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# Cohomology of Fuchsian groups and Fourier interpolation

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We give a new proof of a Fourier interpolation result first proved by Radchenko and Viazovska (2019), deriving it from a vanishing result of the first cohomology of a Fuchsian group with coefficients in the Weil representation.

## 1. Introduction

Let $\mathcal{S}$ be the space of even Schwartz functions on the real line, and $\mathfrak{s}$ the space of sequences of complex numbers $(a_n)_{n\geq 0}$ such that $|a_n| n^k$ is bounded for all $k$; we write $\hat{\phi}(k) = \int_{\mathbb{R}} \phi(x)\, e^{-2\pi i k x}\, dx$ for the Fourier transform of $\phi \in \mathcal{S}$. Radchenko and Viazovska [2019] proved the following beautiful "interpolation formula":

**Theorem 1.1.** *The map*

$$\Psi : \mathcal{S} \to \mathfrak{s} \oplus \mathfrak{s}, \quad \phi \mapsto (\phi(\sqrt{n}), \hat{\phi}(\sqrt{n}))_{n\geq 0}$$

*is an isomorphism onto the codimension* 1 *subspace of* $\mathfrak{s} \oplus \mathfrak{s}$ *cut out by the Poisson summation formula, i.e., the subspace of* $(x_n, y_n)$ *defined by* $\sum_{n\in\mathbb{Z}} x_{n^2} = \sum_{n\in\mathbb{Z}} y_{n^2}$.

This is an abstract interpolation result: The statement implies the existence of a universal formula that computes any value $\phi(x)$ of any even Schwartz function $\phi$ as a linear combination $\sum a_n(x)\, \phi(\sqrt{n}) + \sum \hat{a}_n(x)\, \hat{\phi}(\sqrt{n})$ for some $a_n(x)$, $\hat{a}_n(x)$, but does not specify what those functions are. By contrast, Radchenko and Viazovska first write down this explicit interpolation formula, and then deduce Theorem 1.1 from it. In a sense, what is accomplished in the present paper is to separate the abstract content of this interpolation result from its computational aspect.

The morphism $\Psi$ is in fact a homeomorphism of topological vector spaces with reference to natural topologies. We will give another proof of this theorem. The first step of this proof is to notice that the evaluation points $\sqrt{n}$ occur very naturally in the theory of the *oscillator representation* defined by Segal, Shale and Weil (see [Chan 2012] or [Lion and Vergne 1980] for introductions). Using this observation, the theorem can be reduced to computing the cohomology of a certain Fuchsian group with coefficients in this oscillator representation, and here we prove a more general statement:

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**Theorem 1.2.** *Let $G$ be $\mathrm{SL}_2(\mathbb{R})$ or a finite cover thereof, $\Gamma$ a lattice in $G$, $W$ an irreducible infinite-dimensional $(\mathfrak{g}, K)$-module, and $W^*_{-\infty}$ the distributional globalization of its dual* (see Section 2.4). Then $H^1(\Gamma, W^*_{-\infty})$ is always finite-dimensional, and in fact

$$\dim H^1(\Gamma, W^*_{-\infty}) = \text{multiplicity of } W^{\mathrm{cl}} \text{ in cusp forms on } \Gamma\backslash G, \tag{1}$$

where $W^{\mathrm{cl}}$ is the complementary irreducible representation to $W$ (see Section 2.3).

The theorem can be contrasted with usual Frobenius reciprocity:

$$\dim H^0(\Gamma, W^*_{-\infty}) = \text{multiplicity of } W \text{ in automorphic forms on } \Gamma\backslash G. \tag{2}$$

Note that, in the passage from (1) to (2), "cusp forms" have been replaced by "automorphic forms" and $W^{\mathrm{cl}}$ by $W$. We also emphasize the surprising fact that, in the theorem, the $H^1$ takes *no account* of the topology on $W^*_{-\infty}$: it is simply the usual cohomology of the discrete group $\Gamma$ acting on the abstract vector space $W^*_{-\infty}$. The corresponding determination for finite-dimensional $W$ is the subject of automorphic cohomology and is in particular completely understood, going back to [Eichler 1957].

A variant of Theorem 1.2, computing *all* the cohomology groups $H^i$ when $W$ is a spherical principal series representation, was already proved by Bunke and Olbrich in the 1990s. We were unaware of this work when we first proved Theorem 1.2; our original argument has many points in common with [Bunke and Olbrich 1998], most importantly in our usage of surjectivity of the Laplacian both for analytic and algebraic purposes, but also has some substantial differences of setup and emphasis. We will correspondingly give two proofs: the first based on the results of [Bunke and Olbrich 1998], and the second a shortened version of our original argument.

Some other interpolation consequences of Theorem 1.2, where interpolation is understood in the abstract sense as discussed after Theorem 1.1, arise by replacing $\mathcal{S}$ by other spaces of functions carrying natural representations of $\mathrm{SL}_2(\mathbb{R})$ and its finite covers; we discuss this in Section 6.4. For example, Hedenmalm and Montes-Rodríguez [2011] have shown that the functions $e^{i\pi\alpha nt}$, $e^{i\pi\beta n/t}$ are weakly dense in $L^\infty$ if and only if $\alpha\beta = 1$. We will show that an interpolation result holds at the transition point $\alpha\beta = 1$; we thank the referee for bringing [Hedenmalm and Montes-Rodríguez 2011] to our attention.

**1.1.** ***Theorem 1.2 implies Theorem 1.1.*** Here we give an outline of the argument and refer to Section 6 for details.

We pass first to a dual situation. Denote by $\mathcal{S}^*$ the space of tempered distributions, i.e., the continuous dual of $\mathcal{S}$. For our purposes we regard it as a vector space without topology.

Similarly, we define $\mathfrak{s}^*$ as the continuous dual of $\mathfrak{s}$, where $\mathfrak{s}$ is topologized by means of the norms $\|(b_n)\|_k := \sup_n b_n(1+|n|)^k$; thus, $\mathfrak{s}^*$ may be identified with

sequences $(a_n)$ of complex numbers of polynomial growth, where the pairing of $(a_n) \in \mathfrak{s}^*$ and $(b_n) \in \mathfrak{s}$ is given by the rule $\sum a_n b_n$. With this notation, the map

$$\Psi^* : \mathfrak{s}^* \oplus \mathfrak{s}^* \to \mathcal{S}^*$$

dual to $\Psi$ takes the coordinate functions to the distributions $\delta_n$ and $\hat{\delta}_n$:

$$(a_n, b_n)_{n\geq 0} \mapsto \sum a_n \delta_n + b_n \hat{\delta}_n,$$

where

$$\delta_n(\phi) = \phi(\sqrt{n}), \quad \hat{\delta}_n(\phi) = \hat{\phi}(\sqrt{n}).$$

Then Theorem 1.1 is equivalent to the assertion:

> (Dual interpolation theorem): $\Psi^*$ is surjective and its kernel consists precisely of the "Poisson summation" relation.

The equivalence of this statement and Theorem 1.1 is not a complete formality because of issues of topology: see (52) for an argument that uses a theorem of Banach.

The next key observation is that the space of distributions spanned by $\delta_n$ and by $\hat{\delta}_n$ occur in a natural way in representation theory.

> The closure of the span of $\delta_n$ (respectively, the closure of the span of $\hat{\delta}_n$) coincide with the $e$-fixed and $f$-fixed vectors on the space $\mathcal{S}^*$ of tempered distributions, where
>
> $$e = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \tag{3}$$
>
> act on $\mathcal{S}^*$ according to the oscillator representation (see Section 6.1 for details), namely $e$ and $f$ multiply $\phi$ and $\hat{\phi}$, respectively, by $e^{2\pi i x^2}$, see (51).

Let $\Gamma$ be the group generated by $e$ and $f$ inside $\mathrm{SL}_2(\mathbb{R})$: it is a free group, of index 2 in $\Gamma(2)$, and it lifts to the double cover $G$ of $\mathrm{SL}_2(\mathbb{R})$. As explicated in Section 6, computations of dimensions of modular forms and Theorem 1.2 yield

$$\dim H^0(\Gamma, \mathcal{S}^*) = 1, \quad \dim H^1(\Gamma, \mathcal{S}^*) = 0. \tag{4}$$

The final observation is that:

> The kernel and cokernel of $(\mathcal{S}^*)^e \oplus (\mathcal{S}^*)^f \to \mathcal{S}^*$ compute, respectively, the $H^0$ and $H^1$ of $\Gamma$ acting on $\mathcal{S}^*$.

This follows from a Mayer–Vietoris-type long exact sequence that computes the cohomology of the free group $\Gamma$ [Brown 1982, Chapters II and III], namely,

$$0 \to H^0(\Gamma, \mathcal{S}^*) \to H^0(\langle e\rangle, \mathcal{S}^*) \oplus H^0(\langle f\rangle, \mathcal{S}^*) \to H^0(1, \mathcal{S}^*) \to H^1(\Gamma, \mathcal{S}^*) \to \cdots. \tag{5}$$

Combined with (4), we see that $\mathcal{S}^* = (\mathcal{S}^*)^e + (\mathcal{S}^*)^f$, i.e., the desired surjectivity of $\Psi^*$, and that the intersection of $(\mathcal{S}^*)^e$ and $(\mathcal{S}^*)^f$ is one-dimensional; this corresponds exactly to the Poisson summation formula.

Another way to look at this is the following. The Poisson summation formula is an obstruction to *surjectivity* in Theorem 1.1 and is closely related to the invariance of the distribution $\sum \delta_n \in \mathcal{S}^*$ by $\Gamma$, i.e., the existence of a class in the *zeroth* cohomology of $\Gamma$ on $\mathcal{S}^*$. The above discussion shows a less obvious statement: the obstruction to *injectivity* in Theorem 1.1 is precisely the *first* cohomology of $\Gamma$ on $\mathcal{S}^*$.

**1.2.** ***The proof of Theorem 1.2.*** The analogue of Theorem 1.2 when $W$ is finite-dimensional and $\Gamma\backslash G$ is compact is (by now) a straightforward exercise; as noted, the ideas go back at least to [Eichler 1957], and the general case is documented in [Borel and Wallach 2000]; the noncompact case is less standard but also well known, see, e.g., [Casselman 1984] and [Franke 1998] for a comprehensive treatment.

The main complication of our case is that the coefficients are infinite-dimensional and one might think this renders the question unmanageable. The key point is that $W$ is irreducible as a $G$-module. This says that, "relative to $G$", it is just as good as a finite-dimensional representation.

We will present two proofs of Theorem 1.2:

- The first proof, in Section 3, relies on the work of Bunke and Olbrich [1998], who computed the cohomology of lattices in $\mathrm{SL}_2(\mathbb{R})$ with coefficients in (the distribution globalization of a) principal series representation. We give a sketch of the argument of [Bunke and Olbrich 1998] for the convenience of the reader, and also because their argument as written does not cover the situation we need. To deduce Theorem 1.2 from these results then requires us to pass from a principal series to a subquotient, which we do in a rather ad hoc way.

- The second proof is our original argument prior to learning of the work of Bunke and Olbrich just mentioned. It generalizes the standard way of computing $\Gamma$-cohomology with finite-dimensional coefficients, as given in [Borel and Wallach 2000], to the infinite-dimensional case — at least in cohomological degree 1. Given the content of [Bunke and Olbrich 1998], we have permitted ourselves to abridge some tedious parts of our original argument, and reproduce here in detail the part that is perhaps most distinct from [Bunke and Olbrich 1998] — namely, we express the desired cohomology groups in terms of certain Ext-groups of $(\mathfrak{g}, K)$-modules and then compute these explicitly.

In both arguments the surjectivity of a Laplacian-type operator plays an essential role. Such results are known since the work of Casselman [1984], and in their work, Bunke and Olbrich prove and utilize such a result both at the level of $G$ and $\Gamma\backslash G$. We include a self-contained proof of such a result for $\Gamma\backslash G$ in Section 5.

**1.3.** ***Questions.*** As we have noted, we prove an *abstract* interpolation result. Can one recover the explicit formula for the interpolating functions, as given in [Radchenko and Viazovska 2019], from this approach? It seems to us that our proof is sufficiently explicit that this is, at least, plausible.

It is very interesting to ask about the situation where $\Gamma$ is not a lattice. Indeed, if one were to ask about an interpolation formula with evaluation points $0.9\sqrt{n}$, one is immediately led to similar questions for a discrete but *infinite covolume* subgroup of $\mathrm{SL}_2(\mathbb{R})$, whereas considering $1.1\sqrt{n}$ leads to considering a nondiscrete subgroup. Note that Kulikov, Nazarov and Sodin [Kulikov et al. 2025] have recently shown very general results about Fourier uniqueness that imply, in particular, that evaluating $f$ and $\hat{f}$ at $1.1\sqrt{n}$ do not suffice to determine $f$, but that evaluating them at $0.9\sqrt{n}$ *does*.

Perhaps a more straightforward question is to establish an isomorphism

$$H^i(\Gamma, W_{-\infty}^*) \simeq \mathrm{Ext}^i_{\mathfrak{g},K}(W, \text{ space of automorphic forms for } \Gamma\backslash G), \tag{6}$$

which is valid for general lattices $\Gamma$ in semisimple Lie groups $G$ and general irreducible (smooth, moderate growth) representations $V$ of $G$. Bunke and Olbrich have proved this in the cocompact case, and our original argument proceeded by establishing the case $i = 1$ for general lattices in $\mathrm{SL}_2(\mathbb{R})$. Also, Deitmar and Hilgert [2005, Corollary 3.3] prove a result of this type in great generality, but with the space of automorphic forms replaced by the larger space $C^\infty(\Gamma\backslash G)$ without growth constraints.

## 2. Covering groups of $\mathrm{SL}_2(\mathbb{R})$

Let $q \geq 1$ be a positive integer and let $G$ be the $q$-fold covering of the group $\mathrm{SL}_2(\mathbb{R})$, i.e., $G$ is a connected Lie group equipped with a continuous homomorphism $G \to \mathrm{SL}_2(\mathbb{R})$ with kernel of order $q$. This characterizes $G$ up to unique isomorphism covering the identity of $\mathrm{SL}_2(\mathbb{R})$.

Denote by $\mathfrak{g}$ the shared Lie algebra of $G$ and of $\mathrm{SL}_2(\mathbb{R})$ and $\exp : \mathfrak{g} \to G$ the exponential map. Also denote by $K$ the preimage of $\mathrm{SO}_2(\mathbb{R})$ inside $G$; it is abstractly isomorphic as topological group to $S^1 = \mathbb{R}/\mathbb{Z}$ and we fix such an isomorphism below.

The quotient $G/K$ is identified with the hyperbolic plane $\mathbb{H}$, on which $G$ acts by isometries. Define the norm of $g \in G$ to be $\|g\| := e^{\mathrm{dist}_{\mathbb{H}}(i,gi)}$. Equivalently, we could use $\left\|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right\| = \sqrt{a^2+b^2+c^2+d^2}$ since either of these two norms is bounded by a constant multiple of the other.

**2.1.** ***Lie algebra.*** Let $H, X, Y$ be the standard basis for $\mathfrak{g}$:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also use $\kappa = i(X-Y)$, $2p = H - i(X+Y)$, $2m = H + i(X+Y)$, or, in matrix form

$$\kappa = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad 2p = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad 2m = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}. \tag{7}$$

We have $\kappa = ik$, where $k$ generates the Lie algebra of $K$.

The elements $p, m$ and $\kappa$ satisfy the commutation relations

$$[p, m] = \kappa, \quad [\kappa, p] = 2p, \quad [\kappa, m] = -2m, \tag{8}$$

which say that $p$ and $m$ (shorthand for *plus* and *minus*) raise and lower $\kappa$-weights by 2. The Casimir element $\mathcal{C}$ in the universal enveloping algebra determined by the trace form is given by any of the equivalent formulas:

$$\mathcal{C} = \tfrac{1}{2}H^2 + XY + YX = \tfrac{1}{2}\kappa^2 + pm + mp = \tfrac{1}{2}\kappa^2 + \kappa + 2mp = \tfrac{1}{2}\kappa^2 - \kappa + 2pm. \tag{9}$$

**2.2.** ***Iwasawa decomposition.*** There is a decomposition

$$G = NAK, \tag{10}$$

where $A$ and $N$ are the connected Lie subgroups of $G$ with Lie algebra $\mathbb{R}.H$ and $\mathbb{R}.X$ respectively. We will parameterize elements of $A$ via

$$a_y := \exp\left(\tfrac{1}{2}\log(y)H\right),$$

so that $a_y$ projects to the diagonal element of $\mathrm{SL}_2(\mathbb{R})$ with entries $y^{\pm 1/2}$. We will also write $n_x = \exp(xX)$.

**2.3.** ***$(\mathfrak{g}, K)$-modules.*** Recall that a $(\mathfrak{g}, K)$-module $W$ means a $\mathfrak{g}$-module equipped with a compatible continuous action of $K$. Equivalently, it is described by the following data:

- for each $\zeta \in q^{-1}\mathbb{Z}$, a vector space $W_\zeta$ giving the $\zeta$-weight space of $K$, so that $\kappa$ acts on $W_\zeta$ by $\zeta$;
- maps $p : W_\zeta \to W_{\zeta+2}$ and $m : W_\zeta \to W_{\zeta-2}$ satisfying $[p, m] = \kappa$.

We recall some facts about classification, see [Howe and Tan 1992] for details. Irreducible, infinite-dimensional $(\mathfrak{g}, K)$-modules belong to one of three classes; in each case, the weight spaces $W_\zeta$ have dimension either zero or 1.

- Highest weight modules of weight $\zeta$; these are determined up to isomorphism by the fact that their nonzero weight spaces occur in weights $\{\zeta, \zeta-2, \zeta-4, \dots\}$. $W_\zeta$ is killed by $p$. One computes using (9) that on such modules, the Casimir element $\mathcal{C}$ acts by $\frac{1}{2}\zeta(\zeta+2)$.
- Lowest weight modules of weight $\zeta$; these are determined up to isomorphism by the fact that their nonzero weight spaces occur in weights $\{\zeta, \zeta+2, \zeta+4, \dots\}$. $W_\zeta$ is killed by $m$. Again, (9) shows that the Casimir element $\mathcal{C}$ acts by $\frac{1}{2}\zeta(\zeta-2)$.
- Doubly infinite modules, in which the weights are of the form $\zeta + 2\mathbb{Z}$ for $\zeta \in \frac{1}{q}\mathbb{Z}$.

**Definition 2.1.** For an infinite-dimensional irreducible $(\mathfrak{g}, K)$-module $W$ we define the complementary irreducible representation $W^{\mathrm{cl}}$ to be

$$\begin{cases} \text{the irreducible } (\mathfrak{g}, K)\text{-module with highest weight } \zeta - 2 & \text{if } W \text{ has lowest weight } \zeta, \\ \text{the irreducible } (\mathfrak{g}, K)\text{-module with lowest weight } \zeta + 2 & \text{if } W \text{ has highest weight } \zeta, \\ W & \text{otherwise.} \end{cases}$$

The representation $W^{\mathrm{cl}}$ can be finite-dimensional; this occurs exactly when $W$ is the underlying $(\mathfrak{g}, K)$-module of a discrete series representation on $\mathrm{SL}_2(\mathbb{R})$.

In Section 4 we use the following key fact about $(\mathfrak{g}, K)$-modules.

**Proposition 2.2.** *Let $W$ be an irreducible infinite-dimensional $(\mathfrak{g}, K)$-module with Casimir eigenvalue $\lambda$. Then, for any $(\mathfrak{g}, K)$-module $V$:*

(a) *If $\mathcal{C} - \lambda$ is surjective on $V$, then $\mathrm{Ext}^1_{(\mathfrak{g},K)}(W, V) = 0$.*

(b) *If $V$ is irreducible, $\mathrm{Ext}^1_{(\mathfrak{g},K)}(W, V)$ is one-dimensional if $V \simeq W^{\mathrm{cl}}$, and is zero otherwise.*

*Proof.* We will prove these statements in the case where $W$ is a lowest weight module, which is the case of our main application. The same proof works with slight modifications for $W$ a highest weight or doubly infinite module: in every case, one takes an arbitrary lift of a generating vector, and modifies it using the surjectivity of an appropriate operator.

We prove (a). Take $W$ to be generated by a vector $v_\zeta$ of lowest weight $\zeta$ with $mv_\zeta = 0$. This implies by the classification above that

$$\lambda = \tfrac{1}{2}\zeta(\zeta - 2). \tag{11}$$

Take an extension $V \to E \to W$; to give a splitting we must lift $w_\zeta$ to a vector in $E$ of $K$-type $\zeta$ killed by $m$. Arbitrarily lift $w_\zeta$ to $\widetilde{w}_\zeta \in E_\zeta$. Then $m\widetilde{w}_\zeta \in V_{\zeta-2}$ and it suffices to show that it lies inside the image of $m : V_\zeta \to V_{\zeta-2}$, for we then modify the choice of $\widetilde{w}_\zeta$ by any preimage to get the desired splitting. By (9) and (11) we see that $\mathcal{C} - \lambda : V_{\zeta-2} \to V_{\zeta-2}$ agrees with $2mp$. Since it is surjective, it follows that in particular $m : V_\zeta \to V_{\zeta-2}$ is surjective.

We pass to (b). Suppose $V$ is irreducible; then $\mathrm{Ext}^1_{(\mathfrak{g},K)}(W, V)$ vanishes unless $V$ has the same $\mathcal{C}$-eigenvalue as $W$. The argument above exhibits an injection of

$$\mathrm{Ext}^1_{(\mathfrak{g},K)}(W, V) \hookrightarrow \frac{V_{\zeta-2}}{mV_\zeta}$$

and inspection of $K$-types amongst those irreducibles with the same $\mathcal{C}$-eigenvalue as $V$ shows that this also vanishes unless $V \simeq W^{\mathrm{cl}}$, in which case it is one-dimensional. It remains only to exhibit a nontrivial extension of $W$ by $W^{\mathrm{cl}}$, which is readily done by explicit computation. □

**2.4.** ***Globalizations.*** A globalization of a $(\mathfrak{g}, K)$-module $W$ is any continuous $G$-representation on a topological vector space $\overline{W}$ such that $(\overline{W})_K = W$. We will consider two instances of this: the *smooth*, or *Casselman–Wallach globalization* $W_\infty$, and the *distributional globalization* $W_{-\infty}$.

Following [Casselman 1989], the representation $W_\infty$ is the unique globalization of $W$ as a moderate growth Fréchet $G$-representation. By definition, such a representation is a Fréchet space $F$ (topologized with respect to a family of seminorms) such that for any seminorm $\|\cdot\|_\alpha$, there is an integer $N_\alpha$ and a seminorm $\|\cdot\|_\beta$ for which

$$\|gw\|_\alpha \le \|g\|^{N_\alpha}\|w\|_\beta.$$

The distributional globalization is a dual notion. Indeed, denote by $W^*$ the $K$-finite part of the dual of $W$, equipped with the contragredient $(\mathfrak{g}, K)$-module structure. Then

$$(W_\infty)^* = (W^*)_{-\infty}, \tag{12}$$

where on the left-hand side, the dual is understood as continuous.

We recall an explicit construction of $W_{-\infty}$, see [Bunke and Olbrich 1998, Sections 2 and 3], although it will not be directly used in the rest of the paper: Given $W^*$ as above, let $V^* \subset W^*$ be a finite-dimensional $K$-stable subspace that generates $W^*$ as a $(\mathfrak{g}, K)$-module. Let $(V^*)^* =: V \subset W$ viewed as a $K$-representation, and consider the space

$$\mathcal{E}_V = \{f \in C^\infty(G, V) \mid f(gk) = k^{-1}f(g),\ g \in G,\ k \in K\}.$$

Then the image of $W$ under the map $i : W \to \mathcal{E}_V$ characterized by

$$\langle i(w)(g), v^*\rangle := \langle w, gv^*\rangle \quad (w \in W,\ v^* \in V^*)$$

belongs to the space $\mathcal{A}_V^G$ of sections of moderate growth, i.e., of functions $f \in \mathcal{E}_V$ such that for every $X \in U(\mathfrak{g})$, there is $R = R(f, X)$ for which

$$\|f\|_{X,R} = \sup_{g\in G} \frac{|Xf(g)|}{\|g\|^R} < \infty. \tag{13}$$

We note that this differs from the notion of *uniform* moderate growth, where one requires $R$ to be taken independently of $X$.

The space $\mathcal{A}_V^G$ is topologized as the direct limit of Fréchet spaces with respect to the seminorms $\|\cdot\|_{X,R}$. The map $i$ is injective since $V^*$ generates $W^*$, and the distributional globalization is defined by

$$W_{-\infty} := \overline{i(W)} \subset \mathcal{A}_V^G.$$

## 3. First proof of Theorem 1.2: resolutions of principal series

In this section, we derive Theorem 1.2 from the results of Bunke and Olbrich [1998], adapting the arguments of Section 9 therein to nonspherical principal series. The

two essential ingredients of this argument are the following points established by Bunke and Olbrich, which we shall use as "black boxes":

- acyclicity of $\Gamma$ acting on spaces of moderate growth functions on $G/K$, and
- surjectivity of a Laplace-type operator acting on these spaces.

The first point, at least, is intuitively reasonable: it asserts that moderate growth functions on $G/K$ behave like a free $\Gamma$-module; this is plausible since the $\Gamma$-action on $G/K$ is (at least, virtually) free.

Given these, the idea of the argument for Theorem 1.2 is as follows. We will first show that principal series representations are realized as spaces of moderate growth Laplacian eigenfunctions on $G/K$; by the two points mentioned above, this gives a resolution of the principal series by $\Gamma$-acyclic modules. This permits us to compute cohomology of principal series representations. Finally, every irreducible representation is realized as a subquotient of such a representation, and we will then prove Theorem 1.2 by a study of the associated long exact sequence in cohomology.

**3.1.** ***Setup.*** Fix a Casimir eigenvalue $\lambda$, and a lattice $\Gamma \subset G$. Given $\zeta$ a one-dimensional representation of $K$, define the following spaces of smooth functions (compare with Section 2.4, and see (13) in particular for the notion of moderate growth, which is *not* the same as uniform moderate growth):

$$\begin{aligned} \mathcal{A}^G \text{ (resp. } \mathcal{A}) &= \text{moderate growth functions on } G \text{ (resp. on } \Gamma\backslash G), \\ \mathcal{A}^G_\zeta, \mathcal{A}_\zeta &= \text{subspace with right } K\text{-type } \zeta\colon\ f(gk) = f(g)\zeta(k), \\ \mathcal{A}^G_\zeta(\lambda), \mathcal{A}_\zeta(\lambda) &= \text{subspace with right } K\text{-type } \zeta \text{ and Casimir eigenvalue } \lambda, \\ \mathrm{Cusp}_\zeta(\lambda) &= \text{subspace of } \mathcal{A}_\zeta(\lambda) \text{ consisting of cuspforms.} \end{aligned} \tag{14}$$

We will first prove a variant of Theorem 1.2 for principal series. Let $B$ be the preimage of the upper triangular matrices inside $G$, which we recall is the $q$-fold cover of $\mathrm{SL}_2(\mathbb{R})$; we may write

$$B = MAN,$$

where $A$ and $N$ are as in (10), and $M = Z_K(A) \simeq \mathbb{Z}/2q\mathbb{Z}$. Denote by $\xi \in \mathbb{C}$ the character of $A$ sending $a_y \mapsto y^\xi$. Given a pair of characters $(\sigma, \xi)$ of $K$ and $A$ respectively, let

$$H = \{f \in C^\infty(G) \mid f(mang) = a^{\xi+1}\sigma^{-1}(m)f(g),\ f \ K\text{-finite}\} \tag{15}$$

be the Harish-Chandra module of $K$-finite vectors in the corresponding principal series representation. This depends on $\sigma$ and $\xi$, but to simplify the notation we will not include them explicitly. We denote by $H_{-\infty}$ its distributional completion (see Section 2.4); explicitly, if we identify $H$ as above with functions on $K$ which transform on the left under the character $\sigma^{-1}$, then $H_{-\infty}$ is the corresponding space of *distributions* on $K$.

Let us explicate this in the language of Section 2.3. We will parameterize $\sigma$ by the value of $d\sigma$ at $\kappa$; this is a class in $q^{-1}\mathbb{Z}$ that we will denote by $\zeta_0$. A $K$-basis of $H$ is given by vectors $e_\zeta$ with $\zeta \in \zeta_0 + 2\mathbb{Z}$, normalized to take value 1 at the identity of $G$. The actions of raising and lowering operators are given by

$$pe_\zeta = \tfrac{1}{2}(\zeta + 1 + \xi)\, e_{\zeta+2} \quad \text{and} \quad me_\zeta = \tfrac{1}{2}(-\zeta + 1 + \xi)\, e_{\zeta-2}, \tag{16}$$

and the action of the Casimir on $e_\zeta$ is thereby given by $\frac{1}{2}(\xi^2 - 1)$. From these explicit formulas we readily deduce the following statements:

(a) If $1 + \xi$ does not belong to $\pm\zeta_0 + 2\mathbb{Z}$, then $H$ is irreducible.

(b) If $1 + \xi$ belongs to either $\zeta_0 + 2\mathbb{Z}$ or $-\zeta_0 + 2\mathbb{Z}$ but not both, then $H$ has the structure

$$0 \to \overline{V} \to H \to V \to 0, \tag{17}$$

where $\overline{V}$, $V$ are irreducible $(\mathfrak{g}, K)$-modules; $\overline{V}$ is the module of highest (lowest) weight $\zeta$ according to whether $-\xi - 1$ or $1 + \xi$ belongs to $\zeta_0 + 2\mathbb{Z}$, and $V = \overline{V}^{\mathrm{cl}}$.

(c) If $1 + \xi$ belongs to both[1] $\zeta_0 + 2\mathbb{Z}$ and $-\zeta_0 + 2\mathbb{Z}$, and $\xi \geq 1$, then $H$ has the structure of an extension

$$V^+ \oplus V^- \to H \to F,$$

where $V^-$ is the highest weight representation of weight $-\xi - 1$, and $V^+$ the lowest weight representation of weight $\xi + 1$, whereas $F$ is the finite-dimensional representation of dimension $\xi$ with weights $-\xi + 1, -\xi + 3, \ldots, \xi - 1$. A similar dual description is valid when $\xi \leq 0$, where the finite-dimensional representation now occurs as a subrepresentation.

In the following proposition, we will assume that we are in either cases (a) or (b) of the above classification, that is, $H$ is either irreducible, or decomposes as

$$0 \to \overline{V} \to H \to V \to 0, \tag{18}$$

where both the subrepresentation and quotient are irreducible $(\mathfrak{g}, K)$-modules.

**Proposition 3.1.** *Let $G$ be the degree $q$ connected cover of $\mathrm{SL}_2(\mathbb{R})$. Denote by $\lambda$ the eigenvalue by which $\mathcal{C}$ acts on $H_K$; then there are natural isomorphisms*

$$H^0(\Gamma, H_{-\infty}) \simeq \mathcal{A}_\zeta(\lambda), \quad H^1(\Gamma, H_{-\infty}) \simeq \mathrm{Cusp}_\zeta(\lambda),$$
$$H^i(\Gamma, H_{-\infty}) = 0 \quad \textit{for } i \geq 2,$$

*where $\zeta$ is any $K$-weight generating the dual $(\mathfrak{g}, K)$-module $H^*$.*

The condition on $\zeta$ is automatic when $H$ is irreducible, and in the case when $H$ is reducible is equivalent to asking that $\zeta$ belongs to the $K$-weights of $\overline{V}^*$.

[1] This happens only when $\zeta_0 \in \mathbb{Z}$, and in particular the representation descends to a representation of $\mathrm{SL}_2(\mathbb{R})$.

*Proof.* In Section 9 of [Bunke and Olbrich 1998] this result is proven in the case of $q = 1$ and the trivial $K$-type. We will outline the argument to make clear that it remains valid in the situation where we now work, i.e., permitting a covering of $\mathrm{SL}_2(\mathbb{R})$ and an arbitrary $K$-type.

Fix $v^* \in H^*$ of $K$-type $\zeta$, normalized as in the discussion preceding (16). Then the rule sending $\mathcal{D} \in H_{-\infty}$ to the function $\mathcal{D}(gv^*)$ on $G$ induces an isomorphism

$$H_{-\infty} \simeq \mathcal{A}_\zeta^G(\lambda). \tag{19}$$

We will outline a direct proof of this isomorphism. Injectivity, at least, follows readily: if $\mathcal{D}$ lies in the kernel, it would annihilate the $(\mathfrak{g}, K)$-module generated by $v^*$, which is all of $H^*$, and by continuity $\mathcal{D}$ is then zero.

For surjectivity, one first checks that $K$-finite functions lie in the image of the map — that is to say, a function $f$ of fixed right and left $K$-types, and with a specified Casimir eigenvalue, occurs in the image of the map above. Such an $f$ is uniquely specified up to constants: using the decomposition $G = KAK$, the Casimir eigenvalue amounts to a second-order differential equation for the function $y \mapsto f(a_y)$ for $y \in (1, \infty)$, and of the two-dimensional space of solutions only a one-dimensional subspace extends smoothly over $y = 1$; see [Kitaev 2017, pp. 12 and 13] for an explicit description both of the differential equation and a hypergeometric basis for the solutions.[2] It follows from this uniqueness that $f$ must agree with $\mathcal{D}(gv^*)$ where $\mathcal{D}$ and $v^*$ match the left and right $K$-types of $f$. To pass from surjectivity onto $K$-finite vectors to surjectivity, we take arbitrary $f \in \mathcal{A}_\zeta^G(\lambda)$ and expand it as a sum $\sum_\xi f_\xi$ of left $K$-type. Each $f_\xi$ has a preimage $v_\xi$ according to the previous argument; so one must verify that $\sum_\xi v_\xi$ converges inside $H_{-\infty}$, and for this it is enough to show that $\|v_\xi\|$ grows polynomially with respect to $|\xi|$ (here we compute $\|v_\xi\|$ as the $L^2$-norm restricted to $K$ in (15)). For this we "effectivize" the previous argument: The moderate growth property of $f$ implies a bound of the form $|f_\xi(g)| \le c\|g\|^N$, uniform in $\xi$. On the other hand, $f_\xi = v_\xi(gv^*)$, and such a matrix coefficient always is not too small:

$$|v_\xi(gv^*)| \ge (1+|\xi|)^{-M}\|v_\xi\| \quad \text{for } \textit{some} \text{ choice of } \|g\| \le (1+|\xi|)^M. \tag{20}$$

Such lower estimates on matrix coefficients can be obtained by keeping track of error bounds in asymptotic expressions. They are developed in greater generality in the Casselman–Wallach theory, see, e.g., Corollary 12.4 of [Bernstein and Krötz 2014] for a closely related result. Combining (20) with the upper bound on $f_\xi$ shows that $\|v_\xi\| \le c(1+|\xi|)^{MN+M}$ as desired.

[2] There are other references in the mathematical literature but Kitaev explicitly considers the universal cover.

This concludes our sketch of proof of (19), that is to say, $H_{-\infty}$ is the kernel of

$$\mathcal{A}_\zeta^G \xrightarrow{\mathcal{C}-\lambda} \mathcal{A}_\zeta^G \tag{21}$$

in the notation of (14). We now invoke surjectivity of a Laplace operator: the morphism $\mathcal{C}-\lambda$ of (21) is surjective, by [Bunke and Olbrich 1998, Theorem 2.1]; and consequently (21) is in fact a resolution of $H_{-\infty}$. Moreover, [Bunke and Olbrich 1998, Theorem 5.6] asserts that the higher cohomology of $\Gamma$ acting on $\mathcal{A}_\zeta^G$ vanishes; it is for this argument that Bunke and Olbrich use "moderate growth" rather than "uniform moderate growth". Consequently, the $\Gamma$-cohomology of $H_{-\infty}$ can be computed by taking $\Gamma$-invariants on the complex (21):

$$(\mathcal{A}_\zeta^G)^\Gamma \xrightarrow{\mathcal{C}-\lambda} (\mathcal{A}_\zeta^G)^\Gamma.$$

Clearly, the $H^0$ here coincides with $\mathcal{A}_\zeta(\lambda)$. On the other hand, the image of $\mathcal{C}-\lambda$ contains the orthogonal complement of cusp forms (see [Bunke and Olbrich 1998, Theorem 6.3]; compare Proposition 4.1), and so the $H^1$ coincides with the cokernel of $\mathcal{C}-\lambda$ acting on cusp forms; there we can pass to the orthogonal complement and identify $H^1 \simeq \mathrm{Cusp}_\zeta(\lambda)$ as desired.[3] □

Lemmas 3.2 and 3.3 below will be useful in the sequel. We omit the proof of the first one.

**Lemma 3.2.** *Let $\zeta$ be, as in Proposition 3.1, a $K$-weight on $H^*$ which generates the latter as $(\mathfrak{g}, K)$-module; fix $v_\zeta \in H^*$ nonzero of weight $\zeta$. For any $(\mathfrak{g}, K)$-module $V$, there is an isomorphism*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H^*, V) \to V_\zeta(\lambda), \quad f \mapsto f(v_\zeta), \tag{22}$$

*where $V_\zeta(\lambda)$ is the subspace of $V_\zeta$ killed by $\mathcal{C}-\lambda$.*

The second is a precise statement of Frobenius reciprocity, stated in a less formal way in (2).

**Lemma 3.3.** *Let $V$ be a finite length $(\mathfrak{g}, K)$-module. Then there is an isomorphism*

$$H^0(\Gamma, V_{-\infty}^*) \simeq \mathrm{Hom}_{(\mathfrak{g},K)}(V, \mathcal{A}_K),$$

*where $V_{-\infty}^*$ is the distributional globalization of $V^*$.*

One of the earliest versions of such a statement can be found in [Gelfand et al. 1969, Chapter 1, Section 4]. For completeness we outline the proof, in our language, in Remark 4.3.

For reducible principal series as in (18), we prove:

[3] In fact, $\mathcal{C}-\lambda$ is adjoint to $\mathcal{C}-\bar{\lambda}$, but the kernel of the latter of either is only nonzero if $\lambda$ is real, so we do not keep track of the complex conjugate.

**Proposition 3.4.** *Let $H_{-\infty}$ with Casimir eigenvalue $\lambda$ decompose as in (18). Then the quotient map $H \to V$ induces an isomorphism, after passing to distribution globalizations and $\Gamma$-cohomology,*

$$H^1(\Gamma, V_{-\infty}) \simeq H^1(\Gamma, H_{-\infty}) (\simeq \mathrm{Cusp}_\zeta(\lambda), \textit{ by Proposition 3.1}).$$

*Proof.* The discussion around (17) shows that inverting both $\xi$ and $\sigma$ gives rise to another principal series $\overline{H}$ which fits into the exact sequence

$$0 \to V \to \overline{H} \to \overline{V} \to 0, \tag{23}$$

i.e., for which the roles of subrepresentation and quotient are swapped between $\overline{V}$ and $V$. We will deduce the result by playing off Proposition 3.1 applied to (the distribution globalization of) $H$, and the same Proposition applied to $\overline{H}$.

We first consider the long exact sequence associated to (the distribution globalization of) (18), namely

$$\begin{aligned} 0 \to H^0(\Gamma, \overline{V}_{-\infty}) \to H^0(\Gamma, H_{-\infty}) \xrightarrow{\Omega} H^0(\Gamma, V_{-\infty}) \\ \to H^1(\Gamma, \overline{V}_{-\infty}) \to H^1(\Gamma, H_{-\infty}) \xrightarrow{\Pi} H^1(\Gamma, V_{-\infty}) \to 0. \end{aligned} \tag{24}$$

We have used here that the next group $H^2(\Gamma, \overline{V}_{-\infty})$ of the sequence vanishes: it is isomorphic to $H^3(\Gamma, V_{-\infty})$ by the long exact sequence associated to (23) and Proposition 3.1, and that $H^3$ vanishes always. Indeed, let $\overline{\Gamma}$ be the image of $\Gamma \to \mathrm{PSL}_2(\mathbb{R})$, and $\mu \leqslant \Gamma$ the kernel of $\Gamma \to \overline{\Gamma}$; if $V$ is a $\mathbb{C}[\Gamma]$-module then $H^i(\Gamma, V) = H^i(\overline{\Gamma}, V^\mu)$, and being a lattice in $\mathrm{PSL}_2(\mathbb{R})$, the virtual cohomological dimension of $\overline{\Gamma}$ is at most 2.

We must show that the penultimate map $\Pi$ of (24) is an isomorphism. For this it is enough to show that

$$\dim \mathrm{cokernel}\, \Omega \geq \dim H^1(\Gamma, \overline{V}_{-\infty}).$$

By applying Proposition 3.1 to $\overline{H}$, we find that $H^1(\Gamma, \overline{V}_{-\infty})$ is a quotient of $\mathrm{Cusp}_\chi(\lambda)$, for $\chi$ a weight in $V^*$. It therefore suffices to show that

$$\dim \mathrm{cokernel}\, \Omega \geq \dim\ \mathrm{Cusp}_\chi(\lambda). \tag{25}$$

We will prove this by exhibiting a subspace

$$H^0(\Gamma, V_{-\infty})^{\mathrm{cusp}} \subset H^0(\Gamma, V_{-\infty}) \tag{26}$$

of the codomain of $\Omega$, which does not meet the image of $\Omega$, and whose dimension equals that of $\mathrm{Cusp}_\chi(\lambda)$.

The space $H^0(\Gamma, V_{-\infty})$ is identified, by means of Frobenius reciprocity (see Lemma 3.3) with the space of homomorphisms from the dual $(\mathfrak{g}, K)$-module $V^*$ to the $K$-finite vectors $\mathcal{A}_K$ in the space of automorphic forms. Define $H^0(\Gamma, V_{-\infty})^{\mathrm{cusp}}$

to be the subspace corresponding to homomorphisms $V^* \to \mathcal{A}_K$ that are actually valued in cusp forms. We now show the two properties of this subspace $H^0(\Gamma, V_{-\infty})^{\mathrm{cusp}}$ asserted after (26):

• Its dimension equals that of $\mathrm{Cusp}_\chi(\lambda)$. To see this, apply Lemma 3.2 to $\overline{H}$, with $\zeta = \chi$ and $V$ the $K$-finite vectors of the space of cusp forms; it yields an isomorphism

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H^*, \mathrm{Cusp}_K) \simeq \mathrm{Cusp}_\chi(\lambda).$$

But homomorphisms from $H^*$ to $\mathrm{Cusp}_K$ factor through $V^*$ by semisimplicity of the space of cusp forms (which in turn follows by unitarity). This shows that the space $\mathrm{Hom}_{(\mathfrak{g},K)}(V^*, \mathrm{Cusp}_K)$ has the same dimension as $\mathrm{Cusp}_\chi(\lambda)$, as required.

• It intersects trivially the image of $\Omega$. This amounts to the statement that no homomorphism from $V^*$ to $\mathrm{Cusp}_K$ can be extended to a homomorphism from $H^*$ to $\mathcal{A}_K$. Suppose, then, that $f : H^* \to \mathcal{A}_K$ is a $(\mathfrak{g}, K)$-module homomorphism whose restriction to $V^*$ is nonzero and has cuspidal image. We now make use of the orthogonal projection map from all automorphic forms to cusp forms, which exists because one can sensibly take the inner product of a cusp form with any function of moderate growth. Post-composing $f$ with this projection gives a morphism from $H^*$ to the semisimple $(\mathfrak{g}, K)$-module $\mathrm{Cusp}_K$; since $H^*$ is a nontrivial extension of $\overline{V}^*$ by $V^*$, this morphism is necessarily trivial on the subrepresentation $V^*$, a contradiction. □

Now let us deduce Theorem 1.2. We divide into three cases according to how the representation $W$ of the theorem can be fit into a principal series. Our division corresponds to the division (a), (b), (c) enunciated after (16), and the statements below about the structure of $W$ can all be deduced from the statements given there.

• $W$ is an irreducible principal series, equivalently, $W$ is doubly infinite. In this case, $W^{\mathrm{cl}} = W$, and combining Proposition 3.1 and Lemma 3.2 gives the statement of Theorem 1.2.

• $W$ is an irreducible subquotient of a principal series $H$ with exactly two composition factors. In this case we can suppose that $W = V^*$ with notation as in (18). In that notation we have $W^* = V$ and $W^{\mathrm{cl}} = \overline{V}^*$. Proposition 3.4 gives $H^1(\Gamma, V_{-\infty}) \simeq \mathrm{Cusp}_\zeta(\lambda)$, and Lemma 3.2 shows that $\mathrm{Cusp}_\zeta(\lambda)$ is identified with the space of $(\mathfrak{g}, K)$-homomorphisms from $H^*$ to the space of cusp forms; by semisimplicity of the target such a homomorphism factors through the irreducible quotient $\overline{V}^* = W^{\mathrm{cl}}$. This proves Theorem 1.2 in this case.

• $W$ is an irreducible subquotient of a principal series with more than two composition factors. In this case, $W$ is necessarily a highest- or lowest-weight module factoring through $\mathrm{SL}_2(\mathbb{R})$, and there is an exact sequence

$$F \to H \to \mathcal{D}, \tag{27}$$

where $F$ is finite-dimensional and $\mathcal{D}$ is the sum of $W^*$ and another highest- or lowest-weight module. Here, $W^{\mathrm{cl}} = F^* \simeq F$ and Theorem 1.2 is equivalent to the vanishing of $H^1(\Gamma, W^*_{-\infty})$. In the case of a discrete series that factors through $\mathrm{PSL}_2(\mathbb{R})$, this vanishing follows from [Bunke and Olbrich 1998, Proposition 8.2], and the remaining case of an "odd" discrete series is handled by the same argument. Namely, use the long exact sequence associated to (27); the argument of Proposition 3.1 shows that $H^1(\Gamma, H_{-\infty}) = 0$, and also $H^2(\Gamma, F) = 0$ by Poincaré duality because $F$ is nontrivial. Thus also $H^1(\Gamma, \mathcal{D}_{-\infty}) = 0$ and so its summand $H^1(\Gamma, W^*_{-\infty})$ also vanishes.

## 4. Second proof of Theorem 1.2: extensions of $(\mathfrak{g}, K)$-modules

Our original proof of Theorem 1.2 proceeds by a reduction to a computation in the category of $(\mathfrak{g}, K)$-modules. The two essential ingredients of this argument are:

(a) The Casselman–Wallach theory [Casselman 1989; Wallach 1992] which gives a canonical equivalence between suitable categories of *topological* $G$-representations and *algebraic* $(\mathfrak{g}, K)$-modules.

(b) Surjectivity of a Laplace-type operator acting, now, on spaces of moderate growth functions on $\Gamma\backslash G$.

We will not prove (a), although we will briefly sketch an elementary proof of what we use from it. We will prove (b) in the next section.

Let $\lambda$ be the eigenvalue by which the Casimir $\mathcal{C} \in Z(\mathfrak{g})$ of (9) acts on $W$ (the irreducible $(\mathfrak{g}, K)$-module from the statement of Theorem 1.2). We will use the notation $\mathcal{A}$ from (14) for the space of smooth, *uniform* moderate growth functions $f$ on $\Gamma\backslash G$, i.e., for which there exists $R$ such that for all $X \in \mathfrak{U}$,

$$\|f\|_{X,R} = \sup_{g\in G} \frac{|Xf(g)|}{\|g\|^R} < \infty \tag{28}$$

(compare with (13), and beware that we are using the same notation as in Section 3, but for a slightly different space). We use *uniform* moderate growth because it interfaces more readily with the Casselman–Wallach theory; by contrast, Section 3 used moderate growth because this is used in the acyclicity result mentioned after (21).

Also consider the following subspaces of $\mathcal{A}$:

$$\mathcal{A}_{\lambda-\mathrm{nil}} = K\text{-finite functions on which } \mathcal{C} - \lambda \text{ acts nilpotently},$$
$$\mathrm{Cusp}(\lambda) = \text{subspace of } \mathcal{A}_{\lambda-\mathrm{nil}} \text{ consisting of cusp forms}.$$

The precise form of (b) we will use is this:

**Proposition 4.1.** *The image of the map $\mathcal{C} - \lambda : \mathcal{A}_K \to \mathcal{A}_K$ is precisely the orthogonal complement to* $\mathrm{Cusp}(\lambda)$ *inside $\mathcal{A}_K$.*

This is almost [Bunke and Olbrich 1998, Theorem 6.3], except there the argument is for moderate growth functions rather than uniform moderate growth; they state on page 73 that the same proof remains valid in the uniform moderate growth setting. Also, Cassleman [1984, Theorem 4.4] proves, for the trivial $K$-type, that $\mathcal{C}$ is surjective on spaces of Eisenstein distributions, from which a similar result can be extracted. Since the above statement is in a sense the crux of the argument, and neither reference gives it in precisely this form, we have given a self-contained proof in Section 5. Our proof follows a slightly different strategy and is perhaps of independent interest.

**4.1.** ***Proof of Theorem 1.2: reduction to*** **$(\mathfrak{g}, K)$** ***extensions.*** We begin the proof of Theorem 1.2 assuming Proposition 4.1. This will proceed in three steps:

(i) First, using a topological version of Shapiro's lemma, we make the identification $H^1(\Gamma, W_{-\infty}^*) \simeq \mathrm{Ext}^1_G(W_\infty, \mathcal{A})$.

(ii) Next, we pass from the category of $G$-modules to that of $(\mathfrak{g}, K)$-modules and produce an isomorphism $\mathrm{Ext}^1_G(W_\infty, \mathcal{A}) \simeq \mathrm{Ext}^1_{(\mathfrak{g},K)}(W, \mathcal{A}_{\lambda-\mathrm{nil}})$.

(iii) Finally, we compute that $\mathrm{Ext}^1_{(\mathfrak{g},K)}(W, \mathcal{A}_{\lambda-\mathrm{nil}})$ is isomorphic to the promised space of cuspforms, using the explicit computations from Section 2.3.

In practice, for technical reasons, we carry out (iii) first and then show that the map of (ii) is an isomorphism.

We begin by constructing an isomorphism

$$H^1(\Gamma, W_{-\infty}^*) \simeq \mathrm{Ext}^1_G(W_\infty, \mathcal{A}), \tag{29}$$

where $W_\infty$ is the smooth globalization of $W$.

On the left, we have the ordinary group cohomology of the discrete group $\Gamma$ acting on the vector space $W_{-\infty}^*$, without reference to topology. On the right here we use a *topological* version of Ext defined as follows: present $\mathcal{A}$ as a directed union $\varinjlim \mathcal{A}(R)$ of moderate growth Fréchet $G$-representations (see Section 2.4) obtained by imposing a specific exponent of growth $R$ in (28). The right-hand side is then defined to be the direct limit $\varinjlim \mathrm{Ext}^1_G(W_\infty, \mathcal{A}(R))$, where the elements of each Ext group are represented by isomorphism classes of short exact sequences[4] $\mathcal{A}(R) \to ? \to W_\infty$, with ? a moderate growth Fréchet $G$-representation and the maps are required to be continuous.

The statement (29) is then a version of Shapiro's lemma in group cohomology. Let us spell out the relationship: for $G_1 \leq G_2$ of finite index, and $W$ a finite-dimensional

[4]Here, the notion of exact sequence is the usual one, with no reference to topology: the first map is injective, and its image is the kernel of the second, surjective map.

$G_1$-representation, Shapiro's lemma supplies an isomorphism

$$H^1(G_1, W^*) \overset{\text{(i)}}{\simeq} H^1(G_2, \mathrm{I}_{G_1}^{G_2} W^*) \overset{\text{(ii)}}{\simeq} H^1(G_2, (\mathrm{I}_{G_1}^{G_2}\mathbb{C}) \otimes W^*) \overset{\text{(iii)}}{\simeq} \mathrm{Ext}^1_{G_2}(W, \mathrm{I}_{G_1}^{G_2}\mathbb{C}). \tag{30}$$

Here $I_{G_1}^{G_2}$ is the induction from $G_1$ to $G_2$, and we used in (i) Shapiro's lemma in its standard form [Brown 1982, Chapter 3, Sections 5 and 6]; at step (ii) the projection formula $\mathrm{I}_{G_1}^{G_2} W^* \simeq \mathrm{I}_{G_1}^{G_2}\mathbb{C} \otimes W^*$, and at step (iii) the relationship between group cohomology and Ext-groups which results by deriving the relationship $\mathrm{Hom}_{G_2}(W, V) = (V \otimes W^*)^{G_2}$.

Our statement (29) is precisely analogous to the isomorphism of (30) with $\Gamma$ playing the role of $G_1$, $G$ playing the role of $G_2$, and with topology inserted. It can be proven simply by writing down the explicit maps from far left to far right in (30) and checking that they respect topology and are inverse to one another. There is only one point that is not formal: to prove that there is a well-defined map from left to right, one needs to check that the extension of $G$-representations arising in (iii) by "inflating" a cocycle $j : \Gamma \to W^*_{-\infty}$ indeed has moderate growth. This requires growth bounds on $j$, and these follow simply by writing out $j(\gamma)$, for arbitrary $\gamma \in \Gamma$, in terms of the values of $j$ on a generating set using the cocycle relation. We observe that some "automatic continuity" argument of this nature is needed, because, in the statement of (29), the topology of $W$ figures on the right-hand side but not on the left.

As the next step towards Theorem 1.2, observe that there is a natural map

$$\mathrm{Ext}^1_G(W_\infty, \mathcal{A}) \to \mathrm{Ext}^1_{(\mathfrak{g}, K)}(W, \mathcal{A}_{\lambda-\mathrm{nil}}), \tag{31}$$

where the right-hand side is taken in the category of $(\mathfrak{g}, K)$-modules.

This "natural map" associates to an extension $\mathcal{A} \to E \to W_\infty$ the underlying sequence of $K$-finite vectors in each of $\mathcal{A}, E, W_\infty$ which are annihilated by some power of $\mathcal{C} - \lambda$ (in the case of $W_\infty$, this space is exactly $W$, on which $C - \lambda$ acts trivially). That the resulting sequence remains exact follows from surjectivity of $\mathcal{C} - \lambda$ in the form of Proposition 4.1. We explicate this: one must verify that each element $w \in W$ has a preimage in $E_K$ killed by some power of $(\mathcal{C} - \lambda)$. First, take an arbitrary preimage of $w$ in $E$ and average it over $K$ to produce a preimage $e \in E_K$. Then $(\mathcal{C} - \lambda)\, e$ belongs to the image of $\mathcal{A}_K$, and can be written as $f_1 + f_2$ with $f_1 \in \mathrm{Cusp}(\lambda) \subset \ker(\mathcal{C} - \lambda)$ and $f_2 \in \mathrm{Cusp}(\lambda)^\perp$. Choose, by Proposition 4.1, a class $e' \in \mathcal{A}_K$ with $(\mathcal{C} - \lambda)\, e' = f_2$; then $e - e'$ still lifts $w$ and is now killed by $(\mathcal{C} - \lambda)$.

We will show in Section 4.2 that the right-hand side of (31) has dimension

$$m = \text{the multiplicity of } W^{\mathrm{cl}} \text{ in } \mathrm{Cusp}(\lambda),$$

and in Section 4.3 that (31) is actually an isomorphism. This will conclude the proof, remembering that the left-hand side is identified, by means of (29), with $H^1(\Gamma, W^*_{-\infty})$.

**4.2.** ***Evaluation of the*** **$(\mathfrak{g}, K)$*-ext.*** We compute the $(\mathfrak{g}, K)$-extension on the right-hand side of (33). The space $\mathrm{Cusp}(\lambda)$ decomposes as a finite direct sum of irreducible $(\mathfrak{g}, K)$-modules; this follows from the similar $L^2$ statement, see [Borel 1997, Section 8]. Since each of these irreducible summands has infinitesimal character $\lambda$, their underlying $(\mathfrak{g}, K)$-modules can belong to at most three isomorphism classes, as described in Section 2.3; among these is $W^{\mathrm{cl}}$, the "complementary $(\mathfrak{g}, K)$-module to $W$" from Definition 2.1. Accordingly we decompose

$$\mathcal{A}_{\lambda-\mathrm{nil}} = \mathrm{Cusp}(\lambda)^{\perp} \oplus (W^{\mathrm{cl}})^m \oplus \bigoplus_{\substack{V \subset \mathrm{Cusp}(\lambda) \\ V \not\simeq W^{\mathrm{cl}}}} V, \tag{32}$$

where $\mathrm{Cusp}(\lambda)^{\perp}$ is the orthogonal complement of $\mathrm{Cusp}(\lambda)$ within $\mathcal{A}_{\lambda-\mathrm{nil}}$, and $m$ is the multiplicity of $W^{\mathrm{cl}}$ in $\mathrm{Cusp}(\lambda)$.

The splitting (32) induces a similar direct sum splitting of $\mathrm{Ext}^1_{(\mathfrak{g},K)}(W, \mathcal{A}_{\lambda-\mathrm{nil}})$. But Proposition 4.1 implies that $\mathcal{C} - \lambda$ defines a surjection from $\mathrm{Cusp}(\lambda)^{\perp}$ to itself, and so, applying Proposition 2.2,

$$\mathrm{Ext}^1_{(\mathfrak{g},K)}(W, \mathrm{Cusp}(\lambda)^{\perp}) = 0.$$

The remaining two summands evaluate via the second part of Proposition 2.2 to $\mathbb{C}^m$ and 0 respectively. This yields

$$\mathrm{Ext}^1_{(\mathfrak{g},K)}(W, \mathcal{A}_{\lambda-\mathrm{nil}}) \simeq \mathbb{C}^m.$$

This concludes the proof that the right-hand side of (31) has dimension $m$.

**4.3.** ***Comparison of topology and*** **$(\mathfrak{g}, K)$** ***extensions.*** To conclude, we must show that the map of (31) is in fact an *isomorphism*.

Injectivity of the resulting map on Ext-groups follows using the Casselman–Wallach theory of canonical globalization; the result is formulated in exactly the form we need in [Bernstein and Krötz 2014, Proposition 11.2], namely, a splitting at the level of $(\mathfrak{g}, K)$-modules automatically gives rise to a continuous splitting.[5]

For surjectivity, one cannot directly apply the Casselman–Wallach theory because $\mathcal{A}$ is "too big". However, we saw in Section 4.2 that the right-hand side of (31) actually is generated by the image of $\mathrm{Ext}^1_{(\mathfrak{g},K)}(W, \mathrm{Cusp}(\lambda))$. The space $\mathrm{Cusp}(\lambda)$ has finite length, and then the results of [Casselman 1989] (in the form of the equivalence of categories, see [Wallach 1992, Corollary, Section 11.6.8]) implies that each such extension of $(\mathfrak{g}, K)$-modules arises from an extension of smooth globalizations, which readily implies the desired surjectivity.

[5]We sketch the idea of the argument to emphasize that what we use is relatively elementary: Given an abstract $(\mathfrak{g}, K)$-module splitting $\varphi : W \to \mathcal{A}$ we must show that it does not distort norms too far. Fixing a generating set $w_1, \ldots, w_r$ for $W$, one shows using bounds similar to (20) that any vector $w \in W$ can be written as $\sum h_i \star w_i$ where $h_i$ are bi-$K$-finite functions on $G$ and the norms of the $h_i$ are not too large in terms of the norms of $w$. This permits one to bound the size of $\varphi(w) = \sum h_i \star \varphi(w_i)$.

**Remark 4.2.** Together, the isomorphisms (29) and (31) give an isomorphism

$$H^1(\Gamma, W^*_{-\infty}) \simeq \mathrm{Ext}^1_{(\mathfrak{g},K)}(W, \mathcal{A}_{\lambda-\mathrm{nil}}). \tag{33}$$

The analogous statement in *all* cohomological degrees has been proved for cocompact $\Gamma$ in [Bunke and Olbrich 1997, Theorem 1.4]. However, our argument does not generalize to this case, at least in any routine way: it is not immediately clear to us how to generalize the cocycle growth argument to $H^i$ for $i > 1$.

**Remark 4.3.** For completeness, because we made use of it earlier, we outline the argument for the much easier degree 0 version of (33), i.e., "Frobenius reciprocity":

$$H^0(\Gamma, W^*_{-\infty}) \simeq \mathrm{Hom}_{(\mathfrak{g},K)}(W, \mathcal{A}_{\lambda-\mathrm{nil}}), \tag{34}$$

where we now allow $W$ to be any finite length $(\mathfrak{g}, K)$-module.

The standard construction of Frobenius reciprocity identifies $H^0(\Gamma, W^*_{-\infty})$ with continuous $G$-homomorphisms from $W_\infty$ to $\mathcal{A}$; then, restriction to $K$-finite vectors defines a class in $\mathrm{Hom}_{(\mathfrak{g},K)}(W_K, \mathcal{A}) \simeq \mathrm{Hom}_{(\mathfrak{g},K)}(W_K, \mathcal{A}_{\lambda-\mathrm{nil}})$. This restriction map is an isomorphism by the Casselman–Wallach theory [Wallach 1992, Theorem, Section 11.6.7], taking the target space to be the subspace of $\mathcal{A}$ comprising functions which are (i) by killed by a fixed large power of $(\mathcal{C}-\lambda)$ and (ii) have finite norm (28) for all $X$ and for some fixed large $R$. This proves (34).

Now (34) implies Lemma 3.3: $W$ is annihilated by an ideal of finite codimension in $Z(\mathfrak{g})$; as such, the image of any $(\mathfrak{g}, K)$-homomorphism from $W$ to moderate growth functions automatically has image inside functions of uniform moderate growth [Borel 1997, 5.6], and therefore has image in $\mathcal{A}_{\lambda-\mathrm{nil}}$.

## 5. Surjectivity of Casimir on the space of automorphic forms

The primary analytic ingredient in both proofs is the surjectivity of a Laplacian-type operator; in the first proof this is used on spaces of functions both on $G$ and on $\Gamma\backslash G$, and in the second proof it is used only on $\Gamma\backslash G$. We will now give a self-contained proof of the second version, Proposition 4.1. As noted after that proposition, this statement is essentially in the literature, but given its importance it seemed appropriate to give a self-contained proof.

We follow here the notation of Section 4; in particular, $\mathcal{A}$ is defined using the notion of *uniform* moderate growth. It is enough to show that every function orthogonal to $\mathrm{Cusp}(\lambda)$ occurs in the image of $\mathcal{C}-\lambda : \mathcal{A}_K \to \mathcal{A}_K$. The basic strategy is as follows:

(i) In Section 5.4, we decompose elements of $\mathcal{A}_K$ into functions "near the cusp" and functions of rapid decay.

(ii) In Section 5.5, we construct preimages under $\mathcal{C}-\lambda$ for functions in each subspace. Doing this "near the cusp" amounts to solving an ODE; the construction of preimages for functions of rapid decay is carried out via $L^2$-spectral theory.

Since $\mathcal{C} - \lambda$ commutes with $K$, it suffices to prove Proposition 4.1 with $\mathcal{A}_K$ replaced by its subspace $\mathcal{A}_\zeta$ with $K$-type $\zeta$. In what follows, we will regard $\zeta$ as fixed.

**5.1.** ***Cusps.*** It is convenient to fix once and for all a fundamental domain for $\Gamma\backslash G$: we take

$$\mathfrak{F} = \{z \in \mathbb{H} : d(z, i) \leq d(\gamma z, i) \text{ for all } \gamma \in \Gamma - \{e\}\}, \tag{35}$$

which describes a convex hyperbolic polygon which is (up to boundary) a fundamental domain for $\Gamma$ acting on $\mathbb{H}$; its pullback to $G$ via $g \mapsto g \cdot i$ is a fundamental domain for $\Gamma\backslash G$, which will often be denoted by the same letter. In particular, $\mathfrak{F}$ can be decomposed in the following way, where the sets intersect only along their boundary:

$$\mathfrak{F} = \mathfrak{F}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_h, \tag{36}$$

with $\mathfrak{F}_0$ compact and each $\mathcal{C}_i$ a *cusp*, that is to say, a $G$-translate of a region of the form $\{x + iy : a \leq x \leq b,\ y \geq Y_0\}$. In the Iwasawa coordinates $G = NAK$ of (10), the pullback of $\mathcal{C}_i$ to $G$ therefore has the form

$$\widetilde{\mathcal{C}_i} = g_i \cdot \{n_x a_y k : a \leq x \leq b,\ y \geq Y_0,\ k \in K\}. \tag{37}$$

The map $\widetilde{\mathcal{C}_i} \to \Gamma\backslash G$ is injective on the interior of $\widetilde{\mathcal{C}_i}$. We will often identify $\widetilde{\mathcal{C}_i}$ with its image in $\Gamma\backslash G$.

**5.2.** ***The constant term and moderate growth functions in the cusp.*** Let $f \in \mathcal{A}_\zeta$. Fix a cusp $i$; we write $\Gamma_i^N$ for $\Gamma \cap g_i N g_i^{-1}$. The constant term $f_i^N : g_i N g_i^{-1}\backslash G \to \mathbb{C}$ is defined by the rule

$$f_i^N : x \mapsto \text{average value of } f(g_i n_t g_i^{-1} x) \text{ for } t \in \mathbb{R}. \tag{38}$$

The function $f(g_i n_t g_i^{-1} x)$ is periodic in $t$ and therefore the notion of its average value makes sense. Moreover, the above map is right $G$-equivariant. A basic (and elementary) fact is that $f_i^N$ is asymptotic to $f$ inside $\widetilde{\mathcal{C}_i}$; indeed the function $f - f_i^N$ has rapid decay in $\widetilde{\mathcal{C}_i}$, as proved in [Borel 1997, 7.5]. Here, we say that a function $J : \widetilde{\mathcal{C}_i} \to \mathbb{C}$ has *rapid decay* if, for any $X_1, \ldots, X_r \in \mathfrak{g}$ and any positive integer $N$ we have

$$\sup_{\widetilde{\mathcal{C}_i}} \|g\|^N |X_1 \ldots X_r J(g)| < \infty. \tag{39}$$

Let us consider more generally functions $f$ on $G$ that are left $N$-invariant and have fixed right $K$-type $\zeta$. Such a function may be identified, by means of pullback by $y \mapsto a_y$, with a function $f$ on $\mathbb{R}_+$. The condition of the original $N$-invariant function on $G$ having finite norm under $\|\cdot\|_{X,R}$ for all $X$, with notation as in (28), is equivalent to asking that

$$\left| \left( y \frac{d}{dy} \right)^j f \right| < C_j \cdot (|y|^{-1} + |y|)^R \quad \text{for all } j. \tag{40}$$

That this condition is necessary is seen by applying (28) to $X$ a product of elements in $\mathrm{Lie}(A)$. To see that it is sufficient, we fix $\mathfrak{U}$ belonging to the universal enveloping algebra of $\mathfrak{g}$; now, for any $k \in K$, we may write $\mathfrak{U}$ as a sum of terms $\sum c_i(k)(\mathrm{Ad}(k^{-1})\mathfrak{U}_{N,i})(\mathrm{Ad}(k^{-1})\mathfrak{U}_{A,i})\mathfrak{U}_{K,i}$ where the terms belong to fixed bases for the universal enveloping algebra of $N$, $A$ and $K$ respectively, and the coefficients $c_i(k)$ are bounded independently of $k$. This permits us to bound $\mathfrak{U}f(nak)$ and we see that the bound (40) suffices.

This motivates the following definition: Fix $Y_0 > 0$ and denote by $\mathcal{P}_{\geq Y_0}$ the space of smooth functions on $\mathbb{R}$ supported in $y > Y_0$ satisfying (40) for some $R$. Because of the restriction that $y > Y_0$, this is equivalent to ask that all derivatives are "uniformly" polynomially bounded, i.e., there is $R$ such that, for all $j$, there exists a constant $C_j$ with

$$|d^j f/dy^j| < C_j(2+|y|)^{R-j}. \tag{41}$$

**5.3.** ***The subspace*** $\mathbf{Eis}_\lambda$ ***of Eisenstein series with eigenvalue*** $\boldsymbol{\lambda}$**.** To each cusp $\mathcal{C}_j$, we attach an Eisenstein series $E^j(s)$, which is an $\mathcal{A}_\zeta$-valued meromorphic function of the complex variable $s$, characterized by the fact that for $\mathrm{Re}(s) \gg 1$ it equals

$$E^j(s,g) = \sum_{\gamma \in \Gamma_N^i \backslash \Gamma} H(g_i^{-1}\gamma g)^s,$$

where $H$ is the unique function on $G$ with right $K$-type $\zeta$, invariant on the left by $N$, and on $A$ given by $a_y \mapsto y$.

The resulting vector-valued function is holomorphic when $\mathrm{Re}(s) = \frac{1}{2}$ and we denote its value at $s = \frac{1}{2} + it$ by $E_t^j$. In words, $E_t^j$ is the unitary Eisenstein series of $K$-type $\zeta$ with parameter $t \in \mathbb{R}$ attached to the $j$-th cusp of $\Gamma\backslash G$. Finally, denoting by $\lambda_t$ the eigenvalue of $\mathcal{C}$ on $E_t^j$, let

$$\mathrm{Eis}(\lambda) := \bigoplus_j \{\text{span of all Eisenstein series } E_t^j, \text{ with } t \in \mathbb{R}, \text{ such that } \lambda_t = \lambda\},$$

so that $\mathrm{Eis}(\lambda)$ is a finite-dimensional subspace of $\mathcal{A}_\zeta$ annihilated by $\mathcal{C} - \lambda$. However, if the quadratic function $t \mapsto \lambda_t - \lambda$ happens to have a double zero, we include in the above space the derivative $\frac{d}{dt}E_t^j$, for this is also annihilated by $\mathcal{C} - \lambda$. The Casimir eigenvalue of $E^j(s,g)$ is quadratic in $s$ and therefore the dimension of $\mathrm{Eis}(\lambda)$ is at most twice the number of cusps.

**5.4.** ***Decomposition of*** $\boldsymbol{\mathcal{A}_\zeta}$**.** Consider the subspace of $\mathcal{A}_\zeta$ consisting of $L^2$-eigenfunctions of the Casimir with eigenvalue $\lambda$; call this $\mathrm{Discrete}(\lambda)$.

**Lemma 5.1.** *Let* $\widetilde{\mathcal{C}_i}$ *be the cusps for a fundamental domain for the action of* $\Gamma$ *on* $G$ *as in* (36)*. Then every* $f \in \mathcal{A}_\zeta$*, perpendicular to* $\mathrm{Cusp}(\lambda)$*, can be written as the sum*

$$f = f_s + \sum_i f_{c_i}, \tag{42}$$

*where*:

(i) *The function $f_s$ is smooth, has rapid decay at all the cusps, and is perpendicular to* $\mathrm{Eis}(\lambda) \oplus \mathrm{Discrete}(\lambda)$.

(ii) *Each $f_{c_i}$ is supported in the cusp $\widetilde{\mathcal{C}_i}$ and, with reference to the identification (37)*:

$$\widetilde{\mathcal{C}_i} = g_i \cdot \{n_x a_y k : a \le x \le b,\ y \ge Y_0,\ k \in K\}$$

*has the form*

$$n_x a_y k \mapsto P(y)\zeta(k), \tag{43}$$

*where $P$ belongs to the space $\mathcal{P}_{\ge Y_0}$ described after (41).*

Observe that, although $f$ is only assumed orthogonal to cusp forms, we arrange that $f_s$ is orthogonal also to $\mathrm{Eis}(\lambda)$ and all of $\mathrm{Discrete}(\lambda)$. This is possible because there is a lot of freedom in the decomposition (42). It will be very convenient later.

*Proof.* This is a straightforward cut-off process; the only delicacy is to ensure that $f_s$ is in fact perpendicular to $\mathrm{Eis}(\lambda)$ and $\mathrm{Discrete}(\lambda)$. We start from $f_i^N$, the constant term along the $i$-th cusp as defined in (38). Take $\varphi_i, \psi_i$ smooth functions on $\mathbb{R}_+$ where:

- $\varphi_i = 0$ for $y < Y_0$ and $\varphi_i = 1$ for $y > 2Y_0$.
- $\psi_i$ is supported in $(Y_0, 2Y_0)$.

We consider $\varphi_i$ and $\psi_i$ as functions on $\widetilde{\mathcal{C}_i}$ described by the rules $g_i n_x a_y k \mapsto \varphi_i(y)$ and $g_i n_x a_y k \mapsto \psi_i(y)\zeta(k)$ respectively. Now put $f_s = f - \sum_i (\varphi_i f_i^N + \psi_i)$ so that

$$f = f_s + \sum \underbrace{(\varphi_i f_i^N + \psi_i)}_{f_{c_i}}. \tag{44}$$

We will show that, for suitable choice of $\psi_i$, (44) is the desired splitting of $f$. All the properties except perpendicularity to $\mathrm{Discrete}(\lambda) \oplus \mathrm{Eis}(\lambda)$ follow from general properties of the constant term discussed in Section 5.2. In particular, the uniform bound on the functions $P$ associated — as in (43) — to the various $f_{c_i}$ follow from the condition that $f$ has uniform moderate growth.

Observe that $\varphi_i f_i^N$ and $\psi_i$ are both perpendicular to all cuspidal functions and in particular to $\mathrm{Cusp}(\lambda)$, because they both arise from functions on $g_i N g_i^{-1} \cap \Gamma \backslash G$ which are left invariant by $g_i N g_i^{-1}$. Therefore $f_s$ is also perpendicular to $\mathrm{Cusp}(\lambda)$.

It remains to choose $\psi_i$ in such a way that $f_s$ is indeed perpendicular to the orthogonal complement of $\mathrm{Cusp}(\lambda)$ inside $\mathrm{Discrete}(\lambda) \oplus \mathrm{Eis}(\lambda)$; call this space $\widetilde{\mathrm{Eis}}(\lambda)$, as it is (potentially) a finite-dimensional enlargement of $\mathrm{Eis}(\lambda)$. To do this, for each $\mathcal{E} \in \widetilde{\mathrm{Eis}}(\lambda)$ we should have

$$\left\langle \sum_i f - \varphi_i f_i^N, \mathcal{E} \right\rangle = \sum_i \langle \psi_i, \mathcal{E}_i^N \rangle_{\widetilde{\mathcal{C}_i}}.$$

The right-hand side can be considered as a linear mapping from the vector space of possible $\psi_i$ to the finite-dimensional dual $\widetilde{\mathrm{Eis}}(\lambda)^*$ of the vector space $\widetilde{\mathrm{Eis}}(\lambda)$. It is enough to show this mapping is surjective, and for this it is enough to show that its dual is injective. But the dual map is identified with the constant term

$$\widetilde{\mathrm{Eis}}(\lambda) \to \bigoplus_i C^\infty(T_i, 2T_i), \quad \mathcal{E} \mapsto (\mathcal{E}_i^N)(g_i a_y)$$

and this is injective: if $\mathcal{E}^{N_i}$ vanished in $(T_i, 2T_i)$ then it — being real-analytic — vanishes identically; if this is so for all $i$, then $\mathcal{E}$ would be a cusp form, contradicting the definition of $\widetilde{\mathrm{Eis}}(\lambda)$. □

**5.5.** ***Surjectivity of $\mathcal{C} - \lambda$.*** We now show surjectivity of $\mathcal{C} - \lambda$ on each of the two pieces of $\mathcal{A}_\zeta$ corresponding to the decomposition of Lemma 5.1.

**5.5.1.** *Surjectivity on the cusp.*

**Lemma 5.2.** *The operator $\mathcal{C} - \lambda$ is surjective on the space of functions on $G$ which*

- *are left $N$-invariant and have fixed right $K$-type $\zeta$, and*
- *lie in the space $\mathcal{P}_{\geq Y_0}$ described before (41) when pulled back to $\mathbb{R}_+$ by means of $y \mapsto a_y$.*

*Proof.* Let $f : \mathbb{R}_+ \to \mathbb{C}$ be extended to a function $F : G \to \mathbb{C}$ by left $N$-invariance and with fixed $\kappa$-weight equal to $\zeta$, so that $F$ has the form

$$F(n a_y \exp(\theta k)) = f(y)\, e^{i\zeta\theta}.$$

Observe that for arbitrary $X_1 \in \mathfrak{n} = \mathrm{Lie}(N)$ and $X_2, \ldots, X_k \in \mathfrak{g}$ we have

$$(X_1 \ldots X_k F) \text{ is identically zero on } NA.$$

Indeed, the left-hand side is the partial derivative $\partial_{t_1} \ldots \partial_{t_k}$ of $F(n a e^{t_1 X_1} \ldots e^{t_k X_k})$ evaluated at $t_i = 0$, which vanishes since $F$ is independent of $t_1$. From this observation, it follows that the action of the operator $\mathcal{C} = \frac{1}{2}H^2 - H + 2XY$ on $f$ agrees with the action of $\frac{1}{2}H^2 - H$ on $f(y)$. Since $H$ acts on $f$ via $2y\frac{d}{dy}$, we get that $\mathcal{C} - \lambda$ acts as the differential operator

$$2y^2 \frac{d^2}{dy^2} - \lambda.$$

We show that $\mathcal{C} - \lambda$ is surjective on $\mathcal{P}_{\geq Y_0}$ explicitly, by constructing a $g$ with $(\mathcal{C} - \lambda)\, g = f$ via the method of variation of parameters.

The homogeneous solutions to the equation $\left(2y^2 \frac{d^2}{dy^2} - \lambda\right) g = 0$ are given by $y^{p_1}$, $y^{p_2}$, where the $p_i$ are roots of $2p(1-p) + \lambda = 0$. We assume that $p_1 \neq p_2$, the $p_1 = p_2$ case is similar. A solution to $(\mathcal{C} - \lambda)\, g = f$ can then be found by taking

$$g = b_1(y) y^{p_1} + b_2(y) y^{p_2},$$

where the $b_i$ satisfy

$$\frac{db_i}{dy} = (-1)^i \frac{1/2}{p_1 - p_2} f(y) y^{-p_i - 1}.$$

Taking $f$ as in (41), we take $b_i = \pm\frac{1}{2}(p_i - p_2)\int_{Y_0}^{y} f(y) y^{-p_i-1}$ for $y > Y_0$ and $b_i(y) = 0$ for $y \le Y_0$. By construction, if $f$ belongs to $\mathcal{P}_{\ge Y_0}$ then so does $b_i$ and so also $g$. □

**5.5.2.** *Surjectivity on functions of rapid decay.*

**Proposition 5.3.** *The image of the map $\mathcal{C} - \lambda : \mathcal{A}_\zeta \to \mathcal{A}_\zeta$ contains all functions of rapid decay that are orthogonal to* Eis(λ) *and* Discrete(λ).

*Proof.* Let $f$ be such a function. We fix an orthonormal basis $\{\varphi_i\}$ for the discrete spectrum of $\mathcal{C} - \lambda$ on $L^2(\Gamma\backslash G)_\zeta$, where the subscript means that we restrict to $K$-type $\zeta$. For constants $\mu_j$ depending only on the width of the various cusps, we have, following, e.g., [Borel 1997, Section 13],

$$f = \sum_i \langle f, \varphi_i\rangle \varphi_i + \mu_j \sum_j \int_{t\ge 0} \langle f, E_t^j\rangle E_t^j \, dt. \tag{45}$$

A priori this is an equality inside $L^2$. Let $\lambda_i$ and $\lambda_t$ be, respectively, the eigenvalues of $\mathcal{C} - \lambda$ on $\varphi_i$ and $E_t$; by the assumption on $f$, these are nonvanishing except when $\langle f, \varphi_i\rangle = 0$ or when $\langle f, E_t\rangle = 0$.

