmod 12$*, (iii) *$D_4(S) = 0$ and $k \equiv 4 \bmod 6$.*

If $D_4(S) = 0$, $k \equiv 0, 2 \bmod 6$, $D_3(S) \neq 0$, and case (ii) *does not apply, then $\Delta(S^2,k)^{\min} \neq 0$ and*

$$\operatorname{sgn} \Delta(S^2,k)^{\min} = (-1)^{\delta(k \equiv 6,\, 8 \bmod 12)}\mu(S)$$

for the indicator δ as in (2-1) and the Möbius function μ.

If $D_4(S)=1$ *and* $k\geq 6$, *then apart from the two exceptions* $S_8^{\min}(2^2)=S_6^{\min}(3^2)=0$, $\Delta(S^2,k)^{\min}\neq 0$, *and*

$$\operatorname{sgn}\Delta(S^2,k)^{\min}=(-1)^{\delta(2|S)+k/2}.$$

If $D_4(S)=1$ *and* $k=4$, *then* $\Delta(S^2,4)^{\min}\geq 0$ *for all square-free* $S>1$:

$$\Delta(S^2,4)^{\min}=\begin{cases}\frac{1}{2}\prod_{p|S}(p-1) & \text{if } 2\nmid S,\\ 0 & \text{if } 2|S.\end{cases}$$

Remark. A noteworthy difference between the above and the bias for the full space of newforms is that here for any fixed even $k\geq 6$ there are infinitely many levels S^2 for which $\Delta(S^2,k)^{\min}<0$, whereas by [39, Theorem 1.1(3)], for any fixed even k there are only finitely many levels N for which $\Delta(N,k)^{\text{new}}<0$.

2. Notation and Haar measure

If P is a statement, then we will frequently use the indicator function

$$\delta(P)=\delta_P=\begin{cases}1 & \text{if } P \text{ is true},\\ 0 & \text{if } P \text{ is false}.\end{cases} \tag{2-1}$$

We also use the shorthand

$$e(x)=e^{2\pi ix}.$$

For rings R, we let R^* denote the group of units in R.

Let G be the group $\mathrm{GL}(2)$, and set $\overline{G}=G/Z$, where Z is the center of G. If H is a subgroup of G, then $\overline{H}$ will denote the group $HZ/Z\cong H/(H\cap Z)$. For ℓ prime, we set $Z_\ell=Z(\mathbb{Q}_\ell)$ the center and $K_\ell=G(\mathbb{Z}_\ell)$ the maximal compact subgroup of $G(\mathbb{Q}_\ell)$. Groups $K_0(\mathfrak{p})$, $K_1(\mathfrak{p}^j)$, K' will be defined in Sections 3.1 and 5.2.

Let $\mathbb{A}$ be the adele ring of the rational numbers $\mathbb{Q}$. We give $\overline{G}(\mathbb{A})$ the standard Haar measure for which

$$\operatorname{meas}(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}))=\frac{\pi}{3},$$

with the discrete group $\overline{G}(\mathbb{Q})$ receiving the counting measure. We normalize Haar measure on $\overline{G}(\mathbb{Q}_\ell)$ so that $\overline{K_\ell}$ has measure 1. With this choice, there is a unique Haar measure on $\overline{G}(\mathbb{R})$ for which the above measure on $\overline{G}(\mathbb{A})$ is the restricted product of the measures on $\overline{G}(\mathbb{Q}_\ell)$ for $\ell\leq\infty$. It has the form $dm\,dn\,dk$, where dm is the measure $(dx/|x|)^2$ on the diagonal subgroup $M\cong\mathbb{R}^*\times\mathbb{R}^*$, dn is the measure dx on the unipotent subgroup $N\cong\mathbb{R}$, and dk is the measure on $K_\infty=\mathrm{SO}(2)$ of total measure 1 [23, Corollary 7.45].

For a unitary Hecke character ω, let $L^2(\omega)=L^2(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$ be the space of (classes of) measurable $\mathbb{C}$-valued functions ϕ on $G(\mathbb{A})$ transforming under the center by ω and square integrable modulo $Z(\mathbb{A})G(\mathbb{Q})$. Let $L^1(\bar{\omega})=L^1(G(\mathbb{A}),\bar{\omega})$ be defined in the analogous way; its elements are integrable modulo $Z(\mathbb{A})$.

3. The simple trace formula

3.1. ***Background on supercuspidal representations of*** **GL(2).** Let F be a nonarchimedean local field of characteristic 0 with integer ring $\mathcal{O}$ and prime ideal $\mathfrak{p}$. In this section only, let $G = \mathrm{GL}_2(F)$, $B = B(F)$ the upper-triangular Borel subgroup, $N = N(F)$ the unipotent subgroup of B, M the diagonal subgroup, Z the center, and $K = G(\mathcal{O})$ the standard maximal compact subgroup.

Given a smooth irreducible representation (π, V) of G, it is supercuspidal if it satisfies any of the following equivalent conditions (see, e.g., [7, Sections 9 and 10]):

- V is the span of the vectors of the form $\pi(n)v - v$ for $v \in V$ and $n \in N$.
- The matrix coefficients of π are compactly supported modulo the center.
- π is not principal series or special, i.e., not a subquotient of a representation induced from a character of B.

The following property found by Harish-Chandra is crucial in what follows. We sketch a proof here for the reader's convenience, following [52, Section 2.2].

Proposition 3.1. *Suppose that π is a supercuspidal representation of G, and that $f(g) = \langle \pi(g)v, v'\rangle$ is a matrix coefficient. Then for all $g, h \in G$,*

$$\int_N f(gnh)\, dn = 0. \tag{3-1}$$

Proof. We assume for simplicity that π is unitary, which is always the case if the central character is unitary. Then

$$f(gnh) = \langle \pi(g)\,\pi(n)\,\pi(h)v, v'\rangle = \langle \pi(n)\,\pi(h)v, \pi(g^{-1})v'\rangle,$$

so we can assume without loss of generality that $g = h = 1$. By linearity and the first bullet point above, we may also assume that $v = \pi(n_0)w - w$ for some $w \in V$ and $n_0 \in N$.

Let $N(v)$ be an open compact subgroup of N containing n_0. Then

$$\int_{N(v)} \pi(n)v\, dn = \int_{N(v)} \pi(n)(\pi(n_0)w - w)\, dn = \int_{N(v)} \pi(n)w\, dn - \int_{N(v)} \pi(n)w\, dn = 0.$$

(By smoothness, there exists an open compact subgroup N' of $N(v)$ that fixes w, so the above integrals are really just finite sums.) Since f has compact support, the support of $f|_N$ is contained in some open compact subgroup $N(v)$ as above. Therefore

$$\int_N f(n)\, dn = \int_{N(v)} \langle \pi(n)v, v'\rangle\, dn = \left\langle \int_{N(v)} \pi(n)v\, dn, v' \right\rangle = 0. \qquad \square$$

Corollary 3.2. *If f is a matrix coefficient of a supercuspidal representation, then for any $g, h \in G$ and $m \in M$ (M being the diagonal subgroup),*

$$\int_N f(gn^{-1}mnh)\, dn = 0.$$

Proof. This follows from $n^{-1}mn = mn'$, making a change of variables to integrate over n', and applying the above proposition. □

For any supercuspidal representation σ of G, there exists an open and closed subgroup $H \subseteq G$ containing Z, with H/Z compact, and an irreducible representation ρ of H, such that σ is compactly induced from ρ: $\sigma = \text{c-Ind}_H^G(\rho)$. Let $K_0(\mathfrak{p}) = \left(\begin{smallmatrix} \mathcal{O}^* & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^* \end{smallmatrix}\right)$ be the Iwahori subgroup of G, and fix a prime element ϖ of $\mathcal{O}$. Up to conjugacy, there are two maximal compact-mod-center subgroups of G, namely

$$J = \begin{cases} ZK & \text{(the unramified case)}, \\ ZK_0(\mathfrak{p}) \cup Z\left(\begin{smallmatrix} & 1 \\ \varpi & \end{smallmatrix}\right)K_0(\mathfrak{p}) & \text{(the ramified case)}. \end{cases} \tag{3-2}$$

The latter is the normalizer of $K_0(\mathfrak{p})$. Without loss of generality, one of these contains H, and we call σ *unramified* or *ramified* accordingly.[1] There is a unique ideal $\mathfrak{p}^j$, called the *conductor* of σ, such that the space of vectors in σ fixed by the group

$$K_1(\mathfrak{p}^j) = \begin{pmatrix} \mathcal{O}^* & \mathcal{O} \\ \mathfrak{p}^j & 1+\mathfrak{p}^j \end{pmatrix}$$

is one-dimensional. By [55], $j \geq 2$, and as explained in [6], j is even in the unramified case, and odd in the ramified case.

3.2. ***Simple trace formula.*** Given a unitary Hecke character ω and a function $f \in L^1(\bar{\omega})$, we define the operator $R(f)$ on $L^2(\omega)$ via

$$R(f)\,\phi(x) = \int_{\overline{G}(\mathbb{A})} f(g)\,\phi(xg)\,dg. \tag{3-3}$$

For $k > 2$, let $\mathcal{C}_k$ denote the space of all continuous factorizable functions $f = f_\infty \prod_{\ell<\infty} f_\ell$ on $G(\mathbb{A})$ which transform under the center by $\bar{\omega}$, such that f_ℓ is smooth and compactly supported modulo the center Z_ℓ for all ℓ, there is a finite set S of places of $\mathbb{Q}$ such that for all $\ell \notin S$, f_ℓ is supported on $Z_\ell K_\ell$ and has the value 1 on K_ℓ, and lastly,

$$f_\infty\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \ll_k \frac{(ad-bc)^{k/2}}{(a^2+b^2+c^2+d^2+2|ad-bc|)^{k/2}}.$$

Then $\mathcal{C}_k \subseteq L^1(\bar{\omega})$, and we can consider the operators $R(f)$ for such f.

Recall that $\gamma \in G(\mathbb{Q})$ is *elliptic* if its characteristic polynomial is irreducible. This concept is well defined on conjugacy classes and cosets of the center. We will use the following simple trace formula.

[1]It should be borne in mind that in standard terminology, all supercuspidals are *ramified* in the sense that they have no K-fixed vector. We are using the word in a different sense here, reflecting the nature of the quadratic extension E/F determined by σ [6].

Theorem 3.3. *For $f \in \mathcal{C}_k$, suppose that for some finite place v of $\mathbb{Q}$, f_v is a matrix coefficient of a supercuspidal representation of $G_v = G(\mathbb{Q}_v)$, and therefore by Corollary 3.2 its local hyperbolic orbital integrals vanish identically*:

$$\int_{M_v \backslash G_v} f_v\left(g^{-1}\begin{pmatrix} a & \\ & 1\end{pmatrix} g\right) dg = 0 \tag{3-4}$$

*for all $a \in \mathbb{Q}_v^*$, where M_v is the diagonal subgroup of G_v. Suppose further that (3-4) is also satisfied at a **second** place $w \neq v$ (which may be archimedean). Then*

$$\operatorname{tr} R(f) = \operatorname{meas}(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}))\, f(1) + \sum_{\mathfrak{o} \text{ elliptic in } \overline{G}(\mathbb{Q})} \Phi(\mathfrak{o}, f),$$

where, for an elliptic conjugacy class $\mathfrak{o} \subseteq \overline{G}(\mathbb{Q})$, the orbital integral is defined by

$$\Phi(\mathfrak{o}, f) = \int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} \sum_{\gamma \in \mathfrak{o}} f(g^{-1}\gamma g)\, dg. \tag{3-5}$$

Proof. See [14, Proposition V.2.1 and Theorem V.3.1]. The idea is that the validity of (3-4) at two distinct places kills off the hyperbolic and (nonidentity) unipotent terms on the geometric side of the Arthur–Selberg trace formula, while the stronger condition (3-1) on f_v also forces the operator $R(f)$ to have purely cuspidal image, so the continuous and residual spectral terms vanish as well. In Gelbart's exposition it is assumed that f is compactly supported, but for $f \in \mathcal{C}_k$ everything still converges absolutely as shown in [23], so the same proof is valid. □

3.3. ***Factorization of orbital integrals.*** Here we explain how to compute elliptic orbital integrals locally. The statements and proofs in this section are applicable over an arbitrary number field, though we express everything in terms of $\mathbb{Q}$.

For $\gamma \in G(\mathbb{Q})$, let G_γ be its centralizer. There are two related groups that will be needed. First, since $Z(\mathbb{Q}) \subseteq G_\gamma(\mathbb{Q})$, we may form the quotient, denoted $\overline{G_\gamma(\mathbb{Q})}$. Second, the centralizer of γ (or, more accurately, of the coset $\gamma Z(\mathbb{Q})$) in $\overline{G}(\mathbb{Q})$ is denoted $\overline{G}_\gamma(\mathbb{Q})$. In general these are distinct subgroups of $\overline{G}(\mathbb{Q})$. This will be clarified in the proof of Lemma 3.4 below.

Giving the discrete group $\overline{G_\gamma(\mathbb{Q})}$ the counting measure, define

$$\Phi(\gamma, f) = \int_{\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G}(\mathbb{A})} f(g^{-1}\gamma g)\, dg.$$

For fixed measures on $\overline{G_\gamma(\mathbb{R})}$ and $\overline{G_\gamma(\mathbb{Q}_\ell)}$, we also define the local orbital integrals

$$\Phi(\gamma, f_\infty) = \int_{\overline{G_\gamma(\mathbb{R})}\backslash\overline{G}(\mathbb{R})} f_\infty(g^{-1}\gamma g)\, dg$$

and

$$\Phi(\gamma, f_\ell) = \int_{\overline{G_\gamma(\mathbb{Q}_\ell)}\backslash\overline{G}(\mathbb{Q}_\ell)} f_\ell(g^{-1}\gamma g)\, dg.$$

For compatibility, some care must be taken regarding the normalization of measures. See the statement of Proposition 3.5 below.

Lemma 3.4. *For an elliptic element $\gamma \in G(\mathbb{Q})$, let $\mathfrak{o}$ be its conjugacy class in $\overline{G}(\mathbb{Q})$. Then with notation as above and in (3-5),*

$$\Phi(\mathfrak{o}, f) = \begin{cases} \Phi(\gamma, f) & \text{if } \operatorname{tr}\gamma \neq 0, \\ \frac{1}{2}\Phi(\gamma, f) & \text{if } \operatorname{tr}\gamma = 0. \end{cases}$$

Proof. By definition,

$$\Phi(\mathfrak{o}, f) = \int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} \sum_{\delta\in\overline{G}_\gamma(\mathbb{Q})\backslash\overline{G}(\mathbb{Q})} f(g^{-1}\delta^{-1}\gamma\delta g)\,dg = \int_{\overline{G}_\gamma(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} f(g^{-1}\gamma g)\,dg.$$

Notice that in the definition of $\Phi(\gamma, f)$, the quotient object is $\overline{G_\gamma(\mathbb{Q})}$ rather than $\overline{G}_\gamma(\mathbb{Q})$. The former is a subgroup of the latter, and we claim that

$$[\overline{G}_\gamma(\mathbb{Q}) : \overline{G_\gamma(\mathbb{Q})}] = \begin{cases} 1 & \text{if } \operatorname{tr}\gamma \neq 0, \\ 2 & \text{if } \operatorname{tr}\gamma = 0. \end{cases}$$

The lemma follows immediately from this claim. To prove the claim, note that

$$\overline{G_\gamma(\mathbb{Q})} = \{\delta \in G(\mathbb{Q}) \mid \delta^{-1}\gamma\delta = \gamma\}/Z(\mathbb{Q})$$

and

$$\overline{G}_\gamma(\mathbb{Q}) = \{\delta \in G(\mathbb{Q}) \mid \delta^{-1}\gamma\delta = z\gamma \text{ for some } z \in \mathbb{Q}^*\}/Z(\mathbb{Q}).$$

For any such z, taking determinants we see that $z^2 = 1$, so $z = \pm 1$. We also see that $\operatorname{tr}\gamma = z\operatorname{tr}\gamma$, so $z = 1$ if $\operatorname{tr}\gamma \neq 0$, and in this case the two groups are equal, as claimed. On the other hand, if $\operatorname{tr}\gamma = 0$, then γ is conjugate in $G(\mathbb{Q})$ to its rational canonical form $\left(\begin{smallmatrix} 0 & -\det\gamma \\ 1 & 0 \end{smallmatrix}\right)$, and

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} 0 & -\det\gamma \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} 0 & \det\gamma \\ -1 & 0 \end{pmatrix},$$

from which it follows that $\delta^{-1}\gamma\delta = -\gamma$ has a solution δ. Given any such δ, we find easily that

$$\overline{G}_\gamma(\mathbb{Q}) = \overline{G_\gamma(\mathbb{Q})} \cup \delta\overline{G_\gamma(\mathbb{Q})}. \qquad \square$$

Proposition 3.5. *Let $f \in \mathcal{C}_k$ as defined in Section 3.2, and let $\gamma \in G(\mathbb{Q})$ be an elliptic element. Then for any fixed choice of Haar measures on $\overline{G}(\mathbb{A})$ and $\overline{G_\gamma(\mathbb{A})}$,*

$$\Phi(\gamma, f) = \operatorname{meas}(\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G_\gamma(\mathbb{A})}) \prod_{\ell\leq\infty} \Phi(\gamma, f_\ell), \tag{3-6}$$

where the measures on the groups $\overline{G}(\mathbb{Q}_\ell)$ are chosen (noncanonically) so that the measure on $\overline{G}(\mathbb{A})$ is the restricted product of these local measures relative to the maximal compact subgroups almost everywhere, and likewise the measures on the groups $\overline{G_\gamma(\mathbb{Q}_\ell)}$ are chosen compatibly with the fixed measure on $\overline{G_\gamma(\mathbb{A})} = \prod'_{\ell\leq\infty} \overline{G_\gamma(\mathbb{Q}_\ell)}$.

Remarks. (1) This is well known, but as we have not found a proof in the literature, we include one below. Tate's thesis shows that if the product is absolutely convergent, then the left-hand integral converges absolutely and the equality holds. But here we need a kind of converse: we know a priori that $\Phi(\gamma, f)$ is absolutely convergent.

(2) The specific choice of measures to be used in this paper is summarized in Section 4.7.3, where it is shown that the quotient space $\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G_\gamma(\mathbb{A})}$ is compact, and its measure is computed explicitly in the more general setting with $\mathbb{Q}$ replaced by an arbitrary number field.

Proof. Observe that

$$\begin{aligned}\Phi(\gamma, f) &= \int_{\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G}(\mathbb{A})} f(g^{-1}\gamma g)\,dg\\ &= \mathrm{meas}(\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G_\gamma(\mathbb{A})}) \int_{\overline{G_\gamma(\mathbb{A})}\backslash\overline{G}(\mathbb{A})} f(g^{-1}\gamma g)\,dg\end{aligned}$$

for any choice of Haar measure on $\overline{G_\gamma(\mathbb{A})}$. Absolute convergence is proven for $f \in \mathcal{C}_k$ in [23, Corollary 19.3].

For notational convenience, write $\phi(g) = f(g^{-1}\gamma g)$ (a function on $\overline{G}(\mathbb{A})$), and $\phi_\ell(g_\ell) = f_\ell(g_\ell^{-1}\gamma g_\ell)$ for $\ell \le \infty$, so $\phi(g) = \prod_{\ell\le\infty}\phi_\ell(g_\ell)$. Also, define

$$X = \overline{G_\gamma(\mathbb{A})}\backslash\overline{G}(\mathbb{A}).$$

Then X is the restricted product of the spaces

$$X_\ell = \overline{G_\gamma(\mathbb{Q}_\ell)}\backslash\overline{G}(\mathbb{Q}_\ell),$$

relative to the open compact subsets $H_\ell = \overline{G_\gamma(\mathbb{Q}_\ell)}\backslash\overline{G_\gamma(\mathbb{Q}_\ell)K_\ell} \subseteq X_\ell$. Indeed, the natural map from $\overline{G}(\mathbb{A})$ to $\prod' X_\ell$ is clearly surjective, with kernel $\overline{G_\gamma(\mathbb{A})}$.

Fix Haar measures on each of the local groups $\overline{G}(\mathbb{Q}_\ell)$ and $\overline{G_\gamma(\mathbb{Q}_\ell)}$ compatibly with the fixed Haar measures on $\overline{G}(\mathbb{A})$ and $\overline{G_\gamma(\mathbb{A})}$. This determines a right-$\overline{G}(\mathbb{Q}_\ell)$-invariant measure on X_ℓ with the property that H_ℓ has measure 1 for almost all ℓ. Let S be the finite set of places of $\mathbb{Q}$ outside of which f_ℓ is supported on $Z(\mathbb{Q}_\ell)K_\ell$ with $f_\ell(zk) = \bar{\omega}(z)$. Let S' be a finite set of places outside of which (1) $\gamma \in K_\ell$, and (2) H_ℓ has measure 1. Then setting $S_0 = S \cup S'$, for $\ell \notin S_0$ we have

$$\int_{H_\ell}\phi_\ell(h)\,dh = \int_{H_\ell} f_\ell(k^{-1}\gamma k)\,dk = \mathrm{meas}(H_\ell) = 1.$$

Let

$$S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots$$

be a sequence of finite sets of primes (including ∞) whose union is the full set of primes. Let χ_n be the characteristic function of $X_{S_n} = \prod_{\ell\in S_n} X_\ell \times \prod_{\ell\notin S_n} H_\ell$, and let $\phi_n = \phi \cdot \chi_n$. Note that $\phi_n \to \phi$ pointwise. Since $\phi \in L^1(X)$ as mentioned above, so is ϕ_n. By the dominated convergence theorem,

$$\int_X \phi(x)\,dx = \lim_{n\to\infty} \int_X \phi_n(x)\,dx = \lim_{n\to\infty} \prod_{\ell\in S_n} \int_{X_\ell} \phi_\ell(x_\ell)\,dx_\ell,$$

as needed. □

4. Counting locally supercuspidal newforms

Here we explain how to use the simple trace formula to count cusp forms with prescribed supercuspidal ramification. To set notation, let $N = \prod_{p\mid N} p^{N_p} > 1$ be a positive integer with the property that $N_p \geq 2$ for each prime $p \mid N$. Fix a Dirichlet character ω' modulo N of conductor dividing $\prod_{p\mid N} p^{\lfloor N_p/2\rfloor}$. This requirement comes from the fact that the central character of a supercuspidal representation of conductor p^{N_p} divides $p^{\lfloor N_p/2\rfloor}$ [55, Proposition 3.4]. Let $\omega : \mathbb{A}^* \to \mathbb{C}^*$ be the finite order Hecke character associated to ω' via

$$\mathbb{A}^* = \mathbb{Q}^*(\mathbb{R}^+ \times \hat{\mathbb{Z}}^*) \to \hat{\mathbb{Z}}^*/(1+N\hat{\mathbb{Z}}) \cong (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*, \tag{4-1}$$

where the last arrow is ω'. Letting ω_p be the restriction of ω to $\mathbb{Q}_p^*$, for any prime $p \mid N$ we have

$$\begin{aligned}\omega_p(p) &= \omega(1, \dots, 1, p, 1, \dots)\\ &= \omega(p^{-1}, \dots, p^{-1}, 1, p^{-1}, \dots) = \prod_{\ell\mid N,\, \ell\neq p} \omega_\ell(p^{-1}).\end{aligned} \tag{4-2}$$

Fix an integer $k \geq 2$ satisfying

$$\omega'(-1) = (-1)^k,$$

and let $S_k(N, \omega')$ be the space of cusp forms h satisfying

$$h\left(\frac{az+b}{cz+d}\right) = \omega'(d)^{-1}(cz+d)^k h(z)$$

for $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(N)$. The inverse on $\omega'(d)$ is somewhat nonstandard. It ensures that the adelic cusp form attached to h has central character ω rather than ω^{-1}; see, e.g., [23, Sections 12.2–12.4]. Because we mostly work in the adelic setting, it eases the notation to include the inverse in the classical setting.

For each $p \mid N$, fix a supercuspidal representation σ_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ of conductor p^{N_p} and central character ω_p, and let $\hat{\sigma} = \{\sigma_p\}_{p\mid N}$. We define $H_k(\hat{\sigma})$ to be the set of newforms $h \in S_k(N, \omega')$ whose associated cuspidal representation π_h has the local representation type σ_p at each $p \mid N$. We set $S_k(\hat{\sigma}) = \operatorname{Span} H_k(\hat{\sigma})$. The Dirichlet character ω' is uniquely determined by the tuple $\hat{\sigma}$ via

$$\omega'(d) = \prod_{p\mid N} \omega_p(d) \quad ((d, N) = 1), \tag{4-3}$$

and this justifies our suppression of the central character ω' from the notation $S_k(\hat{\sigma})$. At a certain point we will use the fact that

$$\prod_{p|N} \omega_p(N) = \omega(N^{-1}, \dots, N^{-1}, 1, \dots, 1, N^{-1}, N^{-1}, \dots) = \omega'(1) = 1. \quad (4\text{-}4)$$

4.1. ***Isolating $S_k(\hat{\sigma})$ spectrally.*** For each prime $p \mid N$, we can write $\sigma_p = \text{c-Ind}_{H_p}^{G_p} \rho$, where H_p is contained either in $Z_p K_p$ or the normalizer of an Iwahori subgroup, as in (3-2). By [29, Proposition 2.1], there exists a unit vector w_p in the space of σ_p such that the matrix coefficient $\langle \sigma_p(g) w_p, w_p \rangle$ is supported in H_p. Fix once and for all such a vector w_p for each $p \mid N$. Based on this choice, we define a subspace $A_k(\hat{\sigma}) \subseteq L^2(\omega)$ by

$$A_k(\hat{\sigma}) = \bigoplus_{\pi} \mathbb{C} w_\pi,$$

where π ranges over the cuspidal automorphic representations with central character ω for which $\pi_\infty = \pi_k$, $\pi_p = \sigma_p$ for each $p \mid N$, and π_ℓ is unramified for all finite primes $\ell \nmid N$, and $w_\pi = \otimes w_{\pi_\ell}$ is defined by

$$w_{\pi_\ell} = \begin{cases} \text{unit lowest weight vector} & \text{if } \ell = \infty, \\ \text{unit spherical vector} & \text{if } \ell \nmid N\infty, \\ w_p \text{ (fixed above)} & \text{if } \ell = p \mid N. \end{cases} \quad (4\text{-}5)$$

Here, for almost all ℓ, the spherical vector is the one predetermined by the restricted tensor product $\pi \cong \bigotimes'_{\ell \le \infty} \pi_\ell$. The space $A_k(\hat{\sigma})$ does not consist of adelic newforms in general because at places $p \mid N$, w_p is not necessarily a new vector in the space of the local representation σ_p. Nevertheless, $A_k(\hat{\sigma})$ has the same dimension as the space of newforms $S_k(\hat{\sigma}) = \operatorname{Span} H_k(\hat{\sigma})$.

Using matrix coefficients, we can define a test function $f \in L^1(\bar{\omega})$ for which $R(f)$ is the orthogonal projection of $L^2(\omega)$ onto $A_k(\hat{\sigma})$. Without much extra work, we can incorporate a Hecke operator into the test function.

Fix an integer $\mathtt{n} > 1$ with $\gcd(\mathtt{n}, N) = 1$, and let $T_\mathtt{n}$ be the classical Hecke operator defined by

$$T_\mathtt{n} h(z) = \mathtt{n}^{k-1} \sum_{\substack{ad=\mathtt{n} \\ a>0}} \sum_{r \bmod d} \omega'(a)^{-1} d^{-k} h\left(\frac{az+r}{d}\right) \quad (h \in S_k(N, \omega'),\ z \in \mathbb{H}).$$

When $\mathtt{n} = 1$, $T_\mathtt{n}$ is simply the identity operator.

The operator $T_\mathtt{n}$ can be realized adelically. Let

$$M(\mathtt{n})_\ell = \{ g \in M_2(\mathbb{Z}_\ell) \mid \det g \in \mathtt{n}\mathbb{Z}_\ell^* \}$$

for each prime $\ell \nmid N$. (If working over a larger number field F, one would take $\mathtt{n}$ to be an ideal of the integer ring and set $M(\mathtt{n})_v = \{ g \in M_2(\mathcal{O}_v) \mid (\det g)\mathcal{O}_v = \mathtt{n} \}$ for a place $v < \infty$.) Define a function $f_\ell^{\mathtt{n}} : G(\mathbb{Q}_\ell) \to \mathbb{C}$ by

$$f_\ell^{\mathtt{n}}(g) = \begin{cases} \overline{\omega_\ell(z)} & \text{if } g = zm \text{ for } z \in Z_\ell,\ m \in M(\mathtt{n})_\ell, \\ 0 & \text{if } g \notin Z_\ell M(\mathtt{n})_\ell, \end{cases} \tag{4-6}$$

where ω_ℓ is the local component of the Hecke character ω. Note that $f_\ell^{\mathtt{n}}$ is bi-K_ℓ-invariant, and indeed when $\mathtt{n} \in \mathbb{Z}_\ell^*$, this function is given by

$$f_\ell(g) = \begin{cases} \overline{\omega_\ell(z)} & \text{if } g = zk \in Z_\ell K_\ell, \\ 0 & \text{if } g \notin Z_\ell K_\ell. \end{cases} \tag{4-7}$$

Next, let π_k be the discrete series representation of $\overline{G}(\mathbb{R})$ of weight k, and let v be a lowest weight unit vector in the space of π_k. We define $f_\infty = d_k \overline{\langle \pi_k(g)v, v\rangle}$, where $d_k = \frac{k-1}{4\pi}$ is the formal degree of π_k. Explicitly, with Haar measure on $\overline{G}(\mathbb{R})$ normalized as in Section 2,

$$f_\infty\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \dfrac{k-1}{4\pi}\dfrac{(ad-bc)^{k/2}(2i)^k}{(-b+c+(a+d)i)^k} & \text{if } ad-bc > 0, \\ 0 & \text{otherwise} \end{cases} \tag{4-8}$$

[23, Theorem 14.5]. This function is integrable over $\overline{G}(\mathbb{R})$ exactly when $k > 2$, so the latter will be assumed throughout. It would be possible to treat the $k = 2$ case by using a pseudocoefficient of π_k, but we have not attempted to carry this out (see [43]).

At places $p \mid N$, define

$$f_p(g) = d_{\sigma_p} \overline{\langle \sigma_p(g) w_p, w_p \rangle}, \tag{4-9}$$

where d_{σ_p} is the formal degree and w_p is the unit vector fixed above. The formal degree depends on a choice of Haar measure on $\overline{G}(\mathbb{Q}_p)$, which we normalize as in Section 2. By our choice of w_p, the support of f_p is contained in one of the two groups (3-2), according to whether or not σ_p is ramified.

Finally, we define the global test function

$$f^{\mathtt{n}} = f_\infty \prod_{p \mid N} f_p \prod_{\ell \nmid N} f_\ell^{\mathtt{n}} \tag{4-10}$$

for f_∞ of weight k as in (4-8), f_p as in (4-9), and $f_\ell^{\mathtt{n}}$ as in (4-6).

Proposition 4.1. *With the above definition of $f^{\mathtt{n}}$, the operator $R(f^{\mathtt{n}})$ (defined in (3-3) taking Haar measure on $\overline{G}(\mathbb{A})$ as normalized in Section 2) factors through the orthogonal projection onto the finite dimensional subspace $A_k(\hat{\sigma})$. On this space, $R(f^{\mathtt{n}})$ acts diagonally, with the vectors w_π being eigenvectors. In more detail, given a newform $h \in H_k(\hat{\sigma})$ with $T_{\mathtt{n}} h = a_{\mathtt{n}}(h) h$, let $w \in A_k(\hat{\sigma})$ be the vector associated to π_h as in (4-5). Then*

$$R(f^{\mathtt{n}}) w = \mathtt{n}^{1-(k/2)} a_{\mathtt{n}}(h) w.$$

Consequently,

$$\operatorname{tr}(T_{\mathtt{n}} \mid S_k(\hat{\sigma})) = \mathtt{n}^{(k/2)-1} \operatorname{tr} R(f^{\mathtt{n}}).$$

Remarks. (1) The vector w is defined only up to unitary scaling, but of course the eigenvalue is independent of the choice.

(2) One can also take f_p to be the complex conjugate of the trace of the representation ρ inducing σ_p, if normalized correctly. See Proposition 5.5 and its remark.

Proof. The first statement is proven in [25, Proposition 2.3], but we need to reproduce some of the argument here for the second part. Let $h \in H_k(\hat{\sigma})$, let π be the associated cuspidal representation, and let $w = w_\pi \in A_k(\hat{\sigma})$. For each place $v \mid \infty N$, the test function f_v was chosen so that

$$\pi_v(f_v)\, w_v = w_v$$

[23, Corollary 10.26]. Write

$$w = w_\infty \otimes \bigotimes_{p\mid N} w_p \otimes w' \otimes \bigotimes_{\ell\mid \mathtt{n}} w_\ell,$$

where $w' = \otimes_{\ell\nmid N\mathtt{n}} w_\ell$. We may likewise decompose π as

$$\pi = \pi_\infty \otimes \bigotimes_{p\mid N} \pi_p \otimes \pi' \otimes \bigotimes_{\ell\mid \mathtt{n}} \pi_\ell,$$

where π' is a representation of $G' = \prod'_{p\nmid N\mathtt{n}} G(\mathbb{Q}_p)$. Then letting $f' = \prod_{p\nmid N\mathtt{n}} f_p$, it is elementary to show that $\pi'(f')\, w' = w'$. Therefore (by [23, Proposition 13.17])

$$\begin{aligned} R(f^{\mathtt{n}})\, w &= \pi_\infty(f_\infty)\, w_\infty \otimes \bigotimes_{p\mid N} \pi_p(f_p)\, w_p \otimes \pi'(f')\, w' \otimes \bigotimes_{\ell\mid \mathtt{n}} \pi_\ell(f_\ell^{\mathtt{n}})\, w_\ell \\ &= w_\infty \otimes \bigotimes_{p\mid N} w_p \otimes w' \otimes \bigotimes_{\ell\mid \mathtt{n}} \pi_\ell(f_\ell^{\mathtt{n}})\, w_\ell. \end{aligned}$$

Since w_ℓ is an unramified unit vector in the principal series representation $\pi_\ell = \pi(\chi_1, \chi_2)$ (say), we have $\pi_\ell(f_\ell^{\mathtt{n}})\, w_\ell = \lambda_\ell w_\ell$ for

$$\lambda_\ell = \ell^{a/2} \sum_{j=0}^{a} \chi_1(\ell)^j \chi_2(\ell)^{a-j}, \qquad a = \mathrm{ord}_\ell(\mathtt{n})$$

(see, e.g., [24, Proposition 4.4]). Thus $R(f^{\mathtt{n}})\, w = \lambda w$, where $\lambda = \prod_{\ell\mid \mathtt{n}} \lambda_\ell$. The result now follows by the well-known fact that $\prod_{\ell\mid\mathtt{n}} \lambda_\ell = \mathtt{n}^{1-k/2} a_{\mathtt{n}}(h)$. The latter may be proven as follows. If we let v (denoted φ_h in [23]) be the adelic new vector attached to h, then v is a pure tensor, differing from w only at the places $p \mid N$. A test function $\tilde{f}^{\mathtt{n}}$, say, is used in [23] that differs from $f^{\mathtt{n}}$ only at the places $p \mid N$. By the same argument as above,

$$R(\tilde{f}^{\mathtt{n}})\, v = v_\infty \otimes \bigotimes_{p\mid N} v_p \otimes v' \otimes \bigotimes_{\ell\mid\mathtt{n}} R(f_\ell^{\mathtt{n}})\, v_\ell.$$

Since $v_\ell = w_\ell$ at places $\ell \mid \mathtt{n}$, the eigenvalues are the same, i.e., $R(\tilde{f}^{\mathtt{n}})\, v = \lambda v$. By [23, Theorem 13.14] (which uses a global argument), $\lambda = \mathtt{n}^{1-k/2} a_{\mathtt{n}}(h)$. ☐

4.2. ***First main result: the trace of a Hecke operator.*** We now state our first main theorem, which is a general formula for the trace of $T_{\mathtt{n}}$ on $S_k(\hat{\sigma})$. Its proof will occupy the remainder of Section 4.