Define $\bar{f} \in L^2$ by the rule

$$\bar{f} = \sum_{\lambda_i \ne 0} \frac{\langle f, \varphi_i\rangle}{\lambda_i}\varphi_i + \sum_j \int_{t\in\mathbb{R}} \frac{\langle f, E_t^j\rangle}{\lambda_t} E_t^j \, dt. \tag{46}$$

It is not hard to see that the right-hand side defines an $L^2$-function: The function $\langle f, E_t^j\rangle$ is holomorphic in a neighborhood of $t \in i\mathbb{R}$, as follows from holomorphicity of $t \mapsto E_t^j$ and absolute convergence of the integral defining $\langle f, E_t^j\rangle$. Moreover, by assumption, this holomorphic function vanishes when $\lambda_t = 0$. In particular the function $\langle f, E_t^j\rangle/\lambda_t$ is holomorphic, too; this follows from what we just said if the quadratic function $t \mapsto \lambda_t$ has distinct zeroes, and in the case when it has a double zero $t_0$ we recall that the derivatives $(dE_t^j/dt)|_{t=t_0}$ also belong to Eis(λ). Therefore, the integrand in (46) is locally integrable in $t$, and then its global integrability follows from (45).

We claim that $\bar{f}$ has uniform moderate growth and

$$(\mathcal{C} - \lambda)\bar{f} = f$$

as desired.

In fact, the summation and integrals in both (45) and (46) are absolutely convergent, uniformly on compact sets, and they define functions of uniform moderate growth; moreover, any derivative $X\bar{f}$ coincides with the corresponding summation

inserting $X$ inside the sums and integrals. The proof of these claims follow from nontrivial, but relatively standard, estimates. We summarize these estimates, with references. A convenient general reference for all the analysis required is that of Iwaniec [1995]; he works only with the trivial $K$-type, but analytical issues are exactly the same if we work with a general $K$-type.

We examine the first summand of (46) first. Let $\lambda_i$ be the $(\mathcal{C}-\lambda)$-eigenvalue of $\varphi_i$. Then the easy upper bound in Weyl's law (compare [Iwaniec 1995, (7.11), Corollary 11.2] for the sharp Weyl law in the spherical case; the same proof applies with $K$-type) gives

$$\#\{i : |\lambda_i| \le T\} \le \text{const} \cdot T^2. \tag{47}$$

For any $r \ge 0$ we have an estimate

$$|\langle f, \varphi_i\rangle| \le c_r (1+|\lambda_i|)^{-r}, \tag{48}$$

arising from integration by parts and Cauchy–Schwarz (using $\|\varphi_i\|_{L^2} = 1$). Finally, there is a constant $N$ with the following property: for any invariant differential operator $X \in \mathfrak{U}$ of degree $d$, we have a bound

$$|X\varphi_i(g)| \le (1+|\lambda_i|)^{d+N} \|g\|^N. \tag{49}$$

This is derivable from a Sobolev estimate, again using the normalization $\|\varphi_i\|_{L^2} = 1$; see, e.g., [Bernstein and Reznikov 2002, (3.7)]. These estimates suffice to treat the cuspidal summand of (46).

Now we discuss the integral summand of (46). To examine absolute convergence of the integral, one reasons exactly as for cusp forms, but rather than pointwise estimates in $t$ one only looks at averages over $T \le t \le T+1$. In place of the $L^2$-normalization of $\varphi_i$ we have the estimate

$$\int_T^{T+1} \int_{\mathrm{ht}\le Y} |E_t^j(g)|^2 \ll T^2 + \log(Y),$$

where $\mathrm{ht} \le Y$ means that we integrate over the complement of the set $y \ge Y$ in each cusp. This bound is derived from the Maass–Selberg relations (compare [Iwaniec 1995, Proposition 6.8 and (6.35) and (10.9)]) and average bounds on the scattering matrix (equation (10.13) of the same reference). From this, one obtains in the same way as the cuspidal case bounds on $\int_T^{T+1} |\langle f, E_t^j\rangle|^2$ and $\int_T^{T+1} |X E_t^j|^2$ that are of the same quality as (48) and (49) and the same analysis as for the cuspidal spectrum goes through. □

**5.6.** ***Proof of the proposition.*** We now prove Proposition 4.1, that is to say, that the image of $\mathcal{C}-\lambda$ is the orthogonal complement of cusp forms. Take $f \in \mathcal{A}_\zeta$ and write $f = f_s + \sum f_{c_i}$ as in Lemma 5.1. By Lemma 5.2 and Proposition 5.3 there are functions $g_i, g \in \mathcal{A}_\zeta$ with

$$(\mathcal{C}-\lambda)\, g_i = f_{c_i}, \quad (\mathcal{C}-\lambda)\, g = f_s,$$

where, in the case of $g_i$, we use Lemma 5.2 to produce a function on $\widetilde{\mathcal{C}}_i$, and then extend it by zero to get an element of $\mathcal{A}_\zeta$. Then $g + \sum_i g_i$ is the desired preimage of $f$ under $\mathcal{C} - \lambda$. □

## 6. Interpolation and cohomology

We will recall background on the Segal–Shale–Weil representation (see [Lion and Vergne 1980] for details) necessary to explain why the foregoing results imply the interpolation formula of Radchenko and Viazovska [2019]. We have already outlined the argument in Section 1.1 and what remains is to explain in detail where the actual numbers in (4) come from.

**6.1.** ***The Weil representation.*** Let $L^2(\mathbb{R})_+$ be the Hilbert space of even square integrable functions on $\mathbb{R}$, and let $\mathcal{S}$ be the subspace of even Schwartz functions, i.e., even smooth functions $f$ such that

$$\sup_{x\in\mathbb{R}}\left|x^n \frac{d^m}{dx^m} f(x)\right| < \infty \tag{50}$$

for any pair $(m, n)$ of nonnegative integers. Let $G$ be the degree 2 cover of $\mathrm{SL}_2(\mathbb{R})$. There is a unique unitary representation of $G$ on $L^2(\mathbb{R})_+$, the Weil (or oscillator) representation, for which $\mathcal{S}$ is precisely the subspace of smooth vectors and such that the action of $\mathfrak{g}$ on $\mathcal{S}$ is given by

$$X\cdot\phi(x) = -i\pi x^2\phi(x), \quad Y\cdot\phi(x) = \frac{-i}{4\pi}\frac{\partial^2}{\partial x^2}\phi(x), \quad H\cdot\phi(x) = \left(x\frac{d}{dx} + \frac{1}{2}\right)\phi(x).$$

It then follows that $\kappa = i(X - Y)$ acts by

$$\kappa\cdot\phi(x) = \left(\pi x^2 - \frac{1}{4\pi}\frac{\partial^2}{\partial x^2}\right)\phi(x).$$

The normalization ensures that the action of $G$ is unitary and that the relation $\sigma X\sigma^{-1} = Y$ is preserved, where $\sigma : \mathcal{S} \to \mathcal{S}$ is the Fourier transform

$$\sigma(\phi)(\xi) = \hat{\phi}(\xi) := \int_{\mathbb{R}} \phi(x)\, e^{-2\pi i x\xi}\, dx.$$

Moreover, with respect to the seminorms of (50), the topological vector space $\mathcal{S}$ has the structure of a moderate growth Fréchet representation of $G$.

The vector $v_{1/2} := e^{-\pi x^2}$ has $\kappa$-weight $\frac{1}{2}$ and Casimir eigenvalue $-\frac{3}{8}$. The other $K$-finite vectors in $\mathcal{S}$ are spanned by its Lie algebra translates; they have the form $q(x)\, e^{-\pi x^2}$ for $q$ an even polynomial, and have $\kappa$-weights $\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots$.

**6.2.** ***The lattice*** $\boldsymbol{\Gamma}$**.** If $X \in \mathfrak{g}$ is nilpotent, the projection map identifies $\exp(\mathbb{R}X) \subset G$ with the corresponding 1-parameter subgroup of $\mathrm{SL}_2(\mathbb{R})$. In particular, the map $G \to \mathrm{SL}_2(\mathbb{R})$ splits over any one-parameter unipotent subgroup; thus the groups of upper and lower-triangular matrices have distinguished lifts in $G$.

In particular, the elements $e = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $f = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ defined in (3) have distinguished lifts $\tilde{e}$, $\tilde{f}$ to $G$. They act in the Weil representation by

$$\tilde{e} \cdot \phi(x) = e^{-2\pi i x^2} \phi(x), \quad \tilde{f} \cdot \phi(x) = \sigma \tilde{e} \sigma^{-1} \phi(x). \tag{51}$$

Let $\Gamma \in \mathrm{SL}_2(\mathbb{Z})$ be the subgroup freely generated by $e$ and $f$. It is the subgroup of $\Gamma(2)$ whose diagonal entries are congruent to 1 mod 4, and is conjugate to $\Gamma_1(4)$.

**Lemma 6.1.** *There is a splitting $\Gamma \to G$ which extends the splitting over the two subgroups $\langle e \rangle$ and $\langle f \rangle$. The image of $\Gamma$ in this splitting are precisely the elements of its preimage leaving fixed the distribution $\mathcal{Q} := \sum_{n \in \mathbb{Z}} \delta_{n^2}$; see Section 1.1 for the definition of $\delta_n$.*

*Proof.* The lift $\tilde{e}$ of $e$ to $G$ fixes $\mathcal{Q}$. By Poisson summation, so does the lift $\tilde{f}$ of $f$. The group $\widetilde{\Gamma}$ The group $\widetilde{\Gamma}$ generated by $\tilde{e}$ and $\tilde{f}$ surjects onto $\Gamma$ with kernel of size at most two. But $\widetilde{\Gamma}$ fixes $\mathcal{Q}$, and the two lifts of any $g \in \mathrm{SL}_2(\mathbb{R})$ to $G$ act on $\mathcal{S}$ by different signs, so the map $\widetilde{\Gamma} \to \Gamma$ is injective. □

**6.3.** ***Conclusion of the proof.*** We now fill in the deduction, already sketched in the introduction, of the interpolation Theorem 1.1 from Theorem 1.2.

We first handle a detail of topology from the discussion of Section 1.1, namely, the equivalence between the interpolation statement and its "dual" form. For a Fréchet space $F$ we denote its continuous dual by $F^*$; we regard it as an abstract vector space without topology. Then, for $\eta : E \to F$ a continuous map of Fréchet spaces,

$$\text{if } \eta^* : F^* \to E^* \text{ is bijective, then } \eta \text{ is a homeomorphism.} \tag{52}$$

Indeed, following [Trèves 1967, Theorem 37.2], a continuous homomorphism $\eta : E \to F$ of Fréchet spaces is surjective if $\eta^*$ is injective and its image is weakly closed. Applying this in the situation of (52), we see at least that $\eta$ is surjective. It is injective because the image of $\eta^*$ is orthogonal to the kernel of $\eta$, and then we apply the open mapping theorem to see that it is a homeomorphism.

To verify the equivalence, asserted in Section 1.1, between Theorem 1.1 and its dual version, we apply (52) to the map $\Psi$ of Theorem 1.1, with codomain the closed subspace of $\mathfrak{s} \oplus \mathfrak{s}$ defined by $\sum_{n \in \mathbb{Z}} \phi(n) = \sum_{n \in \mathbb{Z}} \hat{\phi}(n)$.

The other point that was not proved in Section 1.1 was (4), the actual evaluation of $H^0$ and $H^1$ for the dual of the oscillator representation, namely

$$\dim H^0(\Gamma, \mathcal{S}^*) = 1, \quad \dim H^1(\Gamma, \mathcal{S}^*) = 0. \tag{53}$$

Now, $\mathcal{S}^*$ is precisely the distribution globalization of the dual of $\mathcal{S}_K$, i.e., it is the $W^*_{-\infty}$ of the statement of Theorem 1.2 if we take $W$ to be $\mathcal{S}_K$. Therefore Theorem 1.2 reduces us to showing that the multiplicity of $\mathcal{S}_K$ (resp. $\mathcal{S}_K^{\mathrm{cl}}$) in the space of automorphic forms (resp. cusp forms) for $\Gamma$ equals 1 (resp. 0).

From Section 6.1, the $K$-finite vectors $\mathcal{S}_K$ are a realization of the $(\mathfrak{g}, K)$-module of lowest weight $\frac{1}{2}$, whose complementary representation $(\mathcal{S}_K)^{\mathrm{cl}}$ is the $(\mathfrak{g}, K)$-module of highest weight $-\frac{3}{2}$. In general, a homomorphism from a lowest weight $(\mathfrak{g}, K)$-module to any $(\mathfrak{g}, K)$-module $W$ is uniquely specified by the image of the lowest weight vector, which can be an arbitrary element of $W$ killed by $m$; and the dual statement about highest weight modules is also valid.

It follows that $(\mathfrak{g}, K)$-homomorphisms from $\mathcal{S}_K$ (respectively $\mathcal{S}_K^{\mathrm{cl}}$) to the space $\mathcal{A}$ of automorphic forms correspond exactly to holomorphic forms of weight $\frac{1}{2}$ (respectively, antiholomorphic forms of weight $-\frac{3}{2}$); the conditions of being killed by $m$ or $p$ precisely translate to being holomorphic or antiholomorphic. The desired conclusion (53) now follows from:

**Lemma 6.2.** (a) *The space of holomorphic forms for* $\Gamma$ *of weight* $\frac{1}{2}$ *is one-dimensional, and the space of cuspidal holomorphic forms of this weight is trivial.*

(b) *The space of* ***cuspidal*** *holomorphic forms for* $\Gamma$ *of weight* $\frac{3}{2}$ *is trivial; therefore, the space of cuspidal antiholomorphic forms for* $\Gamma$ *of weight* $-\frac{3}{2}$ *is also trivial.*

*Proof.* For (a), the group $\Gamma$ is conjugate to $\Gamma_1(4)$, for which the space of modular forms of weight $\frac{1}{2}$ is spanned by the theta series $\theta_{1/2}(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i z n^2}$ [Serre and Stark 1977].

For (b), we use the fact that multiplication by $\theta$ injects the space of weight $\frac{3}{2}$ forms into the space of weight 2 forms. The space of weight 2 cusp forms for $\Gamma_1(4)$ is, however, trivial; indeed, the compactified modular curve $X_1(4)$ has genus zero. The final assertion follows by complex conjugation. □

**6.4.** ***Variants: odd Schwartz functions, higher dimensions, Heisenberg uniqueness.*** We now show how the same ideas give several other interpolation theorems without changing the group $\Gamma = \langle e, f\rangle$; it may also be of interest to consider $(\infty, p, q)$-triangle groups.

**6.4.1.** *Odd Schwartz functions.* The discussion of Section 6.1 on the even Weil representation $\mathcal{S}$ carries verbatim to its odd counterpart $\mathcal{T}$, whose $(\mathfrak{g}, K)$-module of $K$-finite vectors is spanned by the translates of the lowest weight vector $v_{3/2} = xe^{-\pi x^2}$. As above, we compute using Theorem 1.2, to get

$$H^0(\Gamma, \mathcal{T}^*) = \mathbb{C}, \quad H^1(\Gamma, \mathcal{T}^*) = 0.$$

Indeed, the zeroth cohomology $H^0(\Gamma, \mathcal{T}^*)$ is identified with the space of modular forms of weight $\frac{3}{2}$, a one-dimensional space spanned by $\theta^3$, as can be deduced from [Cohen and Oesterlé 1977]. As for $H^1(\Gamma, \mathcal{T}^*)$, its dimension is equal to the multiplicity of $\mathcal{T}^{\mathrm{cl}}$ in the space of cusp forms on $\Gamma$. The representation $\mathcal{T}^{\mathrm{cl}}$ has highest weight $-\frac{1}{2}$, and the vanishing of $H^1$ results from the absence of holomorphic

cusp forms of weight $\frac{1}{2}$ on $\Gamma$ as in Lemma 6.2. We then deduce an interpolation theorem as in Section 1, noting that in addition to the $\delta_n$ the distributions $\phi \mapsto \phi'(0)$ (resp. $\phi \mapsto \hat{\phi}'(0)$) are also $e$- (resp. $f$-)invariant. Arguing as in Section 1.1 recovers a nonexplicit version of the interpolation theorem of Radchenko and Viazovska for odd Schwartz functions, see [Radchenko and Viazovska 2019, Theorem 7].

**6.4.2.** *Radial Schwartz functions on* $\mathbb{R}^d$. We may, similarly, consider instead the representation $\mathcal{S}_d$ of $\mathrm{SL}_2(\mathbb{R})$ on radial Schwartz functions on $\mathbb{R}^d$. This is, for reasons very similar to that enunciated in Section 6.1, a lowest weight representation of the double cover of $\mathrm{SL}_2(\mathbb{R})$, but now of lowest weight $\frac{d}{2}$ generated by $e^{-\pi(x_1^2+\cdots+x_d^2)}$. We claim that in all cases the corresponding $H^1$ continues to vanish. Indeed, for $d$ even the complementary representation $W^{\mathrm{cl}}$ is finite-dimensional and does not occur in cusp forms; for $d$ odd, occurrences of $W^{\mathrm{cl}}$ in cusp forms correspond just as before to holomorphic cusp forms of weight $\frac{1}{2}(4-d)$ for $\Gamma(2)$, and these do not exist for any odd $d$. Therefore we find that the values of $f$ and $\hat{f}$ at radii $\sqrt{n}$ determine $f$, subject only to a finite-dimensional space of constraints (the dimension is equal to that of weight $\frac{d}{2}$ holomorphic forms for $\Gamma(2)$).

**6.4.3.** *Heisenberg uniqueness.* A result of Hedenmalm and Montes-Rodríguez [2011] asserts that the map

$$L^1(\mathbb{R}) \to \text{sequences}, \quad h \mapsto \int h(t)\, e^{\pi i \alpha n t}\, dt, \int h(t)\, e^{\pi i \beta n/t}\, dt \tag{54}$$

is injective if and only if $\alpha\beta \le 1$. In their terminology, this yields an example of a "Heisenberg uniqueness pair". We thank the referee for bringing this result to our attention. Using our techniques, we show that an abstract interpolation formula — admittedly, on a eccentric function space — holds at the transition point $\alpha\beta = 1$.

**Theorem 6.3.** *Let $\mathcal{H}$ be the space of smooth functions on $\mathbb{R}$ with the property that $x^{-2}h(x^{-1})$ extends from $\mathbb{R}-\{0\}$ to a smooth function on $\mathbb{R}$. Fix $\alpha$, $\beta$ with $\alpha\beta = 1$ and for $n \in \mathbb{Z}$ write $a_n = \int h(t)\, e^{\pi i \alpha n t}\, dt$ and $b_n = \int h(t)\, e^{\pi i \beta n/t}\, dt$. Then the map*

$$h \mapsto \Big((a_n), (b_n), h(0), \lim_{x\to\infty} x^2 h(x)\Big)$$

*defines a linear isomorphism of $\mathcal{H}$ with a codimension 3 subspace $S$ of*[6] $\mathfrak{s}^2 \oplus \mathbb{C}^2$.

In this form, this neither implies nor is implied by the results of [Hedenmalm and Montes-Rodríguez 2011], but it would be interesting to see if our methods can give results closer to theirs, e.g., by considering different completions of the underlying representation.

We obtain Theorem 6.3 in a similar way to Theorem 1.1 — namely, by applying Theorem 1.2 for the same $\Gamma$, but with a different coefficient system. Note that we

[6]Note that integration by parts shows that $a_n, b_n$ indeed belong to the space $\mathfrak{s}$ of sequences with rapid decay, introduced in Section 1.

can and will assume that $\alpha = \beta = 1$ by rescaling. We now consider the space $W$ of smooth 1-forms on $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \infty$, which we may think of equivalently as smooth functions $\Phi(x, y)$ on $\mathbb{R}^2 - \{0\}$ satisfying

$$\Phi(\lambda x, \lambda y) = \lambda^{-2}\Phi(x, y).$$

The 1-form on $\mathbb{P}^1_{\mathbb{R}}$ associated to $\Phi$ is characterized by the fact that, when pulled back to $\mathbb{R}^2 - \{0\}$, it gives the 1-form $\Phi(x, y)(x\,dy - y\,dx)$. Write

$$a_n(\Phi) = \int \Phi(x, 1)\, e^{\pi i n x}\, dx, \quad b_n(\Phi) = \int \Phi(1, y)\, e^{\pi i n y}\, dy. \tag{55}$$

Write $h(x) = \Phi(x, 1)$. We note that $x^{-2}h(1/x) = \Phi(1, x)$, and so extends over 0. The map $\Phi \mapsto h(x) = \Phi(x, 1)$ thus identifies $W$ with the space $\mathcal{H}$ described in the theorem. We are reduced then to proving:

**Claim.** *The rule*

$$\Phi \mapsto \big(a_n, b_n, \Phi(0, 1), \Phi(1, 0)\big) \tag{56}$$

*defines an isomorphism of $W$ with a codimension 3 subspace of $\mathfrak{s}^2 \oplus \mathbb{C}^2$.*

*Proof of Claim.* We apply Theorem 1.2 to the $(\mathfrak{g}, K)$-module $W_K$; the distribution globalization "$W^*_{-\infty}$" that appears in Theorem 1.2 is simply the topological dual $W^*$ to $W$.

To analyze the $e$-invariants on $W^*$, take an arbitrary $e$-invariant distribution $\mathcal{D}$ on $W$. The identification $\Phi \mapsto h$ between the space of $-2$-homogeneous $\Phi$ and $h \in \mathcal{H}$ contains $C_c^\infty(\mathbb{R})$ in its image; thus, we can consider $\mathcal{D}$ as a distribution on the real line, i.e., given any $h \in C_c^\infty(\mathbb{R})$, we form the corresponding $\Phi$ and evaluate $\mathcal{D}$ on it. The result is a periodic distribution under $x \mapsto x + 2$ which must be in the closed subspace spanned by the $a_n$ for $n \in \mathbb{Z}$ — write this distribution $\sum c(n)\, a_n$. Then the difference $\mathcal{D} - \sum c(n)\, a_n$ vanishes on $C_c^\infty(\mathbb{R})$, and is therefore a linear combination of the Taylor coefficients of $\Phi(1, y)$ at $y = 0$; the only such distribution that is invariant under $e$ is $\Phi \mapsto \Phi(1, 0)$. It follows that $(W^*)^e$ is spanned topologically by the $a_n$ and evaluation at $(1, 0)$. Similarly, $(W^*)^f$ is spanned topologically by the $b_n$ and evaluation at $(0, 1)$.

We will now compute the cohomology of $\Gamma$ on $W^*$.

The space $W$ is identified with a reducible principal series of $\mathrm{SL}_2(\mathbb{R})$ which is an extension $D_2^+ \oplus D_2^- \to W \to \mathbb{C}$, where $D_2^\pm$ are the holomorphic and antiholomorphic discrete series of weight 2; the map $W \to \mathbb{C}$ is the integration over $\mathbb{P}^1_{\mathbb{R}}$. Now Theorem 1.2 implies that $H^1(\Gamma, (D_2^\pm)^*)$ vanishes, whereas $H^0(\Gamma, (D_2^\pm)^*)$ has dimension 2 in both the $+$ and $-$ cases. There is therefore an exact sequence

$$0 \to \mathbb{C} \to H^0(\Gamma, W^*) \to \mathbb{C}^4 \to H^1(\Gamma, \mathbb{C}) \to H^1(\Gamma, W^*) \to 0.$$

The map $\mathbb{C}^4 \to H^1(\Gamma, \mathbb{C})$ is surjective with two-dimensional kernel, for it amounts to the map from the four-dimensional space of (holomorphic and antiholomorphic)

Eisenstein series for $\Gamma$ to the two-dimensional cohomology.[7] This proves that $H^1(\Gamma, W^*)$ vanishes, whereas $H^0(\Gamma, W^*)$ is three-dimensional.

We now apply the Mayer–Vietoris sequence (5). In our current context, it implies that $(W^*)^e$ and $(W^*)^f$ span all of $W^*$, and their intersection is precisely three-dimensional. This concludes the proof of the claim. □

It may be of interest to describe the three linear constraints that define this codimension 3 subspace. We follow the notations above. The invariants of $\Gamma$ on $W^*$ have, as basis $A, I, J$ where

$$A(\Phi) = \int_{\mathbb{P}^1} \Phi, \quad I(\Phi) = \sum_{(m,n)\neq(0,0)} \Phi(m,n) - 2\sum_{2|n} \Phi(m,n),$$
$$J(\Phi) = \sum_{(m,n)\neq(0,0)} \Phi(m,n) - 2\sum_{2|m} \Phi(m,n),$$

where in both cases the sum is conditionally convergent (e.g., one can sum over large discs of increasing radii). Then $A$ corresponds to the relation $a_0 = b_0$, whereas both $I$ and $J$ give rise to a relation by expanding the stated intertwiner in two different ways. For example, we compute $I(\Phi)$ in two ways, firstly by summing first over $n$,

$$\sum_{m\neq 0} m^{-2} \sum_n \left(\Phi\left(1, \frac{n}{m}\right) - 2\Phi\left(1, \frac{2n}{m}\right)\right) + 2\left(\sum n^{-2} - 2\sum (2n)^{-2}\right)\Phi(0,1)$$
$$\overset{\text{P.S.}}{=} -\sum_{m\neq 0,\, t\in\mathbb{Z}} |m|^{-1} b_{m(2t+1)} + \frac{\pi^2}{6}\Phi(0,1),$$

where P.S. stands for Poisson summation, and secondly by summing first over $m$,

$$\sum_{n\neq 0}\left(n^{-2}\sum_m \Phi\left(\frac{m}{n}, 1\right) - \sum_m 2(2n)^{-2}\Phi\left(\frac{m}{2n}, 1\right)\right) 2\left(\sum m^{-2} - 2\sum (m)^{-2}\right)\Phi(1,0)$$
$$\overset{\text{P.S.}}{=} \sum |n|^{-1} a_{n(4t+2)} - \frac{\pi^2}{3}\Phi(1,0).$$

Thus we find that the image of $W$ is cut out by the three relations $a_0 = b_0$,

$$\frac{\pi^2}{3}\Phi(1,0) + \frac{\pi^2}{6}\Phi(0,1) = \sum_{m\neq 0,\, t\in\mathbb{Z}} |m|^{-1} b_{m(2t+1)} + \sum_{n\neq 0,\, t\in\mathbb{Z}} |n|^{-1} a_{n(4t+2)},$$

and dually

$$\frac{\pi^2}{6}\Phi(1,0) + \frac{\pi^2}{3}\Phi(0,1) = \sum_{m\neq 0,\, t\in\mathbb{Z}} |m|^{-1} a_{m(2t+1)} + \sum_{n\neq 0,\, t\in\mathbb{Z}} |n|^{-1} b_{n(4t+2)}.$$

[7] Indeed, this map records the obstruction to extending an embedding of $D_2^+ \oplus D_2^-$ into the space of automorphic forms, to the larger space $W$. An embedding of $D_2^+ \oplus D_2^-$ into the space of automorphic forms corresponds to a pair $(f, g)$ of a holomorphic and antiholomorphic 1-form, and it extends to $W$ when $f\,dz + g\,\overline{dz}$ is an exact differential.

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# Harder's denominator problem for $\mathrm{SL}_2(\mathbb{Z})$ and its applications

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We aim to give full details about the proof given by Harder of a theorem on the denominator of the Eisenstein class for $\mathrm{SL}_2(\mathbb{Z})$ and to show that the theorem has some interesting applications including the proof of a recent conjecture by Duke on the integrality of the higher Rademacher symbols. We also present a sharp universal upper bound for the denominators of the values of partial zeta functions associated with narrow ideal classes of real quadratic fields in terms of the denominator of the values of the Riemann zeta function.



## 1. Introduction

**1.1.** ***Background.*** Our main topic are the so-called Eisenstein classes (for $\mathrm{SL}_2(\mathbb{Z})$). These are special elements in the cohomology of the modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ that have many interesting applications in number theory. In this subsection, we briefly review the background and motivation of the theory of Eisenstein classes from an elementary point of view.

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*Keywords:* Eisenstein class, special values of partial zeta functions, real quadratic fields, Duke's conjecture.

**1.1.1.** *Modular forms.* Let $k$ and $N$ be positive integers. A modular form $f(z)$ of weight $k$ and level $\Gamma_1(N)$ is a holomorphic function on the upper half-plane $\mathbb{H}$, satisfying the transformation property

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod N\right\}$, and bounded as $\mathrm{Im}(z)$ goes to $\infty$. By the transformation property, modular forms have a Fourier series expansion: one can write

$$f(z) = \sum_{m=0}^{\infty} a_m(f)\, e^{2\pi i z} = \sum_{m=0}^{\infty} a_m(f)\, q^m, \quad z \in \mathbb{H},\ q := e^{2\pi i z}.$$

If $a_0(f) = 0$, then $f$ is called a cusp form. We denote respectively by $M_k(\Gamma_1(N))$ and $S_k(\Gamma_1(N))$ the spaces of modular forms and cusp forms of weight $k$ and level $\Gamma_1(N)$.

For a modular form $f(z) = \sum_{m=0}^{\infty} a_m(f) q^m \in M_k(\Gamma_1(N))$, we have an associated $L$-function defined by

$$L(f, s) := \sum_{m=1}^{\infty} \frac{a_m(f)}{m^s}$$

which converges for $\mathrm{Re}(s) > k$ and is continued meromorphically to $s \in \mathbb{C}$. (If $f$ is a cusp form then it is known that $L(f, s)$ is holomorphic at all $s \in \mathbb{C}$.)

**Example 1.1.** Suppose that $N = 1$. (In this case $\Gamma_1(1) = \mathrm{SL}_2(\mathbb{Z})$.)

(1) Let $k \in 2\mathbb{Z}_{\geq 2}$. The first examples are the holomorphic Eisenstein series defined by

$$E_k(z) := \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ (c,d)=1}} \frac{1}{(cz+d)^k} = 1 + \frac{2}{\zeta(1-k)} \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m \in M_k(\mathrm{SL}_2(\mathbb{Z})),$$

where $\zeta(s)$ is the Riemann zeta function and $\zeta(1-k)$ is its value at $1-k$, which is known to be a nonzero rational number,[1] and $\sigma_{k-1}(m) := \sum_{0<d|m} d^{k-1}$. In this case the associated $L$-function $L(E_k, s)$ becomes

$$L(E_k, s) = \frac{2}{\zeta(1-k)} \sum_{m=1}^{\infty} \frac{\sigma_{k-1}(m)}{m^s} = 2\frac{\zeta(s)\zeta(s+1-k)}{\zeta(1-k)}.$$

(2) As an example of cusp forms, we have the discriminant function

$$\Delta(z) = \frac{1}{1728}(E_4(z)^3 - E_6(z)^2) \in S_{12}(\mathrm{SL}_2(\mathbb{Z})),$$

[1] Actually, it is well known that $\zeta(1-k) = -B_k/k$, where $B_k$ is the $k$-th Bernoulli number.

which is a cusp form of weight 12. The Fourier expansion of $\Delta(z)$ is of the form

$$\Delta(z) = q\prod_{m=1}^{\infty}(1-q^m)^{24} =: \sum_{m=1}^{\infty}\tau(m)q^m,$$

and the coefficient $\tau(m)$ is called the Ramanujan tau function. The $L$-function $L(\Delta, s) = \sum_{m=1}^{\infty}\tau(m)/m^s$ is known to satisfy the Euler product formula:

$$L(\Delta, s) = \sum_{m=1}^{\infty}\frac{\tau(m)}{m^s} = \prod_{p \text{ prime}}(1-\tau(p)p^{-s}+p^{11-2s})^{-1}.$$

This fact, which was one of the most important discoveries in number theory in the early 20th century, was first observed by Ramanujan, proved by Mordell, and generalized by Hecke to Hecke-eigenforms.

Modular forms are highly geometric and arithmetic objects, of great interest in number theory along with the study of $L$-functions. For example, the famous Shimura–Taniyama conjecture (which is a part of the Langlands program) says that the Hasse–Weil $L$-function $L(E, s)$ associated with an elliptic curve $E$ over $\mathbb{Q}$ coincides with $L(f, s)$ for some modular form $f(z)$ of weight 2. It is well known that Wiles and Taylor [Wiles 1995; Taylor and Wiles 1995] proved the Shimura–Taniyama conjecture for any semistable elliptic curve and this result implies Fermat's last theorem. (Now the Shimura–Taniyama conjecture is proved for any elliptic curve over $\mathbb{Q}$, see [Breuil et al. 2001].) In such a case, when the Hasse–Weil $L$-function of $E$ coincides with the $L$-function of $f$, the Birch and Swinnerton-Dyer conjecture (which is one of the most important problems in current number theory) is concerned with the order of zeros and the leading term of $L(f, s)$ at $s = 1$. (Here the leading term of a holomorphic function $F(s)$ at $s = a \in \mathbb{C}$ means the first nonvanishing coefficient of the Taylor expansion of $F(s)$ at $s = a$.)

More generally, for any "arithmetic object", say $X$ (e.g., number fields, algebraic varieties over number fields, modular forms, automorphic representations, Galois representations, etc.) we can consider its $L$-function $L(X, s)$ with some nice properties, such as Euler product formula, analytic continuation to $s \in \mathbb{C}$, functional equations, etc. Then it is believed that the leading terms of $L(X, s)$ at integers $k \in \mathbb{Z}$ (they are called the special values of the $L$-function $L(X, s)$) are of the form

$$\text{(interesting algebraic number)} \times \text{(interesting transcendental number)}. \qquad (1\text{-}1)$$

This is a very deep and profound philosophical concept in number theory, and the study of arithmetic properties of this algebraic part (its denominator, numerator, $p$-adic interpolation, etc.) and transcendental part (motivicity, irrationality, transcendence, etc.) is a very important research theme. See [Deligne 1979; Beilinson 1984; Kontsevich and Zagier 2001] for more detail.

In the case of modular forms, for a normalized Hecke-eigenform $f \in M_k(\Gamma_1(N))$, i.e., a simultaneous eigenvector of the Hecke operators $T_m$ on $M_k(\Gamma_1(N))$ ($m \in \mathbb{Z}_{\geq 1}$), the special values of its $L$-function $L(f, s)$ have been studied extensively and many things are known. One key point in the study of special values of $L$-functions $L(f, s)$ of modular forms is to consider the cohomological interpretation of the modular forms via the so-called Eichler–Shimura isomorphism.

**1.1.2.** *Eichler–Shimura isomorphism.* In the following, we restrict ourselves to the case of level $\Gamma_1(1) = \mathrm{SL}_2(\mathbb{Z})$ and weight $2k$ with $k \in \mathbb{Z}_{\geq 2}$. Set $\Gamma := \Gamma_1(1) = \mathrm{SL}_2(\mathbb{Z})$. We will explain briefly that there exists a natural Hecke-equivariant isomorphism

$$r : M_{2k}(\Gamma) \oplus \overline{S_{2k}(\Gamma)} \xrightarrow{\sim} H^1(\Gamma, \mathcal{M}_{2k-2}^\vee \otimes \mathbb{C}), \tag{1-2}$$

where

$$\overline{S_{2k}(\Gamma)} := \{\bar{f} \mid f \in S_{2k}(\Gamma)\}$$

is the space of antiholomorphic cusp forms ($\bar{f}$ denotes the function that sends $z \in \mathbb{H}$ to $\overline{f(z)}$),

$$\mathcal{M}_{2k-2} := \{P(X_1, X_2) \in \mathbb{Z}[X_1, X_2] \mid P \text{ is homogeneous of degree } 2k-2\}$$

is the space of homogeneous polynomials of degree $2k-2$ over $\mathbb{Z}$ with a natural action of $\Gamma$ (see Section 2.3), $\mathcal{M}_{2k-2}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}_{2k-2}, \mathbb{Z})$ is the dual $\Gamma$-module of $\mathcal{M}_{2k-2}$, and $H^1(\Gamma, \mathcal{M}_{2k-2}^\vee \otimes \mathbb{C})$ is the first group cohomology of $\Gamma$ with coefficients in $\mathcal{M}_{2k-2}^\vee \otimes \mathbb{C}$. Note that any class $\phi \in H^1(\Gamma, \mathcal{M}_{2k-2}^\vee \otimes \mathbb{C})$ is represented by a 1-cocycle

$$\phi : \Gamma \to \mathcal{M}_{2k-2}^\vee \otimes \mathbb{C} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}_{2k-2}, \mathbb{C}),$$

i.e., $\phi$ is a map satisfying $\phi(\gamma_1\gamma_2) = \gamma_1\phi(\gamma_2) + \phi(\gamma_1)$.

The Eichler–Shimura isomorphism (1-2) is defined by sending $f \in M_{2k}(\Gamma)$, $\bar{g} \in \overline{S_{2k}(\Gamma)}$ to $r(f), r(\bar{g}) \in H^1(\Gamma, \mathcal{M}_{2k-2}^\vee \otimes \mathbb{C})$ represented by cocycles $r(f), r(\bar{g}) \in \mathrm{Map}(\Gamma, \mathcal{M}_{2k-2}^\vee \otimes \mathbb{C})$ such that

$$r(f)(\gamma)(P) = \int_\tau^{\gamma\tau} f(z)P(z, 1)\, dz, \quad r(\bar{g})(\gamma)(P) = \int_\tau^{\gamma\tau} \overline{g(z)}P(\bar{z}, 1)\, d\bar{z}$$

for $\tau \in \mathbb{H}$, $\gamma \in \Gamma$, and $P \in \mathcal{M}_{2k-2}$. Note that $r(f)(\gamma)(P)$ and $r(\bar{g})(\gamma)(P)$ are independent of $\tau$ because $f$ and $g$ are modular forms. It is then known that $r$ is an isomorphism. Moreover, there is a natural action of Hecke operators on $H^1(\Gamma, \mathcal{M}_{2k-2}^\vee \otimes \mathbb{C})$ and $r$ is a Hecke-equivariant isomorphism. See, for example, [Shimura 1994, Chapter 8; Hida 1993, Chapter 6; Bellaïche 2021, Section 5.3].

Now, a remarkable point is that the right-hand side $H^1(\Gamma, \mathcal{M}_{2k-2}^\vee \otimes \mathbb{C})$ of the Eichler–Shimura isomorphism has very rich algebraic and arithmetic structures. For example, it has a natural integral structure $H^1(\Gamma, \mathcal{M}_{2k-2}^\vee)$ that is stable under the Hecke operators. Furthermore, the group cohomology $H^1(\Gamma, \mathcal{M}_{2k-2}^\vee)$ can be

interpreted as a sheaf cohomology $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}^\vee_{2k-2})$ of a modular curve $\Gamma\backslash\mathbb{H}$ with coefficients in the corresponding sheaf $\mathcal{M}^\vee_{2k-2}$ on $\Gamma\backslash\mathbb{H}$. Then since the modular curve $\Gamma\backslash\mathbb{H}$ has the structure of an algebraic curve over $\mathbb{Q}$, the module $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}^\vee_{2k-2})$ admits an action of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$ after tensoring $\mathbb{Z}_\ell$ for a prime number $\ell$.

As an application of such algebraic structures, especially the existence of an integral structure $H^1(\Gamma, \mathcal{M}^\vee_{2k-2})$, let us explain the so-called algebraicity of the special (critical) values of the $L$-function $L(f, s)$ of a normalized Hecke-eigenform $f \in M_{2k}(\Gamma)$, which is an example of a formula of the type (1-1).

First, we consider the case of cusp forms. Let $f \in S_{2k}(\Gamma)$ be a normalized Hecke-eigen-cusp form and let $f^\iota \in \overline{S_{2k}(\Gamma)}$ be an antiholomorphic modular form defined by $f^\iota(z) = -f(-\bar{z})$. (In our case $N = 1$, we actually have $f^\iota = -\bar{f}$, but $f^\iota$ is more natural and useful in the following argument.) Then the Hecke-equivariance of (1-2) and the existence of the integral structure $H^1(\Gamma, \mathcal{M}^\vee_{2k-2})$ (as well as the multiplicity one of Hecke eigenforms) imply that there exist rational Hecke eigenclasses $\phi_+, \phi_- \in H^1(\Gamma, \mathcal{M}^\vee_{2k-2} \otimes \bar{\mathbb{Q}})$ and complex numbers $\omega_+, \omega_- \in \mathbb{C}$ such that

$$\tfrac{1}{2}(r(f) \pm r(f^\iota)) = \omega_\pm \phi_\pm.$$

By evaluating these cocycles at

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad P_j(X_1, X_2) = X_1^j X_2^{2k-2-j}, \quad j = 0, \dots, 2k-2,$$

we find that

$$\begin{aligned} &r(f)(\gamma)(P_j) \\ &\quad = \lim_{\tau \to 0} \int_\tau^{\gamma\tau} f(z) z^j \, dz = \int_0^\infty f(iy)(iy)^j \, d(iy) = \frac{i^{j+1} j!}{(2\pi)^{j+1}} L(f, j+1), \end{aligned} \tag{1-3}$$

and similarly, $r(f^\iota)(\gamma)(P_j) = -((-i)^{j+1} j!/(2\pi)^{j+1}) L(f, j+1)$. Therefore, we obtain

$$\frac{j!}{2(-2\pi i)^{j+1}} (1 \pm (-1)^j) L(f, j+1) = \omega_\pm \phi_\pm(\gamma)(P_j).$$

Hence, if we set $\Lambda(f, s) := (2\pi)^{-s} \Gamma(s) L(f, s)$ (so that $\Lambda(f, s)$ satisfies the functional equation $\Lambda(f, s) = \Lambda(f, 2k-s)$), we have

$$\Lambda(f, j+1) = (-i)^{j+1} \phi_\pm(\gamma)(P_j) \omega_\pm \in i^{j+1} \omega_\pm \bar{\mathbb{Q}},$$

where $\pm = \mathrm{sgn}(-1)^j$. In particular, we see that the special values of $L(f, s)$ at $s = 1, \dots, 2k-1$ are of the form (1-1). (Note that the transcendence of $\omega_\pm$ is an open problem. See [Kohnen 1989] for example.)

**Example 1.2.** Let us consider the case $f(z) = \Delta(z)$ in Example 1.1(2). It is known that $\phi_\pm$ can be taken from rational classes: $\phi_\pm \in H^1(\Gamma, \mathcal{M}^\vee_{2k-2} \otimes \mathbb{Q})$, and $\omega_\pm$ are (up to $\mathbb{Q}^\times$) approximately

$$\omega_+ = i\,0.045751\ldots, \quad \omega_- = 0.046346\ldots.$$

Indeed, for such choices of $\omega_\pm$, we have

$$\begin{aligned}
&\Lambda(\Delta,1) = \Lambda(\Delta,11) = \frac{90\omega_+}{691i}, && \Lambda(\Delta,2) = \Lambda(\Delta,10) = \frac{2\omega_-}{25},\\
&\Lambda(\Delta,3) = \Lambda(\Delta,9) = \frac{\omega_+}{18i}, && \Lambda(\Delta,4) = \Lambda(\Delta,8) = \frac{\omega_-}{24},\\
&\Lambda(\Delta,5) = \Lambda(\Delta,7) = \frac{\omega_+}{28i}, && \Lambda(\Delta,6) = \frac{\omega_-}{30}.
\end{aligned}$$

See [Kontsevich and Zagier 2001, §3.4].

**1.1.3.** *Eisenstein classes.* We consider the case of Eisenstein series, which we will discuss further in this paper. More precisely, let us define the Eisenstein class $\mathrm{Eis}_{2k-2}$ to be the image of the holomorphic Eisenstein series $E_{2k}$ under the Eichler–Shimura isomorphism (1-2):[2]

$$\mathrm{Eis}_{2k-2} := r(E_{2k}) \in H^1(\Gamma, \mathcal{M}^\vee_{2k-2} \otimes \mathbb{C}) \simeq H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}^\vee_{2k-2} \otimes \mathbb{C}).$$

In other words, the Eisenstein class $\mathrm{Eis}_{2k-2}$ can be seen as a cohomological interpretation of the Eisenstein series $E_{2k}$.

The situation is different for the case of cusp forms: namely, if we define $\Lambda(E_{2k}, s) := (2\pi)^{-s}\Gamma(s)L(E_{2k}, s)$ (cf. Example 1.1(1)), then we know that

$$\Lambda(E_{2k}, s) = \sin\left(\tfrac{1}{2}\pi(s+1)\right)^{-1} \frac{\zeta(1-s)\zeta(s+1-2k)}{\zeta(1-2k)}$$

and

$$\Lambda(E_{2k}, 2), \Lambda(E_{2k}, 4), \ldots, \Lambda(E_{2k}, 2k-2) \in \mathbb{Q}$$

by using Euler's formula for the special values of $\zeta(s)$. (Again this is a formula of type (1-1). Note also that $\Lambda(E_{2k}, s) = 0$ for every odd integer $s \in \{3, \ldots, 2k-3\}$.) Then by using a similar argument as in the case of cusp forms in the reverse direction, we see that $\mathrm{Eis}_{2k-2}$ is a rational cohomology class:

$$\mathrm{Eis}_{k-2} \in H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}^\vee_{2k-2} \otimes \mathbb{Q}).$$

See also Corollary 4.18.

Therefore, we can consider the denominator of the Eisenstein class $\mathrm{Eis}_{2k-2}$:

$$\Delta(\mathrm{Eis}_{2k-2}) := \min\{\Delta \in \mathbb{Z}_{>0} \mid \Delta\mathrm{Eis}_{2k-2} \in H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}^\vee_{2k-2})/(\text{torsion})\}.$$

[2]This definition is different from Harder's [2023] definition of $\mathrm{Eis}_{2k-2}$, but we will see further on that they coincide.

Harder has been studying the denominator of the Eisenstein class (actually, he studies Eisenstein classes for more general arithmetic groups) and he proved the following explicit formula for $\Delta(\mathrm{Eis}_{2k-2})$.

**Theorem 1.3** [Harder and Pink 1992, §1, Satz 2; Harder 2023, Theorem 5.1.2, p. 217]. *For any even integer $k \geq 2$, we have*

$$\Delta(\mathrm{Eis}_{2k-2}) = \textit{numerator of } \zeta(1-2k).$$

This very beautiful formula is explained in Harder's book in progress [2023] (a weaker version was obtained by Haberland and Wang, see Remark 2.14), and its analogues for some congruence subgroups, Bianchi groups, Hilbert modular groups, etc. have been studied by many people. See Remark 2.15. One of the interesting points in Theorem 1.3 is that on the Eisenstein parts $\mathbb{Q}E_{2k-2}$ and $\mathbb{Q}\mathrm{Eis}_{2k}$, the Betti integral structure coincides with the de Rham integral structure under the Eichler–Shimura isomorphism. See [Harder 2021, §1.1] and Remark 2.16.

Harder also studies its application to the Galois representations obtained from $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}^{\vee}_{2k-2} \otimes \mathbb{Z}_\ell)$ in [Harder 2023]. In the present paper, we also consider some new applications of Theorem 1.3 to the special values of partial zeta functions of real quadratic fields.

However, currently, the book [Harder 2023] is still under development, and some important arguments and references for the proof of Theorem 1.3 are not given completely. Therefore, it is still a little difficult to access Harder's theory on the denominators of Eisenstein classes.

**1.1.4.** *Aims and features of the present paper.* Taking this situation into account, we aim to

(1) give a fully detailed proof of Theorem 1.3 based on Harder's [2023] argument, and

(2) present some new applications of Theorem 1.3 to the special values of partial zeta functions of real quadratic fields.

We reformulated Harder's original formulation and arguments entirely in terms of classical holomorphic Eisenstein series. We hope this will make Harder's theory on the denominators of Eisenstein classes more accessible to a wider audience. (Note that Harder's original formulation is purely cohomological and has a big advantage when we consider more general arithmetic groups.)

Let us now explain the strategy of the proof of Theorem 1.3.

**1.2.** ***Strategy of the proof of Theorem 1.3.*** Set $n := 2k-2 \geq 2$ and $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. Recall that $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ denotes the upper half plane, on which $\Gamma$ acts by the linear fractional transformation. We denote by $Y := \Gamma\backslash\mathbb{H}$ the modular curve of level $\Gamma$, respectively by $\mathbb{H}^{\mathrm{BS}}$ and $Y^{\mathrm{BS}} = \Gamma\backslash\mathbb{H}^{\mathrm{BS}}$ the Borel–Serre compactifications

of $\mathbb{H}$ and of $Y$, and by $\partial Y^{\mathrm{BS}} := Y^{\mathrm{BS}} - Y$ the Borel–Serre boundary of $Y$. Note that the inclusion map $Y \hookrightarrow Y^{\mathrm{BS}}$ is a homotopy equivalence.

Recall that $\mathcal{M}_n$ is the space of homogeneous polynomials of degree $n$ over $\mathbb{Z}$ with a left $\Gamma$ action and that $\mathcal{M}_n^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}_n, \mathbb{Z})$. Since the $\Gamma$-module $\mathcal{M}_n^\vee$ naturally defines a sheaf on $Y^{\mathrm{BS}}$ (which we also denote by $\mathcal{M}_n^\vee$), we can consider the cohomology groups $H^\bullet(Y^{\mathrm{BS}}, \mathcal{M}_n^\vee)$ and $H^\bullet(\partial Y^{\mathrm{BS}}, \mathcal{M}_n^\vee)$. In addition, since $\mathcal{M}_n^\vee$ has a left action of $M_2(\mathbb{Z})$, these cohomology groups carry the structure of Hecke modules.

The Eisenstein class $\mathrm{Eis}_n \in H^1(Y, \mathcal{M}_n^\vee \otimes \mathbb{Q}) = H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\vee \otimes \mathbb{Q})$ has the important property that it gives rise to a Hecke-equivariant section of the canonical homomorphism $H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\vee \otimes \mathbb{Q}) \to H^1(\partial Y^{\mathrm{BS}}, \mathcal{M}_n^\vee \otimes \mathbb{Q})$ induced by the inclusion map $\partial Y^{\mathrm{BS}} \hookrightarrow Y^{\mathrm{BS}}$. Let us explain this fact. The boundary $\partial Y^{\mathrm{BS}}$ is identified with $\Gamma_\infty \backslash \mathbb{R}$, where $\Gamma_\infty := \left\{ \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbb{Z} \right\}$. It is now easy to see that $\dim_{\mathbb{Q}} H^1(\partial Y^{\mathrm{BS}}, \mathcal{M}_n^\vee \otimes \mathbb{Q}) = 1$ and we have a natural generator $\omega_n \in H^1(\partial Y^{\mathrm{BS}}, \mathcal{M}_n^\vee)/(\text{torsion})$. Harder considered in his book [2023] a unique Hecke-equivariant section

$$H^1(\partial Y^{\mathrm{BS}}, \mathcal{M}_n^\vee \otimes \mathbb{Q}) \to H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\vee \otimes \mathbb{Q})$$

of $H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\vee \otimes \mathbb{Q}) \to H^1(\partial Y^{\mathrm{BS}}, \mathcal{M}_n^\vee \otimes \mathbb{Q})$, and he defined the Eisenstein class

$$\mathrm{Eis}_n^{\mathrm{Harder}} \in H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\vee \otimes \mathbb{Q})$$

to be the image of $\omega_n$ under this section.

**Proposition 1.4** (Lemma 2.8 and Proposition 2.11). *We have* $\mathrm{Eis}_n = \mathrm{Eis}_n^{\mathrm{Harder}}$, *i.e., our definition of the Eisenstein class coincides with Harder's definition.*

We review the strategy of the proof of Theorem 1.3, which is based on Harder's [2023] argument. First, note that for any prime number $p$, we have

$$\begin{aligned}\mathrm{ord}_p(\Delta(\mathrm{Eis}_n)) &= \min\{\delta \in \mathbb{Z}_{\geq 0} \mid p^\delta \mathrm{Eis}_n \in H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\vee \otimes \mathbb{Z}_{(p)})\} \\ &= \min\{\delta \in \mathbb{Z}_{\geq 0} \mid \langle p^\delta \mathrm{Eis}_n, C \rangle \in \mathbb{Z}_{(p)} \text{ for all } C \in H_1(Y^{\mathrm{BS}}, \mathcal{M}_n \otimes \mathbb{Z}_{(p)})\},\end{aligned}$$

where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at $p$. Therefore, it suffices to prove that

$$\begin{aligned}&\mathrm{ord}_p(\text{numerator of } \zeta(-1-n)) \\ &\quad = \min\{\delta \in \mathbb{Z}_{\geq 0} \mid \langle p^\delta \mathrm{Eis}_n, C \rangle \in \mathbb{Z}_{(p)} \text{ for all } C \in H_1(Y^{\mathrm{BS}}, \mathcal{M}_n \otimes \mathbb{Z}_{(p)})\}\end{aligned}$$

for each prime number $p$. The proof consists roughly of the following four parts.

(I) First, in Section 3, we consider 1-chains $C_\nu(\tau) = \{-1/\tau, \tau\} \otimes X_1^\nu X_2^{n-\nu}$ $(\tau \in \mathbb{H}^{\mathrm{BS}})$ for each integer $1 \leq \nu \leq n-1$, where $\{-1/\tau, \tau\}$ is a path from $-1/\tau$ to $\tau$. Note that $C_\nu(\tau)$ is not a cycle: $C_\nu(\tau)$ does not give an element of the cosheaf homology group $H_1(Y^{\mathrm{BS}}, \mathcal{M}_n \otimes \mathbb{Q})$. However, by applying the $p$-th Hecke operator $T_p$ sufficiently many times (namely, $m$ times for $m \geq n$) and modifying them slightly, we obtain

$p$-adically integral homology classes

$$[\widetilde{T_p^m(C_\nu(\tau))}] \in H_1(Y^{\mathrm{BS}}, \mathcal{M}_n \otimes \mathbb{Z}_{(p)}).$$

Note that the homology class $[\widetilde{T_p^m(C_\nu(\tau))}]$ is independent of the choice of $\tau \in \mathbb{H}$. See Lemma 3.15.

(II) Next, in Section 4, we compute the $p$-adic limit of the value of the pairing

$$\lim_{m\to\infty} \langle \mathrm{Eis}_n, [\widetilde{T_p^{m!}(C_\nu(\tau))}]\rangle,$$

where $\langle \cdot, \cdot \rangle : H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\vee) \times H_1(Y^{\mathrm{BS}}, \mathcal{M}_n) \to \mathbb{Z}$ is the pairing induced by $\mathcal{M}_n^\vee \times \mathcal{M}_n \to \mathbb{Z}$; $(f, x) \mapsto f(x)$. More precisely, we will show in Theorem 4.1 and Corollary 7.2 that this $p$-adic limit can be described in terms of the Kubota–Leopoldt $p$-adic $L$-functions: for any integer $\nu \in \{1, \dots, n-1\}$ we obtain the interesting formula

$$\lim_{m\to\infty} \langle \mathrm{Eis}_n, [\widetilde{T_p^{m!}(C_\nu(\tau))}]\rangle = \frac{1-p^{n+1}}{(1-p^\nu)(1-p^{n-\nu})} D_p(n, \nu), \tag{1-4}$$

where

$$D_p(n,\nu) := \frac{L_p(-\nu, \omega^{1+\nu}) L_p(\nu-n, \omega^{n-\nu+1})}{L_p(-1-n, \omega^{n+2})} - L_p(-\nu, \omega^{1+\nu}) - L_p(\nu-n, \omega^{n-\nu+1})$$

and $L_p(s, \omega^a)$ denotes the Kubota–Leopoldt $p$-adic $L$-function associated with the $a$-th power of the Teichmüller character $\omega$.

(III) In Section 5, we introduce the ordinary part of (co)homology groups which are defined by the image of Hida's ordinary projector. More precisely, Hida's ordinary projector $e_{T_p}$ is defined by the $p$-adic limit of the $m!$-th power of the $p$-th Hecke operator (acting on the homology group $H_1(Y^{\mathrm{BS}}, \mathcal{M}_n \otimes \mathbb{Z}_p)$):

$$e_{T_p} := \lim_{m\to\infty} T_p^{m!} \in \mathrm{End}_{\mathbb{Z}_p}(H_1(Y^{\mathrm{BS}}, \mathcal{M}_n \otimes \mathbb{Z}_p)),$$

and then we define the ordinary part

$$H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_n \otimes \mathbb{Z}_p) := e_{T_p} H_1(Y^{\mathrm{BS}}, \mathcal{M}_n \otimes \mathbb{Z}_p).$$

Since $\mathrm{Eis}_n | T_p' = (1+p^{n+1})\mathrm{Eis}_n$,[3] for any homology class $C \in H_1(Y^{\mathrm{BS}}, \mathcal{M}_n \otimes \mathbb{Z}_p)$, we have

$$\begin{aligned}\langle \mathrm{Eis}_n, e_{T_p} C\rangle = \lim_{m\to\infty} \langle \mathrm{Eis}_n, T_p^{m!}(C)\rangle &= \lim_{m\to\infty} \langle \mathrm{Eis}_n | (T_p')^{m!}, C\rangle \\ &= \lim_{m\to\infty} (1+p^{n+1})^{m!} \langle \mathrm{Eis}_n, C\rangle = \langle \mathrm{Eis}_n, C\rangle.\end{aligned}$$

This fact implies that

$$\mathrm{ord}_p(\Delta(\mathrm{Eis}_n)) = \min\{\delta \in \mathbb{Z}_{\geq 0} \mid \langle p^\delta \mathrm{Eis}_n, C\rangle \in \mathbb{Z}_p \text{ for all } C \in H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_n \otimes \mathbb{Z}_p)\}.$$

[3] Here $T_p'$ denotes the $p$-th Hecke operator acting on the cohomology group $H_1(Y^{\mathrm{BS}}, \mathcal{M}_n)$. See Definition 2.3.

Moreover, Proposition 5.7, together with Lemma 3.15(3), shows that

$$H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_n \otimes \mathbb{Z}_p) := H_1(\partial Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{Z}_p + \sum_{\nu=1}^{n-1} \mathbb{Z}_p e_{T_p}[\widetilde{T_p^m(C_\nu(\tau))}]$$

for any integer $m \geq n$. Since $\langle \mathrm{Eis}_n, H_1(\partial Y^{\mathrm{BS}}, \mathcal{M}_n)\rangle = \mathbb{Z}$, we obtain

$$\begin{aligned}&\mathrm{ord}_p(\Delta(\mathrm{Eis}_n))\\ &= \min\{\delta \in \mathbb{Z}_{\geq 0} \mid \langle p^\delta \mathrm{Eis}_n, e_{T_p}[\widetilde{T_p^m(C_\nu(\tau))}]\rangle \in \mathbb{Z}_p \text{ for any integer } 1 \leq \nu \leq n-1\},\end{aligned}$$

for any integer $m \geq n$. Since

$$\langle \mathrm{Eis}_n, e_{T_p}[\widetilde{T_p^m(C_\nu(\tau))}]\rangle = \langle \mathrm{Eis}_n, [\widetilde{T_p^m(C_\nu(\tau))}]\rangle,$$

by replacing $m$ with $m!$ and taking the $p$-adic limit as $m \to \infty$, (1-4) shows that

$$\mathrm{ord}_p(\Delta(\mathrm{Eis}_n)) = \min\{a \in \mathbb{Z}_{\geq 0} \mid p^a D(n, \nu) \in \mathbb{Z}_p \text{ for any integer } 1 \leq \nu \leq n-1\}.$$

See Proposition 8.1.

(IV) In Section 8, we show that the right-hand side of this previous equation is equal to the $p$-adic valuation of the numerator of $\zeta(-1-n)$. We devote Section 6 and Section 7 to the preparation for proving this fact.

### 1.3. *Applications to Duke's conjecture and to the special values of the partial zeta functions of real quadratic fields.*

**1.3.1.** *Duke's conjecture.* Duke [2024] defined a certain map

$$\Psi_k : \Gamma = \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Q}$$

for each integer $k \geq 2$, called the higher Rademacher symbol, which is a generalization of the classical Rademacher symbol, and he conjectured the integrality of the higher Rademacher symbol [Duke 2024, Conjecture, p. 4]. As a first application of Theorem 1.3, we prove this conjecture.

**Theorem 1.5** (Corollary 9.5). *Duke's conjecture holds true, that is, for any integer $k \geq 2$ and matrix $\gamma \in \Gamma$, we have*

$$\Psi_k(\gamma) \in \mathbb{Z}.$$

In fact, Duke [2024, Lemma 6] proved that the higher Rademacher symbols can be written as the integral of the holomorphic Eisenstein series along certain homology cycles (see Proposition 9.4). Therefore, we can derive Theorem 1.5 directly from Theorem 1.3.

**Remark 1.6.** Duke's conjecture was recently also proved by O'Sullivan [2024] using a more direct method.

**1.3.2.** *The denominators of the partial zeta functions of real quadratic fields.* Next, we discuss the denominators of the partial zeta functions associated with narrow ideal classes of orders in real quadratic fields.

Let $F \subset \mathbb{R}$ be a real quadratic field, $\mathcal{O} \subset F$ be an order in $F$, and $\mathcal{A} \in Cl_{\mathcal{O}}^{+}$ be a narrow ideal class of $\mathcal{O}$. Then we have the associated partial zeta function

$$\zeta_{\mathcal{O}}(\mathcal{A}, s) = \sum_{\mathfrak{a} \subset \mathcal{O}, \mathfrak{a} \in \mathcal{A}} \frac{1}{N\mathfrak{a}^s},$$

which can be continued meromorphically to $\mathbb{C}$, and it is known that

$$\zeta_{\mathcal{O}}(\mathcal{A}, 1-k) \in \mathbb{Q}$$

for any integer $k \geq 2$. We also define the positive integer $J_{2k}$ by

$$J_{2k} := \text{denominator of } \zeta(1-2k).$$

Then in Section 9.2, we obtain the following as another consequence of Theorem 1.3.

**Proposition 1.7** (Corollary 9.12). *Let $k \geq 2$ be an integer. Then the integer $J_{2k}$ gives a universal upper bound for the denominator of $\zeta_{\mathcal{O}}(\mathcal{A}, 1-k)$ with respect to orders $\mathcal{O}$ and narrow ideal classes $\mathcal{A} \in Cl_{\mathcal{O}}^{+}$. In other words, we have*

$$J_{2k}\zeta_{\mathcal{O}}(\mathcal{A}, 1-k) \in \mathbb{Z}$$

*for all orders $\mathcal{O}$ in all real quadratic fields and narrow ideal classes $\mathcal{A} \in Cl_{\mathcal{O}}^{+}$.*

In fact, one can construct a natural map $\mathfrak{z}_{\mathcal{O},k} : Cl_{\mathcal{O}}^{+} \to H_1(Y^{\mathrm{BS}}, \mathcal{M}_{2k-2})$ for any integer $k \geq 2$ (see Definition 9.6), and we show in Proposition 9.10 that

$$\langle \mathrm{Eis}_{2k-2}, \mathfrak{z}_{\mathcal{O},k}(\mathcal{A}^{-1}) \rangle = (-1)^k \frac{\zeta_{\mathcal{O}}(\mathcal{A}, 1-k)}{\zeta(1-2k)} = \pm \frac{J_{2k}\zeta_{\mathcal{O}}(\mathcal{A}, 1-k)}{N_{2k}},$$

where $N_{2k} > 0$ denotes the numerator of $\zeta(1-2k)$. Hence Proposition 1.7 follows from Theorem 1.3. See also Remark 9.13 for the relation between Duke's conjecture and Proposition 1.7.

Next, we discuss the sharpness of Proposition 1.7's universal upper bound.

**Theorem 1.8** (Corollary 9.16). *The universal upper bound in Proposition 1.7 is sharp, that is, we have*

$$J_{2k} = \min\left\{ J \in \mathbb{Z}_{>0} \mid J\zeta_{\mathcal{O}}(\mathcal{A}, 1-k) \in \mathbb{Z} \text{ for all orders } \mathcal{O} \text{ in all real quadratic fields and narrow ideal classes } \mathcal{A} \in Cl_{\mathcal{O}}^{+} \right\}.$$

In order to derive Theorem 1.8 from Theorem 1.3, we need to show that the narrow ideal classes of orders in real quadratic fields produce sufficiently large submodules of the homology group $H_1(Y^{\mathrm{BS}}, \mathcal{M}_{2k-2})$, and this will be done in Section 9.3 using some techniques from Hida theory.

**Remark 1.9.** As for the denominator or the integrality of the special values of partial zeta functions of real quadratic fields, or more generally of totally real fields, many works have been done by Coates and Sinnott [1974a; 1974b; 1977], Deligne and Ribet [1980], Cassou-Noguès [1979], Charollois, Dasgupta, and Greenberg [Charollois et al. 2015], Beilinson, Kings, and Levin [Beilinson et al. 2018], Bannai, Hagihara, Yamada, and Yamamoto [Bannai et al. 2022], Bergeron, Charollois, and Garcia [Bergeron et al. 2020], etc., by using a variety of methods including Hilbert modular forms, Shintani [1976] zeta functions, Sczech's [1993] Eisenstein cocycles, etc. Actually, when $\mathcal{O} = \mathcal{O}_F$, the upper bound in Proposition 1.7 follows from these preceding works. More precisely, the results proved by Coates and Sinnott [1977] or Deligne and Ribet [1980] show that for any prime number $p$, we have

$$2^{-1}(1 - p^{2k})\zeta_{\mathcal{O}_F}(\mathcal{A}, 1-k) \in \mathbb{Z}\big[\tfrac{1}{p}\big],$$

which implies that

$$J_{2k}\zeta_{\mathcal{O}_F}(\mathcal{A}, 1-k) \in \mathbb{Z};$$

see [Zagier 1976, pp. 73, 75].

One feature of our method is that by using Theorem 1.3, we capture not only the upper bound for the denominators of the partial zeta functions associated with any orders, but also the sharpness of the upper bound.

**Remark 1.10.** Zagier [1977, p. 149, corollaire] proved a certain formula which explicitly computes the special values $\zeta_{\mathcal{O}}(\mathcal{A}, 1-k)$ of partial zeta functions of orders of real quadratic fields at negative integers in a uniform way. Then by using this formula, he obtained a universal upper bound $d_k$ for the denominators of the values $\zeta_{\mathcal{O}}(\mathcal{A}, 1-k)$ and examined its sharpness briefly. More precisely, he observed that the upper bound $d_k$ is not sharp and discussed how one can improve it when $k = 2, 3$; see [Zagier 1977, pp. 149–150]. Theorem 1.8 can be seen as the complete answer to this problem of determining the sharp universal upper bound for the denominators of $\zeta_{\mathcal{O}}(\mathcal{A}, 1-k)$.

## 2. Preliminaries and the Eisenstein class

In this section, we give the definition of the Eisenstein class and explain Theorem 1.3 (see Theorem 2.13).

Throughout this paper, $n \geq 2$ denotes an even integer.

**2.1.** ***Definitions of modular curve and Borel–Serre compactification.*** Let

$$\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

denote the upper half plane, and let

$$\mathbb{H}^{\mathrm{BS}} := \mathbb{H} \sqcup \bigsqcup_{r \in \mathbb{P}^1(\mathbb{Q})} (\mathbb{P}^1(\mathbb{R}) - \{r\})$$

be the Borel–Serre compactification of $\mathbb{H}$ (see [Goresky 2005] or [Harder 2023, §1, 2 and 7]) and let $\partial\mathbb{H}^{\mathrm{BS}} := \mathbb{H}^{\mathrm{BS}} - \mathbb{H}$. We set

$$\Gamma := \mathrm{SL}_2(\mathbb{Z}).$$

The group $\Gamma$ acts on $\mathbb{H}$ and $\mathbb{H}^{\mathrm{BS}}$ by the linear fractional transformation as usual. We denote respectively by

$$Y := \Gamma\backslash\mathbb{H} \quad \text{and} \quad Y^{\mathrm{BS}} := \Gamma\backslash\mathbb{H}^{\mathrm{BS}}$$

the modular curve of level $\mathrm{SL}_2(\mathbb{Z})$ and its Borel–Serre compactification. We also denote by $\partial Y^{\mathrm{BS}} := Y^{\mathrm{BS}} - Y$ the boundary of $Y^{\mathrm{BS}}$. The boundary $\partial Y^{\mathrm{BS}}$ is homeomorphic to the circle $S^1$ and the fundamental group $\pi_1(\partial Y^{\mathrm{BS}})$ can be identified with $\Gamma_\infty := \left\{\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right) \mid a \in \mathbb{Z}\right\}$.

In the following, $\mathbb{H}^?$ (resp. $Y^?$) refers to either $\mathbb{H}$ or $\mathbb{H}^{\mathrm{BS}}$ (resp. $Y$ or $Y^{\mathrm{BS}}$). Any left $\Gamma$-module $\mathcal{M}$ can be regarded as a (co)sheaf on $Y^?$ in a natural way, and we can consider the homology groups

$$H_\bullet(Y^?, \mathcal{M}), \quad H_\bullet(\partial Y^{\mathrm{BS}}, \mathcal{M}), \quad H_\bullet(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}),$$

which fit into the long exact sequence

$$\cdots \to H_1(\partial Y^{\mathrm{BS}}, \mathcal{M}) \to H_1(Y^{\mathrm{BS}}, \mathcal{M}) \to H_1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}) \to H_0(\partial Y^{\mathrm{BS}}, \mathcal{M}) \to \cdots.$$

Similarly, we have the cohomology groups

$$H^\bullet(Y^?, \mathcal{M}), \quad H^\bullet(\partial Y^{\mathrm{BS}}, \mathcal{M}), \quad H^\bullet(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}),$$

which fit into the long exact sequence

$$\cdots \to H^1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}) \to H^1(Y^{\mathrm{BS}}, \mathcal{M}) \to H^1(\partial Y^{\mathrm{BS}}, \mathcal{M}) \to H^2(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}) \to \cdots.$$

We note that the inclusion map $Y \hookrightarrow Y^{\mathrm{BS}}$ induces isomorphisms

$$H_\bullet(Y, \mathcal{M}) \xrightarrow{\sim} H_\bullet(Y^{\mathrm{BS}}, \mathcal{M}) \quad \text{and} \quad H^\bullet(Y^{\mathrm{BS}}, \mathcal{M}) \xrightarrow{\sim} H^\bullet(Y, \mathcal{M}).$$

If $\mathcal{M}$ has an action of $M_2^+(\mathbb{Z}) := \{\gamma \in M_2(\mathbb{Z}) \mid \det\gamma > 0\}$, then these homology groups (resp. cohomology groups) carry the structure of Hecke modules. In other words, for each prime number $p$ we have a Hecke operator $T_p$ (resp. $T_p'$) on these homology groups (resp. cohomology groups), and the above long exact sequences are compatible with the Hecke operators.

In Section 2.2, we give a way to compute these (co)homology groups, and in Section 2.4, we give an explicit description of the Hecke operators.

**Remark 2.1.** As a sheaf on $Y$, the stalk $\mathcal{M}_x$ at $x \in Y$ coincides with $\mathcal{M}^{\Gamma_{\tilde{x}}}$, where $\tilde{x} \in \mathbb{H}$ is a lift of $x \in Y$ and $\Gamma_{\tilde{x}} := \{\gamma \in \Gamma \mid \gamma\tilde{x} = \gamma\}$. This fact shows that a short

exact sequence of left $\Gamma$-modules does not give a short exact sequence of sheaves on $Y$ in general, that is, the sheafification functor is not exact. However, a short exact sequence of left $\mathbb{Z}\left[\frac{1}{6}\right][\Gamma]$-modules induces a short exact sequence of sheaves on $Y^{\mathrm{BS}}$ since the order of $\Gamma_{\tilde{x}}$ divides 6.

**2.2.** ***Modular symbols and (co)homology.*** Let $X \in \{\mathbb{H}, \mathbb{H}^{\mathrm{BS}}, \partial\mathbb{H}^{\mathrm{BS}}\}$ and $(S_\bullet(X), \partial)$ denote the usual singular chain complex of $X$, i.e., $S_q(X)$ is the free abelian group generated by singular $q$-simplices in $X$ and $\partial : S_q(X) \to S_{q-1}(X)$ is the boundary operator.

The left action of $M_2^+(\mathbb{Z})$ on $X$ induces a left action of $M_2^+(\mathbb{Z})$ on $S_\bullet(X)$, and $(S_\bullet(X), \partial)$ is actually an $M_2^+(\mathbb{Z})$-equivariant complex. Then it is known that for any left $M_2^+(\mathbb{Z})$-module $\mathcal{M}$, which is also seen as a (co)sheaf on $\Gamma\backslash X$, we have natural isomorphisms

$$H_\bullet(\Gamma\backslash X, \mathcal{M}) \cong H_\bullet((S_\bullet(X) \otimes \mathcal{M})_\Gamma),$$
$$H_\bullet(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}\mathcal{M}) \cong H_\bullet((S_\bullet(\mathbb{H}^{\mathrm{BS}})/S_\bullet(\partial\mathbb{H}^{\mathrm{BS}}) \otimes \mathcal{M})_\Gamma),$$

where $(-)_\Gamma$ denotes the $\Gamma$-coinvariant functor. Here the left $M_2^+(\mathbb{Z})$-action on $S_\bullet(X) \otimes \mathcal{M}$ is defined by

$$\gamma \cdot (\sigma \otimes m) := \gamma\sigma \otimes \gamma m,$$

where $\sigma \in S_\bullet(X)$, $m \in \mathcal{M}$, and $\gamma \in M_2^+(\mathbb{Z})$. Set

$$\mathcal{MS}(X) := \mathrm{coker}(S_2(X) \xrightarrow{\partial} S_1(X)).$$

For any elements $\alpha, \beta \in X$, we denote the equivalence class of a path from $\alpha$ to $\beta$ in $\mathcal{MS}(X)$ by

$$\{\alpha, \beta\} \in \mathcal{MS}(X).$$

This is a slight generalization of the usual modular symbols, which will be useful in the following arguments since we don't need to specify paths using this notation. The boundary map $\partial : S_1(X) \to S_0(X)$ induces a $\Gamma$-homomorphism $\partial : \mathcal{MS}(X) \to S_0(X)$, and we have a natural isomorphism

$$H_1(\Gamma\backslash X, \mathcal{M}) \cong \ker\big((\mathcal{MS}(X) \otimes \mathcal{M})_\Gamma \xrightarrow{\partial} (S_0(X) \otimes \mathcal{M})_\Gamma\big).$$

Similarly, we also have natural isomorphisms

$$H^\bullet(\Gamma\backslash X, \mathcal{M}) \cong H^\bullet(\mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X), \mathcal{M})^\Gamma),$$
$$H^\bullet(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}\mathcal{M}) \cong H^\bullet(\mathrm{Hom}_{\mathbb{Z}}(S_\bullet(\mathbb{H}^{\mathrm{BS}})/S_\bullet(\partial\mathbb{H}^{\mathrm{BS}}), \mathcal{M})^\Gamma),$$

where $(-)^\Gamma$ denotes the $\Gamma$-invariant functor. Here the left $M_2^+(\mathbb{Z})$-action on $\mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X), \mathcal{M})$ is defined by

$$(\gamma\phi)(\sigma) := \gamma(\phi(\tilde{\gamma}\sigma)),$$

where $\phi \in \mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X), \mathcal{M})$, $\sigma \in S_\bullet(X)$, and $\tilde{\gamma}$ is the adjugate of $\gamma \in M_2^+(\mathbb{Z})$. Since

$$\ker\big(\mathrm{Hom}_{\mathbb{Z}}(S_1(X), \mathcal{M}) \to \mathrm{Hom}_{\mathbb{Z}}(S_2(X), \mathcal{M})\big) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{MS}(X), \mathcal{M}),$$

we have a natural isomorphism

$$H^1(\Gamma\backslash X, \mathcal{M}) \cong \mathrm{coker}\big(\mathrm{Hom}_{\mathbb{Z}}(S_0(X), \mathcal{M})^\Gamma \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{MS}(X), \mathcal{M})^\Gamma\big).$$

**2.3. *$M_2^+(\mathbb{Z})$-modules $\mathcal{M}_n$ and $\mathcal{M}_n^\flat$.*** For any $2 \times 2$ matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we denote the adjugate of $\gamma$ by

$$\tilde{\gamma} := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Note that if $\gamma \in \Gamma$, then we have $\tilde{\gamma} = \gamma^{-1}$.

Let $\mathbb{Z}[X_1, X_2]$ denote the ring of polynomials of two variables over $\mathbb{Z}$, and we equip $\mathbb{Z}[X_1, X_2]$ with a left action of $M_2^+(\mathbb{Z})$ by

$$(\gamma P)(X_1, X_2) := P(dX_1 - bX_2, -cX_1 + aX_2) = P((X_1, X_2) \cdot {}^t\tilde{\gamma}),$$

where $P \in \mathbb{Z}[X_1, X_2]$ and $\gamma \in M_2^+(\mathbb{Z})$. For each integer $0 \le \nu \le n$, we set

$$e_\nu := X_1^\nu X_2^{n-\nu} \quad \text{and} \quad e_\nu^\flat := (-1)^{n-\nu}\binom{n}{\nu} X_1^{n-\nu} X_2^\nu.$$

We then define submodules $\mathcal{M}_n$ and $\mathcal{M}_n^\flat$ of $\mathbb{Z}[X_1, X_2]$ by

$$\mathcal{M}_n := \bigoplus_{\nu=0}^{n} \mathbb{Z} e_\nu \quad \text{and} \quad \mathcal{M}_n^\flat := \bigoplus_{\nu=0}^{n} \mathbb{Z} e_\nu^\flat.$$

The $\mathbb{Z}$-modules $\mathcal{M}_n$ and $\mathcal{M}_n^\flat$ are closed under the left action of $M_2^+(\mathbb{Z})$ on $\mathbb{Z}[X_1, X_2]$. In particular, both $\mathcal{M}_n$ and $\mathcal{M}_n^\flat$ are left $\Gamma$-modules. We also define the pairing

$$\langle \cdot, \cdot \rangle : \mathcal{M}_n^\flat \times \mathcal{M}_n \to \mathbb{Z}$$

by

$$\langle e_\nu^\flat, e_\mu \rangle = \delta_{\nu,\mu},$$

where $\delta_{\nu,\mu}$ is the Kronecker delta. The pairing $\langle \cdot, \cdot \rangle$ is perfect and $M_2^+(\mathbb{Z})$-equivariant in the sense that for any polynomials $P \in \mathcal{M}_n^\flat$ and $Q \in \mathcal{M}_n$ and matrix $\gamma \in M_2^+(\mathbb{Z})$, we have

$$\langle P, \gamma Q \rangle = \langle \tilde{\gamma} P, Q \rangle.$$

Hence the pairing $\langle \cdot, \cdot \rangle$ induces an $M_2^+(\mathbb{Z})$-equivariant isomorphism

$$\mathcal{M}_n^\flat \xrightarrow{\sim} \mathcal{M}_n^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}_n, \mathbb{Z}), \quad m' \mapsto (m \mapsto \langle m', m \rangle).$$

Here the left action of $M_2^+(\mathbb{Z})$ on $\mathcal{M}_n^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}_n, \mathbb{Z})$ is given by

$$(\gamma\phi)(Q) = \phi(\tilde{\gamma}Q),$$

where $\phi \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}_n, \mathbb{Z})$, $Q \in \mathcal{M}_n$, and $\gamma \in M_2^+(\mathbb{Z})$.

**Remark 2.2.** The left actions of $M_2^+(\mathbb{Z})$ on $\mathcal{M}_n$ and $\mathcal{M}_n^\flat$ are slightly different from the left actions used in Harder's book [2023, Equation (1.57), p. 37]. However, since

$$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

they are isomorphic as left $M_2^+(\mathbb{Z})$-modules. Therefore, there are no essential differences.

**2.4.** ***Hecke operators.*** Let $X \in \{\mathbb{H}, \mathbb{H}^{\mathrm{BS}}, \partial\mathbb{H}\}$ and $\mathcal{M}$ be a left $M_2^+(\mathbb{Z})$-module. Let us define the Hecke operators on $H_\bullet(\Gamma\backslash X, \mathcal{M})$ and $H^\bullet(\Gamma\backslash X, \mathcal{M})$ explicitly.

For each prime number $p$, we have the coset decomposition

$$\Gamma \begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma = \Gamma \begin{pmatrix} p & \\ & 1 \end{pmatrix} \sqcup \bigsqcup_{j=0}^{p-1} \Gamma \begin{pmatrix} 1 & j \\ & p \end{pmatrix}.$$

Hence the endomorphism

$$\mathcal{M} \longrightarrow \mathcal{M}; \quad m \mapsto \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} m + \sum_{j=0}^{p-1} \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} m$$

induces an endomorphism of $\mathcal{M}_\Gamma$. Similarly, the endomorphism

$$\mathcal{M} \longrightarrow \mathcal{M}; \quad m \mapsto \widetilde{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} m + \sum_{j=0}^{p-1} \widetilde{\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}} m$$

induces an endomorphism of $\mathcal{M}^\Gamma$.

**Definition 2.3.** Let $p$ be a prime number.

(1) We define the Hecke operator $T_p$ at $p$ on $S_\bullet(X) \otimes \mathcal{M}$ by

$$T_p(\sigma \otimes P) := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \sigma \otimes \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} P + \sum_{j=0}^{p-1} \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \sigma \otimes \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} P$$

for any simplex $\sigma \in S_\bullet(X)$ and any element $P \in \mathcal{M}$. The operator $T_p$ induces operators on $\mathcal{MS}(X) \otimes \mathcal{M}$ and $H_\bullet(\Gamma\backslash X, \mathcal{M})$, etc., also written as $T_p$.

(2) We define the Hecke operator $T'_p$ at $p$ on $\mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X), \mathcal{M})$ by

$$(\phi|T'_p)(\sigma) := \widetilde{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}\left[\phi\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\sigma\right)\right] + \sum_{j=0}^{p-1} \widetilde{\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}}\left[\phi\left(\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\sigma\right)\right]$$

for any homomorphism $\phi \in \mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X), \mathcal{M}_n^\flat)$ and any simplex $\sigma \in S_\bullet(X)$. The operator $T'_p$ induces operators on $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{MS}(X), \mathcal{M})$ and $H^\bullet(\Gamma\backslash X, \mathcal{M})$, etc., also written as $T'_p$.

For later use, we also define auxiliary operators $U_p$ and $V_p$ on $S_\bullet(X) \otimes \mathcal{M}$ by

$$V_p(\sigma \otimes P) := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\sigma \otimes \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} P, \quad U_p(\sigma \otimes P) := \sum_{j=0}^{p-1} \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\sigma \otimes \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} P,$$

so that $T_p := V_p + U_p$.

**Lemma 2.4.** *The composite $V_pU_p$ acts on $(S_\bullet(X)\otimes\mathcal{M}_n)_\Gamma$ and we have $V_pU_p = p^{n+1}$ as operators on $(S_\bullet(X) \otimes \mathcal{M}_n)_\Gamma$.*

*Proof.* Since diagonal matrices act trivially on $X$, we have

$$V_pU_p(\sigma \otimes P) = \sum_{j=0}^{p-1} \begin{pmatrix} p & pj \\ 0 & p \end{pmatrix}\sigma \otimes \begin{pmatrix} p & pj \\ 0 & p \end{pmatrix} P = p^n \sum_{j=0}^{p-1} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}(\sigma \otimes P)$$

for any simplex $\sigma \in S_\bullet(X)$ and polynomial $P \in \mathcal{M}_n$. Since $\left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ for any integer $j$, the lemma follows from the above equality. □

**2.5.** ***Formal duality.*** Let $X \in \{\mathbb{H}, \mathbb{H}^{\mathrm{BS}}, \partial\mathbb{H}^{\mathrm{BS}}\}$. As explained in Section 2.2, the homology and cohomology groups can be computed as

$$H_\bullet(\Gamma\backslash X, \mathcal{M}_n) \cong H_\bullet((S_\bullet(X)\otimes\mathcal{M}_n)_\Gamma), \quad H^\bullet(\Gamma\backslash X, \mathcal{M}_n^\flat) \cong H_\bullet(\mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X), \mathcal{M}_n^\flat)^\Gamma).$$

The pairing $\langle \cdot, \cdot \rangle : \mathcal{M}^\flat \times \mathcal{M} \to \mathbb{Z}$ induces a pairing

$$\langle \cdot, \cdot \rangle : \mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X), \mathcal{M}_n^\flat) \times S_\bullet(X) \otimes \mathcal{M}_n \to \mathbb{Z},$$

which is computed as

$$\langle \phi, \sigma \otimes P \rangle := \langle \phi(\sigma), P \rangle.$$

Note that for any matrix $\gamma \in M_2^+(\mathbb{Z})$, we have

$$\langle \tilde{\gamma}\phi, \sigma \otimes P \rangle = \langle \tilde{\gamma}\phi(\gamma\sigma), P \rangle = \langle \phi(\gamma\sigma), \gamma P \rangle = \langle \phi, \gamma(\sigma \otimes P) \rangle.$$

Therefore, we have

$$\langle \phi|T'_p, \sigma \otimes P \rangle = \langle \phi, T_p(\sigma \otimes P) \rangle.$$

In particular, we obtain a Hecke-equivariant pairing

$$\langle \cdot, \cdot \rangle : H^\bullet(\Gamma\backslash X, \mathcal{M}_n^\flat) \times H_\bullet(\Gamma\backslash X, \mathcal{M}_n) \to \mathbb{Z},$$

which induces an isomorphism

$$H^\bullet(\Gamma\backslash X, \mathcal{M}_n^\flat)/(\text{torsion}) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(H_\bullet(\Gamma\backslash X, \mathcal{M}_n), \mathbb{Z}).$$

**2.6.** ***Eichler–Shimura homomorphism.*** Let $M_{n+2}(\Gamma)$ denote the space of modular forms of weight $n+2$ and level $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We define a homomorphism

$$r : M_{n+2}(\Gamma) \to \operatorname{Hom}_{\mathbb{Z}}(\mathcal{MS}(\mathbb{H}), \mathcal{M}_n^\flat \otimes \mathbb{C})$$

by

$$r(f)(\{\alpha, \beta\}) := \int_\alpha^\beta f(z)(X_1 - zX_2)^n \, dz$$

for any modular form $f \in M_{n+2}(\Gamma)$ and $\{\alpha, \beta\} \in \mathcal{MS}(\mathbb{H})$. It is well known that

$$r(M_{n+2}(\Gamma)) \subset \operatorname{Hom}_{\mathbb{Z}}(\mathcal{MS}(\mathbb{H}), \mathcal{M}_n^\flat \otimes \mathbb{C})^\Gamma,$$

and the homomorphism $r$ induces an injective homomorphism

$$r : M_{n+2}(\Gamma) \hookrightarrow H^1(Y, \mathcal{M}_n^\flat) \otimes \mathbb{C} = H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\flat) \otimes \mathbb{C},$$

called Eichler–Shimura homomorphism. See [Bellaïche 2021, Section 5.3] for example.

**Remark 2.5.** The definition of the Eichler–Shimura homomorphism shows that, for any element $\sigma = \{\alpha, \beta\} \otimes P \in \mathcal{MS}(\mathbb{H}) \otimes \mathcal{M}_n$ and modular form $f \in M_{n+2}(\Gamma)$, the pairing $\langle r(f), \sigma \rangle$ can be computed as

$$\langle r(f), \sigma \rangle = \int_\alpha^\beta f(z) P(z, 1) \, dz.$$

**Remark 2.6.** For each prime number $p$, the double coset operator $T_p'' := \Gamma \left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right) \Gamma$ acts on the space $M_{n+2}(\Gamma)$ of modular forms from the right by using the weight $n+2$ slash operator[4] $|[\,\cdot\,]_{n+2}$. One can easily show that

$$r(f|[\gamma]_{n+2}) = \tilde{\gamma} \cdot r(f)$$

for any matrix $\gamma \in M_2^+(\mathbb{Z})$. Hence the Eichler–Shimura homomorphism is Hecke-equivariant, that is, for all prime numbers $p$, we have

$$r(f|T_p'') = r(f)|T_p'.$$

In other words, our Hecke operator $T_p'$ coincides with the usual one via the Eichler–Shimura homomorphism.

[4]Here we adopt the normalization $f|[\gamma]_{n+2}(z) := (\det \gamma)^{n+1}(cz+d)^{-n-2} f(\gamma z)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z})$ and $z \in \mathbb{H}$.

The following lemma is well known; see [Bellaïche 2021, Theorem 5.3.27] for example.

**Lemma 2.7.** *The Eichler–Shimura homomorphism induces a Hecke-equivariant isomorphism*

$$M_{n+2}(\Gamma)/S_{n+2}(\Gamma) \xrightarrow{\sim} H^1(\partial Y^{\mathrm{BS}}, \mathcal{M}_n^\flat) \otimes \mathbb{C}.$$

*Here $S_{n+2}(\Gamma)$ denotes the space of cusp forms of weight $n+2$ and level $\Gamma$.*

**2.7.** ***Definition of the Eisenstein class.*** We now define the Eisenstein class $\mathrm{Eis}_n$ and explain its basic properties.

We put $i := \sqrt{-1}$ and we let $\sigma_{n+1}(k)$ denote the sum-of-positive-divisors function, namely, $\sigma_{n+1}(k) := \sum_{0<d|k} d^{n+1}$. Let

$$E_{n+2}(z) := 1 + \frac{2}{\zeta(-1-n)} \sum_{k=1}^{\infty} \sigma_{n+1}(k) e^{2\pi i k z} \in M_{n+2}(\Gamma)$$

denote the normalized holomorphic Eisenstein series of weight $n+2$.

**Lemma 2.8.** (1) *For any element $\tau \in \mathbb{H}$, we have*

$$\langle r(E_{n+2}), \{\tau, \tau+1\} \otimes e_0 \rangle = 1.$$

(2) *For any prime number $p$, we have*

$$r(E_{n+2})|T_p' = (1+p^{n+1}) r(E_{n+2}).$$

*Proof.* Claim (1) follows from the fact that the constant term of $E_{n+2}$ is 1, and Claim (2) follows from the fact that $E_{n+2}|T_p'' = (1+p^{n+1})E_{n+2}$ and the Eichler–Shimura homomorphism is Hecke-equivariant. □

**Definition 2.9.** We define the Eisenstein class $\mathrm{Eis}_n \in H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\flat) \otimes \mathbb{C}$ by

$$\mathrm{Eis}_n := r(E_{n+2}).$$

**Remark 2.10.** Our method of defining the Eisenstein class $\mathrm{Eis}_n$ differs from the method in Harder's book [2023, §3.3.6, Equation (3.85), p. 130]. However, thanks to Lemma 2.8, they coincide.

**2.8.** ***Main theorem.***

**Proposition 2.11.** *The Eisenstein class* $\mathrm{Eis}_n$ *is rational*: $\mathrm{Eis}_n \in H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\flat) \otimes \mathbb{Q}$.

This proposition is proved in Corollary 4.18.

**Definition 2.12.** For any $\Gamma$-module $\mathcal{M}$, we define

$$H^1_{\mathrm{int}}(Y^{\mathrm{BS}}, \mathcal{M}) := \mathrm{im}\big(H^1(Y^{\mathrm{BS}}, \mathcal{M}) \to H^1(Y^{\mathrm{BS}}, \mathcal{M}) \otimes \mathbb{Q}\big).$$

Thanks to Proposition 2.11, we define the denominator $\Delta(\mathrm{Eis}_n)$ of the Eisenstein class $\mathrm{Eis}_n$ with respect to the integral structure $H^1_{\mathrm{int}}(Y^{\mathrm{BS}}, \mathcal{M}_n^\flat)$ by

$$\Delta(\mathrm{Eis}_n) := \min\{\Delta \in \mathbb{Z}_{>0} \mid \Delta\mathrm{Eis}_n \in H^1_{\mathrm{int}}(Y^{\mathrm{BS}}, \mathcal{M}_n^\flat)\}.$$

The following is our main theorem.

**Theorem 2.13** [Harder 2023, Theorem 5.1.2]. *The denominator $\Delta(\mathrm{Eis}_n)$ of the Eisenstein class $\mathrm{Eis}_n$ is equal to the numerator of the special value $\zeta(-1-n)$ of the Riemann zeta function.*

**Remark 2.14.** (1) Haberland [1983, pp. 272–273] proved a weaker version of Theorem 1.3. More precisely, let $p$ be a prime number and assume that $p > n$. Haberland obtained

$$\mathrm{ord}_p(\Delta(\mathrm{Eis}_n)) \le \mathrm{ord}_p(\text{numerator of } \zeta(-1-n)).$$

If we further assume that $p$ divides $\zeta(-1-n)$ and that there exists $\nu \in \{1, \dots, n-1\}$ such that $p \nmid \zeta(-\nu)\zeta(\nu-n)$, he obtained the equality

$$\mathrm{ord}_p(\Delta(\mathrm{Eis}_n)) = \mathrm{ord}_p(\text{numerator of } \zeta(-1-n)).$$

(2) In his dissertation, Wang [1989] proved that $p \mid \Delta(\mathrm{Eis}_n)$ and $p \mid \zeta(-1-n)$ are equivalent.

**Remark 2.15.** For any prime numbers $p$ and $\ell$ with $\ell \nmid p(p-1)$, the denominators over $\mathbb{Z}_\ell$ of Eisenstein classes for $\Gamma_1(p)$ with a character were computed by Kaiser in his diploma thesis [1990] (see also [Mahnkopf 2000] for the study of Eisenstein classes for $\Gamma_1(p^e\ell)$). The $\Lambda$-adic version for congruence subgroups $\Gamma_1(N)$ was studied by [Ohta 2003]. Eisenstein classes for $\mathrm{GL}_2$ over totally real fields have been studied by Maennel in his dissertation [1993]. Eisenstein classes for $\mathrm{GL}_2$ over imaginary quadratic fields have been studied by Harder [1981; 1982], Weselmann [1988], Berger [2008; 2009], and Branchereau [2023].

**Remark 2.16.** (1) Since $H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\flat) \otimes \mathbb{Q} = H^1(Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{Q}$, we have another integral structure $H^1_{\mathrm{int}}(Y^{\mathrm{BS}}, \mathcal{M}_n)$, and we can consider another denominator $\Delta'(\mathrm{Eis}_n)$ of the Eisenstein class $\mathrm{Eis}_n$:

$$\Delta'(\mathrm{Eis}_n) := \min\{\Delta \in \mathbb{Z}_{>0} \mid \Delta\mathrm{Eis}_n \in H^1_{\mathrm{int}}(Y^{\mathrm{BS}}, \mathcal{M}_n)\}.$$

However, we show in Lemma 6.1 that $\Delta(\mathrm{Eis}_n) = \Delta'(\mathrm{Eis}_n)$.

(2) By using the $q$-expansion at the cusp $i\infty$, one can regard $M_{n+2}(\Gamma)$ as a submodule of $\mathbb{C}[\![q]\!]$, and we obtain the de Rham rational structure of $M_{n+2}(\Gamma)$ by $M_{n+2}(\Gamma) \cap \mathbb{Q}[\![q]\!]$. The rationality of the critical values of the $L$-function associated with a cusp form is obtained by studying the gap between the de Rham and Betti rational structures via the Eichler–Shimura homomorphism $r$. However,

Proposition 2.11 shows that the Eisenstein parts of the two rational structures coincide. Moreover, Theorem 2.13 says that the Eisenstein parts of the two integral structures $M_{n+2}(\Gamma)\cap\mathbb{Z}[\![q]\!]$ and $H^1_{\mathrm{int}}(Y^{\mathrm{BS}},\mathcal{M}_n^\flat)$ coincide:

$$r(\mathbb{Q}E_{n+2}(z)\cap\mathbb{Z}[\![q]\!])=\mathbb{Q}\mathrm{Eis}_n\cap H^1_{\mathrm{int}}(Y^{\mathrm{BS}},\mathcal{M}_n^\flat),$$

since 2 is a regular prime. See [Harder 2021, § 1.1].

## 3. Construction of the cycle $\widetilde{T_p^m(C_\nu(\tau))}$

Fix a prime number $p$. In this section, we construct a special homology cycle

$$[\widetilde{T_p^m(C_\nu(\tau))}]\in H_1(Y^{\mathrm{BS}},\mathcal{M}_n\otimes\mathbb{Z}_{(p)})$$

that is used to compute the $p$-part of the denominator $\Delta(\mathrm{Eis}_n)$ of the Eisenstein class $\mathrm{Eis}_n$.

For any integer $1\le\nu\le n-1$ and any element $\tau\in\mathbb{H}^{\mathrm{BS}}$, we set

$$C_\nu(\tau):=\left\{-\frac{1}{\tau},\tau\right\}\otimes e_\nu\in\mathcal{MS}(\mathbb{H}^{\mathrm{BS}})\otimes\mathcal{M}_n,$$

where, as we recall, $e_\nu=X_1^\nu X_2^{n-\nu}$.

**3.1.** ***Computation of $T_p^m(C_\nu(\tau))$.*** Recall also the operators

$$V_p(\sigma\otimes P):=\begin{pmatrix}p&0\\0&1\end{pmatrix}\sigma\otimes\begin{pmatrix}p&0\\0&1\end{pmatrix}P,\quad U_p(\sigma\otimes P):=\sum_{j=0}^{p-1}\begin{pmatrix}1&j\\0&p\end{pmatrix}\sigma\otimes\begin{pmatrix}1&j\\0&p\end{pmatrix}P$$

on $S_\bullet(\mathbb{H}^{\mathrm{BS}})\otimes\mathcal{M}_n$. We have $T_p=V_p+U_p$. For each integer $m\ge0$, set

$$W_m:=\sum_{k=0}^{m}U_p^kV_p^{m-k}.$$

Note that $W_0$ is the identity map. For any (commutative) ring $R$ and cycle $C\in S_\bullet(\mathbb{H}^{\mathrm{BS}})\otimes(\mathcal{M}_n\otimes R)$, we denote by $[C]$ the image of $C$ in $(S_\bullet(\mathbb{H}^{\mathrm{BS}})\otimes(\mathcal{M}_n\otimes R))_\Gamma$.

**Lemma 3.1.** *Let $m\ge1$ be an integer and $C\in S_\bullet(\mathbb{H}^{\mathrm{BS}})\otimes\mathcal{M}_n$.*

(1) $T_p([W_m(C)])=[W_{m+1}(C)]+p^{n+1}[W_{m-1}(C)]$ *and* $T_p([W_0(C)])=[W_1(C)]$.

(2) $$T_p^m([C])=\sum_{A=0}^{\lfloor m/2\rfloor}\mathfrak{c}(m-A,A)p^{(n+1)A}[W_{m-2A}(C)],$$

*where $\lfloor m/2\rfloor$ is the greatest integer less than or equal to $m/2$ and*

$$\mathfrak{c}(A,B):=\left(1-\frac{B}{A+1}\right)\binom{A+B}{B}=\binom{A+B}{B}-\binom{A+B}{B-1}\in\mathbb{Z}.$$

*Here we assume $\binom{a}{b}=0$ if $b<0$.*

*Proof.* Claim (1) follows from the fact that $V_pU_p = p^{n+1}$ proved in Lemma 2.4. Let us prove Claim (2). For notational simplicity, we put

$$w_m := [W_m(C)].$$

Then Claim (1) shows that we can write

$$T_p^m([C]) = \sum_{k=0}^{m} a_k^{(m)} w_{m-k}$$

with $a_k^{(m)} \in \mathbb{Z}$ such that

$$\begin{aligned}
&a_0^{(m)} = a_0^{(m-1)} = \cdots = a_0^{(1)} = a_0^{(0)} = 1,\\
&a_1^{(m)} = a_1^{(m-1)} = \cdots = a_1^{(1)} = 0,\\
&a_k^{(m)} = a_k^{(m-1)} + p^{n+1}a_{k-2}^{(m-1)} \qquad (2 \le k \le m-1),\\
&a_m^{(m)} = p^{n+1}a_{m-2}^{(m-1)}.
\end{aligned}$$

Therefore, we have $a_{2k+1}^{(m)} = 0$ for any integer $0 \le k \le \frac{1}{2}(m-1)$, and hence

$$T_p^m([C]) = \sum_{k=0}^{\lfloor m/2 \rfloor} a_{2k}^{(m)} w_{m-2k}.$$

Let us show that $a_{2k}^{(m)} = \mathfrak{c}(m-k,k)\,p^{k(n+1)}$ for any integer $0 \le k \le \frac{1}{2}m$ by induction on $m$. When $m = 1$ or $k = 0$, this claim is clear. If $m > 1$ and $1 \le k \le \frac{1}{2}(m-1)$, then the induction hypothesis shows that

$$\begin{aligned}
a_{2k}^{(m)} = a_{2k}^{(m-1)} + p^{n+1}a_{2k-2}^{(m-1)} &= \mathfrak{c}(m-1-k,k)\,p^{k(n+1)} + \mathfrak{c}(m-k,k+1)\,p^{k(n+1)}\\
&= \mathfrak{c}(m-k,k)\,p^{k(n+1)}.
\end{aligned}$$

If $m$ is even (let $m = 2t$) then we have

$$a_m^{(m)} = p^{n+1}a_{m-2}^{(m-1)} = \mathfrak{c}(t,t-1)\,p^{t(n+1)} = \frac{1}{t+1}\frac{(2t)!}{(t!)(t!)}p^{t(n+1)} = \mathfrak{c}(t,t)\,p^{t(n+1)}. \quad \square$$

By definition, we have

$$\begin{aligned}
W_{m-2A}(C_\nu(\tau)) &= \sum_{k=0}^{m-2A} U_p^k V_p^{m-2A-k}(C_\nu(\tau))\\
&= \sum_{k=0}^{m-2A} U_p^k \begin{pmatrix} p^{m-2A-k} & 0 \\ 0 & 1 \end{pmatrix} \left( \left\{ -\frac{1}{\tau}, \tau \right\} \otimes e_\nu \right)\\
&= \sum_{k=0}^{m-2A} p^{(n-\nu)(m-2A-k)} U_p^k \left( \left\{ -\frac{p^{m-2A-k}}{\tau}, p^{m-2A-k}\tau \right\} \otimes e_\nu \right).
\end{aligned}$$

**Definition 3.2.** Take elements $\tau_0, \tau_1 \in \mathbb{H}^{\mathrm{BS}}$. For any integers $\nu$, $j$, and $k$ satisfying $1 \le \nu \le n-1$ and $0 \le j \le p^k - 1$, we define

$$\begin{aligned} C_{\nu,k,j}(\tau_0, \tau_1) &:= \begin{pmatrix} 1 & j \\ 0 & p^k \end{pmatrix} \left\{ -\frac{1}{\tau_0}, \tau_1 \right\} \otimes \begin{pmatrix} 1 & j \\ 0 & p^k \end{pmatrix} e_\nu \\ &= \left\{ \frac{-1/\tau_0 + j}{p^k}, \frac{\tau_1 + j}{p^k} \right\} \otimes \begin{pmatrix} 1 & j \\ 0 & p^k \end{pmatrix} e_\nu \in \mathcal{MS}(\mathbb{H}^{\mathrm{BS}}) \otimes \mathcal{M}_n. \end{aligned}$$

Note that we have

$$U_p^k\left( \left\{ -\frac{1}{\tau_0}, \tau_1 \right\} \otimes e_\nu \right) = \sum_{j=0}^{p^k-1} C_{\nu,k,j}(\tau_0, \tau_1).$$

To sum up, we obtain the following corollary.

**Corollary 3.3.** *We have*

$$\begin{aligned} T_p^m([C_\nu(\tau)]) = \sum_{A=0}^{\lfloor m/2 \rfloor} \mathfrak{c}(m-A, A) p^{(n+1)A} \\ \times \sum_{k=0}^{m-2A} p^{(n-\nu)(m-2A-k)} \sum_{j=0}^{p^k-1} \left[ C_{\nu,k,j}\left( \frac{\tau}{p^{m-2A-k}}, p^{m-2A-k}\tau \right) \right]. \end{aligned}$$

**3.2.** ***Computation of the boundary $\partial C_{\nu,k,j}(\tau_0, \tau_1)$.*** Next, we compute the boundary $\partial C_{\nu,k,j}(\tau_0, \tau_1)$.

**Definition 3.4.** (1) For any integers $j$ and $N$ with $p \nmid j$ and $N > 0$, we denote by $d_N(j)$ and $b_N(j)$ the integers uniquely determined by

$$1 \le d_N(j) < p^N \quad \text{and} \quad j d_N(j) - p^N b_N(j) = 1.$$

We also put $d_0(j) := 0$ and $b_0(j) := -1$ for any integer $j$.

(2) For any integers $k$ and $j$, we set

$$l_k(j) := \min\{\mathrm{ord}_p(j), k\}.$$

Note that $l_k(0) = k$. We also put $j' := j/p^{l_k(j)}$.

In the following, for integers $j$ and $k$ with $0 \le j \le p^k - 1$, we often write

$$l := l_k(j), \quad j' := j/p^{l_k(j)}, \quad d := d_{k-l_k(j)}(j'), \quad b := b_{k-l_k(j)}(j') \tag{3-1}$$

for simplicity.

**Definition 3.5.** For any integers $\nu$, $j$, and $k$ with $1 \le \nu \le n-1$ and $0 \le j \le p^k - 1$, we define homogeneous polynomials $E^{(1)}_{\nu,k,j}$ and $E^{(0)}_{\nu,k,j}$ in $\mathcal{M}_n$ by

$$\begin{aligned} E^{(1)}_{\nu,k,j}(X_1, X_2) &:= (p^k X_1 - j X_2)^\nu X_2^{n-\nu}, \\ E^{(0)}_{\nu,k,j}(X_1, X_2) &:= (-1)^{\nu+1} (p^l X_2)^\nu (p^{k-l} X_1 + d X_2)^{n-\nu}. \end{aligned}$$

**Lemma 3.6.** *We have*

$$\partial[C_{\nu,k,j}(\tau_0,\tau_1)] = \left[\left\{\frac{\tau_1+j}{p^k}\right\}\otimes E^{(1)}_{\nu,k,j} + \left\{\frac{p^l\tau_0-d}{p^{k-l}}\right\}\otimes E^{(0)}_{\nu,k,j}\right]$$

*in* $(S_0(\mathbb{H}^{\mathrm{BS}})\otimes\mathcal{M}_n)_\Gamma$.

*Proof.* By definition, we have

$$\partial[C_{\nu,k,j}(\tau_0,\tau_1)] = \left[\begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}\{\tau_1\}\otimes\begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}e_\nu - \begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}\left\{-\frac{1}{\tau_0}\right\}\otimes\begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}e_\nu\right].$$

The definition of $E^{(1)}_{\nu,k,j}$ shows that

$$\begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}\{\tau_1\}\otimes\begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}e_\nu = \left\{\frac{\tau_1+j}{p^k}\right\}\otimes E^{(1)}_{\nu,k,j}.$$

Moreover, we have

$$\begin{aligned}\begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}\left\{-\frac{1}{\tau_0}\right\}\otimes\begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}e_\nu &= \begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\{\tau_0\}\otimes\begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}e_\nu\\ &= \begin{pmatrix}j' & b\\ p^{k-l} & d\end{pmatrix}\begin{pmatrix}p^l & -d\\ 0 & p^{k-l}\end{pmatrix}\{\tau_0\}\otimes\begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}e_\nu.\end{aligned}$$

Since

$$\begin{aligned}&\left[\begin{pmatrix}j' & b\\ p^{k-l} & d\end{pmatrix}\begin{pmatrix}p^l & -d\\ 0 & p^{k-l}\end{pmatrix}\{\tau_0\}\otimes\begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}e_\nu\right]\\ &\qquad= \left[\begin{pmatrix}p^l & -d\\ 0 & p^{k-l}\end{pmatrix}\{\tau_0\}\otimes\begin{pmatrix}d & -b\\ -p^{k-l} & j'\end{pmatrix}\begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}e_\nu\right]\end{aligned}$$

and $\left(\begin{smallmatrix}d & -b\\ -p^{k-l} & j'\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & j\\ 0 & p^k\end{smallmatrix}\right)e_\nu = -E^{(0)}_{\nu,k,j}$, we obtain

$$\left[\begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}\left\{-\frac{1}{\tau_0}\right\}\otimes\begin{pmatrix}1 & j\\ 0 & p^k\end{pmatrix}e_\nu\right] = -\left[\left\{\frac{p^l\tau_0-d}{p^{k-l}}\right\}\otimes E^{(0)}_{\nu,k,j}\right].$$ □

**3.3.** ***A cycle*** $\widetilde{T^m_p(C_\nu(\tau))}$ ***in*** $\mathcal{MS}(\mathbb{H}^{\mathrm{BS}})\otimes(\mathcal{M}_n\otimes\mathbb{Q})$**.** In this subsection, for each integer $1\le\nu\le n-1$ and $m\ge n$, we construct a $p$-adically integral cycle $\widetilde{T^m_p(C_\nu(\tau))}$ in $\mathcal{MS}(\mathbb{H}^{\mathrm{BS}})\otimes(\mathcal{M}_n\otimes\mathbb{Z}_{(p)})$ such that the image of $\widetilde{T^m_p(C_\nu(\tau))}$ in $H_1(Y^{\mathrm{BS}},\partial Y^{\mathrm{BS}},\mathcal{M}_n\otimes\mathbb{Z}_{(p)})$ is $T^m_p([\{0,i\infty\}\otimes e_\nu])$.

**3.3.1.** *Bernoulli polynomials.* Since a key tool for constructing the cycle $\widetilde{T^m_p(C_\nu(\tau))}$ is the Bernoulli polynomials, we briefly recall their basic properties.

Let $t$ be a nonnegative integer. We denote by $B_t(x)$ the $t$-th Bernoulli polynomial and by

$$B_t := B_t(0)$$

the $t$-th Bernoulli number. For notational simplicity, we put

$$\tilde{B}_t(x) := \frac{1}{t}(B_t(x) - B_t).$$

We use the following well-known facts without any notice:

$$B_t(x) = \sum_{\mu=0}^{t} \binom{t}{\mu} B_{t-\mu} x^{\mu}, \quad \int_x^{x+1} B_t(x)\,dx = x^t,$$

$$\int_a^x B_t(x)\,dx = \frac{B_{t+1}(x) - B_{t+1}(a)}{t+1}, \quad \mathrm{ord}_p(B_t) \geq -1.$$

The last fact is called the von Staudt–Clausen theorem. We note that the second and third facts imply that

$$\sum_{j=0}^{x-1} j^t = \tilde{B}_{t+1}(x).$$

**3.3.2.** *$P^\dagger$ and $P^\ddagger$.* Set

$$\mathcal{M}_{n,(p)} := \bigoplus_{\mu=0}^{n} \mathbb{Z}_{(p)} X_1^{\mu} X_2^{n-\mu} \quad \text{and} \quad \mathcal{M}^{\circ}_{n,(p)} := \bigoplus_{\mu=0}^{n-1} \mathbb{Z}_{(p)} X_1^{\mu} X_2^{n-\mu},$$

and let

$$\dagger : \mathcal{M}^{\circ}_{n,(p)} \otimes \mathbb{Q} \to \mathcal{M}_{n,(p)} \otimes \mathbb{Q}; \quad P \mapsto P^\dagger$$

be the $\mathbb{Q}$-linear map defined by

$$(X_1^{\mu} X_2^{n-\mu})^\dagger := X_2^n \frac{B_{\mu+1}(X_1/X_2) - B_{\mu+1}}{\mu+1} = X_2^n \tilde{B}_{\mu+1}(X_1/X_2).$$

For any integer $\mu \in \{0, \ldots, n-1\}$, we have $1 + \mathrm{ord}_p(\mu+1) \leq n$, and hence $p^n X_2^n \tilde{B}_{\mu+1}(X_1/X_2) \in \mathcal{M}_{n,(p)}$. This fact shows that

$$(\mathcal{M}^{\circ}_{n,(p)})^\dagger \subset \frac{1}{p^n} \mathcal{M}_{n,(p)},$$

where $(\mathcal{M}^{\circ}_{n,(p)})^\dagger$ denotes the image of $\mathcal{M}^{\circ}_{n,(p)}$ under the map $\dagger$. Similarly, let

$$\ddagger : \mathcal{M}^{\circ}_{n,(p)} \otimes \mathbb{Q} \to \mathcal{M}_n \otimes \mathbb{Q}; \quad P \mapsto P^\ddagger$$

be the $\mathbb{Q}$-linear map defined by

$$(X_1^{\mu} X_2^{n-\mu})^\ddagger := X_2^n \frac{(X_1/X_2)^{\mu+1} - B_{\mu+1}}{\mu+1}.$$

The following lemma follows from the definitions of $P^\dagger$ and $P^\ddagger$.

**Lemma 3.7.** *For any polynomial $P(X_1, X_2) \in \mathcal{M}_n^\circ \otimes \mathbb{Q}$, we have*

$$P^\dagger(X_1 + X_2, X_2) - P^\dagger(X_1, X_2) = P(X_1, X_2),$$
$$\int_x^{x+1} P^\dagger(z, 1)\, dz = P^\ddagger(x, 1), \quad \int_{x_1}^{x_2} P(z, 1)\, dz = P^\ddagger(x_2, 1) - P^\ddagger(x_1, 1).$$

**3.3.3.** *Definitions of polynomials $P^{(1)}_{\nu,k,j}$ and $P^{(0)}_{\nu,k,j}$.*

**Definition 3.8.** For any integers $\nu$, $j$, and $k$ with $1 \le \nu \le n-1$ and $0 \le j \le p^k - 1$, we define polynomials $P^{(1)}_{\nu,k,j}$ and $P^{(0)}_{\nu,k,j}$ in $\mathcal{M}_n \otimes \mathbb{Q}$ by

$$P^{(1)}_{\nu,k,j} := E^{(1)\dagger}_{\nu,k,j}, \quad P^{(0)}_{\nu,k,j} := E^{(0)\dagger}_{\nu,k,j}.$$

**Lemma 3.9.** *For each $i \in \{0, 1\}$, we have $P^{(i)}_{\nu,k,j} \in p^{\min\{0,k-n\}} \mathcal{M}_{n,(p)}$.*

*Proof.* This follows from the fact that $(\mathcal{M}^\circ_{n,(p)})^\dagger \subset p^{-n}\mathcal{M}_{n,(p)}$ and $(X_2^n)^\dagger = X_1 X_2^{n-1}$ and that the coefficient of $X_1^\mu X_2^{n-\mu}$ in $E^{(i)}_{\nu,k,j}$ is divided by $p^k$ if $\mu \ge 1$. □

**3.3.4.** *Definition of the cycle $\widetilde{T_p^m(C_\nu(\tau))}$.* Now we can define $\tilde{C}_{\nu,k,j}(\tau_0, \tau_1)$ and $\widetilde{T_p^m(C_\nu(\tau))}$.

**Definition 3.10.** Let $\tau_0, \tau_1 \in \mathbb{H}^{\mathrm{BS}}$. For any integers $\nu$, $k$, and $j$ with $1 \le \nu \le n-1$ and $0 \le j \le p^k - 1$, we define an element $\tilde{C}_{\nu,k,j}(\tau_0, \tau_1) \in \mathcal{MS}(\mathbb{H}^{\mathrm{BS}}) \otimes (\mathcal{M}_n \otimes \mathbb{Q})$ by

$$\begin{aligned}&\tilde{C}_{\nu,k,j}(\tau_0, \tau_1)\\ &:= C_{\nu,k,j}(\tau_0, \tau_1) - \left\{\frac{\tau_1 + j}{p^k}, \frac{\tau_1 + j}{p^k} + 1\right\} \otimes P^{(1)}_{\nu,k,j} - \left\{\frac{p^l\tau_0 - d}{p^{k-l}}, \frac{p^l\tau_0 - d}{p^{k-l}} + 1\right\} \otimes P^{(0)}_{\nu,k,j}.\end{aligned}$$

We also put

$$\tilde{C}^{\mathrm{int}}_{\nu,k,j}(\tau_0, \tau_1) := p^{\max\{0,n-k\}} \tilde{C}_{\nu,k,j}(\tau_0, \tau_1).$$

Note that $\tilde{C}^{\mathrm{int}}_{\nu,k,j}(\tau_0, \tau_1) \in \mathcal{MS}(\mathbb{H}^{\mathrm{BS}}) \otimes \mathcal{M}_{n,(p)}$ by Lemma 3.9.

**Lemma 3.11.** *We have*

$$\partial[\tilde{C}^{\mathrm{int}}_{\nu,k,j}(\tau_0, \tau_1)] = 0 \quad \text{in } (S_0(\mathbb{H}^{\mathrm{BS}}) \otimes \mathcal{M}_{n,(p)})_\Gamma.$$

*In particular, $[\tilde{C}^{\mathrm{int}}_{\nu,k,j}(\tau_0, \tau_1)]$ defines a homology class*

$$[\tilde{C}^{\mathrm{int}}_{\nu,k,j}(\tau_0, \tau_1)] \in H_1(Y^{\mathrm{BS}}, \mathcal{M}_{n,(p)}).$$

*Proof.* This follows from Definitions 3.5 and 3.8 and Lemmas 3.6 and 3.7. □

**Lemma 3.12.** *The homology class $[\tilde{C}^{\mathrm{int}}_{\nu,k,j}(\tau_0, \tau_1)]$ is independent of the choices of $\tau_0$ and $\tau_1$.*

*Proof.* Let $\tau_0'$ and $\tau_1'$ be another pair of points in $\mathbb{H}^{\mathrm{BS}}$. We will prove that

$$[\tilde{C}^{\mathrm{int}}_{\nu,k,j}(\tau_0,\tau_1)] = [\tilde{C}^{\mathrm{int}}_{\nu,k,j}(\tau_0',\tau_1')] \quad \text{in } H_1(Y^{\mathrm{BS}}, \mathcal{M}_{n,(p)}).$$

It suffices to construct an element $h \in S_2(\mathbb{H}^{\mathrm{BS}}) \otimes \mathcal{M}_{n,(p)}$ such that

$$\partial[h] = [\tilde{C}^{\mathrm{int}}_{\nu,k,j}(\tau_0,\tau_1)] - [\tilde{C}^{\mathrm{int}}_{\nu,k,j}(\tau_0',\tau_1')] \quad \text{in } (\mathcal{MS}(\mathbb{H}^{\mathrm{BS}}) \otimes \mathcal{M}_{n,(p)})_\Gamma.$$

For notational simplicity, set $q := p^{\max\{0,n-k\}}$. First, since $\mathbb{H}^{\mathrm{BS}}$ is simply connected, there exist elements $h_1, h_2, h_3 \in S_2(\mathbb{H}^{\mathrm{BS}})$ such that

$$\begin{aligned}
\partial h_1 &= \left\{\frac{-1/\tau_0+j}{p^k}, \frac{\tau_1+j}{p^k}\right\} - \left\{\frac{-1/\tau_0'+j}{p^k}, \frac{\tau_1'+j}{p^k}\right\} \\
&\quad - \left\{\frac{\tau_1'+j}{p^k}\frac{\tau_1+j}{p^k}\right\} + \left\{\frac{-1/\tau_0'+j}{p^k}, \frac{-1/\tau_0+j}{p^k}, \right\}, \\
\partial h_2 &= \left\{\frac{\tau_1+j}{p^k}, \frac{\tau_1+j}{p^k}+1\right\} - \left\{\frac{\tau_1'+j}{p^k}, \frac{\tau_1'+j}{p^k}+1\right\} \\
&\quad - \left\{\frac{\tau_1'+j}{p^k}+1, \frac{\tau_1+j}{p^k}+1\right\} + \left\{\frac{\tau_1'+j}{p^k}, \frac{\tau_1+j}{p^k}\right\}, \\
\partial h_3 &= \left\{\frac{p^l\tau_0-d}{p^{k-l}}, \frac{p^l\tau_0-d}{p^{k-l}}+1\right\} - \left\{\frac{p^l\tau_0'-d}{p^{k-l}}, \frac{p^l\tau_0'-d}{p^{k-l}}+1\right\} \\
&\quad - \left\{\frac{p^l\tau_0'-d}{p^{k-l}}+1, \frac{p^l\tau_0-d}{p^{k-l}}+1\right\} + \left\{\frac{p^l\tau_0'-d}{p^{k-l}}, \frac{p^l\tau_0-d}{p^{k-l}}\right\}.
\end{aligned}$$

Then we see that

$$h := q\left(h_1 \otimes \begin{pmatrix} 1 & j \\ 0 & p^k \end{pmatrix} e_\nu - h_2 \otimes P^{(1)}_{\nu,k,j} - h_3 \otimes P^{(0)}_{\nu,k,j}\right)$$

satisfies the desired property. Indeed, we have

$$\partial h = q(\tilde{C}_{\nu,k,j}(\tau_0,\tau_1) - \tilde{C}_{\nu,k,j}(\tau_0',\tau_1') - B^{(1)} - B^{(0)}),$$

where

$$\begin{aligned}
B^{(1)} &:= \left\{\frac{\tau_1'+j}{p^k}\frac{\tau_1+j}{p^k}\right\} \otimes \begin{pmatrix} 1 & j \\ 0 & p^k \end{pmatrix} e_\nu \\
&\quad - \left\{\frac{\tau_1'+j}{p^k}+1, \frac{\tau_1+j}{p^k}+1\right\} \otimes P^{(1)}_{\nu,k,j} + \left\{\frac{\tau_1'+j}{p^k}, \frac{\tau_1+j}{p^k}\right\} \otimes P^{(1)}_{\nu,k,j}, \\
B^{(0)} &:= -\left\{\frac{-1/\tau_0'+j}{p^k}, \frac{-1/\tau_0+j}{p^k}\right\} \otimes \begin{pmatrix} 1 & j \\ 0 & p^k \end{pmatrix} e_\nu \\
&\quad - \left\{\frac{p^l\tau_0'-d}{p^{k-l}}+1, \frac{p^l\tau_0-d}{p^{k-l}}+1\right\} \otimes P^{(0)}_{\nu,k,j} + \left\{\frac{p^l\tau_0'-d}{p^{k-l}}, \frac{p^l\tau_0-d}{p^{k-l}}\right\} \otimes P^{(0)}_{\nu,k,j}.
\end{aligned}$$

Then the same computation as in the proof of Lemma 3.6 shows that

$$\begin{aligned} &qB^{(1)} \\ &\equiv \left\{ \frac{\tau_1' + j}{p^k} \frac{\tau_1 + j}{p^k} \right\} \otimes q\big(E^{(1)}_{\nu,k,j}(X_1, X_2) - P^{(1)}_{\nu,k,j}(X_1 + X_2, X_2) + P^{(1)}_{\nu,k,j}(X_1, X_2)\big) = 0, \\ &qB^{(0)} \equiv \left\{ \frac{p^l \tau_0' - d}{p^{k-l}} + 1, \frac{p^l \tau_0 - d}{p^{k-l}} + 1 \right\} \\ &\qquad \otimes q\big(E^{(0)}_{\nu,k,j}(X_1, X_2) - P^{(0)}_{\nu,k,j}(X_1 + X_2, X_2) + P^{(0)}_{\nu,k,j}(X_1, X_2)\big) = 0, \end{aligned}$$

where $\equiv$ means that it is an equality in the $\Gamma$-coinvariant $(\mathcal{MS}(\mathbb{H}^{\mathrm{BS}}) \otimes \mathcal{M}_{n,(p)})_\Gamma$. Hence we obtain $\partial[h] = [\tilde{C}^{\mathrm{int}}_{\nu,k,j}(\tau_0, \tau_1)] - [\tilde{C}^{\mathrm{int}}_{\nu,k,j}(\tau_0', \tau_1')]$. □

**Definition 3.13.** For any integer $m \geq 0$ and element $\tau \in \mathbb{H}^{\mathrm{BS}}$, we define a cycle $\widetilde{T^m_p(C_\nu(\tau))} \in \mathcal{MS}(\mathbb{H}^{\mathrm{BS}}) \otimes (\mathcal{M}_n \otimes \mathbb{Q})$ by

$$\begin{aligned} \widetilde{T^m_p(C_\nu(\tau))} := \sum_{A=0}^{\lfloor m/2 \rfloor} &\mathfrak{c}(m-A, A) p^{(n+1)A} \\ &\times \sum_{k=0}^{m-2A} p^{(n-\nu)(m-2A-k)} \sum_{j=0}^{p^k-1} \tilde{C}_{\nu,k,j}\left( \frac{\tau}{p^{m-2A-k}}, p^{m-2A-k}\tau \right). \end{aligned}$$

**Definition 3.14.** For any integer $\nu \in \{0, \ldots, n\}$, we define a homology class

$$[C_\nu] \in H_1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_n)$$

to be the element represented by the cycle

$$C_\nu := \{0, i\infty\} \otimes e_\nu,$$

where $i\infty \in \partial\mathbb{H}^{\mathrm{BS}}$ is a point such that

$$i\infty := \lim_{\substack{t \in \mathbb{R}_{>0} \\ t \to \infty}} it.$$

**Lemma 3.15.** *Let $1 \leq \nu \leq n-1$ be an integer.*

(1) $\partial[\widetilde{T^m_p(C_\nu(\tau))}] = 0$ *in* $(S_0(\mathbb{H}^{\mathrm{BS}}) \otimes \mathcal{M}_n \otimes \mathbb{Q})_\Gamma$.

(2) *If $m \geq n$, then we have $\widetilde{T^m_p(C_\nu(\tau))} \in \mathcal{MS}(\mathbb{H}^{\mathrm{BS}}) \otimes \mathcal{M}_{n,(p)}$. Hence $\widetilde{T^m_p(C_\nu(\tau))}$ defines a $p$-integral homology class $[\widetilde{T^m_p(C_\nu(\tau))}]$ in $H_1(Y^{\mathrm{BS}}, \mathcal{M}_{n,(p)})$ which is independent of the choice of $\tau$.*

(3) *If $m \geq n$, then the image of the homology class $[\widetilde{T^m_p(C_\nu(\tau))}]$ under the homomorphism $H_1(Y^{\mathrm{BS}}, \mathcal{M}_{n,(p)}) \to H_1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,(p)})$ is $T^m_p([C_\nu])$.*

*Proof.* Claim (1) follows from Lemma 3.11. Let us show Claim (2). By Lemma 3.9, we have $p^{\max\{0,n-k\}}\tilde{C}_{\nu,k,j}(\tau_0,\tau_1) \in \mathcal{MS}(\mathbb{H}^{\mathrm{BS}}) \otimes \mathcal{M}_{n,(p)}$. Since $1 \le \nu \le n-1$ and $2 \le n$, we have

$$(n+1)A + (n-\nu)(m-2A-k) \ge (n+1)A + m - 2A - k \ge m - k$$

for any nonnegative integer $A$. This fact shows that $\widetilde{T_p^m(C_\nu(\tau))} \in \mathcal{MS}(\mathbb{H}) \otimes \mathcal{M}_{n,(p)}$ if $m \ge n$. It follows from Lemma 3.12 that the homology class $[\widetilde{T_p^m(C_\nu(\tau))}]$ does not depend on the choice of $\tau$. Claim (3) follows from the definition of $\widetilde{T_p^m(C_\nu(\tau))}$ and Lemma 3.1. □

**Remark 3.16.** The above construction of $\widetilde{T_p^m(C_\nu(\tau))}$, especially Definition 3.10, is called the spectacle construction since the cycle $\tilde{C}_{\nu,k,j}(\tau_0,\tau_1)$ looks like a pair of spectacles. See Funke–Millson [2011] for more detail.

## 4. Period

The aim of this section is to compute the value

$$\langle \mathrm{Eis}_n, [\widetilde{T_p^{m!}(C_\nu(\tau))}] \rangle$$

and its $p$-adic limit as $m \to \infty$, that are independent of $\tau \in \mathbb{H}^{\mathrm{BS}}$ by Lemma 3.15(2). In the sequel, the symbol $\lim_{m\to\infty}$ will always mean the $p$-adic limit. The following is the main result of this section.

**Theorem 4.1.** *For any $\tau \in \mathbb{H}^{\mathrm{BS}}$ and any integer $\nu \in \{1,\dots,n-1\}$, we have*

$$\lim_{m\to\infty} \langle \mathrm{Eis}_n, [\widetilde{T_p^{m!}(C_\nu(\tau))}] \rangle = (1-p^{n+1})\left( \frac{1}{1-p^{n+1}} \frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)} - \frac{\zeta(-\nu)}{1-p^{n-\nu}} - \frac{\zeta(\nu-n)}{1-p^{\nu}} \right).$$

In fact, we show in Section 4.3 that

$$\lim_{m\to\infty} \sum_{k=0}^{m} p^{(n-\nu)(m-k)} \sum_{j=0}^{p^k-1} \left\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}\left( \frac{\tau}{p^{m-k}}, p^{m-k}\tau \right) \right\rangle = \frac{1}{1-p^{n+1}} \frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)} - \frac{\zeta(-\nu)}{1-p^{n-\nu}} - \frac{\zeta(\nu-n)}{1-p^{\nu}}. \quad (4\text{-}1)$$

Hence Theorem 4.1 follows from (4-1) and the following lemma.

**Lemma 4.2.** *Suppose that the $p$-adic limit*

$$\lim_{m\to\infty} \sum_{k=0}^{m} p^{(n-\nu)(m-k)} \sum_{j=0}^{p^k-1} \left\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}\left( \frac{\tau}{p^{m-k}}, p^{m-k}\tau \right) \right\rangle$$

*exists. We then have*

$$\lim_{m\to\infty}\langle \mathrm{Eis}_n, [\widetilde{T_p^{m!}(C_\nu(\tau))}]\rangle = (1-p^{n+1})\lim_{m\to\infty}\sum_{k=0}^{m} p^{(n-\nu)(m-k)}\sum_{j=0}^{p^k-1}\left\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}\left(\frac{\tau}{p^{m-k}}, p^{m-k}\tau\right)\right\rangle.$$

*Proof.* For notational simplicity, we put

$$\mathscr{W}^{(m)} := \sum_{k=0}^{m} p^{(n-\nu)(m-k)}\sum_{j=0}^{p^k-1}\left\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}\left(\frac{\tau}{p^{m-k}}, p^{m-k}\tau\right)\right\rangle$$

and $\mathscr{W} := \lim_{m\to\infty}\mathscr{W}^{(m)}$. We then have

$$\langle \mathrm{Eis}_n, [\widetilde{T_p^{m!}(C_\nu(\tau))}]\rangle = \sum_{A=0}^{\frac{1}{2}m!}\mathfrak{c}(m!-A, A)p^{(n+1)A}\mathscr{W}^{(m!-2A)}.$$

Take a positive integer $Q$. There is a positive integer $r$ such that $\mathscr{W}^{(s)} - \mathscr{W} \in p^Q\mathbb{Z}_p$ for any integer $s \geq r$. Hence we have

$$\sum_{A=0}^{\lfloor m/2\rfloor}\mathfrak{c}(m-A, A)p^{(n+1)A}(\mathscr{W}^{(m-2A)} - \mathscr{W}) \equiv \sum_{A=\lfloor (m-r)/2\rfloor+1}^{\lfloor m/2\rfloor}\mathfrak{c}(m-A, A)p^{(n+1)A}(\mathscr{W}^{(m-2A)} - \mathscr{W}) \pmod{p^Q\mathbb{Z}_p}.$$

The sequence $(\mathscr{W}^{(m)} - \mathscr{W})_{m=0}^{\infty}$ is bounded in $\mathbb{Q}_p$, and hence for any sufficiently large integer $m$, we have

$$\sum_{A=\lfloor (m-r)/2\rfloor+1}^{\lfloor m/2\rfloor}\mathfrak{c}(m-A, A)p^{(n+1)A}(\mathscr{W}^{(m-2A)} - \mathscr{W}) \in p^Q\mathbb{Z}_p.$$

This implies that

$$\lim_{m\to\infty}\langle \mathrm{Eis}_n, [\widetilde{T_p^{m!}(C_\nu(\tau))}]\rangle = \lim_{m\to\infty}\sum_{A=0}^{\frac{1}{2}m!}\mathfrak{c}(m!-A, A)p^{(n+1)A}\mathscr{W}.$$

Since $\mathfrak{c}(m!-A, A) = \binom{m!}{A} - \binom{m!}{A-1}$ (note that $\binom{m}{-1} = 0$), we have

$$\sum_{A=0}^{\frac{1}{2}m!}\mathfrak{c}(m!-A, A)p^{(n+1)A} = \sum_{A=0}^{\frac{1}{2}m!}\binom{m!}{A}p^{(n+1)A} - \sum_{A=0}^{\frac{1}{2}m!-1}\binom{m!}{A}p^{(n+1)(A+1)} = (1-p^{n+1})\sum_{A=0}^{\frac{1}{2}m!-1}\binom{m!}{A}p^{(n+1)A} + \binom{m!}{\frac{1}{2}m!}p^{\frac{1}{2}(n+1)m!}.$$

Since

$$\sum_{A=0}^{\frac{1}{2}m!-1}\binom{m!}{A}p^{(n+1)A} \equiv \sum_{A=0}^{m!}\binom{m!}{A}p^{(n+1)A} \pmod{p^{\frac{1}{2}m!}} = (1+p^{n+1})^{m!},$$

we obtain that

$$\lim_{m\to\infty}\sum_{A=0}^{\frac{1}{2}m!}\mathfrak{c}(m!-A, A)p^{(n+1)A} = (1-p^{n+1})\lim_{m\to\infty}(1+p^{n+1})^{m!} = 1-p^{n+1}. \quad \square$$

Therefore, it remains to prove (4-1), and this will be done in Proposition 4.13.

**4.1. $\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}(\tau_0,\tau_1)\rangle$.** We start by computing the value $\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}(\tau_0,\tau_1)\rangle$. In this subsection, we fix integers $k$ and $j$ with $0 \le j \le p^k-1$. Recall that

$$l := l_k(j) \quad \text{and} \quad d := d_{k-l_k(j)}(j') = d_{k-l_k(j)}(j/p^{l_k(j)})$$

are taken as in Definition 3.4 and (3-1) and that

$$\begin{aligned}
&\tilde{C}_{\nu,k,j}(\tau_0,\tau_1)\\
&= \left\{\frac{-1/\tau_0+j}{p^k}, \frac{\tau_1+j}{p^k}\right\}\otimes(p^kX_1-jX_2)^\nu X_2^{n-\nu} - \left\{\frac{\tau_1+j}{p^k}, \frac{\tau_1+j}{p^k}+1\right\}\otimes P^{(1)}_{\nu,k,j}\\
&\qquad - \left\{\frac{p^l\tau_0-d}{p^{k-l}}, \frac{p^l\tau_0-d}{p^{k-l}}+1\right\}\otimes P^{(0)}_{\nu,k,j}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}(\tau_0,\tau_1)\rangle\\
&= \int_{(-1/\tau_0+j)/p^k}^{(\tau_1+j)/p^k} E_{n+2}(z)(p^kz-j)^\nu\,dz - \int_{(\tau_1+j)/p^k}^{(\tau_1+j)/p^k+1} E_{n+2}(z)P^{(1)}_{\nu,k,j}(z,1)\,dz\\
&\qquad - \int_{(p^l\tau_0-d)/p^{k-l}}^{(p^l\tau_0-d)/p^{k-l}+1} E_{n+2}(z)P^{(0)}_{\nu,k,j}(z,1)\,dz\\
&= \frac{1}{p^k}\int_{-1/\tau_0}^{\tau_1} E_{n+2}\left(\frac{z+j}{p^k}\right)z^\nu\,dz - \int_{(\tau_1+j)/p^k}^{(\tau_1+j)/p^k+1} E_{n+2}(z)P^{(1)}_{\nu,k,j}(z,1)\,dz\\
&\qquad - \int_{(p^l\tau_0-d)/p^{k-l}}^{(p^l\tau_0-d)/p^{k-l}+1} E_{n+2}(z)P^{(0)}_{\nu,k,j}(z,1)\,dz. \qquad (4\text{-}2)
\end{aligned}$$

The definition of the Eisenstein series $E_{n+2}$ shows that

$$\begin{aligned}
E_{n+2}\left(\frac{z+j}{p^k}\right) - 1 &= O(e^{-2\pi\,\mathrm{Im}(z)}) && \text{for } \mathrm{Im}(z)\ge 1,\\
E_{n+2}\left(\frac{z+j}{p^k}\right) - \frac{p^{l(n+2)}}{z^{n+2}} &= O\left(\frac{p^{l(n+2)}}{|z|^{n+2}}e^{-2\pi p^{2l-k}/\mathrm{Im}(z)}\right) && \text{for } \mathrm{Im}(z)\le 1,
\end{aligned}$$

where $f(z) = O(g(z))$ means that there is a constant $C$ which does not depend on $k$ and $j$ such that $|f(z)| \leq Cg(z)$. Set

$$\mathscr{L}_{k,j}(s) := \int_0^\infty \left( E_{n+2}\left(\frac{iy+j}{p^k}\right) - 1 \right) y^s \, dy.$$

We have the following Lemma 4.3 and Proposition 4.4, whose proofs will be given in Section 4.1.1 and Section 4.1.3, respectively.

**Lemma 4.3.** *The function $\mathscr{L}_{k,j}(s)$ converges for $\mathrm{Re}(s) > n+1$, and is continued to a meromorphic function on $\mathbb{C}$. Moreover, it has at most simple poles at $s = -1$ and $s = n+1$. In particular, $\mathscr{L}_{k,j}(s)$ is holomorphic at $s = \nu$ for any integer $1 \leq \nu \leq n-1$.*

**Proposition 4.4.** *We have*

$$\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}(\tau_0, \tau_1) \rangle = \frac{i^{\nu+1}}{p^k} \mathscr{L}_{k,j}(\nu) - E^{(1)\ddagger}_{\nu,k,j}\left(\frac{j}{p^k}, 1\right) - E^{(0)\ddagger}_{\nu,k,j}\left(-\frac{d}{p^{k-l}}, 1\right).$$

Note that the value $\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}(\tau_0, \tau_1) \rangle$ does not depend on the choices of $\tau_0$ and $\tau_1$ since $\partial[\tilde{C}_{\nu,k,j}(\tau_0, \tau_1)] = 0$ by Lemma 3.11. Therefore, in the following we take $\tau_0 = it_0$ and $\tau_1 = it_1$ for $t_0, t_1 \in \mathbb{R}_{>0}$.

**4.1.1.** *Computation of the first term of (4-2).* We compute the first term of (4-2):

$$\frac{1}{p^k} \int_{i/t_0}^{it_1} E_{n+2}\left(\frac{z+j}{p^k}\right) z^\nu \, dz.$$

This integral is transformed as follows:

$$\begin{aligned}
&\frac{1}{p^k} \int_{i/t_0}^{it_1} E_{n+2}\left(\frac{z+j}{p^k}\right) z^\nu \, dz \\
&= \frac{1}{p^k} \int_{i/t_0}^{\infty} \left( E_{n+2}\left(\frac{z+j}{p^k}\right) - 1 \right) z^\nu \, dz + \frac{1}{p^k} \int_{i/t_0}^{it_1} z^\nu \, dz \\
&\qquad - \frac{1}{p^k} \int_{it_1}^{\infty} \left( E_{n+2}\left(\frac{z+j}{p^k}\right) - 1 \right) z^\nu \, dz \\
&= \frac{i^{\nu+1}}{p^k} \left\{ \int_{1/t_0}^{\infty} \left( E_{n+2}\left(\frac{iy+j}{p^k}\right) - 1 \right) y^\nu \, dy + \int_{1/t_0}^{t_1} y^\nu \, dy \right. \\
&\qquad \left. - \int_{t_1}^{\infty} \left( E_{n+2}\left(\frac{iy+j}{p^k}\right) - 1 \right) y^\nu \, dy \right\}.
\end{aligned}$$

Set

$$\mathscr{S}_{k,j}(t_0,t_1,s) := \int_{1/t_0}^{\infty}\left(E_{n+2}\left(\frac{iy+j}{p^k}\right)-1\right)y^s\,dy + \int_{1/t_0}^{t_1} y^s\,dy,$$

$$\mathscr{R}^{(1)}_{k,j}(t_1,s) := \int_{t_1}^{\infty}\left(E_{n+2}\left(\frac{iy+j}{p^k}\right)-1\right)y^s\,dy,$$

$$\mathscr{R}^{(0)}_{k,j}(t_0,s) := \int_{0}^{1/t_0}\left(E_{n+2}\left(\frac{iy+j}{p^k}\right)-\frac{p^{l(n+2)}}{(iy)^{n+2}}\right)y^s\,dy.$$

Then we have

$$\frac{1}{p^k}\int_{i/t_0}^{it_1} E_{n+2}\left(\frac{z+j}{p^k}\right)z^\nu\,dz = \frac{i^{\nu+1}}{p^k}(\mathscr{S}_{k,j}(t_0,t_1,\nu)-\mathscr{R}^{(1)}_{k,j}(t_1,\nu)).$$

We see that:

- The first terms of $\mathscr{S}_{k,j}(t_0,t_1,s)$ and $\mathscr{R}^{(i)}_{k,j}(t_i,s)$ converge for all $s\in\mathbb{C}$.
- The second term of $\mathscr{S}_{k,j}(t_0,t_1,s)$ is meromorphic and has at most one simple pole at $s=-1$.

In addition, we also see that

$$\begin{aligned}\mathscr{S}_{k,j}(t_0,t_1,s) &= \mathscr{L}_{k,j}(s)-\mathscr{R}^{(0)}_{k,j}(t_0,s)-\int_0^{1/t_0}\frac{p^{l(n+2)}}{(iy)^{n+2}}y^s\,dy+\int_0^{t_1}y^s\,dy\\ &= \mathscr{L}_{k,j}(s)-\mathscr{R}^{(0)}_{k,j}(t_0,s)-\frac{p^{l(n+2)}}{i^{n+2}}\frac{1}{s-n-1}\frac{1}{t_0^{s-n-1}}+\frac{1}{s+1}t_1^{s+1}.\end{aligned}$$

In particular, all of these functions are meromorphically continued to $s\in\mathbb{C}$ and are holomorphic at $s=\nu$. This proves Lemma 4.3, and we also get the following.

**Lemma 4.5.**

$$\begin{aligned}&\frac{1}{p^k}\int_{i/t_0}^{it_1} E_{n+2}\left(\frac{z+j}{p^k}\right)z^\nu\,dz\\ &\quad=\frac{i^{\nu+1}}{p^k}\left\{\mathscr{L}_{k,j}(\nu)+\frac{t_1^{\nu+1}}{\nu+1}+\frac{p^{l(n+2)}}{i^{n+2}}\frac{t_0^{n-\nu+1}}{n-\nu+1}-\mathscr{R}^{(1)}_{k,j}(t_1,\nu)-\mathscr{R}^{(0)}_{k,j}(t_0,\nu)\right\}.\end{aligned}$$

**4.1.2.** *Computation of the second and the third terms of* (4-2). Here we compute

$$\int_{(\tau_1+j)/p^k}^{(\tau_1+j)/p^k+1} E_{n+2}(z)P^{(1)}_{\nu,k,j}(z,1)\,dz+\int_{(p^l\tau_0-d)/p^{k-l}}^{(p^l\tau_0-d)/p^{k-l}+1} E_{n+2}(z)P^{(0)}_{\nu,k,j}(z,1)\,dz.$$

We put

$$\begin{aligned}\mathscr{T}_{\nu,k,j}(\tau_0,\tau_1) := &\int_{(\tau_1+j)/p^k}^{(\tau_1+j)/p^k+1}(E_{n+2}(z)-1)P^{(1)}_{\nu,k,j}(z,1)\,dz\\ &+\int_{(p^l\tau_0-d)/p^{k-l}}^{(p^l\tau_0-d)/p^{k-l}+1}(E_{n+2}(z)-1)P^{(0)}_{\nu,k,j}(z,1)\,dz.\end{aligned}$$

Note that $\mathscr{T}_{\nu,k,j}(\tau_0,\tau_1)\to 0$ as $\tau_0,\tau_1\to\infty$. On the other hand by Lemma 3.7,

$$\begin{aligned}\int_{(\tau_1+j)/p^k}^{(\tau_1+j)/p^k+1} P^{(1)}_{\nu,k,j}(z,1)\,dz &= E^{(1)\ddagger}_{\nu,k,j}\left(\frac{\tau_1+j}{p^k},1\right)\\ &= E^{(1)\ddagger}_{\nu,k,j}\left(\frac{j}{p^k},1\right)+\int_{j/p^k}^{(\tau_1+j)/p^k} E^{(1)}_{\nu,k,j}(z,1)\,dz\\ &= E^{(1)\ddagger}_{\nu,k,j}\left(\frac{j}{p^k},1\right)+\int_{j/p^k}^{(\tau_1+j)/p^k} (p^kz-j)^{\nu}\,dz\\ &= E^{(1)\ddagger}_{\nu,k,j}\left(\frac{j}{p^k},1\right)+\frac{1}{p^k}\frac{(it_1)^{\nu+1}}{\nu+1}.\end{aligned}$$

Similarly, we have

$$\int_{(p^l\tau_0-d)/p^{k-l}}^{(p^l\tau_0-d)/p^{k-l}+1} P^{(0)}_{\nu,k,j}(z,1)\,dz = E^{(0)\ddagger}_{\nu,k,j}\left(-\frac{d}{p^{k-l}},1\right)+(-1)^{\nu+1}\frac{p^{l(n+2)}}{p^k}\frac{(it_0)^{n-\nu+1}}{n-\nu+1}.$$

**4.1.3.** *Proof of Proposition 4.4.* By combining the computations in Sections 4.1.1 and 4.1.2, we find

$$\begin{aligned}&\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}(\tau_0,\tau_1)\rangle\\ &= \frac{i^{\nu+1}}{p^k}\mathscr{L}_{\nu,k,j}(\nu)+\frac{1}{p^k}\frac{(it_1)^{\nu+1}}{\nu+1}+(-1)^{\nu-1}\frac{p^{l(n+2)}}{p^k}\frac{(it_0)^{n-\nu+1}}{n-\nu+1}-E^{(1)\ddagger}_{\nu,k,j}\left(\frac{j}{p^k},1\right)\\ &\quad-\frac{1}{p^k}\frac{(it_1)^{\nu+1}}{\nu+1}-E^{(0)\ddagger}_{\nu,k,j}\left(-\frac{d}{p^{k-l}},1\right)-(-1)^{\nu+1}\frac{p^{l(n+2)}}{p^k}\frac{(it_0)^{n-\nu+1}}{n-\nu+1}\\ &\quad-\frac{i^{\nu+1}}{p^k}\mathscr{R}^{(1)}_{k,j}(t_1,\nu)-\frac{i^{\nu+1}}{p^k}\mathscr{R}^{(0)}_{k,j}(t_0,\nu)-\mathscr{T}_{k,j}(\tau_0,\tau_1),\end{aligned}$$

and hence

$$\begin{aligned}\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}(\tau_0,\tau_1)\rangle &= \frac{i^{\nu+1}}{p^k}\mathscr{L}_{k,j}(\nu)-E^{(1)\ddagger}_{\nu,k,j}\left(\frac{j}{p^k},1\right)-E^{(0)\ddagger}_{\nu,k,j}\left(-\frac{d}{p^{k-l}},1\right)\\ &\quad-\frac{i^{\nu+1}}{p^k}\mathscr{R}^{(1)}_{k,j}(t_1,\nu)-\frac{i^{\nu+1}}{p^k}\mathscr{R}^{(0)}_{k,j}(t_0,\nu)-\mathscr{T}_{k,j}(\tau_0,\tau_1).\end{aligned}$$

The value $\langle \tilde{C}_{\nu,k,j}(\tau_0,\tau_1), \mathrm{Eis}_n\rangle$ does not depend on $t_0$ and $t_1$ and so we take the limit $t_0,t_1\to\infty$. The last three terms vanish and we obtain the desired identity. □

**4.2.** ***Summation over j.*** In this subsection, we compute the sum

$$\begin{aligned}&\sum_{j=0}^{p^k-1}\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}(\tau_0,\tau_1)\rangle\\ &\qquad=\sum_{j=0}^{p^k-1}\left\{\frac{i^{\nu+1}}{p^k}\mathscr{L}_{k,j}(\nu)-E^{(1)\ddagger}_{\nu,k,j}\left(\frac{j}{p^k},1\right)-E^{(0)\ddagger}_{\nu,k,j}\left(-\frac{d}{p^{k-l}},1\right)\right\}.\end{aligned}$$

We keep using the abbreviation

$$l := l_k(j) \quad \text{and} \quad d := d_{k-l_k(j)}(j') = d_{k-l_k(j)}(j/p^{l_k(j)})$$

that are actually depending on $k$ and $j$. Recall that $\tilde{B}_t(x) = (B_t(x) - B_t)/t$.

**Lemma 4.6.** *The following equalities hold*:

$$(1)\ \frac{i^{\nu+1}}{p^k}\sum_{j=0}^{p^k-1}\mathscr{L}_{k,j}(\nu) = \frac{(1-p^{(n+1)(k+1)}) - (1-p^{(n+1)k})p^{n-\nu}}{1-p^{n+1}}\frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)},$$

$$(2)\ \sum_{j=0}^{p^k-1} E^{(1)\ddagger}_{\nu,k,j}\left(\frac{j}{p^k},1\right) = \frac{(-1)^\nu}{(\nu+1)}\frac{1}{p^k}\tilde{B}_{\nu+2}(p^k) + \sum_{j=0}^{p^k-1} E^{(1),\ddagger}_{\nu,k,j}(0,1),$$

$$(3)\ \sum_{j=0}^{p^k-1} E^{(0)\ddagger}_{\nu,k,j}\left(-\frac{d}{p^{k-l}},1\right)$$
$$= \frac{(-1)^\nu}{n-\nu+1}\frac{1}{p^k}\sum_{l'=0}^{k-1} p^{l'(\nu+1)}\left(\tilde{B}_{n-\nu+2}(p^{k-l'}) - p^{n-\nu+1}\tilde{B}_{n-\nu+2}(p^{k-l'-1})\right)$$
$$+ \sum_{j=0}^{p^k-1} E^{(0)\ddagger}_{\nu,k,j}(0,1).$$

*Proof.* (1) Recall that

$$\mathscr{L}_{k,j}(s) = \int_0^\infty \left(E_{n+2}\left(\frac{iy+j}{p^k}\right) - 1\right) y^s\, dy.$$

Hence we have

$$\begin{aligned}
\frac{1}{p^k}\sum_{j=0}^{p^k-1}\mathscr{L}_{k,j}(s) &= \frac{1}{p^k}\sum_{j=0}^{p^k-1}\frac{2}{\zeta(-1-n)}\sum_{\mu=1}^\infty \sigma_{n+1}(\mu)e^{2\pi i\mu j/p^k}\int_0^\infty e^{-2\pi\mu y/p^k}y^{s+1}\frac{dy}{y}\\
&= \frac{1}{p^k}\frac{2}{\zeta(-1-n)}\sum_{j=0}^{p^k-1}\sum_{\mu=1}^\infty \sigma_{n+1}(\mu)e^{2\pi i\mu j/p^k}\frac{p^{k(s+1)}}{(2\pi\mu)^{s+1}}\Gamma(s+1)\\
&= \frac{2}{\zeta(-1-n)}\frac{\Gamma(s+1)p^{ks}}{(2\pi)^{s+1}}\sum_{\mu=1}^\infty\left(\sum_{j=0}^{p^k-1} e^{2\pi i\mu j/p^k}\right)\frac{\sigma_{n+1}(\mu)}{\mu^{s+1}}\\
&= \frac{2\Gamma(s+1)}{\zeta(-1-n)(2\pi)^{s+1}}\sum_{\mu=1}^\infty\frac{\sigma_{n+1}(p^k\mu)}{\mu^{s+1}}.
\end{aligned}$$

For notational simplicity, we put

$$\mathcal{E}_p(s) := 1 - p^{-s}.$$

We then have

$$\begin{aligned}\sum_{a=0}^{\infty}\frac{\sigma_{n+1}(p^{k+a})}{p^{a(s+1)}} &= \sum_{a=0}^{\infty}\frac{1}{p^{a(s+1)}}\frac{1-p^{(k+a+1)(n+1)}}{1-p^{n+1}}\\ &= \frac{\mathcal{E}_p(1+s)^{-1}-p^{(k+1)(n+1)}\mathcal{E}_p(s-n)^{-1}}{1-p^{n+1}}\\ &= \frac{\mathcal{E}_p(s-n)-p^{(k+1)(n+1)}\mathcal{E}_p(1+s)}{1-p^{n+1}}\mathcal{E}_p(1+s)^{-1}\mathcal{E}_p(s-n)^{-1}.\end{aligned}$$

Hence the well-known relation that $\zeta(1+s)\zeta(s-n)=\sum_{a=1}^{\infty}\sigma_{n+1}(a)a^{-(s+1)}$ implies that

$$\begin{aligned}&\frac{1}{p^k}\sum_{j=0}^{p^k-1}\mathscr{L}_{k,j}(s)\\ &\quad= \frac{2\Gamma(s+1)}{\zeta(-1-n)(2\pi)^{s+1}}\frac{\mathcal{E}_p(s-n)-p^{(k+1)(n+1)}\mathcal{E}_p(1+s)}{1-p^{n+1}}\zeta(s+1)\zeta(s-n).\end{aligned}$$

By setting $s=\nu$ and using the functional equation of the Riemann zeta function (see [Hida 1993, p. 29] for example), we find

$$\begin{aligned}\frac{i^{\nu+1}}{p^k}\sum_{j=0}^{p^k-1}\mathscr{L}_{k,j}(\nu) &= \frac{\mathcal{E}_p(\nu-n)-p^{(k+1)(n+1)}\mathcal{E}_p(\nu+1)}{1-p^{n+1}}\cdot\frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)}\\ &= \frac{(1-p^{(n+1)(k+1)})-(1-p^{(n+1)k})p^{n-\nu}}{1-p^{n+1}}\cdot\frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)}.\end{aligned}$$

(2) By using Lemma 3.7, we find

$$\begin{aligned}\sum_{j=0}^{p^k-1}E_{\nu,k,j}^{(1),\ddagger}\left(\frac{j}{p^k},1\right) &= \sum_{j=0}^{p^k-1}\left(E_{\nu,k,j}^{(1),\ddagger}\left(\frac{j}{p^k},1\right)-E_{\nu,k,j}^{(1),\ddagger}(0,1)\right)+\sum_{j=0}^{p^k-1}E_{\nu,k,j}^{(1),\ddagger}(0,1)\\ &= \sum_{j=0}^{p^k-1}\int_0^{j/p^k}E_{\nu,k,j}^{(1)}(z,1)\,dz+\sum_{j=0}^{p^k-1}E_{\nu,k,j}^{(1),\ddagger}(0,1)\\ &= \sum_{j=0}^{p^k-1}\int_0^{j/p^k}(p^kz-j)^{\nu}\,dz+\sum_{j=0}^{p^k-1}E_{\nu,k,j}^{(1),\ddagger}(0,1)\\ &= \frac{(-1)^{\nu}}{(\nu+1)}\frac{1}{p^k}\sum_{j=0}^{p^k-1}j^{\nu+1}+\sum_{j=0}^{p^k-1}E_{\nu,k,j}^{(1),\ddagger}(0,1).\end{aligned}$$

Therefore, Claim (2) follows from the fact that $\sum_{j=0}^{p^k-1}j^{\nu+1}=\tilde{B}_{\nu+2}(p^k)$.

(3) Lemma 3.7 shows that

$$E^{(0)\ddagger}_{\nu,k,j}\left(-\frac{d}{p^{k-l}},1\right) = -(-1)^\nu p^{l\nu}\int_0^{-d/p^{k-l}} (p^{k-l}z+d)^{n-\nu}\,dz + E^{(0)\ddagger}_{\nu,k,j}(0,1)$$
$$= \frac{(-1)^\nu}{n-\nu+1}\frac{1}{p^k}p^{l(\nu+1)}d^{n-\nu+1} + E^{(0)\ddagger}_{\nu,k,j}(0,1).$$

Since $d_0(0)=0,\ n-\nu+1\geq 2$, and the map $a\mapsto d_N(a)$ induces a permutation on the set $\{1\leq a<p^N \mid (a,p^N)=1\}$, we find

$$\frac{1}{p^k}\sum_{j=0}^{p^k-1} p^{l(\nu+1)}d^{n-\nu+1}$$
$$= \frac{1}{p^k}\sum_{l'=0}^{k-1} p^{l'(\nu+1)}\sum_{\substack{d'=1\\(d',p^{k-l'})=1}}^{p^{k-l'}-1} d'^{n-\nu+1}$$
$$= \frac{1}{p^k}\sum_{l'=0}^{k-1} p^{l'(\nu+1)}(\tilde{B}_{n-\nu+2}(p^{k-l'}) - p^{n-\nu+1}\tilde{B}_{n-\nu+2}(p^{k-l'-1})). \quad \square$$

**Lemma 4.7.** *The following equalities hold*:

(1) $$\sum_{j=0}^{p^k-1} E^{(1)\ddagger}_{\nu,k,j}(0,1) = (-1)^{\nu+1}\sum_{\mu=0}^{\nu}\binom{\nu}{\mu}(-1)^\mu p^{k\mu}\frac{B_{\mu+1}}{\mu+1}\tilde{B}_{\nu-\mu+1}(p^k),$$

(2) $$\sum_{j=0}^{p^k-1} E^{(0)\ddagger}_{\nu,k,j}(0,1)$$
$$= (-1)^\nu p^{k\nu}\frac{B_{n-\nu+1}}{n-\nu+1} + (-1)^\nu\sum_{\mu=0}^{n-\nu}\binom{n-\nu}{\mu}p^{k\mu}\frac{B_{\mu+1}}{\mu+1}\sum_{l'=0}^{k-1}p^{l'(\nu-\mu)}$$
$$\times(\tilde{B}_{n-\nu-\mu+1}(p^{k-l'}) - p^{n-\nu-\mu}\tilde{B}_{n-\nu-\mu+1}(p^{k-l'-1})).$$

*Proof.* (1) Note that $(X_1^\mu X_2^{n-\mu})^\ddagger(0,1) = -B_{\mu+1}/(\mu+1)$. Since $E^{(1)}_{\nu,k,j}(X_1,X_2) = (p^kX_1 - jX_2)^\nu X_2^{n-\nu}$, we have

$$E^{(1)\ddagger}_{\nu,k,j}(0,1) = \sum_{\mu=0}^{\nu}\binom{\nu}{\mu}p^{k\mu}\left(-\frac{B_{\mu+1}}{\mu+1}\right)(-j)^{\nu-\mu}.$$

Hence we have

$$\sum_{j=0}^{p^k-1} E^{(1)\ddagger}_{\nu,k,j}(0,1) = (-1)^{\nu+1}\sum_{\mu=0}^{\nu}\binom{\nu}{\mu}(-1)^\mu p^{k\mu}\frac{B_{\mu+1}}{\mu+1}\tilde{B}_{\nu-\mu+1}(p^k).$$

(2) Since $E^{(0)}_{\nu,k,j}(X_1, X_2) = (-1)^{\nu+1}(p^l X_2)^\nu (p^{k-l}X_1 + dX_2)^{n-\nu}$, we have

$$E^{(0)\ddagger}_{\nu,k,j}(0,1) = (-1)^{\nu+1} p^{l\nu} \sum_{\mu=0}^{n-\nu} \binom{n-\nu}{\mu} p^{\mu(k-l)} \left(-\frac{B_{\mu+1}}{\mu+1}\right) d^{n-\nu-\mu}.$$

First note that in the case where $j = 0$, since $d = d_0(0) = 0$, we find

$$E^{(0)\ddagger}_{\nu,k,0}(0,1) = (-1)^{\nu} p^{k\nu} \frac{B_{n-\nu+1}}{n-\nu+1}.$$

Then by using the same argument as in the proof of Lemma 4.6(3), we also obtain

$$\begin{aligned}\sum_{j=1}^{p^k-1} E^{(0)\ddagger}_{\nu,k,j}(0,1) &= (-1)^{\nu} \sum_{l'=0}^{k-1} \sum_{\substack{d'=1 \\ (d', p^{k-l'})=1}}^{p^{k-l'}-1} p^{l'\nu} \sum_{\mu=0}^{n-\nu} \binom{n-\nu}{\mu} p^{\mu(k-l')} \frac{B_{\mu+1}}{\mu+1} d'^{n-\nu-\mu} \\ &= (-1)^{\nu} \sum_{l'=0}^{k-1} p^{l'\nu} \sum_{\mu=0}^{n-\nu} \binom{n-\nu}{\mu} p^{\mu(k-l')} \frac{B_{\mu+1}}{\mu+1} \\ &\quad \times (\tilde{B}_{n-\nu-\mu+1}(p^{k-l'}) - p^{n-\nu-\mu} \tilde{B}_{n-\nu-\mu+1}(p^{k-l'-1})). \quad \square\end{aligned}$$

**4.3.** ***Summation over k and the p-adic limits.*** We compute the value

$$\mathscr{W}^{(m)} := \sum_{k=0}^{m} p^{(n-\nu)(m-k)} \sum_{j=0}^{p^k-1} \left\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}\left(\frac{\tau}{p^{m-k}}, p^{m-k}\tau\right)\right\rangle$$

and its $p$-adic limit as $m \to \infty$. This enables us to complete the proof of Theorem 4.1.