Theorem 4.2. *Let $k > 2$, let the level N, nebentypus ω', and tuple $\hat{\sigma} = (\sigma_p)_{p\,|\,N}$ of supercuspidals be fixed as at the beginning of Section 4 (ensuring compatibility of central characters with ω'), and let $f = f^{\mathtt{n}}$ as in (4-10). Let T be the product of all primes $p\,|\,N$ with $\operatorname{ord}_p(N)$ odd. Then*

$$\operatorname{tr}(T_{\mathtt{n}}\,|\,S_k(\hat{\sigma})) = \mathtt{n}^{(k/2)-1}\biggl[\overline{\omega'(\mathtt{n}^{1/2})}\frac{1}{12}(k-1)\prod_{p\,|\,N} d_{\sigma_p} + \frac{1}{2}\sum_{M\,|\,T}\Phi\left(\begin{pmatrix} & -\mathtt{n}M \\ 1 & \end{pmatrix}, f\right) + \sum_{M\,|\,T}\sum_{1\le r<\sqrt{4\mathtt{n}/M}}\Phi\left(\begin{pmatrix} 0 & -\mathtt{n}M \\ 1 & rM \end{pmatrix}, f\right)\biggr],$$

where $\omega'(\mathtt{n}^{1/2})$ is taken to be 0 if $\mathtt{n}$ is not a perfect square, d_{σ_p} is the formal degree of σ_p relative to Haar measure fixed in Section 2, and the orbital integrals $\Phi(\gamma, f)$ are defined in Section 3.3.

An orbital integral $\Phi(\gamma, f)$ as above vanishes unless γ is elliptic in $G(\mathbb{Q}_p)$ for each $p\,|\,N$. Assuming this condition is satisfied, let $E = \mathbb{Q}[\gamma]$ be the imaginary quadratic extension of $\mathbb{Q}$ generated by γ, and let $h(E)$, $w(E)$, and d_E be the class number, number of units, and discriminant of E respectively. Then

$$\Phi(\gamma, f) = -\frac{2h(E)}{w(E)\,2^{\omega(d_E)}}\frac{\sin((k-1)\theta_\gamma)}{\sin(\theta_\gamma)}\prod_{p\,|\,\Delta_\gamma N}\Phi(\gamma, f_p), \tag{4-11}$$

where Δ_γ is the discriminant of γ, $\theta_\gamma = \arctan(\sqrt{|\Delta_\gamma|}/\operatorname{tr}\gamma)$ (interpreted as $\frac{\pi}{2}$ if $\operatorname{tr}\gamma = 0$) is the argument of one of the complex eigenvalues of γ, $\omega(d_E)$ is the number of prime factors of d_E, and our choice of measure for $\Phi(\gamma, f_p)$ is summarized in Section 4.7.3 below.

Remarks. (1) For primes $p \nmid N$, the local orbital integrals $\Phi(\gamma, f_p)$ are computed explicitly in Sections 4.4 and 4.5 below. Thus, for the explicit calculation of $\operatorname{tr}(T_{\mathtt{n}}\,|\,S_k(\hat{\sigma}))$ it only remains to calculate the local orbital integrals $\Phi(\gamma, f_p)$ for $p\,|\,N$.

(2) When $\mathtt{n} = 1$, the set of relevant γ is considerably smaller than what appears above if $T > 1$, due to local considerations at $p\,|\,T$. See Theorem 7.1.

The proof of Theorem 4.2 involves results from the rest of Section 4, outlined as follows. First, the test function f satisfies the hypotheses of Theorem 3.3. Indeed, the hyperbolic orbital integrals of f_∞ vanish as shown in [23, Proposition 24.2], and the fact that $f \in \mathcal{C}_k$ is a consequence of the formula for f_∞ (see [23, Lemma 14.2]).

Since we are normalizing measure so that $\operatorname{meas}(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}))=\frac{\pi}{3}$, the identity term in Theorem 3.3 is

$$\frac{\pi}{3}f(1)=\tfrac{1}{12}(k-1)\prod_{p\mid N}d_{\sigma_p}\prod_{\ell\mid\mathtt{n}}f_\ell^{\mathtt{n}}(1).$$

From the definition (4-6) of $f_\ell^{\mathtt{n}}$, we see that $f_\ell^{\mathtt{n}}(1)\neq 0$ only if $1\in Z_\ell M(\mathtt{n})_\ell$, which holds if and only if $\mathtt{n}$ is a perfect square. Assuming this is the case,

$$f_\ell^{\mathtt{n}}(1)=f_\ell^{\mathtt{n}}\left(\begin{pmatrix}\sqrt{\mathtt{n}}&\\&\sqrt{\mathtt{n}}\end{pmatrix}^{-1}\begin{pmatrix}\sqrt{\mathtt{n}}&\\&\sqrt{\mathtt{n}}\end{pmatrix}\right)=\omega_\ell(\sqrt{\mathtt{n}}).$$

Note that by (4-3),

$$\prod_{\ell\mid\mathtt{n}}\omega_\ell(\sqrt{\mathtt{n}})=\prod_{\ell\nmid N}\omega_\ell(\sqrt{\mathtt{n}})=\prod_{\ell\mid N}\overline{\omega_\ell(\sqrt{\mathtt{n}})}=\overline{\omega'(\sqrt{\mathtt{n}})}.$$

Therefore the identity term is

$$\frac{\pi}{3}f(1)=\overline{\omega'(\sqrt{\mathtt{n}})}\,\tfrac{1}{12}(k-1)\prod_{p\mid N}d_{\sigma_p},$$

where it is to be understood that $\omega'(\sqrt{\mathtt{n}})=0$ if $\mathtt{n}$ is not a perfect square.

The structure of the first part of Theorem 4.2 is then immediate from Theorem 3.3, Lemma 3.4, and Proposition 4.1. The set of relevant γ is determined in Section 4.6 below, simply by considering the supports of the local test functions. The vanishing of $\Phi(\gamma,f)$ if γ is hyperbolic in $G(\mathbb{R})$ or $G(\mathbb{Q}_p)$ for some $p\mid N$ is explained in Proposition 4.3 below.

As for (4-11), the first factor is equal to $\operatorname{meas}(\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G_\gamma(\mathbb{A})})$ under our normalization of Haar measures on $G(\mathbb{A})$ and $G_\gamma(\mathbb{A})$. This is shown in Theorem 4.16 below. The second factor of (4-11) (along with the negative sign) is $\Phi(\gamma,f_\infty)$ as in (4-12) below. In Sections 4.4 and 4.5 we explicitly compute the local orbital integrals away from the level, and see in particular that the value is 1 at places not dividing $\Delta_\gamma N$.

The local orbital integrals at the places dividing N of course depend on the choice of supercuspidal representations. The method we use to treat the special cases of simple supercuspidals and depth zero supercuspidals in the second part of this paper is presumably applicable to other cases as well.

4.3. ***Known results about the elliptic terms.*** We record here some basic properties of the elliptic orbital integrals that arise in Theorem 4.2.

Proposition 4.3. *Let γ be elliptic in $G(\mathbb{Q})$. Then for the test function $f=f^{\mathtt{n}}$ of (4-10):*

(1) *$\Phi(\gamma,f)$ is absolutely convergent.*

(2) *$\Phi(\gamma,f)$ depends only on the conjugacy class of γ in $G(\mathbb{A})$ (rather than in $G(\mathbb{Q})$), and likewise for any prime ℓ, $\Phi(\gamma,f_\ell)$ depends only on the $G(\mathbb{Q}_\ell)$-conjugacy class of γ.*

(3) $\Phi(\gamma, f) = 0$ *unless*: $\det\gamma > 0$ *and* γ *is elliptic both in* $G(\mathbb{R})$ *and in* $G(\mathbb{Q}_p)$ *for each* $p \mid M$.

(4) *If* γ *is elliptic in* $G(\mathbb{R})$ *with a complex eigenvalue* $\rho = re^{i\theta}$, *then*

$$\begin{aligned}\Phi(\gamma, f_\infty) &= -r^{2-k}\frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}} \\ &= -\frac{e^{i(k-1)\theta} - e^{-i(k-1)\theta}}{e^{i\theta} - e^{-i\theta}} = -\frac{\sin((k-1)\theta)}{\sin(\theta)}.\end{aligned} \tag{4-12}$$

Remarks. If γ has discriminant $\Delta_\gamma < 0$ and nonzero trace, then we may take $\theta = \arctan(\sqrt{|\Delta_\gamma|}/\operatorname{tr}\gamma)$ in (4-12). If γ has the form $\left(\begin{smallmatrix} & u \\ 1 & \end{smallmatrix}\right)$, then we may take $\theta = \frac{\pi}{2}$, giving

$$\Phi\left(\begin{pmatrix} & u \\ 1 & \end{pmatrix}, f_\infty\right) = -\left[\frac{i^{k-1} - (-i)^{k-1}}{2i}\right] = \begin{cases} (-1)^{k/2} & \text{if } k \text{ is even}, \\ 0 & \text{if } k \text{ is odd}. \end{cases} \tag{4-13}$$

Proof. Nearly everything is proven in [23, pp. 295–302]. The only remaining point is that $\Phi(\gamma, f_p) = 0$ if γ is hyperbolic in $G(\mathbb{Q}_p)$ for some $p \mid M$. For such γ, after conjugating we can take γ diagonal, so $G_\gamma(\mathbb{Q}_p) = M(\mathbb{Q}_p)$. The orbital integral is then taken over $M_p \backslash G_p$ and involves integrating over $N(\mathbb{Q}_p)$ (see (4-15) below). We can use (3-1) to show that it vanishes (as in (3-4)). $\square$

4.4. *Local orbital integrals at primes $\ell \nmid N$: hyperbolic case.* If $\gamma \in G(\mathbb{Q})$ is elliptic, then for each prime ℓ, γ is either hyperbolic or elliptic in $G(\mathbb{Q}_\ell)$. In this section and the next we evaluate the local elliptic orbital integrals at primes $\ell \nmid N$. The methods are standard and the results are presumably not new. For the dimension formulas we require the test function f_ℓ given by (4-7). However, without any extra work we can consider a general local Hecke operator, and consider an arbitrary p-adic field.

Thus, we let F be a p-adic field with valuation v, uniformizer ϖ, ring of integers $\mathcal{O}_F$, maximal ideal $\mathfrak{p} = \varpi\mathcal{O}_F$, and $q_v = |\mathcal{O}_F/\mathfrak{p}|$. Fix an unramified unitary character $\omega_v : F^* \to \mathbb{C}^*$. For an integral ideal $\mathfrak{n}_v \subseteq \mathcal{O}_F$, define

$$M(\mathfrak{n}_v) = \{g \in M_2(\mathcal{O}_F) \mid (\det g)\mathcal{O}_F = \mathfrak{n}_v\}$$

and

$$f^{\mathfrak{n}_v}(g) = \begin{cases} \overline{\omega_v(z)} & \text{if } g = zm \in Z(F)M(\mathfrak{n}_v), \\ 0 & \text{if } g \notin Z(F)M(\mathfrak{n}_v). \end{cases} \tag{4-14}$$

If γ is hyperbolic in $G(F)$, then replacing it by a conjugate if necessary, we can assume that it is diagonal. In this case, $G_\gamma(F) = M(F)$ is the set of invertible diagonal matrices. We may integrate over $\overline{G}(F)$ using the Iwasawa coordinates

$$\int_{\overline{G}(F)} \phi(g)\, dg = \int_{\overline{M}(F)} \int_{N(F)} \int_{K_v} \phi(mnk)\, dm\, dn\, dk,$$

where $K_v = G(\mathcal{O}_F)$. Therefore if ϕ is $M(F)$-invariant,

$$\int_{\overline{G_\gamma(F)}\backslash \overline{G}(F)} \phi(g)\,dg = \int_{N(F)} \int_{K_v} \phi(nk)\,dn\,dk. \tag{4-15}$$

We normalize the measures dn and dk by taking $\mathrm{meas}(N(\mathcal{O}_F)) = \mathrm{meas}(K_v) = 1$.

Proposition 4.4. *For F as above, suppose γ is hyperbolic in $G(F)$. Assuming $\gamma \in M(\mathfrak{n}_v)$, and letting $\Delta_\gamma \in \mathcal{O}_F$ be its discriminant, we have $\Phi(\gamma, f^{\mathfrak{n}_v}) = |\Delta_\gamma|_v^{-1/2}$. In particular, if Δ_γ is a unit, then $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$.*

Proof. We may assume that $\gamma = \left(\begin{smallmatrix} \alpha & \\ & \beta \end{smallmatrix}\right)$ for some distinct $\alpha, \beta \in \mathcal{O}_F$. By (4-15) and the fact that $f^{\mathfrak{n}_v}$ is right K_v-invariant,

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \int_F f^{\mathfrak{n}_v}\left(\begin{pmatrix} 1 & -t \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\right) dt = \int_F f^{\mathfrak{n}_v}\left(\begin{pmatrix} \alpha & t(\alpha-\beta) \\ & \beta \end{pmatrix}\right) dt.$$

Choose $j \geq 0$ so that $\alpha - \beta \in \varpi^j \mathcal{O}_F^*$. By hypothesis, $\alpha, \beta \in \mathcal{O}_F$ and $\alpha\beta\mathcal{O}_F = \mathfrak{n}_v$, so the integrand is nonzero if and only if $t(\alpha-\beta) \in \mathcal{O}_F$, which is equivalent to $t \in \varpi^{-j}\mathcal{O}_F$. Therefore

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \mathrm{meas}(\varpi^{-j}\mathcal{O}_F) = q_v^j = |\alpha-\beta|_v^{-1}.$$

Now let $D = \det\gamma$ and $r = \mathrm{tr}\,\gamma$. Note that

$$4D = 4\alpha\beta = (\alpha+\beta)^2 - (\alpha-\beta)^2 = r^2 - (\alpha-\beta)^2. \tag{4-16}$$

Therefore

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = |\alpha-\beta|_v^{-1} = |r^2 - 4D|_v^{-1/2},$$

as claimed. □

4.5. *Local orbital integrals at primes $\ell \nmid N$: elliptic case.* If γ is elliptic over a field F of characteristic 0, then $E = F[\gamma]$ is a quadratic field extension of F, and

$$G_\gamma(F) = E^*$$

(see [23, Proposition 26.1]). The center $Z(F)$ is isomorphic to F^*.

Proposition 4.5. *Let F be a local field of characteristic* 0, *and suppose γ is elliptic in $G(F)$. Then $G_\gamma(F)/Z(F)$ is compact.*

Proof. If $F = \mathbb{R}$, then $G_\gamma(\mathbb{R}) = \mathbb{R}[\gamma]^* \cong \mathbb{C}^*$, and the map $z \mapsto z/|z|$ gives rise to $\mathbb{C}^*/\mathbb{R}^* \cong \mathrm{SO}(2)/\{\pm 1\}$, which is compact.

Now suppose that F is nonarchimedean, with valuation v and integer ring $\mathcal{O}_F$. Let $E = F[\gamma]$, and choose a prime element $\pi \in \mathcal{O}_E$. Then letting $e \in \{1, 2\}$ be the ramification index of E/F,

$$G_\gamma(F)/Z(F) \cong E^*/F^* = \bigcup_{j=0}^{e-1} \pi^j \mathcal{O}_E^*/\mathcal{O}_F^*, \tag{4-17}$$

which is compact. □

Consider a p-adic field F, with all notation as in the previous subsection. For γ elliptic in $G(F)$, the above leads to the following natural choice of $G(F)$-invariant measure on the quotient space $\overline{G_\gamma(F)}\backslash\overline{G}(F)$. We assign the compact group $\overline{G_\gamma(F)}$ a total volume of 1. We assign $\overline{G}(F)$ the Haar measure for which $\overline{G}(\mathcal{O}_F)$ has measure 1. Together these choices determine the quotient measure via

$$\int_{\overline{G_\gamma(F)}\backslash\overline{G}(F)}\int_{\overline{G_\gamma(F)}}\phi(xy)\,dx\,dy=\int_{\overline{G}(F)}\phi(g)\,dg.$$

In fact, by our normalization, if ϕ is left $G_\gamma(F)$-invariant, then

$$\int_{\overline{G_\gamma(F)}\backslash\overline{G}(F)}\phi(y)\,dy=\int_{\overline{G}(F)}\phi(g)\,dg, \tag{4-18}$$

when γ is elliptic in $G(F)$.

For such γ, $E=F[\gamma]$ is a quadratic extension of F. Fix an F-integral basis $\{1,\varepsilon\}$ for the ring of integers $\mathcal{O}_E$, so

$$\mathcal{O}_E=\mathcal{O}_F+\mathcal{O}_F\,\varepsilon. \tag{4-19}$$

We will need some facts about orders and lattices in E. Recall that an F-*order* in E is a subring containing $\mathcal{O}_F$ which has rank 2 as an $\mathcal{O}_F$-module.

Proposition 4.6. *Let $\mathfrak{O}_{E/F}$ denote the set of all F-orders in E. For $r\ge 0$ and ε as above, define*

$$\mathcal{O}_r=\mathcal{O}_F+\mathfrak{p}^r\varepsilon,$$

where $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}_F$, so in particular $\mathcal{O}_0=\mathcal{O}_E$. Then

$$\mathfrak{O}_{E/F}=\{\mathcal{O}_r\mid r\ge 0\}.$$

Furthermore, letting $e=e(E/F)$ be the ramification index, for $r>0$ we have

$$[\mathcal{O}_E^*:\mathcal{O}_r^*]=\begin{cases}q_v^r & \text{if } e=2,\\ q_v^r+q_v^{r-1} & \text{if } e=1.\end{cases} \tag{4-20}$$

Proof. See also [40, Sections 6.6 and 6.7] for the case $F=\mathbb{Q}_p$. Here we loosely follow Okada [42, Section 2.3]. Clearly $\mathcal{O}_r\in\mathfrak{O}_{E/F}$. Conversely, let $\mathcal{O}\in\mathfrak{O}_{E/F}$. The elements of $\mathcal{O}$ are integral over E [41, Proposition I.2.2] so $\mathcal{O}\subseteq\mathcal{O}_E$. Hence there exists $\alpha\in\mathcal{O}\subseteq\mathcal{O}_E$ such that

$$\mathcal{O}=\mathcal{O}_F+\mathcal{O}_F\,\alpha.$$

Since $\alpha\notin\mathcal{O}_F$, by topological considerations we see that there exists $r\ge 0$ such that $\alpha\in\mathcal{O}_F+\varpi^r\mathcal{O}_E=\mathcal{O}_r$ but $\alpha\notin\mathcal{O}_F+\varpi^{r+1}\mathcal{O}_E=\mathcal{O}_{r+1}$. Hence

$$\mathcal{O}_{r+1}\subsetneq\mathcal{O}\subseteq\mathcal{O}_r.$$

We see easily that $\mathcal{O}_r/\mathcal{O}_{r+1} \cong \mathfrak{p}^r/\mathfrak{p}^{r+1} \cong \mathcal{O}_F/\mathfrak{p}$ as $\mathcal{O}_F$-modules. Since the latter is 1-dimensional as a vector space over $\mathcal{O}_F/\mathfrak{p}$, it has no nonzero proper submodules. It follows that $\mathcal{O} = \mathcal{O}_r$.

For the second part, consider the sequence

$$1 \to \mathcal{O}_F^*/(1+\mathfrak{p}^r) \to \mathcal{O}_E^*/(1+\mathfrak{p}^r\mathcal{O}_E) \to \mathcal{O}_E^*/\mathcal{O}_r^* \to 1,$$

where the maps are the obvious ones. It is straightforward to check that the sequence is exact. Therefore

$$[\mathcal{O}_E^* : \mathcal{O}_r^*] = \frac{|\mathcal{O}_E^*/(1+\mathfrak{p}^r\mathcal{O}_E)|}{|\mathcal{O}_F^*/(1+\mathfrak{p}^r)|}.$$

Let $e = e(E/F)$, so that $\mathfrak{p}\mathcal{O}_E = \mathfrak{P}^e$, where $\mathfrak{P}$ is the maximal ideal of $\mathcal{O}_E$. Then

$$\begin{aligned} |\mathcal{O}_E^*/(1+\mathfrak{p}^r\mathcal{O}_E)| &= |\mathcal{O}_E^*/(1+\mathfrak{P}^{er})| \\ &= [\mathcal{O}_E^* : 1+\mathfrak{P}] \prod_{j=2}^{er} [1+\mathfrak{P}^{j-1} : 1+\mathfrak{P}^j] = (q_E - 1)\, q_E^{er-1} \end{aligned}$$

(see [41, p. 139]). Here,

$$q_E = |\mathcal{O}_E/\mathfrak{P}| = \begin{cases} q_v & \text{if } e = 2, \\ q_v^2 & \text{if } e = 1. \end{cases}$$

Likewise $|\mathcal{O}_F^*/(1+\mathfrak{p}^r)| = (q_v - 1)\, q_v^{r-1}$, and (4-20) follows immediately. $\square$

For the purposes of this subsection, a *lattice* in $F^2 = F \times F$ is an $\mathcal{O}_F$-submodule of rank 2. The group F^* acts by multiplication on the set of lattices, and the orbits are called *lattice classes*. The map $g \mapsto L = g\binom{\mathcal{O}_F}{\mathcal{O}_F}$ from $G(F)$ to the set of lattices in F^2 induces a bijection between $\overline{G}(F)/\overline{K}_v$ and the set of lattice classes, since K_v is the stabilizer of $\binom{\mathcal{O}_F}{\mathcal{O}_F}$.

With notation as in (4-19), we may identify a lattice $L \subseteq F^2$ with the lattice $(1\ \varepsilon)L \subseteq E$, so that in particular $\binom{\mathcal{O}_F}{\mathcal{O}_F}$ is identified with $\mathcal{O}_E$. Given $\eta \in E^*$, it acts by scalar multiplication on the set of lattices in E, and by matrix multiplication (via $E = F[\gamma]$) on the lattices in F^2. In general, these actions are not compatible with the above identification. However, as shown in [23, Lemma 26.20], after possibly replacing γ (or equivalently, ε) by a $G(F)$-conjugate, these two actions do coincide for all $\eta \in E^*$. Explicitly, for any $g \in G(F)$,

$$\eta(1\ \varepsilon)g\begin{pmatrix}\mathcal{O}_F \\ \mathcal{O}_F\end{pmatrix} = (1\ \varepsilon)\eta g\begin{pmatrix}\mathcal{O}_F \\ \mathcal{O}_F\end{pmatrix},$$

where on the left η acts as a scalar via $\eta(1\ \varepsilon) = (\eta\ \eta\varepsilon)$, and on the right it is acting by matrix multiplication. We will assume that γ is chosen in this way, as we may since the value of the orbital integral depends only on γ's conjugacy class in $G(F)$.

We associate to any lattice $L \subseteq E$ the order

$$\mathcal{O}_L = \{\mu \in E \mid \mu L \subseteq L\}.$$

This depends only on the lattice class to which L belongs. Since E is local, every lattice in E is principal in the sense that there exists $y \in E^*$ such that $yL = \mathcal{O}_L$. (One may adapt the proof of [23, Proposition 26.13], which follows [33]).

Given an order $\mathcal{O}$, L is a *proper $\mathcal{O}$-lattice* if $\mathcal{O}_L = \mathcal{O}$. Two proper $\mathcal{O}$-lattices $y\mathcal{O}$ and $z\mathcal{O}$ (for $y, z \in E^*$) are equal if and only if $y/z \in \mathcal{O}^*$. Therefore the set of all proper $\mathcal{O}$-lattices corresponds bijectively with $E^*/\mathcal{O}^*$.

Lemma 4.7. *Suppose* $(\det\gamma)\mathcal{O}_F = \mathfrak{n}_v$ *and* $g \in G(F)$. *Then for* $f^{\mathfrak{n}_v}$ *given by* (4-14), $f^{\mathfrak{n}_v}(g^{-1}\gamma g) \neq 0$ *if and only if* $\gamma \in \mathcal{O}_L$ *for* $L = g\binom{\mathcal{O}_F}{\mathcal{O}_F}$.

Proof. We observe that

$$\gamma \in \mathcal{O}_L \iff \gamma L \subseteq L \iff g^{-1}\gamma g \begin{pmatrix} \mathcal{O}_F \\ \mathcal{O}_F \end{pmatrix} \subseteq \begin{pmatrix} \mathcal{O}_F \\ \mathcal{O}_F \end{pmatrix} \iff g^{-1}\gamma g \in M_2(\mathcal{O}_F).$$

Given that $\operatorname{ord}_v(\det\gamma) = \operatorname{ord}_v(\mathfrak{n}_v)$, the above is equivalent to $g^{-1}\gamma g$ belonging to the support $Z(F)M(\mathfrak{n}_v)$ of $f^{\mathfrak{n}_v}$. $\square$

Proposition 4.8. *Let* $f^{\mathfrak{n}_v}$ *be given by* (4-14). *Then for* $\gamma \in G(F)$ *elliptic, the orbital integral*

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \int_{\overline{G_\gamma(F)}\backslash \bar{G}(F)} f^{\mathfrak{n}_v}(g^{-1}\gamma g)\, dg$$

vanishes unless some conjugate of γ lies in $Z(F)M(\mathfrak{n}_v)$. Taking $\gamma \in M(\mathfrak{n}_v)$, with measure normalized as in (4-18) *we have*

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = e_\gamma \sum_{r=0}^{n_\gamma} [\mathcal{O}_E^* : \mathcal{O}_r^*],$$

where $E = F[\gamma]$ is the associated quadratic extension of F with ramification index $e_\gamma \in \{1, 2\}$ and ring of integers $\mathcal{O}_E = \mathcal{O}_F + \mathcal{O}_F\,\varepsilon$,

$$\mathcal{O}_r = \mathcal{O}_F + \mathfrak{p}^r\varepsilon$$

is the order of index q_v^r inside $\mathcal{O}_E$, and $n_\gamma \geq 0$ is defined by $\mathcal{O}_\gamma = \mathcal{O}_F + \mathcal{O}_F\gamma = \mathcal{O}_r$ for $r = n_\gamma$. In particular, if $\mathcal{O}_\gamma = \mathcal{O}_E$ and $\mathfrak{p}$ is inert in E, then $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$.

Remarks. (1) Let $P_\gamma(X) \in \mathcal{O}_F[X]$ be the characteristic polynomial of γ. If P_γ is irreducible modulo $\mathfrak{p}$, then $e_\gamma = 1$ and $\mathcal{O}_\gamma = \mathcal{O}_E$ [50, p. 18]. Hence $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$ in this case.

(2) The index $[\mathcal{O}_E^* : \mathcal{O}_r^*]$ is given explicitly in (4-20) when $r > 0$ (and is 1 when $r = 0$).

(3) Let $\mathfrak{d}_{E/F} = \det\left(\begin{smallmatrix} 1 & \varepsilon \\ 1 & \bar{\varepsilon} \end{smallmatrix}\right)^2 \mathcal{O}_F$ be the relative discriminant (with the bar denoting Galois conjugation), write $\gamma = s + b\varepsilon$ for $s, b \in \mathcal{O}_F$, and let $\Delta_\gamma = r^2 - 4D$ be the discriminant of γ. Then

$$n_\gamma = \mathrm{ord}_v(b) = \tfrac{1}{2}(\mathrm{ord}_v(\Delta_\gamma) - \mathrm{ord}_v(\mathfrak{d}_{E/F})). \tag{4-21}$$

This follows from the fact that the relative discriminant of

$$\mathcal{O}_\gamma = \mathcal{O}_F + \mathcal{O}_F\gamma = \mathcal{O}_F + \mathcal{O}_F\, b\varepsilon$$

is given on the one hand by

$$\det\begin{pmatrix} 1 & b\varepsilon \\ 1 & b\bar{\varepsilon} \end{pmatrix}^2 \mathcal{O}_F = b^2 \mathfrak{d}_{E/F},$$

and also (using (4-16)) by

$$\det\begin{pmatrix} 1 & \gamma \\ 1 & \bar{\gamma} \end{pmatrix}^2 \mathcal{O}_F = (\gamma - \bar{\gamma})^2 \mathcal{O}_F = \Delta_\gamma \mathcal{O}_F.$$

Further, if F is the completion of a number field L at a place v, $\{1, \varepsilon_L\}$ is an integral basis of $L[\gamma]$ over L, and we write $\gamma = s_L + b_L\varepsilon_L$, then (4-21) also holds with b_L in place of b. Indeed the same argument applies in the global case to give $b_L^2 \mathfrak{d}_{L[\gamma]/L} = \Delta_\gamma \mathcal{O}_L$. By the fact By the fact that the global discriminant is the product of the local ones and (due to γ being elliptic in $G(F)$) there is only one prime of $L[\gamma]$ lying over v, we see that $\mathrm{ord}_v(b_L) = \mathrm{ord}_v(b)$.

(4) If $E = \mathbb{Q}_\ell[\sqrt{d}]$ for $d \in \mathbb{Z}$ square-free, then (see [36, Section 6.10], for example)

$$\mathcal{O}_E = \begin{cases} \mathbb{Z}_2\left[\frac{1}{2}(1+\sqrt{-3})\right] & \text{if } \ell = 2,\ E = \mathbb{Q}_2[\sqrt{-3}], \\ \mathbb{Z}_\ell[\sqrt{d}] & \text{otherwise.} \end{cases} \tag{4-22}$$

In particular, if $\ell > 2$ and the valuation $\alpha = v_\ell(\Delta_\gamma)$ of the discriminant of γ is *odd*, then $e_\gamma = 2$, $n_\gamma = \frac{1}{2}(\alpha - 1)$, and assuming $\gamma \in M(\mathfrak{n})_\ell$,

$$\Phi(\gamma, f_\ell^{\mathfrak{n}}) = 2 \sum_{r=0}^{(\alpha-1)/2} \ell^r. \tag{4-23}$$

Proof of Proposition 4.8. The first statement is clear. Now suppose $\gamma \in M(\mathfrak{n}_v)$. By (4-18),

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \int_{\overline{G}(F)} f^{\mathfrak{n}_v}(g^{-1}\gamma g)\, dg.$$

The integrand is right $\overline{K_v}$-invariant as a function of g. Since $\overline{K_v}$ is open with measure 1, $\overline{G}(F)/\overline{K_v}$ is discrete with the counting measure. Therefore

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \sum_{g \in \overline{G}(F)/\overline{K_v}} f^{\mathfrak{n}_v}(g^{-1}\gamma g).$$

By our earlier remarks, we can view the sum as a sum over the lattice classes, and by Lemma 4.7, $\Phi(\gamma, f^{\mathfrak{n}_v})$ is equal to the number of lattice classes preserved by γ.

Since $\gamma \in E$ is integral over $\mathcal{O}_F$, $\mathcal{O}_\gamma = \mathcal{O}_F + \mathcal{O}_F\gamma$ is an order in E (see [23, Lemma 26.10]). We claim that $\gamma\mathcal{O}_r \subseteq \mathcal{O}_r$ if and only if $0 \le r \le n_\gamma$, where $q_v^{n_\gamma}$ is the index of $\mathcal{O}_\gamma$. Indeed,

$$\gamma\mathcal{O}_r \subseteq \mathcal{O}_r \iff \gamma \in \mathcal{O}_r \iff \mathcal{O}_\gamma \subseteq \mathcal{O}_r \iff r \le n_\gamma.$$

It follows that

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \sum_{r=0}^{n_\gamma}(\#\text{ of classes of proper } \mathcal{O}_r\text{-lattices}).$$

Recall from earlier that the set of proper $\mathcal{O}_r$-lattices corresponds bijectively with $E^*/\mathcal{O}_r^*$. Since we are counting F^*-classes of lattices rather than lattices themselves, we find

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \sum_{0\le r\le n_\gamma} |E^*/F^*\mathcal{O}_r^*| \quad (\mathcal{O}_\gamma = \mathcal{O}_{n_\gamma}).$$

Because $\mathcal{O}_F^* \subseteq \mathcal{O}_r^*$, it follows from (4-17) that $|E^*/F^*\mathcal{O}_r^*| = e_\gamma[\mathcal{O}_E^* : \mathcal{O}_r^*]$, where $e_\gamma \in \{1, 2\}$ is the ramification index of E/F. The result now follows. □

Corollary 4.9. *For $f^{\mathfrak{n}_v}$ as in (4-14), let $\gamma \in M(\mathfrak{n}_v)$ have characteristic polynomial $P_\gamma(X) = X^2 - rX + D \in \mathcal{O}_F[X]$ with discriminant $\Delta_\gamma = r^2 - 4D$. Then if γ is hyperbolic in $G(F)$, $\Phi(\gamma, f^{\mathfrak{n}_v}) = |\Delta_\gamma|_v^{-1/2}$. If γ is elliptic in $G(F)$ and $P_\gamma(X)$ does not have a double root in $\mathcal{O}_F/\mathfrak{p}$, then $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$.*

Consequently, for $\gamma \in M(\mathfrak{n}_v)$ elliptic or hyperbolic in $G(F)$, $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$ if

$$\Delta_\gamma \notin \mathfrak{p}.$$

Proof. The hyperbolic case is just a restatement of Proposition 4.4. Suppose γ is elliptic. If P_γ does not have a double root in $\mathcal{O}_F/\mathfrak{p}$, then it cannot have a simple root either, because otherwise that root would lift to a root in F by Hensel's lemma. By the first remark after Proposition 4.8, $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$.

Furthermore, suppose $\mathfrak{p} \nmid 2$, and note that $P_\gamma'(X) = 2X - r$ vanishes only at $\frac{r}{2} \in \mathcal{O}_F/\mathfrak{p}$. On the other hand,

$$P_\gamma\left(\frac{r}{2}\right) = D - \frac{r^2}{4},$$

which shows that P_γ has a repeated root modulo $\mathfrak{p}$ if and only if $\mathfrak{p} \mid (r^2 - 4D)$. Hence when $\mathfrak{p} \nmid 2$ and $\Delta_\gamma \notin \mathfrak{p}$, $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$.