We keep the notation from the previous sections. Proposition 4.4 shows that

$$\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}(\tau_0, \tau_1)\rangle = \frac{i^{\nu+1}}{p^k} \mathscr{L}_{k,j}(\nu) - E^{(1)\ddagger}_{\nu,k,j}\left(\frac{j}{p^k}, 1\right) - E^{(0)\ddagger}_{\nu,k,j}\left(-\frac{d}{p^{k-l}}, 1\right),$$

and hence

$$\mathscr{W}^{(m)} = \sum_{k=0}^{m} p^{(n-\nu)(m-k)} \sum_{j=0}^{p^k-1} \left(\frac{i^{\nu+1}}{p^k} \mathscr{L}_{k,j}(\nu) - E^{(1)\ddagger}_{\nu,k,j}\left(\frac{j}{p^k}, 1\right) - E^{(0)\ddagger}_{\nu,k,j}\left(-\frac{d}{p^{k-l}}, 1\right)\right).$$

We set

$$\mathscr{W}^{(m)}_1 := \sum_{k=0}^{m} p^{(n-\nu)(m-k)} \sum_{j=0}^{p^k-1} \frac{i^{\nu+1}}{p^k} \mathscr{L}_{k,j}(\nu),$$

$$\mathscr{W}^{(m)}_2 := \sum_{k=0}^{m} p^{(n-\nu)(m-k)} \sum_{j=0}^{p^k-1} E^{(1)\ddagger}_{\nu,k,j}\left(\frac{j}{p^k}, 1\right),$$

$$\mathscr{W}^{(m)}_3 := \sum_{k=0}^{m} p^{(n-\nu)(m-k)} \sum_{j=0}^{p^k-1} E^{(0)\ddagger}_{\nu,k,j}\left(-\frac{d}{p^{k-l}}, 1\right),$$

so that we have $\mathscr{W}^{(m)} = \mathscr{W}_1^{(m)} - \mathscr{W}_2^{(m)} - \mathscr{W}_3^{(m)}$.

**Lemma 4.8.** *We have*

$$\lim_{m\to\infty} \mathscr{W}_1^{(m)} = \frac{1}{1-p^{n+1}} \frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)}.$$

*Proof.* Lemma 4.6(1) shows that

$$\mathscr{W}_1^{(m)} = \frac{1}{1-p^{n+1}} \frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)} \sum_{k=0}^{m} p^{(n-\nu)(m-k)}((1-p^{(n+1)(k+1)})-(1-p^{(n+1)k})p^{n-\nu}).$$

Moreover, we have

$$\begin{aligned}
&\sum_{k=0}^{m} p^{(n-\nu)(m-k)}((1-p^{(n+1)(k+1)})-(1-p^{(n+1)k})p^{n-\nu}) \\
&= p^{(n-\nu)m}(1-p^{n-\nu}) \sum_{k=0}^{m} p^{-(n-\nu)k} - p^{(n-\nu)m}(p^{n+1}-p^{n-\nu}) \sum_{k=0}^{m} p^{(\nu+1)k} \\
&= p^{(n-\nu)m}(1-p^{n-\nu}) \frac{1-p^{-(n-\nu)(m+1)}}{1-p^{-(n-\nu)}} - p^{(n-\nu)m}(p^{n+1}-p^{n-\nu}) \frac{1-p^{(\nu+1)(m+1)}}{1-p^{\nu+1}} \\
&= 1 - p^{(n-\nu)(m+1)} - p^{(n-\nu)m}(p^{n+1}-p^{n-\nu}) \frac{1-p^{(\nu+1)(m+1)}}{1-p^{\nu+1}} \\
&\to 1 \qquad (m\to\infty).
\end{aligned}$$

□

**Lemma 4.9.** *Let $s$ and $t$ be integers with $t > 0$.*

(1) *For any positive integer $u > 0$, we have*

$$\lim_{m\to\infty} \sum_{k=0}^{m} \frac{p^{u(m-k)}}{p^{k-s}} \tilde{B}_t(p^{k-s}) = \frac{B_{t-1}}{1-p^u}.$$

(2) *For any integer $u$ and any $\varepsilon > 0$ with $u + \varepsilon > 0$, we have*

$$\lim_{m\to\infty} p^{m\varepsilon} \sum_{k=0}^{m} \frac{p^{u(m-k)}}{p^{k-s}} \tilde{B}_t(p^{k-s}) = 0.$$

*Proof.* We have

$$\begin{aligned}
\sum_{k=0}^{m} p^{u(m-k)} \frac{1}{p^{k-s}} \tilde{B}_t(p^{k-s}) &= \frac{1}{t} \sum_{k=0}^{m} p^{u(m-k)} \sum_{\mu=1}^{t} \binom{t}{\mu} B_{t-\mu} p^{(\mu-1)(k-s)} \\
&= \frac{1}{t} \sum_{\mu=1}^{t} \binom{t}{\mu} B_{t-\mu} \sum_{k=0}^{m} p^{u(m-k)+(\mu-1)(k-s)}
\end{aligned}$$

$$= \frac{1}{t}\sum_{\mu=1}^{t}\binom{t}{\mu}B_{t-\mu}p^{um-(\mu-1)s}\frac{1-p^{(\mu-1-u)(m+1)}}{1-p^{\mu-1-u}}$$

$$= \frac{1}{t}\sum_{\mu=1}^{t}\binom{t}{\mu}B_{t-\mu}\frac{p^{um-(\mu-1)s}-p^{-(\mu-1)s+(m+1)(\mu-1)-u}}{1-p^{\mu-1-u}}.$$

Hence Claim (2) is clear. Moreover if $u > 0$, then all the terms with $\mu \geq 2$ vanish as $m \to \infty$, and therefore,

$$\lim_{m\to\infty}\sum_{k=0}^{m}\frac{p^{u(m-k)}}{p^{k-s}}\tilde{B}_t(p^{k-s}) = B_{t-1}\frac{-p^{-u}}{1-p^{-u}} = \frac{B_{t-1}}{1-p^u}. \qquad \square$$

**Lemma 4.10.** *We have*

$$\lim_{m\to\infty}\mathscr{W}_2^{(m)} = \frac{(-1)^\nu}{1-p^{n-\nu}}\frac{B_{\nu+1}}{\nu+1} = \frac{1}{1-p^{n-\nu}}\zeta(-\nu).$$

*Proof.* Lemma 4.6(2) shows that

$$\mathscr{W}_2^{(m)} = \sum_{k=0}^{m}p^{(n-\nu)(m-k)}\sum_{j=0}^{p^k-1}E_{\nu,k,j}^{(1)\ddagger}\left(\frac{j}{p^k},1\right)$$

$$= \sum_{k=0}^{m}p^{(n-\nu)(m-k)}\left(\frac{(-1)^\nu}{(\nu+1)}\frac{1}{p^k}\tilde{B}_{\nu+2}(p^k)+\sum_{j=0}^{p^k-1}E_{\nu,k,j}^{(1),\ddagger}(0,1)\right).$$

Hence Lemma 4.9(1) implies that

$$\lim_{m\to\infty}\mathscr{W}_2^{(m)} = \frac{(-1)^\nu}{1-p^{n-\nu}}\frac{B_{\nu+1}}{\nu+1}+\lim_{m\to\infty}\sum_{k=0}^{m}p^{(n-\nu)(m-k)}\sum_{j=0}^{p^k-1}E_{\nu,k,j}^{(1),\ddagger}(0,1).$$

Moreover, by Lemma 4.7, we have

$$\sum_{k=0}^{m}p^{(n-\nu)(m-k)}\sum_{j=0}^{p^k-1}E_{\nu,k,j}^{(1),\ddagger}(0,1)$$

$$= \sum_{k=0}^{m}p^{(n-\nu)(m-k)}\sum_{\mu=0}^{\nu}(-1)^{\nu+\mu+1}\binom{\nu}{\mu}p^{k(\mu+1)}\frac{B_{\mu+1}}{\mu+1}\frac{\tilde{B}_{\nu-\mu+1}(p^k)}{p^k}$$

$$= \sum_{\mu=0}^{\nu}(-1)^{\nu+\mu+1}\binom{\nu}{\mu}\frac{B_{\mu+1}}{\mu+1}p^{m(\mu+1)}\sum_{k=0}^{m}p^{(n-\nu-\mu-1)(m-k)}\frac{\tilde{B}_{\nu-\mu+1}(p^k)}{p^k}.$$

Since $\mu+1+n-\nu-\mu-1 = n-\nu > 0$, Lemma 4.9(2) shows that

$$\lim_{m\to\infty}p^{m(\mu+1)}\sum_{k=0}^{m}p^{(n-\nu-\mu-1)(m-k)}\frac{\tilde{B}_{\nu-\mu+1}(p^k)}{p^k} = 0.$$

Hence we conclude that

$$\lim_{m\to\infty} \mathscr{W}_2^{(m)} = \frac{(-1)^\nu}{1-p^{n-\nu}}\frac{B_{\nu+1}}{\nu+1}. \qquad \square$$

**Lemma 4.11.** *Let $b, s, t$, and $u$ be integers with $s \geq 0$ and $t \geq 0$.*

(1) *If $s > 0$ and $u > 0$, then*

$$\lim_{m\to\infty} \sum_{k=0}^{m} p^{u(m-k)} \sum_{l'=0}^{k-1} p^{l's}\left(\frac{\tilde{B}_{t+1}(p^{k-l'})}{p^{k-l'}} - p^b \frac{\tilde{B}_{t+1}(p^{k-l'-1})}{p^{k-l'-1}}\right) = \frac{B_t}{1-p^s}\frac{1-p^b}{1-p^u}.$$

(2) *For any $\varepsilon > 0$ with $u + \varepsilon > 0$, we have*

$$\lim_{m\to\infty} p^{\varepsilon m} \sum_{k=0}^{m} p^{u(m-k)} \sum_{l'=0}^{k-1} p^{l's}\left(\frac{\tilde{B}_{t+1}(p^{k-l'})}{p^{k-l'}} - p^b \frac{\tilde{B}_{t+1}(p^{k-l'-1})}{p^{k-l'-1}}\right) = 0.$$

*Proof.* We have

$$\begin{aligned}
&\sum_{k=0}^{m} p^{u(m-k)} \sum_{l'=0}^{k-1} p^{l's}\left(\frac{\tilde{B}_{t+1}(p^{k-l'})}{p^{k-l'}} - p^b \frac{\tilde{B}_{t+1}(p^{k-l'-1})}{p^{k-l'-1}}\right)\\
&= \frac{1}{t+1}\sum_{k=0}^{m} p^{u(m-k)} \sum_{l'=0}^{k-1} p^{l's} \sum_{\mu=1}^{t+1} \binom{t+1}{\mu} B_{t+1-\mu} \times (p^{(k-l')(\mu-1)} - p^{(k-l'-1)(\mu-1)+b})\\
&= \frac{1}{t+1}\sum_{\mu=1}^{t+1} \binom{t+1}{\mu} B_{t+1-\mu} \times (1-p^{b-(\mu-1)}) p^{um} \sum_{k=0}^{m} p^{k(\mu-1-u)} \sum_{l'=0}^{k-1} p^{l'(s-\mu+1)}\\
&= \frac{1}{t+1}\sum_{\mu=1}^{t+1} \binom{t+1}{\mu} B_{t+1-\mu} \times (1-p^{b-(\mu-1)}) p^{um} \sum_{k=0}^{m} p^{k(\mu-1-u)} \frac{1-p^{k(s-\mu+1)}}{1-p^{s-\mu+1}}\\
&= \frac{1}{t+1}\sum_{\mu=1}^{t+1} \binom{t+1}{\mu} \frac{B_{t+1-\mu}}{1-p^{s-\mu+1}}\\
&\qquad \times (1-p^{b-(\mu-1)})\left(\frac{p^{um} - p^{(m+1)(\mu-1)-u}}{1-p^{\mu-1-u}} - \frac{p^{um}-p^{(m+1)s-u}}{1-p^{s-u}}\right),
\end{aligned}$$

which implies this lemma in the same way as in the proof of Lemma 4.9. $\square$

**Lemma 4.12.** *We have*

$$\lim_{m\to\infty} \mathscr{W}_3^{(m)} = \frac{(-1)^\nu}{1-p^\nu}\frac{B_{n-\nu+1}}{n-\nu+1} = \frac{1}{1-p^\nu}\zeta(\nu-n).$$

*Proof.* Recall that in Lemma 4.6(3) we have shown that

$$\sum_{j=0}^{p^k-1} E^{(0)\ddagger}_{\nu,k,j}\left(-\frac{d}{p^{k-l}},1\right) = \frac{(-1)^\nu}{n-\nu+1}\frac{1}{p^k}\sum_{l'=0}^{k-1} p^{l'(\nu+1)}(\tilde{B}_{n-\nu+2}(p^{k-l'}) - p^{n-\nu+1}\tilde{B}_{n-\nu+2}(p^{k-l'-1})) + \sum_{j=0}^{p^k-1} E^{(0)\ddagger}_{\nu,k,j}(0,1).$$

Since $1 \le \nu \le n-1$, Lemma 4.11(1) shows that, as $m \to \infty$,

$$\sum_{k=0}^{m} p^{(n-\nu)(m-k)} \sum_{l'=0}^{k-1} p^{l'\nu}\left(\frac{\tilde{B}_{n-\nu+2}(p^{k-l'})}{p^{k-l'}} - p^{n-\nu}\frac{\tilde{B}_{n-\nu+2}(p^{k-l'-1})}{p^{k-l'-1}}\right) \to \frac{B_{n-\nu+1}}{1-p^\nu}.$$

By Lemma 4.7, we have

$$\sum_{j=0}^{p^k-1} E^{(0)\ddagger}_{\nu,k,j}(0,1) = (-1)^\nu p^{k\nu}\frac{B_{n-\nu+1}}{n-\nu+1} + (-1)^\nu \sum_{\mu=0}^{n-\nu}\binom{n-\nu}{\mu} p^{k\mu}\frac{B_{\mu+1}}{\mu+1}\sum_{l'=0}^{k-1} p^{l'(\nu-\mu)} \times (\tilde{B}_{n-\nu-\mu+1}(p^{k-l'}) - p^{n-\nu-\mu}\tilde{B}_{n-\nu-\mu+1}(p^{k-l'-1})).$$

Since $(n-\nu)(m-k)+k\mu = (n-\nu-\mu)(m-k)+m\mu$ and $0 \le \mu \le n-\nu$, all the terms with $\mu \ge 1$ vanish when $m \to \infty$, and hence

$$\begin{aligned}&\lim_{m\to\infty}\sum_{k=0}^{m} p^{(n-\nu)(m-k)}\sum_{j=0}^{p^k-1} E^{(0)\ddagger}_{\nu,k,j}(0,1)\\ &= \lim_{m\to\infty}\sum_{k=0}^{m}(-1)^\nu p^{(n-\nu)(m-k)+k\nu}\frac{B_{n-\nu+1}}{n-\nu+1}\\ &\quad + (-1)^\nu B_1 \lim_{m\to\infty}\sum_{k=0}^{m} p^{(n-\nu)(m-k)}\sum_{l'=0}^{k-1} p^{l'\nu}(\tilde{B}_{n-\nu+1}(p^{k-l'}) - p^{n-\nu}\tilde{B}_{n-\nu+1}(p^{k-l'-1})).\end{aligned}$$

Since

$$\lim_{m\to\infty}\sum_{k=0}^{m} p^{(n-\nu)(m-k)+k\nu} = \lim_{m\to\infty}\frac{p^{(n-\nu)(m+1)} - p^{\nu(m+1)}}{p^{n-\nu}-p^\nu} = 0,$$

the first limit vanishes. Moreover, Lemma 4.11(2) implies that

$$\begin{aligned}&\sum_{k=0}^{m} p^{(n-\nu)(m-k)}\sum_{l'=0}^{k-1} p^{l'\nu}(\tilde{B}_{n-\nu+1}(p^{k-l'}) - p^{n-\nu}\tilde{B}_{n-\nu+1}(p^{k-l'-1}))\\ &= p^m\sum_{k=0}^{m} p^{(n-\nu-1)(m-k)}\sum_{l'=0}^{k-1} p^{l'(\nu-1)}\left(\frac{\tilde{B}_{n-\nu+1}(p^{k-l'})}{p^{k-l'}} - p^{n-\nu-1}\frac{\tilde{B}_{n-\nu+1}(p^{k-l'-1})}{p^{k-l'-1}}\right)\\ &\to 0 \quad (m\to\infty),\end{aligned}$$

which implies this lemma. □

By Lemmas 4.8, 4.10, and 4.12, we obtain the following.

**Proposition 4.13.** *We have*

$$\lim_{m\to\infty}\sum_{k=0}^{m} p^{(n-\nu)(m-k)}\sum_{j=0}^{p^k-1}\left\langle \mathrm{Eis}_n, \tilde{C}_{\nu,k,j}\left(\frac{\tau}{p^{m-k}}, p^{m-k}\tau\right)\right\rangle = \frac{1}{1-p^{n+1}}\frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)} - \frac{\zeta(-\nu)}{1-p^{n-\nu}} - \frac{\zeta(\nu-n)}{1-p^{\nu}}.$$

By the argument in the beginning of Section 4, this completes the proof of Theorem 4.1.

**4.4.** ***Rationality of the Eisenstein class*** $\mathbf{Eis}_n$**.** The computations in (the proofs of) Proposition 4.4 and Lemmas 4.6 and 4.7 imply the following proposition as a special case.

**Proposition 4.14.** *For any integer* $\nu \in \{1, \dots, n-1\}$*, we have*

$$\langle \mathrm{Eis}_n, [\widetilde{C_\nu(\tau)}]\rangle = \frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)} - \zeta(-\nu) - \zeta(\nu-n) \in \mathbb{Q},$$

*where* $[\widetilde{C_\nu(\tau)}]$ *is a special case of* $[\widetilde{T_p^m(C_\nu(\tau))}]$ *for* $m = 0$*.*

*Proof.* By Proposition 4.4, we have

$$\langle \mathrm{Eis}_n, [\widetilde{C_\nu(\tau)}]\rangle = \langle \mathrm{Eis}_n, \tilde{C}_{\nu,0,0}\rangle = i^{\nu+1}\mathscr{L}_{0,0}(\nu) - E^{(1)\ddagger}_{\nu,0,0}(0,1) - E^{(0)\ddagger}_{\nu,0,0}(0,1).$$

By Lemma 4.6(1), we have

$$i^{\nu+1}\mathscr{L}_{0,0}(\nu) = \frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)}.$$

Moreover, in the proof of Lemma 4.7, we showed that

$$E^{(1)\ddagger}_{\nu,0,0}(0,1) = \zeta(-\nu) \quad \text{and} \quad E^{(0)\ddagger}_{\nu,0,0}(0,1) = \zeta(\nu-n). \qquad \square$$

The following lemma will be well known to experts. For instance, Harder [2023, §5.1.3] mentioned that this was proved by Gebertz in her diploma thesis. Here we give a proof for the completeness of the paper.

**Lemma 4.15.** *The relative homology group* $H_1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_n)$ *is generated by the set* $\{[C_\nu] \mid 0 \le \nu \le n\}$*.*

*Proof.* The relative homology group $H_1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_n)$ can be computed as

$$H_1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_n) = \ker\Big(((\mathcal{MS}(\mathbb{H}^{\mathrm{BS}})/\mathcal{MS}(\partial\mathbb{H}^{\mathrm{BS}}))\otimes\mathcal{M}_n)_\Gamma \xrightarrow{\partial} ((S_0(\mathbb{H}^{\mathrm{BS}})/S_0(\partial\mathbb{H}^{\mathrm{BS}}))\otimes\mathcal{M}_n)_\Gamma\Big).$$

Let $[\sigma] \in H_1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_n)$ be a class represented by a 1-chain

$$\sigma = \sum_j \{a_j, b_j\} \otimes P_j \in (\mathcal{MS}(\mathbb{H}^{\mathrm{BS}})/\mathcal{MS}(\partial\mathbb{H}^{\mathrm{BS}})) \otimes \mathcal{M}_n,$$

where $a_j, b_j \in \mathbb{H}^{\mathrm{BS}}$ and $P_j \in \mathcal{M}_n$. The condition that

$$\partial\sigma = 0 \quad \text{in } ((S_0(\mathbb{H}^{\mathrm{BS}})/S_0(\partial\mathbb{H}^{\mathrm{BS}})) \otimes \mathcal{M}_n)_\Gamma$$

implies that

$$\sum_j \{b_j\} \otimes P_j - \{a_j\} \otimes P_j = \sum_k (\gamma_k - 1)(\{d_k\} \otimes Q_k) + \sum_l \{c_l\} \otimes R_l \tag{4-3}$$

in $S_0(\mathbb{H}^{\mathrm{BS}}) \otimes \mathcal{M}_n$ for some $\gamma_k \in \Gamma$, $d_k \in \mathbb{H}^{\mathrm{BS}}$, $c_l \in \partial\mathbb{H}^{\mathrm{BS}}$, and $Q_k, R_l \in \mathcal{M}_n$. Then we can rewrite the identity (4-3) as

$$\sum_{\tau \in \mathbb{H}^{\mathrm{BS}}} \{\tau\} \otimes \left( \sum_{j, b_j = \tau} P_j - \sum_{j, a_j = \tau} P_j - \sum_{k, \gamma_k d_k = \tau} \gamma_k Q_k + \sum_{k, d_k = \tau} Q_k - \sum_{l, c_l = \tau} R_l \right) = 0$$

in $S_0(\mathbb{H}^{\mathrm{BS}}) \otimes \mathcal{M}_n$. Since $S_0(\mathbb{H}^{\mathrm{BS}}) = \bigoplus_{\tau \in \mathbb{H}^{\mathrm{BS}}} \mathbb{Z}\{\tau\}$, this shows that for any $\tau \in \mathbb{H}^{\mathrm{BS}}$, we have

$$\sum_{j, b_j = \tau} P_j - \sum_{j, a_j = \tau} P_j - \sum_{k, \gamma_k d_k = \tau} \gamma_k Q_k + \sum_{k, d_k = \tau} Q_k - \sum_{l, c_l = \tau} R_l = 0 \quad \text{in } \mathcal{M}_n. \tag{4-4}$$

Now, recall that $i\infty \in \partial\mathbb{H}^{\mathrm{BS}}$ denotes the point defined by

$$i\infty := \lim_{\substack{t \in \mathbb{R}_{>0} \\ t \to \infty}} it.$$

Then the identity (4-4) implies that in $\mathcal{MS}(\mathbb{H}^{\mathrm{BS}}) \otimes \mathcal{M}_n$, we have

$$\sum_{\tau \in \mathbb{H}^{\mathrm{BS}}} \{i\infty, \tau\} \otimes \left( \sum_{j, b_j = \tau} P_j - \sum_{j, a_j = \tau} P_j - \sum_{k, \gamma_k d_k = \tau} \gamma_k Q_k + \sum_{k, d_k = \tau} Q_k - \sum_{l, c_l = \tau} R_l \right) = 0.$$

Using this last identity, in $((\mathcal{MS}(\mathbb{H}^{\mathrm{BS}})/\mathcal{MS}(\partial\mathbb{H}^{\mathrm{BS}})) \otimes \mathcal{M}_n)_\Gamma$, we compute

$$\begin{aligned}
&[\sigma] \\
&= \sum_j [\{i\infty, b_j\} \otimes P_j] - [\{i\infty, a_j\} \otimes P_j] \\
&= \sum_k [\{i\infty, \gamma_k d_k\} \otimes \gamma_k Q_k] - \sum_k [\{i\infty, d_k\} \otimes Q_k] + \sum_l [\{i\infty, c_l\} \otimes R_l] \\
&= \sum_k [(\gamma_k - 1)(\{i\infty, d_k\} \otimes Q_k)] + \sum_k [\{i\infty, \gamma_k i\infty\} \otimes \gamma_k Q_k] + \sum_l [\{i\infty, c_l\} \otimes R_l] \\
&= \sum_k [\{i\infty, \gamma_k i\infty\} \otimes \gamma_k Q_k] + \sum_l [\{i\infty, c_l\} \otimes R_l].
\end{aligned}$$

Moreover, as we are considering the relative homology classes, we may replace $c_l$ with any point in the same connected component of $\partial\mathbb{H}^{\mathrm{BS}}$, and in particular, we may replace $c_l$ by $g_l i\infty$ for some $g_l \in \Gamma$. Thus we conclude that the class $[\sigma] \in H_1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_n)$ is represented by a 1-chain of the form

$$\sigma' = \sum_m \{i\infty, \gamma'_m i\infty\} \otimes R'_m$$

for some $\gamma'_m \in \Gamma$ and $R'_m \in \mathcal{M}_n$. The lemma follows from the facts that the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is generated by matrices $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and that

$$[\{i\infty, \gamma_1\gamma_2 i\infty\} \otimes P] = [\{i\infty, \gamma_1 i\infty\} \otimes P] + [\{i\infty, \gamma_2 i\infty\} \otimes \gamma_1^{-1} P]$$

for any $\gamma_1, \gamma_2 \in \Gamma$ and $P \in \mathcal{M}_n$. □

Recall that $\Gamma_\infty$ is the subgroup of $\Gamma$ generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Since $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$, the boundary $\partial Y^{\mathrm{BS}}$ can be identified with $\Gamma_\infty \backslash \mathbb{R}$. Hence we obtain the following lemma.

**Lemma 4.16.** *We have an identification $H_0(\partial Y^{\mathrm{BS}}, \mathcal{M}_n) = (\mathcal{M}_n)_{\Gamma_\infty}$, and hence $H_0(\partial Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{Q}$ is a 1-dimensional $\mathbb{Q}$-vector space generated by $[e_n]$, where $e_n = X_1^n$.*

**Lemma 4.17.** *The kernel of the boundary homomorphism*

$$\partial : H_1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{Q} \to H_0(\partial Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{Q}$$

*is generated by the set $\{[C_\nu] \mid 1 \le \nu \le n-1\}$.*

*Proof.* Let $\sigma \in \ker(H_1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{Q} \to H_0(\partial Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{Q})$. Lemma 4.15 implies that we can write $\sigma = \sum_{\nu=0}^n a_\nu [C_\nu]$ for some numbers $a_0, \dots, a_n \in \mathbb{Q}$. Then by using Lemma 4.16, we find that

$$0 = \partial\sigma = \sum_{\nu=0}^n a_\nu [\{i\infty\} \otimes (e_\nu - (-1)^{n-\nu} e_{n-\nu})] = -(a_0 - a_n)[\{i\infty\} \otimes e_n].$$

Therefore, Lemma 4.16 shows that $a_0 = a_n$. On the other hand we see that

$$[C_0] = -[C_\nu] \quad \text{in } H_1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_n).$$ □

**Corollary 4.18.** *We have* $\mathrm{Eis}_n \in H^1(Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{Q}$.

*Proof.* By Lemma 4.17, the homology group $H_1(Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{Q}$ is generated by the image of $H_1(\partial Y^{\mathrm{BS}}, \mathcal{M}_n)$ and the set $\{[\tilde{C}_\nu(\tau)] \mid 1 \le \nu \le n-1\}$. Therefore, by Lemma 2.8 and Proposition 4.14 we have

$$\langle \mathrm{Eis}_n, H_1(Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{Q} \rangle \subset \mathbb{Q},$$

which implies $\mathrm{Eis}_n \in H^1(Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{Q}$. □

## 5. Denominator of an ordinary cohomology class

In order to study the denominator $\Delta(\mathrm{Eis}_n)$ of the Eisenstein class $\mathrm{Eis}_n$, in this section, we interpret the denominator $\Delta(\mathrm{Eis}_n)$ in terms of the values $\langle \mathrm{Eis}_n, [\widetilde{T_p^m(C_v(\tau))}]\rangle$ of the pairing between the Eisenstein class $\mathrm{Eis}_n$ and the cycles $[\widetilde{T_p^m(C_v(\tau))}] \in H_1(Y^{\mathrm{BS}}, \mathcal{M}_{n,(p)})$.

**5.1.** ***Definition of the ordinary part.*** Let $p$ be a prime number and $M$ a finitely generated $\mathbb{Z}_p$-module with an endomorphism $f : M \to M$. We introduce the notion of the $f$-ordinary part of $M$.

Since $M$ is a finitely generated $\mathbb{Z}_p$-module, the $p$-adic limit

$$e_f := \lim_{m\to\infty} f^{m!} \in \mathrm{End}_{\mathbb{Z}_p}(M)$$

always exists, and $e_f^2 = e_f$. We define the $f$-ordinary part $M_{\mathrm{ord}}$ of $M$ by

$$M_{\mathrm{ord}} := e_f M,$$

and we say that $m \in M$ is ($f$-)ordinary if $m \in M_{\mathrm{ord}}$, that is, $e_f m = m$. We also put $M_{\mathrm{nilp}} := (1 - e_f)M$. We then have $M = M_{\mathrm{ord}} \oplus M_{\mathrm{nilp}}$.

The following lemma follows from the fact that $e_f^2 = e_f$.

**Lemma 5.1.** *The functor $M \mapsto M_{\mathrm{ord}}$ is exact.*

**5.2.** ***Denominator of a cohomology class.*** Recall that

$$H^1_{\mathrm{int}}(Y^{\mathrm{BS}}, \mathcal{M}_n^\flat) = \mathrm{im}(H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\flat) \to H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\flat) \otimes \mathbb{Q}).$$

**Definition 5.2.** For any cohomology class $c \in H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\flat) \otimes \mathbb{Q}$, we define the denominator $\Delta(c) \in \mathbb{Z}_{>0}$ of $c$ by

$$\Delta(c) := \min\{\Delta \in \mathbb{Z}_{>0} \mid \Delta c \in H^1_{\mathrm{int}}(Y^{\mathrm{BS}}, \mathcal{M}_n^\flat)\},$$

and for each prime number $p$, we set

$$\delta_p(c) := \mathrm{ord}_p(\Delta(c)) \quad \text{and} \quad \Delta_p(c) := p^{\delta_p(c)}.$$

**Lemma 5.3.** *Let $c \in H^1(Y^{\mathrm{BS}}, \mathcal{M}_n^\flat) \otimes \mathbb{Q}$ be a cohomology class. We have*

$$\Delta(c) = \min\{\Delta \in \mathbb{Z}_{>0} \mid \Delta\langle c, H_1(Y^{\mathrm{BS}}, \mathcal{M}_n)\rangle \subset \mathbb{Z}\}.$$

*Moreover, for any prime number $p$, we have*

$$\delta_p(c) = \min\{\delta \in \mathbb{Z}_{\geq 0} \mid p^\delta\langle c, H_1(Y^{\mathrm{BS}}, \mathcal{M}_n \otimes \mathbb{Z}_p)\rangle \subset \mathbb{Z}_p\}.$$

*Proof.* This lemma follows immediately from the formal duality (see Section 2.5):

$$H^\bullet(\Gamma\backslash X, \mathcal{M}_n^\flat)/(\text{torsion}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(H_\bullet(\Gamma\backslash X, \mathcal{M}_n), \mathbb{Z}). \qquad \square$$

**5.3.** ***Denominator of an ordinary cohomology class.*** In this subsection, we fix a prime number $p$.

**Definition 5.4.** (1) We put $\mathcal{M}_{n,p} := \mathcal{M}_n \otimes \mathbb{Z}_p$ and $\mathcal{M}^\flat_{n,p} := \mathcal{M}^\flat_n \otimes \mathbb{Z}_p$.

(2) For any finitely generated $\mathbb{Z}_p$-algebra, we put

$$H^{\mathrm{ord}}_\bullet(Y^{\mathrm{BS}}, \mathcal{M}_{n,p} \otimes R) := e_{T_p} H_\bullet(Y^{\mathrm{BS}}, \mathcal{M}_{n,p} \otimes R),$$
$$H^{\mathrm{ord}}_\bullet(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p} \otimes R) := e_{T_p} H_\bullet(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p} \otimes R),$$
$$H^{\mathrm{ord}}_\bullet(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p} \otimes R) := e_{T_p} H_\bullet(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p} \otimes R).$$

(3) For any finitely generated $\mathbb{Z}_p$-algebra $R$, we put

$$H^\bullet_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}^\flat_{n,p} \otimes R) := e_{T'_p} H^\bullet(Y^{\mathrm{BS}}, \mathcal{M}^\flat_{n,p} \otimes R),$$
$$H^\bullet_{\mathrm{ord}}(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}^\flat_{n,p} \otimes R) := e_{T'_p} H^\bullet(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}^\flat_{n,p} \otimes R),$$
$$H^\bullet_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}^\flat_{n,p} \otimes R) := e_{T'_p} H^\bullet(\partial Y^{\mathrm{BS}}, \mathcal{M}^\flat_{n,p} \otimes R).$$

**Lemma 5.5.** *Let $c \in H^1(Y^{\mathrm{BS}}, \mathcal{M}^\flat_{n,p}) \otimes \mathbb{Q}_p$ be a cohomology class. For any homology class $C \in H_1(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes \mathbb{Q}_p$, we have*

$$\langle c, e_{T_p} C\rangle = \lim_{m\to\infty} \langle c, T_p^{m!} C\rangle = \lim_{m\to\infty} \langle c | T_p^{\prime m!}, C\rangle = \langle e_{T'_p} c, C\rangle.$$

*In particular, if $c$ is ordinary, that is, $e_{T'_p} c = c$, then*

$$\langle c, C\rangle = \langle c, e_{T_p} C\rangle,$$

*and hence*

$$\delta_p(c) = \min\{\delta \in \mathbb{Z}_{\geq 0} \mid p^\delta \langle c, H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})\rangle \subset \mathbb{Z}_p\}.$$

*Proof.* This lemma follows from the facts that the pairing $\langle \cdot, \cdot \rangle$ is continuous and $\langle c | T'_p, C\rangle = \langle c, T_p C\rangle$. □

Recall the identification $H_0(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) = (\mathcal{M}_{n,p})_{\Gamma_\infty}$ by Lemma 4.16.

**Lemma 5.6.** *The $\mathbb{Z}_p$-module $H_0^{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) = e_{T_p}(\mathcal{M}_{n,p})_{\Gamma_\infty}$ is free of rank 1 and is generated by $e_{T_p}[e_n]$.*

*Proof.* Let $k \in \{0, \dots, n\}$ be an integer. Since $e_k = X_1^k X_2^{n-k}$, we have

$$T_p([e_k]) = p^{n-k}[e_k] + \sum_{j=0}^{p-1} \sum_{k'=0}^{k} \binom{k}{k'} p^{k'} (-j)^{k-k'} [e_{k'}].$$

In particular, we have $T_p([e_0]) = (p^n + p)[e_0]$, and hence $e_{T_p}[e_0] = 0$. Therefore, inductively, we obtain that $e_{T_p}[e_k] = 0$ for any integer $k \in \{0, \dots, n-1\}$ and that $e_{T_p}[e_n] = [e_n]$, which implies that the $\mathbb{Z}_p$-module $e_{T_p}(\mathcal{M}_{n,p})_{\Gamma_\infty}$ is generated by $e_{T_p}[e_n]$. Hence by Lemma 4.16, $e_{T_p}(\mathcal{M}_{n,p})_{\Gamma_\infty}$ is a free $\mathbb{Z}_p$-module of rank 1. □

Recall that $[C_\nu] = [\{0, i\infty\} \otimes e_\nu] \in H_1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_n)$.

**Proposition 5.7.** *For any integer $m \geq 0$, the $\mathbb{Z}_p$-module $H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$ is generated by (the image of) $H_1^{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$ and a set of lifts of $e_{T_p} T_p^m [C_\nu]$ $(1 \leq \nu \leq n-1)$.*

*Proof.* By definition, we have an exact sequence of Hecke modules

$$0 \to H_1(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \to H_1(Y, \mathcal{M}_{n,p}) \to H_1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \xrightarrow{\partial} H_0(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \to 0.$$

By Lemma 4.15, the ordinary part $H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$ of the relative homology group is generated by the set $\{e_{T_p}[C_\nu] \mid 0 \leq \nu \leq n\}$. Then by the same argument as in Lemma 4.17 using Lemma 5.6 instead of Lemma 4.16, we find that the kernel of the boundary map $\partial : H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \to H_0^{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$ is generated by the set $\{e_{T_p}[C_\nu] \mid 1 \leq \nu \leq n-1\}$. Since the homomorphism

$$T_p : H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \to H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$$

is an isomorphism and the boundary map $\partial$ is Hecke equivariant, it follows that the kernel of the boundary map is generated by the set $\{e_{T_p} T_p^m [C_\nu] \mid 1 \leq \nu \leq n-1\}$. This fact together with the above exact sequence implies the proposition. □

**Corollary 5.8.** *Let $m \geq n$ be an integer. For each integer $\nu \in \{1, \ldots, n-1\}$, recall the cycle $[\widetilde{T_p^m(C_\nu(\tau))}] \in H_1(Y^{\mathrm{BS}}, \mathcal{M}_{n,(p)})$ defined in Definition 3.13 (see also Lemma 3.15(2)). For any ordinary cohomology class $c \in H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes \mathbb{Q}_p$ satisfying $\langle c, H_1(\partial Y^{\mathrm{BS}}, \mathcal{M}_n)\rangle \subset \mathbb{Z}_p$, we have*

$$\delta_p(c) = \min\{\delta \in \mathbb{Z}_{\geq 0} \mid p^\delta \langle c, [\widetilde{T_p^m(C_\nu(\tau))}]\rangle \in \mathbb{Z}_p \text{ for any integer } 1 \leq \nu \leq n-1\}.$$

*Proof.* By Lemma 3.15(3), we see that $e_{T_p}[\widetilde{T_p^m(C_\nu(\tau))}] \in H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$ maps to $e_{T_p} T_p^m [C_\nu(\tau)] \in H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$ under the homomorphism

$$H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \to H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}).$$

Therefore, Proposition 5.7 shows that the $\mathbb{Z}_p$-module $H_1^{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$ is generated by the image of $H_1^{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_n)$ and the set $\{e_{T_p}[\widetilde{T_p^m(C_\nu(\tau))}] \mid 1 \leq \nu \leq n-1\}$. On the other hand, by Lemma 5.5, we have

$$\langle c, e_{T_p}[\widetilde{T_p^m(C_\nu(\tau))}]\rangle = \langle c, [\widetilde{T_p^m(C_\nu(\tau))}]\rangle.$$

Now, since $\langle c, H_1(\partial Y^{\mathrm{BS}}, \mathcal{M}_n)\rangle \subset \mathbb{Z}_p$ by assumption, Lemma 5.5 shows that

$$\delta_p(c) = \min\{\delta \in \mathbb{Z}_{\geq 0} \mid p^\delta \langle c, [\widetilde{T_p^m(C_\nu(\tau))}]\rangle \in \mathbb{Z}_p \text{ for any integer } 1 \leq \nu \leq n-1\}. \quad \square$$

**Corollary 5.9.** *For any integer $m \geq n$, we have*

$$\delta_p(\mathrm{Eis}_n) = \min\{\delta \in \mathbb{Z}_{\geq 0} \mid p^\delta \langle \mathrm{Eis}_n, [\widetilde{T_p^m(C_\nu(\tau))}]\rangle \in \mathbb{Z}_p \text{ for any integer } 1 \leq \nu \leq n-1\}.$$

*Proof.* By Lemma 2.8, we have $\langle \mathrm{Eis}_n, H_1(\partial Y^{\mathrm{BS}}, \mathcal{M}_n)\rangle \subset \mathbb{Z}_p$ and $e_{T_p'} \mathrm{Eis}_n = \mathrm{Eis}_n$. The corollary thus follows from Corollary 5.8. □

## 6. Relation between the denominators of the Eisenstein classes

Recall that $\Delta_p(\mathrm{Eis}_n)$ denotes the $p$-part of the denominator $\Delta(\mathrm{Eis}_n)$ of the Eisenstein class (see Definition 5.2). In this section, we fix a prime number $p \geq 5$ and discuss another expression for the denominator $\Delta(\mathrm{Eis}_n)$ of the Eisenstein class $\mathrm{Eis}_n$. We also study a relation of the denominators $\Delta_p(\mathrm{Eis}_n)$ and $\Delta_p(\mathrm{Eis}_{n'})$ of the Eisenstein classes when $n$ and $n'$ are $p$-adically close.

**6.1.** ***Structure of the ordinary part of cohomology groups.*** We will study the structure of the ordinary part of cohomology groups. Results similar to those obtained in this subsection can be found in [Hida 1986]. Hida [1986; 1988] studied the ordinary part of cohomology groups for $\Gamma'\backslash\mathbb{H}$ in the case that $\Gamma'/\{\pm 1\}$ is torsion-free. However, in the present paper we consider the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ which has torsion elements other than $\pm\left(\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}\right)$. We will therefore provide proofs of all relevant theorems for completeness.

Note that since we assume that $p \geq 5$, any short exact sequence of $\Gamma$-modules induces a long exact sequence in cohomology.

**Lemma 6.1.** *The inclusion map $\mathcal{M}^{\flat}_{n,p} \hookrightarrow \mathcal{M}_{n,p}$ induces an isomorphism*

$$H^{\bullet}_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}^{\flat}_{n,p}) \xrightarrow{\sim} H^{\bullet}_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}).$$

*Proof.* It suffices to prove that $e_{T'_p} H^i(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/\mathcal{M}^{\flat}_{n,p}) = 0$ for any integer $i \geq 0$. Since $X_1^n, X_2^n \in \mathcal{M}^{\flat}_{n,p}$, any element in $\mathcal{M}_{n,p}/\mathcal{M}^{\flat}_{n,p}$ can be represented by a polynomial of the form $X_1X_2f(X_1, X_2)$, where $f \in \mathcal{M}_{n-2,p}$. Hence the fact that

$$\widetilde{\begin{pmatrix} p & \\ & 1 \end{pmatrix}} \cdot X_1X_2f(X_1, X_2) = pX_1X_2f(pX_1, X_2),$$
$$\widetilde{\begin{pmatrix} 1 & j \\ & p \end{pmatrix}} \cdot X_1X_2f(X_1, X_2) = p(X_1 + jX_2)X_2f(X_1 + jX_2, pX_2)$$

shows that $e_{T'_p}c = 0$ for any element $c \in \mathrm{Hom}_{\mathbb{Z}}(S_{\bullet}(\mathbb{H}^{\mathrm{BS}}), \mathcal{M}_{n,p}/\mathcal{M}^{\flat}_{n,p})$. In particular, we have $e_{T'_p} H^i(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/\mathcal{M}^{\flat}_{n,p}) = 0$. □

Thanks to Lemma 6.1, in the following, we focus on the ordinary cohomology groups with coefficient $\mathcal{M}_{n,p}$.

**Lemma 6.2.** *For any polynomial $f(X_1, X_2) \in \mathcal{M}_n/p\mathcal{M}_n$, we have $f|(T'_p)^2 \in \mathbb{F}_p X_2^n$.*

*Proof.* By definition, we have

$$(f|T'_p)(X_1, X_2) = f(0, X_2) + \sum_{j=0}^{p-1} f(X_1 + jX_2, 0).$$

Hence we see that $(f|T'_p)(X_1, 0) = \sum_{j=0}^{p-1} f(X_1, 0) = 0$, and we obtain

$$(f|(T'_p)^2)(X_1, X_2) = (f|T'_p)(0, X_2) \in \mathbb{F}_p X_2^n. \qquad \square$$

The boundary $\partial Y^{\mathrm{BS}}$ is of real dimension 1, and hence $H^2(\partial Y^{\mathrm{BS}}, \mathcal{M})$ vanishes for any $\Gamma$-module $\mathcal{M}$. Therefore, for any integer $r \geq 0$, the short exact sequence $0 \to \mathcal{M}_{n,p} \xrightarrow{\times p^r} \mathcal{M}_{n,p} \to \mathcal{M}_{n,p}/p^r\mathcal{M}_{n,p} \to 0$ induces an isomorphism

$$H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes \mathbb{Z}_p/(p^r) \xrightarrow{\sim} H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p^r\mathcal{M}_{n,p}). \tag{6-1}$$

**Lemma 6.3.** *The ordinary part $H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$ is torsion-free.*

*Proof.* By using the exact sequence of $M_2^+(\mathbb{Z})$-modules

$$0 \to \mathcal{M}_{n,p} \xrightarrow{\times p} \mathcal{M}_{n,p} \to \mathcal{M}_{n,p}/p\mathcal{M}_{n,p} \to 0,$$

we obtain an isomorphism of Hecke modules

$$\mathrm{coker}\big(\mathcal{M}_{n,p}^{\Gamma_\infty}/p\mathcal{M}_{n,p}^{\Gamma_\infty} \to (\mathcal{M}_{n,p}/p\mathcal{M}_{n,p})^{\Gamma_\infty}\big) \xrightarrow{\sim} H^1(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p})[p],$$

where for an abelian group $M$ we write $M[p] := \ker(M \xrightarrow{\times p} M)$ for the subgroup of $p$-torsion elements of $M$. A direct computation shows that $\mathcal{M}_{n,p}^{\Gamma_\infty} = \mathbb{Z}_p X_2^n$. Hence Lemma 6.2 implies that

$$H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p})[p] = \mathrm{coker}\big(\mathbb{F}_p X_2^n \to e_{T'_p}(\mathcal{M}_{n,p}/p\mathcal{M}_{n,p})^{\Gamma_\infty}\big) = 0. \qquad \square$$

**Corollary 6.4.** (1) *The ordinary part $H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$ is a free $\mathbb{Z}_p$-module of rank* 1.

(2) *We have a canonical isomorphism $H^1_{\mathrm{int}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{Z}_p \xrightarrow{\sim} H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$.*

(3) *We have $c|T'_\ell = (1 + \ell^{n+1})c$ for any element $c \in H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$ and prime number $\ell$.*

*Proof.* By Lemma 6.3, we have a surjective homomorphism

$$H^1_{\mathrm{int}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{Z}_p \to H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}).$$

Lemma 2.7 shows that $H^1_{\mathrm{int}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_n) \cong \mathbb{Z}$ and $c|T'_\ell = (1+\ell^{n+1})c$ for any element $c \in H^1_{\mathrm{int}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_n)$ and prime number $\ell$. These facts imply this corollary. $\square$

**Proposition 6.5.** *The ordinary part $H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$ is torsion-free.*

*Proof.* Since $H^0(Y^{\mathrm{BS}}, \mathcal{M}_n) = \mathcal{M}_n^\Gamma = 0$, the exact sequence

$$0 \to \mathcal{M}_{n,p} \xrightarrow{\times p} \mathcal{M}_{n,p} \to \mathcal{M}_{n,p}/p\mathcal{M}_{n,p} \to 0$$

implies that

$$H^0_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p\mathcal{M}_{n,p}) = H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})[p].$$

Since $\mathbb{F}_p X_2^n \cap H^0(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p\mathcal{M}_{n,p}) = (\mathbb{F}_p X_2^n) \cap (\mathcal{M}_{n,p}/p\mathcal{M}_{n,p})^\Gamma = 0$, we get from Lemma 6.2 that the module $H^0_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p\mathcal{M}_{n,p})$ vanishes. □

**Lemma 6.6.** *We have* $H^2_{\mathrm{ord}}(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) = H^2_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) = 0$.

*Proof.* We have that $H^2(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) = 0$ since the boundary $\partial Y^{\mathrm{BS}}$ is homeomorphic to the circle, and hence the canonical homomorphism

$$H^2_{\mathrm{ord}}(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \to H^2_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$$

is surjective. Therefore, we only need to show that $H^2_{\mathrm{ord}}(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) = 0$.

Since $Y$ is a two-dimensional real manifold, we have $H^3(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) = 0$, and hence the short exact sequence $0 \to \mathcal{M}_{n,p} \xrightarrow{\times p} \mathcal{M}_{n,p} \to \mathcal{M}_{n,p}/p\mathcal{M}_{n,p} \to 0$ induces an isomorphism

$$H^2(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes \mathbb{Z}/(p) \xrightarrow{\sim} H^2(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p\mathcal{M}_{n,p}).$$

Therefore, it suffices to prove that $H^2_{\mathrm{ord}}(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p\mathcal{M}_{n,p}) = 0$.

For notational simplicity, set $\overline{\mathcal{M}}_{n,p} := \mathcal{M}_{n,p}/p\mathcal{M}_{n,p}$. Let

$$\mathscr{F} \in S_2(\mathbb{H}^{\mathrm{BS}})$$

be a representative of a fundamental class of $H_2(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \mathbb{Z}) \cong \mathbb{Z}$. Then it is known that the homomorphism $\mathrm{Hom}_{\mathbb{Z}}(S_2(\mathbb{H}^{\mathrm{BS}}), \overline{\mathcal{M}}_{n,p}) \to \overline{\mathcal{M}}_{n,p};\ \phi \mapsto \phi(\mathscr{F})$ induces an isomorphism

$$\mathrm{ev}_{\mathscr{F}} : H^2(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \overline{\mathcal{M}}_{n,p}) \xrightarrow{\sim} (\overline{\mathcal{M}}_{n,p})_\Gamma. \tag{6-2}$$

See [Shimura 1994, Proposition 8.2; Hida 1993, §6.1, Proposition 1] for example.

We will show that for any $[\phi] \in H^2(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}, \overline{\mathcal{M}}_{n,p})$, we have $[\phi]|_{T'_p} = 0$. By (6-2), it suffices to show that $\mathrm{ev}_{\mathscr{F}}([\phi]|_{T'_p}) \equiv 0$. Here we use $\equiv$ to emphasize that it is an identity in $(\overline{\mathcal{M}}_{n,p})_\Gamma$. We then compute

$$\begin{aligned}
\mathrm{ev}_{\mathscr{F}}([\phi]|_{T'_p}) &\equiv \phi|_{T'_p}(\mathscr{F})(X_1, X_2) \\
&= \widetilde{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} \phi\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\mathscr{F}\right)(X_1, X_2) + \sum_{j=0}^{p-1} \widetilde{\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}} \phi\left(\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\mathscr{F}\right)(X_1, X_2) \\
&\equiv \phi\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\mathscr{F}\right)(0, X_2) + \sum_{j=0}^{p-1} \phi\left(\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\mathscr{F}\right)(X_1 + jX_2, 0).
\end{aligned}$$

Put

$$a_p := \phi\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\mathscr{F}\right)(0, 1) \in \mathbb{F}_p, \quad a_j := \phi\left(\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\mathscr{F}\right)(1, 0) \in \mathbb{F}_p.$$

Then we find that

$$\begin{aligned}\mathrm{ev}_{\mathscr{F}}([\phi]|_{T_p'}) &\equiv a_p X_2^n + \sum_{j=0}^{p-1} a_j (X_1 + jX_2)^n \\ &= a_p X_2^n + \sum_{j=0}^{p-1} a_j \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-j} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X_2^n \\ &\equiv a_p X_2^n + \sum_{j=0}^{p-1} a_j X_2^n \\ &= \left(a_p + \sum_{j=0}^{p-1} a_j\right)\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - 1\right) X_1 X_2^{n-1} \\ &\equiv 0. \end{aligned}$$

□

For any $\Gamma$-module $\mathcal{M}$, we define the inner cohomology $H^1_!(Y^{\mathrm{BS}}, \mathcal{M})$ by

$$H^1_!(Y^{\mathrm{BS}}, \mathcal{M}) := \mathrm{im}\big(H^1(Y^{\mathrm{BS}}, \partial Y^{\mathrm{BS}}\mathcal{M}) \to H^1(Y^{\mathrm{BS}}, \mathcal{M})\big)$$

and, when $\mathcal{M}$ is a finitely generated $\mathbb{Z}_p$-module, we put

$$H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}) := e_{T_p'} H^1_!(Y^{\mathrm{BS}}, \mathcal{M}).$$

Then the following corollary follows from Lemma 6.6 and the isomorphism (6-1).

**Corollary 6.7.** *Let $r$ be a nonnegative integer and $\mathcal{M} \in \{\mathcal{M}_{n,p}, \mathcal{M}_{n,p}/p^r\mathcal{M}_{n,p}\}$. Then we have a natural exact sequence of Hecke modules*:

$$0 \to H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}) \to H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}) \to H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}) \to 0.$$

*Proof.* For notational simplicity, set $\mathcal{M}_{n,p}/p^r := \mathcal{M}_{n,p}/p^r\mathcal{M}_{n,p}$. Consider the natural commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) & \longrightarrow & H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) & \longrightarrow & H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p^r) & \to & H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p^r) & \to & H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p^r) \to 0 \end{array}$$

The upper row is exact by Lemma 6.6. Moreover, by (6-1), the right vertical map is surjective, and the bottom row is also exact. □

**Corollary 6.8.** *For any integer $r \geq 0$, the canonical homomorphism $\mathcal{M}_{n,p} \to \mathcal{M}_{n,p}/p^r\mathcal{M}_{n,p}$ induces isomorphisms*

$$\begin{aligned} H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes \mathbb{Z}_p/(p^r) &\xrightarrow{\sim} H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p^r\mathcal{M}_{n,p}), \\ H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes \mathbb{Z}_p/(p^r) &\xrightarrow{\sim} H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p^r\mathcal{M}_{n,p}). \end{aligned}$$

*Proof.* The exact sequence

$$0 \to \mathcal{M}_{n,p} \xrightarrow{\times p^r} \mathcal{M}_{n,p} \to \mathcal{M}_{n,p}/p^r\mathcal{M}_{n,p} \to 0$$

shows that we have an exact sequence

$$0 \to H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes \mathbb{Z}_p/(p^r) \to H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p^r\mathcal{M}_{n,p}) \to H^2_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})[p^r] \to 0.$$

Hence by Lemma 6.6, we obtain the first isomorphism.

By Corollary 6.4(1), we see that $\mathrm{Tor}_1^{\mathbb{Z}_p}(H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}), \mathbb{Z}_p/(p^r)) = 0$, and hence Corollary 6.7 for $\mathcal{M} = \mathcal{M}_{n,p}$ shows that

$$H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes \mathbb{Z}_p/(p^r) = \ker\bigl(H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes \mathbb{Z}_p/(p^r) \to H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes \mathbb{Z}_p/(p^r)\bigr).$$

Hence the second isomorphism follows from the first isomorphism, the isomorphism (6-1), and Corollary 6.7 for $\mathcal{M} = \mathcal{M}_{n,p}/p^r\mathcal{M}_{n,p}$. □

**Theorem 6.9.** *For any positive integers $r$ and $n'$ with $n \equiv n' \pmod{(p-1)p^{r-1}}$, we have the following canonical isomorphism of exact sequences which is $T'_\ell$-equivalent for any prime number $\ell \neq p$:*

$$\begin{array}{ccccccccc} 0 & \to & H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p^r) & \to & H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p^r) & \to & H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p^r) & \to & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \to & H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n',p}/p^r) & \to & H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n',p}/p^r) & \to & H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n',p}/p^r) & \to & 0 \end{array}$$

*where $\mathcal{M}_{n,p}/p^r := \mathcal{M}_{n,p}/p^r\mathcal{M}_{n,p}$.*

*Proof.* Theorem 6.9 follows from the results proved by Hida [1986]; see also [Harder 2011]. In the following, we briefly explain how we derive Theorem 6.9 from Hida's [1986] results.

First, note that since $p \geq 5$, we have canonical isomorphisms between a sheaf cohomology on $Y^{\mathrm{BS}}$ and a group cohomology of $\Gamma$:

$$H^1(Y^{\mathrm{BS}}, \mathcal{M}_{m,p}/p^r) \xrightarrow{\sim} H^1(\Gamma, \mathcal{M}_{m,p}/p^r). \tag{6-3}$$

The inner cohomology group $H^1_!(Y^{\mathrm{BS}}, \mathcal{M}_{m,p}/p^r)$ corresponds to the parabolic subgroup $H^1_P(\Gamma, \mathcal{M}_{m,p}/p^r)$ of $H^1(\Gamma, \mathcal{M}_{m,p}/p^r)$ under the isomorphism (6-3) (see [Hida 1986, Equation (4.1a)] for the definition of the parabolic subgroup).

Let $m \in \{n, n'\}$. Hida [1986, Proposition 4.7] showed that we have isomorphisms

$$\begin{aligned} e_{T_p'} H^1(\Gamma, \mathcal{M}_{m,p}/p^r) &\xrightarrow{\sim} e_{U_p'} H^1(\Gamma_0(p^r), \mathcal{M}_{m,p}/p^r); \quad x \mapsto e_{U_p'}\mathrm{res}(x), \\ e_{T_p'} H^1_P(\Gamma, \mathcal{M}_{m,p}/p^r) &\xrightarrow{\sim} e_{U_p'} H^1_P(\Gamma_0(p^r), \mathcal{M}_{m,p}/p^r); \quad x \mapsto e_{U_p'}\mathrm{res}(x), \end{aligned} \tag{6-4}$$

which are $T'_\ell$-equivariant for any prime number $\ell \neq p$. Here res denotes the restriction map.

Let $L_{m,r}$ denote the $\Gamma_0(p^r)$-module whose underlying abelian group is $\mathbb{Z}_p/(p^r)$ and the $\Gamma_0(p^r)$-action is given by the homomorphism $\Gamma_0(p^r) \to (\mathbb{Z}_p/(p^r))^\times$; $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto a^m \bmod p^r$. Hida also showed [1986, Corollary 4.5 and Equation (6.8)] that the $\Gamma_0(p^r)$-homomorphism $i_r : \mathcal{M}_{m,p}/p^r \to L_{m,r}$; $f(X_1, X_2) \mapsto f(1,0)$ induces Hecke-equivariant isomorphisms

$$\begin{aligned} e_{U'_p} H^1(\Gamma_0(p^r), \mathcal{M}_{m,p}/p^r) &\xrightarrow{\sim} e_{U'_p} H^1(\Gamma_0(p^r), L_{m,r}); \quad x \mapsto i_{r,*}(x), \\ e_{U'_p} H^1_P(\Gamma_0(p^r), \mathcal{M}_{m,p}/p^r) &\xrightarrow{\sim} e_{U'_p} H^1_P(\Gamma_0(p^r), L_{m,r}); \quad x \mapsto i_{r,*}(x). \end{aligned} \tag{6-5}$$

Since $n \equiv n' \pmod{(p-1)p^r}$, we have $L_{n,r} = L_{n',r}$ as $\Gamma_0(p^r)$-modules, by combining the isomorphisms (6-4) and (6-5) for $m = n, n'$, we obtain the commutative diagram

$$\begin{array}{ccc} H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p^r) & \hookrightarrow & H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}/p^r) \\ \cong\downarrow & & \downarrow\cong \\ e_{U'_p} H^1_P(\Gamma_0(p^r), L_{n,r}) & \hookrightarrow & e_{U'_p} H^1_P(\Gamma_0(p^r), L_{n,r}) \\ \| & & \| \\ e_{U'_p} H^1_P(\Gamma_0(p^r), L_{n',r}) & \hookrightarrow & e_{U'_p} H^1_P(\Gamma_0(p^r), L_{n',r}) \\ \cong\uparrow & & \uparrow\cong \\ H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n',p}/p^r) & \hookrightarrow & H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n',p}/p^r) \end{array}$$

where vertical arrows are isomorphisms and $T'_\ell$-equivariant for any prime number $\ell \neq p$. □

**6.2.** ***Another expression for*** $\boldsymbol{\Delta_p(\mathrm{Eis}_n)}$**.** Let $p \geq 5$ be a prime number and take a prime number $\ell \neq p$. Let

$$\mathcal{H}_{\ell,p} := \mathbb{Z}_p[X]$$

be the polynomial ring over $\mathbb{Z}_p$, and by using the Hecke operator $T'_\ell$ at $\ell$, we regard cohomology groups that appear in the present paper as $\mathcal{H}_{\ell,p}$-modules. For notational simplicity, we put

$$x_{\ell,n} := X - (1 + \ell^{n+1}) \quad \text{and} \quad \mathcal{B}_{\ell,p,n} := \mathcal{H}_{\ell,p}/(x_{\ell,n}).$$

Note that by Corollary 6.4, we have $\mathcal{B}_{\ell,p,n} \xrightarrow{\sim} H^1_{\mathrm{ord}}(\partial Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$; $1 \mapsto e_{T'_p}[e_n]$ as $\mathcal{H}_{\ell,p}$-modules.

**Lemma 6.10.** *Let $c \in H^1(Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{C}$ be a cohomology class. If we have that $c|T'_\ell = (1 + \ell^{n+1})c$, then $c|T'_{\ell'} = (1 + \ell'^{n+1})c$ for any prime number $\ell'$, that is, the cohomology class $c$ is a scalar multiple of* $\mathrm{Eis}_n$.

*Proof.* It is well known that one can take a $T'_\ell$-Hecke-eigenbasis $f_1, \dots, f_t \in H^1(Y^{\mathrm{BS}}, \mathcal{M}_n) \otimes \mathbb{C}$ such that $f_1 = \mathrm{Eis}_n$ and that the elements $f_2, \dots, f_t$ correspond to either cusp forms or their complex conjugates via the Eichler–Shimura homomorphism. Then the Ramanujan conjecture proved by Deligne shows that the absolute value of the $T'_\ell$-eigenvalue of $f_i$ $(2 \le i \le t)$ is less than $1 + \ell^{n+1}$, which implies this lemma. □

**Lemma 6.11.** *We have* $H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})[x_{\ell,n}] = \mathbb{Z}_p \Delta_p(\mathrm{Eis}_n)\mathrm{Eis}_n$.

*Proof.* By Proposition 6.5 and Lemma 6.10, $H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})[x_{\ell,n}] \subset \mathbb{Q}_p \cdot \mathrm{Eis}_n$. Hence this lemma follows from the definition of the denominator $\Delta_p(\mathrm{Eis}_n)$ of the Eisenstein class $\mathrm{Eis}_n$ and Lemmas 5.5 and 6.1. □

**Definition 6.12.** We define $[\mathcal{E}_{\ell,p,n}] \in \mathrm{Ext}^1_{\mathcal{H}_{\ell,p}}(\mathcal{B}_{\ell,p,n}, H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}))$ to be the element corresponding to the exact sequence of $\mathcal{H}_{\ell,p}$-modules in Corollary 6.7 for $\mathcal{M} = \mathcal{M}_{n,p}$:

$$0 \to H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \to H^1_{\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \to \mathcal{B}_{\ell,p,n} \to 0.$$

The following lemma follows directly from Lemma 6.11.

**Lemma 6.13.** $\mathrm{Ann}_{\mathbb{Z}_p}([\mathcal{E}_{\ell,p,n}]) = \Delta_p(\mathrm{Eis}_n)\mathbb{Z}_p$.

**Lemma 6.14.** *We have a natural identification*

$$H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes_{\mathcal{H}_{\ell,p}} \mathcal{B}_{\ell,p,n} = \mathrm{Ext}^1_{\mathcal{H}_{\ell,p}}(\mathcal{B}_{\ell,p,n}, H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})).$$

*Proof.* Since $x_{\ell,n}$ is an regular element of $\mathcal{H}_{\ell,p}$, we have an exact sequence of $\mathcal{H}_{\ell,p}$-modules

$$0 \to \mathcal{H}_{\ell,p} \xrightarrow{\times x_\ell} \mathcal{H}_{\ell,p} \to \mathcal{B}_{\ell,p,n} \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_{\mathcal{H}_{\ell,p}}(-, H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}))$ to this short exact sequence, we obtain the desired identification. □

**Lemma 6.15.** *For any positive integers* $r$ *and* $n'$ *with* $n \equiv n' \pmod{(p-1)p^{r-1}}$, *we have a natural isomorphism of* $\mathcal{H}_{\ell,p}$*-modules*

$$H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes_{\mathcal{H}_{\ell,p}} \mathcal{B}_{\ell,p,n}/(p^r) \cong H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n',p}) \otimes_{\mathcal{H}_{\ell,p}} \mathcal{B}_{\ell,p,n'}/(p^r).$$

*Moreover, the image of* $[\mathcal{E}_{\ell,p,n}] \bmod p^r$ *is* $[\mathcal{E}_{\ell,p,n'}] \bmod p^r$ *under this isomorphism* (*and the identification in Lemma 6.14*).

*Proof.* This lemma follows from Theorem 6.9 and Corollary 6.8. □

**Definition 6.16.** We define a polynomial $\Phi_{\ell,n}(t) \in \mathbb{Z}_p[t]$ to be the characteristic polynomial associated with $T'_\ell : H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \to H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$:

$$\Phi_{\ell,n}(t) := \det\bigl(t \cdot \mathrm{id} - T'_\ell \mid H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})\bigr).$$

**Lemma 6.17.** *The $\mathbb{Z}_p$-module $H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes_{\mathcal{H}_{\ell,p}} \mathcal{B}_{\ell,p,n}$ is annihilated by $\Phi_{\ell,n}(1+\ell^{n+1})$.*

*Proof.* By the Cayley–Hamilton theorem, the Hecke module $H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p})$ is annihilated by $\Phi_{\ell,n}(T'_\ell)$. Hence the $\mathbb{Z}_p$-module $H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes_{\mathcal{H}_{\ell,p}} \mathcal{B}_{\ell,p,n}$ is annihilated by $\Phi_{\ell,n}(1+\ell^{n+1})$ since $\mathcal{B}_{\ell,p,n} = \mathcal{H}_{\ell,p}/(X-(1+\ell^{1+n}))$. □

**Lemma 6.18.** *Let $r$ be a positive integer satisfying $r > \mathrm{ord}_p(\Phi_{\ell,n}(1+\ell^{n+1}))$. Then for any even integer $n' \geq 2$ with $n \equiv n' \pmod{(p-1)p^{r-1}}$, we have*

$$\mathrm{ord}_p(\Phi_{\ell,n}(1+\ell^{n+1})) = \mathrm{ord}_p(\Phi_{\ell,n'}(1+\ell^{n'+1})).$$

*Proof.* By Theorem 6.9, we have

$$\Phi_{\ell,n}(t) \equiv \Phi_{\ell,n'}(t) \pmod{p^r}.$$

The fact that $n \equiv n' \pmod{(p-1)p^{r-1}}$ implies that $\ell^{n+1} \equiv \ell^{n'+1} \pmod{p^r}$, and we obtain $\Phi_{\ell,n}(1+\ell^{n+1}) \equiv \Phi_{\ell,n'}(1+\ell^{n'+1}) \pmod{p^r}$. The assumption that $r > \mathrm{ord}_p(\Phi_{\ell,n}(1+\ell^{n+1}))$ shows that $\mathrm{ord}_p(\Phi_{\ell,n}(1+\ell^{n+1})) = \mathrm{ord}_p(\Phi_{\ell,n'}(1+\ell^{n'+1}))$. □

**Proposition 6.19.** *Let $r$ be a positive integer satisfying $r > \mathrm{ord}_p(\Phi_{\ell,n}(1+\ell^{n+1}))$. Then for any even integer $n' \geq 2$ with $n \equiv n' \pmod{(p-1)p^{r-1}}$, we have*

$$\Delta_p(\mathrm{Eis}_n) = \Delta_p(\mathrm{Eis}_{n'}).$$

*Proof.* By Lemmas 6.15, 6.17, and 6.18, we have the natural isomorphism

$$\begin{aligned} H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes_{\mathcal{H}_{\ell,p}} \mathcal{B}_{\ell,p,n} &= H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n,p}) \otimes_{\mathcal{H}_{\ell,p}} \mathcal{B}_{\ell,p,n}/(p^r) \\ &\cong H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n',p}) \otimes_{\mathcal{H}_{\ell,p}} \mathcal{B}_{\ell,p,n'}/(p^r) \\ &= H^1_{!,\mathrm{ord}}(Y^{\mathrm{BS}}, \mathcal{M}_{n',p}) \otimes_{\mathcal{H}_{\ell,p}} \mathcal{B}_{\ell,p,n'}. \end{aligned}$$

Moreover, the image of $[\mathcal{E}_{\ell,p,n}]$ under this isomorphism is $[\mathcal{E}_{\ell,p,n'}]$, and Lemma 6.13 implies that

$$\Delta_p(\mathrm{Eis}_n)\mathbb{Z}_p = \mathrm{Ann}_{\mathbb{Z}_p}([\mathcal{E}_{\ell,p,n}]) = \mathrm{Ann}_{\mathbb{Z}_p}([\mathcal{E}_{\ell,p,n'}]) = \Delta_p(\mathrm{Eis}_{n'})\mathbb{Z}_p.$$ □

## 7. Kubota–Leopoldt $p$-adic $L$-function

Let $p$ be a prime number. In this section, we introduce the Kubota–Leopoldt $p$-adic $L$-functions and prove certain congruence properties that will be used in the proof of Theorem 2.13.

Let $\omega : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$ denote the Teichmüller character, and let $\mathbb{1}$ denote the trivial character. For any Dirichlet character $\chi$, we denote by $L_p(s,\chi) \in \mathbb{C}_p[\![s]\!]$ the Kubota–Leopoldt $p$-adic $L$-function attached to $\chi$.

**Proposition 7.1** [Washington 1997, Theorems 5.11 and 5.12, Exercises 5.11(1)].
(1) *For any Dirichlet character $\chi$, the $p$-adic $L$-function $L_p(s,\chi)$ converges on $\mathbb{Z}_p - \{1\}$. Moreover, for any integer $m \geq 2$, we have*

$$L_p(1-m,\chi) = (1-\chi\omega^{-m}(p)p^{m-1})L(1-m,\chi\omega^{-m}).$$

*In particular, we have* $\mathrm{ord}_p(L_p(1-m,\omega^m)) = \mathrm{ord}_p(\zeta(1-m))$.

(2) *We have*

$$L_p(s,\mathbb{1}) \in \frac{p-1}{p(s-1)} + \mathbb{Z}_p[\![s-1]\!].$$

(3) *If $m \not\equiv 0 \pmod{p-1}$, then we have*

$$L_p(s,\omega^m) \in \mathbb{Z}_p + p\mathbb{Z}_p[\![s-1]\!].$$

By using the Kubota–Leopoldt $p$-adic $L$-functions, Theorem 4.1 can be restated as follows.

**Corollary 7.2.** *If we put*

$$D_p(n,\nu) := \frac{L_p(-\nu,\omega^{1+\nu})L_p(\nu-n,\omega^{n-\nu+1})}{L_p(-1-n,\omega^{n+2})} - L_p(-\nu,\omega^{1+\nu}) - L_p(\nu-n,\omega^{n-\nu+1}),$$

*then for any integer $\nu \in \{1,\dots,n-1\}$ we have*

$$\lim_{m\to\infty}\langle \mathrm{Eis}_n, [\widetilde{T_p^{m!}(C_\nu(\tau))}]\rangle = \frac{1-p^{n+1}}{(1-p^\nu)(1-p^{n-\nu})}D_p(n,\nu).$$

For any even integer $m$, we define a positive integer $N_m$ by

$$N_m := \text{numerator of } \zeta(1-m).$$

**Corollary 7.3.** *Let $m \geq 2$ be an even integer.*

(1) *If $m \not\equiv 0 \pmod{p-1}$, then we have* $\mathrm{ord}_p(\zeta(1-m)) = \mathrm{ord}_p(N_m)$.

(2) *If $m \equiv 0 \pmod{p-1}$, then we have* $\mathrm{ord}_p(N_m) = 0$.

(3) *Let $r$ and $m'$ be positive integers with $m \equiv m' \pmod{(p-1)p^{r-1}}$. If $r > \mathrm{ord}_p(N_m)$, then*

$$\mathrm{ord}_p(N_m) = \mathrm{ord}_p(N_{m'}).$$

*Proof.* Claims (1) and (2) follow immediately from Proposition 7.1. In the case where $m \equiv 0 \pmod{p-1}$, Claim (3) follows from Claim (2). In the case where $m \not\equiv 0 \pmod{p-1}$, Claim (3) follows from Proposition 7.1. □

**Corollary 7.4.** *Let $x$ be an integer with $x \not\equiv 0 \pmod{p-1}$. For any integer $y$, we have*

$$\frac{L_p(1-x,\omega^x)L_p(1-y,\mathbb{1})}{L_p(1-x-y,\omega^x)} \in L_p(1-y,\mathbb{1}) + \frac{\mathbb{Z}_p}{L_p(1-x-y,\omega^x)}.$$

*Proof.* Since $x \not\equiv 0 \pmod{p-1}$, by Proposition 7.1(3), we have

$$L_p(1-x, \omega^x) \in L_p(1-x-y, \omega^x) + py\mathbb{Z}_p.$$

Moreover, by Proposition 7.1(2), we have $pyL_p(1-y, \mathbb{1}) \in 1 + p\mathbb{Z}_p$, which shows

$$\frac{L_p(1-x, \omega^x)L_p(1-y, \mathbb{1})}{L_p(1-x-y, \omega^x)} \in L_p(1-y, \mathbb{1}) + \frac{\mathbb{Z}_p}{L_p(1-x-y, \omega^x)}. \qquad \square$$

**Corollary 7.5.** *For any integers $x$ and $y$, we have*

$$\begin{aligned}\frac{L_p(1-x, \mathbb{1})L_p(1-y, \mathbb{1})}{L_p(1-x-y, \mathbb{1})} &\equiv \frac{p-1}{px} + \frac{p-1}{py} \pmod{\mathbb{Z}_p} \\ &\equiv L_p(1-x, \mathbb{1}) + L_p(1-y, \mathbb{1}) \pmod{\mathbb{Z}_p}.\end{aligned}$$

*Proof.* For notational simplicity, we put

$$R(s) := -\frac{p-1}{ps} \quad \text{and} \quad H(s-1) := L_p(s, \mathbb{1}) - R(1-s).$$

By Proposition 7.1(2), we have $H(s) \in \mathbb{Z}_p[\![s]\!]$, $xR(x)$, $yR(y)$, $(x+y)R(x+y) \in p^{-1}\mathbb{Z}_p^\times$, and $R(x+y)^{-1} = R(x)^{-1} + R(y)^{-1}$. Since $H(s) \in \mathbb{Z}_p[\![s]\!]$, we have

$$H(x) \in H(x+y) + y\mathbb{Z}_p, \quad H(y) \in H(x+y) + x\mathbb{Z}_p.$$

Put $\alpha := 1 + R(x+y)^{-1}H(x+y) \in 1 + p\mathbb{Z}_p$. Then

$$\frac{L_p(1-x, \mathbb{1})L_p(1-y, \mathbb{1})}{L_p(1-x-y, \mathbb{1})} \in \frac{(R(x) + H(x+y) + y\mathbb{Z}_p)(R(y) + H(x+y) + x\mathbb{Z}_p)}{R(x+y) + H(x+y)}$$

and we have

$$\begin{aligned}&\frac{(R(x) + H(x+y) + y\mathbb{Z}_p)(R(y) + H(x+y) + x\mathbb{Z}_p)}{R(x+y) + H(x+y)} \\ &\quad \subset \frac{R(x)^{-1} + R(y)^{-1}}{\alpha}(R(x)R(y) + (R(x) + R(y))H(x+y) + p^{-1}\mathbb{Z}_p) \\ &\quad = R(x) + R(y) + \mathbb{Z}_p. \qquad \square\end{aligned}$$

## 8. Proof of Theorem 2.13

Let $p$ be a prime number. As in Corollary 7.2, for any integer $1 \le \nu \le n-1$, we define

$$D_p(n, \nu) := \frac{L_p(-\nu, \omega^{1+\nu})L_p(\nu-n, \omega^{n-\nu+1})}{L_p(-1-n, \omega^{n+2})} - L_p(-\nu, \omega^{1+\nu}) - L_p(\nu-n, \omega^{n-\nu+1})$$

and set

$$\delta_p(n, \nu) := \max\{-\mathrm{ord}_p(D_p(n, \nu)), 0\}.$$

**Proposition 8.1.** *We have*

$$\delta_p(\mathrm{Eis}_n) = \max_{1\le \nu \le n-1} \delta_p(n,\nu).$$

*Proof.* By Corollary 7.2, we have

$$\lim_{m\to\infty} \langle \mathrm{Eis}_n, \widetilde{[T_p^{m!}(C_\nu(\tau))]}\rangle = \frac{1-p^{n+1}}{(1-p^\nu)(1-p^{n-\nu})} D_p(n,\nu).$$

Hence, for any sufficiently large integer $m$, we have

$$\mathrm{ord}_p\big(\langle \mathrm{Eis}_n, \widetilde{[T_p^{m!}(C_\nu(\tau))]}\rangle\big) = \mathrm{ord}_p(D_p(n,\nu)),$$

and this proposition follows from Corollary 5.9. □

Recall that $N_m$ denotes the numerator of $\zeta(1-m)$.

**Proposition 8.2.** *Let $p$ be a prime number.*

(1) $\delta_p(\mathrm{Eis}_n) \le \mathrm{ord}_p(N_{n+2})$.

(2) *If $p < n$, then $\delta_p(\mathrm{Eis}_n) = \mathrm{ord}_p(N_{n+2})$.*

The proof of Proposition 8.2 is given in Section 8.1. First, we give the proof of Theorem 2.13 assuming Proposition 8.2, that is, we show that $\Delta(\mathrm{Eis}_n) = N_{n+2}$.

*Proof of Theorem 2.13.* Take a prime number $p$. It suffices to show that $\delta_p(\mathrm{Eis}_n) = \mathrm{ord}_p(N_{n+2})$. When $p-1 \mid n+2$, we have $0 \le \delta_p(\mathrm{Eis}_n) \le \mathrm{ord}_p(N_{n+2}) = 0$ by Proposition 8.2, and hence we may assume that $n \not\equiv -2 \pmod{p-1}$. Note that $p \ge 5$ in this case. Take a prime number $\ell \ne p$, and positive integers $r$ and $n'$ satisfying

- $r > \mathrm{ord}_p(\Phi_{\ell,n}(1+\ell^n))$,
- $p < n'$,
- $n \equiv n' \pmod{p^{r-1}(p-1)}$.

Then by Proposition 8.2(2), we have $\delta_p(\mathrm{Eis}_{n'}) = \mathrm{ord}_p(N_{n'+2})$, and Proposition 6.19 implies that

$$\delta_p(\mathrm{Eis}_n) = \delta_p(\mathrm{Eis}_{n'}) = \mathrm{ord}_p(N_{n'+2}).$$

Hence Corollary 7.3 shows that $\delta_p(\mathrm{Eis}_n) = \mathrm{ord}_p(N_{n'+2}) = \mathrm{ord}_p(N_{n+2})$. □

**8.1.** ***Proof of Proposition 8.2.*** In this subsection, we prove Proposition 8.2. The proof is divided into the following two cases:

- $p-1 \nmid n+2$.
- $p-1 \mid n+2$.

**8.1.1.** $p-1 \nmid n+2$.

**Lemma 8.3.** *If $p-1 \nmid n+2$, then we have $\delta_p(\mathrm{Eis}_n) \leq \mathrm{ord}_p(N_{n+2})$.*

*Proof.* Take an integer $\nu \in \{1, \dots, n-1\}$. When $p-1 \nmid 1+\nu$ and $p-1 \nmid n-\nu+1$, both $L_p(-\nu, \omega^{1+\nu})$ and $L_p(\nu-n, \omega^{n-\nu+1})$ are $p$-adic integers, and hence we have

$$\delta_p(n, \nu) \leq \mathrm{ord}_p(L_p(-1-n, \omega^{n+2})) = \mathrm{ord}_p(N_{n+2}).$$

Suppose $p-1 \mid 1+\nu$ (resp. $p-1 \mid n-\nu+1$). Then since $p-1 \nmid n+2$, we see that $n-\nu+1$ (resp. $1+\nu$) is not divisible by $p-1$. Therefore, Corollary 7.4 shows that

$$D_p(n, \nu) \in \frac{\mathbb{Z}_p}{L_p(-1-n, \omega^{n+2})} + \mathbb{Z}_p,$$

which implies that $\delta_p(n, \nu) \leq \mathrm{ord}_p(N_{n+2})$. Hence Proposition 8.1 implies this lemma. □

Moreover, if $p < n$, the result of Carlitz concerning the index of irregularity of a prime shows the following lemma.

**Lemma 8.4.** *If $p-1 \nmid n+2$ and $p < n$, there is an (odd) integer $\nu \in \{1, \dots, n-1\}$ such that $\delta_p(n, \nu) = \mathrm{ord}_p(N_{n+2})$. In particular, we have $\delta_p(\mathrm{Eis}_n) = \mathrm{ord}_p(N_{n+2})$ in this case.*

*Proof.* By Lemma 8.3, for any regular prime $p$ and (odd) integer $\nu \in \{1, \dots, n-1\}$, we have $\delta_p(n, \nu) = \mathrm{ord}_p(N_{n+2}) = 0$. Therefore, we may assume that $p$ is an irregular prime. In particular, $p \geq 37$.

We define the index $d(p)$ of irregularity of the prime number $p$ by

$$d(p) := \#\{1 \leq t \leq p-3 \mid t \in 2\mathbb{Z}, B_t \in p\mathbb{Z}_p\} = \#\{1 \leq t \leq p-3 \mid t \in 2\mathbb{Z}, L_p(1-t, \omega^t) \in p\mathbb{Z}_p\}.$$

Then by using the result of Carlitz [1961, Equation (21)], Skula [1980, Theorem 2.2, Remark 2.3] proved that

$$d(p) < \tfrac{1}{4}(p+3) - \frac{\log 2}{\log p}\big(\tfrac{1}{4}(p-1)\big).$$

Hence if $p \geq 47$, then we have

$$d(p) < \tfrac{1}{4}(p-5).$$

Since the only irregular prime smaller than 47 is 37 and $d(37) = 1 < \frac{1}{4}(37-5)$, the inequality $d(p) < \frac{1}{4}(p-5)$ holds true.

For any integer $a$, we define an integer $[a]_{p-1}$ by

$$0 \leq [a]_{p-1} \leq p-2 \quad \text{and} \quad [a]_{p-1} \equiv a \pmod{p-1}.$$

Since $d(p) < \frac{1}{4}(p-5)$, there is an even integer $t \in \{2, 4, \dots, p-3\}$ with $t \neq [n+2]_{p-1}$ such that

$$L_p(1-t, \omega^t) \in \mathbb{Z}_p^\times \quad \text{and} \quad L_p(1-[n+2-t]_{p-1}, \omega^{[n+2-t]_{p-1}}) \in \mathbb{Z}_p^\times.$$

Furthermore, Proposition 7.1(3) shows that

$$L_p(1-[n+2-t]_{p-1}, \omega^{[n+2-t]_{p-1}}) - L_p(-1+t-n, \omega^{n+2-t}) \in p\mathbb{Z}_p,$$

and hence we have $L_p(-1+t-n, \omega^{n+2-t}) \in \mathbb{Z}_p^\times$. Therefore, we put $\nu := t-1$ and get

$$\delta_p(n, \nu) = \mathrm{ord}_p(N_{n+2}). \qquad \square$$

**8.1.2.** $p-1 \mid n+2$.

**Lemma 8.5.** *If $p-1 \mid n+2$, we have $\delta_p(\mathrm{Eis}_n) = 0 = \mathrm{ord}_p(N_{n+2})$.*

*Proof.* The fact that $\mathrm{ord}_p(N_{n+2}) = 0$ follows from Corollary 7.3(2). Hence by Proposition 8.1, it suffices to show that $\delta_p(n, \nu) = 0$ for any integer $1 \le \nu \le n-1$. Since $p-1 \mid n+2$, we have $\mathrm{ord}_p(L_p(-1-n, \omega^{n+2})) < 0$. If $p-1 \nmid 1+\nu$, then we also have $p-1 \nmid n-\nu+1$, and we get $\delta_p(n, \nu) = 0$ since $L_p(-\nu, \omega^{1+\nu})$ and $L_p(\nu-n, \omega^{n-\nu+1})$ are $p$-adic integers. When $p-1 \mid 1+\nu$, we also have $p-1 \mid n-\nu+1$, and Corollary 7.5 implies that $D_p(n, \nu) \in \mathbb{Z}_p$.We thus obtain $\delta_p(n, \nu) = 0$. $\square$

This completes the proof of Proposition 8.2, and in particular of Theorem 2.13.

## 9. Applications

In this section, we discuss some applications of Theorem 2.13. For notational simplicity, in the following, the (co)homology groups will be denoted by $H^\bullet(Y, \mathcal{M}_n)$ (resp. $H_\bullet(Y, \mathcal{M}_n)$) rather than $H^\bullet(Y^{\mathrm{BS}}, \mathcal{M}_n)$ (resp. $H_\bullet(Y^{\mathrm{BS}}, \mathcal{M}_n)$) since they are naturally isomorphic.

First note that we have the following corollary of Theorem 2.13.

**Corollary 9.1.** *Let $n \ge 2$ be an even integer and $\gamma \in \Gamma$ a matrix. Take a polynomial $P(X_1, X_2) \in \mathcal{M}_n$ such that $\gamma P(X_1, X_2) = P(X_1, X_2)$. Then for any element $\tau \in \mathbb{H}$, we have*

$$N_{n+2} \int_\tau^{\gamma\tau} E_{n+2}(z) P(z, 1)\, dz \in \mathbb{Z}.$$

*Here $N_{n+2} > 0$ is the numerator of $\zeta(-1-n)$.*

*Proof.* Since $\gamma P = P$, we have $\partial(\{\tau, \gamma\tau\} \otimes P) = 0$, and hence $\{\tau, \gamma\tau\} \otimes P$ defines an element in the homology group $H_1(Y, \mathcal{M}_n)$. Therefore, by Theorem 2.13, we obtain

$$N_{n+2} \int_\tau^{\gamma\tau} E_{n+2}(z) P(z, 1)\, dz = \langle N_{n+2}\mathrm{Eis}_n, [\{\tau, \gamma\tau\} \otimes P] \rangle \in \mathbb{Z}. \qquad \square$$

**9.1.** ***Duke's conjecture.*** Duke [2024] defined a certain map called the higher Rademacher symbol

$$\Psi_k : \Gamma \to \mathbb{Q}$$

for each integer $k \in \mathbb{Z}_{\geq 2}$ which is a generalization of the classical Rademacher symbol and gave a conjecture concerning the integrality of the higher Rademacher symbol $\Psi_k$.

**Conjecture 9.2** [Duke 2024, Conjecture, p. 4]. *For any integer $k \in \mathbb{Z}_{\geq 2}$ and matrix $\gamma \in \Gamma$, we have*

$$\Psi_k(\gamma) \in \mathbb{Z}.$$

In the following, we show that Duke's Conjecture 9.2 follows from Theorem 2.13.

**Remark 9.3.** Conjecture 9.2 was also recently proved by O'Sullivan [2024] using a more direct method.

Here, instead of giving the original definition of the higher Rademacher symbols, we recall an integral representation of the higher Rademacher symbols, also given by Duke [2024], which is equivalent to the original definition and more suitable for our purpose.

**Proposition 9.4** [Duke 2024, Definition (2.4) and Lemma 6]. *Let $k \in \mathbb{Z}_{\geq 2}$ be an integer. For any matrix $\gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma - \{\pm \mathrm{id}_{2\times 2}\}$, we define a binary quadratic polynomial $Q_\gamma(X_1, X_2) \in \mathcal{M}_2$ associated with $\gamma$ by*

$$Q_\gamma(X_1, X_2) := -\frac{\mathrm{sgn}(a+d)}{\gcd(c, a-d, b)}(cX_1^2 - (a-d)X_1X_2 - bX_2^2).$$

*We also put $Q_{\pm \mathrm{id}_{2\times 2}}(X_1, X_2) := 0$. Then for any element $\tau \in \mathbb{H}$, we have*

$$\Psi_k(\gamma) = N_{2k} \int_\tau^{\gamma\tau} E_{2k}(z) Q_\gamma(z, 1)^{k-1}\, dz,$$

*where $N_{2k} > 0$ is the numerator of $\zeta(1-2k)$.*

**Corollary 9.5.** *Duke's Conjecture 9.2 holds true.*

*Proof.* By definition, the binary quadratic polynomial $Q_\gamma(X_1, X_2)$ defined in Proposition 9.4 is $\gamma$-invariant. Hence Corollary 9.1 and Proposition 9.4 imply that $\Psi_k(\gamma) \in \mathbb{Z}$. □

**9.2.** ***Partial zeta functions of real quadratic fields.*** We discuss an application to the denominators of the special values of the partial zeta functions of real quadratic fields.

Let $F$ be a real quadratic field, and let $\mathcal{O} \subset F$ be an order of $F$ with discriminant $D_\mathcal{O}$. We denote by $I_\mathcal{O}$ the group of proper fractional $\mathcal{O}$-ideals and $P_\mathcal{O}^+ \subset I_\mathcal{O}$

the subgroup of totally positive principal ideals. We define the narrow ideal class group $Cl_{\mathcal{O}}^+$ of $\mathcal{O}$ by

$$Cl_{\mathcal{O}}^+ := I_{\mathcal{O}}/P_{\mathcal{O}}^+.$$

See [Cox 2013, §7]. We fix an embedding $F \subset \mathbb{R}$, and for any element $\alpha \in F \subset \mathbb{R}$, we denote by $\alpha' \in F \subset \mathbb{R}$ its conjugate over $\mathbb{Q}$.

Let $\mathcal{O}_+^\times$ denote the group of totally positive units in $\mathcal{O}$, and let $\varepsilon_0 \in \mathcal{O}_+^\times$ denote the generator of $\mathcal{O}_+^\times$ such that $\varepsilon_0 > 1$.

**Definition 9.6.** We define a map

$$\mathfrak{z}_{\mathcal{O},k} : Cl_{\mathcal{O}}^+ \to H_1(Y, \mathcal{M}_{2k-2})$$

as follows: let $\mathcal{A} \in Cl_{\mathcal{O}}^+$, and take a representative $\mathfrak{a} \in I_{\mathcal{O}}$ of $\mathcal{A}$. We also take a basis $\alpha_1, \alpha_2 \in \mathfrak{a}$ over $\mathbb{Z}$ such that $\alpha_1\alpha_2' - \alpha_1'\alpha_2 > 0$, and let $\gamma_0 \in \Gamma$ be a matrix such that

$$\gamma_0 \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_0\alpha_1 \\ \varepsilon_0\alpha_2 \end{pmatrix}.$$

Moreover, set

$$N_{\alpha_1,\alpha_2}(X_1, X_2) := -\frac{1}{N\mathfrak{a}}(\alpha_2 X_1 - \alpha_1 X_2)(\alpha_2' X_1 - \alpha_1' X_2).$$

We see that $N_{\alpha_1,\alpha_2}(X_1, X_2) \in \mathcal{M}_2$ and that $\gamma_0 N_{\alpha_1,\alpha_2}(X_1, X_2) = N_{\alpha_1,\alpha_2}(X_1, X_2)$. We then define

$$\mathfrak{z}_{\mathcal{O},k}(\mathcal{A}) := [\{\tau, \gamma_0\tau\} \otimes N_{\alpha_1,\alpha_2}(X_1, X_2)^{k-1}] \in H_1(Y, \mathcal{M}_{2k-2}),$$

where $\tau$ is an arbitrary element in $\mathbb{H}$.

**Lemma 9.7.** *The homology class $\mathfrak{z}_{\mathcal{O},k}(\mathcal{A})$ does not depend on the choices we made.*

*Proof.* The independence of $\tau \in \mathbb{H}$ is clear. Let $\mathfrak{b} \in \mathcal{A}$ be another representative. Then there exists a totally positive element $\alpha \in F^\times$ such that $\mathfrak{b} = \alpha\mathfrak{a}$. Take a basis $\beta_1, \beta_2 \in \mathfrak{a}$ over $\mathbb{Z}$ with $\beta_1\beta_2' - \beta_1'\beta_2 > 0$. Then we obtain a matrix $\gamma_{0,\mathfrak{b}} \in \Gamma$ and a binary quadratic polynomial $N_{\alpha\beta_1,\alpha\beta_2}(X_1, X_2)$ from the basis $\alpha\beta_1, \alpha\beta_2$ of $\mathfrak{b}$. Note that since $\alpha$ is totally positive, we have $(\alpha\beta_1)(\alpha'\beta_2') - (\alpha'\beta_1')(\alpha\beta_2) > 0$. Let $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ be a matrix satisfying

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \gamma \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Then the facts that $\alpha_1\alpha_2' - \alpha_1'\alpha_2 > 0$ and $\beta_1\beta_2' - \beta_1'\beta_2 > 0$ imply that $\gamma \in \Gamma$. Since $\gamma_{0,\mathfrak{b}}\gamma\binom{\alpha_1}{\alpha_2} = \gamma\binom{\varepsilon_0\alpha_1}{\varepsilon_0\alpha_2}$, we have $\gamma^{-1}\gamma_{0,\mathfrak{b}}\gamma = \gamma_0$. Moreover,

$$(X_1\ X_2)^t\tilde{\gamma}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = (X_1\ X_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\gamma\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = (X_1\ X_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix},$$

which implies that $N_{\alpha\beta_1,\alpha\beta_2}(X_1, X_2) = \gamma N_{\alpha_1,\alpha_2}(X_1, X_2)$. Therefore, we have

$$\begin{aligned}[\{\tau, \gamma_{0,\mathfrak{b}}\tau\} \otimes N_{\alpha\beta_1,\alpha\beta_2}(X_1, X_2)^{k-1}] &= [\{\tau, \gamma\gamma_0\gamma^{-1}\tau\} \otimes \gamma N_{\alpha_1,\alpha_2}(X_1, X_2)^{k-1}] \\ &= [\{\gamma^{-1}\tau, \gamma_0\gamma^{-1}\tau\} \otimes N_{\alpha_1,\alpha_2}(X_1, X_2)^{k-1}] \\ &= [\{\tau, \gamma_0\tau\} \otimes N_{\alpha_1,\alpha_2}(X_1, X_2)^{k-1}]\end{aligned}$$

as elements of $H_1(Y, \mathcal{M}_{2k-2})$. □

**Remark 9.8.** (1) Since the matrix $\gamma_0$ in Definition 9.6 is hyperbolic ($|\mathrm{trace}(\gamma_0)| > 2$), we have $\dim_{\mathbb{Q}}\{Q \in \mathcal{M}_2 \otimes \mathbb{Q} \mid \gamma Q = Q\} = 1$. This fact together with [Cox 2013, Equation (7.6)] shows that $N_{\alpha_1,\alpha_2}(X_1, X_2) = Q_{\gamma_0}(X_1, X_2)$.

(2) Gauss's theory concerning binary quadratic forms (see [Cox 2013, Exercise 7.21] for example) shows that for any hyperbolic element $\gamma \in \Gamma$, there is an order $\mathcal{O}$ of a real quadratic field and a narrow ideal class $\mathcal{A} \in Cl^+_{\mathcal{O}}$ such that

$$[\{z, \gamma z\} \otimes Q_\gamma(X_1, X_2)] \in \mathbb{Z}\mathfrak{z}_{\mathcal{O},2}(\mathcal{A}).$$

**Definition 9.9.** For each ideal class $\mathcal{A} \in Cl^+_{\mathcal{O}}$, the partial zeta function $\zeta_{\mathcal{O}}(\mathcal{A}, s)$ associated with $\mathcal{A}$ is defined by

$$\zeta_{\mathcal{O}}(\mathcal{A}, s) := \sum_{\mathfrak{a}\subset\mathcal{O},\mathfrak{a}\in\mathcal{A}} \frac{1}{(N\mathfrak{a})^s} \quad (\mathrm{Re}(s) > 1),$$

and it is well known that $\zeta_{\mathcal{O}}(\mathcal{A}, s)$ can be continued meromorphically to $s \in \mathbb{C}$ and has a simple pole at $s = 1$.

The following integral representation of the special values of the partial zeta function is classically known.

**Proposition 9.10.** *For any integer $k \in \mathbb{Z}_{\geq 2}$ and ideal class $\mathcal{A} \in Cl^+_{\mathcal{O}}$, we have*

$$\langle \mathrm{Eis}_{2k-2}, \mathfrak{z}_{\mathcal{O},k}(\mathcal{A})\rangle = (-1)^k \frac{\zeta_{\mathcal{O}}(\mathcal{A}^{-1}, 1-k)}{\zeta(1-2k)}.$$

Before we give a proof of Proposition 9.10, we recall (a special case of) the so-called Feynman parametrization.

**Lemma 9.11.** *Let $x_1, x_2, a, b \in \mathbb{C}$ be complex numbers such that $x_1a + x_2 \neq 0$ and $x_1b + x_2 \neq 0$. Then for any nonnegative integers $k_1$ and $k_2$, we have*

$$\int_a^b \frac{(b-z)^{k_1}(z-a)^{k_2}}{(x_1z+x_2)^{2+k_1+k_2}}\,dz = \frac{k_1!k_2!}{(k_1+k_2+1)!} \frac{(b-a)^{k_1+k_2+1}}{(x_1a+x_2)^{k_1+1}(x_1b+x_2)^{k_2+1}}.$$

*Proof.* We may assume that $a \neq b$. By setting $y_1 = x_1a + x_2$ and $y_2 = x_1b + x_2$, it suffices to prove that

$$(b-a)\int_a^b \frac{(b-z)^{k_1}(z-a)^{k_2}}{((b-z)y_1+(z-a)y_2)^{2+k_1+k_2}}\,dz = \frac{k_1!k_2!}{(k_1+k_2+1)!} \frac{1}{y_1^{k_1+1}y_2^{k_2+1}}.$$

The case where $k_1 = k_2 = 0$ is clear: we have

$$(b-a)\int_a^b \frac{1}{((b-z)y_1+(z-a)y_2)^2}\,dz = \frac{1}{y_1 y_2}.$$

Then by viewing both sides as holomorphic functions in $(y_1, y_2) \in \mathbb{C}^\times \times \mathbb{C}^\times$ and applying the differential operator $\left(\frac{\partial}{\partial y_1}\right)^{k_1}\left(\frac{\partial}{\partial y_2}\right)^{k_2}$, we obtain the desired identity. □

*Proof of Proposition 9.10.* We use the same notations as in Definition 9.6. Since

$$2\zeta(2k)E_{2k}(z) = \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{(mz+n)^{2k}}$$

and $\mathrm{Eis}_{2k-2} = r(E_{2k})$, for any element $\tau \in \mathbb{H}$ we have

$$2\zeta(2k)\langle \mathrm{Eis}_{2k-2}, \mathfrak{z}_{\mathcal{O},k}(\mathcal{A})\rangle = \int_\tau^{\gamma_0\tau} \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{N_{\alpha_1,\alpha_2}(z,1)^{k-1}}{(mz+n)^{2k}}\,dz.$$

Since $N_{\alpha_1,\alpha_2}(\gamma_0 z, 1) = j(\gamma_0, z)^{-2} N_{\alpha_1,\alpha_2}(z,1)$, where $j\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z\right) := cz+d$ is the factor of automorphy, we fix a complete set $S_{\gamma_0}$ of representatives of $(\mathbb{Z}^2 - \{(0,0)\})/\gamma_0^{\mathbb{Z}}$ to have

$$\begin{aligned}
&\int_\tau^{\gamma_0\tau} \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{N_{\alpha_1,\alpha_2}(z,1)^{k-1}}{(mz+n)^{2k}}\,dz \\
&\qquad = \int_\tau^{\gamma_0\tau} \sum_{l\in\mathbb{Z}} \sum_{(m,n)\in S_{\gamma_0}} \frac{N_{\alpha_1,\alpha_2}(z,1)^{k-1}}{j(\gamma_0^l, z)^{2k}(m(\gamma_0^l z)+n)^{2k}}\,dz \\
&\qquad = \sum_{l\in\mathbb{Z}} \int_{\gamma_0^l\tau}^{\gamma_0^{l+1}\tau} \sum_{(m,n)\in S_{\gamma_0}} \frac{N_{\alpha_1,\alpha_2}(\gamma_0^{-l}z,1)^{k-1}}{j(\gamma_0^l, \gamma_0^{-l}z)^{2k}(mz+n)^{2k}}\,d(\gamma_0^{-l}z) \\
&\qquad = \sum_{l\in\mathbb{Z}} \int_{\gamma_0^l\tau}^{\gamma_0^{l+1}\tau} \sum_{(m,n)\in S_{\gamma_0}} \frac{N_{\alpha_1,\alpha_2}(z,1)^{k-1}}{(mz+n)^{2k}}\,dz.
\end{aligned}$$

Set $\alpha_0 := \alpha_1/\alpha_2 \in F \subset \mathbb{R}$. Then the point $\alpha_0 \in \mathbb{R}$ (resp. $\alpha_0'$) is the attractive fixed point (resp. repelling fixed point) of the hyperbolic matrix $\gamma_0 \in \Gamma$, i.e., we have $\lim_{l\to\infty} \gamma_0^l \tau = \alpha_0$ and $\lim_{l\to\infty} \gamma_0^{-l}\tau = \alpha_0'$ in $\mathbb{P}^1(\mathbb{C})$ for any element $\tau \in \mathbb{H}$. Hence we obtain

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}} \int_{\gamma_0^l\tau}^{\gamma_0^{l+1}\tau} \sum_{(m,n)\in S_{\gamma_0}} \frac{N_{\alpha_1,\alpha_2}(z,1)^{k-1}}{(mz+n)^{2k}}\,dz \\
&\qquad = \int_{\alpha_0'}^{\alpha_0} \sum_{(m,n)\in S_{\gamma_0}} \frac{N_{\alpha_1,\alpha_2}(z,1)^{k-1}}{(mz+n)^{2k}}\,dz \\
&\qquad = \frac{N_{F/\mathbb{Q}}(\alpha_2)^{k-1}}{(N\mathfrak{a})^{k-1}} \sum_{(m,n)\in S_{\gamma_0}} \int_{\alpha_0'}^{\alpha_0} \frac{((\alpha_0-z)(z-\alpha_0'))^{k-1}}{(mz+n)^{2k}}\,dz.
\end{aligned}$$

By using Lemma 9.11, we find

$$\int_{\alpha_0'}^{\alpha_0} \frac{((\alpha_0 - z)(z - \alpha_0'))^{k-1}}{(mz+n)^{2k}}\, dz = \frac{((k-1)!)^2}{(2k-1)!} \frac{(\alpha_0 - \alpha_0')^{2k-1}}{N_{F/\mathbb{Q}}(m\alpha_0 + n)^k}.$$

Note that we have the identity $\alpha_1\alpha_2' - \alpha_1'\alpha_2 = \sqrt{D_{\mathcal{O}}}N\mathfrak{a}$, and this shows that

$$\frac{N_{F/\mathbb{Q}}(\alpha_2)^{k-1}}{(N\mathfrak{a})^{k-1}} \sum_{(m,n)\in S_{\gamma_0}} \frac{(\alpha_0 - \alpha_0')^{2k-1}}{N_{F/\mathbb{Q}}(m\alpha_0+n)^k} = D_{\mathcal{O}}^{k-\frac{1}{2}}(N\mathfrak{a})^k \sum_{\alpha\in(\mathfrak{a}-\{0\})/\mathcal{O}_+^\times} \frac{1}{N_{F/\mathbb{Q}}(\alpha)^k}.$$

For any subset $X \subset F$, we put

$$X_+ := \{\alpha \in X \mid \alpha > 0,\ \alpha' > 0\}, \quad X_- := \{\alpha \in X \mid \alpha > 0,\ \alpha' < 0\}.$$

Let $\mathcal{J} \in Cl_{\mathcal{O}}^+$ denote the ideal class containing the principal ideal $(\sqrt{D_{\mathcal{O}}}) \subset \mathcal{O}$. Note that $\mathcal{J}^{-1} = \mathcal{J}$ in $Cl_{\mathcal{O}}^+$. Then we further compute

$$\begin{aligned}(N\mathfrak{a})^k \sum_{\alpha\in(\mathfrak{a}-\{0\})/\mathcal{O}_+^\times} \frac{1}{N_{F/\mathbb{Q}}(\alpha)^k} &= \sum_{\alpha\in\mathfrak{a}_+/\mathcal{O}_+^\times} \frac{2(N\mathfrak{a})^k}{N_{F/\mathbb{Q}}(\alpha)^k} + \sum_{\alpha\in\mathfrak{a}_-/\mathcal{O}_+^\times} \frac{2(N\mathfrak{a})^k}{N_{F/\mathbb{Q}}(\alpha)^k} \\ &= \sum_{\alpha\in\mathfrak{a}_+/\mathcal{O}_+^\times} \frac{2(N\mathfrak{a})^k}{N_{F/\mathbb{Q}}(\alpha)^k} + (-1)^k \sum_{\alpha\in(\sqrt{D_{\mathcal{O}}}\mathfrak{a})_+/\mathcal{O}_+^\times} \frac{2N(\sqrt{D_{\mathcal{O}}}\mathfrak{a})^k}{N_{F/\mathbb{Q}}(\alpha)^k} \\ &= 2(\zeta_{\mathcal{O}}(\mathcal{A}^{-1},k) + (-1)^k\zeta_{\mathcal{O}}(\mathcal{J}\mathcal{A}^{-1},k)).\end{aligned}$$

We recall the functional equations of the partial zeta functions. Set

$$\begin{aligned}\Lambda_{\mathcal{O}}^+(\mathcal{A}^{-1},s) &:= \pi^{-s}\Gamma\left(\tfrac{1}{2}s\right)^2 D_{\mathcal{O}}^{\frac{1}{2}(s-1)}(\zeta_{\mathcal{O}}(\mathcal{A}^{-1},s) + \zeta_{\mathcal{O}}(\mathcal{A}^{-1}\mathcal{J},s)), \\ \Lambda_{\mathcal{O}}^-(\mathcal{A}^{-1},s) &:= \pi^{-s}\Gamma\left(\tfrac{1}{2}(s+1)\right)^2 D_{\mathcal{O}}^{\frac{1}{2}s}(\zeta_{\mathcal{O}}(\mathcal{A}^{-1},s) - \zeta_{\mathcal{O}}(\mathcal{A}^{-1}\mathcal{J},s)).\end{aligned}$$

Then we have

$$\Lambda_{\mathcal{O}}^+(\mathcal{A}^{-1},s) = \Lambda_{\mathcal{O}}^+(\mathcal{A}^{-1},1-s), \quad \Lambda_{\mathcal{O}}^-(\mathcal{A}^{-1},s) = \Lambda_{\mathcal{O}}^-(\mathcal{A}^{-1},1-s).$$

See [Duke et al. 2018, Equations (59) and (60); Sczech 1993, p. 545] for example. Although [Duke et al. 2018] deals only with the maximal orders, we can apply the same argument to general orders. See also [Siegel 1980; Duke 2024, Equation (4.19); Vlasenko and Zagier 2013, p. 42]. Using these functional equations, we find

$$D_{\mathcal{O}}^{k-\frac{1}{2}}(\zeta_{\mathcal{O}}(\mathcal{A}^{-1},k) + (-1)^k\zeta_{\mathcal{O}}(\mathcal{J}\mathcal{A}^{-1},k)) = \frac{(2\pi)^{2k}}{2((k-1)!)^2}\zeta_{\mathcal{O}}(\mathcal{A}^{-1},1-k).$$

Therefore, by also using the functional equation for the Riemann zeta function (see, for example, [Hida 1993, p. 29]), we obtain

$$\langle \mathrm{Eis}_{2k-2}, \mathfrak{z}_{\mathcal{O},k}(\mathcal{A})\rangle = \frac{(2\pi)^{2k}}{2(2k-1)!\zeta(2k)}\zeta_{\mathcal{O}}(\mathcal{A}^{-1},1-k) = (-1)^k\frac{\zeta_{\mathcal{O}}(\mathcal{A}^{-1},1-k)}{\zeta(1-2k)}. \quad \square$$

We define the positive integer $J_{2k}$ by

$$J_{2k} := \text{denominator of } \zeta(1-2k).$$

**Corollary 9.12.** *Let $F$ be a real quadratic field, $\mathcal{O} \subset F$ be an order in $F$, and let $\mathcal{A} \in Cl_{\mathcal{O}}^{+}$ be a narrow ideal class of $\mathcal{O}$. Then for any integer $k \geq 2$, we have*

$$J_{2k}\zeta_{\mathcal{O}}(\mathcal{A}, 1-k) \in \mathbb{Z}.$$

*Proof.* By Proposition 9.10, we have

$$N_{2k}\langle \mathrm{Eis}_{2k-2}, \mathfrak{z}_{\mathcal{O},k}(\mathcal{A}^{-1})\rangle = \pm N_{2k}\frac{\zeta_{\mathcal{O}}(\mathcal{A}, 1-k)}{\zeta(1-2k)} = \pm J_{2k}\zeta_{\mathcal{O}}(\mathcal{A}, 1-k).$$

Since $N_{2k}\langle \mathrm{Eis}_{2k-2}, \mathfrak{z}_{\mathcal{O},k}(\mathcal{A}^{-1})\rangle \in \mathbb{Z}$ by Theorem 2.13, we obtain $J_{2k}\zeta_{\mathcal{O}}(\mathcal{A}, 1-k) \in \mathbb{Z}$. □

**Remark 9.13.** By Proposition 9.4 [Duke 2024, Lemma 6] and Proposition 9.10, we see that Duke's Conjecture 9.2 is equivalent to Corollary 9.12.

**Remark 9.14.** As for the denominator of the special values of the Dedekind zeta functions of real quadratic fields, or more generally of totally real fields, the same (slightly stronger at $p = 2$) universal upper bound was obtained by Serre [1973, §2, théorème 6]. If we fix a totally real field $F$, then a more refined description for the denominators and even for the numerators of the special values of the Dedekind zeta function of $F$ is obtained from the classical Iwasawa main conjecture proved by Wiles [1990] (see [Kolster 2004]).

**9.3.** ***Sharpness of the universal upper bound in Corollary 9.12.*** Let $k \geq 2$ be an integer. We define a $\mathbb{Z}$-submodule $\mathfrak{Z}_k \subset H_1(Y, \mathcal{M}_{2k-2})$ to be the $\mathbb{Z}$-submodule generated by homology classes of the form $\mathfrak{z}_{\mathcal{O},k}(\mathcal{A})$, that is,

$$\mathfrak{Z}_k := \langle\, \mathfrak{z}_{\mathcal{O},k}(\mathcal{A}) \mid \mathcal{O} \text{ is an order of a real quadratic field and } \mathcal{A} \in Cl_{\mathcal{O}}^{+} \,\rangle_{\mathbb{Z}}.$$

This subsection is devoted to proving the following theorem.

**Theorem 9.15.** *We have $\langle \mathrm{Eis}_{2k-2}, \mathfrak{Z}_k\rangle = (1/N_{2k})\mathbb{Z}$.*

Theorem 9.15 has the following interesting application.

**Corollary 9.16.** *The universal bound in Corollary 9.12 is sharp: for any prime number $p$, there exist an order $\mathcal{O}$ of a real quadratic field and a narrow ideal class $\mathcal{A} \in Cl_{\mathcal{O}}^{+}$ such that*

$$\mathrm{ord}_p(J_{2k}\zeta_{\mathcal{O}}(\mathcal{A}, 1-k)) = 0.$$

*In other words, we have*

$$J_{2k} = \min\big\{ J \in \mathbb{Z}_{>0} \mid J\zeta_{\mathcal{O}}(\mathcal{A}, 1-k) \in \mathbb{Z} \text{ for all orders } \mathcal{O} \text{ in all real quadratic fields and narrow ideal classes } \mathcal{A} \in Cl_{\mathcal{O}}^{+} \big\}.$$

*Proof.* Let $p$ be a prime number. The definition of the module $\mathfrak{Z}_k$ and Theorem 9.15 show that one can find an order $\mathcal{O}$ of a real quadratic field and a narrow ideal class $\mathcal{A} \in Cl_{\mathcal{O}}^{+}$ such that

$$\mathrm{ord}_p(\langle \mathrm{Eis}_{2k-2}, \mathfrak{z}_{\mathcal{O},k}(\mathcal{A}^{-1})\rangle) = -\mathrm{ord}_p(N_{2k}).$$

Since $\zeta(1-2k) = \pm N_{2k}/J_{2k}$, Proposition 9.10 shows that

$$\begin{aligned} 0 &= \mathrm{ord}_p(N_{2k}) + \mathrm{ord}_p(\langle \mathrm{Eis}_{2k-2}, \mathfrak{z}_{\mathcal{O},k}(\mathcal{A}^{-1})\rangle) \\ &= \mathrm{ord}_p(N_{2k}) - \mathrm{ord}_p(\zeta(1-2k)) + \mathrm{ord}_p(\zeta_{\mathcal{O}}(\mathcal{A}, 1-k)) \\ &= \mathrm{ord}_p(J_{2k}\zeta_{\mathcal{O}}(\mathcal{A}, 1-k)). \end{aligned}$$

□

**9.3.1.** *Preparations for proving Theorem 9.15.* Let $N \geq 1$ be an integer and define

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; a-1 \equiv c \equiv d-1 \equiv 0 \pmod{N} \right\}.$$

We also put

$$Y_1(N) := \Gamma_1(N)\backslash\mathbb{H}, \quad Y_1(N)^{\mathrm{BS}} := \Gamma_1(N)\backslash\mathbb{H}^{\mathrm{BS}}, \quad \partial Y_1(N)^{\mathrm{BS}} := Y_1(N)^{\mathrm{BS}} - Y_1(N).$$

We note that the similar facts in Section 2.1 and Section 2.2 hold true for the congruence subgroup $\Gamma_1(N)$. Moreover, the Hecke operators $T_p$ ($p \nmid N$) and $U_p$ ($p \mid N$) act on the homology group $H_1(Y_1(N), \mathcal{M}_{2k-2})$.

Let $k \geq 1$ be an integer. For any hyperbolic matrix $\gamma \in \Gamma_1(N)$ (i.e., $|\mathrm{trace}(\gamma)| > 2$), we set

$$\mathfrak{z}_{\Gamma_1(N),k}(\gamma) := [\{z, \gamma z\} \otimes Q_\gamma(X_1, X_2)^{k-1}] \in H_1(Y_1(N), \mathcal{M}_{2k-2}).$$

**Definition 9.17.** For any integer $k \geq 1$, we define a $\mathbb{Z}$-submodule

$$\mathfrak{Z}_{\Gamma_1(N),k} \subset H_1(Y_1(N), \mathcal{M}_{2k-2})$$

by

$$\mathfrak{Z}_{\Gamma_1(N),k} := \langle \mathfrak{z}_{\Gamma_1(N),k}(\gamma) \mid \gamma \in \Gamma_1(N) \text{ with } |\mathrm{trace}(\gamma)| > 2\rangle_{\mathbb{Z}}.$$

**Remark 9.18.** By Remark 9.8, we have $\mathfrak{Z}_{\Gamma_1(1),k} = \mathfrak{Z}_k$.

**Lemma 9.19.** *For any integer $N \geq 1$ and prime number $p$, we have*

$$\langle [\{z, \gamma z\}] \mid \gamma \in \Gamma_1(Np) - \Gamma(p) \text{ with } |\mathrm{trace}(\gamma)| > 2\rangle_{\mathbb{Z}} = \mathfrak{Z}_{\Gamma_1(Np),1} = H_1(Y_1(Np), \mathbb{Z}).$$

*Proof.* It suffices to show that

$$\langle \gamma \mid \gamma \in \Gamma_1(Np) - \Gamma(p) \text{ with } |\mathrm{trace}(\gamma)| > 2\rangle = \Gamma_1(Np).$$

Here $\langle \cdot \rangle$ means the group generated by the elements inside the bracket. Moreover, since $\Gamma_1(2) = \langle \Gamma_1(6), \Gamma_1(10)\rangle$, we may assume that $Np \geq 3$. Put $\gamma := \left(\begin{smallmatrix} 1 & 1 \\ Np & 1+Np \end{smallmatrix}\right)$. Then the quotient group $\Gamma_1(Np)/(\Gamma(p) \cap \Gamma_1(Np))$ is generated by the image of $\gamma$.

For any matrix $\gamma' \in \Gamma(p) \cap \Gamma_1(Np)$, we have $\gamma'\gamma^{1+ap} \notin \Gamma(p)$ for any integer $a$. Since $\mathrm{trace}(\gamma) = Np+2 > 2$ and $\det \gamma = 1$, the matrix $\gamma$ is hyperbolic, and one can take a matrix $Q \in \mathrm{GL}_2(\mathbb{R})$ such that $Q^{-1}\gamma Q = \left(\begin{smallmatrix} \alpha & \\ & \alpha^{-1} \end{smallmatrix}\right)$. If we put $Q^{-1}\gamma' Q =: \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$, then $\mathrm{trace}(\gamma'\gamma^{1+ap}) = x\alpha^{1+ap} + w\alpha^{-1-ap}$. Since $\mathrm{trace}(\gamma'\gamma^{1+ap}) \equiv 2 \pmod{Np}$ and $Np \geq 3$, we have $\mathrm{trace}(\gamma'\gamma^{1+ap}) \neq 0$. Hence the set $\{\mathrm{trace}(\gamma'\gamma^{1+ap}) \mid a \in \mathbb{Z}\} \subset \mathbb{Z}$ is infinite. Therefore we can find an integer $a$ such that $|\mathrm{trace}(\gamma'\gamma^{1+ap})| > 2$. □

Next, we recall an important result proved by Hida [1986, Corollary 4.5; 1988, Corollary 8.2]. Take an integer $N \geq 1$ and a prime number $p$, and put $q := p^{\mathrm{ord}_p(Np)}$. Then we have a $\Gamma_1(q)$-homomorphism

$$j : \mathbb{Z}/(q) \to \mathcal{M}_{2k-2} \otimes \mathbb{Z}/(q); \quad b \mapsto bX_2^{2k-2},$$

which induces a Hecke-equivariant homomorphism

$$j_* : H_1(Y_1(Np), \mathbb{Z}/(q)) \to H_1(Y_1(Np), \mathcal{M}_{2k-2,p} \otimes \mathbb{Z}/(q)).$$

**Proposition 9.20.** *When $Np \geq 4$, the homomorphism $j$ induces a Hecke-equivariant isomorphism*

$$j_* : H_1^{\mathrm{ord}}(Y_1(Np), \mathbb{Z}/(q)) \xrightarrow{\sim} H_1^{\mathrm{ord}}(Y_1(Np), \mathcal{M}_{2k-2,p} \otimes \mathbb{Z}/(q)).$$

*Here we define $H_1^{\mathrm{ord}}(Y_1(Np), -) := e_{U_p} H_1(Y_1(Np), -)$.*

*Proof.* The proof of this proposition is essentially the same as that of [Hida 1993, §7.2, Theorem 2] for cohomology groups. We note that $\Gamma_1(Np)$ is torsion-free since $Np \geq 4$. Hence any short exact sequence of $\Gamma_1(Np)$-modules induces a long exact sequence in homology.

If we put

$$\mathcal{C} := (\mathcal{M}_{2k-2}/\mathbb{Z}X_2^{2k-2}) \otimes \mathbb{Z}/(q),$$

then the short exact sequence $0 \to \mathbb{Z}/(q) \xrightarrow{j} \mathcal{M}_{2k-2} \otimes \mathbb{Z}/(q) \to \mathcal{C} \to 0$ induces an exact sequence of $\mathbb{Z}/(q)$-modules

$$H_2(Y_1(Np), \mathcal{C}) \to H_1(Y_1(Np), \mathbb{Z}/(q)) \xrightarrow{j_*} H_1(Y_1(N), \mathcal{M}_{2k-2} \otimes \mathbb{Z}/(q)) \to H_1(Y_1(Np), \mathcal{C}).$$

Since the operator $U_p$ is defined by $U_p = \sum_{u=0}^{p-1} \left(\begin{smallmatrix} 1 & u \\ 0 & p \end{smallmatrix}\right)$, we have

$$\left(\begin{smallmatrix} 1 & u \\ 0 & p \end{smallmatrix}\right) \cdot P(X_1, X_2) \equiv P(-uX_2, X_2) \pmod{p} \in \mathbb{F}_p X_2^{2k-2}$$

for any polynomial $P \in \mathcal{M}_{2k-2}$. In other words, we have

$$\left(\begin{smallmatrix} 1 & u \\ 0 & p \end{smallmatrix}\right)(\mathcal{M}_{2k-2}/\mathbb{Z}X_2^{2k-1}) \subset p(\mathcal{M}_{2k-2}/\mathbb{Z}X_2^{2k-1}),$$

and this shows that $H_1^{\mathrm{ord}}(Y_1(N), \mathcal{C}) = H_2^{\mathrm{ord}}(Y_1(N), \mathcal{C}) = 0$. □

**Lemma 9.21.** *Let $N$ be an integer and let $p$ be a prime number. Set $q := p^{\mathrm{ord}_p(Np)}$. Then for any matrix $\gamma \in \Gamma_1(Np) - \Gamma(p)$, we have*

$$j_*(\mathfrak{z}_{\Gamma_1(Np),1}(\gamma) \bmod q) \in (\mathbb{Z}/(q))^\times \cdot \mathfrak{z}_{\Gamma_1(Np),k}(\gamma).$$

*Proof.* If we put $\gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then we have

$$Q_\gamma(X_1, X_2) = -\frac{\mathrm{sgn}(a+d)}{\gcd(c, a-d, b)}(cX_1^2 - (a-d)X_1X_2 - bX_2^2).$$

Since $\gamma \in \Gamma_1(Np) - \Gamma(p)$, we have $c \equiv a - d \equiv 0 \pmod q$ and $b \not\equiv 0 \pmod p$, which shows that

$$Q_\gamma(X_1, X_2) \equiv \pm\frac{b}{\gcd(c, a-d, b)} X_2^2 \pmod q$$

and

$$j_*(\mathfrak{z}_{\Gamma_1(Np),1}(\gamma) \bmod q) = \left(\pm\frac{\gcd(c, a-d, b)}{b}\right)^{k-1} \mathfrak{z}_{\Gamma_1(Np),k}(\gamma) \pmod q$$
$$\in (\mathbb{Z}/(q))^\times \cdot \mathfrak{z}_{\Gamma_1(Np),k}(\gamma). \quad \square$$

**Theorem 9.22.** *For any integer $N \geq 1$ and prime number $p$ satisfying $Np \geq 4$, we have*

$$e_{U_p}(\mathfrak{Z}_{\Gamma_1(Np),k} \otimes \mathbb{Z}_p) = H_1^{\mathrm{ord}}(Y_1(Np), \mathcal{M}_{2k-2,p}).$$

*Proof.* We first note that $e_{U_p}H_0(Y_1(Np), \mathcal{M}_{2k-2,p}) = e_{U_p}((\mathcal{M}_{2k-2,p})_{\Gamma_1(Np)})$ vanishes since $U_p \cdot X_2^{2k-2} = pX_2^{2k-2}$ and $\left(\begin{smallmatrix} 1 & u \\ 0 & p \end{smallmatrix}\right) \cdot P(X_1, X_2) \equiv P(-uX_2, X_2) \pmod p$. This fact implies that

$$H_1^{\mathrm{ord}}(Y_1(Np), \mathcal{M}_{2k-2,p}) \otimes \mathbb{Z}/(q) \xrightarrow{\sim} H_1^{\mathrm{ord}}(Y_1(Np), \mathcal{M}_{2k-2,p} \otimes \mathbb{Z}/(q)).$$

Here $q := p^{\mathrm{ord}_p(Np)}$. Hence this theorem follows from Proposition 9.20 and Lemmas 9.19 and 9.21. $\square$

**9.3.2.** *Proof of Theorem 9.15.* Let $k \geq 2$ be an integer. For any positive integers $M$ and $N$ with $M \mid N$, we denote by

$$\pi_*^{N,M} : H_1(Y_1(N), \mathcal{M}_{2k-2}) \to H_1(Y_1(M), \mathcal{M}_{2k-2}),$$
$$\pi_{M,N}^* : H^1(Y_1(M), \mathcal{M}_{2k-2}) \to H^1(Y_1(N), \mathcal{M}_{2k-2}),$$

the homomorphisms induced by the natural projection $Y_1(N) \to Y_1(M)$; $z \mapsto z$.

**Corollary 9.23.** *For any integer $N \geq 1$ and prime number $p$, we have*

$$\mathbb{Z}_p \subset \langle \mathrm{Eis}_{2k-2}, \pi_*^{Np,1}(e_{U_p}(\mathfrak{Z}_{\Gamma_1(Np),k} \otimes \mathbb{Z}_p)) \rangle.$$

*Proof.* Take an element $\tau \in \mathbb{H}$. Since

$$U_p([\{\tau, \tau+1\} \otimes X_2^{2k-2}]) = \left[\left\{\frac{\tau}{p}, \frac{\tau}{p}+1\right\} \otimes X_2^{2k-2}\right] = [\{\tau, \tau+1\} \otimes X_2^{2k-2}],$$

we have

$$[\{\tau, \tau+1\} \otimes X_2^{2k-2}] \in H_1^{\mathrm{ord}}(\partial Y_1(Np)^{\mathrm{BS}}, \mathcal{M}_{2k-2,p}) \subset H_1^{\mathrm{ord}}(Y_1(Np), \mathcal{M}_{2k-2,p}).$$

On the other hand, since

$$\pi_*^{Np,1}([\{\tau, \tau+1\} \otimes X_2^{2k-2}]) = [\{\tau, \tau+1\} \otimes X_2^{2k-2}] \in H_1(Y, \mathcal{M}_{2k-2}),$$

we have

$$\langle \mathrm{Eis}_{2k-2}, \pi_*^{Np,1}([\{\tau, \tau+1\} \otimes X_2^{2k-2}]) \rangle = 1.$$

Therefore, when $Np \geq 4$, Theorem 9.22 implies this lemma. When $Np \leq 3$, we have $N = 1$ and $p \mid 6$. Then this case follows from the case $N = 3$ and $p \mid 6$ since $\pi_*^{3p,1}(e_{U_p}(\mathfrak{Z}_{\Gamma_1(3p),k} \otimes \mathbb{Z}_p)) \subset \pi_*^{p,1}(e_{U_p}(\mathfrak{Z}_{\Gamma_1(p),k} \otimes \mathbb{Z}_p))$. □

**Lemma 9.24.** *Let $N \geq 1$ be an integer and let $p$ be a prime number. Then for any homology class $x \in H_1(Y_1(Np), \mathcal{M}_{2k-2}) \otimes \mathbb{C}_p$ we have*

$$e_{T_p} \pi_*^{Np,1}(e_{U_p} x) = \pi_*^{Np,1}(e_{U_p} x).$$

*Proof.* By using the formal duality, it suffices to show that

$$e_{U'_p} \pi_{1,Np}^*(e_{T'_p} y) = e_{U'_p} \pi_{1,Np}^*(y)$$

for any cohomology class $y \in H^1(Y, \mathcal{M}_{2k-2}) \otimes \mathbb{C}_p$. This claim is well known; see [Gouvêa 1992, Lemma 2] for example. □

**Corollary 9.25.** *For any prime number $p \geq 5$, we have*

$$\langle \mathrm{Eis}_{2k-2}, \pi_*^{p,1}(e_{U_p}(\mathfrak{Z}_{\Gamma_1(p),k} \otimes \mathbb{Z}_p)) \rangle = \frac{1}{N_{2k}} \mathbb{Z}_p.$$

*Proof.* Take a prime number $p \geq 5$. Then Theorem 9.22 shows that

$$\langle \mathrm{Eis}_{2k-2}, \pi_*^{p,1}(e_{U_p}(\mathfrak{Z}_{\Gamma_1(p),k} \otimes \mathbb{Z}_p)) \rangle = \langle \mathrm{Eis}_{2k-2}, \pi_*^{p,1}(H_1^{\mathrm{ord}}(Y_1(p), \mathcal{M}_{2k-2,p})) \rangle.$$

By Lemma 9.24, we have a natural homomorphism

$$\pi_*^{p,1} : H_1^{\mathrm{ord}}(Y_1(p), \mathcal{M}_{2k-2,p})/(\text{torsion}) \to H_1^{\mathrm{ord}}(Y, \mathcal{M}_{2k-2,p})/(\text{torsion}),$$

which is the dual of the homomorphism

$$H^1_{\mathrm{ord}}(Y, \mathcal{M}_{2k-2,p}) \to H^1_{\mathrm{ord}}(Y_1(p), \mathcal{M}_{2k-2,p}); \quad y \mapsto e_{U'_p} \pi_{1,p}^*(y).$$

The fact that the index $[\Gamma_0(p) : \Gamma_1(p)] = p - 1$ is relatively prime to $p$ together with the isomorphism (6-4) implies that $\pi_*^{p,1} : H_1^{\mathrm{ord}}(Y_1(p), \mathcal{M}_{2k-2,p})/(\text{torsion}) \to H_1^{\mathrm{ord}}(Y, \mathcal{M}_{2k-2,p})/(\text{torsion})$ is surjective. Hence we have

$$\langle \mathrm{Eis}_{2k-2}, \pi_*^{p,1}(H_1^{\mathrm{ord}}(Y_1(p), \mathcal{M}_{2k-2,p})) \rangle = \langle \mathrm{Eis}_{2k-2}, H_1^{\mathrm{ord}}(Y, \mathcal{M}_{2k-2,p}) \rangle.$$

Therefore, this corollary follows from Theorem 2.13 and Lemma 5.5. □

We note that for any prime number $p$, the operators $V_p := \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $V_p' := \widetilde{\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)}$ induce homomorphisms

$$V_p : H_1(Y_1(Np), \mathcal{M}_{2k-2}) \to H_1(Y_1(N), \mathcal{M}_{2k-2}),$$
$$V_p' : H^1(Y_1(N), \mathcal{M}_{2k-2}) \to H^1(Y_1(Np), \mathcal{M}_{2k-2}).$$

**Lemma 9.26.** *For any prime number $p$ and integer $N \geq 1$, we have*

$$e_{U_p'}(\pi_{1,Np}^*(\mathrm{Eis}_{2k-2})) = \frac{1}{1-p^{2k-1}}(\pi_{1,Np}^*(\mathrm{Eis}_{2k-2}) - \pi_{1,N}^*(\mathrm{Eis}_{2k-2})|V_p').$$

*Proof.* We put

$$\mathrm{Eis}_{2k-2}^{(1)} := \frac{1}{1-p^{2k-1}}(\pi_{1,Np}^*(\mathrm{Eis}_{2k-2}) - \pi_{1,N}^*(\mathrm{Eis}_{2k-2})|V_p'),$$
$$\mathrm{Eis}_{2k-2}^{(p^{2k-1})} := \frac{1}{1-p^{2k-1}}(-p^{2k-1}\pi_{1,Np}^*(\mathrm{Eis}_{2k-2}) + \pi_{1,N}^*(\mathrm{Eis}_{2k-2})|V_p').$$

Since $\pi_{1,Np}^* T_p' = U_p' \pi_{1,Np}^* + V_p' \pi_{1,N}^*$ and $U_p' V_p' \pi_{1,N}^* = p^{2k-1}\pi_{1,Np}^*$, the relation $\mathrm{Eis}_{2k-2}|T_p' = (1+p^{2k-1})\mathrm{Eis}_{2k-2}$ shows that

$$\mathrm{Eis}_{2k-2}^{(1)}|U_p' = \mathrm{Eis}_{2k-2}^{(1)} \quad \text{and} \quad \mathrm{Eis}_{2k-2}^{(p^{2k-1})}|U_p' = p^{2k-1}\mathrm{Eis}_{2k-2}^{(p^{2k-1})}.$$

Since $\pi_{1,Np}^*(\mathrm{Eis}_{2k-2}) = \mathrm{Eis}_{2k-2}^{(1)} + \mathrm{Eis}_{2k-2}^{(p^{2k-1})}$, these facts imply the lemma. □

**Lemma 9.27.** *For any integer $N \geq 1$ and a prime number $p$, we have*

$$\langle \mathrm{Eis}_{2k-2}, \pi_*^{Np,1}(e_{U_p}(\mathfrak{Z}_{\Gamma_1(Np),k} \otimes \mathbb{Z}_p))\rangle \subset \langle \mathrm{Eis}_{2k-2}, \mathfrak{Z}_{\Gamma_1(1),k}\rangle \mathbb{Z}_p.$$

*Proof.* Lemma 9.26 implies that

$$\begin{aligned}\langle e_{U_p'}\pi_{Np,1}^*(\mathrm{Eis}_{2k-2}), \mathfrak{Z}_{\Gamma_1(Np),k}\rangle &\subset \langle \pi_{1,Np}^*(\mathrm{Eis}_{2k-2}), \mathfrak{Z}_{\Gamma_1(Np),k}\rangle \mathbb{Z}_p + \langle V_p'\pi_{1,N}^*(\mathrm{Eis}_{2k-2}), \mathfrak{Z}_{\Gamma_1(Np),k}\rangle \mathbb{Z}_p \\ &\subset \langle \mathrm{Eis}_{2k-2}, \mathfrak{Z}_{\Gamma_1(1),k}\rangle \mathbb{Z}_p + \langle \mathrm{Eis}_{2k-2}, \pi_*^{N,1}(V_p \mathfrak{Z}_{\Gamma_1(Np),k})\rangle \mathbb{Z}_p.\end{aligned}$$

Hence we have

$$\begin{aligned}&\langle \mathrm{Eis}_{2k-2}, \pi_*^{Np,1}(e_{U_p}(\mathfrak{Z}_{\Gamma_1(Np),k} \otimes \mathbb{Z}_p))\rangle \\ &\qquad = \langle e_{U_p'}\pi_{1,Np}^*(\mathrm{Eis}_{2k-2}), \mathfrak{Z}_{\Gamma_1(N),k}\rangle \mathbb{Z}_p \\ &\qquad \subset \langle \mathrm{Eis}_{2k-2}, \mathfrak{Z}_{\Gamma_1(1),k}\rangle \mathbb{Z}_p + \langle \mathrm{Eis}_{2k-2}, \pi_*^{N,1}(V_p \mathfrak{Z}_{\Gamma_1(Np),k})\rangle \mathbb{Z}_p.\end{aligned}$$

Therefore, it suffices to show that $V_p \mathfrak{Z}_{\Gamma_1(Np),k} \subset \mathfrak{Z}_{\Gamma_1(N),k}$. Let $\gamma \in \Gamma_1(Np)$ be a matrix. Then we have $\gamma_p := \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\gamma\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \in \Gamma_1(N)$. By the definitions of $\mathfrak{z}_{\Gamma_1(N),k}$ and $\mathfrak{z}_{\Gamma_1(Np),k}$, we obtain

$$V_p \cdot \mathfrak{z}_{\Gamma_1(Np),k}(\gamma) \in \mathbb{Z}\mathfrak{z}_{\Gamma_1(N),k}(\gamma_p).$$

In particular, we have $V_p \mathfrak{Z}_{\Gamma_1(Np),k} \subset \mathfrak{Z}_{\Gamma_1(N),k}$. □

*Proof of Theorem 9.15.* By Theorem 2.13 and Remark 9.18, we only need to show that

$$\frac{1}{N_{2k}}\mathbb{Z}_p \subset \langle \mathrm{Eis}_{2k-2}, \mathfrak{Z}_{\Gamma_1(1),k}\rangle \mathbb{Z}_p$$

for any prime number $p$. When $p \geq 5$, this claim follows from Corollary 9.25 and Lemma 9.27 applied to $N = 1$. Suppose $p = 2$ or $p = 3$. Then since the primes 2 and 3 are regular, these primes does not divide $N_{2k}$. Hence, Corollary 9.23 and Lemma 9.27 show that

$$\frac{1}{N_{2k}}\mathbb{Z}_p = \mathbb{Z}_p \subset \langle \mathrm{Eis}_{2k-2}, \mathfrak{Z}_{\Gamma_1(1),k}\rangle \mathbb{Z}_p. \qquad \square$$

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# The Bombieri–Pila determinant method

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We give a concise and accessible introduction to the real-analytic determinant method for counting integral points on algebraic curves, based on the classic 1989 paper of Bombieri and Pila.

## 1. Introduction

Consider the problem of counting the number of integral points on an algebraic curve in a box; that is, solutions $(x, y) \in \{1, \dots, N\}^2$ to the equation $F(x, y) = 0$ for an irreducible polynomial $F \in \mathbb{R}[x, y]$. In breakthrough work in Diophantine geometry, Bombieri and Pila [3] obtained an essentially sharp quantitative upper bound for this problem.

**Theorem 1** (Bombieri–Pila [3], Pila [20]). *Let $F \in \mathbb{R}[x, y]$ be an irreducible polynomial of degree $d \geq 2$. If $N$ is sufficiently large, depending only on $d$, then*

$$\big|\{(x, y) \in \{1, \dots, N\}^2 : F(x, y) = 0\}\big| \leq (\log N)^{O(d)} N^{1/d}.$$

A particularly striking feature of the bound is that the implicit constants depend only on $d = \deg F$, not on the coefficients of $F$ themselves (which may be arbitrarily large).

Bombieri and Pila's original method [3] gave a bound of the shape $N^{1/d+o(1)}$ with a weaker explicit form for $N^{o(1)}$. Pila [20] refined the method to yield the stronger bound $(\log N)^{O(d)} N^{1/d}$. It is possible that the strong bound $O(N^{1/d})$ may hold. If so, such a bound is best possible, as witnessed by the simple example $x = y^d$.

In this note we give an accessible proof of Theorem 1. We follow [3; 20], but simplify the presentation for the sake of clarity and to bring the key features of the method to the fore. This note is essentially self-contained, appropriate for an undergraduate level reader with basic familiarity with calculus (of a single variable) and linear algebra. The only results cited without proof are Bézout's theorem and the implicit function theorem.

**1.1.** ***Features and scope of the exposition.*** Our primary aim is to present the key ideas of the Bombieri–Pila real-analytic determinant method, in the setting of

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curves. We will not discuss the ideas behind subsequent works, such as the $p$-adic determinant method of Heath-Brown [14] and a global version of Salberger [23]. The $p$-adic method is stronger than its real-analytic predecessor in several respects, for example, in its treatment of singularities. However the real-analytic determinant method does retain some advantages, for instance, to count points on transcendental (higher-dimensional) varieties (see, for example, the work of Pila and Wilkie [21]), or for problems of a more Archimedean nature, such as counting points 'near' (i.e., using the Archimedean metric) curves and higher-dimensional varieties. We give an alternative application along these lines, to counting integral points on convex curves, in Section 5.

To appreciate the bounds under study, we stress two key features: Firstly, the determinant method obtains strong bounds when the degree $d$ is large compared to the number of variables $n$. In particular, as in [3] and some of [14], the bounds improve as the degree $d$ increases. Whereas in the opposite regime, when the degree $d$ is fixed and the number of variables $n$ sufficiently large (depending on $d$), the determinant method yields weaker results as compared to the circle method, for example (as shown in spectacular fashion by Birch [2]).

We give a conceptual heuristic for this (which we outline more precisely in Section 3): When $n = 2$, the Bombieri–Pila determinant method essentially constructs a vector space of polynomials $F \in \mathbb{Z}[x, y]$, with dimension growing in $d$. The goal is then to find one such polynomial which satisfies certain properties, including to vanish on the integer points of our given curve. So as $d$ grows, we have increasing degrees of freedom to construct our desired polynomial. More generally, when $n \geq 3$, Heath-Brown constructs a certain vector space of polynomials $F \in \mathbb{Z}[x_1, \dots, x_n]$. This turns out to work well when $(n-1)/d^{1/(n-2)}$ is small, in particular when $d$ is large compared to $n$.

Secondly, the bounds under study are uniform in $F$. In particular, such uniformity is essential for applications to higher-dimensional varieties, since then one may induct on the dimension via 'slicing' arguments, applying uniform bounds at each successive dimension. In [14], Heath-Brown introduced a post hoc trick to obtain uniformity, which has recently been used in other contexts (see [4; 5]). If one does not care about uniformity in $F$, then often alternative methods perform far better (for example, if the points correspond to projective points on a curve of genus $\geq 2$, then Faltings' theorem [10] implies that there are only finitely many integral points).

We hope this note will foster a wider understanding of the determinant method, with large potential to spur further applications. Just in 2024, the determinant method has been applied by Greenfeld, Iliopoulou, Peluse [11] (via [8]) to bound integer distance sets, and by Browning, Lichtman, Teräväinen [7] (via [3] and [14]) to bound the exceptional set in the $abc$ conjecture. Such recent examples highlight the timeliness of our exposition.

**1.2.** ***Further work.*** To offer a bit of broader context, in this section we will very briefly highlight some related results involving counting integral points and the determinant method. Since our primary goal is to give a short, elementary exposition of the Bombieri–Pila method, we will not attempt by any means to give a complete survey of the literature.

Let us introduce some convenient notation: for any $n, d \geq 2$, we write

$$X^{\mathbb{A}}_{n,d}(N) := \sup_F \big|\{\mathbf{x} = (x_1, \dots, x_n) \in \{1, \dots, N\}^n : F(\mathbf{x}) = 0\}\big|,$$

where the supremum ranges over all irreducible $F \in \mathbb{Q}[x_1, \dots, x_n]$ of degree $d$. In this notation, the estimate in Theorem 1 of Bombieri–Pila gives

$$X^{\mathbb{A}}_{2,d}(N) \ll_d N^{1/d+o(1)}.$$

Importantly, the implied constants depend only on the degree $d$, but are independent of $N$. Pila [19] extended this result to higher dimensions by a slicing argument, showing for any $n \geq 2$,

$$X^{\mathbb{A}}_{n,d}(N) \ll_{n,d} N^{n-2+1/d+o(1)}. \tag{1}$$

Here the implied constants depend only on $n, d$, but not on $N$. Again, the example $x_1 = x_2^d$ shows that this exponent is the best possible.

Most of the subsequent progress has occurred in the projective setting, where the aim is to provide bounds for

$$X^{\mathbb{P}}_{n,d}(N) = \sup_F \big|\{\boldsymbol{x} = (x_0, \dots, x_n) \in \{1, \dots, N\}^{n+1} : \gcd(\boldsymbol{x}) = 1,\ F(\boldsymbol{x}) = 0\}\big|.$$

Here the supremum ranges over all *homogeneous* irreducible $F \in \mathbb{Q}[x_0, \dots, x_n]$ of degree $d$. Note that (1) immediately implies $X^{\mathbb{P}}_{n,d}(N) \ll N^{n-1+1/d+o(1)}$.

Many results in the projective setting use a $p$-adic variant of the determinant method, developed by Heath-Brown [14], which is inspired by the real-analytic method of Bombieri–Pila presented here. As mentioned above, since our focus is on the real-analytic method we will not discuss Heath-Brown's $p$-adic determinant method (and its subsequent generalisations) here. We restrict ourselves to very briefly highlight the state of the art.

Notably, in [14] Heath-Brown obtained a variety of estimates, including

$$X^{\mathbb{P}}_{2,d}(N) \ll_d N^{2/d+o(1)} \quad \text{and} \quad X^{\mathbb{P}}_{3,d}(N) \ll_d N^{2+o(1)}.$$

This result is sharp, as the example $F(x_0, x_1, x_2, x_3) = x_0^d + x_1^d - x_2^d - x_3^d$ shows. However, if we remove the 'trivial' solutions, then more can be said. In particular, if we remove the points that lie on any lines on the surface $F = 0$ then Heath-Brown improves this estimate to $\ll_d N^{1+3/\sqrt{d}+o(1)}$, with an even stronger result if $F$ is assumed to be nonsingular.

Building on the global $p$-adic determinant method of Salberger [23] and others, Walsh [28] removed the $N^{o(1)}$ factor, proving

$$X^{\mathbb{P}}_{2,d}(N) \ll_d N^{2/d}, \tag{2}$$

which is sharp for $d$ fixed. For explicit dependence on $d$, Castryck, Cluckers, Dittmann, Nguyen [8] obtained

$$X^{\mathbb{P}}_{2,d}(N) \ll d^4 N^{2/d}, \quad X^{\mathbb{A}}_{2,d}(N) \ll d^3(d+\log N)N^{1/d}. \tag{3}$$

Recently Binyamini, Cluckers, and Novikov [1] used real-analytic methods to prove

$$X^{\mathbb{P}}_{2,d}(N) \ll d^2(\log N)^{O(1)}N^{2/d}, \quad X^{\mathbb{A}}_{2,d}(N) \ll d^2(\log N)^{O(1)}N^{1/d}, \tag{4}$$

which they show is sharp in $N$ and $d$, up to the $(\log N)^{O(1)}$ factors. Notably, in (4) the implicit constants are absolute, and so the bound (4) is preferred to (2) when $d$ is large compared to $N$.

In higher dimensions $n \geq 3$, recall (1) implies that $X^{\mathbb{P}}_{n,d}(N) \ll_n N^{n-1+1/d+o(1)}$. It is believed that, uniformly for all $d \geq 2$,

$$X^{\mathbb{P}}_{n,d}(N) \ll_{n,d} N^{n-1+o(1)}. \tag{5}$$

Heath-Brown's work establishes this for $n = 3$, and furthermore the elementary theory of quadratic forms may establish this bound for $d = 2$ and all $n$. Interestingly, Marmon [17] has shown that an extension of the real-analytic method of Bombieri–Pila can be used to recover the main results of Heath-Brown, rather than the $p$-adic method employed there.

Further, (5) is a case of the uniform dimension growth conjecture, attributed to Heath-Brown [14], which posits a bound of $O_{n,d}(N^{\dim X+o(1)})$ for any integral projective variety $X \subset \mathbb{P}^n$ of degree $d$. This was resolved for $d \geq 4$ in [23]; also see [6]. (In fact, [23] fully resolves the 'nonuniform' dimension growth conjecture for all $n, d \geq 3$, where the implied constant may depend on $F$.) Again, one may ask whether the factor of $N^{o(1)}$ in (5) may be sharpened, or simply removed. This was achieved in [9], showing for $d \geq 5$,

$$X^{\mathbb{P}}_{n,d}(N) \ll_n d^7 N^{n-1}.$$

For $d = 2$, at least a factor of $(\log N)$ is required, as the example $F(x_0, x_1, x_2, x_3) = x_0x_1 - x_2x_3$ shows; see Serre [25, p. 178].

As mentioned in Section 1.1, determinant methods work best when $d$ is larger, and by their nature are less suited to smaller $d$ as compared to other approaches [2]. In particular, when $d = 3$ the best known bound is $X^{\mathbb{P}}_{n,3}(N) \ll_n N^{n-1+1/7+o(1)}$ from [22]. As such, it is a key question for contemporary analytic number theory to remove the factor $N^{1/7}$, so to obtain the conjectured bound (5) when $d = 3$. All these results in turn are (type-I) cases of Serre's conjecture for thin sets; see

[25; 26] for further details. Finally, we mention that the determinant method may be viewed as an example of 'polynomial methods'. See [13] for an introductory survey of this much broader topic, which has seen wide applications.

## 2. Covering integer points by curves

In this section we present the key ingredient of the proof of Theorem 1. The main idea is that, given some smooth function $f$ which has rapidly decaying derivatives, we can cover the integer points on the graph $(x, f(x))$ by a small number of curves drawn from a certain specified set. This quickly leads to efficient upper bounds for the number of such points via an application of Bézout's theorem.

Let $\mathcal{M} \subseteq \mathbb{R}[x, y]$ be a finite set of monomials. We write $\langle \mathcal{M} \rangle$ for the span of $\mathcal{M}$, that is, those polynomials in $\mathbb{R}[x, y]$ whose monomials are all in $\mathcal{M}$. We will sometimes abuse notation and write $\boldsymbol{j} = (j_1, j_2) \in \mathcal{M}$ to mean $x^{j_1} y^{j_2} \in \mathcal{M}$. Let

$$p = p_{\mathcal{M}} := \sum_{x^{j_1} y^{j_2} \in \mathcal{M}} j_1 \quad \text{and} \quad q = q_{\mathcal{M}} := \sum_{x^{j_1} y^{j_2} \in \mathcal{M}} j_2. \tag{6}$$

The main example to keep in mind is the case when $\mathcal{M} = \{x^{j_1} y^{j_2} : j_1 + j_2 \le d\}$, the set of all monomials of degree at most $d$. In this case we have

$$D = |\mathcal{M}| = \tfrac{1}{2}(d+1)(d+2) \quad \text{and} \quad p = q = \tfrac{1}{3} d D,$$

and $\langle \mathcal{M} \rangle$ is simply the set of polynomials with degree at most $d$. The need to work with the more general situation is because we will apply Bézout's theorem to count integer points on some curve $F(x, y) = 0$, and therefore need to make sure that we are not constructing some $G \in \mathbb{R}[x, y]$ with $F \mid G$. The reader may like to look ahead to Section 4 to see how to choose $\mathcal{M}$ to avoid this, but on first reading they should just take $\mathcal{M}$ to be all monomials of degree at most $d$.

The driving force of the Bombieri–Pila determinant method is the following lemma, which states that if a function $f$ has rapidly decaying derivatives then the integer points on any sufficiently short segment of the graph $(x, f(x))$ can be covered by a single curve in $\langle \mathcal{M} \rangle$. If one does not care about the quantitative aspects, some result like this is trivial, since any $D - 1$ points are contained in a curve in $\langle \mathcal{M} \rangle$ by linear algebra. For sufficiently smooth functions, however, the following bound is far superior.

**Lemma 2** (Bombieri–Pila). *Let $\mathcal{M}$ be a finite set of monomials of size $D = |\mathcal{M}|$ and let $p, q$ be defined as in (6). Let $I \subseteq [0, N]$ be a closed interval and $f \in C^{D-1}(I)$. Suppose that $X > 0$ and $\delta \ge 1/N$ are such that, for all $0 \le i < D$ and $x \in I$,*

$$\left| \frac{f^{(i)}(x)}{i!} \right| \le X \delta^i.$$

*If*

$$|I| < \tfrac{1}{4}\delta^{-1}((2N)^p(DX)^q)^{-1/\binom{D}{2}}$$

*then* $\{(x, f(x)) : x \in I\} \cap \mathbb{Z}^2$ *is contained in some curve in* $\langle \mathcal{M} \rangle$ (*that is, the zero set of some* $F \in \langle \mathcal{M} \rangle$).

For applications, the following lemma is often more convenient to apply. It covers longer pieces of the graph by (not too many) curves in $\langle \mathcal{M} \rangle$. The proof is a simple greedy application of Lemma 2.

**Lemma 3** (Bombieri–Pila). *Let* $\mathcal{M}$ *be a finite set of monomials of size* $D = |\mathcal{M}|$ *and let* $p, q$ *be defined as in (6). Let* $I \subseteq [0, N]$ *be a closed interval and* $f \in C^{D-1}(I)$. *Suppose that* $X > 0$ *and* $\delta \geq 1/N$ *are such that, for all* $0 \leq i < D$ *and* $x \in I$,

$$\left| \frac{f^{(i)}(x)}{i!} \right| \leq X\delta^i.$$

*The integer points* $\{(x, f(x)) : x \in I\} \cap \mathbb{Z}^2$ *are contained in the union of at most*

$$4\delta|I|((2N)^p(DX)^q)^{1/\binom{D}{2}} + 1$$

*many curves in* $\langle \mathcal{M} \rangle$.

We can trivially cover such integer points by at most $|I| + 1$ many curves, and so this lemma gives a significant saving over this trivial upper bound roughly when $\delta \ll (N^p D^q X^q)^{-1/\binom{D}{2}}$.

*Proof.* Let $\{(x, f(x)) : x \in I\} \cap \mathbb{Z}^2 = \{z_1, \dots, z_t\}$, say, arranged in increasing order of their $x$-coordinates. Define a sequence of integers $n_0, n_1, n_2, \dots$ by $n_0 = 0$ and recursively let $n_\ell$ denote the largest index for which the points

$$\{z_i : n_{\ell-1} \leq i < n_\ell\}$$

are contained in a single curve in $\langle \mathcal{M} \rangle$. Suppose the sequence $n_0, n_1, \dots, n_m$ terminates after $m + 1$ elements. For $0 \leq \ell < m$ the set $\{z_{n_\ell}, \dots, z_{n_{\ell+1}}\}$ is not contained in a single curve in $\langle \mathcal{M} \rangle$, and so Lemma 2 implies that the length of the interval $[x_{n_\ell}, x_{n_{\ell+1}}]$ is

$$x_{n_{\ell+1}} - x_{n_\ell} \geq \tfrac{1}{4}\delta^{-1}((2N)^p(DX)^q)^{-1/\binom{D}{2}}.$$

On the other hand, since $x_{n_0}, \dots, x_{n_m} \in I$, we have

$$|I| \geq x_{n_m} - x_{n_0} = \sum_{0 \leq \ell < m} (x_{n_{\ell+1}} - x_{n_\ell}) \geq \tfrac{1}{4}m\delta^{-1}((2N)^p(DX)^q)^{-1/\binom{D}{2}}.$$

Rearranging gives the desired bound on $m + 1$, the number of curves in $\mathcal{M}$. □

## 3. The determinant method

In this section we will prove Lemma 2, and explain the main idea behind the (real) determinant method. The basic structure of the proof is as follows:

(0) We would like to find some polynomial $F \in \langle \mathcal{M} \rangle$ which vanishes on all points in $S = \{(x, f(x)) : x \in I\} \cap \mathbb{Z}^2$.

(1) Consider the matrix $A$ with entries $z^{\boldsymbol{j}} = (x^{j_1}, f(x)^{j_2})$, for $z \in S$ and $\boldsymbol{j} \in \mathcal{M}$. If $A$ has rank $< |S|$, then there is some linear dependency between the rows of $A$, which will mean some $F \in \langle \mathcal{M} \rangle$ vanishes on $S$, as desired (see Lemma 4).

(2) For sake of contradiction, suppose this does not happen. Then there is some nonsingular $|S| \times |S|$ submatrix $A'$. Importantly, this submatrix $A'$ has integer values, and hence its determinant will be at least 1 in absolute value.

(3) We force a contradiction by proving $|\det A'| < 1$ directly. To do this, we control $\det(x^{j_1} f(x)^{j_2})$ by a determinant involving the derivatives of $f$ which we then bound trivially, using the Leibniz determinant formula and the assumption that the derivatives of $f$ decay rapidly. This is shown in Lemma 6.

We begin with step (1), converting the problem into one concerning a matrix of monomials. By interpolation, any $D-1$ points in the plane lie on a common curve in $\langle \mathcal{M} \rangle$. The key insight driving the Bombieri–Pila method (and most instances of the so-called polynomial method) is that this common curve can cover even more points, assuming the rank of the associated monomial matrix is not maximal.

**Lemma 4.** *Let $\mathcal{M} \subset \mathbb{R}[x, y]$ be a finite set of monomials. For any $z_1, \dots, z_t \in \mathbb{R}^2$ if*

$$\operatorname{rank}(z_i^{\boldsymbol{j}})_{\substack{i \le t \\ \boldsymbol{j} \in \mathcal{M}}} < |\mathcal{M}|,$$

*then $z_1, \dots, z_t$ are contained in a curve in $\langle \mathcal{M} \rangle$.*

**Remark 5.** We shall not need it, but the converse of Lemma 4 also holds: if $z_1, \dots, z_t$ are contained in a curve in $\langle \mathcal{M} \rangle$, then the rank is less than $|\mathcal{M}|$.

*Proof.* Let $\mathcal{N} \subseteq \mathcal{M}$ and $S \subseteq \{1, \dots, t\}$ be such that $z_S^{\mathcal{N}} := (z_i^{\boldsymbol{j}})_{i \in S, \boldsymbol{j} \in \mathcal{N}}$ is an $r \times r$ minor of maximal rank. By assumption $r < |\mathcal{M}|$, and so there exists $\boldsymbol{k} \in \mathcal{M} \setminus \mathcal{N}$.

Consider $f \in \mathbb{R}[x, y]$ given by

$$f(x, y) := \det \begin{pmatrix} (x, y)^{\mathcal{N}} & (x, y)^{\boldsymbol{k}} \\ z_S^{\mathcal{N}} & z_S^{\boldsymbol{k}} \end{pmatrix} = \sum_{\boldsymbol{j} \in \mathcal{N} \cup \{\boldsymbol{k}\}} \Big( \epsilon_{\boldsymbol{j}} \det(z_i^{\boldsymbol{l}})_{\substack{i \in S \\ \boldsymbol{l} \in \mathcal{N} \cup \{\boldsymbol{k}\} \setminus \{\boldsymbol{j}\}}} \Big) x^{j_1} y^{j_2}, \quad (7)$$

for some $\epsilon_{\boldsymbol{j}} \in \{\pm 1\}$. Here in the second equality we have used the definition of determinant. From the right-hand side of (7), we see $f(x, y)$ defines a curve in $\langle \mathcal{M} \rangle$. (In fact this curve is in the $\mathbb{Z}$-linear span of $\mathcal{M}$ if $z_1, \dots, z_t \in \mathbb{Z}^2$.) It remains to note that $f(z_i) = 0$ for all $1 \le i \le t$. Indeed, if $i \in S$, the matrix in (7) above has repeated rows, while if $i \notin S$, the determinant $f(x, y)$ is zero by maximality of $r$. □

To profitably apply Lemma 4, we will need to be able to bound the determinant in a nontrivial way. This is accomplished, in the Bombieri–Pila method, via the following lemma.

**Lemma 6.** *Let $n \geq 1$ and $I \subset \mathbb{R}$ be some closed interval with $x_1, \dots, x_n \in I$. Suppose $f_1, \dots, f_n \in C^{n-1}(I)$ and $A_{ij} \geq 0$ are such that for $1 \leq i, j \leq n$ and all $x \in I$,*

$$\left| \frac{f_j^{(i-1)}(x)}{(i-1)!} \right| \leq A_{ij}.$$

*Then we have*

$$|\det(f_j(x_i))| \leq \left( \prod_{i>j} |x_i - x_j| \right) \sum_{\sigma \in S_n} \prod_{1 \leq i \leq n} A_{i\sigma(i)},$$

*where $S_n$ is the standard group of bijections $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$.*

*Proof.* We first claim that for every $1 \leq i, j \leq n$ there exists some $\xi_{ij} \in I$ such that

$$\frac{f_j^{(i-1)}(\xi_{ij})}{(i-1)!} = \sum_{\ell \leq i} f_j(x_\ell) \prod_{\substack{m \leq i \\ m \neq \ell}} \frac{1}{x_\ell - x_m}. \tag{8}$$

This fact is a consequence of Lagrange interpolation: consider the polynomial, of degree $< i$,

$$g_{ij}(\xi) = \sum_{\ell \leq i} f_j(x_\ell) \prod_{\substack{m \leq i \\ m \neq \ell}} \frac{\xi - x_m}{x_\ell - x_m}.$$

By construction, $g_{ij}$ agrees with $f_j$ at $i$ many points (namely $f_j(x_k) = g_{ij}(x_k)$ for $k \leq i$). That is, the function $f_j - g_{ij}$ has $i$ many zeros in $I$, and hence by repeated applications of Rolle's theorem, the $(i-1)$-fold derivative will have some zero in $I$. That is, there is some $\xi_{ij} \in I$ such that $f_j^{(i-1)}(\xi_{ij}) = g_{ij}^{(i-1)}(\xi_{ij})$. Hence (8) follows, since the right-hand side is precisely the derivative $g_{ij}^{(i-1)}$ (which is constant, since $\deg g_{ij} < i$).