If $\mathfrak{p} \mid 2$ and $(r^2 - 4D) \notin \mathfrak{p}$, then $r \in \mathcal{O}_F^*$, and therefore $P_\gamma'(X) = 2X - r$ is nonzero mod $\mathfrak{p}$. Hence P_γ does not have a repeated root, and $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$ in this case as well. □

Although the result of Proposition 4.8 appears complicated, it is not so hard to evaluate it by hand, using the remarks that follow the proposition and standard results about quadratic extensions of p-adic fields.

Example 4.10. Let ℓ be a prime not dividing D, and let $\gamma = \left(\begin{smallmatrix} 0 & -D \\ 1 & 0 \end{smallmatrix}\right)$. Then for f_ℓ as in (4-7),

$$\Phi(\gamma, f_\ell) = \begin{cases} 2 & \text{if } \ell = 2 \text{ and } D \equiv 1, 5, 7 \bmod 8, \\ 4 & \text{if } \ell = 2 \text{ and } D \equiv 3 \bmod 8, \\ 1 & \text{if } \ell \neq 2. \end{cases}$$

Remark. Some additional examples are given in Section 7.5.

Proof. First suppose $\ell \neq 2$. Since the discriminant $-4D$ of $P_\gamma(X) = X^2 + D$ is not divisible by ℓ, $\Phi(\gamma, f_\ell) = 1$ by Corollary 4.9.

Now suppose $\ell = 2$, so D is odd since $\ell \nmid D$. Recall that the squares of $\mathbb{Q}_2^*$ are exactly the elements of the set $2^{2\mathbb{Z}}(1 + 8\mathbb{Z}_2)$ [49, Theorem II.4]. Thus $-D$ is a square in $\mathbb{Q}_2^*$ if and only if

$$D \equiv 7 \bmod 8.$$

When this congruence is satisfied, γ is hyperbolic, and by Corollary 4.9,

$$\Phi(\gamma, f_2) = |-4D|_2^{-1/2} = 2.$$

Now suppose that $-D$ is not a square in $\mathbb{Q}_2$, i.e., it is not $1 \bmod 8$. We recall some facts about the quadratic extensions of $\mathbb{Q}_2$ (see, e.g., [36, Chapter 6]). There are exactly seven such extensions, namely $\mathbb{Q}_2[\sqrt{d}]$ for

$$d = -1, \pm 3, \pm 2, \pm 6,$$

with $\mathbb{Q}_2[\sqrt{-3}]$ being the unique unramified quadratic extension. With the exception of $d = -3$, the ring of integers is $\mathbb{Z}_2[\sqrt{d}]$. For $d = -3$, the ring of integers is $\mathbb{Z}_2\left[\frac{1}{2}(1+\sqrt{-3})\right]$. Under the given hypothesis, $-D \equiv d \bmod 8$, where $d \in \{-1, \pm 3\}$. So $-D = dx$ for some $x \in 1+8\mathbb{Z}_2$, and hence $-D = dy^2$ for some $y \in \mathbb{Z}_2^*$. Therefore, writing $E = \mathbb{Q}_2[\sqrt{-D}]$, we have $\mathcal{O}_E = \mathcal{O}_\gamma$ unless $d = -3$. In the former case, $E/\mathbb{Q}_2$ is ramified, so by Proposition 4.8,

$$\Phi(\gamma, f_2) = 2 \quad (D \equiv 1, 5 \bmod 8).$$

If $D \equiv 3 \bmod 8$, then $\mathcal{O}_E = \mathbb{Z}_2 + \mathbb{Z}_2\varepsilon$ for $\varepsilon = \frac{1}{2}(1 + \sqrt{-3})$. Hence

$$\mathcal{O}_\gamma = \mathbb{Z}_2 + \mathbb{Z}_2\sqrt{-D} = \mathbb{Z}_2 + \mathbb{Z}_2\sqrt{-3} = \mathbb{Z}_2 + \mathbb{Z}_2 2\varepsilon.$$

So in the notation of Proposition 4.8, $n_\gamma = 1$. Since $E/\mathbb{Q}_2$ is unramified, using (4-20), we have

$$\Phi(\gamma, f_2) = [\mathcal{O}_E^* : \mathcal{O}_E^*] + [\mathcal{O}_E^* : \mathcal{O}_\gamma^*] = 1 + 3 = 4. \qquad \square$$

4.6. ***The set of relevant γ.*** Here we determine explicitly the finite set of conjugacy classes in $\overline{G}(\mathbb{Q})$ that can have a nonzero contribution to the trace of $R(f)$ for f as in (4-10). Writing $N=\prod_{p\mid N}p^{N_p}$, define the square-free integers

$$S=\prod_{p\mid N,\,N_p\text{ even}}p,\quad T=\prod_{p\mid N,\,N_p\text{ odd}}p.$$

We say that an elliptic element $\gamma\in G(\mathbb{Q}_p)$ is *unramified* (at p) if $v_p(\det\gamma)$ is even, and *ramified* otherwise.

Lemma 4.11. *Let $\gamma\in G(\mathbb{Q})$ be elliptic, and suppose $\Phi(\gamma,f)\neq 0$ for $f=f^{\mathtt{n}}$ as in (4-10). Then there exists a unique positive divisor $M\mid T$ and a scalar $z\in\mathbb{Q}^*$ such that* $\operatorname{tr}(\gamma z)\geq 0$ *is an integer and*

$$\det(z\gamma)=\mathtt{n}M.$$

In particular, the rational canonical form of $z\gamma$ lies in $M_2(\mathbb{Z})$.

Proof. If $p\mid S$, then γ is unramified at p since f_p is supported in Z_pK_p. For $p\mid T$, the support of f_p has both ramified and unramified elements (see (3-2)). Let M be the product of those primes $p\mid T$ at which γ is ramified. For each prime $\ell\nmid N$, some conjugate of γ must lie in $\operatorname{Supp}(f_\ell^{\mathtt{n}})=Z_\ell M(\mathtt{n})_\ell$ since otherwise the integrand of $\Phi(\gamma,f)$ vanishes. It follows that $v_p(\det\gamma/\mathtt{n}M)$ is even for *all* primes p, where v_p is the p-adic valuation. Hence $\det\gamma\in\pm\mathtt{n}M\mathbb{Q}^{*2}$, where $\mathbb{Q}^{*2}$ is the set of squares in $\mathbb{Q}^*$. Because f_∞ is supported on $G(\mathbb{R})^+$, there is a scalar $z\in\mathbb{Q}^*$ such that $\det(z\gamma)=\mathtt{n}M$, as claimed. Because $\Phi(z\gamma,f)=\Phi(\gamma,f)\neq 0$, some $G(\mathbb{A}_{\text{fin}})$-conjugate of $z\gamma$ lies in

$$\prod_{p\mid M}\begin{pmatrix} & 1\\ p & \end{pmatrix}K_p\times\prod_{p\mid(ST/M)}K_p\times\prod_{\ell\nmid N}M(\mathtt{n})_\ell\subseteq M_2(\hat{\mathbb{Z}})\tag{4-24}$$

(recall that f_p is supported in the group J of (3-2)). In particular, $\operatorname{tr}(z\gamma)\in\hat{\mathbb{Z}}\cap\mathbb{Q}=\mathbb{Z}$. Scaling z by -1 if necessary, we may arrange further that $\operatorname{tr}(z\gamma)\geq 0$. □

Lemma 4.12. *Let F be a p-adic field, and γ an elliptic element of $G(F)$ with* $\operatorname{tr}\gamma\in\mathcal{O}_F$ *and* $\det\gamma\in\mathfrak{p}$. *Then* $\operatorname{tr}\gamma\in\mathfrak{p}$.

Proof. Denote the characteristic polynomial of γ by

$$P_\gamma(X)=X^2-dX+\det\gamma,$$

where $d=\operatorname{tr}\gamma$. Notice that $P_\gamma(0)\equiv 0\bmod\mathfrak{p}$. Furthermore, $P'_\gamma(0)\equiv -d\bmod\mathfrak{p}$. If d is nonzero modulo $\mathfrak{p}$, then by Hensel's lemma, P_γ has a root in $\mathfrak{p}$, contradicting the fact that γ is elliptic in $G(F)$. Hence $d\in\mathfrak{p}$. □

Proposition 4.13. *For $\gamma\in\overline{G}(\mathbb{Q})$ elliptic, and $f=f^{\mathtt{n}}$ the test function defined in (5-21), $\Phi(\gamma,f)=0$ unless the conjugacy class of γ has a representative in $G(\mathbb{Q})$ of the form $\left(\begin{smallmatrix}0 & -\mathtt{n}M\\ 1 & rM\end{smallmatrix}\right)$ for some $M\mid T$ and $0\leq r<\sqrt{4\mathtt{n}/M}$.*

Remark. If the characteristic polynomial of $\gamma = \left(\begin{smallmatrix} 0 & -\mathtt{n}M \\ 1 & rM \end{smallmatrix}\right)$ has a root in $\mathbb{Q}_p$, then $\Phi(\gamma, f) = 0$ by Proposition 4.3.

Proof. Let $\mathfrak{o}$ be an elliptic conjugacy class in $\overline{G}(\mathbb{Q})$ with $\Phi(\mathfrak{o}, f) \neq 0$. By Lemma 4.11, $\mathfrak{o}$ has a unique representative $\gamma \in G(\mathbb{Q})$ with characteristic polynomial of the form

$$P_\gamma(X) = X^2 - dX + \mathtt{n}M \in \mathbb{Z}[X],$$

where $d = \operatorname{tr}\gamma \geq 0$ and $M|T$. By Proposition 4.3, we know that γ is elliptic in $G(\mathbb{Q}_p)$ for each $p|N$ and also in $G(\mathbb{R})$. It follows by Lemma 4.12 that $M|d$. Write $d = rM$. Given that γ is elliptic in $G(\mathbb{R})$, we have $d^2 < 4\mathtt{n}M$, i.e.,

$$r^2 M < 4\mathtt{n}.$$

So, taking γ in rational canonical form as we may, it has the form

$$\gamma = \begin{pmatrix} 0 & -\mathtt{n}M \\ 1 & rM \end{pmatrix}, \quad 0 \leq r < \sqrt{4\mathtt{n}/M}. \qquad \square$$

4.7. ***The measure of $\overline{G_\gamma(F)}\backslash\overline{G_\gamma(\mathbb{A}_F)}$.*** Let F be a number field with adele ring $\mathbb{A}_F$, and let γ be an elliptic element of $G(F)$. With G_γ the centralizer of γ in G, here we will compute the measure of $\overline{G_\gamma(F)}\backslash\overline{G_\gamma(\mathbb{A}_F)}$. The result is given in Theorem 4.16 below. A related discussion can be found in [16, Section 5].

The basic idea is straightforward: we know that $G_\gamma(\mathbb{A}_F) = \mathbb{A}_F[\gamma]^* = \mathbb{A}_E^*$, where $E = F[\gamma]$ is a quadratic extension of F. (The proof of this fact given in [23, Proposition 26.1] for $F = \mathbb{Q}$ applies to any number field.) The center of $G(\mathbb{A}_F)$ is isomorphic to $\mathbb{A}_F^*$, so

$$\overline{G_\gamma(\mathbb{A}_F)} \cong \mathbb{A}_F^*\backslash\mathbb{A}_E^* \tag{4-25}$$

topologically and algebraically. Finally, $G_\gamma(F) \cong F[\gamma]^* = E^*$ by loc. cit., so

$$\overline{G_\gamma(F)}\backslash\overline{G_\gamma(\mathbb{A}_F)} = \mathbb{A}_F^* E^*\backslash\mathbb{A}_E^* \cong (F^*\backslash\mathbb{A}_F^*)\backslash(E^*\backslash\mathbb{A}_E^*) \cong (F^*\backslash\mathbb{A}_F^1)\backslash(E^*\backslash\mathbb{A}_E^1),$$

where the superscript 1 indicates ideles of norm 1, and the latter isomorphism comes from modding out by an embedded copy of $\mathbb{R}^+$ in $\mathbb{A}_F^* \subseteq \mathbb{A}_E^*$. For any number field L the measure of $L^*\backslash\mathbb{A}_L^1$ is computed in Tate's thesis under suitable normalization, which we may use with $L = E$, F to obtain the measure of the above space. However, as will be seen, we need to be very careful about the normalization of measures, particularly in the last step.

4.7.1. *Quotient measure.* Recall that if $H < G$ are unimodular locally compact groups with Haar measures μ_H and μ_G and H closed in G, there is a unique left G-invariant quotient measure $\mu_{G/H}$ on G/H satisfying

$$\int_{G/H}\left[\int_H f(gh)\,d\mu_H(h)\right]d\mu_{G/H}(g) = \int_G f(g)\,d\mu_G(g) \quad \text{for all } f \in C_c(G).$$

Lemma 4.14. *Let H, K and T be unimodular locally compact groups, with Haar measures μ_T, μ_H, μ_K, respectively. Assume that $H < K$, and let $G = T \times K$ and $J = T \times H$. Then relative to the product measures $\mu_G = \mu_T \times \mu_K$ and $\mu_J = \mu_T \times \mu_H$, we have $\mu_{G/J} = \mu_{K/H}$ on the group $G/J \cong K/H$.*

Proof. For $f \in C_c(G)$,

$$\int_{K/H}\left[\int_J f(xy)\,d\mu_J(y)\right]d\mu_{K/H}(x)$$
$$= \int_{K/H}\left[\int_H \int_T f(xht)\,d\mu_T(t)\,d\mu_H(h)\right]d\mu_{K/H}(x)$$
$$= \int_K\left[\int_T f(kt)\,d\mu_T(t)\right]d\mu_K(k) = \int_G f(g)\,d\mu_G(g). \qquad \square$$

4.7.2. *A volume from Tate's thesis.* Let L be a number field with adele ring $\mathbb{A}_L = \prod_v' L_v$, where v ranges over the places of L. In Tate's thesis, measures μ_v on the local multiplicative groups L_v^* are normalized as follows. If v is real,

$$d\mu_v(x) = \frac{dx}{|x|} \tag{4-26}$$

for $x \in \mathbb{R}^*$. If v is complex,

$$d\mu_v(z) = 2\frac{dx\,dy}{x^2+y^2} = \frac{2}{r}dr\,d\theta \tag{4-27}$$

for $z = x+iy = re^{i\theta} \in \mathbb{C}^*$. Finally, at a nonarchimedean place v, μ_v is the Haar measure on L_v^* satisfying

$$\mu_v(\mathcal{O}_v^*) = (\mathbb{N}\mathfrak{D}_v)^{-1/2}, \tag{4-28}$$

where $\mathcal{O}_v$ is the ring of integers of L_v, $\mathfrak{D}_v$ is the different of L_v and $\mathbb{N}\mathfrak{D}_v = |\mathcal{O}_v/\mathfrak{D}_v|$. Taking the restricted product of the above local measures, we obtain a Haar measure

$$\mu_L = \prod_v{}' \mu_v \quad \text{on } \mathbb{A}_L^*.$$

Let $L_\infty^* = \prod_{v|\infty} L_v^*$; we embed it into $\mathbb{A}_L^*$ by taking 1's at the nonarchimedean components. We embed $\mathbb{R}^+$ into L_∞^* and hence into $\mathbb{A}_L^*$ via

$$\lambda(t) = (t^{1/n}, t^{1/n}, \dots, t^{1/n}),$$

where $n = n_L = [L : \mathbb{Q}]$. Then if L has r_1 real embeddings and $2r_2$ complex embeddings, for $t \in \mathbb{R}^+$ we have

$$|\lambda(t)|_{\mathbb{A}_L} = \prod_{v|\infty} |t|_v^{1/n} = t^{(r_1+2r_2)/n} = t$$

(recall that in the ideles we take the square of the usual absolute value at the complex places).

Let $T \cong \mathbb{R}^+$ denote the image of the map λ. We give it the Haar measure dt/t. We have

$$\mathbb{A}_L^* \cong T \times \mathbb{A}_L^1, \tag{4-29}$$

where $\mathbb{A}_L^1$ is the subgroup consisting of ideles of norm 1. There is a unique measure μ_L^1 on $\mathbb{A}_L^1 \cong \mathbb{A}_L^*/T$ such that

$$\mu_L = \frac{dt}{t} \times \mu_L^1.$$

The multiplicative group L^* embeds diagonally in $\mathbb{A}_L^*$ as a discrete subgroup, and by the product formula, $L^* \subseteq \mathbb{A}_L^1$.

Theorem 4.15 [54, Theorem 4.3.2]. *The group L^* is discrete and cocompact in $\mathbb{A}_L^1$. Giving L^* the counting measure, for μ_L^1 as above we have*

$$\mu_L^1(L^* \backslash \mathbb{A}_L^1) = \frac{2^{r_1}(2\pi)^{r_2} h(L) R_L}{|d_L|^{1/2} w_L},$$

where $h(L)$, R_L, d_L and w_L are the class number, regulator, discriminant, and number of roots of unity of L, respectively.

Remark. This is the residue of the Dedekind zeta function of L at $s = 1$.

4.7.3. *Haar measure for orbital integrals.* Let $\gamma \in G(F)$ be an elliptic element. Here we define a Haar measure η on $\overline{G_\gamma(\mathbb{A}_F)}$ which is convenient to use for computing the elliptic orbital integrals. Given a nonarchimedean place v of F, γ is necessarily either elliptic or hyperbolic in $G(F_v)$. We select a compact open subgroup H_v of $\overline{G_\gamma(F_v)} = Z(F_v)\backslash G_\gamma(F_v)$ as follows. If γ is elliptic in $G(F_v)$, then the full group is compact by Proposition 4.5, and we take $H_v = \overline{G_\gamma(F_v)}$. If γ is hyperbolic in $G(F_v)$, then $G_\gamma(F_v)$ is conjugate to the diagonal subgroup $M(F_v)$. In this case we define H_v to be the subgroup of $\overline{G_\gamma(F_v)}$ taken by this conjugation to $\overline{M}(\mathcal{O}_v) \cong \mathcal{O}_v^*$, where $\mathcal{O}_v$ is the ring of integers of F_v.

Next, we choose a local Haar measure η_v on $\overline{G_\gamma(F_v)}$ for each place v of F as follows. If $v \nmid \infty$, we normalize η_v so that $\eta_v(H_v) = 1$. If $v|\infty$ is a real place of F and γ is elliptic over F_v, we take $\eta_v(\overline{G_\gamma(F_v)}) = 1$. If $v|\infty$ and γ is hyperbolic over F_v, then $\overline{G_\gamma(F_v)} \cong M(F_v)/F_v^* \cong F_v^*$, and we give it the measure $d\eta_v(x) = d\mu_v(x)$ for μ_v as in (4-26) or (4-27).[2]

Note that

$$\overline{G_\gamma(\mathbb{A}_F)} = \prod_v{}' \overline{G_\gamma(F_v)},$$

where the product is restricted relative to the subgroups H_v. We let η denote the Haar measure on $\overline{G_\gamma(\mathbb{A}_F)}$ which is the restricted product of the above local measures η_v.

[2]With $F = \mathbb{Q}$, these are the measures that are used in the local orbital integral calculations in the present paper. See Sections 4.4 and 4.5 for finite $\ell \nmid N$ and [23, Section 26.2] for the $\ell = \infty$ calculation yielding (4-12). For $\ell | N$, in Section 6 we will use the same measure used in Section 4.5.

As explained in (4-25), for $E = F[\gamma]$ we have

$$\overline{G_\gamma(\mathbb{A}_F)} \cong \mathbb{A}_E^*/\mathbb{A}_F^*.$$

So another natural measure on $\overline{G_\gamma(\mathbb{A}_F)}$ is the quotient measure $\mu_{E/F}$ coming from the Haar measures μ_E and μ_F on $\mathbb{A}_E^*$ and $\mathbb{A}_F^*$ obtained by taking $L = E$ and $L = F$ respectively in Section 4.7.2.

Let us next determine the constant relating the two measures η and $\mu_{E/F}$. For a place v of F and a place w of E lying over v, we have defined the measures μ_v and μ_w on F_v^* and E_w^* in Section 4.7.2. We let $\mu_v' = \prod_{w|v} \mu_w$ be the product measure on $E_v^* = \prod_{w|v} E_w^*$, and define $\bar{\mu}_v'$ to be the corresponding quotient measure on $E_v^*/F_v^* \cong \overline{G_\gamma(F_v)}$. Then $\mu_{E/F} = \prod_v' \bar{\mu}_v'$ where v runs over the places of F. For each v we need to find the constant relating η_v to $\bar{\mu}_v'$.

Let v be a nonarchimedean place of F. Suppose γ is hyperbolic in $G(F_v)$, so that $\eta_v(\overline{M}(\mathcal{O}_v)) = 1$. Let $w, \overline{w}$ be the primes of E lying over v. Then

$$E_v := E \otimes F_v \cong E_w \oplus E_{\overline{w}},$$

and $\mu_v' = \mu_w \times \mu_{\overline{w}}$ on

$$G_\gamma(F_v) \cong E_v^* \cong E_w^* \times E_{\overline{w}}^* \cong F_v^* \times F_v^*.$$

Hence

$$\mu_v'(\mathcal{O}_w^* \times \mathcal{O}_{\overline{w}}^*) = \mu_w(\mathcal{O}_w^*)\mu_{\overline{w}}(\mathcal{O}_{\overline{w}}^*) = (\mathbb{N}\mathfrak{D}_w)^{-1/2}(\mathbb{N}\mathfrak{D}_{\overline{w}})^{-1/2}$$

by (4-28). (This is in fact equal to $\mathbb{N}\mathfrak{D}_v$, but we prefer to leave it unsimplified for global reasons.) Likewise, the diagonally embedded subgroup $F_v^* \subseteq E_v^*$ has measure $\mu_v(\mathcal{O}_v^*) = (\mathbb{N}\mathfrak{D}_v)^{-1/2}$. Therefore the quotient measure $\bar{\mu}_v'$ on $E_v^*/F_v^* \cong \overline{G_\gamma(F_v)}$ gives the open subgroup $(\mathcal{O}_w^* \times \mathcal{O}_{\overline{w}}^*)/\mathcal{O}_v^* \cong \overline{M}(\mathcal{O}_v)$ the measure $\frac{(\mathbb{N}\mathfrak{D}_w)^{-1/2}(\mathbb{N}\mathfrak{D}_{\overline{w}})^{-1/2}}{(\mathbb{N}\mathfrak{D}_v)^{-1/2}}$. Consequently,

$$\eta_v = \frac{(\mathbb{N}\mathfrak{D}_w)^{1/2}(\mathbb{N}\mathfrak{D}_{\overline{w}})^{1/2}}{(\mathbb{N}\mathfrak{D}_v)^{1/2}}\bar{\mu}_v'$$

for such v.

Now suppose γ is elliptic in $G(F_v)$ (again with v nonarchimedean). Then there is a unique valuation w of E extending v, and $E_w = F_v[\gamma]$ is a quadratic extension of F_v. Let $\mathcal{O}_w$ be its ring of integers, with a uniformizer ϖ. Then for the ramification index $e_v = e(w/v) \in \{1, 2\}$,

$$\overline{G_\gamma(F_v)} \cong E_w^*/F_v^* = \bigcup_{j=0}^{e_v-1} \varpi^j \mathcal{O}_w^*/\mathcal{O}_v^*$$

as in (4-17). By definition of the local component μ_w of μ_E, $\mu_w(\mathcal{O}_w^*) = (\mathbb{N}\mathfrak{D}_w)^{-1/2}$. The local component of μ_F at v gives $\mathrm{meas}(\mathcal{O}_v^*) = (\mathbb{N}\mathfrak{D}_v)^{-1/2}$. Therefore the quotient measure $\bar{\mu}'_v$ satisfies

$$\bar{\mu}'_v(\mathcal{O}_w^*/\mathcal{O}_v^*) = \frac{(\mathbb{N}\mathfrak{D}_w)^{-1/2}}{(\mathbb{N}\mathfrak{D}_v)^{-1/2}}.$$

Since $\eta_v(\overline{G_\gamma(F_v)}) = 1$, it follows that

$$\eta_v = \frac{1}{e_v}\frac{(\mathbb{N}\mathfrak{D}_w)^{1/2}}{(\mathbb{N}\mathfrak{D}_v)^{1/2}}\bar{\mu}'_v$$

for such v.

Suppose $F_v = \mathbb{R}$ and γ is elliptic in $G(F_v)$. Then $E_w = \mathbb{C}^*$ and $\overline{G_\gamma(F_v)} = \mathbb{C}^*/\mathbb{R}^*$. A set of representatives in $\mathbb{C}^*$ is $\{e^{i\theta} \mid \theta \in [0, \pi)\}$. Since the measure $\mu_v(x) = dx/|x|$ on $\mathbb{R}^*$ matches the factor dr/r in $\mu_w(z) = (2dr\, d\theta)/r$ given in (4-27), it follows that

$$\bar{\mu}'_v(\mathbb{C}^*/\mathbb{R}^*) = 2\pi.$$

Since $\eta_v(\overline{G_\gamma(\mathbb{R})}) = 1$,

$$\eta_v = \frac{1}{2\pi}\bar{\mu}'_v$$

for such v.

If $F_v = \mathbb{R}$ or $\mathbb{C}$ and γ is hyperbolic in $G(F_v)$, then as in the analogous nonarchimedean case,

$$E_v^* = E_w \times E_{\bar{w}} \cong F_v^* \times F_v^*,$$

and the quotient measure on $\overline{G_\gamma(F_v)} \cong E_v^*/F_v^* \cong F_v^*$ is $\bar{\mu}'_v(x) = \mu_v(x)$. In such cases we have likewise defined $\eta_v = \mu_v$. So $\eta_v = \bar{\mu}'_v$ for such v.

Putting everything together, we have shown that

$$\eta = \Bigg[\prod_{v\nmid\infty}\frac{1}{e_v}\frac{\prod_{w|v}(\mathbb{N}\mathfrak{D}_w)^{1/2}}{(\mathbb{N}\mathfrak{D}_v)^{1/2}}\Bigg]\Bigg[\prod_{v|\infty,\, \gamma \text{ elliptic in } G(F_v)}\frac{1}{2\pi}\Bigg]\mu_{E/F}.$$

We can simplify using three well-known facts from algebraic number theory (see, e.g., [41, Section III.2]):

(1) $e_v = 2$ if and only if $\mathfrak{p}_v \mid \mathfrak{d}_{E/F}$ where $\mathfrak{d}_{E/F}$ is the relative discriminant.

(2) The absolute discriminant of a local field is the absolute norm of the different.

(3) The product of the local discriminants is the global discriminant.

It follows that taking $d_F, d_E \in \mathbb{Z}$ to be the discriminants of F and E respectively,

$$\eta = \frac{|d_E|^{1/2}}{|d_F|^{1/2}}\frac{1}{2^{\omega_F(\mathfrak{d}_{E/F})}}\frac{1}{(2\pi)^{\alpha_\gamma}}\mu_{E/F}, \tag{4-30}$$

where $\omega_F(\mathfrak{d}_{E/F})$ is the number of distinct prime factors of $\mathfrak{d}_{E/F}$ in $\mathcal{O}_F$, and α_γ is the number of (real) archimedean places v of F for which γ is elliptic in $G(F_v)$.

4.7.4. *The quotient measure.* We turn now to the quotient space whose measure we need to compute, namely $\overline{G_\gamma(F)}\backslash\overline{G_\gamma(\mathbb{A}_F)} \cong E^*\mathbb{A}_F^*\backslash\mathbb{A}_E^* \cong \mathbb{A}_E^*/\mathbb{A}_F^*E^*$. We have defined the quotient measure $\mu_{E/F}$ on $\mathbb{A}_E^*/\mathbb{A}_F^*$. By (4-29), we have

$$\mathbb{A}_F^* = T \times \mathbb{A}_F^1, \quad \mathbb{A}_E^* = T \times \mathbb{A}_E^1.$$

We regard $\mathbb{A}_F^*$ as a subset of $\mathbb{A}_E^*$, so T is the set

$$T = \{(a, a, \dots, a) \in E_\infty^* \mid a > 0\} \subseteq \mathbb{A}_E^*.$$

We will use Lemma 4.14 to relate $\mu_{E/F}$ to the quotient measure on $\mathbb{A}_E^1/\mathbb{A}_F^1$ coming from the measures μ_E^1 and μ_F^1 defined below (4-29). Recall that T is given the measure $d\mu_T(t) = dt/t$, where $t^{1/n_E} = a$ for $n_E = [E : \mathbb{Q}]$. In terms of the parameter a,

$$d\mu_T(t) = n_E \frac{da}{a}.$$

Notice that this is *not* the measure given to $\mathbb{R}^+$ upon taking $L = F$ in (4-29), which is $n_F(da/a) = (n_F/n_E)\, d\mu_T(t)$. In other words, for μ_T normalized as above, μ_F^1 is defined by

$$\mu_F = \frac{1}{[E:F]}\mu_T \times \mu_F^1.$$

Therefore

$$\mu_F = \mu_T \times \frac{1}{[E:F]}\mu_F^1 = \mu_T \times \tfrac{1}{2}\mu_F^1.$$

Hence by Lemma 4.14, the quotient measure $\mu_{E/F}$ on $\mathbb{A}_E^*/\mathbb{A}_F^* \cong \mathbb{A}_E^1/\mathbb{A}_F^1$ is the same as the quotient measure coming from μ_E^1 and $\frac{1}{2}\mu_F^1$. We denote this quotient measure by $\mu_{E/F}^1$.

Finally, taking the quotient by the discrete subgroup E^* we have

$$\mu_{E/F}(\mathbb{A}_E^*/E^*\mathbb{A}_F^*) = \mu_{E/F}^1\left(\frac{(\mathbb{A}_E^1/E^*)}{(\mathbb{A}_F^1E^*/E^*)}\right) = \frac{\mu_E^1(\mathbb{A}_E^1/E^*)}{\frac{1}{2}\mu_F^1(\mathbb{A}_F^1/F^*)}. \tag{4-31}$$

As a technical point, the measure on the disjoint union

$$\mathbb{A}_F^1E^* = \bigcup_{\alpha\in E^*/F^*} \mathbb{A}_F^1\alpha$$

is simply $\frac{1}{2}\mu_F^1$ on each component since E^* is given the counting measure. This explains why the quotient measure on $\mathbb{A}_F^1E^*/E^*$ is the same as $\frac{1}{2}\mu_F^1$ on $\mathbb{A}_F^1/F^*$. Applying Theorem 4.15 and (4-30) to (4-31), we immediately obtain the following.

Theorem 4.16. *Let $\gamma \in G(F)$ be an elliptic element, and let η be the measure introduced in Section 4.7.3. Then for $E = F[\gamma]$,*

$$\eta(\overline{G_\gamma(F)}\backslash\overline{G_\gamma(\mathbb{A}_F)}) = \frac{2^{r_1(E)}(2\pi)^{r_2(E)}h(E)R_E}{2^{r_1(F)}(2\pi)^{r_2(F)}h(F)R_F} \cdot \frac{w_F}{w_E} \cdot \frac{2}{2^{\omega_F(\mathfrak{d}_{E/F})}(2\pi)^{\alpha_\gamma}},$$

with notation as in Theorem 4.15, where $\omega_F(\mathfrak{d}_{E/F})$ is the number of distinct prime *ideals of $\mathcal{O}_F$ dividing the relative discriminant $\mathfrak{d}_{E/F}$, and α_γ is the number of (real) archimedean places v of F for which γ is elliptic in $G(F_v)$.*

In the special case where $F=\mathbb{Q}$ and $E=\mathbb{Q}[\gamma]$ is quadratic imaginary, we have $\alpha_\gamma=1,\ w_F=2,\ h(F)=R_E=R_F=1$, so

$$\eta(\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G_\gamma(\mathbb{A})})=\frac{2h(E)}{w_E\,2^{\omega(d_E)}},\tag{4-32}$$

where $\omega(d_E)$ is the number of distinct prime factors of the discriminant d_E.

With the above in place, the proof of Theorem 4.2 is complete.

5. The case $N = S^2T^3$: proof of Theorem 1.1

Henceforth, we will focus on the case where $N=S^2T^3$ for S and T relatively prime square-free integers. In order to prove Theorem 1.1, by Theorem 4.2 we just need to compute the orbital integrals at the primes dividing N. We begin in Sections 5.1 and 5.2 by reviewing the construction of supercuspidals of conductor p^2 (depth zero case) and of conductor p^3 (simple case), giving explicit formulas for the local test functions to be used. In Section 5.3 we outline the global setup, and then compute the required orbital integrals in Section 6 to complete the proof.

5.1. ***Depth zero supercuspidal representations.*** Let F be a p-adic field, with ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{p}=\varpi\mathcal{O}$, and residue field $\Bbbk=\mathcal{O}/\mathfrak{p}$ of size q. The supercuspidal representations of $G(F)$ of minimal conductor are the so-called depth zero supercuspidals, with conductor $\mathfrak{p}^2$. They have the form $\sigma=\text{c-Ind}_{ZK}^{G(F)}(\rho)$, where ρ is a $(q-1)$-dimensional representation of $K=G(\mathcal{O})$ inflated from a cuspidal representation of $G(\Bbbk)$, and c-Ind denotes compact induction. Some of their properties are summarized below (see, e.g., [29] for more detail).

Temporarily, write $G=G(\Bbbk)$. Let L be the unique quadratic extension of $\Bbbk$. The multiplicative group L^* embeds as a nonsplit torus $\mathbb{T}\subseteq G$, with $\Bbbk^*$ mapping onto the center $Z\subseteq G$. A character $\nu: L^*\to\mathbb{C}^*$ is *primitive* (or regular) if $\nu\neq\nu^q$, or equivalently, if ν is not of the form $\chi\circ N_{\Bbbk}^L$ for a character χ of $\Bbbk^*$, where $N_{\Bbbk}^L$ is the norm map. There are $q(q-1)$ primitive characters of L^*. Given a character ω of $\Bbbk^*$, let $[\omega]$ denote the set of primitive characters ν satisfying $\nu|_{\Bbbk^*}=\omega$. By [29, Proposition 2.3], the cardinality of $[\omega]$ is

$$P_\omega=\begin{cases} q-1 & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is trivial,}\\ q+1 & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is nontrivial,}\\ q & \text{if } q \text{ is even.}\end{cases}\tag{5-1}$$

Let $U=\left(\begin{smallmatrix}1&\Bbbk\\0&1\end{smallmatrix}\right)$ be the upper triangular unipotent subgroup of G. A representation of G is *cuspidal* if it does not contain a U-fixed vector. Fix a nontrivial additive character

$$\psi:\Bbbk\to\mathbb{C}^*.$$

We will always take $\psi(x) = e(x/p) = e^{2\pi i x/p}$ if $\Bbbk = \mathbb{Z}/p\mathbb{Z}$. We may view ψ as a character of U in the obvious way.