Let $G_\ell(x_1, \dots, x_i) = (i-1)! \prod_{\substack{m \leq i \\ m \neq \ell}} \frac{1}{x_\ell - x_m}$. By (8) it follows that

$$\begin{aligned} \det(f_j^{(i-1)}(\xi_{ij})) &= \det\left( \sum_{\ell \leq i} f_j(x_\ell) G_\ell(x_1, \dots, x_i) \right) \\ &= \left( \prod_{1 \leq i \leq n} G_i(x_1, \dots, x_i) \right) \det(f_j(x_i)). \end{aligned} \tag{9}$$

Here we used the general fact that $\det\left(\sum_{\ell \leq i} a_{j\ell} b_{\ell i}\right) = \left(\prod_i b_{ii}\right) \det(a_{ji})$, since the matrix on the left can be factored as $(a_{j\ell})$ times the lower triangular matrix $(b_{\ell i})_{\ell \leq i}$, the determinant of which is equal to the product of its diagonal entries.

Therefore, since $G_i(x_1, \dots, x_i)^{-1} = \frac{1}{(i-1)!} \prod_{m<i}(x_i - x_m)$, the identity (9) implies

$$\begin{aligned}
|\det(f_j(x_i))| &\le \Bigg( \prod_{1\le i\le n} |G_i(x_1, \dots, x_i)| \Bigg)^{-1} |\det(f_j^{(i-1)}(\xi_{ij}))| \\
&\le \Bigg( \prod_{i>j} |x_i - x_j| \Bigg) \sum_{\sigma\in S_n} \prod_{i\le n} \frac{|f_{\sigma(i)}^{(i-1)}(\xi_{i\sigma(i)})|}{(i-1)!} \\
&\le \Bigg( \prod_{i>j} |x_i - x_j| \Bigg) \sum_{\sigma\in S_n} \prod_{1\le i\le n} A_{i\sigma(i)}
\end{aligned}$$

by the Leibniz formula and assumed bound on the derivatives. □

We will apply the previous lemma with $f_{\boldsymbol{j}}(x) = x^{j_1} f(x)^{j_2}$, and so it will be necessary to bound the derivatives of such functions. This lemma is the only place we use the assumption that $I \subseteq [0, N]$.

**Lemma 7.** *Let $k \ge 1$, $I \subseteq [0, N]$ a closed interval, and $f \in C^k(I)$. Suppose that $X > 0$ and $\delta \ge 1/N$ are such that, for all $0 \le i \le k$ and $x \in I$,*

$$\left| \frac{f^{(i)}(x)}{i!} \right| \le X\delta^i.$$

*For any integer pair $\boldsymbol{j} = (j_1, j_2) \in \mathbb{Z}^2_{\ge 0}$, the function $f_{\boldsymbol{j}}(x) = x^{j_1} f(x)^{j_2}$ satisfies*

$$\left| \frac{f_{\boldsymbol{j}}^{(i-1)}(x)}{(i-1)!} \right| \le (2N)^{j_1} (iX)^{j_2} \delta^{i-1}$$

*for all $1 \le i \le k$ and $x \in I$.*

*Proof.* For any $1 \le i \le k$, the product rule gives

$$\frac{f_{\boldsymbol{j}}^{(i-1)}(x)}{(i-1)!} = \sum_{i_0+i_1+\cdots+i_{j_2}=i-1} \binom{j_1}{i_0} x^{j_1-i_0} \frac{f^{(i_1)}(x)}{i_1!} \cdots \frac{f^{(i_{j_2})}(x)}{i_{j_2}!}.$$

Using the assumed derivative bounds on $f^{(i)}$, since $\binom{j_1}{i_0} \le 2^{j_1}$ and $|x| \le N$ for $x \in I$,

$$\begin{aligned}
\left| \frac{f_{\boldsymbol{j}}^{(i-1)}(x)}{(i-1)!} \right| &\le \sum_{i_0+i_1+\cdots+i_{j_2}=i-1} \binom{j_1}{i_0} |x|^{j_1-i_0} (X\delta^{i_1}) \cdots (X\delta^{i_{j_2}}) \\
&\le \sum_{i_0+i_1+\cdots+i_{j_2}=i-1} 2^{j_1} N^{j_1-i_0} X^{j_2} \delta^{i-i_0-1} \\
&\le i^{j_2} 2^{j_1} N^{j_1} X^{j_2} \delta^{i-1},
\end{aligned}$$

using $\delta N \ge 1$, and bounding the number of partitions of $i-1$ into $m$ parts by $i^{m-1}$. □

We now have enough tools to prove Lemma 2.

*Proof of Lemma 2.* Let $\{(x, f(x)) : x \in I\} \cap \mathbb{Z}^2 = \{z_1, \dots, z_t\}$, say. By Lemma 4, it suffices to show that the matrix $M = (z_i^{\boldsymbol{j}})_{1 \le i \le t,\, \boldsymbol{j} \in \mathcal{M}}$ has rank $< D$. If not, then $M$ must have maximal rank $D$, and hence there is a subset of $D$ indices $S \subseteq \{1, \dots, t\}$ such that

$$\Delta := \det(z_i^{\boldsymbol{j}})_{\substack{i \in S \\ \boldsymbol{j} \in \mathcal{M}}} \neq 0. \tag{10}$$

Since each $z_i \in \mathbb{Z}^2$ we must have $\Delta \in \mathbb{Z}$, and so in particular $|\Delta| \ge 1$. Relabelling if necessary, we can assume that $S = \{1, \dots, D\}$.

Let $f_{\boldsymbol{j}}(x) = x^{j_1} f(x)^{j_2}$, so that $f_{\boldsymbol{j}}(x_i) = z_i^{\boldsymbol{j}}$. By Lemma 7 and the assumption on the derivatives of $f$, we have, for all $1 \le i, j \le D$ and $x \in I$,

$$\left| \frac{f_{\boldsymbol{j}}^{(i-1)}(x)}{(i-1)!} \right| \le (2N)^{j_1} (DX)^{j_2} \delta^{i-1}.$$

Thus by Lemma 6 with $f_{\boldsymbol{j}}(x) := x^{j_1} f(x)^{j_2}$ for $\boldsymbol{j} = (j_1, j_2) \in \mathcal{M}$ and $A_{i\boldsymbol{j}} = (2N)^{j_1} (DX)^{j_2} \delta^{i-1}$,

$$\begin{aligned} 1 \le |\Delta| = |\det(f_{\boldsymbol{j}}(x_i))_{\substack{i \in S \\ \boldsymbol{j} \in \mathcal{M}}}| &\le \left( \prod_{i>j} |x_i - x_j| \right) \sum_{\sigma} \prod_{i \in S} A_{i\sigma(i)} \\ &\le \prod_{i>j} |I| \sum_{\sigma} \prod_{i \in S} (2N)^{\sigma(i)_1} (DX)^{\sigma(i)_2} \delta^{i-1} \\ &\le |I|^{\binom{D}{2}} D! \, (2N)^p (DX)^q \delta^{\binom{D}{2}}, \end{aligned} \tag{11}$$

recalling the quantities $p$ and $q$ from (6). Here $\sigma$ ranges over all bijections from $S \to \mathcal{M}$. Using the crude bounds $D! \le D^D$ and $D^{2/(D-1)} \le 4$ for all $D \ge 2$, isolating $|I|$ in (11) above gives

$$\begin{aligned} |I| &\ge D^{-D/\binom{D}{2}} [(2N)^p (DX)^q]^{-1/\binom{D}{2}} \delta^{-1} \\ &= D^{-2/(D-1)} [(2N)^p (DX)^q]^{-1/\binom{D}{2}} \delta^{-1} \\ &\ge \tfrac{1}{4} \delta^{-1} [(2N)^p (DX)^q]^{-1/\binom{D}{2}}, \end{aligned}$$

which contradicts our assumption. Hence $\operatorname{rank}(M) < D$, and thus an application of Lemma 4 covers $\{z_1, \dots, z_t\}$ with a single curve in $\mathcal{M}$, as desired. □

## 4. Application of the key lemma

We now show how to use the determinant method (more precisely, its consequence in the form of Lemma 3) to bound the number of integral points on curves. This uses Bézout's theorem.

**Theorem 8** (Bézout's theorem). *Let $F, G \in \mathbb{R}[x, y]$ be nonconstant polynomials with no common divisor in $\mathbb{R}[x, y]$. There are at most $\deg F \cdot \deg G$ many points $(x, y) \in \mathbb{R}^2$, counted with multiplicity, such that $F(x, y) = G(x, y) = 0$.*

We will only require Bézout's theorem for irreducible $F$, where it takes the following form. Recall that $F \in \mathbb{R}[x, y]$ is irreducible if it cannot be factored into the product of two nonconstant polynomials in $\mathbb{R}[x, y]$.

**Corollary 9.** *Let $F \in \mathbb{R}[x, y]$ be an irreducible polynomial and $G \in \mathbb{R}[x, y]$ such that $F \nmid G$. There are at most $\deg F \cdot \deg G$ many points $(x, y) \in \mathbb{R}^2$ such that $F(x, y) = G(x, y) = 0$.*

We use Bézout's theorem as our fundamental tool to count integer points on arbitrary curves, by covering such points with other curves of bounded degree.

**Lemma 10.** *Let $\ell \geq d \geq 2$. Let $I \subseteq [0, N]$ be a closed interval, and $f \in C^\infty(I)$ with $F(x, f) = 0$ for some irreducible polynomial $F \in \mathbb{R}[x, y]$ of degree $d \geq 2$. Suppose that $N > 0$ and $\delta \geq 1/|I|$ satisfy*

$$\left| \frac{f^{(i)}(x)}{i!} \right| \leq N\delta^i \quad \text{for all } x \in I,$$

*for every $0 \leq i < D = d(\ell - d + 1)$. Then we have*

$$\big|\{(x, f(x)) : x \in I\} \cap \mathbb{Z}^2\big| \ll (d\ell)^2 N^{1/d+O(1/\ell)}\delta|I|.$$

In particular if the $i$-th derivative of $f$ decays like $|I|^{-i}N^{1+o(1)}$ (as we will shortly show can be arranged in practice) then Lemma 10 gives the desired upper bound of $N^{1/d+o(1)}$, taking $\ell = O(\log N)$.

*Proof.* Let $i_F = i$ be the maximal index such that $x^{d-i_F}y^{i_F}$ is a monomial in $F$. Define

$$\mathcal{M} = \{x^{j_1}y^{j_2} : d \leq j_1 + j_2 \leq \ell \text{ and } x^{d-i_F}y^{i_F} \nmid x^{j_1}y^{j_2}\}.$$

Next for each integer $h \in [d, \ell]$, we observe

$$\big|\{(j_1, j_2) \in \mathcal{M} : h = j_1 + j_2\}\big| = d.$$

Indeed, such $\boldsymbol{j}$ are of the form $\boldsymbol{j} = (j_1, h - j_1)$ for $j_1 \in [0, h]$. The condition $x^{d-i_F}y^{i_F} \nmid x^{j_1}y^{j_2}$ implies $d - i_F > j_1$ or $i_F > h - j_1$. The first case gives $j_1 \in [0, d - i_F)$, and the second case gives $j_1 \in (h - i_F, h]$. Since $h \geq d$ these combine for $d - i_F + i_F = d$ choices of $j_1$. Hence $\big|\{(j_1, j_2) \in \mathcal{M} : h = j_1 + j_2\}\big| = d$.

Thus we obtain

$$D = |\mathcal{M}| = \sum_{d \leq h \leq \ell} d = d(\ell - d + 1) = d\ell + O(d^2) \tag{12}$$

and

$$p+q = \sum_{d\le h\le \ell} hd = \tfrac{1}{2}d(\ell(\ell+1)-d(d-1))$$
$$\le \tfrac{1}{2}\ell^2 d + O(\ell d) = \frac{D^2}{2d} + O\left(\frac{D^2}{\ell}\right) \tag{13}$$

for $p, q$ as in (6). In particular,

$$\frac{p+q}{\binom{D}{2}} = \frac{2(p+q)}{D(D-1)} \le \frac{\frac{D}{d}+O\left(\frac{D}{\ell}\right)}{D-1} \le \frac{1}{d} + O\left(\frac{1}{\ell}\right). \tag{14}$$

Now we claim that $F \nmid G$ for all $G = G(x, y) \in \langle \mathcal{M} \rangle$. Indeed, if not then $G = FH$, where $H$ has degree $d'$, say. Let $i_H$ be maximal such that $x^{d'-i_H} y^{i_H}$ is a monomial in $H$. Then $x^{d+d'-i_F-i_H} y^{i_F+i_H}$ is a monomial in $G$ (by maximality of $i_F$ and $i_H$ no other monomial of degree $d+d'$ can cancel it) which is divisible by $x^{d-i_F} y^{i_F}$, and hence is not in $\mathcal{M}$, a contradiction. Hence $F \nmid G$.

Therefore, by Bézout's theorem in the form of Corollary 9, each curve $G(x, y) \in \langle \mathcal{M} \rangle$ can contain at most $d\ell$ many points in the graph $\Gamma = \{(x, f(x)) : x \in \mathbb{R}\}$. We may take

$$\le 4\delta|I|((2N)^p (DN)^q)^{1/\binom{D}{2}} + 1$$

many curves $G(x, y) \in \langle \mathcal{M} \rangle$ to cover $\Gamma \cap \mathbb{Z}^2$, by Lemma 3. Hence we conclude

$$\begin{aligned} |\Gamma \cap \mathbb{Z}^2| &\ll d\ell(\delta|I|(N^{p+q} D^q)^{1/\binom{D}{2}} + 1) \\ &\ll d\ell(\delta|I|(ND)^{1/d+O(1/\ell)} + 1) \\ &\ll (d\ell)^2 \delta|I| N^{1/d+O(1/\ell)} \end{aligned}$$

using $1/\delta \le |I|$ and (14). □

To deduce Theorem 1 from Lemma 10 it remains to divide the curve inside $\{1, \dots, N\}^2$ into a small number of pieces which locally look like a graph $(x, f(x))$ for some $f$ with sufficiently rapidly decaying derivatives. This can be done in several ways; in particular Pila and Wilkie [21] have shown that this can be efficiently done using a lemma of Gromov [12] and Yomdin [29]. This method was used (and simplified) by Marmon [17] in his recent extension of the real-analytic method. Here we will follow Bombieri and Pila and use a less efficient but more elementary approach, greedily dividing the curve into pieces with small derivatives aside from a small number of (very short) exceptional intervals on which the derivative is too large.

The following technical lemma is preparation for such a division.

**Lemma 11.** *Suppose $F(x, y) \in \mathbb{R}[x, y]$ is an irreducible polynomial of degree $d$. Let $I$ be an interval and $f \in C^\infty(I)$ satisfy $F(x, f(x)) = 0$. Then for any $k \ge 1$ and $c \in \mathbb{R}\setminus\{0\}$,*

$$|\{x \in I : f^{(k)}(x) = c\}| \ll kd^2.$$

*Proof.* Suppose first that $f$ is a polynomial, necessarily of degree $\le d$. Then $f^{(k)}$ is a polynomial of degree $\le d-k$, whence $f^{(k)}(x)=c$ has at most $d-k$ many solutions.

We may now assume that $f$ is not a polynomial. We claim that for each $1\le k\le d$, there is a polynomial $H_k\in\mathbb{R}[x,y]$ of degree at most $d_k:=(k-1)(2d-3)+d-1$ such that, for all $x\in I$,

$$H_k(x,f)+F_y(x,f)^{2k-1}f^{(k)}(x)=0, \tag{15}$$

where we write $F_y(x,y)=\frac{\partial}{\partial y}F(x,y)$, which is a polynomial of degree $\le d-1$.

We prove (15) by induction on $k$. For $k=1$, differentiating $F(x,f)=0$ with respect to $x$ gives $F_x(x,f)+F_y(x,f)f'(x)=0$, with $H_1=F_x$ of degree $d-1$. For the inductive step, assuming (15) holds for $k\ge 1$, differentiating with respect to $x$ gives

$$\begin{aligned}0=(H_k)_x(x,f)+(H_k)_y(x,f)f'(x)+F_y(x,f)^{2k-1}f^{(k+1)}(x)\\ +(2k-1)F_y(x,f)^{2k-2}f^{(k)}(x)\big(F_{xy}(x,f)+F_{yy}(x,f)f'(x)\big).\end{aligned}$$

We then multiply this equation by $F_y^2$, giving

$$0=F_y^2\big((H_k)_x+(H_k)_yf'\big)+F_y^{2k+1}f^{(k+1)}+(2k-1)F_y^{2k}f^{(k)}(F_{xy}+F_{yy}f').$$

Eliminating $f^{(k)}$ and $f'$, using $F_y^{2k-1}f^{(k)}=-H_k$ from (15) and $F_y\,f'=-F_x$,

$$\begin{aligned}0&=F_y^2(H_k)_x-F_yF_x(H_k)_y-(2k-1)H_k(F_yF_{xy}-F_xF_{yy})+F_y^{2k+1}f^{(k+1)}\\ &=:H_{k+1}(x,f)+F_y^{2k+1}f^{(k+1)},\end{aligned}$$

where $H_{k+1}$ has degree at most $d_k+2d-3=d_{k+1}$. This completes the induction for (15).

It follows that, for any constant $c\in\mathbb{R}$, the solutions $x\in I$ to $f^{(k)}(x)=c$ must satisfy $R_c(x,f(x))=0$, where $R_c(x,y)=H_k(x,y)+F_y(x,y)^{2k-1}c$. Since they must also satisfy $F(x,f(x))=0$, it suffices to show $F\nmid R_c$, whence Bézout's theorem in the form of Corollary 9 bounds the number of common points $x$ by $(\deg F)(\deg R_c)\le d(2kd)\ll kd^2$, as desired.

It remains to derive a contradiction if $F\mid R_c$. Note that this means that $F(x,y)=0$ implies $R_c(x,y)=0$ for any $x,y\in\mathbb{R}$. In particular, our assumption $F(x,f(x))=0$ for all $x\in I$ implies

$$\begin{aligned}0=R_c(x,f(x))&=H_k(x,f(x))+F_y(x,f(x))^{2k-1}c\\ &=F_y(x,f(x))^{2k-1}(c-f^{(k)}(x))\end{aligned} \tag{16}$$

for all $x\in I$, recalling (15). Since $F_y(x,f(x))$ is a polynomial of degree $\le d-1$, by Corollary 9 there are $O(d^2)$ (in particular finitely many) $x\in I$ such that $F_y(x,f(x))=0$. Thus by (16), $f^{(k)}(x)=c$ holds, except for at most finitely

many values $x \in I$. Since $f$ is assumed to be $C^\infty$ this implies $f^{(k)}$ is identically a constant, so that $f$ is a polynomial of degree at most $k$, but this is a contradiction. Hence $F \nmid R_c$, and the proof is complete. □

We now apply Lemma 11 to obtain our desired division into subintervals with control on the derivatives in each subinterval.

**Lemma 12.** *Suppose $F(x, y) \in \mathbb{R}[x, y]$ is an irreducible polynomial of degree $d \geq 2$. Let $I$ be an interval and $f(x) \in C^\infty(I)$ satisfy $F(x, f) = 0$. Let $A_1, \dots, A_k > 0$. We may partition $I$ into $O(k^2d^2)$ many subintervals $I_\nu$ such that, for each interval $I_\nu$ and each $1 \leq i \leq k$,*

(i) $|f^{(i)}(x)| \leq A_i$ *for all* $x \in I_\nu$, or

(ii) $|f^{(i)}(x)| \geq A_i$ *for all* $x \in I_\nu$.

*Proof.* By Lemma 11 there are $O(kd^2)$ solutions to $f^{(i)}(x) = \pm A_i$ for each $1 \leq i \leq k$. Let $x_1, \dots, x_r$ be the union of all such solutions for $1 \leq i \leq k$ (so $r = O(k^2d^2)$), ordered such that $x_0 \leq x_1 < \cdots < x_r \leq x_{r+1}$, writing $I = [x_0, x_{r+1}]$. By construction $f$ satisfies (i) or (ii) on each of the $r$ subintervals $I_\nu = [x_\nu, x_{\nu+1}]$ for $\nu \leq r$. The number of subintervals is $r = O(k^2d^2)$, as required. □

The previous lemma allows a division into subintervals where we can control the size of the derivatives. For an application of Lemma 10 we specifically require that the derivatives be *small*, and hence will need a different method to handle the contribution from subintervals where the derivative remains large. Following Bombieri and Pila this is managed by the following lemma, which shows that such a subinterval must at least be very short, and then a trivial bound will suffice.

**Lemma 13.** *Let $k \geq 1$, $0 < \delta < 1$, and $X > 0$. Let $I$ be an interval and $f \in C^k(I)$. If*

$$\left|\frac{f^{(i)}(x)}{i!}\right| \leq X\delta^i \quad \text{for all } 0 \leq i < k \qquad \text{while} \qquad \left|\frac{f^{(k)}(x)}{k!}\right| \geq X\delta^k$$

*for every $x \in I$, then $|I| \leq 2/\delta$.*

*Proof.* Write $I = [a, b]$. Considering the Taylor expansion at $x = a$, there exists $\xi \in I$ for which

$$f(b) - f(a) = \sum_{1 \leq i < k} \frac{f^{(i)}(a)}{i!}(b-a)^i + \frac{f^{(k)}(\xi)}{k!}(b-a)^k.$$

Since $|f(b)|, |f(a)| \leq X$ by assumption, isolating the remainder term gives

$$X\delta^k|I|^k \leq \left|\frac{f^{(k)}(\xi)}{k!}(b-a)^k\right| \leq 2X + X\sum_{1 \leq i < k} \delta^i|I|^i.$$

Defining $\lambda = \delta|I|$, it follows that

$$\lambda^k \le 2 + \sum_{1 \le i < k} \lambda^i = 2 + \frac{\lambda^k - \lambda}{\lambda - 1}.$$

We can assume $\lambda > 1$ or we are done, and then we deduce $\lambda^k(\lambda - 2) \le \lambda - 2$, whence $\lambda \le 2$ as required. □

We can now use the previous lemmas to deduce a bound for the number of integral points on the graph of a sufficiently smooth function along a curve of bounded degree.

**Theorem 14.** *Let $d \ge 2$ and $N \ge 1$ be some integer sufficiently large in terms of $d$, and let $I$ be some interval of length $N$. Let $f \in C^\infty(I)$ be such that $|f'(x)| \le 1$ for all $x \in I$. If there exists some irreducible polynomial $F(x, y) \in \mathbb{R}[x, y]$ of degree $d \ge 2$ such that $F(x, f(x)) = 0$, identically as a function of $x$, then*

$$\big|\{(x, f(x)) : x \in I\} \cap \mathbb{Z}^2\big| \le (\log N)^{O(d)} N^{1/d}.$$

*Proof.* For any real $N > 0$ define

$$G(N) := \sup_{|I| \le N} \sup_{\substack{f \in C^\infty(I) \\ |f'| \le 1}} \big|\{(x, f(x)) : x \in I\} \cap \mathbb{Z}^2\big|.$$

We shall prove that for any integer $\ell \ge d$ there exists some $K = K(d, \ell)$ satisfying $2 \le K \le \ell^{O(1)}$ such that for any $N > 0$ we have

$$G(N) \le K^{O(d)} N^{1/d + O(1/\ell)} + KG(K^{-2d} N). \tag{17}$$

We first show how (17) implies Theorem 14. Fix some large $N > 0$ and $\ell \ge d$ (which will depend on $N$). Iteratively applying (17), since $K(K^{-2d})^{1/d} = K^{-1} \le \frac{1}{2}$ (assuming $\ell$ is large enough), it follows by induction that, for any $n \ge 1$,

$$G(N) \le K^{O(d)} N^{1/d + O(1/\ell)} \sum_{0 \le j < n} 2^{-j} + K^n G(K^{-2nd} N).$$

In particular, if we choose $n$ large enough such that $K^{2nd} \in [N, K^{2d} N)$, we have $K^n \le K N^{1/2d}$ and $G(K^{-2nd} N) \le G(1) \le 1$, so that

$$G(N) \ll K^{O(d)} N^{1/d + O(1/\ell)} + K N^{1/2d} \ll \ell^{O(d)} N^{1/d + O(1/\ell)} \ll (\log N)^{O(d)} N^{1/d},$$

recalling $K \le \ell^{O(1)}$ and choosing $\ell = \lceil \log N \rceil$. Thus (17) implies the result.

Now to prove (17) itself, we fix some interval $I$ of length $N$ and $f \in C^\infty(I)$ with $|f'| \le 1$. Without loss of generality, translating the graph of $f$ by an integer if necessary, we may assume that $I = [0, N]$ and $|f(x)| \le N$ for all $x \in I$. Fix some $\ell \ge d$ and let $\delta = \delta(d, \ell) \in (2/N, 1)$ be some quantity to be chosen later.

We shall apply Lemma 12 with $A_i = N\delta^i$, and recall $D = |\mathcal{M}| \ll d\ell$ from (12). Indeed, by Lemma 12 we may partition $I$ into at most $O(d^2D^2)$ subintervals $I_\nu$ such that, for each $I_\nu$ and each $1 \leq i < D$, either

(i) $\left|\frac{f^{(i)}(x)}{i!}\right| \leq N\delta^i$ for all $x \in I_\nu$, or

(ii) $\left|\frac{f^{(i)}(x)}{i!}\right| \geq N\delta^i$ for all $x \in I_\nu$.

Suppose first that $I_\nu$ satisfies $|I_\nu| \geq 1/\delta$ and (i) holds for every $1 \leq i < D$. Then applying Lemma 10 we have

$$\left|\{(x, f(x)) : x \in I_\nu\} \cap \mathbb{Z}^2\right| \ll (d\ell)^2 N^{1/d+O(1/\ell)}\delta|I_\nu|.$$

Otherwise, either $|I_\nu| \leq 1/\delta$, or there exists a (minimal) $1 \leq k < D$ such that (ii) holds for $k$, but (i) holds for all $i < k$. In this case, Lemma 13 implies $|I_\nu| \leq 2\delta^{-1}$.

Summing the contribution from each $I_\nu$ we deduce that

$$\begin{aligned}\left|\{(x, f(x)) : x \in I\} \cap \mathbb{Z}^2\right| &= \sum_{\nu \ll d^2D^2} \left|\{(x, f(x)) : x \in I_\nu\} \cap \mathbb{Z}^2\right| \\ &\ll (d\ell)^2 N^{1/d+O(1/\ell)}\delta \sum_{\nu \ll d^2D^2} |I_\nu| + d^2D^2G(2\delta^{-1}) \\ &\ll d^4\ell^2(N^{1/d+O(1/\ell)}\delta N + G(2\delta^{-1})),\end{aligned}$$

using the fact that $\sum_\nu |I_\nu| = |I| \leq N$. This gives (17) for some $K \ll d^4\ell^2 \leq \ell^{O(1)}$, after choosing $\delta = 2K^{2d}/N$. □

We are now prepared to conclude the main result. This follows from the previous theorem and some elementary algebraic geometry, coupled with the inverse function theorem, to divide the curve into $O(1)$ many pieces with flat first derivatives.

*Proof of Theorem 1.* Let $C = \{(x, y) \in \mathbb{R}^2 : F(x, y) = 0\}$. We first claim that $C \cap [0, N]^2$ has $O(d^2)$ many connected components. To show this, each connected component of $C \cap [0, N]^2$ forms either a loop, or a path from one boundary point of $[0, N]^2$ to another. In the case of a path, $C$ may intersect each of the boundary lines $[0, N]^2$ at $O(d)$ many points, by Corollary 9, and hence $O(d^2)$ many such paths. In the case of a loop, it must contain a point $(x, y)$ such that $F_x(x, y) = 0$ or $F_y(x, y) = 0$, where $F_x = \frac{\partial}{\partial x}F(x, y)$ and $F_y = \frac{\partial}{\partial y}F(x, y)$. And since both $F_x$ and $F_y$ are polynomials in $\mathbb{R}[x, y]$ of degree $\leq d - 1$, by Corollary 9 there are at most $O(d^2)$ many points $(x, y) \in C$ such that $F_x(x, y) = 0$ or $F_y(x, y) = 0$. Hence in total there are $O(d^2)$ components. We fix one of these components, and call this $C_i$.

We now claim that there are $O(d^2)$ many points in $[0, N]^2$, which we remove, and $t = O(d^2)$ many open sets $U_1, \dots, U_t$ which cover the remainder of $C_i$, such that

$$F_x(x, y) \neq 0 \quad \text{and} \quad F_y(x, y) \neq 0 \qquad \text{for all } (x, y) \in \bigcup_i U_i,$$

and for all $1 \le i \le t$ either

(i) $|F_x(x, y)| \le |F_y(x, y)|$ for all $(x, y) \in U_i$, or

(ii) $|F_y(x, y)| \ge |F_x(x, y)|$ for all $(x, y) \in U_i$.

As above there are $O(d^2)$ many points where $F_x = 0$ or $F_y = 0$, which we remove. If $F_x = \pm F_y$ then (i) and (ii) automatically hold. Otherwise, $F_x \mp F_y$ is a nonzero polynomial of degree $\le d-1$, and hence by Corollary 9 there are $O(d^2)$ many $(x, y) \in C_i$ such that $F_x(x, y) = \pm F_y(x, y)$. We can remove all such points by excising a further $O(d^2)$ many points, and then by continuity, either (i) or (ii) must hold along each segment of $C_i$ that remains after removing any of these $O(d^2)$ many points. The claim now follows, letting $U_i$ be some open narrow tube around each curve segment.

Now for each open $U_i \subseteq (0, N)^2$ as above, without loss of generality, (i) holds: $|F_x| \le |F_y|$ on $U_i$. By the implicit function theorem [16, Theorem 3.16], there is an interval $I_i \subseteq [0, N]$ and a smooth function $f_i : I_i \to \mathbb{R}$ such that

$$C \cap U_i = \{(x, f_i(x)) : x \in I_i\}$$

and

$$f_i'(x) = -\frac{F_x(x, f_i(x))}{F_y(x, f_i(x))} \quad \text{for all } x \in I_i.$$

In particular $|f_i'(x)| \le 1$ for all $x \in I_i$ by (i). Hence by Theorem 14,

$$|C \cap U_i \cap \{1, \dots, N\}^2| \ll (\log N)^{O(d)} N^{1/d}. \tag{18}$$

Since the open $U_i$ cover $C$, we conclude $|C \cap \{1, \dots, N\}^2| \ll d^2 (\log N)^{O(d)} N^{1/d}$. □

## 5. Integer points on convex graphs

In this final section we give another application of Bombieri and Pila's work [3]. This application highlights the versatility of the real-analytic determinant method; in particular this application is beyond the scope of the $p$-adic determinant method.

We say a function $f : \mathbb{R} \to \mathbb{R}$ is *strictly convex* if, between any two points on its graph, the line between those points lies strictly above the graph. That is, for any $x, y \in \mathbb{R}$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) < tf(x) + (1-t)f(y).$$

If $f$ is differentiable, this is equivalent to the derivative of $f$ being strictly increasing; if $f$ is twice differentiable, this is equivalent to $f''(x) > 0$ pointwise.

It is natural question how many integer points lie on the graph of a strictly convex function, inside $[0, N]^2$ say. The example of $f(x) = x^2$ shows that $\gg N^{1/2}$ is possible. Some experimentation may suggest that this lower bound could be sharp, but this turns out to be false. As we shall prove, Jarnik [15] constructed a strictly convex function with $\gg N^{2/3}$ many points in $[0, N]^2$, which is best possible.

**Theorem 15** (Jarnik [15]). *If $f:[0,N]\to[0,N]$ is a strictly convex function, then*

$$\left|\{(x,f(x)) : x\in[0,N]\}\cap\mathbb{Z}^2\right| \ll N^{2/3}.$$

*Moreover, for all large $N$ there exists a strictly convex function $f:[0,N]\to[0,N]$ such that*

$$\left|\{(x,f(x)) : x\in[0,N]\}\cap\mathbb{Z}^2\right| \gg N^{2/3}.$$

In particular, the implied constants are absolute and do not depend on $f$.

*Proof.* We first prove the upper bound. Suppose there are $0\le n_1<\cdots<n_t\le N$ such that $f(n_i)\in\mathbb{Z}$ for all $1\le i\le t$. Let $a_i=n_{i+1}-n_i$ and $b_i=f(n_{i+1})-f(n_i)$. Since $\sum_i a_i\le N$ there are at least $\frac{3}{4}t$ many indices $i$ such that $a_i\le 4N/t$, and similarly there are at least $\frac{3}{4}t$ many indices $i$ such that $b_i\le 4N/t$. It follows there are at least $\frac{1}{2}t$ many indices $i$ such that $\max(a_i,b_i)\le 4N/t$. We note, however, that each $n_i$ gives rise to a distinct pair $(a_i,b_i)$, since by strict convexity we have, if $n_i<n_j$,

$$\frac{b_i}{a_i}=\frac{f(n_{i+1})-f(n_i)}{n_{i+1}-n_i}<\frac{f(n_{j+1})-f(n_j)}{n_{j+1}-n_j}=\frac{b_j}{a_j}.$$

Thus $\frac{1}{2}t\le \#\{(a_i,b_i):\max(a_i,b_i)\le 4N/t\}\le(4N/t)^2$; hence $t\ll N^{2/3}$ as claimed.

We now construct the function for the lower bound. Let $H$ be some parameter to be chosen later. We consider all integer vectors $\boldsymbol{v}=(q,a)$ with $\gcd(a,q)=1$ and $1\le a,q\le H$, and order them as $\boldsymbol{v}_1,\dots,\boldsymbol{v}_t$ such that $a_i/q_i<a_{i+1}/q_{i+1}$. Let $A_i=\sum_{1\le j\le i}a_j$ and $Q_i=\sum_{1\le j\le i}q_j$. Let $\tilde{f}:[0,Q_t]\to[0,A_t]$ be the piecewise linear function connecting the points $(Q_i,A_i)$ for $0\le i\le t$. Notice that

$$Q_t=\sum_{1\le j\le t}q_j=\sum_{1\le q\le H}\sum_{\substack{1\le a\le H\\(a,q)=1}}q\le H^3,$$

(by symmetry, $A_t=Q_t\le H^3$) and the number of integer points on the graph of $\tilde{f}$ is

$$\ge t=\sum_{\substack{1\le a,q\le H\\(a,q)=1}}1\gg H^2.$$

Since the gradient of the line segments is strictly increasing (the gradient between $(Q_i,A_i)$ and $(Q_{i+1},A_{i+1})$ is precisely $a_{i+1}/q_{i+1}$) the graph of $\tilde{f}$ is strictly convex if we consider only pairs of points on different line segments. We can make the entire graph strictly convex if we replace each line segment $\tilde{f}$ by a slight curve $f_\epsilon$. To make this explicit, one would compute $\tilde{f}(x)$ by linear interpolation through the endpoints $(Q_i,A_i)$ and $(Q_{i+1},A_{i+1})$. Then for $\overline{Q}=\frac{1}{2}Q_i+Q_{i+1}$, one defines $\tilde{f}_\epsilon(x)$ by Lagrange interpolation through the endpoints with the additional point $(\overline{Q},\tilde{f}(\overline{Q})+\epsilon)$. Each parabolic segment of $\tilde{f}_\epsilon$ now lies above its secant line, hence strictly convex, for $\epsilon=\epsilon_{N,\tilde{f}}>0$ sufficiently small.

This creates the graph of a strictly convex function $f:[0,H^3]\to[0,H^3]$ with $t\gg H^2$ many integer points $(Q_i,A_i)$. Setting $H=\lfloor N^{1/3}\rfloor$ completes the proof. □

It may seem that this is the end of the story: we have lower and upper bounds of the same order of magnitude (and in fact Jarnik even gave refined bounds that match exactly up to lower-order terms). Note, however, that Jarnik's construction of a strictly convex function with many integer points was piecemeal, and in particular was not smooth. Indeed, it is not even in $C^1$. One may hope, therefore, that the upper bound for the number of integral points can be improved granted additional smoothness hypotheses.

Swinnerton-Dyer [27] showed this is indeed true, proving an upper bound of $O_f(N^{3/5+o(1)})$ provided $f \in C^3$. A uniform (with no dependence on $f$) version of this result was proved by Schmidt [24], under the additional assumption that $f^{(3)} \neq 0$ in $[0, N]$. Schmidt conjectured that the $\frac{3}{5}$ here could be improved to $\frac{1}{2}$ under the same assumptions.

In [3] Bombieri and Pila gave, as another application of their determinant method, a proof that an upper bound of the strength $N^{1/2+o(1)}$ can be achieved provided $f$ is sufficiently smooth. As above, the example $f(x) = x^2$ shows that an exponent of $\frac{1}{2}$ is the best possible here. The proof is very similar to that of Theorem 14, except that the assumption of strict convexity plays the role of Bézout's theorem, when bounding the number of integral points on the graph intersected with a line.

**Theorem 16** (Bombieri–Pila [3]). *Let $d \geq 2$ and set $D = (d+1)(d+2)/2$. Let $f : [0, N] \to [0, N]$ be a strictly convex function. If $f \in C^D([0, N])$ and $f^{(D)}(x) \neq 0$ for all $x \in [0, N]$ then*

$$\left|\{(x, f(x)) : x \in [0, N]\} \cap \mathbb{Z}^2\right| \ll_d N^{\frac{1}{2}+\frac{8}{3(d+3)}+o(1)}.$$

In particular, the implied constant depends only on $d$ but not on $f$.

*Proof.* For any real $N > 0$ define

$$G(N) := \sup_{|I| \leq N} \sup_{f} \left|\{(x, f(x)) : x \in I\} \cap \mathbb{Z}^2\right|,$$

where the second supremum is over all strictly convex functions $f : I \to [0, N]$ such that $f \in C^D(I)$. We shall prove that there exists some $K = K(d)$ satisfying $2 \leq K \leq d^{O(1)}$ such that for any $N > 0$ we have

$$G(N) \leq K^{O(d)} N^{\frac{1}{2}+\frac{8}{3(d+3)}} + KG(K^{-2d}N). \tag{19}$$

This implies the result by an identical argument to that in the proof of Theorem 14. To prove (19), as in the proof of Theorem 14, we let $\delta \geq 1/N$ be some parameter to be chosen later and want to divide $[0, N]$ into $d^{O(1)}$ many subintervals $I_\nu$ such that, for each $I_\nu$ and each $1 \leq i < D$, either

(i) $\left|\frac{f^{(i)}(x)}{i!}\right| \leq N\delta^i$ for all $x \in I_\nu$, or

(ii) $\left|\frac{f^{(i)}(x)}{i!}\right| \geq N\delta^i$ for all $x \in I_\nu$.

This time we do not have an assumption like $F(x, f) = 0$ where $F$ has bounded degree, and so Lemma 11 is not available. Instead, we note that the assumption that $f^{(D)} \neq 0$ directly implies $f^{(i)}(x) = c$ has at most $D - i$ solutions $x \in I$, for any $c \in \mathbb{R}$ and $1 \leq i < D$. This can be used in place of Lemma 11, and hence such a subdivision can be found proceeding as in the proof of Lemma 12.

The length of an interval such that (ii) holds for some $1 \leq i < D$ is, by Lemma 13, at most $2/\delta$. If (i) holds for all $1 \leq i < D$, then by Lemma 3 we can cover the integer points from $I_\nu$ by

$$\ll_d \delta |I_\nu| N^{\frac{2dD}{3\binom{D}{2}}} + 1 = \delta |I_\nu| N^{\frac{8}{3(d+3)}} + 1$$

many integral curves of degree $\leq d$, where we choose $\mathcal{M}$ to be the set of all monomials of degree $\leq d$ (so that $D = (d+1)(d+2)/2$ and $p = q = dD/3$). We now note that, by strict convexity, any line intersects the graph of $f(x)$ in at most two points. The number of integer points on any curve of degree $d \geq 2$ inside $[0, N]^2$ is, by Theorem 1, at most $N^{1/2+o(1)}$. It follows that

$$G(N) \leq d^{O(1)} \delta |I| N^{\frac{1}{2} + \frac{8}{3(d+3)} + o(1)} + G\left(\frac{2}{\delta}\right).$$

The conclusion follows choosing $K = d^C$ and $\delta = K^{C'd}/N$ for some constants $C, C'$. □

We close by restating the conjecture of Schmidt that asks for the same quality bound, while assuming a much weaker smoothness condition on $f$.

**Conjecture 17** (Schmidt [24]). *If $f : [0, N] \to [0, N]$ is a strictly convex function such that $f \in C^3([0, N])$ and $f^{(3)}(x) \neq 0$ for all $x \in [0, N]$ then*

$$\left|\{(x, f(x)) : x \in I\} \cap \mathbb{Z}^2\right| \ll N^{\frac{1}{2}+o(1)}.$$

While still very much open, Schmidt's conjecture was stated as a central motivation for Bombieri and Pila's work [3]. Using a strengthening of the real-analytic determinant method presented here, Pila [18] has proved that such a bound holds if $f^{(105)}$ exists and does not vanish, as well as nonvanishing determinant of a $3 \times 3$ matrix involving the first five derivatives of $f$ — this may be viewed as an 'enhanced convexity' condition.

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# Counting locally supercuspidal newforms

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The trace formula is a versatile tool for computing sums of spectral data across families of automorphic forms. Using specialized test functions, one can treat small families with refined spectral properties. This has proven fruitful in analytic applications. We detail such methodology here, with the aim of counting newforms in certain small families. The result is a general formula for the number of holomorphic newforms of weight $k$ and level $N$ whose local representation type at each $p \mid N$ is a fixed supercuspidal representation $\sigma_p$ of $\mathrm{GL}_2(\mathbb{Q}_p)$. This is given in terms of local elliptic orbital integrals attached to matrix coefficients of the $\sigma_p$. We evaluate the formula explicitly in the case where each $\sigma_p$ has conductor $\leq p^3$. The technical heart of the paper is the explicit calculation of elliptic orbital integrals attached to such $\sigma_p$. We also compute the traces of Hecke operators on the span of these newforms. Some applications are given to biases among root numbers of newforms.



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## 1. Introduction

**1.1.** ***Overview.*** Modular forms are holomorphic functions on the complex upper half-plane $\mathbb{H}$ that obey a type of symmetry under the action of $\mathrm{SL}_2(\mathbb{Z})$ (or a congruence subgroup) on $\mathbb{H}$ by linear fractional transformations. They belong to the realm of analysis, but this symmetry embodies a deep link with number theory and algebra. Indeed, Langlands' famous functoriality conjecture predicts that there is a precise connection between the algebraic structure of the field $\mathbb{Q}$ of rational numbers (as captured by representations of its absolute Galois group) and spectral properties of automorphic forms (the latter being simultaneous eigenfunctions of the Laplace operator and its $p$-adic analogs, the Hecke operators) [13]. This connection is expressed as an equality of $L$-functions.

Automorphic forms can be elusive, and for most purposes it is not feasible to study them and their $L$-functions one at a time. The trace formula is a technique that provides access to averages of spectral data across families of forms, where the family is determined by a choice of test function. For instance, by choosing a test function with certain invariance properties, one obtains a sum of Hecke eigenvalues $\lambda_n(h)$ for all eigenforms $h$ of a given level and weight, i.e., the trace of the Hecke operator $T_n$ on $S_k(N)$ (see, for example, [23]).

The trace formula and its relative cousins have seen widespread use in analytic number theory, with applications to such problems as estimating moments of $L$-functions with consequent subconvexity bounds for a single $L$-function, determining

the asymptotic distribution of the Hecke eigenvalues of a growing family of cusp forms (vertical Sato–Tate laws), and finding densities of low-lying zeros of families $L$-functions (Katz–Sarnak philosophy). See [3] for a recent survey of these and other applications.

Our aim in the present article is to train the trace formula microscope more narrowly through the use of specialized test functions, thereby providing access to thinner families in the automorphic spectrum. This is achieved using the "simple trace formula", variants of which have been in use since the 1970s [15, (7.21)]. Our motivation (described in the next section) is to count cusp forms in these thin families. But the explicit and flexible local-to-global techniques detailed here for GL(2) can be used in many other applications.

Counterintuitively, by considering smaller families, in some situations one obtains simpler trace formulas and stronger analytic results. We mention here some examples that illustrate this. First, Hu [18] and Hu, Petrow and Young [19] have recently developed Fourier relative trace formulas for newforms with certain prescribed local representation types. This is used to estimate thin averages of Rankin–Selberg $L$-functions, leading to improved hybrid subconvexity bounds.

In a different direction, in 2007 Booker and Strömbergsson [4] used the Selberg trace formula to provide evidence for Selberg's conjecture that the first Laplace eigenvalue in the cuspidal spectrum of $\Gamma\backslash\mathbb{H}$ for a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is $\geq \frac{1}{4}$. In verifying the conjecture for $\Gamma = \Gamma_1(N)$ for square-free $N < 857$, they observed that the trace formula simplifies upon sieving out the contribution of oldforms in this case. They were also able to restrict to the even (or odd) part of the spectrum. With Lee in [5], they subsequently extended this work to remove the square-free hypothesis on $N$. However, in this case removing the oldform contribution introduces further complication. To proceed, they developed a novel method to sieve the spectrum down further to twist-minimal newforms, arriving at a simpler formula. In both papers, working with a thinner family extended the reach of their numerical computations.

A general discussion about the value of isolating small families of automorphic forms is given in [45, Section 1.5]. In the breakthrough papers [44; 45], Petrow and Young established Weyl-type subconvexity bounds for Dirichlet $L$-functions using a family of Maass forms that is locally principal series at all finite places.

**1.2.** ***Description of main results.*** Given an integer

$$N = \prod_{p \mid N} p^{N_p} > 1,$$

let $H_k(N)$ be the set of cuspidal Hecke newforms of level $N$ and weight $k$. Each $h \in H_k(N)$ corresponds to a cuspidal automorphic representation $\pi_h$ of $G(\mathbb{A}) = G(\mathbb{R}) \prod_p' G(\mathbb{Q}_p)$ where $G = \mathrm{PGL}_2$. The representation $\pi_h$ factors as a restricted

tensor product

$$\pi_h \cong \bigotimes_{p\le\infty}{}' \pi_{h,p}$$

of infinite-dimensional irreducible admissible representations of the local groups. We know that $\pi_{h,\infty} = \pi_k$ is the weight $k$ discrete series representation, that for each prime $p \nmid N$, $\pi_{h,p}$ is an unramified principal series representation with Satake parameters determined by the $p$-th Hecke eigenvalue of $h$, and that for each $p \mid N$, $\pi_{h,p}$ is ramified of conductor $p^{N_p}$ (see, for example, [12]).

There is an algorithm, due to Loeffler and Weinstein [35], to determine the isomorphism class of each ramified $\pi_{h,p}$ given $h$. Here we consider the opposite problem, namely to understand the cusp forms $h$ with prescribed local ramification behavior. To this end, we define the following spaces of newforms. For each $p \mid N$, fix an irreducible admissible representation $\sigma_p$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$ of conductor $p^{N_p}$, and let $\hat{\sigma} = (\sigma_p)_{p\mid N}$. We then let $H_k(\hat{\sigma})$ be the set of weight $k$ newforms of level $N$ having the local representation type $\sigma_p$ at each $p \mid N$:

$$H_k(\hat{\sigma}) = \{h \in H_k(N) \mid \pi_{h,p} \cong \sigma_p \text{ for all } p \mid N\}.$$

Defining

$$S_k(\hat{\sigma}) = \mathrm{Span}\, H_k(\hat{\sigma}), \qquad S_k^{\mathrm{new}}(N) = \mathrm{Span}\, H_k(N),$$

we have

$$S_k^{\mathrm{new}}(N) = \bigoplus_{\hat{\sigma}} S_k(\hat{\sigma}), \tag{1-1}$$

where $\hat{\sigma}$ ranges over all tuples as above.

The dimensions of the spaces $S_k^{\mathrm{new}}(N)$ have been computed by Greg Martin in [37], by sieving the well-known dimension formulas for the full spaces $S_k(N)$. It is an open problem to refine these dimension formulas by computing $\dim S_k(\hat{\sigma}) = |H_k(\hat{\sigma})|$ for each tuple $\hat{\sigma}$. More generally one can ask for the traces of Hecke operators on $S_k(\hat{\sigma})$. A complete solution to this problem seems well out of reach, but even special cases are of great interest. For example, such information would enable investigations into the effect of the underlying representation type on various statistical properties of cusp forms.

In some special cases, asymptotic results about $|H_k(\hat{\sigma})|$ are known. When $p$ is a finite prime, the representation $\sigma_p$ of $G(\mathbb{Q}_p)$ is either principal series, special, or supercuspidal [7, Section 9.11]. Only the latter two types are square-integrable (assuming unitary central character), and these are amenable to study via the trace formula. Kim, Shin and Templier [21] gave asymptotics for automorphic representations with prescribed supercuspidal local behavior in a quite general setting. In the case of $\mathrm{PGL}_2(\mathbb{Q})$, their work shows that if each $\sigma_p$ is supercuspidal,

$$|H_k(\hat{\sigma})| \sim \tfrac{1}{12}(k-1)\prod_{p\mid N} d_{\sigma_p} \tag{1-2}$$

as $k, N \to \infty$, where $d_{\sigma_p}$ is the formal degree of $\sigma_p$, suitably normalized. They use the trace formula, and the main technical input is a bound for the elliptic orbital integrals attached to supercuspidal matrix coefficients. In a related earlier work, Weinstein [56] gave asymptotics for cusp forms with prescribed local inertial types, concluding that the set of types lacking global realization is finite. Fixing inertial type is weaker than fixing the local representation, but this result includes types which are not square-integrable. This is discussed further in a recent paper of Dieulefait, Pacetti and Tsaknias [10].

We remark that in Corollary 7.2 we will show that the asymptotic (1-2) is in fact an equality when $k \geq 3$ is odd (so in particular the nebentypus is nontrivial) and $N$ has a prime factor $p > 3$ with $\mathrm{ord}_p(N)$ odd.

When $N$ is square-free, each $\sigma_p$ is necessarily special. Going beyond asymptotics, Kimball Martin [38] computed $|H_k(\hat{\sigma})|$ explicitly in this case, by applying Yamauchi's trace formula for Atkin–Lehner operators. As an interesting consequence, he discovered that there is a bias among newforms of square-free level, favoring root number $+1$: letting $S_k^{\pm}(N)$ denote the span of the newforms of root number $\pm 1$, we have

$$\dim S_k^+(N) - \dim S_k^-(N) \geq 0,$$

when $N$ is square-free, with the inequality being strict with finitely many explicit exceptions. For example, if $N > 3$ and $k > 2$,

$$\dim S_k^+(N) - \dim S_k^-(N) = c_N h(-N), \tag{1-3}$$

where $c_N \in \left\{\frac{1}{2}, 1, 2\right\}$ is a constant depending on the equivalence class of $N$ modulo 8, and $h(-N)$ is the class number of $\mathbb{Q}(\sqrt{-N})$.

In the present paper, we further investigate the case where each $\sigma_p$ is supercuspidal. Our first main result is Theorem 4.2 giving, for such tuples $\hat{\sigma}$, a general formula for the trace of a Hecke operator $T_{\mathfrak{n}}$ on $S_k(\hat{\sigma})$ as a main term plus a finite sum of elliptic orbital integrals $\Phi(\gamma, f)$. This theorem is obtained from the adelic $\mathrm{GL}_2$ trace formula using a test function $f$ built using supercuspidal matrix coefficients at the ramified places. In Section 3.3 we show how each global elliptic orbital integral can be factorized into a product of local ones, multiplied by a global measure term that is computed in Theorem 4.16. This global measure is the source of the class numbers of quadratic number fields that appear in classical trace formulas. The local orbital integrals at primes not dividing the level are evaluated explicitly over an arbitrary local field of characteristic 0 in Sections 4.4 and 4.5. We have kept these calculations as general as possible in order that they may find use in other applications of the trace formula.

Theorem 4.2 thereby reduces explicit evaluation of $\mathrm{tr}(T_{\mathfrak{n}} \,|\, S_k(\hat{\sigma}))$ to the calculation of certain local elliptic orbital integrals at the places dividing the level. We

demonstrate proof of concept in Sections 5 and 6 by carrying out the latter in the special case where each $\sigma_p$ has conductor $\le p^3$. As recalled in Section 3.1, the supercuspidals come in two series: the unramified supercuspidals, of conductor $p^{2r}$, and the ramified supercuspidals, of conductor $p^{2r+1}$. We thus treat the first ($r = 1$) family in each series. The result is the following explicit formula for $\mathrm{tr}(T_{\mathfrak{n}} \,|\, S_k(\hat{\sigma}))$ under this restriction. We allow nontrivial nebentypus, which requires the tuple $\hat{\sigma}$ to satisfy a global central character constraint described in Section 5.3. Of course, when $\dim S_k(\hat{\sigma}) = 1$ as sometimes happens when $k$ and $N$ are small, it provides a direct way to compute the Fourier coefficients of the associated newform.

**Theorem 1.1.** *Let $N = S^2T^3 > 1$ for $S, T$ relatively prime and square-free, and let $\omega'$ be a Dirichlet character of level $N$ and conductor dividing $ST$. Let $\hat{\sigma} = (\sigma_p)_{p|N}$ be a tuple of supercuspidal representations, with $\sigma_p$ of conductor $p^2$ (resp. $p^3$) if $p \,|\, S$ (resp. $p \,|\, T$), chosen compatibly with $\omega'$ as in Section 5.3. For $k > 2$ satisfying $\omega'(-1) = (-1)^k$, let $S_k(\hat{\sigma}) \subseteq S_k^{\mathrm{new}}(N, \omega')$ be the associated space of newforms. Then for $(\mathfrak{n}, N) = 1$ and $T_{\mathfrak{n}}$ the usual Hecke operator defined in Section 4.1,*

$$\begin{aligned}&\mathrm{tr}(T_{\mathfrak{n}} \,|\, S_k(\hat{\sigma}))\\ &= \mathfrak{n}^{(k/2)-1}\Bigg[\overline{\omega'(\sqrt{\mathfrak{n}})}\,\tfrac{1}{12}(k-1)\prod_{p|S}(p-1)\prod_{p|T}\tfrac{1}{2}(p^2-1)\\ &\qquad + \frac{1}{2}\sum_{M|T}\Phi\left(\begin{pmatrix} & -\mathfrak{n}M \\ 1 & \end{pmatrix}\right) + \sum_{M|T}\sum_{1\le r<\sqrt{4\mathfrak{n}/M}}\Phi\left(\begin{pmatrix} 0 & -\mathfrak{n}M \\ 1 & rM \end{pmatrix}\right)\Bigg],\end{aligned}$$

*where $\omega'(\sqrt{\mathfrak{n}})$ is taken to be 0 if $\mathfrak{n}$ is not a perfect square. Each orbital integral $\Phi(\gamma)$ as above may be evaluated explicitly using*

$$\Phi(\gamma) = \frac{2h(E)}{w_E\, 2^{\omega(d_E)}}\Phi_\infty(\gamma)\prod_{p|N}\Phi_p(\gamma)\prod_{\ell\,|\,\Delta_\gamma,\,\ell\nmid N}\Phi_\ell(\gamma). \tag{1-4}$$

*Here, $\ell$ and $p$ denote prime numbers, $\Delta_\gamma$ is the discriminant of the characteristic polynomial of $\gamma$, $E = \mathbb{Q}[\gamma]$ is an imaginary quadratic field with class number $h(E)$, discriminant $d_E$ (with $\omega(d_E)$ distinct prime factors) and $w_E$ roots of unity. Given*

$$\gamma = \begin{pmatrix} 0 & -\mathfrak{n}M \\ 1 & rM \end{pmatrix}$$

*for $0 \le r < \sqrt{4\mathfrak{n}/M}$, the factors in (1-4) are given explicitly as follows.*

*Taking $\theta_\gamma = \arctan(\sqrt{|\Delta_\gamma|}/rM)$ $\big($interpreted as $\frac{\pi}{2}$ if $r = 0\big)$,*

$$\Phi_\infty(\gamma) = -\frac{\sin((k-1)\theta_\gamma)}{\sin(\theta_\gamma)}$$

(as in Proposition 4.3).

*Suppose $\ell \mid \Delta_\gamma$ and $\ell \nmid N$. Then if $\gamma$ is hyperbolic in $G(\mathbb{Q}_\ell)$,*

$$\Phi_\ell(\gamma) = |\Delta_\gamma|_\ell^{-1/2}$$

(as in Proposition 4.4). *If $\gamma$ is elliptic in $G(\mathbb{Q}_\ell)$, then* (as in Proposition 4.8 and (4-20), (4-21))

$$\Phi_\ell(\gamma) = e_\gamma(\ell) \sum_{j=0}^{\operatorname{ord}_\ell(b)} \ell^j \left(1 + \frac{2 - e_\gamma(\ell)}{\ell} \delta_{j>0}\right),$$

*where $\delta_{j>0}$ is an indicator function, $e_\gamma(\ell) \in \{1, 2\}$ is 2 if and only if $\ell$ ramifies in $E$, and $b$ is defined by $\Delta_\gamma = b^2 d_E$ for $d_E$ the discriminant of $E$.*

*Suppose $p \mid N$. If $\gamma$ is hyperbolic in $G(\mathbb{Q}_p)$, then $\Phi_p(\gamma) = 0$. So we will assume that $\gamma$ is elliptic in $G(\mathbb{Q}_p)$. We consider the three cases $p \mid M$, $p \mid (T/M)$, and $p \mid S$ separately. If $p \mid M$, then $\Phi_p(\gamma) = 0$ unless there exists $y$ such that $y^2 \equiv -pt_p/\mathtt{n}M \bmod p$, where $t_p$ is the parameter of the fixed supercuspidal representation $\sigma_p = \sigma_{t_p}^{\zeta_p}$ of conductor $p^3$* (see Section 5.2). *In this case,*

$$\Phi_p\left(\begin{pmatrix} 0 & -\mathtt{n}M \\ 1 & rM \end{pmatrix}\right) = \overline{\zeta_p}\left[e\left(-\frac{yrM}{p^2}\right)\omega_p(y) + \delta(p \neq 2)\, e\left(\frac{yrM}{p^2}\right)\omega_p(-y)\right]$$

(as in Proposition 6.4), *where $\zeta_p$ and $\omega_p$ are the root number and central character of $\sigma_p$ respectively, $e(x) = e^{2\pi i x}$ and $\delta$ is an indicator function.*

*If $p \mid (T/M)$, then $\Phi_p(\gamma) = 0$ unless the characteristic polynomial $P_\gamma$ of $\gamma$ has a nonzero double root modulo $p$, say*

$$P_\gamma(X) \equiv (X - z)^2 \bmod p \quad \text{for some } z \in (\mathbb{Z}/p\mathbb{Z})^*. \tag{1-5}$$

*Under this condition, we have* (as in Proposition 6.5 and its remarks)

$$\Phi_p(\gamma) = \frac{\overline{\omega_p(z)}}{p} \sum_{n=1}^{\operatorname{ord}_p(\Delta_\gamma)-1} \sum_{c \bmod p} \mathcal{N}_\gamma(c, n) \sum_{y=1}^{p-1} e\left(\frac{yc}{zp}\right) e\left(-\frac{t_p}{yzp}\right)^{\delta(n=1)},$$

*where $t_p \in (\mathbb{Z}/p\mathbb{Z})^*$ is the parameter of $\sigma_p = \sigma_{t_p}^{\zeta_p}$, $\omega_p$ is its central character, $e(x) = e^{2\pi i x}$, and*

$$\mathcal{N}_\gamma(c, n) = \#\{b \bmod p^{n+1} \mid P_\gamma(b) \equiv cp^n \bmod p^{n+1}\}.$$

*Finally, suppose $p \mid S$. If (1-5) is satisfied, then* (as in Proposition 6.8),

$$\Phi_p(\gamma) = -\overline{\omega_p(z)} + \frac{\overline{\omega_p(z)}}{p} \sum_{n=1}^{\operatorname{ord}_p(\Delta_\gamma)-1} \left[(p-1)\,\mathcal{N}_\gamma(0, n) - \sum_{c=1}^{p-1} \mathcal{N}_\gamma(c, n)\right]$$

*for $\mathcal{N}_\gamma(c, n)$ as above. On the other hand, if $P_\gamma$ is irreducible modulo $p$, then*

$$\Phi_p(\gamma) = -\overline{\nu(\gamma)} - \overline{\nu^p(\gamma)},$$

*where $\nu$ is the primitive character of $\mathbb{F}_{p^2}^*$ attached to the fixed supercuspidal $\sigma_p$ of conductor $p^2$* (see Section 5.1), *$\omega_p = \nu|_{\mathbb{F}_p^*}$, and we interpret the above to mean $-\overline{\nu(x)} - \overline{\nu^p(x)}$ if $x \in \mathbb{F}_{p^2}^*$ has the same minimum polynomial over $\mathbb{F}_p$ as the reduction of $\gamma$ mod $p$.*

**Remarks.** (1) What we call $S_k(N, \omega')$ is usually called $S_k(N, \omega'^{-1})$. See the beginning of Section 4 for explanation. The reason we assume that the conductor of $\omega'$ divides $ST$ is that this is necessary for the existence of tuples $\hat{\sigma}$ given the conductor hypotheses, by [55, Proposition 3.4].

(2) The theorem contains various simple conditions under which an orbital integral as in (1-4) vanishes. These are summarized and established in Proposition 5.6.

(3) Analytic applications often require uniform bounds for the orbital integrals appearing on the geometric side. Such bounds were established in a much more general context by Kim, Shin and Templier [21, (1.5), (1.6), (1.8)]. Using these, they proved a vertical (fixed $p$) equidistribution result for $p$-th Hecke eigenvalues in $S_k(\hat{\sigma})$ as $N \to \infty$, refining the result of Serre [51]. Their paper includes several helpful examples to explain their results in the setting of PGL(2). The explicit formulas for local orbital integrals developed in the present paper illustrate their bounds; see the remarks after Proposition 6.5, for example.

(4) Although we describe some interesting consequences of Theorem 1.1 below in Section 1.3, perhaps the main utility of this article is the methodology leading to the theorem, rather than this particular trace formula. Indeed, there are any number of variants that one could pursue simply by doing some additional local computations and updating the set of relevant global $\gamma$'s on the geometric side:

- One could capture newforms with prescribed representation type at some places, and, less restrictively, prescribed local conductor at some other places. For the latter places, the local elliptic orbital integral calculation is carried out in [28].
- We have excluded the case where $\mathrm{ord}_p(N) = 1$ at a prime $p$ for the same reason that we impose $k > 2$: the matrix coefficients of the local representations in such cases are square-integrable but not integrable [23, Proposition 14.3; 53]. So these functions cannot be used directly in the trace formula. One could incorporate these representation types either by using pseudocoefficients [21, Example 6.6; 30; 43], or, via the Jacquet–Langlands correspondence, by computing the corresponding local orbital integrals on a quaternion algebra [22].
- One could capture Maass newforms with prescribed local behavior by taking the archimedean component $f_\infty$ of the test function to be bi-SO(2)-invariant, as described, for example, in [26, Chapters 3 and 4]. In this case, the inclusion of a supercuspidal matrix coefficient at some place $p$ will annihilate the continuous and residual spectra, but at least two such places would be needed in order to annihilate

the hyperbolic and unipotent terms on the geometric side of the trace formula, as explained in Theorem 3.3 below. Further, in this case $\gamma$ need no longer be elliptic in $G(\mathbb{R})$ in order to contribute nontrivially, so there are more relevant $\gamma$'s that would have to be considered.

- The nonarchimedean local calculations in the present paper are all carried out over arbitrary $p$-adic fields, so with some additional global considerations one could work over a number field.

The technical heart of the paper is Section 6, in which we calculate local elliptic orbital integrals attached to the supercuspidal representations of conductor $\leq p^3$. Character values of supercuspidal representations on various groups appear in many places, but the orbital integral calculations in Section 6 are new. Some related calculations were made by Palm in his doctoral thesis [43, Section 9.11]. Although there are some errors in that work, the methods have been adapted for our computations.

In Section 7 we illustrate Theorem 1.1 by computing dimension formulas and some examples of $\mathrm{tr}(T_{\mathfrak{n}} \mid S_k(\hat{\sigma}))$ for $\mathfrak{n} > 1$.

**1.3.** ***Dimension formulas and root number bias.*** Upon taking $\mathfrak{n} = 1$ in Theorem 4.2, we obtain a general formula for $\dim S_k(\hat{\sigma})$, given in Theorem 7.1. As shown there, the list of relevant $\gamma$ can be narrowed considerably when $\mathfrak{n} = 1$; only $M = T, \frac{T}{2}$ contribute to the formula when $T > 3$. We will state some special cases below, but first we provide some additional motivation.

Simple supercuspidals are the representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ with conductor $p^3$. Assuming trivial central character, they can be parametrized by the pairs $(t, \zeta)$ where $t \in (\mathbb{Z}/p\mathbb{Z})^*$ and $\zeta \in \{\pm 1\}$. There are thus $2(p-1)$ such representations, denoted $\sigma_t^{\zeta}$, and each is constructed in the same way via compact induction from a character $\chi_{t,\zeta}$ of a certain open compact-mod-center subgroup $H_t'$ of $\mathrm{GL}_2(\mathbb{Q}_p)$.

An interesting question is whether each member of such a local family has the same global multiplicity, in the following sense. For $T > 1$ square-free, consider $N = T^3$ in (1-1), with $\hat{\sigma}$ running over all tuples $(\sigma_{t_p}^{\zeta_p})_{p \mid T}$. (We assume trivial central character for the moment, though the general case is considered in the main body of this paper.) In this case we have the dimension formula

$$\dim S_k^{\mathrm{new}}(T^3) = \tfrac{1}{12}(k-1) \prod_{p \mid T} (p^2 - 1)(p - 1) \tag{1-6}$$

as in [37]. Since there is no immediately apparent reason for nature favoring one simple supercuspidal over another, one might surmise that the subspaces $S_k(\hat{\sigma})$ all have the same dimension, i.e., that the asymptotic (1-2), which in the present situation becomes

$$\dim S_k(\hat{\sigma}) \sim \tfrac{1}{12}(k-1) \prod_{p \mid T} \tfrac{1}{2}(p^2 - 1), \tag{1-7}$$

is an equality. (Note that the right-hand side of (1-7) results from dividing (1-6) by the number $2(p-1)$ of simple supercuspidals at each place $p \mid T$.) This would be consistent with a 2011 calculation of Gross [17, p. 1255], who fixed the tuple of parameters $(t_p)_{p \mid N}$ and allowed the $\zeta_p$ parameters to vary. Using the trace formula he showed

$$\sum_{(\zeta_p)_{p \mid T}} \dim S_k((\sigma_{t_p}^{\zeta_p})_{p \mid T}) = \tfrac{1}{12}(k-1) \prod_{p \mid T} (p^2 - 1), \tag{1-8}$$

which is what one would expect, upon dividing (1-6) by the number of tuples $(t_p)_{p \mid T}$.

However, equation (1-7) is *not* in fact an equality in general, for the simple reason that, as we spell out at (5-22), the right-hand side of (1-7) fails to be an integer for infinitely many values of $T$. This is manifested in recent work of Pi and Qi [46], who considered a sum different from that treated by Gross, namely, varying the $t_p$ and $\zeta_p$ parameters subject to the constraint $(-1)^{k/2} \prod_{p \mid T} \zeta_p = \epsilon$ for fixed $\epsilon \in \{\pm 1\}$. This amounts to counting the newforms with root number $\epsilon$. They found, for $k \geq 4$ even and square-free $T > 3$, that

$$\dim S_k^{\mathrm{new}}(T^3)^{\pm} = \tfrac{1}{24}(k-1) \prod_{p \mid T} (p^2 - 1)(p-1) \pm \tfrac{1}{2} c_T \, \varphi(T) h(-T), \tag{1-9}$$

where $c_T$ and $h$ are as in (1-3) and $\varphi$ is Euler's $\varphi$-function. This shows that, just as in the case of square-free level, there is a bias in favor of positive root number. Instead of the Arthur–Selberg trace formula, they used a Petersson formula obtained using the simple supercuspidal new vector matrix coefficient from [27].

By evaluating the $S = \mathtt{n} = 1$ case of Theorem 1.1, in Section 7.4 we obtain an explicit formula for $\dim S_k(\hat{\sigma})$ that refines each of the above results. For example, we have the following.

**Theorem 1.2.** *Let $N = T^3$ for $T > 3$ odd and square-free, let $k > 2$ be even, and let $\hat{\sigma} = (\sigma_{t_p}^{\zeta_p})_{p \mid N}$ be a tuple of simple supercuspidal representations with trivial central characters. Then*

$$\dim S_k(\hat{\sigma}) = \tfrac{1}{12}(k-1) \prod_{p \mid N} \tfrac{1}{2}(p^2 - 1) + \Delta(\hat{t}) \, \epsilon(k, \hat{\zeta}) \, b_T \, h(-T), \tag{1-10}$$

*where $\Delta(\hat{t}) \in \{0, 1\}$ is nonzero if and only if $-pt_p/T$ is a square modulo $p$ for each $p \mid T$, $\epsilon(k, \hat{\zeta}) = (-1)^{k/2} \prod_{p \mid N} \zeta_p$ is the common global root number of the newforms comprising $H_k(\hat{\sigma})$, $b_T$ is a certain power of 2 depending on $T \bmod 8$, and $h(-T)$ is the class number of $\mathbb{Q}[\sqrt{-T}]$.*

This is a special case of Theorem 7.17, which also allows for $T$ even. The presence of $\Delta(\hat{t})$ demonstrates that the dimension is not simply a function of the weight, level and root number (even when the right-hand side of (1-7) is an integer). Indeed, as described in [6] for example, each $\sigma_p$ has attached a ramified quadratic extension of $\mathbb{Q}_p$, namely $E_{\sigma_p} = \mathbb{Q}_p(\sqrt{t_p p})$, which depends only on the Legendre

symbol $(t_p/p)$. So $\dim S_k(\hat{\sigma})$ depends only on $T, k$, the fields $E_{\sigma_p}$, and the global root number. (If $T$ is even, the dimension also depends on the local root number $\zeta_2$.)

The second term in (1-10) comes from an elliptic orbital integral. These do not appear in (1-8), but combine to form the error term in (1-9). Indeed, the local root number already appears as a coefficient in our local test function at the places dividing $T$, so the global root number naturally appears in the elliptic orbital integral that yields the error term in (1-10). This helps explain the positive bias of the root number in this situation.

At the end of Section 7.4, we indicate how our results recover the dimension formula (1-6) and the root number bias (1-9). In Theorem 7.16 we find that the root numbers of newforms of level 27 have a strict bias toward $-1$ (among the possibilities $\pm 1, \pm i$) when $k \equiv 5 \bmod 6$ and the nebentypus has conductor 3.

In a more recent paper, K. Martin [39] addressed the question of root number bias in $S_k^{\mathrm{new}}(N)$ for arbitrary levels. He showed that there is a bias towards root number $+1$ with one exception: when $N = S^2$ for a square-free number $S$ and $(-1)^{k/2} = -\prod_{p|S}(-1)$, then the root number has a strict negative bias when $k$ is sufficiently large. In discussing why the exceptions arise, he noted that the picture is obscured by the existence of newforms of level $S^2$ which are twists of forms of lower level. (No such forms exist in the $N = T^3$ case discussed above.)

Theorem 1.1 allows us to investigate this further, since the subspace $S_k^{\mathrm{min}}(S^2) \subseteq S_k^{\mathrm{new}}(S^2)$ spanned by the newforms which are not twists of newforms of lower level is the direct sum

$$S_k^{\mathrm{min}}(S^2) = \bigoplus_{\hat{\sigma}} S_k(\hat{\sigma}),$$

ranging over all $\hat{\sigma} = (\sigma_p)_{p|S}$ with each $\sigma_p$ a supercuspidal representation of conductor $p^2$ (a "depth zero" supercuspidal) and trivial central character.

In fact, even without using a specialized trace formula, we can infer the existence of negative bias for the root numbers in $S_k^{\mathrm{min}}(S^2)$ for many pairs $(S, k)$ by the following heuristic coming from finite fields (see Section 5.1 for more detail and a summary of the construction of depth zero supercuspidals). Given an odd prime $p$, there are $p-1$ primitive characters of $\mathbb{F}_{p^2}^*$ with trivial restriction to $\mathbb{F}_p^*$. It follows that the number of $\sigma_p$ as above is $\frac{1}{2}(p-1)$. If $p \equiv 3 \bmod 4$, this number is odd, so the set of such $\sigma_p$ contains a preponderance either of local root number $\epsilon_p = +1$ or $\epsilon_p = -1$. So if $S$ is a product of such primes, then for some integer $c \geq 1$ there are $c$ more tuples $\hat{\sigma}$ with one nonarchimedean sign $\epsilon_{\mathrm{fin}} = \prod_{p|S} \epsilon_p$ than the other. By (1-2), the spaces $S_k(\hat{\sigma})$ all have roughly the same dimension $\frac{1}{12}(k-1)\prod_{p|S}(p-1)$, up to variations of lower magnitude when $k+S$ is sufficiently large. Then with $k/2$ of the appropriate parity, there is a bias towards root number $\epsilon = (-1)^{k/2}\epsilon_{\mathrm{fin}} = -1$, with roughly $c\frac{1}{12}(k-1)\prod_{p|S}(p-1)$ more forms of global root number $-1$ than $+1$. (We will show that in fact $c = 1$; see Proposition 7.6.)

To make a precise statement, we first apply Theorem 1.1 with $\mathfrak{n} = T = 1$ to obtain the following.

**Theorem 1.3.** *Let $N = S^2$ for $S > 1$ square-free, let $k > 2$ be even, and let $\hat{\sigma} = (\sigma_{\nu_p})_{p|N}$ be a tuple of depth zero supercuspidal representations with trivial central characters, with $\nu_p$ the primitive character of $\mathbb{F}_{p^2}^*$ associated to $\sigma_p$. Then*

$$\dim S_k(\hat{\sigma}) = \tfrac{1}{12}(k-1)\prod_{p|S}(p-1) + D_4(S)\,\tfrac{1}{4}\epsilon(k,\hat{\sigma}) \prod_{\text{odd } p|S} 2 + D_3(S)\,b(k)\,\tfrac{1}{3}(-1)^{\delta_{3|S}} \prod_{p|S,\,p\neq 3} B(\nu_p), \quad (1\text{-}11)$$

*where $\epsilon(k,\hat{\sigma})$ is the common global root number of the newforms in $S_k(\hat{\sigma})$, $D_4(S) \in \{0,1\}$ is 0 if and only if $S$ is divisible by a prime $p \equiv 1 \bmod 4$, $D_3(S) \in \{0,1\}$ is 0 if and only if $S$ is divisible by a prime $p \equiv 1 \bmod 3$, $\delta$ is the indicator function defined in (2-1),*

$$b(k) = \begin{cases} 1 & \text{if } 6|k, \\ -1 & \text{if } k \equiv 2 \bmod 6, \\ 0 & \text{otherwise}, \end{cases} \quad (1\text{-}12)$$

*and, for $p \equiv 2 \bmod 3$,*

$$B(\nu_p) = \begin{cases} -2 & \text{if the order of } \nu_p \text{ (in the character group of } \mathbb{F}_{p^2}^*) \text{ divides } \tfrac{1}{3}(p+1), \\ 1 & \text{otherwise}. \end{cases}$$

The above is a special case of Theorem 7.3, which allows for nontrivial nebentypus and $k$ odd. We will use Theorem 1.3 to derive an explicit formula for the bias

$$\Delta(S^2,k)^{\min} = \dim S_k^{\min}(S^2)^+ - \dim S_k^{\min}(S^2)^-$$

for $k > 2$ even and $S > 1$ square-free. This is given in Proposition 7.6. For the time being, we just state the following consequence, which is somewhat different from the behavior observed for the larger spaces of newforms of level $S^2$ appearing in [39, Theorem 1.1(3) and Proposition 1.3].

**Proposition 1.4.** *Assume $k \geq 4$ is even. With notation as above, $\Delta(S^2,k)^{\min} = 0$ in each of the following situations*: (i) $D_4(S) = D_3(S) = 0$, (ii) *$S$ is divisible by some prime $p \equiv 5 \bmod 12$*, (iii) *$D_4(S) = 0$ and $k \equiv 4 \bmod 6$.*

*If $D_4(S) = 0$, $k \equiv 0, 2 \bmod 6$, $D_3(S) \neq 0$, and case* (ii) *does not apply, then $\Delta(S^2,k)^{\min} \neq 0$ and*

$$\operatorname{sgn} \Delta(S^2,k)^{\min} = (-1)^{\delta(k \equiv 6,\,8 \bmod 12)}\mu(S)$$

*for the indicator $\delta$ as in (2-1) and the Möbius function $\mu$.*

*If $D_4(S)=1$ and $k\geq 6$, then apart from the two exceptions $S_8^{\min}(2^2)=S_6^{\min}(3^2)=0$, $\Delta(S^2,k)^{\min}\neq 0$, and*

$$\operatorname{sgn}\Delta(S^2,k)^{\min}=(-1)^{\delta(2\mid S)+k/2}.$$

*If $D_4(S)=1$ and $k=4$, then $\Delta(S^2,4)^{\min}\geq 0$ for all square-free $S>1$:*

$$\Delta(S^2,4)^{\min}=\begin{cases}\frac{1}{2}\prod_{p\mid S}(p-1) & \text{if } 2\nmid S,\\ 0 & \text{if } 2\mid S.\end{cases}$$

**Remark.** A noteworthy difference between the above and the bias for the full space of newforms is that here for any fixed even $k\geq 6$ there are infinitely many levels $S^2$ for which $\Delta(S^2,k)^{\min}<0$, whereas by [39, Theorem 1.1(3)], for any fixed even $k$ there are only finitely many levels $N$ for which $\Delta(N,k)^{\text{new}}<0$.

## 2. Notation and Haar measure

If $P$ is a statement, then we will frequently use the indicator function

$$\delta(P)=\delta_P=\begin{cases}1 & \text{if } P \text{ is true},\\ 0 & \text{if } P \text{ is false}.\end{cases} \tag{2-1}$$

We also use the shorthand

$$e(x)=e^{2\pi i x}.$$

For rings $R$, we let $R^*$ denote the group of units in $R$.

Let $G$ be the group $\mathrm{GL}(2)$, and set $\overline{G}=G/Z$, where $Z$ is the center of $G$. If $H$ is a subgroup of $G$, then $\overline{H}$ will denote the group $HZ/Z\cong H/(H\cap Z)$. For $\ell$ prime, we set $Z_\ell=Z(\mathbb{Q}_\ell)$ the center and $K_\ell=G(\mathbb{Z}_\ell)$ the maximal compact subgroup of $G(\mathbb{Q}_\ell)$. Groups $K_0(\mathfrak{p})$, $K_1(\mathfrak{p}^j)$, $K'$ will be defined in Sections 3.1 and 5.2.

Let $\mathbb{A}$ be the adele ring of the rational numbers $\mathbb{Q}$. We give $\overline{G}(\mathbb{A})$ the standard Haar measure for which

$$\operatorname{meas}(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}))=\frac{\pi}{3},$$

with the discrete group $\overline{G}(\mathbb{Q})$ receiving the counting measure. We normalize Haar measure on $\overline{G}(\mathbb{Q}_\ell)$ so that $\overline{K_\ell}$ has measure 1. With this choice, there is a unique Haar measure on $\overline{G}(\mathbb{R})$ for which the above measure on $\overline{G}(\mathbb{A})$ is the restricted product of the measures on $\overline{G}(\mathbb{Q}_\ell)$ for $\ell\leq\infty$. It has the form $dm\,dn\,dk$, where $dm$ is the measure $(dx/|x|)^2$ on the diagonal subgroup $M\cong\mathbb{R}^*\times\mathbb{R}^*$, $dn$ is the measure $dx$ on the unipotent subgroup $N\cong\mathbb{R}$, and $dk$ is the measure on $K_\infty=\mathrm{SO}(2)$ of total measure 1 [23, Corollary 7.45].