Given a primitive character ν of $\mathbb{T}$, there is a unique irreducible cuspidal representation ρ_ν of dimension $q-1$ satisfying

$$\mathrm{Ind}_{ZU}^{G}(\Bbbk)(\nu \otimes \psi) = \rho_\nu \oplus \mathrm{Ind}_{\mathbb{T}}^{G}\, \nu.$$

Every cuspidal representation arises in this way, and $\rho_\nu \cong \rho_{\nu'}$ if and only if $\nu' \in \{\nu, \nu^q\}$.

We have the following well-known formula for the character of ρ_ν. For $x \in G(\Bbbk)$,

$$\operatorname{tr} \rho_\nu(x) = \begin{cases} (q-1)\,\nu(x) & \text{if } x \in Z, \\ -\nu(z) & \text{if } x = zu,\ z \in Z,\ u \in U,\ u \neq 1, \\ -\nu(x) - \nu^q(x) & \text{if } x \in \mathbb{T},\ x \notin Z, \\ 0 & \text{if no conjugate of } x \text{ belongs to } \mathbb{T} \cup ZU. \end{cases} \tag{5-2}$$

Because $\nu(c^{-1}xc) = \nu(x^q)$ for all $c \in N_G(\mathbb{T}) - \mathbb{T}$, there is no ambiguity evaluating $\operatorname{tr} \rho_\nu(y)$ using the third row above if y is conjugate in $G(\Bbbk)$ to $x \in \mathbb{T}$.

Working now in the group $G(F)$, given the surjection $K \to G(\Bbbk)$ obtained by reduction modulo $\mathfrak{p}$, we may view ρ_ν as a representation of K. Its central character is given by $z \mapsto \nu(z(1+\mathfrak{p}))$ for $z \in \mathcal{O}^*$. By choosing a complex number $\nu(\varpi)$ of norm 1, we may extend ρ_ν to a representation of ZK, and then

$$\sigma_\nu = \text{c-Ind}_{ZK}^{G(F)}(\rho_\nu)$$

is an irreducible unitary supercuspidal representation of conductor $\mathfrak{p}^2$. Its formal degree under the normalization $\mathrm{meas}(K) = 1$ is

$$d_{\sigma_\nu} = \dim \rho_\nu = q - 1. \tag{5-3}$$

The only equivalences among the representations σ_ν are $\sigma_\nu \cong \sigma_{\nu^q}$ (provided $\nu^q(\varpi)$ is defined to be the same complex number as $\nu(\varpi)$).

We define the test function $f_{\mathfrak{p}} : G(F) \to \mathbb{C}$ by

$$f_{\mathfrak{p}}(g) = \begin{cases} \overline{\operatorname{tr} \rho_\nu(g)} & \text{if } g \in ZK, \\ 0 & \text{otherwise,} \end{cases} \tag{5-4}$$

where $\operatorname{tr} \rho_\nu$ is given in (5-2).

Proposition 5.1. *Suppose σ_ν has trivial central character. Then its root number is given by*

$$\epsilon_\nu = \epsilon\big(\tfrac{1}{2}, \sigma_\nu, \psi\big) = \begin{cases} -(-1)^{(q+1)/r} & \text{if } q \text{ is odd}, \\ -1 & \text{if } q \text{ is even}, \end{cases} \tag{5-5}$$

where r is the order of ν in the character group of L^*. Suppose further that q is odd and $4 \nmid (q-1)$ so that $\alpha^2 = -1$ for some $\alpha \in L^* - \Bbbk^*$. Then

$$\epsilon_\nu = -\nu(\alpha). \tag{5-6}$$

Remark. Under the hypothesis, $\nu|_{\mathbb{k}^*}$ is trivial, which is equivalent to $r\,|\,(q+1)$ when q is odd.

Proof. The root number coincides with the Atkin–Lehner sign of the representation [48, 3.2.2 Theorem]. We will show that it is a Gauss sum for ν, which can be evaluated explicitly. The Atkin–Lehner sign ϵ_ν is defined by

$$\sigma_\nu\left(\begin{pmatrix} & 1\\ \varpi^2 & \end{pmatrix}\right)\varphi = \epsilon_\nu\,\varphi,$$

where φ is a new vector in the space of σ_ν. Note that $\epsilon_\nu^2 = 1$ since $\sigma\left(\left(\begin{smallmatrix} & 1\\ \varpi^2 & \end{smallmatrix}\right)^2\right) = \sigma\left(\left(\begin{smallmatrix} \varpi^2 & \\ & \varpi^2\end{smallmatrix}\right)\right)$ acts trivially under the hypothesis of trivial central character.

A model for ρ_ν on the space $\mathbb{C}[\mathbb{k}^*]$ of complex-valued functions on $\mathbb{k}^*$ is described in [29], following [47]. In terms of this model, the new space $(\text{c-Ind}_{ZK}^{G(F)}(\rho_\nu))^{K_1(\mathfrak{p}^2)}$ is spanned by the function $\varphi : G(F) \to \mathbb{C}[\mathbb{k}^*]$ supported on the coset $ZK\left(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right)K_1(\mathfrak{p}^2)$ and defined by

$$\varphi\left(zk\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}\right) = \rho_\nu(zk)\,w \quad (z\in Z,\ k\in K),$$

where $w \in \mathbb{C}[\mathbb{k}^*]$ is the constant function 1 [29, Proposition 3.1]. In particular,

$$\varphi\left(\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}\right)(1) = w(1) = 1.$$

Therefore the Atkin–Lehner eigenvalue is given by

$$\begin{aligned}
\epsilon_\nu = \left[\sigma_\nu\left(\begin{pmatrix} & 1\\ \varpi^2 & \end{pmatrix}\right)\varphi\right]\left(\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}\right)(1) &= \varphi\left(\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}\begin{pmatrix} & 1\\ \varpi^2 & \end{pmatrix}\right)(1)\\
&= \varphi\left(\begin{pmatrix}\varpi & \\ & \varpi\end{pmatrix}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}\right)(1)\\
&= \nu(\varpi)\left(\rho_\nu\left(\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\right)w\right)(1)\\
&= \left(\rho_\nu\left(\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\begin{pmatrix}-1 & \\ & 1\end{pmatrix}\right)w\right)(1),
\end{aligned}$$

since we are assuming $\nu|_{F^*} = 1$. Let $f_a \in \mathbb{C}[\mathbb{k}^*]$ be the characteristic function of $a \in \mathbb{k}^*$, so that $w = \sum_{a\in\mathbb{k}^*} f_a$. Using [29, (2-11)] we see that $\rho_\nu\left(\left(\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix}\right)\right)w = w$, and just below (2-16) of the same reference, we have

$$\left(\rho_\nu\left(\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\right)f_a\right)(1) = -\frac{1}{q}\nu(a^{-1})\sum_{\substack{u\in L^*\\ N(u)=a}}\psi(\mathrm{tr}_{\mathbb{k}}^{L}(u))\,\nu(u) \quad \text{for all } a\in\mathbb{k}^*.$$

We are assuming that $\nu|_{\Bbbk^*} = 1$, so $\nu(a^{-1}) = 1$, and summing over $a \in \Bbbk^*$ we have

$$\epsilon_\nu = -\frac{1}{q} \sum_{u \in L^*} \psi(\mathrm{tr}_{\Bbbk}^{L}(u))\,\nu(u).$$

This Gauss sum can be evaluated explicitly by an elementary calculation, giving (5-5); see [2, Theorem 11.6.1] for details.

Now suppose q is odd and $4 \nmid (q-1)$, and let t be a generator of the cyclic group L^*, so in particular $t^{(q^2-1)/2} = -1$. If ν has order r, there exists j with $\gcd(j, r) = 1$ such that $\nu(t) = e(j/r)$. Taking $\alpha = t^{(q^2-1)/4}$,

$$\nu(\alpha) = e\left(\frac{j(q+1)(q-1)}{4r}\right) = (-1)^{\frac{j(q+1)}{r}\frac{q-1}{2}} = (-1)^{\frac{j(q+1)}{r}},$$

since $\frac{1}{2}(q-1)$ is odd by hypothesis. The above is equal to $(-1)^{(q+1)/r} = -\epsilon_\nu$, since r is odd when j is even, and $2\,|\,(q+1)$. This proves (5-6). $\square$

Corollary 5.2. *Fix $\epsilon \in \{\pm 1\}$. Then the number of depth zero supercuspidal representations of $G(F)$ with trivial central character and root number ϵ is*

$$\begin{cases} \frac{1}{4}(q-1) & \text{if } q \equiv 1 \bmod 4, \\ \frac{1}{4}(q+1) & \text{if } q \equiv 3 \bmod 4 \text{ and } \epsilon = 1, \\ \frac{1}{4}(q-3) & \text{if } q \equiv 3 \bmod 4 \text{ and } \epsilon = -1, \\ 0 & \text{if } q \text{ is even and } \epsilon = 1, \\ \frac{1}{2}q & \text{if } q \text{ is even and } \epsilon = -1. \end{cases}$$

Proof. With notation as in (5-1), the number of supercuspidals with a given central character ω is $\frac{1}{2}P_\omega$. (We divide by 2 to account for the fact that ν and ν^q induce the same supercuspidal.) So the assertion for q even is immediate from (5-1) and (5-5).

Let q be odd, and let t be a generator of the cyclic group L^*. Then t^{q+1} is a generator of $\Bbbk^*$. The characters of L^* are the maps ν_m defined by

$$\nu_m(t) = e\left(\frac{m}{q^2-1}\right)$$

for $0 \le m < q^2 - 1$. We consider only those characters satisfying $\nu_m|_{\Bbbk^*} = 1$, i.e., $(q-1)\,|\,m$. Notice that ν_m is imprimitive if and only if $\nu_m^{q-1} = 1$, which holds if and only if $(q+1)\,|\,m$. So we consider the values $m = k(q-1)$ (for $1 \le k < (q+1)$) which are not multiples of $q+1$, i.e., $k \ne \frac{1}{2}(q+1)$.

The order of ν_m is

$$\frac{q^2-1}{\gcd(m, q^2-1)} = \frac{q+1}{\gcd(k, q+1)}. \tag{5-7}$$

By (5-5), σ_{ν_m} has root number

$$\epsilon_{\nu_m} = -(-1)^{\gcd(k,q+1)} = -(-1)^k, \tag{5-8}$$

since $q+1$ is even. Notice that the removed value $\frac{1}{2}(q+1)$ of k is odd if and only if $q \equiv 1 \mod 4$. So in this case, among the remaining $q-1$ values of k, half are odd and half are even. If $q \equiv 3 \mod 4$, then $\frac{1}{2}(q-1)+1 = \frac{1}{2}(q+1)$ of the remaining values of k are odd, and $\frac{1}{2}(q-1)-1 = \frac{1}{2}(q-3)$ are even.

To count supercuspidal representations, we divide the number of relevant k's by 2 since the distinct characters ν_m and ν_m^q induce the same representation. □

5.2. ***Simple supercuspidal representations.*** With notation as in the previous section, we recall here the construction of the supercuspidal representations of $G(F)$ of conductor $\mathfrak{p}^3$. The central character of any such representation is at most tamely ramified. So we begin by fixing a character $\omega_{\mathfrak{p}}$ of the center $Z = Z(F) \cong F^*$ of $G(F)$, trivial on $1+\mathfrak{p}$.

Define the following compact open subgroup of $G(F)$:

$$K' = \begin{pmatrix} 1+\mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}.$$

Fix a nontrivial character

$$\psi : \mathbb{k} \to \mathbb{C}^*,$$

which we also regard as a character of $\mathcal{O}$ trivial on $\mathfrak{p}$. Given $t \in \mathbb{k}^*$, define a character $\chi = \chi_t : K' \to \mathbb{C}^*$ by

$$\chi\left(\begin{pmatrix} a & b \\ c\varpi & d \end{pmatrix}\right) = \psi(b+tc). \tag{5-9}$$

The matrix

$$g_t = g_\chi = \begin{pmatrix} & t \\ \varpi & \end{pmatrix}$$

normalizes K', and furthermore

$$\chi(g_\chi^{-1} k g_\chi) = \chi(k) \quad \text{for all } k \in K'. \tag{5-10}$$

Given χ as above, let

$$H' = ZK' \cup g_\chi ZK'. \tag{5-11}$$

Although it is not reflected in the notation H', this set depends on both t and the fixed choice of ϖ. Given that $g_\chi^2 = t\varpi$, we may extend χ to a character χ_ζ of H' via

$$\chi_\zeta(g_\chi^d zk) = \zeta^d \omega_{\mathfrak{p}}(z)\chi(k) \tag{5-12}$$

for $z \in Z$ and $k \in K'$, where ζ is a fixed complex number satisfying

$$\zeta^2 = \omega_{\mathfrak{p}}(t\varpi). \tag{5-13}$$

Proposition 5.3. *The compactly induced representation $\sigma_\chi^\zeta = \text{c-Ind}_{H'}^{G(F)}(\chi_\zeta)$ is an irreducible supercuspidal representation of conductor $\mathfrak{p}^3$, with root number*

$$\epsilon\left(\tfrac{1}{2}, \sigma_\chi^\zeta, \psi\right) = \zeta.$$

Conversely, every irreducible admissible representation of $G(F)$ *of conductor* $\mathfrak{p}^3$ *with central character trivial on* $1+\mathfrak{p}$ *arises in this way.*

Proof. See [31]. For a more recent treatment using the above notation (but on GL_n), see [27, Sections 4 and 5 and Proposition 7.2]. The root number is computed in [1, Corollary 3.12]. □

We will also use the notation

$$\sigma_t^\zeta = \sigma_\chi^\zeta$$

for t, χ as in (5-9), though it should be borne in mind that the representation depends also on the choice of additive character ψ and uniformizer ϖ. When $F=\mathbb{Q}_p$, we will always take $\varpi = p$ and

$$\psi(x) = e(x/p) = e^{2\pi i x/p} \quad \text{for } x \in \mathbb{Z}/p\mathbb{Z}.$$

Henceforth we assume that $\omega_\mathfrak{p}$, and hence also σ_t^ζ, is unitary. Under the normalization $\mathrm{meas}(\overline{G(\mathcal{O})}) = 1$, the formal degree of σ_χ^ζ is

$$d_\chi = \tfrac{1}{2}(q^2-1). \tag{5-14}$$

This is seen, e.g., from (6.4) of [27] and the last line of the proof of Corollary 6.5 of the same paper.

We define the matrix coefficient $f_\mathfrak{p} : G(F) \to \mathbb{C}$ by

$$f_\mathfrak{p}(g) = d_\chi \overline{\left\langle \sigma_t^\zeta(g) \frac{\phi}{\|\phi\|}, \frac{\phi}{\|\phi\|} \right\rangle},$$

where $\phi \in \text{c-Ind}_{H'}^{G(F)}(\chi_\zeta)$ is the function

$$\phi(g) = \begin{cases} \chi_\zeta(g) & \text{if } g \in H', \\ 0 & \text{otherwise.} \end{cases} \tag{5-15}$$

Note that

$$\|\phi\|^2 = \int_{\overline{G}(F)} |\phi(g)|^2 \, dg = \mathrm{meas}(\overline{H}'). \tag{5-16}$$

Likewise,

$$\begin{aligned} \langle \sigma_t^\zeta(g)\,\phi, \phi\rangle = \int_{\overline{G}(F)} \phi(xg)\,\overline{\phi(x)}\,dx &= \int_{\overline{H}'} \phi(xg)\,\overline{\chi_\zeta(x)}\,dx \\ &= \begin{cases} \mathrm{meas}(\overline{H}')\chi_\zeta(g) & \text{if } g \in H', \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5-17}$$

By (5-14), (5-16), and (5-17), we have

$$f_\mathfrak{p}(g) = \begin{cases} \frac{1}{2}(q^2-1)\overline{\chi_\zeta(g)} & \text{if } g \in H', \\ 0 & \text{otherwise.} \end{cases} \tag{5-18}$$

5.3. ***Global setup.*** Fix square-free integers $S, T > 0$ with $ST > 1$ and $\gcd(S, T) = 1$, and let $k > 2$. Set $N = S^2T^3$, and let ω' be a Dirichlet character of modulus N satisfying

$$\omega'(-1) = (-1)^k. \tag{5-19}$$

Let ω be the Hecke character attached to ω' in (4-1). We assume in addition that for each $p\,|\,N$, ω_p is trivial on $1 + p\mathbb{Z}_p$, since this is true of the central character of every supercuspidal representation of conductor $\le p^3$. Equivalently, the conductor of ω' divides ST.

Proposition 5.4. *If $N = 2^2$ or 2^3 and k is odd, there is no such character.*

Proof. If N is a power of 2, then by (4-3) and (5-19), $(-1)^k = \omega'(-1) = \omega_2(-1) = 1$ since ω_2 is trivial on $\mathbb{Z}_2^* = 1 + 2\mathbb{Z}_2$. So k must be even. □

Under the stated hypotheses, for each $p\,|\,S$, ω_p is trivial on $1 + p\mathbb{Z}_p$. We may thus view ω_p as a character of $(\mathbb{Z}_p/p\mathbb{Z}_p)^* = \mathbb{F}_p^*$. For each such p, fix a primitive character ν_p of $\mathbb{F}_{p^2}^*$ such that $\nu_p|_{\mathbb{F}_p^*} = \omega_p$. Recall that the number $P_{\omega_p} > 0$ of such primitive characters is given in (5-1). We define $\nu_p(p) = \omega_p(p)$ and extend multiplicatively so that ν_p can also be viewed as a character of $\mathbb{Q}_p^*$, which allows us to view ρ_{ν_p} as a representation of Z_pK_p with central character ω_p. We let

$$\sigma_p = \sigma_{\nu_p} = \text{c-Ind}_{Z_pK_p}^{G(\mathbb{Q}_p)}(\rho_{\nu_p})$$

be the associated supercuspidal representation of $G(\mathbb{Q}_p)$. The number of isomorphism classes of supercuspidal representations of conductor p^2 and central character ω_p is $\frac{1}{2}P_{\omega_p}$.

For each prime $p\,|\,T$, fix a simple supercuspidal representation $\sigma_p = \sigma_{t_p}^{\zeta_p}$ of $G(\mathbb{Q}_p)$ with central character ω_p, where $t_p \in (\mathbb{Z}/p\mathbb{Z})^*$ and $\zeta_p^2 = \omega_p(t_p p)$. When the prime p is understood, we sometimes write t, ζ instead of t_p, ζ_p. By (4-2),

$$\zeta_p^2 = \omega_p(t_p p) = \omega_p(t_p) \prod_{\ell\,|\,N,\, \ell \neq p} \omega_\ell(p^{-1}). \tag{5-20}$$

In particular, when $N = p^3$ for p prime, $\zeta_p^2 = \omega_p(t_p)$.

Having made the above choices, we let $\hat{\sigma} = (\sigma_p)_{p\,|\,N}$ denote this tuple of local representations. Then $S_k(\hat{\sigma}) \subseteq S_k^{\text{new}}(S^2T^3, \omega')$.

Now consider the test function

$$f = f^{\text{n}} = f_\infty \prod_{p\,|\,N} f_p \prod_{\ell \nmid N} f_\ell^{\text{n}} \tag{5-21}$$

as in (4-10) with $N = S^2T^3$, where, for $p\,|\,S$ (resp. $p\,|\,T$), f_p is the chosen test function given in (5-4) (resp. in (5-18)).

The above setup is slightly different from that used in (4-10) and Proposition 4.1 since f_p is not a single matrix coefficient when $p\,|\,S$, but a certain sum of matrix coefficients, and without the formal degree coefficient. Nevertheless, the conclusions of Proposition 4.1 do hold for the above test function, as the next result shows.

Proposition 5.5. *With f defined above*, $\operatorname{tr}(T_{\mathtt{n}}\,|\,S_k(\hat{\sigma})) = \mathtt{n}^{(k/2)-1}\operatorname{tr} R(f)$.

Remark. This is not special to depth zero supercuspidals. By [29, Proposition 1.2], the proof below applies with any unramified (even power conductor) supercuspidals σ_p at $p\,|\,S$, using $d_{\sigma_p} = \dim\rho$ in place of $p-1$, where $\sigma_p = \text{c-Ind}_{ZK}^{G}(\rho)$. (Ramified supercuspidals may be induced from a *character* of an appropriately chosen open compact-mod-center subgroup, so for these, one can use a test function analogous to (5-18).)

Proof. In the proof of Proposition 4.1, we used the fact [23, Corollary 10.26] that for $\sigma = \sigma_p$, the operator $\sigma(d_\sigma \overline{\langle\sigma(g)\,w, w\rangle})$ is the orthogonal projection of the space of σ onto $\mathbb{C}w$. For f_p in (5-4), by [29, Proposition 1.1], there is an orthonormal set $\{w_1, \dots, w_{p-1}\}$ of vectors in the space of σ such that

$$f_p(g) = \sum_{j=1}^{p-1} \overline{\langle\sigma(g)\,w_j, w_j\rangle}.$$

So $\sigma(d_\sigma f_p) = \sigma((p-1) f_p)$ is the orthogonal projection onto $\operatorname{Span}\{w_1, \dots, w_{p-1}\}$. Therefore using this local test function in the proof of Proposition 4.1 would give us a block sum of $p-1$ copies of the matrix for $\mathtt{n}^{1-(k/2)}T_{\mathtt{n}}$. To get the correct trace, we would need to divide by $p-1$, which is achieved by simply taking f_p instead of $(p-1)f_p$. $\square$

Noting that $f_p(1) = \dim\rho_v = p-1 = d_{\sigma_p}$ for $p\,|\,S$, the identity term in the formula for $\operatorname{tr} R(f)$ is

$$\overline{\omega'(\mathtt{n}^{1/2})}\,\tfrac{1}{12}(k-1)\prod_{p\,|\,S}(p-1)\prod_{p\,|\,T}\tfrac{1}{2}(p^2-1), \tag{5-22}$$

as seen between the brackets in Theorem 4.2. We remark that this is not always an integer when $\mathtt{n} = 1$. For example consider the case where $S = 1$. For $p \geq 3$ prime,

$$v_2(p^2-1) = v_2(p-1) + v_2(p+1) \geq 3,$$

with equality holding precisely when $p \equiv 3, 5 \bmod 8$. (Here, v_2 is the 2-adic valuation.) It follows easily that when $\mathtt{n} = 1$, the identity term $\frac{1}{12}(k-1)\prod_{p\,|\,T}\frac{1}{2}(p^2-1)$ fails to be an integer in exactly the following situations:

- $T = 2$ and $k \not\equiv 1 \bmod 8$.
- $T = 3$ and $k \not\equiv 1 \bmod 3$.
- $T = 2p$ for some $p \equiv 3, 5 \bmod 8$, and k is even.

So in such instances, when $S = \mathtt{n} = 1$ the elliptic contribution to $|H_k(\hat{\sigma})|$ in Theorem 4.2 must be nonzero for this simple reason.

The list of relevant matrices in the trace formula of Theorem 4.2 can be refined in certain situations.

Proposition 5.6. *Let $N = S^2T^3$ as above, let $f = f^{\mathtt{n}}$ be the test function defined in (5-21), let $M \mid T$, and $0 \le r < \sqrt{4\mathtt{n}/M}$. Then $\Phi\big(\left(\begin{smallmatrix} 0 & -\mathtt{n}M \\ 1 & rM \end{smallmatrix}\right), f\big) = 0$ in each of the following situations*:

- *$r = 0$ and k is odd.*
- *There exists $p \mid N$ such that $X^2 - rMX + \mathtt{n}M$ has a root in $\mathbb{Q}_p$.*
- *There exists $p \mid M$ such that $-pt_p/\mathtt{n}M$ is not a square modulo p, where t_p is the parameter of the local representation $\sigma_{t_p}^{\zeta_p}$.*
- *There exists $p \mid (T/M)$ such that $X^2 - rMX + \mathtt{n}M \equiv (X - z)^2 \bmod p$ has no solution $z \in (\mathbb{Z}/p\mathbb{Z})^*$.*

Remark. For the case $\mathtt{n} = 1$, we can refine the list of relevant γ even further (see Proposition 7.9 below).

Proof. The first bullet point follows from (4-13).

Let $\gamma = \left(\begin{smallmatrix} & -\mathtt{n}M \\ 1 & rM \end{smallmatrix}\right)$, and suppose that $\Phi(\gamma, f) \ne 0$. Then by Proposition 4.3, γ is elliptic in $G(\mathbb{Q}_p)$, which gives the second bullet point.

For the third bullet point, suppose $p \mid M$. Write $\det\gamma = up$ for some $u \in \mathbb{Z}_p^*$. Assuming the local orbital integral $\Phi(\gamma, f_p)$ is nonzero, $f_p(g^{-1}\gamma g) \ne 0$ for some $g \in G(\mathbb{Q}_p)$. Then $g^{-1}\gamma g$ belongs to the ramified component of $\mathrm{Supp}(f_p)$, i.e., writing $t = t_p$,

$$g^{-1}\gamma g = z \begin{pmatrix} & t \\ p & \end{pmatrix} \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in Z g_{\chi_p} K'$$

for some $b, c \in \mathbb{Z}_p$, $a, d \in 1 + p\mathbb{Z}_p$, and $z \in \mathbb{Z}_p^*$. Taking determinants, we have

$$up = -tpz^2(ad - pbc),$$

and hence

$$u \equiv -tz^2 \bmod p. \tag{5-23}$$

This shows that $-t/u$ is a quadratic residue modulo p.

Finally, if $p \mid (N/M)$, then $\det\gamma \in \mathbb{Z}_p^*$ so if $\Phi(\gamma, f_p) \ne 0$, some conjugate $g^{-1}\gamma g$ lies in the unramified component of $\mathrm{Supp}(f_p)$:

$$g^{-1}\gamma g = z \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in ZK'$$

for z, a, b, c, d as above. Taking determinants, $\det\gamma \equiv z^2 \bmod p$. Taking the trace, $\operatorname{tr}\gamma \equiv 2z \bmod p$. Hence $P_\gamma(X) \equiv X^2 - 2zX + z^2 \equiv (X - z)^2 \bmod p$. $\square$

6. Local orbital integrals at primes $p \mid N$ for $N = S^2T^3$

Our goal here is to compute

$$\Phi(\gamma, f_p) = \int_{\overline{G_\gamma(\mathbb{Q}_p)}\backslash \overline{G}(\mathbb{Q}_p)} f_p(g^{-1}\gamma g)\, dg$$

taking for f_p the test functions given in (5-4) and (5-18), and for γ the matrices given in Theorem 4.2, and using the quotient measure defined in Section 4.5, so

$$\Phi(\gamma, f_p) = \int_{\overline{G}(\mathbb{Q}_p)} f_p(g^{-1}\gamma g)\, dg.$$

With these calculations, Theorem 1.1 will follow immediately from Theorem 4.2.

We will use the strategy adopted by Palm in [43, Proposition 9.11.3] which avoids the use of lattices or buildings. There are errors in the statement and proof of his proposition, so we cannot simply quote the result. However, the basic method is sound and can be adapted to give the result in the cases of interest to us here.

The following lemma will allow us to rewrite the integral in such a way as to exploit the structure of the support of f_p.

Lemma 6.1 [43, Lemma 6.4.10]. *Let G be a unimodular locally compact group, and suppose I_1, I_2 are two open compact subgroups of G, each given total Haar measure 1. Then for any choice of Haar measure on G we have*

$$\int_G \phi(g)\, dg = \sum_{x \in I_1\backslash G/I_2} \mathrm{meas}_G(I_1 x I_2) \int_{I_1}\int_{I_2} \phi(i_1 x i_2)\, di_2\, di_1 \tag{6-1}$$

for all $\phi \in C_c(G)$.

Proof. For $\phi \in C_c(G)$, we see that

$$\int_G \phi(g)\, dg = \int_G \int_{I_1}\int_{I_2} \phi(i_1 g i_2)\, di_2\, di_1\, dg$$

by changing the order of integration and using the bi-invariance of dg. The inner double integral defines a compactly supported function F of $g \in G$ which is constant on double cosets $I_1 g I_2$, and is therefore a finite linear combination of characteristic functions of such double cosets. The identity (6-1) clearly holds for the characteristic function of a double coset. By linearity it holds for F as well, so

$$\begin{aligned}
\int_G \phi(g)\, dg &= \int_G F(g)\, dg \\
&= \sum_{x \in I_1\backslash G/I_2} \mathrm{meas}_G(I_1 x I_2) \int_{I_1}\int_{I_2} F(i_1 x i_2)\, di_2\, di_1 \\
&= \sum_{x \in I_1\backslash G/I_2} \mathrm{meas}_G(I_1 x I_2) F(x) \\
&= \sum_{x \in I_1\backslash G/I_2} \mathrm{meas}_G(I_1 x I_2) \int_{I_1}\int_{I_2} \phi(i_1 x i_2)\, di_2\, di_1. \qquad \square
\end{aligned}$$

6.1. ***Preliminaries when $p\,|\,T$.*** Throughout much of this section, we will work over a p-adic field F with notation as in Section 5.2, and write G for $G(F)$, and $\overline{G}$ for G/Z. Having fixed a simple supercuspidal representation σ_t^ζ of G with unitary central character $\omega_{\mathfrak{p}}$, we take $f_{\mathfrak{p}}$ to be the test function given in (5-18).

Applying Lemma 6.1 to (4-18), we have

$$\begin{aligned}\Phi(\gamma, f_{\mathfrak{p}}) &= \int_{\overline{G}} f_{\mathfrak{p}}(g^{-1}\gamma g)\,dg\\ &= \sum_{x\in \overline{K}'\backslash\overline{G}/\overline{K}'} \operatorname{meas}_{\overline{G}}(\overline{K}'x\overline{K}')\int_{\overline{K}'}\int_{\overline{K}'} f_{\mathfrak{p}}(h_2^{-1}x^{-1}h_1^{-1}\gamma h_1 x h_2)\,dh_1\,dh_2,\end{aligned}$$

where each dh_i is normalized to have total measure 1. Since $f_{\mathfrak{p}}|_{K'}$ is a character, h_2 has no effect, and we obtain

$$\Phi(\gamma, f_{\mathfrak{p}}) = \sum_{x\in \overline{K}'\backslash\overline{G}/\overline{K}'} \operatorname{meas}_{\overline{G}}(\overline{K}'x\overline{K}')\int_{\overline{K}'} f_{\mathfrak{p}}(x^{-1}h^{-1}\gamma h x)\,dh. \tag{6-2}$$

In order to compute the above, we need a few preparations. First, recall the affine Bruhat decomposition

$$G = K'MK' \cup K'MwK' = K'MK' \cup K'Mg_\chi K',$$

where $w = \left(\begin{smallmatrix} & -1\\ 1 & \end{smallmatrix}\right)$ and M is the diagonal subgroup [7, Proposition 17.1]. Accordingly, we may take as a set of representatives $x \in \overline{K}'\backslash\overline{G}/\overline{K}'$ the elements $x = m$ and $x = mg_\chi$ for

$$m \in \left\{\begin{pmatrix} y & \\ & 1\end{pmatrix}, \begin{pmatrix} y & \\ & \varpi^j\end{pmatrix}, \begin{pmatrix} \varpi^n & \\ & y\end{pmatrix} \,\middle|\, j>0,\ n>0,\ y\in(\mathcal{O}/\mathfrak{p})^*\right\}. \tag{6-3}$$

For each such x we need to compute the integral in (6-2), which we denote by

$$K_\gamma(x) = \int_{\overline{K}'} f_{\mathfrak{p}}(x^{-1}h^{-1}\gamma h x)\,dh.$$

By (5-10),

$$f_{\mathfrak{p}}(g_\chi^{-1} g g_\chi) = f_{\mathfrak{p}}(g) \quad \text{for all } g.$$

Therefore $K_\gamma(xg_\chi) = K_\gamma(x)$. Furthermore, since g_χ normalizes K', the measure of $\overline{K}'x\overline{K}'$ is unchanged if x is replaced by xg_χ. It follows that

$$\Phi(\gamma, f_{\mathfrak{p}}) = 2\sum_{x \text{ in (6-3)}} \operatorname{meas}_{\overline{G}}(\overline{K}'x\overline{K}')K_\gamma(x). \tag{6-4}$$

Lemma 6.2. *Let $x = \left(\begin{smallmatrix}\varpi^n & \\ & y\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} y & \\ & \varpi^n\end{smallmatrix}\right)$ for $n \ge 0$ and $y \in \mathcal{O}^*$. Then with measure on $\overline{G}$ normalized so that* $\operatorname{meas}(\overline{K}) = 1$,

$$\operatorname{meas}_{\overline{G}}(\overline{K}'x\overline{K}') = \frac{q^n}{q^2-1}. \tag{6-5}$$

Proof. We may assume that $y = 1$ since, for example,

$$\operatorname{meas}\left(\overline{K}'\begin{pmatrix}\varpi^n & \\ & y\end{pmatrix}\overline{K}'\right) = \operatorname{meas}\left(\overline{K}'\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}\overline{K}'\begin{pmatrix}1 & \\ & y\end{pmatrix}\right) = \operatorname{meas}\left(\overline{K}'\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}\overline{K}'\right).$$

Likewise, since g_χ normalizes K' and $g_\chi^{-1}\left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right)g_\chi = \left(\begin{smallmatrix}1 & \\ & \varpi^n\end{smallmatrix}\right)$, we may assume that $x = \left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right)$.

We claim that for $n \geq 0$,

$$K'\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}K' = \bigcup_{b \in \mathcal{O}/\mathfrak{p}^n}\begin{pmatrix}\varpi^n & b \\ 0 & 1\end{pmatrix}K', \tag{6-6}$$

a disjoint union. The union is disjoint since

$$\begin{pmatrix}\varpi^n & b_1 \\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}\varpi^n & b_2 \\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & \frac{b_2 - b_1}{\varpi^n} \\ 0 & 1\end{pmatrix},$$

which is in K' if and only if $b_1 \equiv b_2 \bmod \mathfrak{p}^n$. The inclusion $\supseteq$ in (6-6) follows from

$$\begin{pmatrix}\varpi^n & b \\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & b \\ 0 & 1\end{pmatrix}\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix} \in K'\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}.$$

The reverse inclusion follows from

$$\begin{pmatrix}a & b \\ c & d\end{pmatrix}\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix} = \begin{pmatrix}\varpi^n & bd^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix}a - cbd^{-1} & 0 \\ c\varpi^n & d\end{pmatrix}.$$

By the decomposition (6-6),

$$\operatorname{meas}\left(\overline{K}'\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}\overline{K}'\right) = q^n \operatorname{meas}(\overline{K}') = \frac{q^n}{q^2 - 1},$$

since $\operatorname{meas}(\overline{K}') = 1/(q^2 - 1)$ when $\operatorname{meas}(\overline{K}) = 1$, as shown in the proof of [27, Corollary 6.5]. □

If $x = \left(\begin{smallmatrix}y & \\ & 1\end{smallmatrix}\right)$, then $K_\gamma(x) = f_\mathfrak{p}(\gamma^y)$ where $\gamma^y = \left(\begin{smallmatrix}y^{-1} & \\ & 1\end{smallmatrix}\right)\gamma\left(\begin{smallmatrix}y & \\ & 1\end{smallmatrix}\right)$, since $f_\mathfrak{p}$ is a character of K', $\left(\begin{smallmatrix}y & \\ & 1\end{smallmatrix}\right)$ normalizes K', and we give $\overline{K}'$ measure 1. Thus, in view of the above lemma, (6-4) now becomes

$$\Phi(\gamma, f_\mathfrak{p}) = \frac{2}{q^2 - 1}\sum_{y \in (\mathcal{O}/\mathfrak{p})^*} f_\mathfrak{p}(\gamma^y) + 2\sum_{n=1}^{\infty}\frac{q^n}{q^2 - 1}\sum_{y \in (\mathcal{O}/\mathfrak{p})^*}\left[K_\gamma\left(\begin{pmatrix}\varpi^n & \\ & y\end{pmatrix}\right) + K_\gamma\left(\begin{pmatrix}y & \\ & \varpi^n\end{pmatrix}\right)\right]. \tag{6-7}$$

To compute $K_\gamma(x)$, we fix coordinates on $\overline{K}'$ with the following.