For a unitary Hecke character $\omega$, let $L^2(\omega)=L^2(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$ be the space of (classes of) measurable $\mathbb{C}$-valued functions $\phi$ on $G(\mathbb{A})$ transforming under the center by $\omega$ and square integrable modulo $Z(\mathbb{A})G(\mathbb{Q})$. Let $L^1(\bar{\omega})=L^1(G(\mathbb{A}),\bar{\omega})$ be defined in the analogous way; its elements are integrable modulo $Z(\mathbb{A})$.

## 3. The simple trace formula

**3.1.** ***Background on supercuspidal representations of*** **GL(2).** Let $F$ be a nonarchimedean local field of characteristic 0 with integer ring $\mathcal{O}$ and prime ideal $\mathfrak{p}$. In this section only, let $G = \mathrm{GL}_2(F)$, $B = B(F)$ the upper-triangular Borel subgroup, $N = N(F)$ the unipotent subgroup of $B$, $M$ the diagonal subgroup, $Z$ the center, and $K = G(\mathcal{O})$ the standard maximal compact subgroup.

Given a smooth irreducible representation $(\pi, V)$ of $G$, it is supercuspidal if it satisfies any of the following equivalent conditions (see, e.g., [7, Sections 9 and 10]):

- $V$ is the span of the vectors of the form $\pi(n)\,v - v$ for $v \in V$ and $n \in N$.
- The matrix coefficients of $\pi$ are compactly supported modulo the center.
- $\pi$ is not principal series or special, i.e., not a subquotient of a representation induced from a character of $B$.

The following property found by Harish-Chandra is crucial in what follows. We sketch a proof here for the reader's convenience, following [52, Section 2.2].

**Proposition 3.1.** *Suppose that $\pi$ is a supercuspidal representation of $G$, and that $f(g) = \langle \pi(g)\,v, v' \rangle$ is a matrix coefficient. Then for all $g, h \in G$,*

$$\int_N f(gnh)\,dn = 0. \tag{3-1}$$

*Proof.* We assume for simplicity that $\pi$ is unitary, which is always the case if the central character is unitary. Then

$$f(gnh) = \langle \pi(g)\,\pi(n)\,\pi(h)\,v, v' \rangle = \langle \pi(n)\,\pi(h)\,v, \pi(g^{-1})\,v' \rangle,$$

so we can assume without loss of generality that $g = h = 1$. By linearity and the first bullet point above, we may also assume that $v = \pi(n_0)\,w - w$ for some $w \in V$ and $n_0 \in N$.

Let $N(v)$ be an open compact subgroup of $N$ containing $n_0$. Then

$$\int_{N(v)} \pi(n)\,v\,dn = \int_{N(v)} \pi(n)(\pi(n_0)\,w - w)\,dn = \int_{N(v)} \pi(n)\,w\,dn - \int_{N(v)} \pi(n)\,w\,dn = 0.$$

(By smoothness, there exists an open compact subgroup $N'$ of $N(v)$ that fixes $w$, so the above integrals are really just finite sums.) Since $f$ has compact support, the support of $f|_N$ is contained in some open compact subgroup $N(v)$ as above. Therefore

$$\int_N f(n)\,dn = \int_{N(v)} \langle \pi(n)\,v, v' \rangle\,dn = \left\langle \int_{N(v)} \pi(n)\,v\,dn, v' \right\rangle = 0. \qquad \square$$

**Corollary 3.2.** *If $f$ is a matrix coefficient of a supercuspidal representation, then for any $g, h \in G$ and $m \in M$ ($M$ being the diagonal subgroup),*

$$\int_N f(gn^{-1}mnh)\,dn = 0.$$

*Proof.* This follows from $n^{-1}mn = mn'$, making a change of variables to integrate over $n'$, and applying the above proposition. □

For any supercuspidal representation $\sigma$ of $G$, there exists an open and closed subgroup $H \subseteq G$ containing $Z$, with $H/Z$ compact, and an irreducible representation $\rho$ of $H$, such that $\sigma$ is compactly induced from $\rho$: $\sigma = \text{c-Ind}_H^G(\rho)$. Let $K_0(\mathfrak{p}) = \left(\begin{smallmatrix} \mathcal{O}^* & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^* \end{smallmatrix}\right)$ be the Iwahori subgroup of $G$, and fix a prime element $\varpi$ of $\mathcal{O}$. Up to conjugacy, there are two maximal compact-mod-center subgroups of $G$, namely

$$J = \begin{cases} ZK & \text{(the unramified case)}, \\ ZK_0(\mathfrak{p}) \cup Z\left(\begin{smallmatrix} & 1 \\ \varpi & \end{smallmatrix}\right) K_0(\mathfrak{p}) & \text{(the ramified case)}. \end{cases} \tag{3-2}$$

The latter is the normalizer of $K_0(\mathfrak{p})$. Without loss of generality, one of these contains $H$, and we call $\sigma$ *unramified* or *ramified* accordingly.[1] There is a unique ideal $\mathfrak{p}^j$, called the *conductor* of $\sigma$, such that the space of vectors in $\sigma$ fixed by the group

$$K_1(\mathfrak{p}^j) = \begin{pmatrix} \mathcal{O}^* & \mathcal{O} \\ \mathfrak{p}^j & 1 + \mathfrak{p}^j \end{pmatrix}$$

is one-dimensional. By [55], $j \geq 2$, and as explained in [6], $j$ is even in the unramified case, and odd in the ramified case.

**3.2.** ***Simple trace formula.*** Given a unitary Hecke character $\omega$ and a function $f \in L^1(\bar{\omega})$, we define the operator $R(f)$ on $L^2(\omega)$ via

$$R(f)\,\phi(x) = \int_{\bar{G}(\mathbb{A})} f(g)\,\phi(xg)\,dg. \tag{3-3}$$

For $k > 2$, let $\mathcal{C}_k$ denote the space of all continuous factorizable functions $f = f_\infty \prod_{\ell < \infty} f_\ell$ on $G(\mathbb{A})$ which transform under the center by $\bar{\omega}$, such that $f_\ell$ is smooth and compactly supported modulo the center $Z_\ell$ for all $\ell$, there is a finite set $S$ of places of $\mathbb{Q}$ such that for all $\ell \notin S$, $f_\ell$ is supported on $Z_\ell K_\ell$ and has the value 1 on $K_\ell$, and lastly,

$$f_\infty\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \ll_k \frac{(ad-bc)^{k/2}}{(a^2+b^2+c^2+d^2+2|ad-bc|)^{k/2}}.$$

Then $\mathcal{C}_k \subseteq L^1(\bar{\omega})$, and we can consider the operators $R(f)$ for such $f$.

Recall that $\gamma \in G(\mathbb{Q})$ is *elliptic* if its characteristic polynomial is irreducible. This concept is well defined on conjugacy classes and cosets of the center. We will use the following simple trace formula.

[1] It should be borne in mind that in standard terminology, all supercuspidals are *ramified* in the sense that they have no $K$-fixed vector. We are using the word in a different sense here, reflecting the nature of the quadratic extension $E/F$ determined by $\sigma$ [6].

**Theorem 3.3.** *For $f \in \mathcal{C}_k$, suppose that for some finite place $v$ of $\mathbb{Q}$, $f_v$ is a matrix coefficient of a supercuspidal representation of $G_v = G(\mathbb{Q}_v)$, and therefore by Corollary 3.2 its local hyperbolic orbital integrals vanish identically*:

$$\int_{M_v \backslash G_v} f_v\left(g^{-1}\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right) dg = 0 \tag{3-4}$$

*for all $a \in \mathbb{Q}_v^*$, where $M_v$ is the diagonal subgroup of $G_v$. Suppose further that (3-4) is also satisfied at a* ***second*** *place $w \neq v$ (which may be archimedean). Then*

$$\operatorname{tr} R(f) = \operatorname{meas}(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}))\, f(1) + \sum_{\mathfrak{o} \text{ elliptic in } \overline{G}(\mathbb{Q})} \Phi(\mathfrak{o}, f),$$

*where, for an elliptic conjugacy class $\mathfrak{o} \subseteq \overline{G}(\mathbb{Q})$, the orbital integral is defined by*

$$\Phi(\mathfrak{o}, f) = \int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} \sum_{\gamma \in \mathfrak{o}} f(g^{-1}\gamma g)\, dg. \tag{3-5}$$

*Proof.* See [14, Proposition V.2.1 and Theorem V.3.1]. The idea is that the validity of (3-4) at two distinct places kills off the hyperbolic and (nonidentity) unipotent terms on the geometric side of the Arthur–Selberg trace formula, while the stronger condition (3-1) on $f_v$ also forces the operator $R(f)$ to have purely cuspidal image, so the continuous and residual spectral terms vanish as well. In Gelbart's exposition it is assumed that $f$ is compactly supported, but for $f \in \mathcal{C}_k$ everything still converges absolutely as shown in [23], so the same proof is valid. □

**3.3.** ***Factorization of orbital integrals.*** Here we explain how to compute elliptic orbital integrals locally. The statements and proofs in this section are applicable over an arbitrary number field, though we express everything in terms of $\mathbb{Q}$.

For $\gamma \in G(\mathbb{Q})$, let $G_\gamma$ be its centralizer. There are two related groups that will be needed. First, since $Z(\mathbb{Q}) \subseteq G_\gamma(\mathbb{Q})$, we may form the quotient, denoted $\overline{G_\gamma(\mathbb{Q})}$. Second, the centralizer of $\gamma$ (or, more accurately, of the coset $\gamma Z(\mathbb{Q})$) in $\overline{G}(\mathbb{Q})$ is denoted $\overline{G}_\gamma(\mathbb{Q})$. In general these are distinct subgroups of $\overline{G}(\mathbb{Q})$. This will be clarified in the proof of Lemma 3.4 below.

Giving the discrete group $\overline{G_\gamma(\mathbb{Q})}$ the counting measure, define

$$\Phi(\gamma, f) = \int_{\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G}(\mathbb{A})} f(g^{-1}\gamma g)\, dg.$$

For fixed measures on $\overline{G_\gamma(\mathbb{R})}$ and $\overline{G_\gamma(\mathbb{Q}_\ell)}$, we also define the local orbital integrals

$$\Phi(\gamma, f_\infty) = \int_{\overline{G_\gamma(\mathbb{R})}\backslash\overline{G}(\mathbb{R})} f_\infty(g^{-1}\gamma g)\, dg$$

and

$$\Phi(\gamma, f_\ell) = \int_{\overline{G_\gamma(\mathbb{Q}_\ell)}\backslash\overline{G}(\mathbb{Q}_\ell)} f_\ell(g^{-1}\gamma g)\, dg.$$

For compatibility, some care must be taken regarding the normalization of measures. See the statement of Proposition 3.5 below.

**Lemma 3.4.** *For an elliptic element $\gamma \in G(\mathbb{Q})$, let $\mathfrak{o}$ be its conjugacy class in $\overline{G}(\mathbb{Q})$. Then with notation as above and in (3-5),*

$$\Phi(\mathfrak{o}, f) = \begin{cases} \Phi(\gamma, f) & \text{if } \operatorname{tr}\gamma \neq 0, \\ \frac{1}{2}\Phi(\gamma, f) & \text{if } \operatorname{tr}\gamma = 0. \end{cases}$$

*Proof.* By definition,

$$\Phi(\mathfrak{o}, f) = \int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} \sum_{\delta \in \overline{G}_\gamma(\mathbb{Q})\backslash\overline{G}(\mathbb{Q})} f(g^{-1}\delta^{-1}\gamma\,\delta g)\,dg = \int_{\overline{G}_\gamma(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} f(g^{-1}\gamma g)\,dg.$$

Notice that in the definition of $\Phi(\gamma, f)$, the quotient object is $\overline{G_\gamma(\mathbb{Q})}$ rather than $\overline{G}_\gamma(\mathbb{Q})$. The former is a subgroup of the latter, and we claim that

$$[\overline{G}_\gamma(\mathbb{Q}) : \overline{G_\gamma(\mathbb{Q})}] = \begin{cases} 1 & \text{if } \operatorname{tr}\gamma \neq 0, \\ 2 & \text{if } \operatorname{tr}\gamma = 0. \end{cases}$$

The lemma follows immediately from this claim. To prove the claim, note that

$$\overline{G_\gamma(\mathbb{Q})} = \{\delta \in G(\mathbb{Q}) \mid \delta^{-1}\gamma\,\delta = \gamma\}/Z(\mathbb{Q})$$

and

$$\overline{G}_\gamma(\mathbb{Q}) = \{\delta \in G(\mathbb{Q}) \mid \delta^{-1}\gamma\,\delta = z\gamma \text{ for some } z \in \mathbb{Q}^*\}/Z(\mathbb{Q}).$$

For any such $z$, taking determinants we see that $z^2 = 1$, so $z = \pm 1$. We also see that $\operatorname{tr}\gamma = z\operatorname{tr}\gamma$, so $z = 1$ if $\operatorname{tr}\gamma \neq 0$, and in this case the two groups are equal, as claimed. On the other hand, if $\operatorname{tr}\gamma = 0$, then $\gamma$ is conjugate in $G(\mathbb{Q})$ to its rational canonical form $\left(\begin{smallmatrix} 0 & -\det\gamma \\ 1 & 0 \end{smallmatrix}\right)$, and

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} 0 & -\det\gamma \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} 0 & \det\gamma \\ -1 & 0 \end{pmatrix},$$

from which it follows that $\delta^{-1}\gamma\,\delta = -\gamma$ has a solution $\delta$. Given any such $\delta$, we find easily that

$$\overline{G}_\gamma(\mathbb{Q}) = \overline{G_\gamma(\mathbb{Q})} \cup \delta\overline{G_\gamma(\mathbb{Q})}. \qquad \square$$

**Proposition 3.5.** *Let $f \in \mathcal{C}_k$ as defined in Section 3.2, and let $\gamma \in G(\mathbb{Q})$ be an elliptic element. Then for any fixed choice of Haar measures on $\overline{G}(\mathbb{A})$ and $\overline{G_\gamma(\mathbb{A})}$,*

$$\Phi(\gamma, f) = \operatorname{meas}(\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G_\gamma(\mathbb{A})}) \prod_{\ell \leq \infty} \Phi(\gamma, f_\ell), \tag{3-6}$$

*where the measures on the groups $\overline{G}(\mathbb{Q}_\ell)$ are chosen (noncanonically) so that the measure on $\overline{G}(\mathbb{A})$ is the restricted product of these local measures relative to the maximal compact subgroups almost everywhere, and likewise the measures on the groups $\overline{G_\gamma(\mathbb{Q}_\ell)}$ are chosen compatibly with the fixed measure on $\overline{G_\gamma(\mathbb{A})} = \prod'_{\ell \leq \infty} \overline{G_\gamma(\mathbb{Q}_\ell)}$.*

**Remarks.** (1) This is well known, but as we have not found a proof in the literature, we include one below. Tate's thesis shows that if the product is absolutely convergent, then the left-hand integral converges absolutely and the equality holds. But here we need a kind of converse: we know a priori that $\Phi(\gamma, f)$ is absolutely convergent.

(2) The specific choice of measures to be used in this paper is summarized in Section 4.7.3, where it is shown that the quotient space $\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G_\gamma(\mathbb{A})}$ is compact, and its measure is computed explicitly in the more general setting with $\mathbb{Q}$ replaced by an arbitrary number field.

*Proof.* Observe that

$$\begin{aligned}\Phi(\gamma, f) &= \int_{\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G}(\mathbb{A})} f(g^{-1}\gamma g)\,dg \\ &= \operatorname{meas}(\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G_\gamma(\mathbb{A})}) \int_{\overline{G_\gamma(\mathbb{A})}\backslash\overline{G}(\mathbb{A})} f(g^{-1}\gamma g)\,dg\end{aligned}$$

for any choice of Haar measure on $\overline{G_\gamma(\mathbb{A})}$. Absolute convergence is proven for $f \in \mathcal{C}_k$ in [23, Corollary 19.3].

For notational convenience, write $\phi(g) = f(g^{-1}\gamma g)$ (a function on $\overline{G}(\mathbb{A})$), and $\phi_\ell(g_\ell) = f_\ell(g_\ell^{-1}\gamma g_\ell)$ for $\ell \le \infty$, so $\phi(g) = \prod_{\ell\le\infty}\phi_\ell(g_\ell)$. Also, define

$$X = \overline{G_\gamma(\mathbb{A})}\backslash\overline{G}(\mathbb{A}).$$

Then $X$ is the restricted product of the spaces

$$X_\ell = \overline{G_\gamma(\mathbb{Q}_\ell)}\backslash\overline{G}(\mathbb{Q}_\ell),$$

relative to the open compact subsets $H_\ell = \overline{G_\gamma(\mathbb{Q}_\ell)}\backslash\overline{G_\gamma(\mathbb{Q}_\ell)K_\ell} \subseteq X_\ell$. Indeed, the natural map from $\overline{G}(\mathbb{A})$ to $\prod' X_\ell$ is clearly surjective, with kernel $\overline{G_\gamma(\mathbb{A})}$.

Fix Haar measures on each of the local groups $\overline{G}(\mathbb{Q}_\ell)$ and $\overline{G_\gamma(\mathbb{Q}_\ell)}$ compatibly with the fixed Haar measures on $\overline{G}(\mathbb{A})$ and $\overline{G_\gamma(\mathbb{A})}$. This determines a right-$\overline{G}(\mathbb{Q}_\ell)$-invariant measure on $X_\ell$ with the property that $H_\ell$ has measure 1 for almost all $\ell$. Let $S$ be the finite set of places of $\mathbb{Q}$ outside of which $f_\ell$ is supported on $Z(\mathbb{Q}_\ell)K_\ell$ with $f_\ell(zk) = \bar{\omega}(z)$. Let $S'$ be a finite set of places outside of which (1) $\gamma \in K_\ell$, and (2) $H_\ell$ has measure 1. Then setting $S_0 = S \cup S'$, for $\ell \notin S_0$ we have

$$\int_{H_\ell} \phi_\ell(h)\,dh = \int_{H_\ell} f_\ell(k^{-1}\gamma k)\,dk = \operatorname{meas}(H_\ell) = 1.$$

Let

$$S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots$$

be a sequence of finite sets of primes (including $\infty$) whose union is the full set of primes. Let $\chi_n$ be the characteristic function of $X_{S_n} = \prod_{\ell\in S_n} X_\ell \times \prod_{\ell\notin S_n} H_\ell$, and

let $\phi_n = \phi \cdot \chi_n$. Note that $\phi_n \to \phi$ pointwise. Since $\phi \in L^1(X)$ as mentioned above, so is $\phi_n$. By the dominated convergence theorem,

$$\int_X \phi(x)\,dx = \lim_{n\to\infty} \int_X \phi_n(x)\,dx = \lim_{n\to\infty} \prod_{\ell\in S_n} \int_{X_\ell} \phi_\ell(x_\ell)\,dx_\ell,$$

as needed. □

## 4. Counting locally supercuspidal newforms

Here we explain how to use the simple trace formula to count cusp forms with prescribed supercuspidal ramification. To set notation, let $N = \prod_{p\,|\,N} p^{N_p} > 1$ be a positive integer with the property that $N_p \geq 2$ for each prime $p\,|\,N$. Fix a Dirichlet character $\omega'$ modulo $N$ of conductor dividing $\prod_{p\,|\,N} p^{\lfloor N_p/2\rfloor}$. This requirement comes from the fact that the central character of a supercuspidal representation of conductor $p^{N_p}$ divides $p^{\lfloor N_p/2\rfloor}$ [55, Proposition 3.4]. Let $\omega : \mathbb{A}^* \to \mathbb{C}^*$ be the finite order Hecke character associated to $\omega'$ via

$$\mathbb{A}^* = \mathbb{Q}^*(\mathbb{R}^+ \times \hat{\mathbb{Z}}^*) \to \hat{\mathbb{Z}}^*/(1 + N\hat{\mathbb{Z}}) \cong (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*, \tag{4-1}$$

where the last arrow is $\omega'$. Letting $\omega_p$ be the restriction of $\omega$ to $\mathbb{Q}_p^*$, for any prime $p\,|\,N$ we have

$$\begin{aligned}\omega_p(p) &= \omega(1, \dots, 1, p, 1, \dots)\\ &= \omega(p^{-1}, \dots, p^{-1}, 1, p^{-1}, \dots) = \prod_{\ell\,|\,N,\,\ell\neq p} \omega_\ell(p^{-1}).\end{aligned} \tag{4-2}$$

Fix an integer $k \geq 2$ satisfying

$$\omega'(-1) = (-1)^k,$$

and let $S_k(N, \omega')$ be the space of cusp forms $h$ satisfying

$$h\left(\frac{az+b}{cz+d}\right) = \omega'(d)^{-1}(cz+d)^k h(z)$$

for $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(N)$. The inverse on $\omega'(d)$ is somewhat nonstandard. It ensures that the adelic cusp form attached to $h$ has central character $\omega$ rather than $\omega^{-1}$; see, e.g., [23, Sections 12.2–12.4]. Because we mostly work in the adelic setting, it eases the notation to include the inverse in the classical setting.

For each $p\,|\,N$, fix a supercuspidal representation $\sigma_p$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ of conductor $p^{N_p}$ and central character $\omega_p$, and let $\hat{\sigma} = \{\sigma_p\}_{p\,|\,N}$. We define $H_k(\hat{\sigma})$ to be the set of newforms $h \in S_k(N, \omega')$ whose associated cuspidal representation $\pi_h$ has the local representation type $\sigma_p$ at each $p\,|\,N$. We set $S_k(\hat{\sigma}) = \mathrm{Span}\, H_k(\hat{\sigma})$. The Dirichlet character $\omega'$ is uniquely determined by the tuple $\hat{\sigma}$ via

$$\omega'(d) = \prod_{p\,|\,N} \omega_p(d) \quad ((d, N) = 1), \tag{4-3}$$

and this justifies our suppression of the central character $\omega'$ from the notation $S_k(\hat{\sigma})$. At a certain point we will use the fact that

$$\prod_{p\,|\,N} \omega_p(N) = \omega(N^{-1}, \ldots, N^{-1}, 1, \ldots, 1, N^{-1}, N^{-1}, \ldots) = \omega'(1) = 1. \quad (4\text{-}4)$$

**4.1.** ***Isolating $S_k(\hat{\sigma})$ spectrally.*** For each prime $p\,|\,N$, we can write $\sigma_p = \text{c-Ind}_{H_p}^{G_p} \rho$, where $H_p$ is contained either in $Z_p K_p$ or the normalizer of an Iwahori subgroup, as in (3-2). By [29, Proposition 2.1], there exists a unit vector $w_p$ in the space of $\sigma_p$ such that the matrix coefficient $\langle \sigma_p(g)\, w_p, w_p \rangle$ is supported in $H_p$. Fix once and for all such a vector $w_p$ for each $p\,|\,N$. Based on this choice, we define a subspace $A_k(\hat{\sigma}) \subseteq L^2(\omega)$ by

$$A_k(\hat{\sigma}) = \bigoplus_{\pi} \mathbb{C} w_\pi,$$

where $\pi$ ranges over the cuspidal automorphic representations with central character $\omega$ for which $\pi_\infty = \pi_k$, $\pi_p = \sigma_p$ for each $p\,|\,N$, and $\pi_\ell$ is unramified for all finite primes $\ell \nmid N$, and $w_\pi = \otimes w_{\pi_\ell}$ is defined by

$$w_{\pi_\ell} = \begin{cases} \text{unit lowest weight vector} & \text{if } \ell = \infty, \\ \text{unit spherical vector} & \text{if } \ell \nmid N\infty, \\ w_p \text{ (fixed above)} & \text{if } \ell = p\,|\,N. \end{cases} \quad (4\text{-}5)$$

Here, for almost all $\ell$, the spherical vector is the one predetermined by the restricted tensor product $\pi \cong \bigotimes'_{\ell \le \infty} \pi_\ell$. The space $A_k(\hat{\sigma})$ does not consist of adelic newforms in general because at places $p\,|\,N$, $w_p$ is not necessarily a new vector in the space of the local representation $\sigma_p$. Nevertheless, $A_k(\hat{\sigma})$ has the same dimension as the space of newforms $S_k(\hat{\sigma}) = \text{Span}\, H_k(\hat{\sigma})$.

Using matrix coefficients, we can define a test function $f \in L^1(\bar{\omega})$ for which $R(f)$ is the orthogonal projection of $L^2(\omega)$ onto $A_k(\hat{\sigma})$. Without much extra work, we can incorporate a Hecke operator into the test function.

Fix an integer $\mathfrak{n} > 1$ with $\gcd(\mathfrak{n}, N) = 1$, and let $T_\mathfrak{n}$ be the classical Hecke operator defined by

$$T_\mathfrak{n} h(z) = \mathfrak{n}^{k-1} \sum_{\substack{ad=\mathfrak{n} \\ a>0}} \sum_{r \bmod d} \omega'(a)^{-1} d^{-k} h\left(\frac{az+r}{d}\right) \quad (h \in S_k(N, \omega'),\ z \in \mathbb{H}).$$

When $\mathfrak{n} = 1$, $T_\mathfrak{n}$ is simply the identity operator.

The operator $T_\mathfrak{n}$ can be realized adelically. Let

$$M(\mathfrak{n})_\ell = \{g \in M_2(\mathbb{Z}_\ell) \mid \det g \in \mathfrak{n}\mathbb{Z}_\ell^*\}$$

for each prime $\ell \nmid N$. (If working over a larger number field $F$, one would take $\mathfrak{n}$ to be an ideal of the integer ring and set $M(\mathfrak{n})_v = \{g \in M_2(\mathcal{O}_v) \mid (\det g)\,\mathcal{O}_v = \mathfrak{n}\}$ for a

place $v < \infty$.) Define a function $f_\ell^{\mathfrak{n}} : G(\mathbb{Q}_\ell) \to \mathbb{C}$ by

$$f_\ell^{\mathfrak{n}}(g) = \begin{cases} \overline{\omega_\ell(z)} & \text{if } g = zm \text{ for } z \in Z_\ell,\ m \in M(\mathfrak{n})_\ell, \\ 0 & \text{if } g \notin Z_\ell M(\mathfrak{n})_\ell, \end{cases} \tag{4-6}$$

where $\omega_\ell$ is the local component of the Hecke character $\omega$. Note that $f_\ell^{\mathfrak{n}}$ is bi-$K_\ell$-invariant, and indeed when $\mathfrak{n} \in \mathbb{Z}_\ell^*$, this function is given by

$$f_\ell(g) = \begin{cases} \overline{\omega_\ell(z)} & \text{if } g = zk \in Z_\ell K_\ell, \\ 0 & \text{if } g \notin Z_\ell K_\ell. \end{cases} \tag{4-7}$$

Next, let $\pi_k$ be the discrete series representation of $\overline{G}(\mathbb{R})$ of weight $k$, and let $v$ be a lowest weight unit vector in the space of $\pi_k$. We define $f_\infty = d_k \overline{\langle \pi_k(g)\, v, v\rangle}$, where $d_k = \frac{k-1}{4\pi}$ is the formal degree of $\pi_k$. Explicitly, with Haar measure on $\overline{G}(\mathbb{R})$ normalized as in Section 2,

$$f_\infty\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \dfrac{k-1}{4\pi} \dfrac{(ad-bc)^{k/2}(2i)^k}{(-b+c+(a+d)i)^k} & \text{if } ad-bc > 0, \\ 0 & \text{otherwise} \end{cases} \tag{4-8}$$

[23, Theorem 14.5]. This function is integrable over $\overline{G}(\mathbb{R})$ exactly when $k > 2$, so the latter will be assumed throughout. It would be possible to treat the $k = 2$ case by using a pseudocoefficient of $\pi_k$, but we have not attempted to carry this out (see [43]).

At places $p \mid N$, define

$$f_p(g) = d_{\sigma_p} \overline{\langle \sigma_p(g)\, w_p, w_p\rangle}, \tag{4-9}$$

where $d_{\sigma_p}$ is the formal degree and $w_p$ is the unit vector fixed above. The formal degree depends on a choice of Haar measure on $\overline{G}(\mathbb{Q}_p)$, which we normalize as in Section 2. By our choice of $w_p$, the support of $f_p$ is contained in one of the two groups (3-2), according to whether or not $\sigma_p$ is ramified.

Finally, we define the global test function

$$f^{\mathfrak{n}} = f_\infty \prod_{p \mid N} f_p \prod_{\ell \nmid N} f_\ell^{\mathfrak{n}} \tag{4-10}$$

for $f_\infty$ of weight $k$ as in (4-8), $f_p$ as in (4-9), and $f_\ell^{\mathfrak{n}}$ as in (4-6).

**Proposition 4.1.** *With the above definition of $f^{\mathfrak{n}}$, the operator $R(f^{\mathfrak{n}})$ (defined in (3-3) taking Haar measure on $\overline{G}(\mathbb{A})$ as normalized in Section 2) factors through the orthogonal projection onto the finite dimensional subspace $A_k(\hat{\sigma})$. On this space, $R(f^{\mathfrak{n}})$ acts diagonally, with the vectors $w_\pi$ being eigenvectors. In more detail, given a newform $h \in H_k(\hat{\sigma})$ with $T_{\mathfrak{n}} h = a_{\mathfrak{n}}(h)\, h$, let $w \in A_k(\hat{\sigma})$ be the vector associated to $\pi_h$ as in (4-5). Then*

$$R(f^{\mathfrak{n}})\, w = \mathfrak{n}^{1-(k/2)} a_{\mathfrak{n}}(h)\, w.$$

*Consequently,*

$$\operatorname{tr}(T_{\mathfrak{n}} \mid S_k(\hat{\sigma})) = \mathfrak{n}^{(k/2)-1} \operatorname{tr} R(f^{\mathfrak{n}}).$$

**Remarks.** (1) The vector $w$ is defined only up to unitary scaling, but of course the eigenvalue is independent of the choice.

(2) One can also take $f_p$ to be the complex conjugate of the trace of the representation $\rho$ inducing $\sigma_p$, if normalized correctly. See Proposition 5.5 and its remark.

*Proof.* The first statement is proven in [25, Proposition 2.3], but we need to reproduce some of the argument here for the second part. Let $h \in H_k(\hat{\sigma})$, let $\pi$ be the associated cuspidal representation, and let $w = w_\pi \in A_k(\hat{\sigma})$. For each place $v \mid \infty N$, the test function $f_v$ was chosen so that

$$\pi_v(f_v)\, w_v = w_v$$

[23, Corollary 10.26]. Write

$$w = w_\infty \otimes \bigotimes_{p \mid N} w_p \otimes w' \otimes \bigotimes_{\ell \mid \mathfrak{n}} w_\ell,$$

where $w' = \otimes_{\ell \nmid N\mathfrak{n}} w_\ell$. We may likewise decompose $\pi$ as

$$\pi = \pi_\infty \otimes \bigotimes_{p \mid N} \pi_p \otimes \pi' \otimes \bigotimes_{\ell \mid \mathfrak{n}} \pi_\ell,$$

where $\pi'$ is a representation of $G' = \prod'_{p \nmid N\mathfrak{n}} G(\mathbb{Q}_p)$. Then letting $f' = \prod_{p \nmid N\mathfrak{n}} f_p$, it is elementary to show that $\pi'(f')\, w' = w'$. Therefore (by [23, Proposition 13.17])

$$\begin{aligned} R(f^{\mathfrak{n}})\, w &= \pi_\infty(f_\infty)\, w_\infty \otimes \bigotimes_{p \mid N} \pi_p(f_p)\, w_p \otimes \pi'(f')\, w' \otimes \bigotimes_{\ell \mid \mathfrak{n}} \pi_\ell(f_\ell^{\mathfrak{n}})\, w_\ell \\ &= w_\infty \otimes \bigotimes_{p \mid N} w_p \otimes w' \otimes \bigotimes_{\ell \mid \mathfrak{n}} \pi_\ell(f_\ell^{\mathfrak{n}})\, w_\ell. \end{aligned}$$

Since $w_\ell$ is an unramified unit vector in the principal series representation $\pi_\ell = \pi(\chi_1, \chi_2)$ (say), we have $\pi_\ell(f_\ell^{\mathfrak{n}})\, w_\ell = \lambda_\ell w_\ell$ for

$$\lambda_\ell = \ell^{a/2} \sum_{j=0}^{a} \chi_1(\ell)^j \chi_2(\ell)^{a-j}, \quad a = \mathrm{ord}_\ell(\mathfrak{n})$$

(see, e.g., [24, Proposition 4.4]). Thus $R(f^{\mathfrak{n}})\, w = \lambda w$, where $\lambda = \prod_{\ell \mid \mathfrak{n}} \lambda_\ell$. The result now follows by the well-known fact that $\prod_{\ell \mid \mathfrak{n}} \lambda_\ell = \mathfrak{n}^{1-k/2} a_{\mathfrak{n}}(h)$. The latter may be proven as follows. If we let $v$ (denoted $\varphi_h$ in [23]) be the adelic new vector attached to $h$, then $v$ is a pure tensor, differing from $w$ only at the places $p \mid N$. A test function $\tilde{f}^{\mathfrak{n}}$, say, is used in [23] that differs from $f^{\mathfrak{n}}$ only at the places $p \mid N$. By the same argument as above,

$$R(\tilde{f}^{\mathfrak{n}})\, v = v_\infty \otimes \bigotimes_{p \mid N} v_p \otimes v' \otimes \bigotimes_{\ell \mid \mathfrak{n}} R(f_\ell^{\mathfrak{n}})\, v_\ell.$$

Since $v_\ell = w_\ell$ at places $\ell \mid \mathfrak{n}$, the eigenvalues are the same, i.e., $R(\tilde{f}^{\mathfrak{n}})\, v = \lambda v$. By [23, Theorem 13.14] (which uses a global argument), $\lambda = \mathfrak{n}^{1-k/2} a_{\mathfrak{n}}(h)$. □

**4.2.** ***First main result: the trace of a Hecke operator.*** We now state our first main theorem, which is a general formula for the trace of $T_{\mathfrak{n}}$ on $S_k(\hat{\sigma})$. Its proof will occupy the remainder of Section 4.

**Theorem 4.2.** *Let $k > 2$, let the level $N$, nebentypus $\omega'$, and tuple $\hat{\sigma} = (\sigma_p)_{p|N}$ of supercuspidals be fixed as at the beginning of Section 4 (ensuring compatibility of central characters with $\omega'$), and let $f = f^{\mathfrak{n}}$ as in (4-10). Let $T$ be the product of all primes $p|N$ with $\operatorname{ord}_p(N)$ odd. Then*

$$\operatorname{tr}(T_{\mathfrak{n}} \mid S_k(\hat{\sigma})) = \mathfrak{n}^{(k/2)-1}\Bigg[\overline{\omega'(\mathfrak{n}^{1/2})}\frac{1}{12}(k-1)\prod_{p|N} d_{\sigma_p} + \frac{1}{2}\sum_{M|T}\Phi\left(\begin{pmatrix} & -\mathfrak{n}M \\ 1 & \end{pmatrix}, f\right) + \sum_{M|T}\sum_{1\le r<\sqrt{4\mathfrak{n}/M}}\Phi\left(\begin{pmatrix} 0 & -\mathfrak{n}M \\ 1 & rM \end{pmatrix}, f\right)\Bigg],$$

*where $\omega'(\mathfrak{n}^{1/2})$ is taken to be $0$ if $\mathfrak{n}$ is not a perfect square, $d_{\sigma_p}$ is the formal degree of $\sigma_p$ relative to Haar measure fixed in Section 2, and the orbital integrals $\Phi(\gamma, f)$ are defined in Section 3.3.*

*An orbital integral $\Phi(\gamma, f)$ as above vanishes unless $\gamma$ is elliptic in $G(\mathbb{Q}_p)$ for each $p|N$. Assuming this condition is satisfied, let $E = \mathbb{Q}[\gamma]$ be the imaginary quadratic extension of $\mathbb{Q}$ generated by $\gamma$, and let $h(E)$, $w(E)$, and $d_E$ be the class number, number of units, and discriminant of $E$ respectively. Then*

$$\Phi(\gamma, f) = -\frac{2h(E)}{w(E)2^{\omega(d_E)}}\frac{\sin((k-1)\theta_\gamma)}{\sin(\theta_\gamma)}\prod_{p|\Delta_\gamma N}\Phi(\gamma, f_p), \tag{4-11}$$

*where $\Delta_\gamma$ is the discriminant of $\gamma$, $\theta_\gamma = \arctan(\sqrt{|\Delta_\gamma|}/\operatorname{tr}\gamma)$ (interpreted as $\frac{\pi}{2}$ if $\operatorname{tr}\gamma = 0$) is the argument of one of the complex eigenvalues of $\gamma$, $\omega(d_E)$ is the number of prime factors of $d_E$, and our choice of measure for $\Phi(\gamma, f_p)$ is summarized in Section 4.7.3 below.*

**Remarks.** (1) For primes $p \nmid N$, the local orbital integrals $\Phi(\gamma, f_p)$ are computed explicitly in Sections 4.4 and 4.5 below. Thus, for the explicit calculation of $\operatorname{tr}(T_{\mathfrak{n}} \mid S_k(\hat{\sigma}))$ it only remains to calculate the local orbital integrals $\Phi(\gamma, f_p)$ for $p|N$.

(2) When $\mathfrak{n} = 1$, the set of relevant $\gamma$ is considerably smaller than what appears above if $T > 1$, due to local considerations at $p|T$. See Theorem 7.1.

The proof of Theorem 4.2 involves results from the rest of Section 4, outlined as follows. First, the test function $f$ satisfies the hypotheses of Theorem 3.3. Indeed, the hyperbolic orbital integrals of $f_\infty$ vanish as shown in [23, Proposition 24.2], and the fact that $f \in \mathcal{C}_k$ is a consequence of the formula for $f_\infty$ (see [23, Lemma 14.2]).

Since we are normalizing measure so that $\mathrm{meas}(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})) = \frac{\pi}{3}$, the identity term in Theorem 3.3 is

$$\frac{\pi}{3} f(1) = \tfrac{1}{12}(k-1) \prod_{p \mid N} d_{\sigma_p} \prod_{\ell \mid \mathtt{n}} f_\ell^{\mathtt{n}}(1).$$

From the definition (4-6) of $f_\ell^{\mathtt{n}}$, we see that $f_\ell^{\mathtt{n}}(1) \neq 0$ only if $1 \in Z_\ell M(\mathtt{n})_\ell$, which holds if and only if $\mathtt{n}$ is a perfect square. Assuming this is the case,

$$f_\ell^{\mathtt{n}}(1) = f_\ell^{\mathtt{n}}\left(\begin{pmatrix} \sqrt{\mathtt{n}} & \\ & \sqrt{\mathtt{n}} \end{pmatrix}^{-1} \begin{pmatrix} \sqrt{\mathtt{n}} & \\ & \sqrt{\mathtt{n}} \end{pmatrix}\right) = \omega_\ell(\sqrt{\mathtt{n}}).$$

Note that by (4-3),

$$\prod_{\ell \mid \mathtt{n}} \omega_\ell(\sqrt{\mathtt{n}}) = \prod_{\ell \nmid N} \omega_\ell(\sqrt{\mathtt{n}}) = \prod_{\ell \mid N} \overline{\omega_\ell(\sqrt{\mathtt{n}})} = \overline{\omega'(\sqrt{\mathtt{n}})}.$$

Therefore the identity term is

$$\frac{\pi}{3} f(1) = \overline{\omega'(\sqrt{\mathtt{n}})}\, \tfrac{1}{12}(k-1) \prod_{p \mid N} d_{\sigma_p},$$

where it is to be understood that $\omega'(\sqrt{\mathtt{n}}) = 0$ if $\mathtt{n}$ is not a perfect square.

The structure of the first part of Theorem 4.2 is then immediate from Theorem 3.3, Lemma 3.4, and Proposition 4.1. The set of relevant $\gamma$ is determined in Section 4.6 below, simply by considering the supports of the local test functions. The vanishing of $\Phi(\gamma, f)$ if $\gamma$ is hyperbolic in $G(\mathbb{R})$ or $G(\mathbb{Q}_p)$ for some $p \mid N$ is explained in Proposition 4.3 below.

As for (4-11), the first factor is equal to $\mathrm{meas}(\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G_\gamma(\mathbb{A})})$ under our normalization of Haar measures on $G(\mathbb{A})$ and $G_\gamma(\mathbb{A})$. This is shown in Theorem 4.16 below. The second factor of (4-11) (along with the negative sign) is $\Phi(\gamma, f_\infty)$ as in (4-12) below. In Sections 4.4 and 4.5 we explicitly compute the local orbital integrals away from the level, and see in particular that the value is 1 at places not dividing $\Delta_\gamma N$.

The local orbital integrals at the places dividing $N$ of course depend on the choice of supercuspidal representations. The method we use to treat the special cases of simple supercuspidals and depth zero supercuspidals in the second part of this paper is presumably applicable to other cases as well.

**4.3.** ***Known results about the elliptic terms.*** We record here some basic properties of the elliptic orbital integrals that arise in Theorem 4.2.

**Proposition 4.3.** *Let $\gamma$ be elliptic in $G(\mathbb{Q})$. Then for the test function $f = f^{\mathtt{n}}$ of (4-10):*

(1) *$\Phi(\gamma, f)$ is absolutely convergent.*

(2) *$\Phi(\gamma, f)$ depends only on the conjugacy class of $\gamma$ in $G(\mathbb{A})$ (rather than in $G(\mathbb{Q})$), and likewise for any prime $\ell$, $\Phi(\gamma, f_\ell)$ depends only on the $G(\mathbb{Q}_\ell)$-conjugacy class of $\gamma$.*

(3) $\Phi(\gamma, f) = 0$ *unless*: $\det\gamma > 0$ *and* $\gamma$ *is elliptic both in* $G(\mathbb{R})$ *and in* $G(\mathbb{Q}_p)$ *for each* $p \mid M$.

(4) *If* $\gamma$ *is elliptic in* $G(\mathbb{R})$ *with a complex eigenvalue* $\rho = re^{i\theta}$, *then*

$$\begin{aligned}\Phi(\gamma, f_\infty) &= -r^{2-k}\frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}} \\ &= -\frac{e^{i(k-1)\theta} - e^{-i(k-1)\theta}}{e^{i\theta} - e^{-i\theta}} = -\frac{\sin((k-1)\theta)}{\sin(\theta)}.\end{aligned} \tag{4-12}$$

**Remarks.** If $\gamma$ has discriminant $\Delta_\gamma < 0$ and nonzero trace, then we may take $\theta = \arctan(\sqrt{|\Delta_\gamma|}/\operatorname{tr}\gamma)$ in (4-12). If $\gamma$ has the form $\left(\begin{smallmatrix} & u \\ 1 & \end{smallmatrix}\right)$, then we may take $\theta = \frac{\pi}{2}$, giving

$$\Phi\left(\begin{pmatrix} & u \\ 1 & \end{pmatrix}, f_\infty\right) = -\left[\frac{i^{k-1} - (-i)^{k-1}}{2i}\right] = \begin{cases}(-1)^{k/2} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.}\end{cases} \tag{4-13}$$

*Proof.* Nearly everything is proven in [23, pp. 295–302]. The only remaining point is that $\Phi(\gamma, f_p) = 0$ if $\gamma$ is hyperbolic in $G(\mathbb{Q}_p)$ for some $p \mid M$. For such $\gamma$, after conjugating we can take $\gamma$ diagonal, so $G_\gamma(\mathbb{Q}_p) = M(\mathbb{Q}_p)$. The orbital integral is then taken over $M_p\backslash G_p$ and involves integrating over $N(\mathbb{Q}_p)$ (see (4-15) below). We can use (3-1) to show that it vanishes (as in (3-4)). □

**4.4.** ***Local orbital integrals at primes*** $\ell \nmid N$***: hyperbolic case.*** If $\gamma \in G(\mathbb{Q})$ is elliptic, then for each prime $\ell$, $\gamma$ is either hyperbolic or elliptic in $G(\mathbb{Q}_\ell)$. In this section and the next we evaluate the local elliptic orbital integrals at primes $\ell \nmid N$. The methods are standard and the results are presumably not new. For the dimension formulas we require the test function $f_\ell$ given by (4-7). However, without any extra work we can consider a general local Hecke operator, and consider an arbitrary $p$-adic field.

Thus, we let $F$ be a $p$-adic field with valuation $v$, uniformizer $\varpi$, ring of integers $\mathcal{O}_F$, maximal ideal $\mathfrak{p} = \varpi\mathcal{O}_F$, and $q_v = |\mathcal{O}_F/\mathfrak{p}|$. Fix an unramified unitary character $\omega_v : F^* \to \mathbb{C}^*$. For an integral ideal $\mathfrak{n}_v \subseteq \mathcal{O}_F$, define

$$M(\mathfrak{n}_v) = \{g \in M_2(\mathcal{O}_F) \mid (\det g)\mathcal{O}_F = \mathfrak{n}_v\}$$

and

$$f^{\mathfrak{n}_v}(g) = \begin{cases}\overline{\omega_v(z)} & \text{if } g = zm \in Z(F)M(\mathfrak{n}_v), \\ 0 & \text{if } g \notin Z(F)M(\mathfrak{n}_v).\end{cases} \tag{4-14}$$

If $\gamma$ is hyperbolic in $G(F)$, then replacing it by a conjugate if necessary, we can assume that it is diagonal. In this case, $G_\gamma(F) = M(F)$ is the set of invertible diagonal matrices. We may integrate over $\overline{G}(F)$ using the Iwasawa coordinates

$$\int_{\overline{G}(F)} \phi(g)\,dg = \int_{\overline{M}(F)}\int_{N(F)}\int_{K_v} \phi(mnk)\,dm\,dn\,dk,$$

where $K_v = G(\mathcal{O}_F)$. Therefore if $\phi$ is $M(F)$-invariant,

$$\int_{\overline{G_\gamma(F)}\backslash \overline{G}(F)} \phi(g)\, dg = \int_{N(F)} \int_{K_v} \phi(nk)\, dn\, dk. \tag{4-15}$$

We normalize the measures $dn$ and $dk$ by taking $\mathrm{meas}(N(\mathcal{O}_F)) = \mathrm{meas}(K_v) = 1$.

**Proposition 4.4.** *For $F$ as above, suppose $\gamma$ is hyperbolic in $G(F)$. Assuming $\gamma \in M(\mathfrak{n}_v)$, and letting $\Delta_\gamma \in \mathcal{O}_F$ be its discriminant, we have $\Phi(\gamma, f^{\mathfrak{n}_v}) = |\Delta_\gamma|_v^{-1/2}$. In particular, if $\Delta_\gamma$ is a unit, then $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$.*

*Proof.* We may assume that $\gamma = \left(\begin{smallmatrix} \alpha & \\ & \beta \end{smallmatrix}\right)$ for some distinct $\alpha, \beta \in \mathcal{O}_F$. By (4-15) and the fact that $f^{\mathfrak{n}_v}$ is right $K_v$-invariant,

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \int_F f^{\mathfrak{n}_v}\left(\begin{pmatrix} 1 & -t \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\right) dt = \int_F f^{\mathfrak{n}_v}\left(\begin{pmatrix} \alpha & t(\alpha-\beta) \\ & \beta \end{pmatrix}\right) dt.$$

Choose $j \geq 0$ so that $\alpha - \beta \in \varpi^j \mathcal{O}_F^*$. By hypothesis, $\alpha, \beta \in \mathcal{O}_F$ and $\alpha\beta\mathcal{O}_F = \mathfrak{n}_v$, so the integrand is nonzero if and only if $t(\alpha-\beta) \in \mathcal{O}_F$, which is equivalent to $t \in \varpi^{-j}\mathcal{O}_F$. Therefore

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \mathrm{meas}(\varpi^{-j}\mathcal{O}_F) = q_v^j = |\alpha - \beta|_v^{-1}.$$

Now let $D = \det \gamma$ and $r = \operatorname{tr} \gamma$. Note that

$$4D = 4\alpha\beta = (\alpha+\beta)^2 - (\alpha-\beta)^2 = r^2 - (\alpha-\beta)^2. \tag{4-16}$$

Therefore

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = |\alpha - \beta|_v^{-1} = |r^2 - 4D|_v^{-1/2},$$

as claimed. □

**4.5.** ***Local orbital integrals at primes $\ell \nmid N$: elliptic case.*** If $\gamma$ is elliptic over a field $F$ of characteristic 0, then $E = F[\gamma]$ is a quadratic field extension of $F$, and

$$G_\gamma(F) = E^*$$

(see [23, Proposition 26.1]). The center $Z(F)$ is isomorphic to $F^*$.

**Proposition 4.5.** *Let $F$ be a local field of characteristic 0, and suppose $\gamma$ is elliptic in $G(F)$. Then $G_\gamma(F)/Z(F)$ is compact.*

*Proof.* If $F = \mathbb{R}$, then $G_\gamma(\mathbb{R}) = \mathbb{R}[\gamma]^* \cong \mathbb{C}^*$, and the map $z \mapsto z/|z|$ gives rise to $\mathbb{C}^*/\mathbb{R}^* \cong \mathrm{SO}(2)/\{\pm 1\}$, which is compact.

Now suppose that $F$ is nonarchimedean, with valuation $v$ and integer ring $\mathcal{O}_F$. Let $E = F[\gamma]$, and choose a prime element $\pi \in \mathcal{O}_E$. Then letting $e \in \{1, 2\}$ be the ramification index of $E/F$,

$$G_\gamma(F)/Z(F) \cong E^*/F^* = \bigcup_{j=0}^{e-1} \pi^j \mathcal{O}_E^*/\mathcal{O}_F^*, \tag{4-17}$$

which is compact. □

Consider a $p$-adic field $F$, with all notation as in the previous subsection. For $\gamma$ elliptic in $G(F)$, the above leads to the following natural choice of $G(F)$-invariant measure on the quotient space $\overline{G_\gamma(F)}\backslash\overline{G}(F)$. We assign the compact group $\overline{G_\gamma(F)}$ a total volume of 1. We assign $\overline{G}(F)$ the Haar measure for which $\overline{G}(\mathcal{O}_F)$ has measure 1. Together these choices determine the quotient measure via

$$\int_{\overline{G_\gamma(F)}\backslash\overline{G}(F)} \int_{\overline{G_\gamma(F)}} \phi(xy)\,dx\,dy = \int_{\overline{G}(F)} \phi(g)\,dg.$$

In fact, by our normalization, if $\phi$ is left $G_\gamma(F)$-invariant, then

$$\int_{\overline{G_\gamma(F)}\backslash\overline{G}(F)} \phi(y)\,dy = \int_{\overline{G}(F)} \phi(g)\,dg, \tag{4-18}$$

when $\gamma$ is elliptic in $G(F)$.

For such $\gamma$, $E = F[\gamma]$ is a quadratic extension of $F$. Fix an $F$-integral basis $\{1, \varepsilon\}$ for the ring of integers $\mathcal{O}_E$, so

$$\mathcal{O}_E = \mathcal{O}_F + \mathcal{O}_F\,\varepsilon. \tag{4-19}$$

We will need some facts about orders and lattices in $E$. Recall that an *$F$-order* in $E$ is a subring containing $\mathcal{O}_F$ which has rank 2 as an $\mathcal{O}_F$-module.

**Proposition 4.6.** *Let $\mathfrak{O}_{E/F}$ denote the set of all $F$-orders in $E$. For $r \geq 0$ and $\varepsilon$ as above, define*

$$\mathcal{O}_r = \mathcal{O}_F + \mathfrak{p}^r\,\varepsilon,$$

*where $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}_F$, so in particular $\mathcal{O}_0 = \mathcal{O}_E$. Then*

$$\mathfrak{O}_{E/F} = \{\mathcal{O}_r \mid r \geq 0\}.$$

*Furthermore, letting $e = e(E/F)$ be the ramification index, for $r > 0$ we have*

$$[\mathcal{O}_E^* : \mathcal{O}_r^*] = \begin{cases} q_v^r & \text{if } e = 2, \\ q_v^r + q_v^{r-1} & \text{if } e = 1. \end{cases} \tag{4-20}$$

*Proof.* See also [40, Sections 6.6 and 6.7] for the case $F = \mathbb{Q}_p$. Here we loosely follow Okada [42, Section 2.3]. Clearly $\mathcal{O}_r \in \mathfrak{O}_{E/F}$. Conversely, let $\mathcal{O} \in \mathfrak{O}_{E/F}$. The elements of $\mathcal{O}$ are integral over $E$ [41, Proposition I.2.2] so $\mathcal{O} \subseteq \mathcal{O}_E$. Hence there exists $\alpha \in \mathcal{O} \subseteq \mathcal{O}_E$ such that

$$\mathcal{O} = \mathcal{O}_F + \mathcal{O}_F\,\alpha.$$

Since $\alpha \notin \mathcal{O}_F$, by topological considerations we see that there exists $r \geq 0$ such that $\alpha \in \mathcal{O}_F + \varpi^r\mathcal{O}_E = \mathcal{O}_r$ but $\alpha \notin \mathcal{O}_F + \varpi^{r+1}\mathcal{O}_E = \mathcal{O}_{r+1}$. Hence

$$\mathcal{O}_{r+1} \subsetneq \mathcal{O} \subseteq \mathcal{O}_r.$$

We see easily that $\mathcal{O}_r/\mathcal{O}_{r+1} \cong \mathfrak{p}^r/\mathfrak{p}^{r+1} \cong \mathcal{O}_F/\mathfrak{p}$ as $\mathcal{O}_F$-modules. Since the latter is 1-dimensional as a vector space over $\mathcal{O}_F/\mathfrak{p}$, it has no nonzero proper submodules. It follows that $\mathcal{O} = \mathcal{O}_r$.

For the second part, consider the sequence

$$1 \to \mathcal{O}_F^*/(1+\mathfrak{p}^r) \to \mathcal{O}_E^*/(1+\mathfrak{p}^r\mathcal{O}_E) \to \mathcal{O}_E^*/\mathcal{O}_r^* \to 1,$$

where the maps are the obvious ones. It is straightforward to check that the sequence is exact. Therefore

$$[\mathcal{O}_E^* : \mathcal{O}_r^*] = \frac{|\mathcal{O}_E^*/(1+\mathfrak{p}^r\mathcal{O}_E)|}{|\mathcal{O}_F^*/(1+\mathfrak{p}^r)|}.$$

Let $e = e(E/F)$, so that $\mathfrak{p}\mathcal{O}_E = \mathfrak{P}^e$, where $\mathfrak{P}$ is the maximal ideal of $\mathcal{O}_E$. Then

$$\begin{aligned}|\mathcal{O}_E^*/(1+\mathfrak{p}^r\mathcal{O}_E)| &= |\mathcal{O}_E^*/(1+\mathfrak{P}^{er})| \\ &= [\mathcal{O}_E^* : 1+\mathfrak{P}] \prod_{j=2}^{er} [1+\mathfrak{P}^{j-1} : 1+\mathfrak{P}^j] = (q_E - 1)\, q_E^{er-1}\end{aligned}$$

(see [41, p. 139]). Here,

$$q_E = |\mathcal{O}_E/\mathfrak{P}| = \begin{cases} q_v & \text{if } e = 2, \\ q_v^2 & \text{if } e = 1. \end{cases}$$

Likewise $|\mathcal{O}_F^*/(1+\mathfrak{p}^r)| = (q_v - 1)\, q_v^{r-1}$, and (4-20) follows immediately. □

For the purposes of this subsection, a *lattice* in $F^2 = F \times F$ is an $\mathcal{O}_F$-submodule of rank 2. The group $F^*$ acts by multiplication on the set of lattices, and the orbits are called *lattice classes*. The map $g \mapsto L = g\binom{\mathcal{O}_F}{\mathcal{O}_F}$ from $G(F)$ to the set of lattices in $F^2$ induces a bijection between $\overline{G}(F)/\overline{K}_v$ and the set of lattice classes, since $K_v$ is the stabilizer of $\binom{\mathcal{O}_F}{\mathcal{O}_F}$.

With notation as in (4-19), we may identify a lattice $L \subseteq F^2$ with the lattice $(1\ \varepsilon)L \subseteq E$, so that in particular $\binom{\mathcal{O}_F}{\mathcal{O}_F}$ is identified with $\mathcal{O}_E$. Given $\eta \in E^*$, it acts by scalar multiplication on the set of lattices in $E$, and by matrix multiplication (via $E = F[\gamma]$) on the lattices in $F^2$. In general, these actions are not compatible with the above identification. However, as shown in [23, Lemma 26.20], after possibly replacing $\gamma$ (or equivalently, $\varepsilon$) by a $G(F)$-conjugate, these two actions do coincide for all $\eta \in E^*$. Explicitly, for any $g \in G(F)$,

$$\eta(1\ \varepsilon)g\begin{pmatrix}\mathcal{O}_F \\ \mathcal{O}_F\end{pmatrix} = (1\ \varepsilon)\eta g\begin{pmatrix}\mathcal{O}_F \\ \mathcal{O}_F\end{pmatrix},$$

where on the left $\eta$ acts as a scalar via $\eta(1\ \varepsilon) = (\eta\ \eta\varepsilon)$, and on the right it is acting by matrix multiplication. We will assume that $\gamma$ is chosen in this way, as we may since the value of the orbital integral depends only on $\gamma$'s conjugacy class in $G(F)$.

We associate to any lattice $L \subseteq E$ the order

$$\mathcal{O}_L = \{\mu \in E \mid \mu L \subseteq L\}.$$

This depends only on the lattice class to which $L$ belongs. Since $E$ is local, every lattice in $E$ is principal in the sense that there exists $y \in E^*$ such that $yL = \mathcal{O}_L$. (One may adapt the proof of [23, Proposition 26.13], which follows [33]).

Given an order $\mathcal{O}$, $L$ is a *proper $\mathcal{O}$-lattice* if $\mathcal{O}_L = \mathcal{O}$. Two proper $\mathcal{O}$-lattices $y\mathcal{O}$ and $z\mathcal{O}$ (for $y, z \in E^*$) are equal if and only if $y/z \in \mathcal{O}^*$. Therefore the set of all proper $\mathcal{O}$-lattices corresponds bijectively with $E^*/\mathcal{O}^*$.

**Lemma 4.7.** *Suppose* $(\det \gamma)\mathcal{O}_F = \mathfrak{n}_v$ *and* $g \in G(F)$. *Then for* $f^{\mathfrak{n}_v}$ *given by* (4-14), $f^{\mathfrak{n}_v}(g^{-1}\gamma g) \neq 0$ *if and only if* $\gamma \in \mathcal{O}_L$ *for* $L = g\binom{\mathcal{O}_F}{\mathcal{O}_F}$.

*Proof.* We observe that

$$\gamma \in \mathcal{O}_L \iff \gamma L \subseteq L \iff g^{-1}\gamma g \begin{pmatrix} \mathcal{O}_F \\ \mathcal{O}_F \end{pmatrix} \subseteq \begin{pmatrix} \mathcal{O}_F \\ \mathcal{O}_F \end{pmatrix} \iff g^{-1}\gamma g \in M_2(\mathcal{O}_F).$$

Given that $\mathrm{ord}_v(\det \gamma) = \mathrm{ord}_v(\mathfrak{n}_v)$, the above is equivalent to $g^{-1}\gamma g$ belonging to the support $Z(F)M(\mathfrak{n}_v)$ of $f^{\mathfrak{n}_v}$. □

**Proposition 4.8.** *Let* $f^{\mathfrak{n}_v}$ *be given by* (4-14). *Then for* $\gamma \in G(F)$ *elliptic, the orbital integral*

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \int_{\overline{G_\gamma(F)}\backslash \overline{G}(F)} f^{\mathfrak{n}_v}(g^{-1}\gamma g)\, dg$$

*vanishes unless some conjugate of* $\gamma$ *lies in* $Z(F)M(\mathfrak{n}_v)$. *Taking* $\gamma \in M(\mathfrak{n}_v)$, *with measure normalized as in* (4-18) *we have*

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = e_\gamma \sum_{r=0}^{n_\gamma} [\mathcal{O}_E^* : \mathcal{O}_r^*],$$

*where* $E = F[\gamma]$ *is the associated quadratic extension of* $F$ *with ramification index* $e_\gamma \in \{1, 2\}$ *and ring of integers* $\mathcal{O}_E = \mathcal{O}_F + \mathcal{O}_F\,\varepsilon$,

$$\mathcal{O}_r = \mathcal{O}_F + \mathfrak{p}^r \varepsilon$$

*is the order of index* $q_v^r$ *inside* $\mathcal{O}_E$, *and* $n_\gamma \geq 0$ *is defined by* $\mathcal{O}_\gamma = \mathcal{O}_F + \mathcal{O}_F\,\gamma = \mathcal{O}_r$ *for* $r = n_\gamma$. *In particular, if* $\mathcal{O}_\gamma = \mathcal{O}_E$ *and* $\mathfrak{p}$ *is inert in* $E$, *then* $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$.

**Remarks.** (1) Let $P_\gamma(X) \in \mathcal{O}_F[X]$ be the characteristic polynomial of $\gamma$. If $P_\gamma$ is irreducible modulo $\mathfrak{p}$, then $e_\gamma = 1$ and $\mathcal{O}_\gamma = \mathcal{O}_E$ [50, p. 18]. Hence $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$ in this case.

(2) The index $[\mathcal{O}_E^* : \mathcal{O}_r^*]$ is given explicitly in (4-20) when $r > 0$ (and is 1 when $r = 0$).

(3) Let $\mathfrak{d}_{E/F} = \det\left(\begin{smallmatrix} 1 & \varepsilon \\ 1 & \bar{\varepsilon} \end{smallmatrix}\right)^2 \mathcal{O}_F$ be the relative discriminant (with the bar denoting Galois conjugation), write $\gamma = s + b\varepsilon$ for $s, b \in \mathcal{O}_F$, and let $\Delta_\gamma = r^2 - 4D$ be the discriminant of $\gamma$. Then

$$n_\gamma = \mathrm{ord}_v(b) = \tfrac{1}{2}(\mathrm{ord}_v(\Delta_\gamma) - \mathrm{ord}_v(\mathfrak{d}_{E/F})). \tag{4-21}$$

This follows from the fact that the relative discriminant of

$$\mathcal{O}_\gamma = \mathcal{O}_F + \mathcal{O}_F\,\gamma = \mathcal{O}_F + \mathcal{O}_F\,b\varepsilon$$

is given on the one hand by

$$\det\begin{pmatrix} 1 & b\varepsilon \\ 1 & b\bar{\varepsilon} \end{pmatrix}^2 \mathcal{O}_F = b^2 \mathfrak{d}_{E/F},$$

and also (using (4-16)) by

$$\det\begin{pmatrix} 1 & \gamma \\ 1 & \bar{\gamma} \end{pmatrix}^2 \mathcal{O}_F = (\gamma - \bar{\gamma})^2 \mathcal{O}_F = \Delta_\gamma \mathcal{O}_F.$$

Further, if $F$ is the completion of a number field $L$ at a place $v$, $\{1, \varepsilon_L\}$ is an integral basis of $L[\gamma]$ over $L$, and we write $\gamma = s_L + b_L\varepsilon_L$, then (4-21) also holds with $b_L$ in place of $b$. Indeed the same argument applies in the global case to give $b_L^2 \mathfrak{d}_{L[\gamma]/L} = \Delta_\gamma \mathcal{O}_L$. By the fact By the fact that the global discriminant is the product of the local ones and (due to $\gamma$ being elliptic in $G(F)$) there is only one prime of $L[\gamma]$ lying over $v$, we see that $\mathrm{ord}_v(b_L) = \mathrm{ord}_v(b)$.

(4) If $E = \mathbb{Q}_\ell[\sqrt{d}]$ for $d \in \mathbb{Z}$ square-free, then (see [36, Section 6.10], for example)

$$\mathcal{O}_E = \begin{cases} \mathbb{Z}_2\left[\frac{1}{2}(1+\sqrt{-3})\right] & \text{if } \ell = 2,\ E = \mathbb{Q}_2[\sqrt{-3}], \\ \mathbb{Z}_\ell[\sqrt{d}] & \text{otherwise.} \end{cases} \tag{4-22}$$

In particular, if $\ell > 2$ and the valuation $\alpha = v_\ell(\Delta_\gamma)$ of the discriminant of $\gamma$ is *odd*, then $e_\gamma = 2$, $n_\gamma = \frac{1}{2}(\alpha - 1)$, and assuming $\gamma \in M(\mathfrak{n})_\ell$,

$$\Phi(\gamma, f_\ell^{\mathfrak{n}}) = 2 \sum_{r=0}^{(\alpha-1)/2} \ell^r. \tag{4-23}$$

*Proof of Proposition 4.8.* The first statement is clear. Now suppose $\gamma \in M(\mathfrak{n}_v)$. By (4-18),

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \int_{\overline{G}(F)} f^{\mathfrak{n}_v}(g^{-1}\gamma g)\, dg.$$

The integrand is right $\overline{K_v}$-invariant as a function of $g$. Since $\overline{K_v}$ is open with measure 1, $\overline{G}(F)/\overline{K_v}$ is discrete with the counting measure. Therefore

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \sum_{g \in \overline{G}(F)/\overline{K_v}} f^{\mathfrak{n}_v}(g^{-1}\gamma g).$$

By our earlier remarks, we can view the sum as a sum over the lattice classes, and by Lemma 4.7, $\Phi(\gamma, f^{\mathfrak{n}_v})$ is equal to the number of lattice classes preserved by $\gamma$.

Since $\gamma \in E$ is integral over $\mathcal{O}_F$, $\mathcal{O}_\gamma = \mathcal{O}_F + \mathcal{O}_F\gamma$ is an order in $E$ (see [23, Lemma 26.10]). We claim that $\gamma\mathcal{O}_r \subseteq \mathcal{O}_r$ if and only if $0 \le r \le n_\gamma$, where $q_v^{n_\gamma}$ is the index of $\mathcal{O}_\gamma$. Indeed,

$$\gamma\mathcal{O}_r \subseteq \mathcal{O}_r \iff \gamma \in \mathcal{O}_r \iff \mathcal{O}_\gamma \subseteq \mathcal{O}_r \iff r \le n_\gamma.$$

It follows that

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \sum_{r=0}^{n_\gamma} (\#\text{ of classes of proper } \mathcal{O}_r\text{-lattices}).$$

Recall from earlier that the set of proper $\mathcal{O}_r$-lattices corresponds bijectively with $E^*/\mathcal{O}_r^*$. Since we are counting $F^*$-classes of lattices rather than lattices themselves, we find

$$\Phi(\gamma, f^{\mathfrak{n}_v}) = \sum_{0 \le r \le n_\gamma} |E^*/F^*\mathcal{O}_r^*| \quad (\mathcal{O}_\gamma = \mathcal{O}_{n_\gamma}).$$

Because $\mathcal{O}_F^* \subseteq \mathcal{O}_r^*$, it follows from (4-17) that $|E^*/F^*\mathcal{O}_r^*| = e_\gamma[\mathcal{O}_E^* : \mathcal{O}_r^*]$, where $e_\gamma \in \{1, 2\}$ is the ramification index of $E/F$. The result now follows. □

**Corollary 4.9.** *For $f^{\mathfrak{n}_v}$ as in (4-14), let $\gamma \in M(\mathfrak{n}_v)$ have characteristic polynomial $P_\gamma(X) = X^2 - rX + D \in \mathcal{O}_F[X]$ with discriminant $\Delta_\gamma = r^2 - 4D$. Then if $\gamma$ is hyperbolic in $G(F)$, $\Phi(\gamma, f^{\mathfrak{n}_v}) = |\Delta_\gamma|_v^{-1/2}$. If $\gamma$ is elliptic in $G(F)$ and $P_\gamma(X)$ does not have a double root in $\mathcal{O}_F/\mathfrak{p}$, then $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$.*

*Consequently, for $\gamma \in M(\mathfrak{n}_v)$ elliptic or hyperbolic in $G(F)$, $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$ if*

$$\Delta_\gamma \notin \mathfrak{p}.$$

*Proof.* The hyperbolic case is just a restatement of Proposition 4.4. Suppose $\gamma$ is elliptic. If $P_\gamma$ does not have a double root in $\mathcal{O}_F/\mathfrak{p}$, then it cannot have a simple root either, because otherwise that root would lift to a root in $F$ by Hensel's lemma. By the first remark after Proposition 4.8, $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$.

Furthermore, suppose $\mathfrak{p} \nmid 2$, and note that $P_\gamma'(X) = 2X - r$ vanishes only at $\frac{r}{2} \in \mathcal{O}_F/\mathfrak{p}$. On the other hand,

$$P_\gamma\left(\frac{r}{2}\right) = D - \frac{r^2}{4},$$

which shows that $P_\gamma$ has a repeated root modulo $\mathfrak{p}$ if and only if $\mathfrak{p} \mid (r^2 - 4D)$. Hence when $\mathfrak{p} \nmid 2$ and $\Delta_\gamma \notin \mathfrak{p}$, $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$.

If $\mathfrak{p} \mid 2$ and $(r^2 - 4D) \notin \mathfrak{p}$, then $r \in \mathcal{O}_F^*$, and therefore $P_\gamma'(X) = 2X - r$ is nonzero mod $\mathfrak{p}$. Hence $P_\gamma$ does not have a repeated root, and $\Phi(\gamma, f^{\mathfrak{n}_v}) = 1$ in this case as well. □

Although the result of Proposition 4.8 appears complicated, it is not so hard to evaluate it by hand, using the remarks that follow the proposition and standard results about quadratic extensions of $p$-adic fields.

**Example 4.10.** Let $\ell$ be a prime not dividing $D$, and let $\gamma = \left(\begin{smallmatrix} 0 & -D \\ 1 & 0 \end{smallmatrix}\right)$. Then for $f_\ell$ as in (4-7),

$$\Phi(\gamma, f_\ell) = \begin{cases} 2 & \text{if } \ell = 2 \text{ and } D \equiv 1, 5, 7 \bmod 8, \\ 4 & \text{if } \ell = 2 \text{ and } D \equiv 3 \bmod 8, \\ 1 & \text{if } \ell \neq 2. \end{cases}$$

**Remark.** Some additional examples are given in Section 7.5.

*Proof.* First suppose $\ell \neq 2$. Since the discriminant $-4D$ of $P_\gamma(X) = X^2 + D$ is not divisible by $\ell$, $\Phi(\gamma, f_\ell) = 1$ by Corollary 4.9.

Now suppose $\ell = 2$, so $D$ is odd since $\ell \nmid D$. Recall that the squares of $\mathbb{Q}_2^*$ are exactly the elements of the set $2^{2\mathbb{Z}}(1 + 8\mathbb{Z}_2)$ [49, Theorem II.4]. Thus $-D$ is a square in $\mathbb{Q}_2^*$ if and only if

$$D \equiv 7 \bmod 8.$$

When this congruence is satisfied, $\gamma$ is hyperbolic, and by Corollary 4.9,

$$\Phi(\gamma, f_2) = |-4D|_2^{-1/2} = 2.$$

Now suppose that $-D$ is not a square in $\mathbb{Q}_2$, i.e., it is not 1 mod 8. We recall some facts about the quadratic extensions of $\mathbb{Q}_2$ (see, e.g., [36, Chapter 6]). There are exactly seven such extensions, namely $\mathbb{Q}_2[\sqrt{d}]$ for

$$d = -1, \pm 3, \pm 2, \pm 6,$$

with $\mathbb{Q}_2[\sqrt{-3}]$ being the unique unramified quadratic extension. With the exception of $d = -3$, the ring of integers is $\mathbb{Z}_2[\sqrt{d}]$. For $d = -3$, the ring of integers is $\mathbb{Z}_2\left[\frac{1}{2}(1+\sqrt{-3})\right]$. Under the given hypothesis, $-D \equiv d \bmod 8$, where $d \in \{-1, \pm 3\}$. So $-D = dx$ for some $x \in 1+8\mathbb{Z}_2$, and hence $-D = dy^2$ for some $y \in \mathbb{Z}_2^*$. Therefore, writing $E = \mathbb{Q}_2[\sqrt{-D}]$, we have $\mathcal{O}_E = \mathcal{O}_\gamma$ unless $d = -3$. In the former case, $E/\mathbb{Q}_2$ is ramified, so by Proposition 4.8,

$$\Phi(\gamma, f_2) = 2 \quad (D \equiv 1, 5 \bmod 8).$$

If $D \equiv 3 \bmod 8$, then $\mathcal{O}_E = \mathbb{Z}_2 + \mathbb{Z}_2\,\varepsilon$ for $\varepsilon = \frac{1}{2}(1 + \sqrt{-3})$. Hence

$$\mathcal{O}_\gamma = \mathbb{Z}_2 + \mathbb{Z}_2\sqrt{-D} = \mathbb{Z}_2 + \mathbb{Z}_2\sqrt{-3} = \mathbb{Z}_2 + \mathbb{Z}_2\,2\varepsilon.$$

So in the notation of Proposition 4.8, $n_\gamma = 1$. Since $E/\mathbb{Q}_2$ is unramified, using (4-20), we have

$$\Phi(\gamma, f_2) = [\mathcal{O}_E^* : \mathcal{O}_E^*] + [\mathcal{O}_E^* : \mathcal{O}_\gamma^*] = 1 + 3 = 4. \qquad \square$$

**4.6.** ***The set of relevant $\gamma$.*** Here we determine explicitly the finite set of conjugacy classes in $\overline{G}(\mathbb{Q})$ that can have a nonzero contribution to the trace of $R(f)$ for $f$ as in (4-10). Writing $N = \prod_{p|N} p^{N_p}$, define the square-free integers

$$S = \prod_{p\,|\,N,\, N_p \text{ even}} p, \quad T = \prod_{p\,|\,N,\, N_p \text{ odd}} p.$$

We say that an elliptic element $\gamma \in G(\mathbb{Q}_p)$ is *unramified* (at $p$) if $v_p(\det\gamma)$ is even, and *ramified* otherwise.

**Lemma 4.11.** *Let $\gamma \in G(\mathbb{Q})$ be elliptic, and suppose $\Phi(\gamma, f) \neq 0$ for $f = f^{\mathfrak{n}}$ as in (4-10). Then there exists a unique positive divisor $M \,|\, T$ and a scalar $z \in \mathbb{Q}^*$ such that $\operatorname{tr}(\gamma z) \geq 0$ is an integer and*

$$\det(z\gamma) = \mathfrak{n}M.$$

*In particular, the rational canonical form of $z\gamma$ lies in $M_2(\mathbb{Z})$.*

*Proof.* If $p \,|\, S$, then $\gamma$ is unramified at $p$ since $f_p$ is supported in $Z_p K_p$. For $p \,|\, T$, the support of $f_p$ has both ramified and unramified elements (see (3-2)). Let $M$ be the product of those primes $p \,|\, T$ at which $\gamma$ is ramified. For each prime $\ell \nmid N$, some conjugate of $\gamma$ must lie in $\operatorname{Supp}(f_\ell^{\mathfrak{n}}) = Z_\ell M(\mathfrak{n})_\ell$ since otherwise the integrand of $\Phi(\gamma, f)$ vanishes. It follows that $v_p(\det\gamma/\mathfrak{n}M)$ is even for *all* primes $p$, where $v_p$ is the $p$-adic valuation. Hence $\det\gamma \in \pm\mathfrak{n}M\mathbb{Q}^{*2}$, where $\mathbb{Q}^{*2}$ is the set of squares in $\mathbb{Q}^*$. Because $f_\infty$ is supported on $G(\mathbb{R})^+$, there is a scalar $z \in \mathbb{Q}^*$ such that $\det(z\gamma) = \mathfrak{n}M$, as claimed. Because $\Phi(z\gamma, f) = \Phi(\gamma, f) \neq 0$, some $G(\mathbb{A}_{\text{fin}})$-conjugate of $z\gamma$ lies in

$$\prod_{p\,|\,M} \begin{pmatrix} & 1 \\ p & \end{pmatrix} K_p \times \prod_{p\,|\,(ST/M)} K_p \times \prod_{\ell \nmid N} M(\mathfrak{n})_\ell \subseteq M_2(\hat{\mathbb{Z}}) \tag{4-24}$$

(recall that $f_p$ is supported in the group $J$ of (3-2)). In particular, $\operatorname{tr}(z\gamma) \in \hat{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$. Scaling $z$ by $-1$ if necessary, we may arrange further that $\operatorname{tr}(z\gamma) \geq 0$. □

**Lemma 4.12.** *Let $F$ be a $p$-adic field, and $\gamma$ an elliptic element of $G(F)$ with $\operatorname{tr}\gamma \in \mathcal{O}_F$ and $\det\gamma \in \mathfrak{p}$. Then $\operatorname{tr}\gamma \in \mathfrak{p}$.*

*Proof.* Denote the characteristic polynomial of $\gamma$ by

$$P_\gamma(X) = X^2 - dX + \det\gamma,$$

where $d = \operatorname{tr}\gamma$. Notice that $P_\gamma(0) \equiv 0 \bmod \mathfrak{p}$. Furthermore, $P'_\gamma(0) \equiv -d \bmod \mathfrak{p}$. If $d$ is nonzero modulo $\mathfrak{p}$, then by Hensel's lemma, $P_\gamma$ has a root in $\mathfrak{p}$, contradicting the fact that $\gamma$ is elliptic in $G(F)$. Hence $d \in \mathfrak{p}$. □

**Proposition 4.13.** *For $\gamma \in \overline{G}(\mathbb{Q})$ elliptic, and $f = f^{\mathfrak{n}}$ the test function defined in (5-21), $\Phi(\gamma, f) = 0$ unless the conjugacy class of $\gamma$ has a representative in $G(\mathbb{Q})$ of the form $\left(\begin{smallmatrix} 0 & -\mathfrak{n}M \\ 1 & rM \end{smallmatrix}\right)$ for some $M \,|\, T$ and $0 \leq r < \sqrt{4\mathfrak{n}/M}$.*

**Remark.** If the characteristic polynomial of $\gamma = \left(\begin{smallmatrix} 0 & -\mathrm{n}M \\ 1 & rM \end{smallmatrix}\right)$ has a root in $\mathbb{Q}_p$, then $\Phi(\gamma, f) = 0$ by Proposition 4.3.

*Proof.* Let $\mathfrak{o}$ be an elliptic conjugacy class in $\overline{G}(\mathbb{Q})$ with $\Phi(\mathfrak{o}, f) \neq 0$. By Lemma 4.11, $\mathfrak{o}$ has a unique representative $\gamma \in G(\mathbb{Q})$ with characteristic polynomial of the form

$$P_\gamma(X) = X^2 - dX + \mathrm{n}M \in \mathbb{Z}[X],$$

where $d = \operatorname{tr}\gamma \geq 0$ and $M \mid T$. By Proposition 4.3, we know that $\gamma$ is elliptic in $G(\mathbb{Q}_p)$ for each $p \mid N$ and also in $G(\mathbb{R})$. It follows by Lemma 4.12 that $M \mid d$. Write $d = rM$. Given that $\gamma$ is elliptic in $G(\mathbb{R})$, we have $d^2 < 4\mathrm{n}M$, i.e.,

$$r^2 M < 4\mathrm{n}.$$

So, taking $\gamma$ in rational canonical form as we may, it has the form

$$\gamma = \begin{pmatrix} 0 & -\mathrm{n}M \\ 1 & rM \end{pmatrix}, \quad 0 \leq r < \sqrt{4\mathrm{n}/M}. \qquad \square$$

**4.7.** ***The measure of $\overline{G_\gamma(F)}\backslash\overline{G_\gamma(\mathbb{A}_F)}$.*** Let $F$ be a number field with adele ring $\mathbb{A}_F$, and let $\gamma$ be an elliptic element of $G(F)$. With $G_\gamma$ the centralizer of $\gamma$ in $G$, here we will compute the measure of $\overline{G_\gamma(F)}\backslash\overline{G_\gamma(\mathbb{A}_F)}$. The result is given in Theorem 4.16 below. A related discussion can be found in [16, Section 5].