Lemma 6.3. *Let G, H, K be compact topological groups, with $G = HK$ and $H \cap K = \{1\}$. Let dh and dk be the respective Haar measures on H, K of total measure* 1. *Then the Haar measure on G of measure* 1 *is given by*

$$\int_G f(g)\,dg = \int_H \int_K f(hk)\,dk\,dh.$$

Proof. This is a special case of [23, Lemma 7.13]. □

We will use the Iwahori decomposition [7, (7.3.1)] of K'. Letting $M(1+\mathfrak{p}) = \left(\begin{smallmatrix} 1+\mathfrak{p} & \\ & 1+\mathfrak{p}\end{smallmatrix}\right)$, $N(\mathcal{O}) = \left(\begin{smallmatrix} 1 & \mathcal{O} \\ 0 & 1\end{smallmatrix}\right)$, and $N'(\mathfrak{p}) = \left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p} & 1\end{smallmatrix}\right)$, the decomposition

$$K' = N(\mathcal{O}) \cdot N'(\mathfrak{p}) \cdot M(1+\mathfrak{p})$$

is a (topological) direct product, and the same is true for any ordering of the three factors. We will take $\operatorname{meas}(\overline{K}') = \operatorname{meas}(K') = 1$, so that applying the above lemma, this Haar measure on $\overline{K}'$ is given by both of the following:

$$\int_{\overline{K}'} \phi(k)\,dk = \int_{\mathcal{O}} \int_{\mathcal{O}} \int_{M(1+\mathfrak{p})} \phi\left(\begin{pmatrix} 1 & 0 \\ \varpi c & 1\end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} m\right) dm\,db\,dc \tag{6-8}$$

$$= \int_{\mathcal{O}} \int_{\mathcal{O}} \int_{M(1+\mathfrak{p})} \phi\left(\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi c & 1\end{pmatrix} m\right) dm\,dc\,db, \tag{6-9}$$

where dm, db, dc each have total measure 1.

6.2. ***The case where $p\,|\,T$ and γ is ramified.*** The aim here is to compute $\Phi(\gamma, f_p)$ when $p|N$ and γ is ramified at p, i.e., $v_p(\det\gamma)$ is odd. As above we work over a p-adic field F with uniformizer ϖ, and a fixed supercuspidal representation σ_t^ζ of $G(F)$ as in Section 5.2. We can assume that $v_{\mathfrak{p}}(\det\gamma) = 1$, and further, by Lemma 4.12, that $v_{\mathfrak{p}}(\operatorname{tr}\gamma) \geq 1$. So we will consider matrices in the F-rational canonical form

$$\gamma = \begin{pmatrix} 0 & -u\varpi \\ 1 & v\varpi\end{pmatrix} = w \begin{pmatrix} 1 & v\varpi \\ 0 & u\varpi\end{pmatrix} \tag{6-10}$$

for $u \in \mathcal{O}^*$, $v \in \mathcal{O}$, and $w = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$.

Proposition 6.4. *For γ as in (6-10) and $f_{\mathfrak{p}}$ as in (5-18), $\Phi(\gamma, f_{\mathfrak{p}}) = 0$ unless $-t/u$ is a square modulo $\mathfrak{p}$. If the latter condition holds and $y^2 \equiv -t/u \bmod \mathfrak{p}$, then*

$$\Phi\left(\begin{pmatrix} 0 & -u\varpi \\ 1 & v\varpi\end{pmatrix}, f_{\mathfrak{p}}\right) = \bar{\zeta}\left(\overline{\psi(yv)}\,\omega_{\mathfrak{p}}(y) + \delta(\mathfrak{p}\nmid 2)\,\overline{\psi(-yv)}\,\omega_{\mathfrak{p}}(-y)\right),$$

where ψ is the nontrivial character of $\mathcal{O}/\mathfrak{p}$ used in (5-9). Thus, in the case of trivial central character (so $\zeta^2 = 1$), we have

$$\Phi(\gamma, f_{\mathfrak{p}}) = \begin{cases} 2\zeta\,\mathrm{Re}(\psi(yv)) & \textit{if } \mathfrak{p}\nmid 2, \\ \overline{\psi(yv)}\,\zeta & \textit{if } \mathfrak{p}\,|\,2.\end{cases}$$

When the central character is trivial, $F=\mathbb{Q}_p$, and $v\in\mathbb{Z}$, this gives

$$\Phi(\gamma, f_p)=\begin{cases}2\zeta\cos\left(\frac{2\pi yv}{p}\right) & \text{if } p\neq 2,\\ (-1)^v\zeta & \text{if } p=2.\end{cases} \tag{6-11}$$

Proof. We need to compute each term of (6-7). First, note that for $y\in(\mathcal{O}/\mathfrak{p})^*$,

$$\gamma^y=\begin{pmatrix}0 & -u\varpi/y\\ y & v\varpi\end{pmatrix}=\begin{pmatrix} & t\\ \varpi & \end{pmatrix}\begin{pmatrix}y/\varpi & v\\ & -u\varpi/ty\end{pmatrix}$$

does not belong to the support of $f_\mathfrak{p}$. Hence $f_\mathfrak{p}(\gamma^y)=K_\gamma\left(\left(\begin{smallmatrix}y & \\ & 1\end{smallmatrix}\right)\right)=0$.

Next, suppose $x=\left(\begin{smallmatrix}y & \\ & \varpi^j\end{smallmatrix}\right)$ with $j>0$ and $y\in\mathcal{O}^*$. Then we use the measure in (6-8):

$$K_\gamma(x)$$
$$=\int_\mathcal{O}\int_\mathcal{O}\int_{M(1+\mathfrak{p})} f_\mathfrak{p}\left(x^{-1}m^{-1}\begin{pmatrix}1 & -b\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\varpi c & 1\end{pmatrix}\gamma\begin{pmatrix}1 & 0\\ \varpi c & 1\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}mx\right)dm\,db\,dc.$$

Note that m commutes with x, and lies in the kernel of $f_\mathfrak{p}$. Therefore the integration over $M(1+\mathfrak{p})$ has no effect, and

$$K_\gamma(x)=\int_\mathcal{O}\int_\mathcal{O} f_\mathfrak{p}\left(x^{-1}\begin{pmatrix}1 & -b\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\varpi c & 1\end{pmatrix}\gamma\begin{pmatrix}1 & 0\\ \varpi c & 1\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}x\right)db\,dc.$$

Likewise

$$\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}x=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & \\ & \varpi^j\end{pmatrix}=\begin{pmatrix}y & \\ & \varpi^j\end{pmatrix}\begin{pmatrix}1 & b\varpi^j/y\\ & 1\end{pmatrix}.$$

Note that the matrix on the right lies in K' since $j>0$, and in fact it is in the kernel of $f_\mathfrak{p}$. Therefore the integral over b also has no effect, and

$$K_\gamma(x)=\int_\mathfrak{p} f_\mathfrak{p}\left(x^{-1}\begin{pmatrix}1 & 0\\ -r & 1\end{pmatrix}\gamma\begin{pmatrix}1 & 0\\ r & 1\end{pmatrix}x\right)dr, \tag{6-12}$$

where dr gives $\mathfrak{p}$ the measure 1.

Taking $\gamma=w\left(\begin{smallmatrix}1 & v\varpi\\ 0 & u\varpi\end{smallmatrix}\right)$ as in (6-10),

$$\begin{aligned}\begin{pmatrix}1 & 0\\ -r & 1\end{pmatrix}w\begin{pmatrix}1 & v\varpi\\ 0 & u\varpi\end{pmatrix}\begin{pmatrix}1 & 0\\ r & 1\end{pmatrix}&=w\begin{pmatrix}1 & r\\ 0 & 1\end{pmatrix}\begin{pmatrix}1+rv\varpi & v\varpi\\ ru\varpi & u\varpi\end{pmatrix}\\ &=w\begin{pmatrix}1+r^2u\varpi+rv\varpi & (v+ru)\varpi\\ ru\varpi & u\varpi\end{pmatrix}.\end{aligned}$$

Writing the above as $w\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in w\left(\begin{smallmatrix} 1+\mathfrak{p}^2 & \mathfrak{p} \\ \mathfrak{p}^2 & \varpi\mathcal{O}^*\end{smallmatrix}\right)$,

$$\begin{aligned} x^{-1}\begin{pmatrix} 1 & 0 \\ -r & 1\end{pmatrix}\gamma\begin{pmatrix} 1 & 0 \\ r & 1\end{pmatrix}x &= \begin{pmatrix} y^{-1} & \\ & \varpi^{-j}\end{pmatrix} w \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} y & \\ & \varpi^{j}\end{pmatrix} \\ &= w\begin{pmatrix} \varpi^{-j} & \\ & y^{-1}\end{pmatrix}\begin{pmatrix} ay & b\varpi^{j} \\ cy & d\varpi^{j}\end{pmatrix} \\ &= w\begin{pmatrix} ay/\varpi^{j} & b \\ c & d\varpi^{j}/y\end{pmatrix} \\ &= g_\chi\begin{pmatrix} \varpi^{-1} & \\ & -t^{-1}\end{pmatrix}\begin{pmatrix} ay/\varpi^{j} & b \\ c & d\varpi^{j}/y\end{pmatrix} \\ &= g_\chi\begin{pmatrix} ay/\varpi^{j+1} & b/\varpi \\ -c/t & -d\varpi^{j}/ty\end{pmatrix}. \end{aligned}$$

Since the determinant is $u\varpi$, this belongs to the support of $f_\mathfrak{p}$ if and only if the matrix on the right belongs to $\mathcal{O}^*K'$. But this would require $j+1=0$, which is impossible since $j>0$. Hence

$$K_\gamma\left(\begin{pmatrix} y & \\ & \varpi^{j}\end{pmatrix}\right) = 0$$

for all $j>0$ and all $y \in \mathcal{O}^*$.

Lastly, consider $x = \left(\begin{smallmatrix} \varpi^{n} & \\ & y\end{smallmatrix}\right)$ for $n>0$ and $y\in\mathcal{O}^*$. We proceed in just the same way, only this time using the coordinates given in (6-9). Taking $-b$ in place of b for convenience, and eliminating the integral over $M(1+\mathfrak{p})$ with the same justification as before,

$$K_\gamma(x) = \int_\mathcal{O}\int_\mathcal{O} f_\mathfrak{p}\left(x^{-1}\begin{pmatrix} 1 & 0 \\ -\varpi c & 1\end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix} 1 & -b \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi c & 1\end{pmatrix}x\right)db\,dc.$$

Now

$$\begin{pmatrix} 1 & 0 \\ c\varpi & 1\end{pmatrix}x = \begin{pmatrix} 1 & 0 \\ c\varpi & 1\end{pmatrix}\begin{pmatrix} \varpi^{n} & \\ & y\end{pmatrix} = \begin{pmatrix} \varpi^{n} & \\ & y\end{pmatrix}\begin{pmatrix} 1 & \\ c\varpi^{1+n}/y & 1\end{pmatrix}.$$

The matrix on the right lies in the kernel of $f_\mathfrak{p}$. Therefore

$$K_\gamma(x) = \int_\mathcal{O} f_\mathfrak{p}\left(x^{-1}\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix} 1 & -b \\ 0 & 1\end{pmatrix}x\right)db. \tag{6-13}$$

Taking $\gamma = w\left(\begin{smallmatrix} 1 & v\varpi \\ 0 & u\varpi\end{smallmatrix}\right)$, we have

$$\begin{aligned} \begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} w \begin{pmatrix} 1 & v\varpi \\ & u\varpi\end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & 1\end{pmatrix} &= w\begin{pmatrix} 1 & 0 \\ -b & 1\end{pmatrix}\begin{pmatrix} 1 & -b+v\varpi \\ 0 & u\varpi\end{pmatrix} \\ &= w\begin{pmatrix} 1 & -b+v\varpi \\ -b & b^2 - v\varpi b + u\varpi\end{pmatrix}. \end{aligned}$$

Thus letting $P_\gamma(X)$ denote the characteristic polynomial of γ,

$$\begin{aligned}
x^{-1}\begin{pmatrix}1&b\\0&1\end{pmatrix}\gamma\begin{pmatrix}1&-b\\0&1\end{pmatrix}x &= \begin{pmatrix}\varpi^{-n}&\\&y^{-1}\end{pmatrix} w \begin{pmatrix}1&-b+v\varpi\\-b&P_\gamma(b)\end{pmatrix}\begin{pmatrix}\varpi^{n}&\\&y\end{pmatrix}\\
&= w\begin{pmatrix}y^{-1}&\\&\varpi^{-n}\end{pmatrix}\begin{pmatrix}\varpi^{n}&y(-b+v\varpi)\\-b\varpi^{n}&yP_\gamma(b)\end{pmatrix}\\
&= w\begin{pmatrix}\varpi^{n}/y&-b+v\varpi\\-b&yP_\gamma(b)/\varpi^{n}\end{pmatrix}\\
&= g_\chi\begin{pmatrix}\varpi^{-1}&\\&-t^{-1}\end{pmatrix}\begin{pmatrix}\varpi^{n}/y&-b+v\varpi\\-b&yP_\gamma(b)/\varpi^{n}\end{pmatrix}\\
&= g_\chi\begin{pmatrix}\varpi^{n-1}/y&v-b/\varpi\\b/t&-yP_\gamma(b)/t\varpi^{n}\end{pmatrix}. \qquad (6\text{-}14)
\end{aligned}$$

Since the determinant is $u\varpi$, the above belongs to the support of $f_{\mathfrak{p}}$ if and only if the matrix on the right belongs to $\mathcal{O}^*K'$. This means in particular that $n=1$ and $b\in\mathfrak{p}$. Make the change of variables $b=c\varpi$, $db=|\varpi|dc=q^{-1}dc$. Then

$$P_\gamma(b)=\varpi(u-vc\varpi+c^2\varpi),$$

and

$$K_\gamma\left(\begin{pmatrix}\varpi&\\&y\end{pmatrix}\right)=q^{-1}\int_{\mathcal{O}} f_{\mathfrak{p}}\left(g_\chi y^{-1}\begin{pmatrix}1&yv-cy\\cy\varpi/t&-y^2(u-vc\varpi+c^2\varpi)/t\end{pmatrix}\right)dc.$$

From the definition of K', the integrand is nonzero if and only if $y^2\equiv -t/u \bmod \mathfrak{p}$. (We have already seen in Proposition 5.6 that $-t/u$ must be a square mod $\mathfrak{p}$.) Assuming this to be the case, then from (5-9), (5-12), and (5-18), we have

$$\begin{aligned}
K_\gamma(x) &= q^{-1}\overline{\omega_{\mathfrak{p}}(y^{-1})\zeta}\,d_\chi\int_{\mathcal{O}}\overline{\psi(yv-cy+cy)}\,dc\\
&= q^{-1}\omega_{\mathfrak{p}}(y)\bar\zeta d_\chi\overline{\psi(yv)}=\frac{q^2-1}{2q}\omega_{\mathfrak{p}}(y)\bar\zeta\,\overline{\psi(yv)}. \qquad (6\text{-}15)
\end{aligned}$$

To recap, for $\gamma=\left(\begin{smallmatrix}0&-u\varpi\\1&v\varpi\end{smallmatrix}\right)$, $K_\gamma(x)=0$ unless $x=\left(\begin{smallmatrix}\varpi&\\&y\end{smallmatrix}\right)$ for $y^2\equiv -t/u \bmod \mathfrak{p}$, so (6-7) becomes

$$\Phi(\gamma,f_{\mathfrak{p}})=\frac{2q}{q^2-1}\sum_{y\in(\mathcal{O}/\mathfrak{p})^*}K_\gamma\left(\begin{pmatrix}\varpi&\\&y\end{pmatrix}\right)=\bar\zeta\sum_{\varepsilon\in\{\pm1 \bmod \mathfrak{p}\}}\overline{\psi(\varepsilon y_0 v)}\,\omega_{\mathfrak{p}}(\varepsilon y_0),$$

where y_0 is any fixed solution to $y_0^2\equiv -t/u \bmod \mathfrak{p}$. Note that when $\mathfrak{p}\,|\,2$, we can take $\varepsilon=1$, and if $F=\mathbb{Q}_2$ we can also take $y_0=1$. □

6.3. ***The case where p | T and γ is unramified.*** We adopt the same notation used in the previous subsection. Suppose γ is unramified, i.e., $\mathrm{ord}_{\mathfrak{p}}(\det\gamma)$ is even. Scaling if needed, we may assume that $\det\gamma\in\mathcal{O}^*$. For the nonvanishing of $\Phi(\gamma,f_{\mathfrak{p}})$, it is necessary that some conjugate of γ belong to the unramified component of the support of $f_{\mathfrak{p}}$, namely $ZK'_{\mathfrak{p}}$. Given that $u=\det\gamma\in\mathcal{O}^*$, this means that $\operatorname{tr}\gamma$ must also be integral. So we may take γ in rational canonical form

$$\gamma=\begin{pmatrix}0 & -u\\ 1 & v\end{pmatrix} \tag{6-16}$$

for some $u\in\mathcal{O}^*$ and $v\in\mathcal{O}$.

Proposition 6.5. *For γ elliptic in $G(F)$ and of the form (6-16), $\Phi(\gamma,f_{\mathfrak{p}})=0$ unless the characteristic polynomial P_γ has a nonzero double root modulo $\mathfrak{p}$:*

$$P_\gamma(X)\equiv(X-z)^2 \bmod \mathfrak{p} \tag{6-17}$$

for some $z\in(\mathcal{O}/\mathfrak{p})^*$. Under this condition,

$$\Phi(\gamma,f_{\mathfrak{p}})=\frac{\overline{\omega_{\mathfrak{p}}(z)}}{q}\sum_{n=1}^{\infty}\sum_{c\bmod\mathfrak{p}}\mathcal{N}_\gamma(c,n)\sum_{y\in(\mathcal{O}/\mathfrak{p})^*}\psi\left(\frac{yc}{z}\right)\psi\left(-\frac{t}{yz}\right)^{\delta(n=1)}, \tag{6-18}$$

where ψ is the nontrivial character of $\mathcal{O}/\mathfrak{p}$ used in (5-9), $t\in(\mathcal{O}/\mathfrak{p})^*$ is the parameter of σ_t^ζ, and

$$\mathcal{N}_\gamma(c,n)=\#\{b \bmod \mathfrak{p}^{n+1}\mid P_\gamma(b)\equiv c\varpi^n \bmod \mathfrak{p}^{n+1}\}.$$

Remarks. (1) Since P_γ is irreducible over F, there exists r such that $P_\gamma(X)\equiv 0 \bmod \mathfrak{p}^r$ has no solution, and hence $\mathcal{N}_\gamma(c,n)=0$ for all pairs (c,n) with $n\geq r$. So the series is actually a finite sum.

(2) When $n=1$ the sum over y is a Kloosterman sum. When $n>1$,

$$\sum_y\psi\left(\frac{yc}{z}\right)=\begin{cases}q-1 & \text{if } c\equiv 0 \bmod \mathfrak{p},\\ -1 & \text{otherwise.}\end{cases}$$

(3) When $F=\mathbb{Q}_p$, the integer $\mathcal{N}_\gamma(c,n)$ is given explicitly in [26, Lemma 9.6], and presumably there is a version of that lemma for an arbitrary p-adic field. In particular, $N_\gamma(c,n)=0$ unless $n\leq\operatorname{ord}_p(\Delta_\gamma)-1$, and for such n,

$$N_\gamma(c,n)\leq p^{\lfloor(n+1)/2\rfloor}$$

assuming γ is elliptic in $G(\mathbb{Q}_p)$ and satisfies (6-17). This gives the following bound for the orbital integral: setting $\delta=\operatorname{ord}_p(\Delta_\gamma)$,

$$\begin{aligned}|\Phi(\gamma,f_p)|\leq\sum_{n=1}^{\delta-1}(p-1)(p^{1/2})^{n+1}&=p(p-1)\sum_{n=0}^{\delta-2}(p^{1/2})^n\\ &=p(p-1)\frac{(p^{1/2})^{\delta-1}-1}{p^{1/2}-1}\\ &=p(p^{1/2}+1)(p^{-1/2}p^{\delta/2}-1)\leq 2p|\Delta_\gamma|_p^{-1/2}.\end{aligned}$$

This illustrates the general bound given in [21, (1.8) and Theorem 3.11], according to which

$$|\Phi(\gamma, f_{\mathfrak{p}})| \le C \cdot (d_{\sigma_{\mathfrak{p}}})^{\eta} |\Delta_\gamma|_{\mathfrak{p}}^{-1/2},$$

where $C > 0$ and $\eta < 1$ depend only on $G(F)$.

Proof of Proposition 6.5. The first statement was proven in Proposition 5.6. Suppose $\Phi(\gamma, f_{\mathfrak{p}}) \neq 0$ for γ as in (6-16). We will compute each term of (6-7). It is not hard to check that $f_{\mathfrak{p}}(\gamma^y) = 0$ and $K_\gamma\left(\left(\begin{smallmatrix} y & \\ & \varpi^n \end{smallmatrix}\right)\right) = 0$, since the matrices involved do not intersect the support of $f_{\mathfrak{p}}$. Therefore

$$\Phi(\gamma, f_{\mathfrak{p}}) = 2\sum_{n=1}^{\infty} \frac{q^n}{q^2-1} \sum_{y \in (\mathcal{O}/\mathfrak{p})^*} K_\gamma\left(\begin{pmatrix} \varpi^n & \\ & y \end{pmatrix}\right). \tag{6-19}$$

Now fix $n \ge 1$ and $y \in \mathcal{O}^*$ and let $x = \left(\begin{smallmatrix} \varpi^n & \\ & y \end{smallmatrix}\right)$. As in (6-13), we have

$$K_\gamma(x) = \int_{\mathcal{O}} f_{\mathfrak{p}}\left(x^{-1}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} x\right) db.$$

By a quick calculation (see (6-14) with u, v in place of $u\varpi, v\varpi$),

$$x^{-1}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} x = \begin{pmatrix} b & -yP_\gamma(b)/\varpi^n \\ \varpi^n/y & v-b \end{pmatrix}. \tag{6-20}$$

Since the determinant is $u \in \mathcal{O}^*$, this belongs to the support of $f_{\mathfrak{p}}$ only if it belongs to $\mathcal{O}^* K'$. In particular, $b \in \mathcal{O}^*$ and $P_\gamma(b) \equiv 0 \mod \mathfrak{p}^n$. Therefore $b \in z + \mathfrak{p}$ for z as in (6-17). From (6-17) we see that $v \equiv 2z \mod \mathfrak{p}$ so $v - b \in z + \mathfrak{p}$ as well. Therefore, pulling out a factor of z from the above matrix,

$$K_\gamma\left(\begin{pmatrix} \varpi^n & \\ & y \end{pmatrix}\right) = \overline{\omega_{\mathfrak{p}}(z)} \int_{z+\mathfrak{p}} f_{\mathfrak{p}}\left(\begin{pmatrix} 1 & -yP_\gamma(b)/z\varpi^n \\ \varpi^n/yz & 1 \end{pmatrix}\right) db.$$

Writing $P_\gamma(b) \equiv c\varpi^n \mod \mathfrak{p}^{n+1}$ for some $c \in \mathcal{O}/\mathfrak{p}$, by (5-18) the integrand becomes

$$f_{\mathfrak{p}}\left(\begin{pmatrix} 1 & -yc/z \\ \varpi^n/yz & 1 \end{pmatrix}\right) = \tfrac{1}{2}(q^2-1)\overline{\psi\left(-\frac{yc}{z} + \frac{t\varpi^{n-1}}{yz}\right)}.$$

This depends (via c) only on the coset $b + \mathfrak{p}^{n+1}$ (in fact it depends only on $b + \mathfrak{p}^n$ but we will not use this). Each such coset has measure $q^{-(n+1)}$. Therefore if we let

$$\mathcal{N}_\gamma(c, n) = \#\{b \mod \mathfrak{p}^{n+1} \mid P_\gamma(b) \equiv c\varpi^n \mod \mathfrak{p}^{n+1}\}$$

for $c \in \mathcal{O}/\mathfrak{p}$, we find that

$$K_\gamma(x) = \overline{\omega_{\mathfrak{p}}(z)}\frac{q^2-1}{2q^{n+1}} \sum_{c \bmod \mathfrak{p}} \psi\left(\frac{yc}{z}\right)\psi\left(-\frac{t}{yz}\right)^{\delta(n=1)} \mathcal{N}_\gamma(c, n).$$

Inserting this into (6-19) gives the result. □

Example 6.6. For $M \in \mathbb{Z}_2^*$ and f_2 as in (5-18),

$$\Phi\left(\begin{pmatrix} & -M \\ 1 & \end{pmatrix}, f_2\right) = \begin{cases} 1 & \text{if } M \equiv 1 \bmod 4, \\ -3 & \text{if } M \equiv 3 \bmod 8, \\ 0 & \text{if } M \equiv 7 \bmod 8. \end{cases}$$

Proof. First, $\gamma = \left(\begin{smallmatrix} & -M \\ 1 & \end{smallmatrix}\right)$ is hyperbolic in $G(\mathbb{Q}_2)$ if and only if $-M$ is a square in $\mathbb{Q}_2$, which holds if and only if $M \equiv 7 \bmod 8$. In this case, $\Phi(\gamma, f_2) = 0$ by (3-4).

Assuming $M \not\equiv 7 \bmod 8$, we may apply Proposition 6.5. We need to determine

$$\mathcal{N}_\gamma(0, n) = \text{number of solutions to } x^2 \equiv -M \bmod 2^{n+1}$$

and

$$\mathcal{N}_\gamma(1, n) = \text{ number of solutions to } x^2 \equiv 2^n - M \bmod 2^{n+1}.$$

Given any odd integer D, the number of solutions to $x^2 \equiv D \bmod 2^j$ is

$$\begin{cases} 1, & j = 1, \\ 2, & j = 2,\ D \equiv 1 \bmod 4, \\ 0, & j = 2,\ D \equiv 3 \bmod 4, \\ 4, & j > 2,\ D \equiv 1 \bmod 8, \\ 0, & j > 2,\ D \not\equiv 1 \bmod 8 \end{cases}$$

[32, Theorem 87]. Therefore

$$\mathcal{N}_\gamma(0, n) = \begin{cases} 2 & \text{if } n = 1 \text{ and } M \equiv 3 \bmod 4, \\ 0 & \text{if } n = 1 \text{ and } M \equiv 1 \bmod 4, \\ 4 & \text{if } n \geq 2 \text{ and } M \equiv 7 \bmod 8, \\ 0 & \text{if } n \geq 2 \text{ and } M \not\equiv 7 \bmod 8, \end{cases}$$

$$\mathcal{N}_\gamma(1, n) = \begin{cases} 0 & \text{if } n = 1 \text{ and } M \equiv 3 \bmod 4, \\ 2 & \text{if } n = 1 \text{ and } M \equiv 1 \bmod 4, \\ 4 & \text{if } n = 2 \text{ and } M \equiv 3 \bmod 8, \\ 0 & \text{if } n = 2 \text{ and } M \not\equiv 3 \bmod 8, \\ 4 & \text{if } n \geq 3 \text{ and } M \equiv 7 \bmod 8, \\ 0 & \text{if } n \geq 3 \text{ and } M \not\equiv 7 \bmod 8. \end{cases}$$

By definition, $\psi_2(x) = (-1)^x$ for $x \in \mathbb{Z}$, and ω_2 is trivial on $1 + 2\mathbb{Z}_2 = \mathbb{Z}_2^*$. So by (6-18) and the above,

$$\begin{aligned} \Phi(\gamma, f_2) &= \tfrac{1}{2}[\mathcal{N}_\gamma(0, 1)\,\psi_2(0)\,\psi_2(1) + \mathcal{N}_\gamma(1, 1)\,\psi_2(1)^2] \\ &\qquad + \tfrac{1}{2}[\mathcal{N}_\gamma(0, 2)\,\psi_2(0) + \mathcal{N}_\gamma(1, 2)\,\psi_2(1)] \\ &= \begin{cases} \tfrac{1}{2}[0 + 2] + \tfrac{1}{2}[0 + 0] = 1 & \text{if } M \equiv 1 \bmod 4, \\ \tfrac{1}{2}[-2 + 0] + \tfrac{1}{2}[0 - 4] = -3 & \text{if } M \equiv 3 \bmod 8. \end{cases} \end{aligned}$$

$\square$

Example 6.7. For f_3 as in (5-18) and any $m \in \mathbb{Z}_3^*$,

$$\Phi\left(\begin{pmatrix} 0 & -m^2 \\ 1 & m \end{pmatrix}, f_3\right) = \omega_3(-m)\, t \cdot 2_{t=1},$$

where $2_{t=1}$ is a factor of 2 which is present only when $t=1$. Here, $t \in \{\pm 1\} = (\mathbb{Z}/3\mathbb{Z})^*$ is the parameter of the fixed simple supercuspidal representation σ_t^ζ of $G(\mathbb{Q}_3)$.

Remark. When $N = 3$, we have $\omega_3(-1) = \omega'(-1) = (-1)^k$, so

$$\omega_3(-m) = \begin{cases} (-1)^k & \text{if } m \in 1 + 3\mathbb{Z}_3, \\ 1 & \text{if } m \in -1 + 3\mathbb{Z}_3. \end{cases}$$

Proof. We will apply Proposition 6.5. First note that

$$P_\gamma(X) = X^2 - mX + m^2 \equiv (X+m)^2 \bmod 3,$$

so we can take $z = -m$ in (6-17). We need to find

$$\mathcal{N}_\gamma(c, n) = \#\{b \mid b^2 - mb + m^2 \equiv c3^n \bmod 3^{n+1}\}.$$

If $b \in \mathbb{Z}_3^*$ is a double root of P_γ modulo 3, then we may write $b = -m + 3d$, so

$$P_\gamma(b) = (-m+3d)^2 - m(-m+3d) + m^2 = 3m^2 + 9(d^2 - md) \in 3\mathbb{Z}_3^*.$$

Thus, $\mathrm{ord}_3(P_\gamma(b)) = 1$, which means that $\mathcal{N}_\gamma(c, n) = 0$ for all $n \geq 2$, and also $\mathcal{N}_\gamma(0, 1) = 0$. Some elementary calculations show that independently of m, $\mathcal{N}_\gamma(-1, 1) = 0$ and $\mathcal{N}_\gamma(1, 1) = 3$. In view of (6-18), this means

$$\begin{aligned}
\Phi(\gamma, f_3) &= \tfrac{1}{3}(\overline{\omega_3(-m)})\, \mathcal{N}_\gamma(1,1)\left(\psi_3\left(\frac{1}{-m}\right)\psi_3\left(\frac{-t}{-m}\right) + \psi_3\left(\frac{1}{m}\right)\psi_3\left(\frac{-t}{m}\right)\right) \\
&= \overline{\omega_3(-m)}\left(e\left(\frac{1-t}{-3m}\right) + e\left(\frac{1-t}{3m}\right)\right) \\
&= \omega_3(-m)\left[e\left(\frac{t-1}{3}\right) + e\left(\frac{1-t}{3}\right)\right].
\end{aligned}$$

When $t = 1$ (resp. $t = -1$), the expression in the brackets equals 2 (resp. -1). $\square$

6.4. ***The case where $p \mid S$.*** When $p \mid S$, the support of f_p is contained in $Z_p K_p$, so the orbital integral vanishes unless γ is unramified. We again work over a p-adic field F, with the usual notation, and fix a depth zero supercuspidal representation σ_ν of $G = G(F)$ for ν a primitive character of $\mathbb{F}_{q^2}^*$.

Proposition 6.8. *Let $f_\mathfrak{p}$ be the test function defined in (5-4), and let $\gamma = \left(\begin{smallmatrix} & -u \\ 1 & v \end{smallmatrix}\right)$ be an elliptic element of $G(F)$, where $u \in \mathcal{O}^*$ and $v \in \mathcal{O}$. If there exists $z \in (\mathcal{O}/\mathfrak{p})^*$ such that*

$$P_\gamma(X) \equiv (X - z)^2 \bmod \mathfrak{p},$$

then

$$\Phi(\gamma, f_{\mathfrak{p}}) = -\overline{\omega_{\mathfrak{p}}(z)} + \frac{\overline{\omega_{\mathfrak{p}}(z)}}{q} \sum_{n=1}^{\infty} \Bigg[(q-1)\,\mathcal{N}_\gamma(0,n) - \sum_{c \in (\mathcal{O}/\mathfrak{p})^*} \mathcal{N}_\gamma(c,n) \Bigg], \tag{6-21}$$

where $\mathcal{N}_\gamma(c,n) = \#\{b \bmod \mathfrak{p}^{n+1} \mid P_\gamma(b) \equiv c\varpi^n \bmod \mathfrak{p}^{n+1}\}$.

If $P_\gamma(X)$ *is irreducible modulo* $\mathfrak{p}$, *then*

$$\Phi(\gamma, f_{\mathfrak{p}}) = -\overline{\nu(\gamma)} - \overline{\nu^q(\gamma)}, \tag{6-22}$$

where we interpret the above to mean $-\overline{\nu(x)} - \overline{\nu^q(x)}$ *if* $x \in \mathbb{F}_{q^2}^*$ *has the same minimum polynomial over* $\mathbb{F}_q$ *as the reduction of* γ *mod* $\mathfrak{p}$, *i.e.*, γ *is conjugate to* $x \in \mathbb{T}$.

Remarks. (1) The remaining possibility where $P_\gamma(X)$ has two distinct roots mod $\mathfrak{p}$ cannot occur due to Hensel's Lemma, since we are assuming that γ is elliptic in $G(F)$.

(2) See the remarks after Proposition 6.5 regarding $\mathcal{N}_\gamma(c,n)$. In particular, the sum in (6-21) is finite, and when $F = \mathbb{Q}_p$ we find $|\Phi(\gamma, f_p)| \le 1 + 4|\Delta_\gamma|_p^{-1/2}$.

Proof. In this proof we write $\overline{G}$ for $\overline{G}(F)$, Z for $Z(F)$, and K for $G(\mathcal{O})$. By Lemma 6.1,

$$\begin{aligned}\Phi(\gamma, f_{\mathfrak{p}}) &= \int_{\overline{G}} f_{\mathfrak{p}}(g^{-1}\gamma g)\, dg \\ &= \sum_{x \in \overline{K}\backslash \overline{G}/\overline{K}} \mathrm{meas}_{\overline{G}}(\overline{K}x\overline{K}) \int_{\overline{K}} \int_{\overline{K}} f_{\mathfrak{p}}(h_2^{-1}x^{-1}h_1^{-1}\gamma h_1 x h_2)\, dh_1\, dh_2,\end{aligned}$$

with dh_1 and dh_2 each having total measure 1. The integrand is nonzero only if $x^{-1}h_1^{-1}\gamma h_1 x \in ZK$. Therefore, since $f_{\mathfrak{p}}$ is a trace, h_2 has no effect, so

$$\Phi(\gamma, f_{\mathfrak{p}}) = \sum_{x \in \overline{K}\backslash \overline{G}/\overline{K}} \mathrm{meas}_{\overline{G}}(\overline{K}x\overline{K}) \int_{\overline{K}} f_{\mathfrak{p}}(x^{-1}h\gamma h^{-1}x)\, dh.$$

(For convenience in what follows, we have set $h = h_1^{-1}$.)

By the Cartan decomposition of G, a set of representatives for $\overline{K}\backslash\overline{G}/\overline{K}$ is given by

$$\left\{ \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} \,\middle|\, n \ge 0 \right\}.$$

Arguing as in [36, Lemma 4.5.6(2)], for $x = \left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right)$,

$$|K\backslash KxK| = \begin{cases} q^{n-1}(q+1) & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

Therefore $\text{meas}_{\overline{G}}(\overline{K}x\overline{K}) = q^{n-1}(q+1)$ when $n > 0$, so

$$\Phi(\gamma, f_{\mathfrak{p}}) = f_{\mathfrak{p}}(\gamma) + \sum_{n=1}^{\infty} q^{n-1}(q+1)K_\gamma(n), \tag{6-23}$$

where

$$K_\gamma(n) = \int_K f_{\mathfrak{p}}\left(\begin{pmatrix} \varpi^{-n} & \\ & 1 \end{pmatrix} h\gamma h^{-1} \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix}\right) dh \quad (n > 0).$$

(We may integrate over K since both K and $\overline{K}$ have measure 1.) Write $h\gamma h^{-1} = \left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right) \in K$. Then

$$\begin{pmatrix} \varpi^{-n} & \\ & 1 \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} = \begin{pmatrix} w & x/\varpi^n \\ y\varpi^n & z \end{pmatrix}. \tag{6-24}$$

This belongs to the support of $f_{\mathfrak{p}}$ only if $x \in \mathfrak{p}^n$.

In the integrand above, we can freely multiply h by a diagonal element of K since such an element commutes with x and can be eliminated since $f_{\mathfrak{p}}$ is a trace. In particular, we can assume $\det h = 1$. Write $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\det h = 1$. Then

$$h\gamma h^{-1} = \begin{pmatrix} * & -b^2 + abv - a^2u \\ * & * \end{pmatrix}.$$

If $a \in \mathfrak{p}$ then we must have $b \in \mathcal{O}^*$ since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K$. But then the upper-right entry above cannot belong to $\mathfrak{p}^n$, so the integrand vanishes by (6-24). Therefore we may assume $a \in \mathcal{O}^*$, i.e., $h \in A$, where

$$A = \begin{pmatrix} \mathcal{O}^* & * \\ * & * \end{pmatrix} \cap K.$$

Let's find the measure of A. Let $K(\mathfrak{p}) = 1 + M_2(\mathfrak{p}) \subseteq A$. This is the kernel of the reduction mod $\mathfrak{p}$ map $K \to G(\mathcal{O}/\mathfrak{p})$. Since $|G(\mathcal{O}/\mathfrak{p})| = (q^2-1)(q^2-q)$, we see that $\text{meas}(K(\mathfrak{p})) = 1/((q^2-1)(q^2-q))$. Let $\overline{A} = A \bmod K(\mathfrak{p})$. Thinking of $\overline{A}$ as a set of matrices in $G(\mathcal{O}/\mathfrak{p})$, we see that

$$|\overline{A}| = (q-1)\,q(q^2-q).$$

(There are $(q-1)\,q$ possible top rows, and then q^2-q remaining choices for the bottom row.) Hence

$$\text{meas}(A) = \frac{(q-1)\,q(q^2-q)}{(q^2-1)(q^2-q)} = \frac{q}{q+1}.$$

It is not hard to show that

$$A = \begin{pmatrix} \mathcal{O}^* & \\ & \mathcal{O}^* \end{pmatrix}\begin{pmatrix} 1 & \\ \mathcal{O} & 1 \end{pmatrix}\begin{pmatrix} 1 & \mathcal{O} \\ & 1 \end{pmatrix},$$

and that the corresponding decomposition of any element of A is unique. Therefore by Lemma 6.3 we can use the above as a coordinate system for integration over A. Since, as noted above, the diagonal component has no effect on the value of the integral, we have

$$K_\gamma(n) = \frac{q}{q+1}\int_{\mathcal{O}}\int_{\mathcal{O}} f_{\mathfrak{p}}\left(x^{-1}\begin{pmatrix}1 & \\ c & 1\end{pmatrix}\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\gamma\begin{pmatrix}1 & -b\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ -c & 1\end{pmatrix}x\right) db\, dc,$$

where $x = \left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right)$, and db and dc each have total measure 1. The integral over c can be eliminated, since $\left(\begin{smallmatrix}1 & \\ -c & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \\ -\varpi^n c & 1\end{smallmatrix}\right)$, and the rightmost matrix belongs to K. Therefore

$$\begin{aligned} K_\gamma(n) &= \frac{q}{q+1}\int_{\mathcal{O}} f_{\mathfrak{p}}\left(\begin{pmatrix}\varpi^{-n} & \\ & 1\end{pmatrix}\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\begin{pmatrix} & -u\\ 1 & v\end{pmatrix}\begin{pmatrix}1 & -b\\ & 1\end{pmatrix}\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}\right) db \\ &= \frac{q}{q+1}\int_{\mathcal{O}^*} f_{\mathfrak{p}}\left(\begin{pmatrix}b & -P_\gamma(b)/\varpi^n\\ 0 & v-b\end{pmatrix}\right) db \end{aligned}$$

as in (6-20). (As a reminder, db is additive measure.) We have replaced the lower-left entry by 0, using the fact that by definition (see (5-4)), $f_{\mathfrak{p}}$ is sensitive only to the reduction of its argument mod $\mathfrak{p}$. Further, the integrand is nonzero only if $P_\gamma(b) \in \mathfrak{p}^n$. Under this condition, given that the characteristic polynomial of the matrix in the integrand is $P_\gamma(X)$, and this cannot have distinct roots mod $\mathfrak{p}$ as γ is elliptic in $G(\mathbb{Q}_p)$, there exists $z \in (\mathcal{O}/\mathfrak{p})^*$ such that $b \equiv v - b \equiv z \bmod \mathfrak{p}$. In particular, the matrix (viewed modulo $\mathfrak{p}$) belongs to ZU, with notation as in (5-2). Write $P_\gamma(b) \equiv c\varpi^n \bmod \mathfrak{p}^{n+1}$, for $c \in \mathcal{O}/\mathfrak{p}$. The integrand becomes

$$\overline{\omega_{\mathfrak{p}}(z)}\, f_{\mathfrak{p}}\left(\begin{pmatrix}1 & -c/z\\ 0 & 1\end{pmatrix}\right) = \begin{cases}\overline{\omega_{\mathfrak{p}}(z)}(q-1) & \text{if } c = 0,\\ -\overline{\omega_{\mathfrak{p}}(z)} & \text{if } c \in (\mathcal{O}/\mathfrak{p})^*.\end{cases}$$

This depends (via c) only on the coset $b + \mathfrak{p}^{n+1}$, which has measure $q^{-(n+1)}$. Therefore

$$K_\gamma(n) = \overline{\omega_{\mathfrak{p}}(z)} \cdot \frac{q}{q+1} \cdot \frac{1}{q^{n+1}}\left[(q-1)\,\mathcal{N}_\gamma(0,n) - \sum_{c \in (\mathcal{O}/\mathfrak{p})^*} \mathcal{N}_\gamma(c,n)\right].$$

Plugging the above into (6-23), equation (6-21) follows. (Note that $f_{\mathfrak{p}}(\gamma) = -\overline{\omega_{\mathfrak{p}}(z)}$ in this case, since $\gamma - z = \left(\begin{smallmatrix}-z & -u\\ 1 & v-z\end{smallmatrix}\right) \not\equiv 0 \bmod \mathfrak{p}$, so γ is conjugate mod $\mathfrak{p}$ to zu for some $1 \neq u \in U$).

By the above discussion $K_\gamma(n) = 0$ for all $n > 0$ if $P_\gamma(X)$ is irreducible mod $\mathfrak{p}$. So in this case (6-23) gives $\Phi(\gamma, f_{\mathfrak{p}}) = f_{\mathfrak{p}}(\gamma) = -\overline{\nu(\gamma)} - \overline{\nu^q(\gamma)}$ by (5-2) and (5-4). □

7. General dimension formula, and examples with $N = S^2T^3$

When $\mathtt{n} = 1$, the list of relevant γ in Theorem 4.2 can be simplified. The result is the following general dimension formula.

Theorem 7.1. *Let $N = \prod_{p|N} p^{N_p} > 1$ with $N_p \geq 2$ for all $p|N$. Fix $k > 2$ and a tuple $\hat\sigma = (\sigma_p)_{p|N}$ of supercuspidal representations so that $S_k(\hat\sigma) \subseteq S_k^{\mathrm{new}}(N, \omega')$ for a Dirichlet character ω', as detailed at the beginning of Section 4. Let T be the product of all primes $p|N$ with N_p odd. Let $f = f^1$ be the test function defined in (4-10) with $\mathtt{n} = 1$ but with f_p chosen as in (7-10) below for all $p|T$. Then*

$$\begin{aligned}\dim S_k(\hat\sigma) = \frac{1}{12}(k-1)\prod_{p|N} d_{\sigma_p} &+ \frac{1}{2}\Phi\left(\begin{pmatrix} & -T \\ 1 & \end{pmatrix}, f\right) \\ &+ \frac{1}{2}\delta_{T\in 2\mathbb{Z}^+}\,\Phi\left(\begin{pmatrix} & -T/2 \\ 1 & \end{pmatrix}, f\right) + \delta_{T=2}\,\Phi\left(\begin{pmatrix} 0 & -2 \\ 1 & 2\end{pmatrix}, f\right) \\ &+ \delta_{T=3}\,\Phi\left(\begin{pmatrix} 0 & -3 \\ 1 & 3\end{pmatrix}, f\right) + \delta_{T\in\{1,3\}}\,\Phi\left(\begin{pmatrix} 0 & -1 \\ 1 & 1\end{pmatrix}, f\right).\end{aligned}$$

Here, d_{σ_p} is the formal degree of σ_p relative to the Haar measure fixed in Section 2, and the orbital integrals $\Phi(\gamma, f)$ are given as in Theorem 4.2.

Proof. The case where $T = 1$ is already contained in Theorem 4.2 by taking $\mathtt{n} = 1$. The simplifications when $T > 1$ are proven in Proposition 7.9 below. □

As with Theorem 1.1, using the results of Section 6 we can compute the above explicitly in any case of interest when $N = S^2T^3$ with S and T square-free relatively prime positive integers. Although there is not a particularly nice formula for all such levels, as an illustration we will work everything out in the two special cases where $N = S^2$ and $N = T^3$. These results are stated in Sections 7.1 and 7.4 respectively. In Section 7.5 we give some examples to illustrate Theorem 1.1 with $\mathtt{n} > 1$.

First, we highlight the following consequence of Theorem 7.1.

Corollary 7.2. *In the setting of Theorem 7.1 above, suppose that the weight k is odd, so $\omega'(-1) = -1$. For T as in Theorem 7.1, if $T > 3$ the elliptic terms vanish, so*

$$\dim S_k(\hat\sigma) = \tfrac{1}{12}(k-1)\prod_{p|N} d_{\sigma_p} \quad (k > 2 \text{ odd},\ T > 3).$$

Remark. If $N = 2^2$ or $N = 2^3$, then $S_k(\hat\sigma)$ is undefined when k is odd since by Proposition 5.4 there is no appropriate nebentypus.

Proof. If $\gamma = \left(\begin{smallmatrix} & -T \\ 1 & \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} & -T/2 \\ 1 & \end{smallmatrix}\right)$, then $\Phi(\gamma, f_\infty) = 0$ when k is odd, by (4-13). □

7.1. ***Dimension formula and root number bias when $N = S^2$.*** When we set $T = 1$ and take $N = S^2$, the formula in Theorem 7.1 gives the following.

Theorem 7.3. *Let $N = S^2$ for S square-free, $k > 2$, ω' a Dirichlet character of modulus N and conductor dividing S, and let $\hat{\sigma} = (\sigma_p)_{p\mid S}$ be a tuple of depth zero supercuspidal representations chosen compatibly with ω' as in Section 5.3, with $T = 1$. Then the subspace $S_k(\hat{\sigma}) \subseteq S_k^{\mathrm{new}}(S^2, \omega')$ has dimension*

$$\dim S_k(\hat{\sigma}) = \tfrac{1}{12}(k-1)\prod_{p\mid N}(p-1) + A_1 + A_2,$$

where

$$A_1 = \tfrac{1}{4}(-1)^{S+1+(k/2)}\,\delta_{k\in 2\mathbb{Z}} \prod_{\text{odd } p\mid N} \big(-\overline{\nu_p(\alpha)} - \overline{\nu_p^p(\alpha)}\big)\,\delta_{p\equiv 3 \bmod 4}, \tag{7-1}$$

where ν_p is the primitive character of $\mathbb{F}_{p^2}^*$ defining σ_p and $\alpha \in \mathbb{F}_{p^2}^*$ satisfies $\alpha^2 = -1$, and

$$A_2 = \tfrac{1}{3}(\delta_{k\equiv 0,2 \bmod 3})\,(-1)^{\delta_{k\equiv 2,3 \bmod 6}}(-\omega_3(-1))^{\delta(3\mid N)} \prod_{p\mid N,\, p\neq 3} \big(-\overline{\nu_p(\beta)} - \overline{\nu_p^p(\beta)}\big)\,\delta_{p\equiv 2 \bmod 3}, \tag{7-2}$$

where $\beta \in \mathbb{F}_{p^2}^*$ satisfies $\beta^2 - \beta + 1 = 0$.

Remarks. (1) Note that $A_1 = A_2 = 0$ in each of the following situations: (i) $k \equiv 1 \bmod 6$, (ii) there exist primes $p, q \mid N$ (which could be equal) such that $p \equiv 1 \bmod 4$ and $q \equiv 1 \bmod 3$, (iii) k is odd and $p \equiv 1 \bmod 3$ for some $p \mid N$, (iv) $k \equiv 1 \bmod 3$ and $p \equiv 1 \bmod 4$ for some $p \mid N$.

(2) By summing the above formula over all tuples $\hat{\sigma}$, one obtains a formula for the dimension of the space $S_k^{\min}(S^2, \omega')$ of twist-minimal newforms. See Proposition 7.7.

(3) Theorem 1.3 from the introduction follows from the above by taking ω' trivial. We will prove this after first proving the above result.

Proof. Taking $T = 1$ in Theorem 7.1, we have

$$\dim S_k(\hat{\sigma}) = \frac{1}{12}(k-1)\prod_{p\mid N}(p-1) + \frac{1}{2}\Phi\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, f\right) + \Phi\left(\begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix}, f\right).$$

Consider $\gamma = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$. Its discriminant is $\Delta_\gamma = -4$, and we adopt the shorthand

$$\Phi(\gamma) = m\Phi_\infty\Phi_2 \prod_{\text{odd } p\mid N} \Phi_p$$

for (1-4), where $m = 2h(E)/(w(E)2^{\omega(d_E)})$ for $E = \mathbb{Q}[\gamma]$. We find that $m = \frac{1}{4}$ and $\Phi_\infty = (-1)^{k/2}\delta_{k\in 2\mathbb{Z}}$. If S is odd, then $\Phi_2 = 2$ by Example 4.10. If S is even, Φ_2 is given by (6-21). Here, $\mathcal{N}_\gamma(c, n) = 0$ for all $n \geq 2$, $\mathcal{N}_\gamma(0, 1) = 0$ and $\mathcal{N}_\gamma(1, 1) = 2$. So $\Phi_2 = -1 + \frac{1}{2}(-2) = -2$. Thus in both cases, $\Phi_2 = 2(-1)^{S+1}$. Finally, for odd $p \mid S$, γ is elliptic in $G(\mathbb{Q}_p)$ if and only if -1 is not a square in $\mathbb{Q}_p$, i.e., $p \equiv 3 \bmod 4$.

In such cases, $P_\gamma(X)$ is irreducible modulo p, so by (6-22), $\Phi_p = -\overline{\nu_p(\gamma)} - \overline{\nu_p^p(\gamma)}$. Multiplying everything together, we see that $\frac{1}{2}\Phi\big(\big(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\big), f\big)$ gives (7-1).

Now consider $\gamma = \big(\begin{smallmatrix} & -1 \\ 1 & 1 \end{smallmatrix}\big)$. Then $\Delta_\gamma = -3$, so

$$\Phi(\gamma) = m\Phi_\infty \Phi_3 \prod_{p\,|\,N,\, p\neq 3} \Phi_p.$$

We find that $m = \frac{1}{6}$, and

$$\Phi_\infty = -\frac{\sin((k-1)\,\pi/3)}{\sin(\pi/3)} = \begin{cases} 0 & \text{if } k \equiv 1 \bmod 3, \\ 1 & \text{if } k \equiv 0, 5 \bmod 6, \\ -1 & \text{if } k \equiv 2, 3 \bmod 6. \end{cases}$$

If $3 \nmid N$, then by Proposition 4.8, $\Phi_3 = 2$ since 3 is ramified in $\mathbb{Q}_3[\gamma] = \mathbb{Q}_3[\sqrt{-3}]$ and $\mathcal{O}_\gamma = \mathbb{Z}_3\big[\frac{1}{2}(1+\sqrt{-3})\big] = \mathbb{Z}_3[\sqrt{-3}]$ is the full ring of integers. If $3\,|\,N$, then Φ_3 is given by (6-21) with $z = -1$. We find that $\mathcal{N}_\gamma(c, n) = 0$ for all $n \geq 2$, $\mathcal{N}_\gamma(1, 1) = 3$, and $\mathcal{N}_\gamma(0, 1) = \mathcal{N}_\gamma(2, 1) = 0$. So

$$\Phi_3 = -\overline{\omega_3(-1)} + \tfrac{1}{3}\overline{\omega_3(-1)}(-3) = -2\omega_3(-1).$$

For $p\,|\,N$ with $p \neq 3$, $P_\gamma(X) = X^2 - X + 1$ is irreducible in $\mathbb{Q}_p$ if and only if -3 is not a square in $\mathbb{Q}_p$, or equivalently, $p \equiv 2 \bmod 3$ (see [23, Lemma 27.4]). For such p, Φ_p is given by (6-22). Multiplying these factors together gives (7-2), and the theorem follows. □

Now suppose ω' is trivial, so $k > 2$ is even. In this case we can simplify the expressions for A_1 and A_2 to obtain Theorem 1.3, as follows.

Proof of Theorem 1.3. Recall that by (5-6), when $p \equiv 3 \bmod 4$ and ω_p is trivial, $-\nu_p(\alpha) = -\nu_p^p(\alpha) = \epsilon_p$ is the root number of σ_p. Likewise, by (5-5), $(-1)^{S+1} = \epsilon_2$ when S is even (and 1 otherwise). So in this case, we simply have

$$A_1 = \tfrac{1}{4}(\epsilon(k, \hat{\sigma}))\, D_4(S) \prod_{\text{odd } p\,|\,S} 2, \tag{7-3}$$

where $\epsilon(k, \hat{\sigma}) = (-1)^{k/2} \prod_{p\,|\,S} \epsilon_p$ is the common global root number of the newforms in $S_k(\hat{\sigma})$, and $D_4(S) \in \{0, 1\}$ vanishes exactly when S is divisible by a prime $p \equiv 1 \bmod 4$.

Turning to (7-2), if $p \equiv 2 \bmod 3$, the polynomial $X^2 - X + 1$ is irreducible over $\mathbb{F}_p$. So $L = \mathbb{F}_{p^2}$ has a root $\beta \in L^* - \mathbb{F}_p^*$. Let t be a generator of the cyclic group L^*. The dual group of L^* is the set $\{\nu_m \mid 1 \leq m \leq p^2 - 1\}$, where $\nu_m = \nu_{p,m}$ is defined by

$$\nu_m(t) = \nu_{p,m}(t) = e\bigg(\frac{m}{p^2 - 1}\bigg).$$

Suppose p is odd. As shown in the proof of Corollary 5.2, the list of depth zero supercuspidal representations of $G(\mathbb{Q}_p)$ with trivial central character is

$$\{\sigma_{\nu_{p-1}}, \sigma_{\nu_{2(p-1)}}, \dots \sigma_{\nu_{\frac{p-1}{2}(p-1)}}\}.$$

So there exists $m = k(p-1)$ such that the primitive character ν_p of $\mathbb{F}_{p^2}^*$ fixed in Theorem 1.3 is given by

$$\nu_p = \nu_{p,m} = \nu_m.$$

Hopefully this conflict of notation ($\nu_p = \nu_m$) will cause no confusion, since m cannot equal p.

Noting that $\beta^3 = -1$, we can take $\beta = t^{(p^2-1)/6}$. Then for $m = k(p-1)$,

$$\nu_m(\beta) = e\left(\frac{k(p-1)(p^2-1)}{6(p^2-1)}\right) = e\left(\frac{k(p-1)}{6}\right) = \begin{cases} 1 & \text{if } 3\,|\,k, \\ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} & \text{otherwise.} \end{cases}$$

Therefore

$$B(\nu_p) := -\overline{\nu_p(\beta)} - \overline{\nu_p(\beta^p)} = -2\,\mathrm{Re}(\nu_m(\beta)) = \begin{cases} -2 & \text{if } 3\,|\,k, \\ 1 & \text{if } 3\nmid k. \end{cases} \tag{7-4}$$

When $p = 2$, there is only one supercuspidal, corresponding to $m = k = 1$, we can take $t = \beta$, and (7-4) holds as well. By (5-7), $3\,|\,k$ if and only if the order of ν_m divides $\frac{1}{3}(p+1)$. So the above coincides with $B(\nu_p)$ defined in Theorem 1.3, and the theorem follows from (7-3) and (7-4). □

Next we will use Theorem 1.3 to count the locally supercuspidal newforms of level S^2 with a given global root number. (What we will actually compute is the bias in global root number, but the count for each sign could be determined easily by following the proof of Proposition 7.6.)

To understand the impact of the local root numbers on the product of $B(\nu_p)$ in (1-11), the primes of interest are equivalent to 2 mod 3, so aside from $p = 2$, we have $p \equiv 5 \bmod 6$. It is helpful to look at two typical examples:

	ν	ν_{10}	ν_{20}	ν_{30}	ν_{40}	ν_{50}			
$p=11$	AL	$+$	$-$	$+$	$-$	$+$			
	$B(\nu)$	1	1	-2	1	1			
	ν	ν_{16}	$\nu_{2\cdot16}$	$\nu_{3\cdot16}$	$\nu_{4\cdot16}$	$\nu_{5\cdot16}$	$\nu_{6\cdot16}$	$\nu_{7\cdot16}$	$\nu_{8\cdot16}$
$p=17$	AL	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$
	$B(\nu)$	1	1	-2	1	1	-2	1	1

(7-5)

The Atkin–Lehner sign (AL) in the second row comes from (5-8), and the third row is from (7-4).

Lemma 7.4. *Given $S > 1$ square-free, let H_S^+ (resp. H_S^-) denote the set of tuples $\hat{\sigma} = (\sigma_p)_{p|S}$ satisfying*:

- *For each $p \mid S$, σ_p has trivial central character and conductor p^2.*
- $\prod_{p \mid S} \epsilon_p = 1$ *(resp.* -1*), where ϵ_p is the root number of σ_p.*

For ν_p the primitive character of $\mathbb{F}_{p^2}^*$ attached to σ_p, and $B(\nu_p)$ defined in (7-4), define

$$\mathcal{B}(S)^{\pm} = \sum_{\hat{\sigma} \in H_S^{\pm}} \prod_{p \mid S,\, p \neq 3} B(\nu_p).$$

Suppose $D_3(S) = 1$ (in the notation of Theorem 1.3), and let $\omega(S)$ denote the number of prime factors of S. Then if $\gcd(S, 6) = 1$,

$$\mathcal{B}(S)^{+} = \begin{cases} 2^{\omega(S)-1} & \text{if there exists } p \mid S \text{ with } p \equiv 5 \bmod 12, \\ 2^{\omega(S)} & \text{if } \omega(S) \text{ is even and } p \equiv 11 \bmod 12 \text{ for all } p \mid S, \\ 0 & \text{if } \omega(S) \text{ is odd and } p \equiv 11 \bmod 12 \text{ for all } p \mid S, \end{cases} \tag{7-6}$$

and $\mathcal{B}(S)^{-}$ is the same but with "even" and "odd" interchanged, i.e., $\mathcal{B}(S)^{-} = 2^{\omega(S)} - \mathcal{B}(S)^{+}$.

If S is odd and $3 \mid S$, then $\mathcal{B}(S)^{\pm} = \mathcal{B}\big(\frac{S}{3}\big)^{\pm}$ if $S > 3$, and $\mathcal{B}(3)^{+} = 1,\ \mathcal{B}(3)^{-} = 0$.

If S is even, then $\mathcal{B}(S)^{\pm} = \mathcal{B}\big(\frac{S}{2}\big)^{\mp}$ if $S > 2$, and $\mathcal{B}(2)^{+} = 0,\ \mathcal{B}(2)^{-} = 1$.

Proof. Suppose $\gcd(S, 6) = 1$. We prove (7-6) by induction on $\omega(S)$. For the base case, we take $S = p$ for a prime $p \equiv 5 \bmod 6$. As in (7-5), there are $\frac{1}{3}(p+1)$ representations with $B(\nu_p) = 1$, of which $\frac{1}{6}(p+1)$ have $\epsilon_p = 1$ and $\frac{1}{6}(p+1)$ have $\epsilon_p = -1$. There are $\frac{1}{6}(p-5)$ representations with $B(\nu_p) = -2$, of which $\big\lceil \frac{1}{12}(p-5) \big\rceil$ have $\epsilon_p = 1$, and $\big\lfloor \frac{1}{12}(p-5) \big\rfloor$ have $\epsilon_p = -1$. Therefore

$$\mathcal{B}(p)^{+} = \sum_{\sigma_p \in H_p^{+}} B(\nu_p) = \tfrac{1}{6}(p+1) - 2\big\lceil \tfrac{1}{12}(p-5) \big\rceil = \begin{cases} 1 & \text{if } p \equiv 5 \bmod 12, \\ 0 & \text{if } p \equiv 11 \bmod 12. \end{cases}$$

Likewise

$$\mathcal{B}(p)^{-} = \sum_{\sigma_p \in H_p^{-}} B(\nu_p) = \tfrac{1}{6}(p+1) - 2\big\lfloor \tfrac{1}{12}(p-5) \big\rfloor = \begin{cases} 1 & \text{if } p \equiv 5 \bmod 12, \\ 2 & \text{if } p \equiv 11 \bmod 12. \end{cases}$$

This proves the base case. Suppose (7-6) holds for some $S > 1$ with $\gcd(S, 6) = 1$, and $\ell \equiv 5 \bmod 6$ is a prime not dividing S. Then the result follows, by considering cases, from the fact that

$$\mathcal{B}(S\ell)^{+} = \mathcal{B}(S)^{+} \cdot \mathcal{B}(\ell)^{+} + \mathcal{B}(S)^{-} \cdot \mathcal{B}(\ell)^{-}$$

and

$$\mathcal{B}(S\ell)^{-} = \mathcal{B}(S)^{+} \cdot \mathcal{B}(\ell)^{-} + \mathcal{B}(S)^{-} \cdot \mathcal{B}(\ell)^{+}.$$

When $3 \mid S$, the claim follows from the fact that there is a unique depth zero supercuspidal representation of $\mathrm{PGL}_2(\mathbb{Q}_3)$, and it has root number $+1$ (see Corollary 5.2). When $2 \mid S$, the claim follows from the fact that there is a unique depth zero supercuspidal representation σ_ν of $\mathrm{PGL}(\mathbb{Q}_2)$, and it has $B(\nu) = 1$ and root number -1. $\square$

Lemma 7.5. *Let H_S^+ and H_S^- be defined as in Lemma 7.4 above. As in (1-11), define $D_4(S) \in \{0, 1\}$ to be 0 if and only if S is divisible by a prime $p \equiv 1 \bmod 4$. Then*

$$|H_S^{\pm}| = \begin{cases} \frac{1}{2}\prod_{odd\ p|S} \frac{1}{2}(p-1) & \text{if } D_4(S) = 0, \\ \frac{1}{2}\prod_{odd\ p|S} \frac{1}{2}(p-1) \pm \frac{1}{2}(-1)^{\delta(2|S)} & \text{if } D_4(S) = 1. \end{cases}$$

Proof. For each odd prime p, there are $\frac{1}{2}(p-1)$ depth zero supercuspidals with trivial central character (see Section 5.1). For $p = 2$, there is only one. Therefore for all square-free $S > 1$, the total number of tuples $\hat{\sigma} = (\sigma_p)_{p|S}$ with each σ_p having depth zero and trivial central character is

$$|H_S^+| + |H_S^-| = \prod_{\text{odd } p|S} \tfrac{1}{2}(p-1). \tag{7-7}$$

Now suppose S is divisible by a prime $p_0 \equiv 1 \bmod 4$. Fix $\epsilon_{\text{fin}} = \pm 1$. By the above, the number of tuples $(\sigma_p)_{p|(S/p_0)}$ is $\prod_{\text{odd } p|(S/p_0)} \frac{1}{2}(p-1)$. Having fixed one such tuple, by Corollary 5.2 there are then $\frac{1}{4}(p_0 - 1)$ choices for σ_{p_0} subject to $\prod_{p|S} \epsilon_{\sigma_p} = \epsilon_{\text{fin}}$. This proves the result when $D_4(S) = 0$.

Now suppose $p \equiv 3 \bmod 4$ for all odd $p|S$. For this case, in view of (7-7), the given formula is equivalent to $|H_S^+| - |H_S^-| = (-1)^{\delta(2|S)}$. We will prove the latter by induction on the number $\omega(S)$ of primes dividing S. If $S = 2$, the given formula holds since there is just one representation σ_2, and it has $\epsilon_{\sigma_2} = -1$. If $S = p \equiv 3 \bmod 4$, the given formula holds by Corollary 5.2. Having established the base case, suppose now that the given formula holds for some S satisfying $D_4(S) = 1$, and that $p_0 \nmid S$ is a prime satisfying $p_0 \equiv 3 \bmod 4$. We construct a tuple $\hat{\sigma} = (\sigma_p)_{p|Sp_0}$ by first choosing the components at $p|S$, and then at p_0. Let $P = |H_S^+|$ and $Q = |H_{p_0}^+|$, so $|H_S^-| = P - (-1)^{\delta(2|S)}$ and $|H_{p_0}^-| = Q - 1$ by the inductive hypothesis. Then

$$|H_{Sp_0}^+| = PQ + (P - (-1)^{\delta(2|S)})(Q-1) = 2PQ - P - (-1)^{\delta(2|S)}Q + (-1)^{\delta(2|S)},$$

and

$$|H_{Sp_0}^-| = P(Q-1) + (P - (-1)^{\delta(2|S)})Q = 2PQ - P - (-1)^{\delta(2|S)}Q.$$

Subtracting,

$$|H_{Sp_0}^+| - |H_{Sp_0}^-| = (-1)^{\delta(2|Sp_0)},$$

as needed. □

Proposition 7.6. *For $S > 1$ square-free, let*

$$\Delta(S^2, k)^{\min} = \dim S_k^{\min}(S^2)^+ - \dim S_k^{\min}(S^2)^-.$$

Then for $k > 2$ even,

$$\Delta(S^2, k)^{\min} = \Delta_M + \Delta_{A_1} + \Delta_{A_2}, \tag{7-8}$$

where

$$\Delta_M = D_4(S)(-1)^{k/2+\delta(2\,|\,S)}\tfrac{1}{12}(k-1)\prod_{p\,|\,S}(p-1), \quad \Delta_{A_1} = \tfrac{1}{4}(D_4(S))\prod_{p\,|\,S}(p-1)$$

for $D_4(S)$ *as in Lemma 7.5 above, and*

$$\Delta_{A_2} = \delta(k \equiv 0, 2 \bmod 6)\tfrac{1}{3}(D_3(S))(-1)^{\delta(k\equiv 6,8 \bmod 12)}\mu(S)\,\Omega_0(S'),$$

where $D_3(S) \in \{0,1\}$ *is* 0 *if and only if* $p \equiv 1 \bmod 3$ *for some* $p\,|\,S$, $\mu(S) = \prod_{p\,|\,S}(-1)$ *is the Möbius function, and for* $S' = S/\gcd(S,6)$,

$$\Omega_0(S') = \begin{cases} 0 & \text{if there exists } p\,|\,S' \text{ such that } p \equiv 5 \bmod 12, \\ 2^{\omega(S')} & \text{if } p \equiv 11 \bmod 12 \text{ for all } p\,|\,S', \end{cases}$$

where $\omega(S') = \sum_{p\,|\,S'} 1$. *(Note that* $\Omega_0(1) = 1$.*)*

Remark. Proposition 1.4, which summarizes the conditions under which $\Delta(S^2,k)^{\min}$ vanishes, is positive, or is negative, follows easily. The claim in the third paragraph of Proposition 1.4 is due to the fact that when $D_4(S) = 1$,

$$\begin{aligned} |\Delta_{A_1} + \Delta_{A_2}| &\le \frac{1}{4}\prod_{p\,|\,S}(p-1) + \frac{1}{3}\prod_{p\,|\,S'} 2 \\ &= \Big[\frac{1}{4} + \frac{1}{3}\prod_{p\,|\,S'}\frac{2}{p-1}\prod_{p\,|\,\gcd(S,6)}\frac{1}{p-1}\Big]\prod_{p\,|\,S}(p-1) \le \frac{7}{12}\prod_{p\,|\,S}(p-1), \end{aligned}$$

where the last inequality is strict if $S > 2$. So if $k \ge 10$, or $k = 8$ and $S > 2$, it follows that $|\Delta_{A_1} + \Delta_{A_2}| < |\Delta_M|$, and hence the sign of Δ_M is the sign of the bias. One checks by hand (or LMFDB) that $S_8^{\min}(2^2) = 0$. The case $k = 6$ follows similarly, replacing the rightmost inequality by $< \frac{5}{12}\prod_{p\,|\,S}(p-1)$ when $S > 6$ and $D_4(S) = 1$, and checking the $S\,|\,6$ cases by hand.

Proof of Proposition 7.6. We have

$$\Delta(S^2,k)^{\min} = \sum_{\hat\sigma:\epsilon(k,\hat\sigma)=1} \dim S_k(\hat\sigma) - \sum_{\hat\sigma:\epsilon(k,\hat\sigma)=-1} \dim S_k(\hat\sigma). \tag{7-9}$$

Applying Theorem 1.3 to each summand, we get a sum of three terms as in (7-8). Since the archimedean factor of the global root number is $(-1)^{k/2}$ (see [20, Theorem 14.17] and [9]), the set of tuples $\hat\sigma$ with global root number ϵ is $H_S^{(-1)^{k/2}\epsilon}$, with notation as in Lemma 7.4. Therefore the contribution of the main term is

$$\Delta_M = \tfrac{1}{12}(k-1)\prod_{p\,|\,S}(p-1)(|H_S^{(-1)^{k/2}}| - |H_S^{-(-1)^{k/2}}|),$$

and using Lemma 7.5 we obtain the formula given for Δ_M.

Likewise, the contribution of the A_1 term of Theorem 1.3 to (7-9) is

$$\begin{aligned}\Delta_{A_1} &= |H_S^{(-1)^{k/2}}|\tfrac{1}{4}D_4(S)\cdot 1 \prod_{\text{odd } p\mid S} 2 - |H_S^{-(-1)^{k/2}}|\tfrac{1}{4}D_4(S)\cdot(-1)\prod_{\text{odd } p\mid S} 2\\ &= \tfrac{1}{4}D_4(S)(|H_S^+|+|H_S^-|)\prod_{\text{odd } p\mid S} 2,\end{aligned}$$

and the given formula follows from (7-7).

In the notation of Theorem 1.3 and Lemma 7.4, the contribution of A_2 to (7-9) is

$$\begin{aligned}\Delta_{A_2} &= \tfrac{1}{3}(D_3(S)\,b(k)(-1)^{\delta(3\mid S)})(\mathcal{B}(S)^{(-1)^{k/2}} - \mathcal{B}(S)^{-(-1)^{k/2}})\\ &= \tfrac{1}{3}(D_3(S)\,b(k)(-1)^{\delta(3\mid S)+k/2})(\mathcal{B}(S)^{+} - \mathcal{B}(S)^{-}).\end{aligned}$$

By considering possibilities for $\gcd(6, S)$, it is easy to check using Lemma 7.4 that

$$\mathcal{B}(S)^{+} - \mathcal{B}(S)^{-} = (-1)^{\delta(2\mid S)}\mu(S')\,\Omega_0(S').$$

The result then follows from $(-1)^{\delta(2\mid S)+\delta(3\mid S)}\mu(S') = \mu(S)$ and the fact that

$$(-1)^{k/2}b(k) = \begin{cases} 1 & \text{if } k\equiv 0, 2 \bmod 12,\\ -1 & \text{if } k\equiv 6, 8 \bmod 12,\\ 0 & \text{if } k\equiv 4 \bmod 6.\end{cases}$$ □

By similar arguments, we obtain the dimension of the space of twist-minimal forms of level S^2.

Proposition 7.7. *For $S > 1$ square-free and $k > 2$ even,*

$$\begin{aligned}\dim S_k^{\min}(S^2) = \tfrac{1}{12}(k-1)\prod_{\text{odd } p\mid S}\tfrac{1}{2}(p-1)^2 + \tfrac{1}{4}(D_4(S))(-1)^{\delta(2\mid S)+k/2}\prod_{\text{odd } p\mid S} 2\\ + \tfrac{1}{3}(D_3(S)\,b(k))(-1)^{\delta(3\mid S)}\prod_{p\mid (S/\gcd(6,S))} 2\end{aligned}$$

for

$$b(k) = \begin{cases} 1 & \textit{if } 6\mid k,\\ -1 & \textit{if } k\equiv 2 \bmod 6,\\ 0 & \textit{otherwise}.\end{cases}$$

Remark. Although we have assumed $k > 2$, the above formula is valid when $k = 2$ as well. More generally, the dimension of $S_k^{\min}(N, \chi)$ has been computed by Child [8, Section 5.1].

Proof. We have

$$\dim S_k(S^2)^{\min} = d_M + d_{A_1} + d_{A_2},$$

where

$$d_M = \tfrac{1}{12}(k-1)\prod_{p\,|\,S}(p-1)(|H_S^{(-1)^{k/2}}| + |H_S^{-(-1)^{k/2}}|),$$
$$d_{A_1} = \tfrac{1}{4}(D_4(S))(-1)^{k/2}(|H_S^+| - |H_S^-|)\prod_{\text{odd } p\,|\,S} 2,$$

and

$$d_{A_2} = \tfrac{1}{3}(D_3(S)\,b(k)(-1)^{\delta(3|S)})(\mathcal{B}(S)^+ + \mathcal{B}(S)^-).$$

The result follows upon applying (7-7) to d_M, Lemma 7.5 to d_{A_1}, and the fact that $\mathcal{B}(S)^+ + \mathcal{B}(S)^- = 2^{\omega(S')} = \prod_{p\,|\,S'} 2$, for $S' = S/\gcd(6, S)$. □

7.2. *Simplification when* $\mathtt{n} = 1$ *and* $T > 1$. We return to the general setting of Theorem 4.2 with no constraint on the conductor exponents of the σ_p. Our aim here is to cull the list of matrices that appear in Theorem 4.2 when $\mathtt{n} = 1$ and $T > 1$. The result is Proposition 7.9, from which Theorem 7.1 follows.

Recall that for $p\,|\,T$, σ_p is a supercuspidal representation whose conductor is of the form p^n with $n \ge 3$ odd. It is well known (see, e.g., [6, Section A.3.8]) that there is a ramified quadratic extension $E/\mathbb{Q}_p$ with E^* embedded in $G(\mathbb{Q}_p)$ such that σ_p is compactly induced from a character χ of $J_n = E^* U^{(n-1)/2}$, where $U^r = 1 + \left(\begin{smallmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p \end{smallmatrix}\right)^r$ is an open compact subgroup of $G(\mathbb{Q}_p)$ and $\chi|_{F^*} = \omega_p$. In the notation of Section 5.2, U^1 coincides with K', J_3 with H', and in general $J_n \subseteq H'$. We use the local test function defined for $g \in G(\mathbb{Q}_p)$ by

$$f_p(g) = \begin{cases} d_{\sigma_p}\overline{\chi(g)} & \text{if } g \in J_n, \\ 0 & \text{otherwise,} \end{cases} \tag{7-10}$$

where d_{σ_p} is the formal degree (depending only on the conductor). This coincides with (5-18) when $n = 3$.

If $p\,|\,T$, the support of f_p is the disjoint union of its unramified and ramified elements:

$$\mathrm{Supp}(f_p) = J_n = (J_n \cap ZK') \cup (J_n \cap \pi_E ZK'), \tag{7-11}$$

where π_E is a prime element of E whose square is a prime element of $\mathbb{Q}_p$. We may decompose f_p as $f_p = f_u + f_r$, a sum of two functions supported on the unramified and ramified elements of J_n respectively. In the paper of Gross [17, p. 1240], discussed in Section 1.3, $n = 3$ and the local test function used is a multiple of f_u. The following is largely contained in [17, Proposition 5.1].

Proposition 7.8. *Let $f^1 = f^{\mathtt{n}}$ for $\mathtt{n} = 1$. Suppose γ is elliptic in $G(\mathbb{Q})$ and unramified at some prime $p\,|\,T$. Then either γ has p-torsion in $\overline{G}(\mathbb{Q})$ and $p \in \{2, 3\}$, or $\Phi(\gamma, f^1) = 0$. As a result, $\Phi(\gamma, f^1) = 0$ in each of the following situations*:

(1) *γ is unramified at some prime $p\,|\,T$ with $p > 3$.*

(2) *γ is unramified at $3\,|\,T$ and $T \neq 3$.*

Proof. Write $f = f^1$. Suppose $\Phi(\gamma, f) \neq 0$. By Proposition 4.3, γ is elliptic in $G(\mathbb{R})$ and $\det\gamma > 0$. Hence it belongs to a compact-mod-center subgroup U_∞ of $G(\mathbb{R})$ (U_∞ being some conjugate of $\mathbb{R}^* \cdot \mathrm{SO}(2)$). Likewise, at every finite place v, the support of f_v is a compact-mod-center subgroup J_v of G_v (here is where we use $\mathtt{n} = 1$), and γ belongs to some conjugate U_v of J_v. (In fact since $\gamma \in K_v$ a.e., we can take $U_v = K_v$ a.e.) Hence γ belongs to a compact-mod-center subgroup $\prod_v U_v$ of $G(\mathbb{A})$. Identifying γ with its image modulo the center, we have

$$\gamma \in \overline{G}(\mathbb{Q}) \cap \prod_v \overline{U}_v.$$

This is a *finite* group since $\overline{G}(\mathbb{Q})$ is discrete in $\overline{G}(\mathbb{A})$ [23, Section 7.11]. In particular, γ is a torsion element of $\overline{G}(\mathbb{Q})$, i.e., some power of γ lies in the center $Z(\mathbb{Q})$.

Since γ is unramified at $p \mid T$, some conjugate of γ belongs to the unramified part of the support of f_p, which is a subset of the pro-p group $\overline{K}'$. (Recall that K' is the pro-p-Sylow subgroup of the Iwahori subgroup of $G(\mathbb{Q}_p)$). It follows that the order of γ in $\overline{G}(\mathbb{Q})$ is a power of p. However, it is known that any torsion element of $\overline{G}(\mathbb{Q})$ has order 1, 2, 3, 4, or 6 [11, Lemma 1]. Since $\gamma \neq 1$, we conclude that $p \leq 3$. This proves (1).

The 3-torsion elements of $\overline{G}(\mathbb{Q})$ comprise a single conjugacy class containing $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ [11, Lemma 1]. Therefore, if $p = 3$, γ is conjugate in $G(\mathbb{Q})$ to a matrix of the form $\begin{pmatrix} 0 & -z \\ z & z \end{pmatrix}$ and is hence everywhere unramified. By the above, this means T is not divisible by any prime $p > 3$. It is also odd, because otherwise γ would somehow simultaneously have 3-torsion and 2-power torsion. Hence $T = 3$, which proves (2).

By the same reference, the 4-torsion elements of $\overline{G}(\mathbb{Q})$ are all conjugate to $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. But such an element is ramified at 2. Hence γ has 2-torsion if $p = 2$. □

Proposition 7.9. *With notation as in Section 4.1, let T be the product of the primes p for which $\mathrm{ord}_p(N)$ is odd, and for $p \mid T$ take f_p as in (7-10). Then for $\gamma \in \overline{G}(\mathbb{Q})$, $\Phi(\gamma, f^1) = 0$ unless either $\gamma = 1$ or the conjugacy class of γ has a representative in $G(\mathbb{Q})$ of one of the forms given in the table below:*

form of T	*list of relevant elliptic γ for $\mathtt{n} = 1$*
even $T \neq 2$	$\begin{pmatrix} & -T \\ 1 & \end{pmatrix}, \begin{pmatrix} & -T/2 \\ 1 & \end{pmatrix}$
$T = 2$	$\begin{pmatrix} & -2 \\ 1 & \end{pmatrix}, \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}$
odd $T > 3$	$\begin{pmatrix} & -T \\ 1 & \end{pmatrix}$
$T = 3$	$\begin{pmatrix} & -3 \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & -3 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$
$T = 1$	$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

Remark. When $T/2 \equiv 7 \bmod 8$, the matrix $\begin{pmatrix} & -T/2 \\ 1 & \end{pmatrix}$ is hyperbolic (rather than elliptic) in $G(\mathbb{Q}_2)$, so its orbital integral vanishes. All other entries in the above table are elliptic in $G(\mathbb{Q}_p)$ for each $p\,|\,T$, but for $p\,|\,S$ this needs to be checked on a case-by-case basis.

Proof. The case where $T=1$ is already contained in Theorem 4.2, taking $\mathtt{n}=1$. So suppose $T>1$ and $\Phi(\gamma, f)\neq 0$. By Proposition 4.13, we may take $\gamma=\left(\begin{smallmatrix}0 & -M\\ 1 & rM\end{smallmatrix}\right)$ for some $M\,|\,T$ and $0\le r<\sqrt{4/M}$. Notice that if $M>3$ then $r=0$. Suppose first that $T\neq 3$. By Proposition 7.8, γ must be ramified at all odd primes dividing T, so $M=T$ or $M=T/2$. If T is odd, this means $M=T$ and we obtain the third row of the above table. Suppose T is even and $M=T/2$. By Proposition 7.8, γ has 2-torsion in $\overline{G}(\mathbb{Q})$. Note that

$$\gamma^2=\begin{pmatrix}-M & -rM^2\\ rM & r^2M^2-M\end{pmatrix}$$

is a scalar matrix if and only if $r=0$. Therefore $\gamma=\left(\begin{smallmatrix}0 & -M\\ 1 & 0\end{smallmatrix}\right)$. This establishes the top two rows of the table. (When $M=T=2$, $r=1$ is admissible, and for $\gamma=\left(\begin{smallmatrix}0 & -2\\ 1 & 2\end{smallmatrix}\right)$, $P_\gamma(X)=X^2-2X+2$ is an Eisenstein polynomial for the prime 2, which is indeed irreducible in $\mathbb{Q}_2[X]$ [50, p. 19].)

Now suppose $T=3$. Then $M=1$ or $M=3$. In the latter case, $\gamma=\left(\begin{smallmatrix}0 & -3\\ 1 & 3r\end{smallmatrix}\right)$ for $r=0,1$. If $M=1$, then $\gamma=\left(\begin{smallmatrix}0 & -1\\ 1 & r\end{smallmatrix}\right)$ for $r=0,1$, and γ is unramified at 3. If $r=0$, this matrix has 2-torsion, in violation of Proposition 7.8. Hence $\gamma=\left(\begin{smallmatrix}0 & -1\\ 1 & 1\end{smallmatrix}\right)$. (In this case, $P_\gamma(X)=X^2-X+1$ has discriminant -3, which is not a square in $\mathbb{Q}_3$, and hence γ is indeed elliptic in $G(\mathbb{Q}_3)$.) $\square$

7.3. ***Global orbital integrals for*** $\mathtt{n}=\mathbf{1}$, $\boldsymbol{N=T^3}$. Here we will evaluate the global elliptic orbital integrals of Theorem 7.1 explicitly when $N=T^3>1$ for T square-free. We must consider

$$\gamma=\begin{pmatrix} & -T\\ 1 & \end{pmatrix}, \begin{pmatrix} & -T/2\\ 1 & \end{pmatrix}_{(T\text{ even})}, \begin{pmatrix}0 & -2\\ 1 & 2\end{pmatrix}_{(T=2)}, \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}_{(T=3)}, \begin{pmatrix}0 & -3\\ 1 & 3\end{pmatrix}_{(T=3)}$$

as appearing in Proposition 7.9.

We introduce some notation before stating the global results. Given our tuple $\hat\sigma=(\sigma_{t_p}^{\zeta_p})_{p|T}$ of simple supercuspidal representations, for $k>2$ define

$$\epsilon(k,\hat\sigma)=i^k\prod_{p|N}\zeta_p. \tag{7-12}$$

This is the common global root number of the cusp forms comprising $H_k(\hat\sigma)$ (see Proposition 5.3 and [9; 20, Theorem 14.17]). Throughout this section $f=f^1$ as in (5-21).

Proposition 7.10. *For $N=T^3$, with notation as above, suppose that for each odd prime factor p of the square-free integer $T>1$, $-pt_p/T$ is a square modulo p.*

Then for $k \geq 4$ even,

$$\Phi\left(\begin{pmatrix} & -T \\ 1 & \end{pmatrix}, f\right) = \frac{\overline{\epsilon(k, \hat{\sigma})}\, 2_7\, 4_3\, h(-T)}{3_{T=3}\, 2^{\omega(T)}} \sum_y \omega'(y),$$

where numbers with subscripts are present only when T falls into the subscript's equivalence class modulo 8, $3_{T=3}$ is a factor of 3 which is present only when $T = 3$, and y ranges over all integers modulo T that satisfy $y^2 \equiv -pt_p/T \bmod p$ for all $p \mid T$. If the central character is trivial, the above simplifies to

$$\Phi\left(\begin{pmatrix} & -T \\ 1 & \end{pmatrix}, f\right) = \frac{\epsilon(k, \hat{\sigma})\, h(-T)\, w_T}{3_{T=3}}, \tag{7-13}$$

where

$$w_T = \begin{cases} \frac{1}{2} & \text{if } T \text{ is even,} \\ 1 & \text{if } T \equiv 1 \bmod 4, \\ 2 & \text{if } T \equiv 7 \bmod 8, \\ 4 & \text{if } T \equiv 3 \bmod 8. \end{cases}$$

Remark. If the first hypothesis is not satisfied or k is odd, then $\Phi(\gamma, f) = 0$; see Proposition 5.6.

Proof. Take $\gamma = \begin{pmatrix} & -T \\ 1 & \end{pmatrix}$, $\Delta_\gamma = -4T$, and let M be the odd part of T, so that $T = 2^a M$ for some $a \in \{0, 1\}$. Corresponding to (1-4), write

$$\Phi(\gamma, f) = m\Phi_\infty \Phi_2 \prod_{p \mid M} \Phi_p = m(-1)^{k/2} \Phi_2 \prod_{p \mid M} \overline{\zeta_p} \sum_{y_p} \omega_p(y_p),$$

where we have applied (4-13) and Proposition 6.4, with y_p running over the two (since p is odd) solutions to $y_p^2 \equiv -pt_p/T \bmod p$. We can exchange the sum and product. To each of the $2^{\omega(M)}$ tuples $(y_p)_{p \mid M}$, the Chinese remainder theorem assigns a unique integer y modulo T satisfying $y \equiv y_p \bmod p$ for all $p \mid T$, where we take $y_2 = 1$ if T is even. Further,

$$\omega'(y) = \prod_{p \mid T} \omega_p(y) = \prod_{p \mid T} \omega_p(y_p) = \prod_{p \mid M} \omega_p(y_p).$$

The first equality holds because $\gcd(y, T) = 1$ (see [23, (12.4)]); the second holds since each ω_p is trivial on $1 + p\mathbb{Z}_p$. By Example 4.10 (for T odd) or Proposition 6.4 (for T even),

$$\Phi_2 = \begin{cases} \overline{\zeta_2} & \text{if } T \text{ is even,} \\ 2 & \text{if } T \equiv 1, 5, 7 \bmod 8, \\ 4 & \text{if } T \equiv 3 \bmod 8. \end{cases}$$

It follows that

$$\Phi(\gamma, f) = \frac{2h(E)}{w_E 2^{\omega(d_E)}} \overline{\epsilon(k, \hat{\sigma})}\, a_T \sum_y \omega'(y)$$

for y as in the statement of the proposition,

$$a_T = \begin{cases} 1 & \text{if } T \text{ is even,} \\ 2 & \text{if } T \equiv 1, 5, 7 \bmod 8, \\ 4 & \text{if } T \equiv 3 \bmod 8, \end{cases}$$

and $E = \mathbb{Q}(\sqrt{-T})$. Since $T > 1$, we know that

$$w_E = |\mathcal{O}_E^*| = \begin{cases} 6 & \text{if } T = 3, \\ 2 & \text{otherwise.} \end{cases}$$

So $\frac{1}{2} w_E = 3_{T=3}$ and $2h(E)/w_E = h(-T)/3_{T=3}$. Recall that

$$d_E = \begin{cases} -4T, & -T \equiv 2, 3 \bmod 4, \\ -T, & -T \equiv 1 \bmod 4. \end{cases}$$

Therefore, placing the congruence condition on T rather than $-T$,

$$2^{\omega(d_E)} = \begin{cases} 2 \cdot 2^{\omega(T)} & \text{if } T \equiv 1 \bmod 4, \\ 2^{\omega(T)} & \text{if } T \equiv 2, 3 \bmod 4. \end{cases}$$

Hence using the definition of a_T in the following numerator,

$$\Phi(\gamma, f) = \overline{\epsilon(k, \hat{\sigma})}\, h(-T) \frac{2_{1,5,7} \cdot 4_3}{3_{T=3} \cdot 2_{1,5} \cdot 2^{\omega(T)}} \sum_y \omega'(y),$$

where numbers with subscripts are only present when T falls into one of the subscript equivalence classes modulo 8. The general result now follows.

If ω' is trivial, the sum over y equals the number of terms, namely $2^{\omega(M)}$. Then (7-13) follows from

$$\frac{2^{\omega(M)}}{2^{\omega(T)}} = \begin{cases} 1 & \text{if } T \text{ is odd,} \\ \frac{1}{2} & \text{if } T \text{ is even} \end{cases}$$

and the fact that $\epsilon(k, \hat{\sigma}) \in \{\pm 1\}$ is real in this case. □

Proposition 7.11. *For $N = T^3$, suppose that the square-free integer $T = 2M$ is even, and that for each prime factor p of T, $-pt_p/M$ is a square modulo p. Then for even $k \geq 4$,*

$$\Phi\left(\begin{pmatrix} & -M \\ 1 & \end{pmatrix}, f\right) = h(-M) \frac{\overline{\epsilon(k, \hat{\sigma})}}{\zeta_2} \cdot \frac{z_M}{2_{M=1}\, 3_{M=3}\, 2^{\omega(M)}} \sum_y \omega'(y),$$

where $2_{M=1}$ is a factor of 2 which is present only when $M = 1$, $3_{M=3}$ is defined similarly,

$$z_M = \begin{cases} \frac{1}{2} & \textit{if } M \equiv 1 \bmod 4, \\ -3 & \textit{if } M \equiv 3 \bmod 8, \\ 0 & \textit{if } M \equiv 7 \bmod 8, \end{cases}$$

and y ranges over all elements modulo M that satisfy $y^2 \equiv -pt_p/M \bmod p$ for each $p\,|\,M$. If ω' is trivial, the sum over y simply cancels with the factor of $2^{\omega(M)}$. (Again, if the condition on the t_p fails to hold or k is odd, the orbital integral vanishes.)

Proof. We use the same proof as for the previous proposition, with minor modifications. First, by Example 6.6,

$$\Phi\left(\begin{pmatrix} & -M \\ 1 & \end{pmatrix}, f_2\right) = \begin{cases} 1 & \text{if } M \equiv 1 \bmod 4, \\ -3 & \text{if } M \equiv 3 \bmod 8, \\ 0 & \text{if } M \equiv 7 \bmod 8. \end{cases}$$

Taking $E = \mathbb{Q}[\sqrt{-M}]$ we have

$$2^{\omega(d_E)} = \begin{cases} 2 \cdot 2^{\omega(M)} & \text{if } M \equiv 1 \bmod 4, \\ 2^{\omega(M)} & \text{if } M \equiv 3 \bmod 4 \end{cases}$$

as in the previous proof, and $2h(E)/w_E = h(-M)/(3_{M=3}\, 2_{M=1})$ since $\mathbb{Q}[\sqrt{-1}]$ has unit group of order 4 when $M = 1$. Hence (assuming $M \not\equiv 7 \bmod 8$)

$$\Phi\left(\begin{pmatrix} & -M \\ 1 & \end{pmatrix}, f\right) = \frac{h(-M)(-3)_3}{3_{M=3}\, 2_{M=1}\, 2_{1,5}\, 2^{\omega(M)}} \frac{\overline{\epsilon(k,\hat{\sigma})}}{\zeta_2} \sum_y \omega'(y),$$

where numerical subscripts refer to the congruence class of M modulo 8. □

Proposition 7.12. *Suppose $N = 2^3$, $\zeta \in \{\pm 1\}$ and $\sigma = \sigma^\zeta$ is our fixed simple supercuspidal representation of $G(\mathbb{Q}_2)$ (the parameter t must equal 1 when $p = 2$). Then*

$$\Phi\left(\begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}, f\right) = \tfrac{1}{4}\epsilon(k,\sigma)\, g_8(k),$$

where $g_8(k) = -1$ if $k \equiv 0, 2 \bmod 8$, and $g_8(k) = 1$ if $k \equiv 4, 6 \bmod 8$.

Remark. In view of Proposition 5.4, we assume that k is even.

Proof. Given that γ has characteristic polynomial $X^2 - 2X + 2$ with discriminant $\Delta_\gamma = -4$, we find $E = \mathbb{Q}[\gamma] = \mathbb{Q}[i]$. Hence $h(E) = 1$, $w_E = |\mathcal{O}_E^*| = 4$, and $d_E = -4$. By (1-4),

$$\Phi(\gamma, f) = m\Phi_\infty \Phi_2 = \tfrac{1}{4}\Phi_\infty\Phi_2.$$

Applying Proposition 6.4 with $p = 2$ and $v = 1$, we have $\Phi_2 = -\zeta$. So

$$\Phi(\gamma, f) = -\tfrac{1}{4}\Phi_\infty\, \zeta. \tag{7-14}$$

The complex eigenvalues of γ are $1 \pm i$, so we apply (4-12) with $\theta = \frac{\pi}{4}$ to get

$$\Phi_\infty = -\sqrt{2}\sin\left(\tfrac{1}{4}(k-1)\,\pi\right) = \begin{cases} 1 & \text{if } k \equiv 0, 6 \bmod 8, \\ -1 & \text{if } k \equiv 2, 4 \bmod 8. \end{cases}$$

Multiplying this by -1 as in (7-14) yields $(-1)^{k/2} g_8(k)$ with g_8 as given. □

Proposition 7.13. *Suppose $T = 3$ so $N = 3^3$, and let $\sigma = \sigma_t^\zeta$ be our fixed simple supercuspidal representation of $G(\mathbb{Q}_3)$, for $t = \pm 1$. Then*

$$\Phi\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, f\right) = \frac{t}{2_{t=-1}} c_3(k),$$

where $c_3(k) = \frac{1}{3} + \left\lfloor \frac{k}{3} \right\rfloor - \frac{k}{3}$.

Proof. Let $E = \mathbb{Q}[\gamma] = \mathbb{Q}[\sqrt{-3}]$. Then $h(E) = 1$, $d_E = -3$, and $w_E = |\mathcal{O}_E^*| = 6$. By (1-4) and taking $m = 1$ in Example 6.7 and its remark,

$$\Phi(\gamma, f) = \Phi_\infty \cdot \tfrac{1}{6}((-1)^k t \cdot 2_{t=1}) = \tfrac{1}{3}((-1)^k \Phi_\infty) \frac{t}{2_{t=-1}}.$$

By (4-12), we find that

$$(-1)^k \Phi(\gamma, f_\infty) = (-1)^{k+1} \frac{\sin\left(\frac{(k-1)\pi}{3}\right)}{\sin(\pi/3)} = \begin{cases} 1 & \text{if } k \equiv 0 \bmod 3, \\ 0 & \text{if } k \equiv 1 \bmod 3, \\ -1 & \text{if } k \equiv 2 \bmod 3. \end{cases} \tag{7-15}$$

Using the above, we see that $\frac{1}{3}((-1)^k \Phi(\gamma, f_\infty)) = \frac{1}{3} + \left\lfloor \frac{k}{3} \right\rfloor - \frac{k}{3}$. □

Proposition 7.14. *Suppose $N = 3^3$, and let $\sigma = \sigma_t^\zeta$ be a fixed simple supercuspidal representation of $G(\mathbb{Q}_3)$. Then*

$$\Phi\left(\begin{pmatrix} 0 & -3 \\ 1 & 3 \end{pmatrix}, f\right) = \begin{cases} 0 & \textit{if } t = 1, \\ \epsilon(k, \sigma)\, g_6(k) & \textit{if } t = -1, \end{cases}$$

where

$$g_6(k) = \begin{cases} 0 & \textit{if } k \equiv 1 \bmod 6, \\ -\frac{1}{6} & \textit{if } k \equiv 0, 2 \bmod 6, \\ \frac{1}{2} & \textit{if } k \equiv 3 \bmod 6, \\ \frac{1}{3} & \textit{if } k \equiv 4 \bmod 6, \\ -\frac{1}{2} & \textit{if } k \equiv 5 \bmod 6. \end{cases}$$

Proof. Let $\gamma = \left(\begin{smallmatrix} 0 & -3 \\ 1 & 3 \end{smallmatrix}\right)$, so $\Delta_\gamma = -3$. We have $E = \mathbb{Q}[\gamma] = \mathbb{Q}[\sqrt{-3}]$, so the measure factor is $\frac{1}{6}$ as in the previous proof. Therefore as in (1-4), we may write

$$\Phi(\gamma, f) = \tfrac{1}{6} \Phi_\infty \Phi_3. \tag{7-16}$$

By Proposition 6.4, $\Phi_3 = 0$ unless $-t$ is a square modulo 3, i.e., unless $t = -1$. Assuming this holds, we have

$$\begin{aligned} \Phi_3 &= \bar{\zeta} \cdot \left(\overline{\psi(1)}\, \omega_3(1) + \overline{\psi(-1)}\, \omega_3(-1)\right) \\ &= \bar{\zeta} \cdot (e^{-2\pi i/3} + (-1)^k e^{2\pi i/3}) = -\bar{\zeta}\, [i\sqrt{3}]_{k \text{ odd}}, \end{aligned}$$

where the factor of $i\sqrt{3}$ is present only when k is odd.

By (4-4) with $N=3$, $\omega_3(3)=1$. So $\zeta^2=\omega_3(t)=\omega_3(-1)=(-1)^k$, so $\zeta=\pm(i^k)$. In particular, the global root number $\varepsilon(\sigma,\zeta)=i^k\zeta$ is real and $\bar{\zeta}=(-1)^k\zeta$.

The complex roots of $P_\gamma(X)=X^2-3X+3$ are $\frac{1}{2}(3\pm i\sqrt{3})=\sqrt{3}\big(\frac{1}{2}(\sqrt{3}\pm i)\big)$, so in (4-12) we can take $\theta=\frac{\pi}{6}$ and $\Phi(\gamma,f_\infty)=-2\sin\big(\frac{1}{6}(k-1)\pi\big)$. Hence (7-16) becomes

$$\Phi(\gamma,f)=\tfrac{1}{3}(-1)^k\zeta\sin\big(\tfrac{1}{6}(k-1)\pi\big)[i\sqrt{3}]_{k\text{ odd}}=\begin{cases}\zeta/3 & \text{if } k\equiv 4 \bmod 12,\\ -i\zeta/2 & \text{if } k\equiv 3,5 \bmod 12,\\ \zeta/6 & \text{if } k\equiv 2,6 \bmod 12,\\ 0 & \text{if } k\equiv 1,7 \bmod 12,\\ -\zeta/6 & \text{if } k\equiv 0,8 \bmod 12,\\ i\zeta/2 & \text{if } k\equiv 9,11 \bmod 12,\\ -\zeta/3 & \text{if } k\equiv 10 \bmod 12.\end{cases}$$

Upon factoring out $\epsilon(k,\sigma)=i^k\zeta$, we obtain $g_6(k)$ as given. □

7.4. *Dimension formulas when $N=T^3$.* Here we put everything together to compute $|H_k(\hat{\sigma})|=\dim S_k(\hat{\sigma})$ for $\hat{\sigma}=(\sigma_p)_{p\mid N}$ a tuple of simple supercuspidal representations of $G(\mathbb{Q}_p)$ as in Theorem 7.1 with $S=1$.

We begin with the case $N=2^3$, where the central character is necessarily trivial due to (4-2) and Proposition 5.4.

Theorem 7.15. *Let $N=2^3$, fix $\zeta\in\{\pm1\}$, and let $\sigma=\sigma_\zeta$ be the associated simple supercuspidal representation of $G(\mathbb{Q}_2)$ with trivial central character. Then*

$$|H_k(\sigma)|=\begin{cases}0 & \textit{if } k \textit{ is odd},\\ \lfloor\frac{k}{8}\rfloor & \textit{if } k\equiv 0,2 \bmod 8,\\ \lfloor\frac{k}{8}\rfloor+\frac{1}{2}(1+\epsilon(k,\sigma)) & \textit{if } k\equiv 4,6 \bmod 8,\end{cases}$$

where $\epsilon(k,\sigma)=(-1)^{k/2}\zeta$ is the global root number.

Proof. When k is odd, the assertion follows from Proposition 5.4. Suppose k is even. By Theorem 7.1,

$$|H_k(\sigma)|=\tfrac{1}{12}(k-1)\cdot\tfrac{3}{2}+\tfrac{1}{2}\Phi\left(\begin{pmatrix} & -2\\ 1 & \end{pmatrix},f\right)+\tfrac{1}{2}\Phi\left(\begin{pmatrix} & -1\\ 1 & \end{pmatrix},f\right)+\Phi\left(\begin{pmatrix}0 & -2\\ 1 & 2\end{pmatrix},f\right).$$

Applying the results of Section 7.3 using $h(-2)=h(-1)=1$, we find

$$|H_k(\sigma)|=\tfrac{1}{8}(k-1)+\tfrac{1}{4}((-1)^{k/2}\zeta)+\tfrac{1}{8}((-1)^{k/2})+\tfrac{1}{4}((-1)^{k/2}\zeta)\,g_8(k)$$

for

$$g_8(k)=\begin{cases}-1 & \text{if } k\equiv 0,2 \bmod 8,\\ 1 & \text{if } k\equiv 4,6 \bmod 8.\end{cases}$$

The result follows upon simplifying each of the cases. □

Theorem 7.16. *Let $N = 3^3$, fix $t \in \{\pm 1\}$, a character ω_3 of $\mathbb{Q}_3^*$ trivial on $1 + 3\mathbb{Z}_3$, $\zeta \in \mathbb{C}$ with $\zeta^2 = \omega_3(t)$* (see (4-2)), *and let $\sigma = \sigma_t^\zeta$ be the associated simple supercuspidal representation of $G(\mathbb{Q}_3)$ with central character ω_3. Then for $k > 2$, setting $\epsilon = i^k\zeta$, we have*

$$|H_k(\sigma)| = \begin{cases} \lfloor \frac{k}{3} \rfloor + \frac{1}{2}(\epsilon - 1) & \text{if } k \equiv 0 \bmod 3 \text{ and } t = -1, \\ \lfloor \frac{k}{3} \rfloor & \text{if } k \equiv 1 \bmod 6 \text{ or } t = 1, \\ \lfloor \frac{k}{3} \rfloor + \frac{1}{2}(\epsilon + 1) & \text{if } k \equiv 2 \bmod 6 \text{ and } t = -1, \\ \lfloor \frac{k}{3} \rfloor + \epsilon & \text{if } k \equiv 4 \bmod 6 \text{ and } t = -1, \\ \lfloor \frac{k}{3} \rfloor + \frac{1}{2}(1 - \epsilon) & \text{if } k \equiv 5 \bmod 6 \text{ and } t = -1. \end{cases}$$

Remarks. (1) If $t = -1$, then $\zeta^2 = \omega_3(-1) = (-1)^k$, so $\zeta = \pm i^k$, as noted earlier. Therefore $\epsilon \in \{\pm 1\}$ when $t = -1$. When $t = 1$ and k is odd, $\epsilon = \pm i$.

(2) There is one more newform with $\epsilon = -1$ than with $\epsilon = 1, i$, or $-i$ when $k \equiv 5 \bmod 6$, i.e., the root number has a slight bias toward -1 in this case. For example, when $k = 5$ and ω' is the Dirichlet character of conductor 3, there are five newforms of level 27, with respective root numbers $1, -1, -1, i, -i$. These newforms are discussed further in Section 7.5.

Proof. By Theorem 7.1,

$$\begin{aligned} |H_k(\sigma)| &= \frac{1}{12}(k-1)\cdot\frac{8}{2} + \frac{1}{2}\Phi\left(\begin{pmatrix} & -3 \\ 1 & \end{pmatrix}, f\right) + \Phi\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, f\right) + \Phi\left(\begin{pmatrix} 0 & -3 \\ 1 & 3 \end{pmatrix}, f\right) \\ &= \frac{1}{3}(k-1) + \frac{2\epsilon}{3}\delta_{t=-1}\cdot\delta_{k\in 2\mathbb{Z}} + \frac{t}{2_{t=-1}}c_3(k) + \epsilon g_6(k)\,\delta_{t=-1}, \end{aligned}$$

where we have applied Propositions 7.10, 7.13, and 7.14, and $c_3(k)$, $g_6(k)$ are recalled below. (For nonvanishing of $\Phi\left(\left(\begin{smallmatrix} & -3 \\ 1 & \end{smallmatrix}\right), f\right)$, the hypothesis in Proposition 7.10 requires that $-t$ be a square modulo 3, i.e., $t = -1$, and k even. Then $\bar{\epsilon} = \epsilon$ and the sum over y in that result is $1 + (-1)^k = 2$.)

If $t = 1$, then because $c_3(k) = \frac{1}{3}(1 - k) + \lfloor \frac{k}{3} \rfloor$, the above simplifies to $\lfloor \frac{k}{3} \rfloor$, as needed.

Now suppose $t = -1$, and write $k = a + 6c$ for some $0 \le a \le 5$. If k is odd, then

$$|H_k(\sigma)| = \tfrac{1}{3}(k-1) - \tfrac{1}{2}\left(\tfrac{1}{3}(1-k) + \lfloor \tfrac{k}{3} \rfloor\right) + \epsilon g_6(k) = \tfrac{1}{2}(k-1) - \tfrac{1}{2}\lfloor \tfrac{k}{3} \rfloor + \epsilon g_6(k).$$

Using the fact that $g_6(k) = 0, \frac{1}{2}, -\frac{1}{2}$ when $a = 1, 3, 5$ respectively, we get

$$|H_k(\sigma)| = \begin{cases} 2c = \lfloor \frac{k}{3} \rfloor & \text{if } a = 1, \\ 2c + 1 + \frac{1}{2}(\epsilon - 1) = \lfloor \frac{k}{3} \rfloor + \frac{1}{2}(\epsilon - 1) & \text{if } a = 3, \\ 2c + 1 + \frac{1}{2}(1 - \epsilon) = \lfloor \frac{k}{3} \rfloor + \frac{1}{2}(1 - \epsilon) & \text{if } a = 5. \end{cases}$$

If k is even, then there is one extra term, namely $\frac{2\epsilon}{3}$, so

$$|H_k(\sigma)| = \tfrac{1}{2}(k-1) - \tfrac{1}{2}\left\lfloor \tfrac{k}{3} \right\rfloor + \epsilon\left(\tfrac{2}{3} + g_6(k)\right).$$

Here, $g_6(k) = -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}$ when $a = 0, 2, 4$ respectively. Upon simplifying,

$$|H_k(\sigma)| = \begin{cases} 2c + \frac{1}{2}(\epsilon - 1) = \left\lfloor \frac{k}{3} \right\rfloor + \frac{1}{2}(\epsilon - 1) & \text{if } a = 0, \\ 2c + \frac{1}{2}(1+\epsilon) = \left\lfloor \frac{k}{3} \right\rfloor + \frac{1}{2}(1+\epsilon) & \text{if } a = 2, \\ 2c + 1 + \epsilon = \left\lfloor \frac{k}{3} \right\rfloor + \varepsilon & \text{if } a = 4. \end{cases}$$ □

Theorem 7.17. *Suppose $N = T^3$ with $T > 3$ square-free, $M = \frac{T}{2}$, $k \geq 4$ is even, and $\hat{\sigma} = (\sigma_{t_p}^{\zeta_p})_{p|N}$ is a tuple of simple supercuspidal representations with trivial central characters. Then*

$$|H_k(\hat{\sigma})| = \tfrac{1}{12}(k-1)\prod_{p|T}\tfrac{1}{2}(p^2-1) + \Delta_1(\hat{t})\,\epsilon(k,\hat{\sigma})\,b_T\,h(-T) + \Delta_2(\hat{t})\frac{\epsilon(k,\hat{\sigma})\,j_M\,h(-M)}{\zeta_2\,3_{M=3}}, \tag{7-17}$$

where $\epsilon(k,\hat{\sigma}) \in \{\pm 1\}$ is the common global root number of the newforms in $H_k(\hat{\sigma})$ given in (7-12),

$$b_T = \begin{cases} \frac{1}{4} & \textit{if } T \textit{ is even}, \\ \frac{1}{2} & \textit{if } T \equiv 1 \bmod 4, \\ 1 & \textit{if } T \equiv 7 \bmod 8, \\ 2 & \textit{if } T \equiv 3 \bmod 8, \end{cases} \qquad j_M = \begin{cases} \frac{1}{4} & \textit{if } M \equiv 1 \bmod 4, \\ -\frac{3}{2} & \textit{if } M \equiv 3 \bmod 8, \\ 0 & \textit{if } M \equiv 7 \bmod 8, \end{cases}$$

$h(d)$ is the class number of $\mathbb{Q}[\sqrt{-d}]$, and $\Delta_i(\hat{t}) \in \{0, 1\}$ is nonzero if and only if (i) *T is even in the case $i = 2$, and* (ii) *$-2^{i-1}pt_p/T$ is a square modulo p for each odd $p|T$.*

Remarks. To keep the formula simple, we have restricted ourselves to the case of trivial central character; the general case is obtained similarly. Even in the general case, one may restrict to k even because by Corollary 7.2,

$$|H_k(\hat{\sigma})| = \tfrac{1}{12}(k-1)\prod_{p|N}\tfrac{1}{2}(p^2-1) \quad (T > 3,\ k \text{ odd}). \tag{7-18}$$

Proof. This follows from Theorem 7.1 and Propositions 7.10 and 7.11. □

As a corollary, we recover the following dimension formulas of [37].

Corollary 7.18. *For $T = 2, 3$ and $k \geq 4$ even,*

$$\dim S_k^{\mathrm{new}}(8) = \left\lfloor \tfrac{k}{4} \right\rfloor, \quad \dim S_k^{\mathrm{new}}(27) = k - 1 + \left\lfloor \tfrac{k}{3} \right\rfloor.$$

For $T > 3$ square-free, and $k \geq 4$ even,

$$\dim S_k^{\mathrm{new}}(T^3) = \tfrac{1}{12}(k-1)\prod_{p|T}(p-1)^2(p+1). \tag{7-19}$$

Remarks. As shown in [37], the formula is also valid for $k = 2$. When k is odd and ω' has conductor dividing T, $\dim S_k^{\mathrm{new}}(T^3, \omega')$ is also equal to (7-19). This follows from (7-18).

Proof. For $T = 2$, by Theorem 7.15,

$$|H_k(2^3)| = |H_k(\sigma^+)| + |H_k(\sigma^-)| = \begin{cases} 2\lfloor \frac{k}{8} \rfloor & \text{if } k \equiv 0, 2 \bmod 8, \\ 2\lfloor \frac{k}{8} \rfloor + 1 & \text{if } k \equiv 4, 6 \bmod 8. \end{cases}$$

This is easily seen to be the same as $\lfloor \frac{k}{4} \rfloor$.

For $T = 3$, for fixed k we add the formula in Theorem 7.16 over all $t, \zeta \in \{\pm 1\}$, or equivalently, $t, \epsilon \in \{\pm 1\}$. Writing the $t = 1$ contribution first, we obtain

$$|H_k(3^3)| = 2\lfloor \tfrac{k}{3} \rfloor + \begin{cases} 2\lfloor \frac{k}{3} \rfloor - 1 & \text{if } k \equiv 0 \bmod 3, \\ 2\lfloor \frac{k}{3} \rfloor & \text{if } k \equiv 1 \bmod 3, \\ 2\lfloor \frac{k}{3} \rfloor + 1 & \text{if } k \equiv 2 \bmod 3. \end{cases}$$

The above is easily seen to equal $k - 1 + \lfloor \frac{k}{3} \rfloor$, as required.

For $T > 3$ we have

$$\dim S_k^{\mathrm{new}}(T^3) = |H_k(T^3)| = \sum_{\hat\sigma} |H_k(\hat\sigma)|,$$

where $\hat\sigma$ ranges over the $\prod_{p|T} 2(p-1)$ tuples (t_p, ζ_p), with trivial central character. By (7-17), this equals

$$\tfrac{1}{12}(k-1) \prod_{p|T} \tfrac{1}{2}(p^2-1)\, 2(p-1) + \sum_{\hat\sigma} \Delta_1(\hat t)\, \epsilon(k, \hat\sigma)\, b_T\, h(-T) + \sum_{\hat\sigma} \Delta_2(\hat t) \frac{\epsilon(k, \hat\sigma)\, j_M\, h(-M)}{\zeta_2\, 3_{M=3}}.$$

Clearly, from (7-12), exactly half of the $\hat\sigma$ satisfying $\Delta_1(\hat t) = 1$ have $\epsilon(k, \hat\sigma) = +1$, and half have $\epsilon(k, \hat\sigma) = -1$. So the first sum over $\hat\sigma$ vanishes. Likewise if T is even, $\epsilon(k, \hat\sigma)/\zeta_2 = +1$ (resp. -1) exactly half of the time since T is divisible by at least one prime different from 2, so the second sum also vanishes. $\square$

Next, we compute the dimension of the subspace of forms with a given root number, which recovers the main result (1-9) of [46].

Corollary 7.19 [46]. *For $T > 3$ square-free and $k \geq 4$ even, the subspace of $S_k^{\mathrm{new}}(T^3)$ with root number ± 1 has dimension*

$$|H_k^{\pm}(T^3)| = \tfrac{1}{24}(k-1) \prod_{p|T} (p-1)^2(p+1) \pm \tfrac{1}{2}(c_T\, h(-T)) \prod_{p|T} (p-1),$$

where $c_T = b_T$ if T is odd, and $c_T = 2b_T$ if T is even, i.e.,

$$c_T = \begin{cases} \frac{1}{2} & \text{if } T \equiv 1, 2 \bmod 4, \\ 1 & \text{if } T \equiv 7 \bmod 8, \\ 2 & \text{if } T \equiv 3 \bmod 8. \end{cases} \tag{7-20}$$

Proof. Given $\hat{\sigma} = (\sigma_{t_p}^{\zeta_p})_{p|T}$, let $\hat{t} = (t_p)_{p|T}$ and $\hat{\zeta} = (\zeta_p)_{p|T}$. The root number is determined by $\hat{\zeta}$ and k. Let $A_k^{\pm}$ be the set of all tuples $\hat{\zeta}$ for which $(-1)^{k/2}\prod_{p|T}\zeta_p = \pm 1$. Then

$$|A_k^+| = |A_k^-| = \tfrac{1}{2}\prod_{p|T} 2. \tag{7-21}$$

By (7-17), we see that

$$\begin{aligned} &|H_k^{\pm}(T^3)| \\ &\quad = \sum_{\hat{\zeta}\in A_k^{\pm}} \sum_{\hat{t}} |H_k(\hat{\sigma})| \\ &\quad = \sum_{\hat{\zeta}\in A_k^{\pm}} \sum_{\hat{t}} \Big(\tfrac{1}{12}(k-1)\prod_{p|T}\tfrac{1}{2}(p^2-1) \pm b_T\, h(-T)\Delta_1(\hat{t}) \pm \zeta_2 \frac{j_M\, h(-M)}{3_{M=3}}\Delta_2(\hat{t})\Big), \end{aligned}$$

where M is the odd part of T. Recall that $\Delta_2(\hat{t}) = 0$ if T odd. If T is even, upon summing over $\zeta_2 = \pm 1$ the last term will be eliminated, so we can ignore it henceforth. For any given odd prime p, exactly half of the elements $t_p \in (\mathbb{Z}/p\mathbb{Z})^*$ have the property that $-pt_p/T$ is a square. Therefore, the number of tuples $\hat{t}$ for which $\Delta_1(\hat{t}) \neq 0$ is $\prod_{p|M}\frac{1}{2}(p-1)$. The total number of tuples $\hat{t}$ is $\prod_{p|T}(p-1) = \prod_{p|M}(p-1)$. It follows that

$$|H_k^{\pm}(T^3)| = \sum_{\hat{\zeta}\in A_k^{\pm}} \Big(\tfrac{1}{12}(k-1)\prod_{p|T}\tfrac{1}{2}(p^2-1)(p-1) \pm b_T\, h(-T)\prod_{p|M}\tfrac{1}{2}(p-1)\Big).$$

By (7-21), we obtain

$$|H_k^{\pm}(T^3)| = \tfrac{1}{24}(k-1)\prod_{p|T}(p-1)^2(p+1) \pm \tfrac{1}{2}(2_T b_T\, h(-T))\prod_{p|T}(p-1),$$

where 2_T is a factor of 2 which is only present when T is even. We see immediately that $2_T b_T = c_T$ as given. □

7.5. *Some examples with* n > 1. In this section we illustrate Theorem 1.1 with some examples. (A different set of examples is given in the earliest version of this paper posted on the arXiv.) We will compare with the Galois orbits of newforms tabulated in LMFDB [34]. Though $S_k(\hat{\sigma})$ occasionally forms a Galois orbit, typically the orbit is a direct sum of more than one such space. It also happens that a space $S_k(\hat{\sigma})$ decomposes as a direct sum of more than one Galois orbit.

Examples of these phenomena can be found in $S_4^{\min}(23^2)$, where Theorem 1.3 gives $\dim S_4(\hat{\sigma}) = \frac{1}{2}(11+\epsilon) \in \{5, 6\}$, but the twist-minimal Galois orbits can have dimensions 1, 2, 5, 6, 12 or 24.

7.5.1. We first consider an example with odd weight. Take $N = 3^3$, $k = 5$, and ω' the Dirichlet character of modulus 27 and conductor 3. We consider simple supercuspidal representations σ_t^{ζ}, where $t \in \{\pm 1\}$ and $\zeta^2 = \omega'(t)$. In LMFDB [34] we find the following data for the space $S_5(27, \omega')$:

LMFDB label	ϵ	dim	$\operatorname{tr} T_4$	$\operatorname{tr} T_7$	(ζ, t)
27.5.b.a	1	1	16	71	$(-i, -1)$
27.5.b.b	-1	2	-76	34	$(i, -1)$
27.5.b.c	$\pm i$	2	14	-38	$(1, 1) \oplus (-1, 1)$

The final column, using the shorthand $(\zeta, t) = S_5(\sigma_t^{\zeta})$, is immediate upon comparing Theorem 7.16 with the ϵ and dim columns. Using Theorem 7.17 we find the following, which refines the above.

Example 7.20. With notation as above,

$$\operatorname{tr}(T_4 \mid S_5(\sigma_t^{\zeta})) = \tfrac{1}{2}(37t - 23) + 46i\zeta \cdot \delta_{t=-1},$$
$$\operatorname{tr}(T_7 \mid S_5(\sigma_t^{\zeta})) = \tfrac{1}{4}(67 - 143t) + \tfrac{1}{2}(37i\zeta)\delta_{t=-1}.$$

We will give an indication of the proof of the above formulas. The calculations for $\mathtt{n} = 7$ are a little bit more interesting, so we start with this case. By Theorem 1.1,

$$\operatorname{tr}(T_7 \mid S_5(\sigma_t^{\zeta}))$$
$$= 7^{3/2}\left[\Phi\left(\begin{pmatrix} & -21 \\ 1 & 3\end{pmatrix}\right) + \Phi\left(\begin{pmatrix} & -21 \\ 1 & 6\end{pmatrix}\right) + \Phi\left(\begin{pmatrix} & -21 \\ 1 & 9\end{pmatrix}\right) + \sum_{r=1}^{5}\Phi\left(\begin{pmatrix} & -7 \\ 1 & r\end{pmatrix}\right)\right].$$

We have used (4-13) to eliminate the trace zero matrices, since k is odd. The matrix $\left(\begin{smallmatrix} & -7 \\ 1 & 3\end{smallmatrix}\right)$ is unramified at $p = 3$ but has no double characteristic root mod 3. So its orbital integral vanishes by Proposition 5.6. The first three integrals vanish unless

$$y^2 \equiv -\frac{t}{7} \equiv -t \mod 3$$

has a solution, i.e., $t = -1$. In this case, applying Proposition 6.4 to $\gamma = \left(\begin{smallmatrix} & -21 \\ 1 & 9\end{smallmatrix}\right)$ and $p = 3$, we see that $v = 3$ so the local integral has the value $\overline{\zeta_5}(\omega_3(1) + \omega_3(-1)) = 0$. Hence this γ can be discarded. We compute the remaining orbital integrals locally as summarized in the following table, where $m = 2h(E)/(w(E)\,2^{\omega(d_E)})$ is the global measure factor for $E = \mathbb{Q}[\gamma]$, and ℓ denotes a prime factor of the discriminant Δ_γ other than 3 (if such exists). The global orbital integral is then $\Phi = m\Phi_\infty\Phi_3\Phi_\ell$. The factor

$$\Phi_\infty = -\frac{\sin(4\arctan(\sqrt{|\Delta_\gamma|}/\operatorname{tr}\gamma))}{\sin(\arctan(\sqrt{|\Delta_\gamma|}/\operatorname{tr}\gamma))}$$

was computed using software.

γ	Δ_γ	ℓ	m	Φ_∞	Φ_3	Φ_ℓ
$\left(\begin{smallmatrix}& -21\\ 1 & 3\end{smallmatrix}\right)$	$-3\cdot 5^2$	5	$\frac{1}{6}$	$11\sqrt{3}\cdot 7^{-3/2}$	$-i\bar{\zeta}\sqrt{3}\cdot\delta_{t=-1}$	7
$\left(\begin{smallmatrix}& -21\\ 1 & 6\end{smallmatrix}\right)$	$-2^4\cdot 3$	2	$\frac{1}{6}$	$4\sqrt{3}\cdot 7^{-3/2}$	$i\bar{\zeta}\sqrt{3}\cdot\delta_{t=-1}$	10
$\left(\begin{smallmatrix}& -7\\ 1 & 1\end{smallmatrix}\right)$	-3^3		$\frac{1}{6}$	$13\cdot 7^{-3/2}$	4	
$\left(\begin{smallmatrix}& -7\\ 1 & 2\end{smallmatrix}\right)$	$-2^3\cdot 3$	2	$\frac{1}{2}$	$20\cdot 7^{-3/2}$	$\frac{1}{2}(1-3t)$	2
$\left(\begin{smallmatrix}& -7\\ 1 & 4\end{smallmatrix}\right)$	$-2^2\cdot 3$	2	$\frac{1}{6}$	$-8\cdot 7^{-3/2}$	$-\frac{1}{2}(3t+1)$	4
$\left(\begin{smallmatrix}& -7\\ 1 & 5\end{smallmatrix}\right)$	-3		$\frac{1}{6}$	$-55\cdot 7^{-3/2}$	$\frac{1}{2}(3t+1)$	

The formula for $\operatorname{tr} T_7$ in Example 7.20 follows upon simplifying. Most of the entries in the above table are straightforward, but we highlight a few. For example, $\gamma = \left(\begin{smallmatrix}& -21\\ 1 & 6\end{smallmatrix}\right)$ is elliptic in $G(\mathbb{Q}_2)$, and by the quadratic formula,

$$\mathbb{Z}_2[\gamma] = \mathbb{Z}_2\big[\tfrac{1}{2}(6+2^2\sqrt{-3})\big] = \mathbb{Z}_2 + \mathbb{Z}_2 2^2\varepsilon,$$

where $\varepsilon = \frac{1}{2}(1+\sqrt{-3})$. So $n_\gamma = 2$ and $\Phi_2(\gamma) = 1+(2+1)+(4+2) = 10$ by Proposition 4.8 and (4-20).

The matrix $\gamma = \left(\begin{smallmatrix}& -7\\ 1 & 1\end{smallmatrix}\right)$ is unramified at $p=3$, so $\Phi_3(\gamma)$ is computed using Proposition 6.5. We find (using software) that $\mathcal{N}_\gamma(0,1) = \mathcal{N}_\gamma(0,2) = 3$, $\mathcal{N}_\gamma(1,2) = 6$, $\mathcal{N}_\gamma(1,3) = 9$, and $\mathcal{N}_\gamma(c,n) = 0$ for all other pairs (c,n). Since $P_\gamma(X) \equiv (X+1)^2 \bmod 3$, we take $z=-1$, so, using the third remark after Proposition 6.5, for $t=\pm 1$ we have

$$\Phi_3\left(\begin{pmatrix}& -7\\ 1 & 1\end{pmatrix}\right) = \frac{-1}{3}\left[3\left(e\left(\frac{t}{3}\right)+e\left(\frac{-t}{3}\right)\right)+3(2)+6(-1)+9(-1)\right] = 4.$$

Finally, $\gamma = \left(\begin{smallmatrix}& -7\\ 1 & 2\end{smallmatrix}\right)$ is unramified at $p=3$ and $\mathcal{N}_\gamma(-1,1) = 3$ is the only nonzero value of $\mathcal{N}_\gamma(c,n)$. We take $z=1$ in Proposition 6.5 to get

$$\Phi_3\left(\begin{pmatrix}& -7\\ 1 & 2\end{pmatrix}\right) = \frac{1}{3}\cdot 3\left[e\left(\frac{-1-t}{3}\right)+e\left(\frac{1+t}{3}\right)\right] = 2\cos\frac{2\pi(1+t)}{3} = \begin{cases}-1 & \text{if } t=1,\\ 2 & \text{if } t=-1.\end{cases}$$

This equals $\frac{1}{2}(1-3t)$ for $t=\pm 1$. The remaining entries in the above T_7 table are found in a similar fashion.

For $\operatorname{tr} T_4$, in the identity term we have $\omega'(\sqrt{4}) = -1$. So

$$\begin{aligned}\operatorname{tr}(T_4\,|\,S_5(\sigma_t^\zeta)) = 8\Bigg[-\frac{4}{3} &+ \Phi\left(\begin{pmatrix}& -12\\ 1 & 3\end{pmatrix}\right) + \Phi\left(\begin{pmatrix}& -12\\ 1 & 6\end{pmatrix}\right)\\ &+ \Phi\left(\begin{pmatrix}& -4\\ 1 & 1\end{pmatrix}\right) + \Phi\left(\begin{pmatrix}& -4\\ 1 & 2\end{pmatrix}\right) + \Phi\left(\begin{pmatrix}& -4\\ 1 & 3\end{pmatrix}\right)\Bigg].\end{aligned}$$

The last term can be eliminated since it is unramified at $p = 3$ and it has no characteristic root modulo 3. The remaining orbital integrals are computed locally as follows, and the formula for $\operatorname{tr} T_4$ in Example 7.20 follows upon simplification.

γ	Δ_γ	ℓ	m	Φ_∞	Φ_3	Φ_ℓ
$\left(\begin{smallmatrix} & -12 \\ 1 & 3\end{smallmatrix}\right)$	$-3\cdot 13$	13	1	$5\sqrt{3}\cdot 8^{-1}$	$-i\bar{\zeta}\sqrt{3}\cdot\delta_{t=-1}$	2
$\left(\begin{smallmatrix} & -12 \\ 1 & 6\end{smallmatrix}\right)$	$-2^2\cdot 3$	2	$\frac{1}{6}$	$-\sqrt{3}$	$i\bar{\zeta}\sqrt{3}\cdot\delta_{t=-1}$	4
$\left(\begin{smallmatrix} & -4 \\ 1 & 1\end{smallmatrix}\right)$	$-3\cdot 5$	5	$\frac{1}{2}$	$7\cdot 8^{-1}$	$\frac{1}{2}(3t-1)$	2
$\left(\begin{smallmatrix} & -4 \\ 1 & 2\end{smallmatrix}\right)$	$-2^2\cdot 3$	2	$\frac{1}{6}$	1	$\frac{1}{2}(3t+1)$	4

7.5.2. Let $N = 2^3 11^2$ and $k = 6$, and let σ^ζ be a simple supercuspidal representation of $\mathrm{PGL}_2(\mathbb{Q}_2)$ and σ_ν a depth zero supercuspidal representation of $\mathrm{PGL}_2(\mathbb{Q}_{11})$. Here, $\zeta \in \{\pm 1\}$, and ν is one of the five primitive characters of L^* listed in (7-5), where $L = \mathbb{F}_{11^2}$ and we take the generator t of L^* to be a root of the polynomial $X^2 + 7X + 2 \in \mathbb{F}_{11}[X]$. Let $\hat{\sigma}$ be the associated tuple. Then by Theorem 7.1,

$$\dim S_6(\hat{\sigma}) = \frac{25}{4} + \frac{1}{2}\Phi\left(\begin{pmatrix} & -2 \\ 1 & \end{pmatrix}, f^1\right) + \frac{1}{2}\Phi\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, f^1\right) + \Phi\left(\begin{pmatrix} & -2 \\ 1 & 2\end{pmatrix}, f^1\right).$$

Over $\mathbb{F}_{11}$, $X^2 + 2 = (x+3)(x-3)$, so $\left(\begin{smallmatrix} & -2 \\ 1 & \end{smallmatrix}\right)$ is hyperbolic in $G(\mathbb{Q}_{11})$ by Hensel's lemma, and therefore its orbital integral vanishes. Using Example 6.6 and the argument at (7-3),

$$\frac{1}{2}\Phi\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\right) = \frac{1}{2} m \Phi_\infty \Phi_2 \Phi_{11} = \frac{1}{2}\cdot\frac{1}{4}\cdot(-1)^{6/2}\cdot 1\cdot 2\epsilon_{11} = -\frac{1}{4}\epsilon_{11}.$$

Taking $\gamma = \left(\begin{smallmatrix} & -2 \\ 1 & 2\end{smallmatrix}\right)$, $P_\gamma(X) = X^2 - 2X + 2$ is irreducible over $\mathbb{F}_{11}$, so by (6-22),

$$\Phi_{11} = -\overline{\nu(\gamma)} - \overline{\nu^{11}(\gamma)}.$$

For $L^* = \langle t\rangle$ as above, we find (using software) that t^{51} has minimum polynomial $P_\gamma(X)$. Therefore, if $\nu = \nu_m$ for $m = 10w \in \{10, 20, 30, 40, 50\}$ as in (7-5) where $\mu_m(t) = e\left(\frac{m}{120}\right)$, we have

$$\nu_m(\gamma) = e\left(\frac{51m}{120}\right) = e\left(\frac{17w}{4}\right) = e\left(\frac{w}{4}\right) = i^w.$$

Using this, $\Phi_{11}(\gamma)$ is given by

ν	ν_{10}	ν_{20}	ν_{30}	ν_{40}	ν_{50}
ϵ_{11}	$+$	$-$	$+$	$-$	$+$
Φ_{11}	0	2	0	-2	0

(7-22)

As in the proof of Proposition 7.12, $m = \frac{1}{4}$, $\Phi_\infty = 1$ (since $k = 6$), and $\Phi_2 = -\zeta$. Hence $\Phi(\gamma) = -\frac{1}{4}\zeta\Phi_{11}$ for Φ_{11} as above. Thus

$$\dim S_6(\hat{\sigma}) = \tfrac{25}{4} - \tfrac{1}{4}\epsilon_{11} - \tfrac{1}{4}\zeta\Phi_{11} = \begin{cases} 6 & \text{if } \epsilon_{11} = 1, \text{ or } \zeta = 1 \text{ and } \nu = \nu_{20}, \\ & \text{or } \zeta = -1 \text{ and } \nu = \nu_{40}, \\ 7 & \text{if } \zeta = 1 \text{ and } \nu = \nu_{40}, \\ & \text{or } \zeta = -1 \text{ and } \nu = \nu_{20}. \end{cases} \tag{7-23}$$

We would like to match the above spaces to Galois orbits of twist-minimal newforms in $S_6^{\text{new}}(2^3 11^2)$. In the table below, the first five columns show LMFDB [34] data, with AL entries corresponding to the Atkin–Lehner signs at $p = 2, 11$. These are equal to ζ and ϵ_{11} respectively. The dim column gives the size of the orbit.

LMFDB label	dim	$\operatorname{tr} T_7$	AL 2	AL 11	(ζ, ν)
968.6.a.f	6	-124	$-$	$-$	$(-1, \nu_{40})$
968.6.a.g	6	124	$+$	$-$	$(1, \nu_{20})$
968.6.a.h	6	-88	$+$	$+$	$(1, \nu_{30})$
968.6.a.i	6	88	$-$	$+$	$(-1, \nu_{30})$
968.6.a.j	7	-62	$-$	$-$	$(-1, \nu_{20})$
968.6.a.k	7	62	$+$	$-$	$(1, \nu_{40})$
968.6.a.l	6	-206	$+$	$+$	$(1, \nu_{10}) \oplus (1, \nu_{50})$
968.6.a.m	6	206	$-$	$+$	$(-1, \nu_{10}) \oplus (-1, \nu_{50})$

In the final column we have adopted the notation $S_6(\hat{\sigma}) = (\zeta, \nu)$. This column was obtained as follows. Comparing (7-22) and (7-23) with the AL and dim columns, we immediately infer the entries with $\epsilon_{11} = -1$, i.e., with ν_{20} and ν_{40}. We can distinguish the remaining entries by looking at Hecke eigenvalues. For this we apply Theorem 1.1 to compute $\operatorname{tr}(T_7 \mid S_6(\hat{\sigma}))$. The result is the following.

Example 7.21. Let $N = 2^3 11^2$ and $\hat{\sigma} = (\sigma^\zeta, \sigma_\nu)$ be a tuple of supercuspidal representations of conductors 2^3 and 11^2 respectively, as above. Then

$$\operatorname{tr}(T_7 \mid S_6(\hat{\sigma})) = -98\zeta\epsilon_{11} - 5\zeta X_{11} - 31Y_{11},$$

where ϵ_{11}, X_{11} and Y_{11} are given as follows:

ν	ν_{10}	ν_{20}	ν_{30}	ν_{40}	ν_{50}
ϵ_{11}	$+$	$-$	$+$	$-$	$+$
X_{11}	1	1	-2	1	1
Y_{11}	$\sqrt{3}$	-1	0	1	$-\sqrt{3}$

For example, in the notation used above,

$$\operatorname{tr}(T_7 \mid (1, \nu_{10})) = -103 - 31\sqrt{3}, \quad \operatorname{tr}(T_7 \mid (1, \nu_{50})) = -103 + 31\sqrt{3}.$$

We sketch the proof as follows. By Theorem 1.1,

$$\begin{aligned}&\operatorname{tr}(T_7 \mid S_6(\hat{\sigma})) \\ &= 7^2\left[\tfrac{1}{2}\Phi\left(\begin{pmatrix} & -7 \\ 1 & \end{pmatrix}\right) + \tfrac{1}{2}\Phi\left(\begin{pmatrix} & -14 \\ 1 & \end{pmatrix}\right) + \sum_{r=1}^{5}\Phi\left(\begin{pmatrix} & -7 \\ 1 & r \end{pmatrix}\right) + \sum_{r=1}^{3}\Phi\left(\begin{pmatrix} & -14 \\ 1 & 2r \end{pmatrix}\right)\right].\end{aligned}$$

All but three of the orbital integrals vanish for simple reasons. The matrices $\left(\begin{smallmatrix} & -7 \\ 1 & \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} & -7 \\ 1 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} & -7 \\ 1 & 3 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} & -14 \\ 1 & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} & -14 \\ 1 & 2 \end{smallmatrix}\right)$ are hyperbolic in $G(\mathbb{Q}_{11})$, since their characteristic polynomials have two distinct roots modulo 11. The matrices $\left(\begin{smallmatrix} & -7 \\ 1 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} & -7 \\ 1 & 5 \end{smallmatrix}\right)$ are unramified at $p = 2$ but do not have characteristic roots modulo 2. So the associated orbital integrals vanish by Proposition 5.6, and

$$\operatorname{tr}(T_7 \mid S_6(\hat{\sigma})) = 7^2\left[\tfrac{1}{2}\Phi\left(\begin{pmatrix} & -14 \\ 1 & \end{pmatrix}\right) + \Phi\left(\begin{pmatrix} & -7 \\ 1 & 4 \end{pmatrix}\right) + \Phi\left(\begin{pmatrix} & -14 \\ 1 & 6 \end{pmatrix}\right)\right].$$

The formula in Example 7.21 follows upon computing each of these terms locally. The local results are shown in the following table, with notation as in the previous $N = 27$ example. The global orbital integral for a given row is $\Phi = m\Phi_\infty\Phi_2\Phi_{11}\Phi_\ell$.

γ	Δ_γ	ℓ	m	Φ_∞	Φ_2	Φ_{11}	Φ_ℓ
$\left(\begin{smallmatrix} & -14 \\ 1 & \end{smallmatrix}\right)$	$-2^3\cdot 7$	7	1	-1	ζ	$2\epsilon_{11}$	2
$\left(\begin{smallmatrix} & -14 \\ 1 & 6 \end{smallmatrix}\right)$	$-2^2\cdot 5$	5	$\frac{1}{2}$	$\frac{5}{7^2}$	$-\zeta$	X_{11}	2
$\left(\begin{smallmatrix} & -7 \\ 1 & 4 \end{smallmatrix}\right)$	$-2^2\cdot 3$	3	$\frac{1}{6}$	$\frac{31}{7^2}$	-3	Y_{11}	2

The Φ_{11} column was determined as follows. As described earlier, $\mathbb{F}_{11^2}^* = \langle t\rangle$ where $t^2+7t+2=0$. For each γ as above, there is a power t^j whose minimum polynomial over $\mathbb{F}_{11}$ is $P_\gamma(X)$. The power j was found with software, and is given as follows:

γ	$\left(\begin{smallmatrix} & -14 \\ 1 & \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} & -14 \\ 1 & 6 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} & -7 \\ 1 & 4 \end{smallmatrix}\right)$
j	18	8	17

In each case, (6-22) implies that

$$\Phi_{11} = -\overline{\nu(\gamma)} - \overline{\nu^{11}(\gamma)} = -\overline{\nu(t^j)} - \overline{\nu(t^{11j})}.$$

By definition, $\nu_m(t) = e\left(\frac{m}{120}\right)$, so if $\nu = \nu_m$ for $m = 10w$,

$$\Phi_{11}(\gamma) = -e\left(-\frac{jw}{12}\right) - e\left(-\frac{11jw}{12}\right),$$

which can be evaluated by hand or using software to obtain the Φ_{11} column in the above table.

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