The basic idea is straightforward: we know that $G_\gamma(\mathbb{A}_F) = \mathbb{A}_F[\gamma]^* = \mathbb{A}_E^*$, where $E = F[\gamma]$ is a quadratic extension of $F$. (The proof of this fact given in [23, Proposition 26.1] for $F = \mathbb{Q}$ applies to any number field.) The center of $G(\mathbb{A}_F)$ is isomorphic to $\mathbb{A}_F^*$, so

$$\overline{G_\gamma(\mathbb{A}_F)} \cong \mathbb{A}_F^*\backslash\mathbb{A}_E^* \tag{4-25}$$

topologically and algebraically. Finally, $G_\gamma(F) \cong F[\gamma]^* = E^*$ by loc. cit., so

$$\overline{G_\gamma(F)}\backslash\overline{G_\gamma(\mathbb{A}_F)} = \mathbb{A}_F^* E^*\backslash\mathbb{A}_E^* \cong (F^*\backslash\mathbb{A}_F^*)\backslash(E^*\backslash\mathbb{A}_E^*) \cong (F^*\backslash\mathbb{A}_F^1)\backslash(E^*\backslash\mathbb{A}_E^1),$$

where the superscript 1 indicates ideles of norm 1, and the latter isomorphism comes from modding out by an embedded copy of $\mathbb{R}^+$ in $\mathbb{A}_F^* \subseteq \mathbb{A}_E^*$. For any number field $L$ the measure of $L^*\backslash\mathbb{A}_L^1$ is computed in Tate's thesis under suitable normalization, which we may use with $L = E, F$ to obtain the measure of the above space. However, as will be seen, we need to be very careful about the normalization of measures, particularly in the last step.

**4.7.1.** *Quotient measure.* Recall that if $H < G$ are unimodular locally compact groups with Haar measures $\mu_H$ and $\mu_G$ and $H$ closed in $G$, there is a unique left $G$-invariant quotient measure $\mu_{G/H}$ on $G/H$ satisfying

$$\int_{G/H}\left[\int_H f(gh)\,d\mu_H(h)\right]d\mu_{G/H}(g) = \int_G f(g)\,d\mu_G(g) \quad \text{for all } f \in C_c(G).$$

**Lemma 4.14.** *Let $H$, $K$ and $T$ be unimodular locally compact groups, with Haar measures $\mu_T, \mu_H, \mu_K$, respectively. Assume that $H < K$, and let $G = T \times K$ and $J = T \times H$. Then relative to the product measures $\mu_G = \mu_T \times \mu_K$ and $\mu_J = \mu_T \times \mu_H$, we have $\mu_{G/J} = \mu_{K/H}$ on the group $G/J \cong K/H$.*

*Proof.* For $f \in C_c(G)$,

$$\begin{aligned}\int_{K/H}\left[\int_J f(xy)\,d\mu_J(y)\right]d\mu_{K/H}(x)&\\ &= \int_{K/H}\left[\int_H\int_T f(xht)\,d\mu_T(t)\,d\mu_H(h)\right]d\mu_{K/H}(x)\\ &= \int_K\left[\int_T f(kt)\,d\mu_T(t)\right]d\mu_K(k) = \int_G f(g)\,d\mu_G(g).\end{aligned}$$

□

**4.7.2.** *A volume from Tate's thesis.* Let $L$ be a number field with adele ring $\mathbb{A}_L = \prod_v' L_v$, where $v$ ranges over the places of $L$. In Tate's thesis, measures $\mu_v$ on the local multiplicative groups $L_v^*$ are normalized as follows. If $v$ is real,

$$d\mu_v(x) = \frac{dx}{|x|} \tag{4-26}$$

for $x \in \mathbb{R}^*$. If $v$ is complex,

$$d\mu_v(z) = 2\frac{dx\,dy}{x^2+y^2} = \frac{2}{r}dr\,d\theta \tag{4-27}$$

for $z = x + iy = re^{i\theta} \in \mathbb{C}^*$. Finally, at a nonarchimedean place $v$, $\mu_v$ is the Haar measure on $L_v^*$ satisfying

$$\mu_v(\mathcal{O}_v^*) = (\mathbb{N}\mathfrak{D}_v)^{-1/2}, \tag{4-28}$$

where $\mathcal{O}_v$ is the ring of integers of $L_v$, $\mathfrak{D}_v$ is the different of $L_v$ and $\mathbb{N}\mathfrak{D}_v = |\mathcal{O}_v/\mathfrak{D}_v|$. Taking the restricted product of the above local measures, we obtain a Haar measure

$$\mu_L = \prod_v{}'\mu_v \quad \text{on } \mathbb{A}_L^*.$$

Let $L_\infty^* = \prod_{v|\infty} L_v^*$; we embed it into $\mathbb{A}_L^*$ by taking 1's at the nonarchimedean components. We embed $\mathbb{R}^+$ into $L_\infty^*$ and hence into $\mathbb{A}_L^*$ via

$$\lambda(t) = (t^{1/n}, t^{1/n}, \ldots, t^{1/n}),$$

where $n = n_L = [L : \mathbb{Q}]$. Then if $L$ has $r_1$ real embeddings and $2r_2$ complex embeddings, for $t \in \mathbb{R}^+$ we have

$$|\lambda(t)|_{\mathbb{A}_L} = \prod_{v|\infty}|t|_v^{1/n} = t^{(r_1+2r_2)/n} = t$$

(recall that in the ideles we take the square of the usual absolute value at the complex places).

Let $T \cong \mathbb{R}^+$ denote the image of the map $\lambda$. We give it the Haar measure $dt/t$. We have

$$\mathbb{A}_L^* \cong T \times \mathbb{A}_L^1, \tag{4-29}$$

where $\mathbb{A}_L^1$ is the subgroup consisting of ideles of norm 1. There is a unique measure $\mu_L^1$ on $\mathbb{A}_L^1 \cong \mathbb{A}_L^*/T$ such that

$$\mu_L = \frac{dt}{t} \times \mu_L^1.$$

The multiplicative group $L^*$ embeds diagonally in $\mathbb{A}_L^*$ as a discrete subgroup, and by the product formula, $L^* \subseteq \mathbb{A}_L^1$.

**Theorem 4.15** [54, Theorem 4.3.2]. *The group $L^*$ is discrete and cocompact in $\mathbb{A}_L^1$. Giving $L^*$ the counting measure, for $\mu_L^1$ as above we have*

$$\mu_L^1(L^*\backslash\mathbb{A}_L^1) = \frac{2^{r_1}(2\pi)^{r_2}h(L)R_L}{|d_L|^{1/2}w_L},$$

*where $h(L)$, $R_L$, $d_L$ and $w_L$ are the class number, regulator, discriminant, and number of roots of unity of $L$, respectively.*

**Remark.** This is the residue of the Dedekind zeta function of $L$ at $s = 1$.

**4.7.3.** *Haar measure for orbital integrals.* Let $\gamma \in G(F)$ be an elliptic element. Here we define a Haar measure $\eta$ on $\overline{G_\gamma(\mathbb{A}_F)}$ which is convenient to use for computing the elliptic orbital integrals. Given a nonarchimedean place $v$ of $F$, $\gamma$ is necessarily either elliptic or hyperbolic in $G(F_v)$. We select a compact open subgroup $H_v$ of $\overline{G_\gamma(F_v)} = Z(F_v)\backslash G_\gamma(F_v)$ as follows. If $\gamma$ is elliptic in $G(F_v)$, then the full group is compact by Proposition 4.5, and we take $H_v = \overline{G_\gamma(F_v)}$. If $\gamma$ is hyperbolic in $G(F_v)$, then $G_\gamma(F_v)$ is conjugate to the diagonal subgroup $M(F_v)$. In this case we define $H_v$ to be the subgroup of $\overline{G_\gamma(F_v)}$ taken by this conjugation to $\overline{M}(\mathcal{O}_v) \cong \mathcal{O}_v^*$, where $\mathcal{O}_v$ is the ring of integers of $F_v$.

Next, we choose a local Haar measure $\eta_v$ on $\overline{G_\gamma(F_v)}$ for each place $v$ of $F$ as follows. If $v \nmid \infty$, we normalize $\eta_v$ so that $\eta_v(H_v) = 1$. If $v \mid \infty$ is a real place of $F$ and $\gamma$ is elliptic over $F_v$, we take $\eta_v(\overline{G_\gamma(F_v)}) = 1$. If $v \mid \infty$ and $\gamma$ is hyperbolic over $F_v$, then $\overline{G_\gamma(F_v)} \cong M(F_v)/F_v^* \cong F_v^*$, and we give it the measure $d\eta_v(x) = d\mu_v(x)$ for $\mu_v$ as in (4-26) or (4-27).[2]

Note that

$$\overline{G_\gamma(\mathbb{A}_F)} = \prod_v{}' \overline{G_\gamma(F_v)},$$

where the product is restricted relative to the subgroups $H_v$. We let $\eta$ denote the Haar measure on $\overline{G_\gamma(\mathbb{A}_F)}$ which is the restricted product of the above local measures $\eta_v$.

[2]With $F = \mathbb{Q}$, these are the measures that are used in the local orbital integral calculations in the present paper. See Sections 4.4 and 4.5 for finite $\ell \nmid N$ and [23, Section 26.2] for the $\ell = \infty$ calculation yielding (4-12). For $\ell \mid N$, in Section 6 we will use the same measure used in Section 4.5.

As explained in (4-25), for $E = F[\gamma]$ we have

$$\overline{G_\gamma(\mathbb{A}_F)} \cong \mathbb{A}_E^*/\mathbb{A}_F^*.$$

So another natural measure on $\overline{G_\gamma(\mathbb{A}_F)}$ is the quotient measure $\mu_{E/F}$ coming from the Haar measures $\mu_E$ and $\mu_F$ on $\mathbb{A}_E^*$ and $\mathbb{A}_F^*$ obtained by taking $L = E$ and $L = F$ respectively in Section 4.7.2.

Let us next determine the constant relating the two measures $\eta$ and $\mu_{E/F}$. For a place $v$ of $F$ and a place $w$ of $E$ lying over $v$, we have defined the measures $\mu_v$ and $\mu_w$ on $F_v^*$ and $E_w^*$ in Section 4.7.2. We let $\mu_v' = \prod_{w|v} \mu_w$ be the product measure on $E_v^* = \prod_{w|v} E_w^*$, and define $\bar{\mu}_v'$ to be the corresponding quotient measure on $E_v^*/F_v^* \cong \overline{G_\gamma(F_v)}$. Then $\mu_{E/F} = \prod_v' \bar{\mu}_v'$ where $v$ runs over the places of $F$. For each $v$ we need to find the constant relating $\eta_v$ to $\bar{\mu}_v'$.

Let $v$ be a nonarchimedean place of $F$. Suppose $\gamma$ is hyperbolic in $G(F_v)$, so that $\eta_v(\overline{M}(\mathcal{O}_v)) = 1$. Let $w$, $\overline{w}$ be the primes of $E$ lying over $v$. Then

$$E_v := E \otimes F_v \cong E_w \oplus E_{\overline{w}},$$

and $\mu_v' = \mu_w \times \mu_{\overline{w}}$ on

$$G_\gamma(F_v) \cong E_v^* \cong E_w^* \times E_{\overline{w}}^* \cong F_v^* \times F_v^*.$$

Hence

$$\mu_v'(\mathcal{O}_w^* \times \mathcal{O}_{\overline{w}}^*) = \mu_w(\mathcal{O}_w^*)\mu_{\overline{w}}(\mathcal{O}_{\overline{w}}^*) = (\mathbb{N}\mathfrak{D}_w)^{-1/2}(\mathbb{N}\mathfrak{D}_{\overline{w}})^{-1/2}$$

by (4-28). (This is in fact equal to $\mathbb{N}\mathfrak{D}_v$, but we prefer to leave it unsimplified for global reasons.) Likewise, the diagonally embedded subgroup $F_v^* \subseteq E_v^*$ has measure $\mu_v(\mathcal{O}_v^*) = (\mathbb{N}\mathfrak{D}_v)^{-1/2}$. Therefore the quotient measure $\bar{\mu}_v'$ on $E_v^*/F_v^* \cong \overline{G_\gamma(F_v)}$ gives the open subgroup $(\mathcal{O}_w^* \times \mathcal{O}_{\overline{w}}^*)/\mathcal{O}_v^* \cong \overline{M}(\mathcal{O}_v)$ the measure $\frac{(\mathbb{N}\mathfrak{D}_w)^{-1/2}(\mathbb{N}\mathfrak{D}_{\overline{w}})^{-1/2}}{(\mathbb{N}\mathfrak{D}_v)^{-1/2}}$. Consequently,

$$\eta_v = \frac{(\mathbb{N}\mathfrak{D}_w)^{1/2}(\mathbb{N}\mathfrak{D}_{\overline{w}})^{1/2}}{(\mathbb{N}\mathfrak{D}_v)^{1/2}} \bar{\mu}_v'$$

for such $v$.

Now suppose $\gamma$ is elliptic in $G(F_v)$ (again with $v$ nonarchimedean). Then there is a unique valuation $w$ of $E$ extending $v$, and $E_w = F_v[\gamma]$ is a quadratic extension of $F_v$. Let $\mathcal{O}_w$ be its ring of integers, with a uniformizer $\varpi$. Then for the ramification index $e_v = e(w/v) \in \{1, 2\}$,

$$\overline{G_\gamma(F_v)} \cong E_w^*/F_v^* = \bigcup_{j=0}^{e_v-1} \varpi^j \mathcal{O}_w^*/\mathcal{O}_v^*$$

as in (4-17). By definition of the local component $\mu_w$ of $\mu_E$, $\mu_w(\mathcal{O}_w^*) = (\mathbb{N}\mathfrak{D}_w)^{-1/2}$. The local component of $\mu_F$ at $v$ gives $\mathrm{meas}(\mathcal{O}_v^*) = (\mathbb{N}\mathfrak{D}_v)^{-1/2}$. Therefore the

quotient measure $\bar{\mu}'_v$ satisfies

$$\bar{\mu}'_v(\mathcal{O}_w^*/\mathcal{O}_v^*) = \frac{(\mathbb{N}\mathfrak{D}_w)^{-1/2}}{(\mathbb{N}\mathfrak{D}_v)^{-1/2}}.$$

Since $\eta_v(\overline{G_\gamma(F_v)}) = 1$, it follows that

$$\eta_v = \frac{1}{e_v}\frac{(\mathbb{N}\mathfrak{D}_w)^{1/2}}{(\mathbb{N}\mathfrak{D}_v)^{1/2}}\bar{\mu}'_v$$

for such $v$.

Suppose $F_v = \mathbb{R}$ and $\gamma$ is elliptic in $G(F_v)$. Then $E_w = \mathbb{C}^*$ and $\overline{G_\gamma(F_v)} = \mathbb{C}^*/\mathbb{R}^*$. A set of representatives in $\mathbb{C}^*$ is $\{e^{i\theta} \mid \theta \in [0, \pi)\}$. Since the measure $\mu_v(x) = dx/|x|$ on $\mathbb{R}^*$ matches the factor $dr/r$ in $\mu_w(z) = (2dr\, d\theta)/r$ given in (4-27), it follows that

$$\bar{\mu}'_v(\mathbb{C}^*/\mathbb{R}^*) = 2\pi.$$

Since $\eta_v(\overline{G_\gamma(\mathbb{R})}) = 1$,

$$\eta_v = \frac{1}{2\pi}\bar{\mu}'_v$$

for such $v$.

If $F_v = \mathbb{R}$ or $\mathbb{C}$ and $\gamma$ is hyperbolic in $G(F_v)$, then as in the analogous nonarchimedean case,

$$E_v^* = E_w \times E_{\bar{w}} \cong F_v^* \times F_v^*,$$

and the quotient measure on $\overline{G_\gamma(F_v)} \cong E_v^*/F_v^* \cong F_v^*$ is $\bar{\mu}'_v(x) = \mu_v(x)$. In such cases we have likewise defined $\eta_v = \mu_v$. So $\eta_v = \bar{\mu}'_v$ for such $v$.

Putting everything together, we have shown that

$$\eta = \left[\prod_{v\nmid\infty}\frac{1}{e_v}\frac{\prod_{w\mid v}(\mathbb{N}\mathfrak{D}_w)^{1/2}}{(\mathbb{N}\mathfrak{D}_v)^{1/2}}\right]\left[\prod_{v\mid\infty,\,\gamma \text{ elliptic in } G(F_v)}\frac{1}{2\pi}\right]\mu_{E/F}.$$

We can simplify using three well-known facts from algebraic number theory (see, e.g., [41, Section III.2]):

(1) $e_v = 2$ if and only if $\mathfrak{p}_v \mid \mathfrak{d}_{E/F}$ where $\mathfrak{d}_{E/F}$ is the relative discriminant.

(2) The absolute discriminant of a local field is the absolute norm of the different.

(3) The product of the local discriminants is the global discriminant.

It follows that taking $d_F, d_E \in \mathbb{Z}$ to be the discriminants of $F$ and $E$ respectively,

$$\eta = \frac{|d_E|^{1/2}}{|d_F|^{1/2}}\frac{1}{2^{\omega_F(\mathfrak{d}_{E/F})}}\frac{1}{(2\pi)^{\alpha_\gamma}}\mu_{E/F}, \tag{4-30}$$

where $\omega_F(\mathfrak{d}_{E/F})$ is the number of distinct prime factors of $\mathfrak{d}_{E/F}$ in $\mathcal{O}_F$, and $\alpha_\gamma$ is the number of (real) archimedean places $v$ of $F$ for which $\gamma$ is elliptic in $G(F_v)$.

**4.7.4.** *The quotient measure.* We turn now to the quotient space whose measure we need to compute, namely $\overline{G_\gamma(F)}\backslash\overline{G_\gamma(\mathbb{A}_F)} \cong E^*\mathbb{A}_F^*\backslash\mathbb{A}_E^* \cong \mathbb{A}_E^*/\mathbb{A}_F^*E^*$. We have defined the quotient measure $\mu_{E/F}$ on $\mathbb{A}_E^*/\mathbb{A}_F^*$. By (4-29), we have

$$\mathbb{A}_F^* = T \times \mathbb{A}_F^1, \quad \mathbb{A}_E^* = T \times \mathbb{A}_E^1.$$

We regard $\mathbb{A}_F^*$ as a subset of $\mathbb{A}_E^*$, so $T$ is the set

$$T = \{(a, a, \dots, a) \in E_\infty^* \mid a > 0\} \subseteq \mathbb{A}_E^*.$$

We will use Lemma 4.14 to relate $\mu_{E/F}$ to the quotient measure on $\mathbb{A}_E^1/\mathbb{A}_F^1$ coming from the measures $\mu_E^1$ and $\mu_F^1$ defined below (4-29). Recall that $T$ is given the measure $d\mu_T(t) = dt/t$, where $t^{1/n_E} = a$ for $n_E = [E : \mathbb{Q}]$. In terms of the parameter $a$,

$$d\mu_T(t) = n_E \frac{da}{a}.$$

Notice that this is *not* the measure given to $\mathbb{R}^+$ upon taking $L = F$ in (4-29), which is $n_F(da/a) = (n_F/n_E)\, d\mu_T(t)$. In other words, for $\mu_T$ normalized as above, $\mu_F^1$ is defined by

$$\mu_F = \frac{1}{[E:F]}\mu_T \times \mu_F^1.$$

Therefore

$$\mu_F = \mu_T \times \frac{1}{[E:F]}\mu_F^1 = \mu_T \times \tfrac{1}{2}\mu_F^1.$$

Hence by Lemma 4.14, the quotient measure $\mu_{E/F}$ on $\mathbb{A}_E^*/\mathbb{A}_F^* \cong \mathbb{A}_E^1/\mathbb{A}_F^1$ is the same as the quotient measure coming from $\mu_E^1$ and $\frac{1}{2}\mu_F^1$. We denote this quotient measure by $\mu_{E/F}^1$.

Finally, taking the quotient by the discrete subgroup $E^*$ we have

$$\mu_{E/F}(\mathbb{A}_E^*/E^*\mathbb{A}_F^*) = \mu_{E/F}^1\left(\frac{(\mathbb{A}_E^1/E^*)}{(\mathbb{A}_F^1E^*/E^*)}\right) = \frac{\mu_E^1(\mathbb{A}_E^1/E^*)}{\frac{1}{2}\mu_F^1(\mathbb{A}_F^1/F^*)}. \tag{4-31}$$

As a technical point, the measure on the disjoint union

$$\mathbb{A}_F^1E^* = \bigcup_{\alpha \in E^*/F^*} \mathbb{A}_F^1\alpha$$

is simply $\frac{1}{2}\mu_F^1$ on each component since $E^*$ is given the counting measure. This explains why the quotient measure on $\mathbb{A}_F^1E^*/E^*$ is the same as $\frac{1}{2}\mu_F^1$ on $\mathbb{A}_F^1/F^*$. Applying Theorem 4.15 and (4-30) to (4-31), we immediately obtain the following.

**Theorem 4.16.** *Let $\gamma \in G(F)$ be an elliptic element, and let $\eta$ be the measure introduced in Section 4.7.3. Then for $E = F[\gamma]$,*

$$\eta(\overline{G_\gamma(F)}\backslash\overline{G_\gamma(\mathbb{A}_F)}) = \frac{2^{r_1(E)}(2\pi)^{r_2(E)}h(E)R_E}{2^{r_1(F)}(2\pi)^{r_2(F)}h(F)R_F} \cdot \frac{w_F}{w_E} \cdot \frac{2}{2^{\omega_F(\mathfrak{d}_{E/F})}(2\pi)^{\alpha_\gamma}},$$

*with notation as in Theorem 4.15, where $\omega_F(\mathfrak{d}_{E/F})$ is the number of distinct prime*

*ideals of $\mathcal{O}_F$ dividing the relative discriminant $\mathfrak{d}_{E/F}$, and $\alpha_\gamma$ is the number of (real) archimedean places $v$ of $F$ for which $\gamma$ is elliptic in $G(F_v)$.*

*In the special case where $F=\mathbb{Q}$ and $E=\mathbb{Q}[\gamma]$ is quadratic imaginary, we have $\alpha_\gamma=1,\ w_F=2,\ h(F)=R_E=R_F=1$, so*

$$\eta(\overline{G_\gamma(\mathbb{Q})}\backslash\overline{G_\gamma(\mathbb{A})})=\frac{2h(E)}{w_E\,2^{\omega(d_E)}},\tag{4-32}$$

*where $\omega(d_E)$ is the number of distinct prime factors of the discriminant $d_E$.*

With the above in place, the proof of Theorem 4.2 is complete.

## 5. The case $N=S^2T^3$: proof of Theorem 1.1

Henceforth, we will focus on the case where $N=S^2T^3$ for $S$ and $T$ relatively prime square-free integers. In order to prove Theorem 1.1, by Theorem 4.2 we just need to compute the orbital integrals at the primes dividing $N$. We begin in Sections 5.1 and 5.2 by reviewing the construction of supercuspidals of conductor $p^2$ (depth zero case) and of conductor $p^3$ (simple case), giving explicit formulas for the local test functions to be used. In Section 5.3 we outline the global setup, and then compute the required orbital integrals in Section 6 to complete the proof.

**5.1.** ***Depth zero supercuspidal representations.*** Let $F$ be a $p$-adic field, with ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{p}=\varpi\mathcal{O}$, and residue field $\Bbbk=\mathcal{O}/\mathfrak{p}$ of size $q$. The supercuspidal representations of $G(F)$ of minimal conductor are the so-called depth zero supercuspidals, with conductor $\mathfrak{p}^2$. They have the form $\sigma=\text{c-Ind}_{ZK}^{G(F)}(\rho)$, where $\rho$ is a $(q-1)$-dimensional representation of $K=G(\mathcal{O})$ inflated from a cuspidal representation of $G(\Bbbk)$, and c-Ind denotes compact induction. Some of their properties are summarized below (see, e.g., [29] for more detail).

Temporarily, write $G=G(\Bbbk)$. Let $L$ be the unique quadratic extension of $\Bbbk$. The multiplicative group $L^*$ embeds as a nonsplit torus $\mathbb{T}\subseteq G$, with $\Bbbk^*$ mapping onto the center $Z\subseteq G$. A character $\nu: L^*\to\mathbb{C}^*$ is *primitive* (or regular) if $\nu\neq\nu^q$, or equivalently, if $\nu$ is not of the form $\chi\circ N_{\Bbbk}^L$ for a character $\chi$ of $\Bbbk^*$, where $N_{\Bbbk}^L$ is the norm map. There are $q(q-1)$ primitive characters of $L^*$. Given a character $\omega$ of $\Bbbk^*$, let $[\omega]$ denote the set of primitive characters $\nu$ satisfying $\nu|_{\Bbbk^*}=\omega$. By [29, Proposition 2.3], the cardinality of $[\omega]$ is

$$P_\omega=\begin{cases}q-1 & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is trivial,}\\ q+1 & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is nontrivial,}\\ q & \text{if } q \text{ is even.}\end{cases}\tag{5-1}$$

Let $U=\left(\begin{smallmatrix}1 & \Bbbk\\ 0 & 1\end{smallmatrix}\right)$ be the upper triangular unipotent subgroup of $G$. A representation of $G$ is *cuspidal* if it does not contain a $U$-fixed vector. Fix a nontrivial additive character

$$\psi:\Bbbk\to\mathbb{C}^*.$$

We will always take $\psi(x) = e(x/p) = e^{2\pi i x/p}$ if $\Bbbk = \mathbb{Z}/p\mathbb{Z}$. We may view $\psi$ as a character of $U$ in the obvious way.

Given a primitive character $\nu$ of $\mathbb{T}$, there is a unique irreducible cuspidal representation $\rho_\nu$ of dimension $q-1$ satisfying

$$\operatorname{Ind}_{ZU}^{G}(\Bbbk)(\nu \otimes \psi) = \rho_\nu \oplus \operatorname{Ind}_{\mathbb{T}}^{G} \nu.$$

Every cuspidal representation arises in this way, and $\rho_\nu \cong \rho_{\nu'}$ if and only if $\nu' \in \{\nu, \nu^q\}$.

We have the following well-known formula for the character of $\rho_\nu$. For $x \in G(\Bbbk)$,

$$\operatorname{tr} \rho_\nu(x) = \begin{cases} (q-1)\,\nu(x) & \text{if } x \in Z, \\ -\nu(z) & \text{if } x = zu,\ z \in Z,\ u \in U,\ u \neq 1, \\ -\nu(x) - \nu^q(x) & \text{if } x \in \mathbb{T},\ x \notin Z, \\ 0 & \text{if no conjugate of } x \text{ belongs to } \mathbb{T} \cup ZU. \end{cases} \tag{5-2}$$

Because $\nu(c^{-1}xc) = \nu(x^q)$ for all $c \in N_G(\mathbb{T}) - \mathbb{T}$, there is no ambiguity evaluating $\operatorname{tr} \rho_\nu(y)$ using the third row above if $y$ is conjugate in $G(\Bbbk)$ to $x \in \mathbb{T}$.

Working now in the group $G(F)$, given the surjection $K \to G(\Bbbk)$ obtained by reduction modulo $\mathfrak{p}$, we may view $\rho_\nu$ as a representation of $K$. Its central character is given by $z \mapsto \nu(z(1+\mathfrak{p}))$ for $z \in \mathcal{O}^*$. By choosing a complex number $\nu(\varpi)$ of norm 1, we may extend $\rho_\nu$ to a representation of $ZK$, and then

$$\sigma_\nu = \text{c-}\operatorname{Ind}_{ZK}^{G(F)}(\rho_\nu)$$

is an irreducible unitary supercuspidal representation of conductor $\mathfrak{p}^2$. Its formal degree under the normalization $\operatorname{meas}(K) = 1$ is

$$d_{\sigma_\nu} = \dim \rho_\nu = q - 1. \tag{5-3}$$

The only equivalences among the representations $\sigma_\nu$ are $\sigma_\nu \cong \sigma_{\nu^q}$ (provided $\nu^q(\varpi)$ is defined to be the same complex number as $\nu(\varpi)$).

We define the test function $f_\mathfrak{p} : G(F) \to \mathbb{C}$ by

$$f_\mathfrak{p}(g) = \begin{cases} \overline{\operatorname{tr} \rho_\nu(g)} & \text{if } g \in ZK, \\ 0 & \text{otherwise,} \end{cases} \tag{5-4}$$

where $\operatorname{tr} \rho_\nu$ is given in (5-2).

**Proposition 5.1.** *Suppose $\sigma_\nu$ has trivial central character. Then its root number is given by*

$$\epsilon_\nu = \epsilon\left(\tfrac{1}{2}, \sigma_\nu, \psi\right) = \begin{cases} -(-1)^{(q+1)/r} & \text{if } q \text{ is odd}, \\ -1 & \text{if } q \text{ is even}, \end{cases} \tag{5-5}$$

*where $r$ is the order of $\nu$ in the character group of $L^*$. Suppose further that $q$ is odd and $4 \nmid (q-1)$ so that $\alpha^2 = -1$ for some $\alpha \in L^* - \Bbbk^*$. Then*

$$\epsilon_\nu = -\nu(\alpha). \tag{5-6}$$

**Remark.** Under the hypothesis, $\nu|_{\Bbbk^*}$ is trivial, which is equivalent to $r\,|\,(q+1)$ when $q$ is odd.

*Proof.* The root number coincides with the Atkin–Lehner sign of the representation [48, 3.2.2 Theorem]. We will show that it is a Gauss sum for $\nu$, which can be evaluated explicitly. The Atkin–Lehner sign $\epsilon_\nu$ is defined by

$$\sigma_\nu\left(\begin{pmatrix} & 1 \\ \varpi^2 & \end{pmatrix}\right)\varphi = \epsilon_\nu\,\varphi,$$

where $\varphi$ is a new vector in the space of $\sigma_\nu$. Note that $\epsilon_\nu^2 = 1$ since $\sigma\left(\begin{pmatrix} & 1 \\ \varpi^2 & \end{pmatrix}^2\right) = \sigma\left(\begin{pmatrix} \varpi^2 & \\ & \varpi^2 \end{pmatrix}\right)$ acts trivially under the hypothesis of trivial central character.

A model for $\rho_\nu$ on the space $\mathbb{C}[\Bbbk^*]$ of complex-valued functions on $\Bbbk^*$ is described in [29], following [47]. In terms of this model, the new space $(\text{c-Ind}_{ZK}^{G(F)}(\rho_\nu))^{K_1(\mathfrak{p}^2)}$ is spanned by the function $\varphi : G(F) \to \mathbb{C}[\Bbbk^*]$ supported on the coset $ZK\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}K_1(\mathfrak{p}^2)$ and defined by

$$\varphi\left(zk\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}\right) = \rho_\nu(zk)\,w \quad (z \in Z,\ k \in K),$$

where $w \in \mathbb{C}[\Bbbk^*]$ is the constant function 1 [29, Proposition 3.1]. In particular,

$$\varphi\left(\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}\right)(1) = w(1) = 1.$$

Therefore the Atkin–Lehner eigenvalue is given by

$$\begin{aligned}
\epsilon_\nu = \left[\sigma_\nu\left(\begin{pmatrix} & 1 \\ \varpi^2 & \end{pmatrix}\right)\varphi\right]\left(\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}\right)(1) &= \varphi\left(\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ \varpi^2 & \end{pmatrix}\right)(1) \\
&= \varphi\left(\begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}\right)(1) \\
&= \nu(\varpi)\left(\rho_\nu\left(\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}\right)w\right)(1) \\
&= \left(\rho_\nu\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\right)w\right)(1),
\end{aligned}$$

since we are assuming $\nu|_{F^*} = 1$. Let $f_a \in \mathbb{C}[\Bbbk^*]$ be the characteristic function of $a \in \Bbbk^*$, so that $w = \sum_{a\in\Bbbk^*} f_a$. Using [29, (2-11)] we see that $\rho_\nu\left(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\right)w = w$, and just below (2-16) of the same reference, we have

$$\left(\rho_\nu\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right)f_a\right)(1) = -\frac{1}{q}\nu(a^{-1})\sum_{\substack{u\in L^* \\ N(u)=a}}\psi(\mathrm{tr}_{\Bbbk}^{L}(u))\,\nu(u) \quad \text{for all } a \in \Bbbk^*.$$

We are assuming that $\nu|_{\Bbbk^*} = 1$, so $\nu(a^{-1}) = 1$, and summing over $a \in \Bbbk^*$ we have

$$\epsilon_\nu = -\frac{1}{q} \sum_{u \in L^*} \psi(\mathrm{tr}_{\Bbbk}^{L}(u))\, \nu(u).$$

This Gauss sum can be evaluated explicitly by an elementary calculation, giving (5-5); see [2, Theorem 11.6.1] for details.

Now suppose $q$ is odd and $4 \nmid (q-1)$, and let $t$ be a generator of the cyclic group $L^*$, so in particular $t^{(q^2-1)/2} = -1$. If $\nu$ has order $r$, there exists $j$ with $\gcd(j, r) = 1$ such that $\nu(t) = e(j/r)$. Taking $\alpha = t^{(q^2-1)/4}$,

$$\nu(\alpha) = e\left(\frac{j(q+1)(q-1)}{4r}\right) = (-1)^{\frac{j(q+1)}{r}\frac{q-1}{2}} = (-1)^{\frac{j(q+1)}{r}},$$

since $\frac{1}{2}(q-1)$ is odd by hypothesis. The above is equal to $(-1)^{(q+1)/r} = -\epsilon_\nu$, since $r$ is odd when $j$ is even, and $2 | (q+1)$. This proves (5-6). □

**Corollary 5.2.** *Fix $\epsilon \in \{\pm 1\}$. Then the number of depth zero supercuspidal representations of $G(F)$ with trivial central character and root number $\epsilon$ is*

$$\begin{cases} \frac{1}{4}(q-1) & \text{if } q \equiv 1 \bmod 4, \\ \frac{1}{4}(q+1) & \text{if } q \equiv 3 \bmod 4 \text{ and } \epsilon = 1, \\ \frac{1}{4}(q-3) & \text{if } q \equiv 3 \bmod 4 \text{ and } \epsilon = -1, \\ 0 & \text{if } q \text{ is even and } \epsilon = 1, \\ \frac{1}{2}q & \text{if } q \text{ is even and } \epsilon = -1. \end{cases}$$

*Proof.* With notation as in (5-1), the number of supercuspidals with a given central character $\omega$ is $\frac{1}{2}P_\omega$. (We divide by 2 to account for the fact that $\nu$ and $\nu^q$ induce the same supercuspidal.) So the assertion for $q$ even is immediate from (5-1) and (5-5).

Let $q$ be odd, and let $t$ be a generator of the cyclic group $L^*$. Then $t^{q+1}$ is a generator of $\Bbbk^*$. The characters of $L^*$ are the maps $\nu_m$ defined by

$$\nu_m(t) = e\left(\frac{m}{q^2-1}\right)$$

for $0 \le m < q^2 - 1$. We consider only those characters satisfying $\nu_m|_{\Bbbk^*} = 1$, i.e., $(q-1) | m$. Notice that $\nu_m$ is imprimitive if and only if $\nu_m^{q-1} = 1$, which holds if and only if $(q+1) | m$. So we consider the values $m = k(q-1)$ (for $1 \le k < (q+1)$) which are not multiples of $q+1$, i.e., $k \ne \frac{1}{2}(q+1)$.

The order of $\nu_m$ is

$$\frac{q^2-1}{\gcd(m, q^2-1)} = \frac{q+1}{\gcd(k, q+1)}. \tag{5-7}$$

By (5-5), $\sigma_{\nu_m}$ has root number

$$\epsilon_{\nu_m} = -(-1)^{\gcd(k,q+1)} = -(-1)^k, \tag{5-8}$$

since $q+1$ is even. Notice that the removed value $\frac{1}{2}(q+1)$ of $k$ is odd if and only if $q \equiv 1 \bmod 4$. So in this case, among the remaining $q-1$ values of $k$, half are odd and half are even. If $q \equiv 3 \bmod 4$, then $\frac{1}{2}(q-1)+1 = \frac{1}{2}(q+1)$ of the remaining values of $k$ are odd, and $\frac{1}{2}(q-1)-1 = \frac{1}{2}(q-3)$ are even.

To count supercuspidal representations, we divide the number of relevant $k$'s by 2 since the distinct characters $\nu_m$ and $\nu_m^q$ induce the same representation. □

**5.2.** ***Simple supercuspidal representations.*** With notation as in the previous section, we recall here the construction of the supercuspidal representations of $G(F)$ of conductor $\mathfrak{p}^3$. The central character of any such representation is at most tamely ramified. So we begin by fixing a character $\omega_\mathfrak{p}$ of the center $Z = Z(F) \cong F^*$ of $G(F)$, trivial on $1+\mathfrak{p}$.

Define the following compact open subgroup of $G(F)$:

$$K' = \begin{pmatrix} 1+\mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}.$$

Fix a nontrivial character

$$\psi : \Bbbk \to \mathbb{C}^*,$$

which we also regard as a character of $\mathcal{O}$ trivial on $\mathfrak{p}$. Given $t \in \Bbbk^*$, define a character $\chi = \chi_t : K' \to \mathbb{C}^*$ by

$$\chi\left(\begin{pmatrix} a & b \\ c\varpi & d \end{pmatrix}\right) = \psi(b+tc). \tag{5-9}$$

The matrix

$$g_t = g_\chi = \begin{pmatrix} & t \\ \varpi & \end{pmatrix}$$

normalizes $K'$, and furthermore

$$\chi(g_\chi^{-1} k g_\chi) = \chi(k) \quad \text{for all } k \in K'. \tag{5-10}$$

Given $\chi$ as above, let

$$H' = ZK' \cup g_\chi ZK'. \tag{5-11}$$

Although it is not reflected in the notation $H'$, this set depends on both $t$ and the fixed choice of $\varpi$. Given that $g_\chi^2 = t\varpi$, we may extend $\chi$ to a character $\chi_\zeta$ of $H'$ via

$$\chi_\zeta(g_\chi^d zk) = \zeta^d \omega_\mathfrak{p}(z)\chi(k) \tag{5-12}$$

for $z \in Z$ and $k \in K'$, where $\zeta$ is a fixed complex number satisfying

$$\zeta^2 = \omega_\mathfrak{p}(t\varpi). \tag{5-13}$$

**Proposition 5.3.** *The compactly induced representation* $\sigma_\chi^\zeta = \text{c-Ind}_{H'}^{G(F)}(\chi_\zeta)$ *is an irreducible supercuspidal representation of conductor* $\mathfrak{p}^3$, *with root number*

$$\epsilon\left(\tfrac{1}{2}, \sigma_\chi^\zeta, \psi\right) = \zeta.$$

*Conversely, every irreducible admissible representation of $G(F)$ of conductor $\mathfrak{p}^3$ with central character trivial on $1+\mathfrak{p}$ arises in this way.*

*Proof.* See [31]. For a more recent treatment using the above notation (but on $\mathrm{GL}_n$), see [27, Sections 4 and 5 and Proposition 7.2]. The root number is computed in [1, Corollary 3.12]. □

We will also use the notation

$$\sigma_t^\zeta = \sigma_\chi^\zeta$$

for $t$, $\chi$ as in (5-9), though it should be borne in mind that the representation depends also on the choice of additive character $\psi$ and uniformizer $\varpi$. When $F = \mathbb{Q}_p$, we will always take $\varpi = p$ and

$$\psi(x) = e(x/p) = e^{2\pi i x/p} \quad \text{for } x \in \mathbb{Z}/p\mathbb{Z}.$$

Henceforth we assume that $\omega_\mathfrak{p}$, and hence also $\sigma_t^\zeta$, is unitary. Under the normalization $\mathrm{meas}(\overline{G(\mathcal{O})}) = 1$, the formal degree of $\sigma_\chi^\zeta$ is

$$d_\chi = \tfrac{1}{2}(q^2-1). \tag{5-14}$$

This is seen, e.g., from (6.4) of [27] and the last line of the proof of Corollary 6.5 of the same paper.

We define the matrix coefficient $f_\mathfrak{p} : G(F) \to \mathbb{C}$ by

$$f_\mathfrak{p}(g) = d_\chi \overline{\left\langle \sigma_t^\zeta(g)\frac{\phi}{\|\phi\|}, \frac{\phi}{\|\phi\|}\right\rangle},$$

where $\phi \in \text{c-Ind}_{H'}^{G(F)}(\chi_\zeta)$ is the function

$$\phi(g) = \begin{cases} \chi_\zeta(g) & \text{if } g \in H', \\ 0 & \text{otherwise.} \end{cases} \tag{5-15}$$

Note that

$$\|\phi\|^2 = \int_{\overline{G}(F)} |\phi(g)|^2\, dg = \mathrm{meas}(\overline{H}'). \tag{5-16}$$

Likewise,

$$\begin{aligned}\langle \sigma_t^\zeta(g)\,\phi, \phi\rangle = \int_{\overline{G}(F)} \phi(xg)\,\overline{\phi(x)}\,dx &= \int_{\overline{H}'} \phi(xg)\,\overline{\chi_\zeta(x)}\,dx \\ &= \begin{cases} \mathrm{meas}(\overline{H}')\chi_\zeta(g) & \text{if } g \in H', \\ 0 & \text{otherwise.} \end{cases}\end{aligned} \tag{5-17}$$

By (5-14), (5-16), and (5-17), we have

$$f_\mathfrak{p}(g) = \begin{cases} \frac{1}{2}(q^2-1)\overline{\chi_\zeta(g)} & \text{if } g \in H', \\ 0 & \text{otherwise.} \end{cases} \tag{5-18}$$

**5.3.** ***Global setup.*** Fix square-free integers $S, T > 0$ with $ST > 1$ and $\gcd(S, T) = 1$, and let $k > 2$. Set $N = S^2T^3$, and let $\omega'$ be a Dirichlet character of modulus $N$ satisfying

$$\omega'(-1) = (-1)^k. \tag{5-19}$$

Let $\omega$ be the Hecke character attached to $\omega'$ in (4-1). We assume in addition that for each $p \mid N$, $\omega_p$ is trivial on $1 + p\mathbb{Z}_p$, since this is true of the central character of every supercuspidal representation of conductor $\leq p^3$. Equivalently, the conductor of $\omega'$ divides $ST$.

**Proposition 5.4.** *If $N = 2^2$ or $2^3$ and $k$ is odd, there is no such character.*

*Proof.* If $N$ is a power of 2, then by (4-3) and (5-19), $(-1)^k = \omega'(-1) = \omega_2(-1) = 1$ since $\omega_2$ is trivial on $\mathbb{Z}_2^* = 1 + 2\mathbb{Z}_2$. So $k$ must be even. □

Under the stated hypotheses, for each $p \mid S$, $\omega_p$ is trivial on $1 + p\mathbb{Z}_p$. We may thus view $\omega_p$ as a character of $(\mathbb{Z}_p/p\mathbb{Z}_p)^* = \mathbb{F}_p^*$. For each such $p$, fix a primitive character $\nu_p$ of $\mathbb{F}_{p^2}^*$ such that $\nu_p|_{\mathbb{F}_p^*} = \omega_p$. Recall that the number $P_{\omega_p} > 0$ of such primitive characters is given in (5-1). We define $\nu_p(p) = \omega_p(p)$ and extend multiplicatively so that $\nu_p$ can also be viewed as a character of $\mathbb{Q}_p^*$, which allows us to view $\rho_{\nu_p}$ as a representation of $Z_pK_p$ with central character $\omega_p$. We let

$$\sigma_p = \sigma_{\nu_p} = \text{c-Ind}_{Z_pK_p}^{G(\mathbb{Q}_p)}(\rho_{\nu_p})$$

be the associated supercuspidal representation of $G(\mathbb{Q}_p)$. The number of isomorphism classes of supercuspidal representations of conductor $p^2$ and central character $\omega_p$ is $\frac{1}{2}P_{\omega_p}$.

For each prime $p \mid T$, fix a simple supercuspidal representation $\sigma_p = \sigma_{t_p}^{\zeta_p}$ of $G(\mathbb{Q}_p)$ with central character $\omega_p$, where $t_p \in (\mathbb{Z}/p\mathbb{Z})^*$ and $\zeta_p^2 = \omega_p(t_p p)$. When the prime $p$ is understood, we sometimes write $t, \zeta$ instead of $t_p, \zeta_p$. By (4-2),

$$\zeta_p^2 = \omega_p(t_p p) = \omega_p(t_p) \prod_{\ell \mid N, \ell \neq p} \omega_\ell(p^{-1}). \tag{5-20}$$

In particular, when $N = p^3$ for $p$ prime, $\zeta_p^2 = \omega_p(t_p)$.

Having made the above choices, we let $\hat{\sigma} = (\sigma_p)_{p \mid N}$ denote this tuple of local representations. Then $S_k(\hat{\sigma}) \subseteq S_k^{\text{new}}(S^2T^3, \omega')$.

Now consider the test function

$$f = f^{\mathfrak{n}} = f_\infty \prod_{p \mid N} f_p \prod_{\ell \nmid N} f_\ell^{\mathfrak{n}} \tag{5-21}$$

as in (4-10) with $N = S^2T^3$, where, for $p \mid S$ (resp. $p \mid T$), $f_p$ is the chosen test function given in (5-4) (resp. in (5-18)).

The above setup is slightly different from that used in (4-10) and Proposition 4.1 since $f_p$ is not a single matrix coefficient when $p \mid S$, but a certain sum of matrix

coefficients, and without the formal degree coefficient. Nevertheless, the conclusions of Proposition 4.1 do hold for the above test function, as the next result shows.

**Proposition 5.5.** *With $f$ defined above,* $\operatorname{tr}(T_{\mathfrak{n}} \mid S_k(\hat{\sigma})) = \mathfrak{n}^{(k/2)-1} \operatorname{tr} R(f)$.

**Remark.** This is not special to depth zero supercuspidals. By [29, Proposition 1.2], the proof below applies with any unramified (even power conductor) supercuspidals $\sigma_p$ at $p \mid S$, using $d_{\sigma_p} = \dim \rho$ in place of $p-1$, where $\sigma_p = \text{c-Ind}_{ZK}^{G}(\rho)$. (Ramified supercuspidals may be induced from a *character* of an appropriately chosen open compact-mod-center subgroup, so for these, one can use a test function analogous to (5-18).)

*Proof.* In the proof of Proposition 4.1, we used the fact [23, Corollary 10.26] that for $\sigma = \sigma_p$, the operator $\sigma(d_\sigma \overline{\langle \sigma(g)w, w\rangle})$ is the orthogonal projection of the space of $\sigma$ onto $\mathbb{C}w$. For $f_p$ in (5-4), by [29, Proposition 1.1], there is an orthonormal set $\{w_1, \ldots, w_{p-1}\}$ of vectors in the space of $\sigma$ such that

$$f_p(g) = \sum_{j=1}^{p-1} \overline{\langle \sigma(g) w_j, w_j \rangle}.$$

So $\sigma(d_\sigma f_p) = \sigma((p-1) f_p)$ is the orthogonal projection onto $\operatorname{Span}\{w_1, \ldots, w_{p-1}\}$. Therefore using this local test function in the proof of Proposition 4.1 would give us a block sum of $p-1$ copies of the matrix for $\mathfrak{n}^{1-(k/2)} T_{\mathfrak{n}}$. To get the correct trace, we would need to divide by $p-1$, which is achieved by simply taking $f_p$ instead of $(p-1) f_p$. □

Noting that $f_p(1) = \dim \rho_v = p - 1 = d_{\sigma_p}$ for $p \mid S$, the identity term in the formula for $\operatorname{tr} R(f)$ is

$$\overline{\omega'(\mathfrak{n}^{1/2})} \, \tfrac{1}{12}(k-1) \prod_{p \mid S} (p-1) \prod_{p \mid T} \tfrac{1}{2}(p^2-1), \tag{5-22}$$

as seen between the brackets in Theorem 4.2. We remark that this is not always an integer when $\mathfrak{n} = 1$. For example consider the case where $S = 1$. For $p \geq 3$ prime,

$$v_2(p^2-1) = v_2(p-1) + v_2(p+1) \geq 3,$$

with equality holding precisely when $p \equiv 3, 5 \bmod 8$. (Here, $v_2$ is the 2-adic valuation.) It follows easily that when $\mathfrak{n} = 1$, the identity term $\frac{1}{12}(k-1) \prod_{p \mid T} \frac{1}{2}(p^2-1)$ fails to be an integer in exactly the following situations:

- $T = 2$ and $k \not\equiv 1 \bmod 8$.
- $T = 3$ and $k \not\equiv 1 \bmod 3$.
- $T = 2p$ for some $p \equiv 3, 5 \bmod 8$, and $k$ is even.

So in such instances, when $S = \mathtt{n} = 1$ the elliptic contribution to $|H_k(\hat{\sigma})|$ in Theorem 4.2 must be nonzero for this simple reason.

The list of relevant matrices in the trace formula of Theorem 4.2 can be refined in certain situations.

**Proposition 5.6.** *Let $N = S^2T^3$ as above, let $f = f^{\mathtt{n}}$ be the test function defined in (5-21), let $M \mid T$, and $0 \le r < \sqrt{4\mathtt{n}/M}$. Then $\Phi\left(\left(\begin{smallmatrix} 0 & -\mathtt{n}M \\ 1 & rM \end{smallmatrix}\right), f\right) = 0$ in each of the following situations*:

- *$r = 0$ and $k$ is odd.*
- *There exists $p \mid N$ such that $X^2 - rMX + \mathtt{n}M$ has a root in $\mathbb{Q}_p$.*
- *There exists $p \mid M$ such that $-pt_p/\mathtt{n}M$ is not a square modulo $p$, where $t_p$ is the parameter of the local representation $\sigma_{t_p}^{\zeta_p}$.*
- *There exists $p \mid (T/M)$ such that $X^2 - rMX + \mathtt{n}M \equiv (X - z)^2 \bmod p$ has no solution $z \in (\mathbb{Z}/p\mathbb{Z})^*$.*

**Remark.** For the case $\mathtt{n} = 1$, we can refine the list of relevant $\gamma$ even further (see Proposition 7.9 below).

*Proof.* The first bullet point follows from (4-13).

Let $\gamma = \left(\begin{smallmatrix} 0 & -\mathtt{n}M \\ 1 & rM \end{smallmatrix}\right)$, and suppose that $\Phi(\gamma, f) \neq 0$. Then by Proposition 4.3, $\gamma$ is elliptic in $G(\mathbb{Q}_p)$, which gives the second bullet point.

For the third bullet point, suppose $p \mid M$. Write $\det \gamma = up$ for some $u \in \mathbb{Z}_p^*$. Assuming the local orbital integral $\Phi(\gamma, f_p)$ is nonzero, $f_p(g^{-1}\gamma g) \neq 0$ for some $g \in G(\mathbb{Q}_p)$. Then $g^{-1}\gamma g$ belongs to the ramified component of $\mathrm{Supp}(f_p)$, i.e., writing $t = t_p$,

$$g^{-1}\gamma g = z \begin{pmatrix} & t \\ p & \end{pmatrix} \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in Z g_{\chi_p} K'$$

for some $b, c \in \mathbb{Z}_p$, $a, d \in 1 + p\mathbb{Z}_p$, and $z \in \mathbb{Z}_p^*$. Taking determinants, we have

$$up = -tpz^2(ad - pbc),$$

and hence

$$u \equiv -tz^2 \bmod p. \tag{5-23}$$

This shows that $-t/u$ is a quadratic residue modulo $p$.

Finally, if $p \mid (N/M)$, then $\det \gamma \in \mathbb{Z}_p^*$ so if $\Phi(\gamma, f_p) \neq 0$, some conjugate $g^{-1}\gamma g$ lies in the unramified component of $\mathrm{Supp}(f_p)$:

$$g^{-1}\gamma g = z \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in Z K'$$

for $z, a, b, c, d$ as above. Taking determinants, $\det \gamma \equiv z^2 \bmod p$. Taking the trace, $\mathrm{tr}\, \gamma \equiv 2z \bmod p$. Hence $P_\gamma(X) \equiv X^2 - 2zX + z^2 \equiv (X - z)^2 \bmod p$. □

## 6. Local orbital integrals at primes $p \mid N$ for $N = S^2T^3$

Our goal here is to compute

$$\Phi(\gamma, f_p) = \int_{\overline{G_\gamma(\mathbb{Q}_p)}\backslash \bar{G}(\mathbb{Q}_p)} f_p(g^{-1}\gamma g)\, dg$$

taking for $f_p$ the test functions given in (5-4) and (5-18), and for $\gamma$ the matrices given in Theorem 4.2, and using the quotient measure defined in Section 4.5, so

$$\Phi(\gamma, f_p) = \int_{\bar{G}(\mathbb{Q}_p)} f_p(g^{-1}\gamma g)\, dg.$$

With these calculations, Theorem 1.1 will follow immediately from Theorem 4.2.

We will use the strategy adopted by Palm in [43, Proposition 9.11.3] which avoids the use of lattices or buildings. There are errors in the statement and proof of his proposition, so we cannot simply quote the result. However, the basic method is sound and can be adapted to give the result in the cases of interest to us here.

The following lemma will allow us to rewrite the integral in such a way as to exploit the structure of the support of $f_p$.

**Lemma 6.1** [43, Lemma 6.4.10]. *Let $G$ be a unimodular locally compact group, and suppose $I_1$, $I_2$ are two open compact subgroups of $G$, each given total Haar measure* 1. *Then for any choice of Haar measure on $G$ we have*

$$\int_G \phi(g)\, dg = \sum_{x \in I_1\backslash G/I_2} \mathrm{meas}_G(I_1 x I_2) \int_{I_1}\int_{I_2} \phi(i_1 x i_2)\, di_2\, di_1 \tag{6-1}$$

*for all $\phi \in C_c(G)$.*

*Proof.* For $\phi \in C_c(G)$, we see that

$$\int_G \phi(g)\, dg = \int_G \int_{I_1}\int_{I_2} \phi(i_1 g i_2)\, di_2\, di_1\, dg$$

by changing the order of integration and using the bi-invariance of $dg$. The inner double integral defines a compactly supported function $F$ of $g \in G$ which is constant on double cosets $I_1 g I_2$, and is therefore a finite linear combination of characteristic functions of such double cosets. The identity (6-1) clearly holds for the characteristic function of a double coset. By linearity it holds for $F$ as well, so

$$\begin{aligned}
\int_G \phi(g)\, dg &= \int_G F(g)\, dg \\
&= \sum_{x \in I_1\backslash G/I_2} \mathrm{meas}_G(I_1 x I_2) \int_{I_1}\int_{I_2} F(i_1 x i_2)\, di_2\, di_1 \\
&= \sum_{x \in I_1\backslash G/I_2} \mathrm{meas}_G(I_1 x I_2) F(x) \\
&= \sum_{x \in I_1\backslash G/I_2} \mathrm{meas}_G(I_1 x I_2) \int_{I_1}\int_{I_2} \phi(i_1 x i_2)\, di_2\, di_1.
\end{aligned}$$

□

**6.1.** ***Preliminaries when $p|T$.*** Throughout much of this section, we will work over a $p$-adic field $F$ with notation as in Section 5.2, and write $G$ for $G(F)$, and $\overline{G}$ for $G/Z$. Having fixed a simple supercuspidal representation $\sigma_t^\zeta$ of $G$ with unitary central character $\omega_\mathfrak{p}$, we take $f_\mathfrak{p}$ to be the test function given in (5-18).

Applying Lemma 6.1 to (4-18), we have

$$\begin{aligned}\Phi(\gamma, f_\mathfrak{p}) &= \int_{\overline{G}} f_\mathfrak{p}(g^{-1}\gamma g)\, dg \\ &= \sum_{x \in \overline{K}'\backslash \overline{G}/\overline{K}'} \mathrm{meas}_{\overline{G}}(\overline{K}' x \overline{K}') \int_{\overline{K}'} \int_{\overline{K}'} f_\mathfrak{p}(h_2^{-1} x^{-1} h_1^{-1} \gamma h_1 x h_2)\, dh_1\, dh_2,\end{aligned}$$

where each $dh_i$ is normalized to have total measure 1. Since $f_\mathfrak{p}|_{K'}$ is a character, $h_2$ has no effect, and we obtain

$$\Phi(\gamma, f_\mathfrak{p}) = \sum_{x \in \overline{K}'\backslash \overline{G}/\overline{K}'} \mathrm{meas}_{\overline{G}}(\overline{K}' x \overline{K}') \int_{\overline{K}'} f_\mathfrak{p}(x^{-1} h^{-1} \gamma h x)\, dh. \tag{6-2}$$

In order to compute the above, we need a few preparations. First, recall the affine Bruhat decomposition

$$G = K'MK' \cup K'MwK' = K'MK' \cup K'Mg_\chi K',$$

where $w = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$ and $M$ is the diagonal subgroup [7, Proposition 17.1]. Accordingly, we may take as a set of representatives $x \in \overline{K}'\backslash \overline{G}/\overline{K}'$ the elements $x = m$ and $x = mg_\chi$ for

$$m \in \left\{ \begin{pmatrix} y & \\ & 1 \end{pmatrix}, \begin{pmatrix} y & \\ & \varpi^j \end{pmatrix}, \begin{pmatrix} \varpi^n & \\ & y \end{pmatrix} \,\middle|\, j > 0,\ n > 0,\ y \in (\mathcal{O}/\mathfrak{p})^* \right\}. \tag{6-3}$$

For each such $x$ we need to compute the integral in (6-2), which we denote by

$$K_\gamma(x) = \int_{\overline{K}'} f_\mathfrak{p}(x^{-1} h^{-1} \gamma h x)\, dh.$$

By (5-10),

$$f_\mathfrak{p}(g_\chi^{-1} g g_\chi) = f_\mathfrak{p}(g) \quad \text{for all } g.$$

Therefore $K_\gamma(x g_\chi) = K_\gamma(x)$. Furthermore, since $g_\chi$ normalizes $K'$, the measure of $\overline{K}' x \overline{K}'$ is unchanged if $x$ is replaced by $x g_\chi$. It follows that

$$\Phi(\gamma, f_\mathfrak{p}) = 2 \sum_{x \text{ in (6-3)}} \mathrm{meas}_{\overline{G}}(\overline{K}' x \overline{K}') K_\gamma(x). \tag{6-4}$$

**Lemma 6.2.** *Let $x = \left(\begin{smallmatrix} \varpi^n & \\ & y \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} y & \\ & \varpi^n \end{smallmatrix}\right)$ for $n \geq 0$ and $y \in \mathcal{O}^*$. Then with measure on $\overline{G}$ normalized so that* $\mathrm{meas}(\overline{K}) = 1$,

$$\mathrm{meas}_{\overline{G}}(\overline{K}' x \overline{K}') = \frac{q^n}{q^2 - 1}. \tag{6-5}$$

*Proof.* We may assume that $y = 1$ since, for example,

$$\operatorname{meas}\left(\overline{K}'\begin{pmatrix}\varpi^n & \\ & y\end{pmatrix}\overline{K}'\right) = \operatorname{meas}\left(\overline{K}'\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}\overline{K}'\begin{pmatrix}1 & \\ & y\end{pmatrix}\right)$$
$$= \operatorname{meas}\left(\overline{K}'\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}\overline{K}'\right).$$

Likewise, since $g_\chi$ normalizes $K'$ and $g_\chi^{-1}\left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right)g_\chi = \left(\begin{smallmatrix}1 & \\ & \varpi^n\end{smallmatrix}\right)$, we may assume that $x = \left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right)$.

We claim that for $n \geq 0$,

$$K'\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}K' = \bigcup_{b\in\mathcal{O}/\mathfrak{p}^n}\begin{pmatrix}\varpi^n & b\\ 0 & 1\end{pmatrix}K', \tag{6-6}$$

a disjoint union. The union is disjoint since

$$\begin{pmatrix}\varpi^n & b_1\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}\varpi^n & b_2\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & \frac{b_2-b_1}{\varpi^n}\\ 0 & 1\end{pmatrix},$$

which is in $K'$ if and only if $b_1 \equiv b_2 \bmod \mathfrak{p}^n$. The inclusion $\supseteq$ in (6-6) follows from

$$\begin{pmatrix}\varpi^n & b\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix} \in K'\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}.$$

The reverse inclusion follows from

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix} = \begin{pmatrix}\varpi^n & bd^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}a-cbd^{-1} & 0\\ c\varpi^n & d\end{pmatrix}.$$

By the decomposition (6-6),

$$\operatorname{meas}\left(\overline{K}'\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}\overline{K}'\right) = q^n \operatorname{meas}(\overline{K}') = \frac{q^n}{q^2-1},$$

since $\operatorname{meas}(\overline{K}') = 1/(q^2-1)$ when $\operatorname{meas}(\overline{K}) = 1$, as shown in the proof of [27, Corollary 6.5]. □

If $x = \left(\begin{smallmatrix}y & \\ & 1\end{smallmatrix}\right)$, then $K_\gamma(x) = f_\mathfrak{p}(\gamma^y)$ where $\gamma^y = \left(\begin{smallmatrix}y^{-1} & \\ & 1\end{smallmatrix}\right)\gamma\left(\begin{smallmatrix}y & \\ & 1\end{smallmatrix}\right)$, since $f_\mathfrak{p}$ is a character of $K'$, $\left(\begin{smallmatrix}y & \\ & 1\end{smallmatrix}\right)$ normalizes $K'$, and we give $\overline{K}'$ measure 1. Thus, in view of the above lemma, (6-4) now becomes

$$\Phi(\gamma, f_\mathfrak{p}) = \frac{2}{q^2-1}\sum_{y\in(\mathcal{O}/\mathfrak{p})^*} f_\mathfrak{p}(\gamma^y)$$
$$+ 2\sum_{n=1}^{\infty}\frac{q^n}{q^2-1}\sum_{y\in(\mathcal{O}/\mathfrak{p})^*}\left[K_\gamma\left(\begin{pmatrix}\varpi^n & \\ & y\end{pmatrix}\right) + K_\gamma\left(\begin{pmatrix}y & \\ & \varpi^n\end{pmatrix}\right)\right]. \tag{6-7}$$

To compute $K_\gamma(x)$, we fix coordinates on $\overline{K}'$ with the following.

**Lemma 6.3.** *Let $G, H, K$ be compact topological groups, with $G = HK$ and $H \cap K = \{1\}$. Let $dh$ and $dk$ be the respective Haar measures on $H, K$ of total measure 1. Then the Haar measure on $G$ of measure 1 is given by*

$$\int_G f(g)\,dg = \int_H \int_K f(hk)\,dk\,dh.$$

*Proof.* This is a special case of [23, Lemma 7.13]. □

We will use the Iwahori decomposition [7, (7.3.1)] of $K'$. Letting $M(1+\mathfrak{p}) = \left(\begin{smallmatrix} 1+\mathfrak{p} & \\ & 1+\mathfrak{p} \end{smallmatrix}\right)$, $N(\mathcal{O}) = \left(\begin{smallmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{smallmatrix}\right)$, and $N'(\mathfrak{p}) = \left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p} & 1 \end{smallmatrix}\right)$, the decomposition

$$K' = N(\mathcal{O}) \cdot N'(\mathfrak{p}) \cdot M(1+\mathfrak{p})$$

is a (topological) direct product, and the same is true for any ordering of the three factors. We will take $\mathrm{meas}(\overline{K}') = \mathrm{meas}(K') = 1$, so that applying the above lemma, this Haar measure on $\overline{K}'$ is given by both of the following:

$$\int_{\overline{K}'} \phi(k)\,dk = \int_{\mathcal{O}} \int_{\mathcal{O}} \int_{M(1+\mathfrak{p})} \phi\left(\begin{pmatrix} 1 & 0 \\ \varpi c & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} m\right) dm\,db\,dc \tag{6-8}$$

$$= \int_{\mathcal{O}} \int_{\mathcal{O}} \int_{M(1+\mathfrak{p})} \phi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi c & 1 \end{pmatrix} m\right) dm\,dc\,db, \tag{6-9}$$

where $dm, db, dc$ each have total measure 1.

**6.2.** ***The case where $p \mid T$ and $\gamma$ is ramified.*** The aim here is to compute $\Phi(\gamma, f_p)$ when $p \mid N$ and $\gamma$ is ramified at $p$, i.e., $v_p(\det \gamma)$ is odd. As above we work over a $p$-adic field $F$ with uniformizer $\varpi$, and a fixed supercuspidal representation $\sigma_t^\zeta$ of $G(F)$ as in Section 5.2. We can assume that $v_{\mathfrak{p}}(\det \gamma) = 1$, and further, by Lemma 4.12, that $v_{\mathfrak{p}}(\mathrm{tr}\,\gamma) \geq 1$. So we will consider matrices in the $F$-rational canonical form

$$\gamma = \begin{pmatrix} 0 & -u\varpi \\ 1 & v\varpi \end{pmatrix} = w \begin{pmatrix} 1 & v\varpi \\ 0 & u\varpi \end{pmatrix} \tag{6-10}$$

for $u \in \mathcal{O}^*$, $v \in \mathcal{O}$, and $w = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$.

**Proposition 6.4.** *For $\gamma$ as in (6-10) and $f_{\mathfrak{p}}$ as in (5-18), $\Phi(\gamma, f_{\mathfrak{p}}) = 0$ unless $-t/u$ is a square modulo $\mathfrak{p}$. If the latter condition holds and $y^2 \equiv -t/u \mod \mathfrak{p}$, then*

$$\Phi\left(\begin{pmatrix} 0 & -u\varpi \\ 1 & v\varpi \end{pmatrix}, f_{\mathfrak{p}}\right) = \bar{\zeta}\left(\overline{\psi(yv)}\,\omega_{\mathfrak{p}}(y) + \delta(\mathfrak{p} \nmid 2)\,\overline{\psi(-yv)}\,\omega_{\mathfrak{p}}(-y)\right),$$

*where $\psi$ is the nontrivial character of $\mathcal{O}/\mathfrak{p}$ used in (5-9). Thus, in the case of trivial central character (so $\zeta^2 = 1$), we have*

$$\Phi(\gamma, f_{\mathfrak{p}}) = \begin{cases} 2\zeta\,\mathrm{Re}(\psi(yv)) & \text{if } \mathfrak{p} \nmid 2, \\ \overline{\psi(yv)}\,\zeta & \text{if } \mathfrak{p} \mid 2. \end{cases}$$

*When the central character is trivial, $F = \mathbb{Q}_p$, and $v \in \mathbb{Z}$, this gives*

$$\Phi(\gamma, f_p) = \begin{cases} 2\zeta \cos\left(\frac{2\pi y v}{p}\right) & \text{if } p \neq 2, \\ (-1)^v \zeta & \text{if } p = 2. \end{cases} \tag{6-11}$$

*Proof.* We need to compute each term of (6-7). First, note that for $y \in (\mathcal{O}/\mathfrak{p})^*$,

$$\gamma^y = \begin{pmatrix} 0 & -u\varpi/y \\ y & v\varpi \end{pmatrix} = \begin{pmatrix} & t \\ \varpi & \end{pmatrix} \begin{pmatrix} y/\varpi & v \\ & -u\varpi/ty \end{pmatrix}$$

does not belong to the support of $f_\mathfrak{p}$. Hence $f_\mathfrak{p}(\gamma^y) = K_\gamma\left(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)\right) = 0$.

Next, suppose $x = \left(\begin{smallmatrix} y & \\ & \varpi^j \end{smallmatrix}\right)$ with $j > 0$ and $y \in \mathcal{O}^*$. Then we use the measure in (6-8):

$$K_\gamma(x) = \int_\mathcal{O}\int_\mathcal{O}\int_{M(1+\mathfrak{p})} f_\mathfrak{p}\left(x^{-1}m^{-1}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\varpi c & 1 \end{pmatrix}\gamma\begin{pmatrix} 1 & 0 \\ \varpi c & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} mx\right) dm\, db\, dc.$$

Note that $m$ commutes with $x$, and lies in the kernel of $f_\mathfrak{p}$. Therefore the integration over $M(1+\mathfrak{p})$ has no effect, and

$$K_\gamma(x) = \int_\mathcal{O}\int_\mathcal{O} f_\mathfrak{p}\left(x^{-1}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\varpi c & 1 \end{pmatrix}\gamma\begin{pmatrix} 1 & 0 \\ \varpi c & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} x\right) db\, dc.$$

Likewise

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & \\ & \varpi^j \end{pmatrix} = \begin{pmatrix} y & \\ & \varpi^j \end{pmatrix}\begin{pmatrix} 1 & b\varpi^j/y \\ & 1 \end{pmatrix}.$$

Note that the matrix on the right lies in $K'$ since $j > 0$, and in fact it is in the kernel of $f_\mathfrak{p}$. Therefore the integral over $b$ also has no effect, and

$$K_\gamma(x) = \int_\mathfrak{p} f_\mathfrak{p}\left(x^{-1}\begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix}\gamma\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} x\right) dr, \tag{6-12}$$

where $dr$ gives $\mathfrak{p}$ the measure 1.

Taking $\gamma = w\left(\begin{smallmatrix} 1 & v\varpi \\ 0 & u\varpi \end{smallmatrix}\right)$ as in (6-10),

$$\begin{aligned}\begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix} w \begin{pmatrix} 1 & v\varpi \\ 0 & u\varpi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} &= w\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+rv\varpi & v\varpi \\ ru\varpi & u\varpi \end{pmatrix} \\ &= w\begin{pmatrix} 1+r^2u\varpi+rv\varpi & (v+ru)\varpi \\ ru\varpi & u\varpi \end{pmatrix}.\end{aligned}$$

Writing the above as $w\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in w\left(\begin{smallmatrix} 1+\mathfrak{p}^2 & \mathfrak{p} \\ \mathfrak{p}^2 & \varpi\mathcal{O}^* \end{smallmatrix}\right)$,

$$\begin{aligned} x^{-1}\begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix}\gamma\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}x &= \begin{pmatrix} y^{-1} & \\ & \varpi^{-j} \end{pmatrix} w \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y & \\ & \varpi^{j} \end{pmatrix} \\ &= w\begin{pmatrix} \varpi^{-j} & \\ & y^{-1} \end{pmatrix}\begin{pmatrix} ay & b\varpi^{j} \\ cy & d\varpi^{j} \end{pmatrix} \\ &= w\begin{pmatrix} ay/\varpi^{j} & b \\ c & d\varpi^{j}/y \end{pmatrix} \\ &= g_\chi\begin{pmatrix} \varpi^{-1} & \\ & -t^{-1} \end{pmatrix}\begin{pmatrix} ay/\varpi^{j} & b \\ c & d\varpi^{j}/y \end{pmatrix} \\ &= g_\chi\begin{pmatrix} ay/\varpi^{j+1} & b/\varpi \\ -c/t & -d\varpi^{j}/ty \end{pmatrix}. \end{aligned}$$

Since the determinant is $u\varpi$, this belongs to the support of $f_\mathfrak{p}$ if and only if the matrix on the right belongs to $\mathcal{O}^*K'$. But this would require $j+1=0$, which is impossible since $j>0$. Hence

$$K_\gamma\left(\begin{pmatrix} y & \\ & \varpi^{j} \end{pmatrix}\right) = 0$$

for all $j>0$ and all $y\in\mathcal{O}^*$.

Lastly, consider $x = \left(\begin{smallmatrix} \varpi^{n} & \\ & y \end{smallmatrix}\right)$ for $n>0$ and $y\in\mathcal{O}^*$. We proceed in just the same way, only this time using the coordinates given in (6-9). Taking $-b$ in place of $b$ for convenience, and eliminating the integral over $M(1+\mathfrak{p})$ with the same justification as before,

$$K_\gamma(x) = \int_\mathcal{O}\int_\mathcal{O} f_\mathfrak{p}\left(x^{-1}\begin{pmatrix} 1 & 0 \\ -\varpi c & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\gamma\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi c & 1 \end{pmatrix}x\right) db\, dc.$$

Now

$$\begin{pmatrix} 1 & 0 \\ c\varpi & 1 \end{pmatrix}x = \begin{pmatrix} 1 & 0 \\ c\varpi & 1 \end{pmatrix}\begin{pmatrix} \varpi^{n} & \\ & y \end{pmatrix} = \begin{pmatrix} \varpi^{n} & \\ & y \end{pmatrix}\begin{pmatrix} 1 & \\ c\varpi^{1+n}/y & 1 \end{pmatrix}.$$

The matrix on the right lies in the kernel of $f_\mathfrak{p}$. Therefore

$$K_\gamma(x) = \int_\mathcal{O} f_\mathfrak{p}\left(x^{-1}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\gamma\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}x\right) db. \tag{6-13}$$

Taking $\gamma = w\left(\begin{smallmatrix} 1 & v\varpi \\ 0 & u\varpi \end{smallmatrix}\right)$, we have

$$\begin{aligned} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & v\varpi \\ & u\varpi \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} &= w\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}\begin{pmatrix} 1 & -b+v\varpi \\ 0 & u\varpi \end{pmatrix} \\ &= w\begin{pmatrix} 1 & -b+v\varpi \\ -b & b^2 - v\varpi b + u\varpi \end{pmatrix}. \end{aligned}$$

Thus letting $P_\gamma(X)$ denote the characteristic polynomial of $\gamma$,

$$\begin{aligned} x^{-1}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}1 & -b\\ 0 & 1\end{pmatrix}x &= \begin{pmatrix}\varpi^{-n} & \\ & y^{-1}\end{pmatrix} w \begin{pmatrix}1 & -b+v\varpi\\ -b & P_\gamma(b)\end{pmatrix}\begin{pmatrix}\varpi^n & \\ & y\end{pmatrix}\\ &= w\begin{pmatrix}y^{-1} & \\ & \varpi^{-n}\end{pmatrix}\begin{pmatrix}\varpi^n & y(-b+v\varpi)\\ -b\varpi^n & yP_\gamma(b)\end{pmatrix}\\ &= w\begin{pmatrix}\varpi^n/y & -b+v\varpi\\ -b & yP_\gamma(b)/\varpi^n\end{pmatrix}\\ &= g_\chi\begin{pmatrix}\varpi^{-1} & \\ & -t^{-1}\end{pmatrix}\begin{pmatrix}\varpi^n/y & -b+v\varpi\\ -b & yP_\gamma(b)/\varpi^n\end{pmatrix}\\ &= g_\chi\begin{pmatrix}\varpi^{n-1}/y & v-b/\varpi\\ b/t & -yP_\gamma(b)/t\varpi^n\end{pmatrix}. \end{aligned} \tag{6-14}$$

Since the determinant is $u\varpi$, the above belongs to the support of $f_\mathfrak{p}$ if and only if the matrix on the right belongs to $\mathcal{O}^*K'$. This means in particular that $n=1$ and $b\in\mathfrak{p}$. Make the change of variables $b=c\varpi$, $db=|\varpi|dc=q^{-1}dc$. Then

$$P_\gamma(b)=\varpi(u-vc\varpi+c^2\varpi),$$

and

$$K_\gamma\left(\begin{pmatrix}\varpi & \\ & y\end{pmatrix}\right)=q^{-1}\int_{\mathcal{O}} f_\mathfrak{p}\left(g_\chi y^{-1}\begin{pmatrix}1 & yv-cy\\ cy\varpi/t & -y^2(u-vc\varpi+c^2\varpi)/t\end{pmatrix}\right)dc.$$

From the definition of $K'$, the integrand is nonzero if and only if $y^2\equiv -t/u \bmod \mathfrak{p}$. (We have already seen in Proposition 5.6 that $-t/u$ must be a square mod $\mathfrak{p}$.) Assuming this to be the case, then from (5-9), (5-12), and (5-18), we have

$$\begin{aligned} K_\gamma(x)&=q^{-1}\overline{\omega_\mathfrak{p}(y^{-1})\zeta}\,d_\chi\int_{\mathcal{O}}\overline{\psi(yv-cy+cy)}\,dc\\ &=q^{-1}\omega_\mathfrak{p}(y)\bar\zeta\,d_\chi\overline{\psi(yv)}=\frac{q^2-1}{2q}\omega_\mathfrak{p}(y)\overline{\zeta\psi(yv)}. \end{aligned} \tag{6-15}$$

To recap, for $\gamma=\left(\begin{smallmatrix}0 & -u\varpi\\ 1 & v\varpi\end{smallmatrix}\right)$, $K_\gamma(x)=0$ unless $x=\left(\begin{smallmatrix}\varpi & \\ & y\end{smallmatrix}\right)$ for $y^2\equiv -t/u \bmod \mathfrak{p}$, so (6-7) becomes

$$\Phi(\gamma,f_\mathfrak{p})=\frac{2q}{q^2-1}\sum_{y\in(\mathcal{O}/\mathfrak{p})^*}K_\gamma\left(\begin{pmatrix}\varpi & \\ & y\end{pmatrix}\right)=\bar\zeta\sum_{\varepsilon\in\{\pm1 \bmod \mathfrak{p}\}}\overline{\psi(\varepsilon y_0 v)}\,\omega_\mathfrak{p}(\varepsilon y_0),$$

where $y_0$ is any fixed solution to $y_0^2\equiv -t/u \bmod \mathfrak{p}$. Note that when $\mathfrak{p}\mid 2$, we can take $\varepsilon=1$, and if $F=\mathbb{Q}_2$ we can also take $y_0=1$. □

**6.3.** ***The case where $p\mid T$ and $\gamma$ is unramified.*** We adopt the same notation used in the previous subsection. Suppose $\gamma$ is unramified, i.e., $\mathrm{ord}_\mathfrak{p}(\det\gamma)$ is even. Scaling if needed, we may assume that $\det\gamma\in\mathcal{O}^*$. For the nonvanishing of $\Phi(\gamma,f_\mathfrak{p})$, it is

necessary that some conjugate of $\gamma$ belong to the unramified component of the support of $f_{\mathfrak{p}}$, namely $ZK'_{\mathfrak{p}}$. Given that $u = \det\gamma \in \mathcal{O}^*$, this means that $\operatorname{tr}\gamma$ must also be integral. So we may take $\gamma$ in rational canonical form

$$\gamma = \begin{pmatrix} 0 & -u \\ 1 & v \end{pmatrix} \tag{6-16}$$

for some $u \in \mathcal{O}^*$ and $v \in \mathcal{O}$.

**Proposition 6.5.** *For $\gamma$ elliptic in $G(F)$ and of the form (6-16), $\Phi(\gamma, f_{\mathfrak{p}}) = 0$ unless the characteristic polynomial $P_\gamma$ has a nonzero double root modulo $\mathfrak{p}$:*

$$P_\gamma(X) \equiv (X - z)^2 \bmod \mathfrak{p} \tag{6-17}$$

*for some $z \in (\mathcal{O}/\mathfrak{p})^*$. Under this condition,*

$$\Phi(\gamma, f_{\mathfrak{p}}) = \frac{\overline{\omega_{\mathfrak{p}}(z)}}{q} \sum_{n=1}^{\infty} \sum_{c \bmod \mathfrak{p}} \mathcal{N}_\gamma(c, n) \sum_{y \in (\mathcal{O}/\mathfrak{p})^*} \psi\left(\frac{yc}{z}\right) \psi\left(-\frac{t}{yz}\right)^{\delta(n=1)}, \tag{6-18}$$

*where $\psi$ is the nontrivial character of $\mathcal{O}/\mathfrak{p}$ used in (5-9), $t \in (\mathcal{O}/\mathfrak{p})^*$ is the parameter of $\sigma_t^\zeta$, and*

$$\mathcal{N}_\gamma(c, n) = \#\{b \bmod \mathfrak{p}^{n+1} \mid P_\gamma(b) \equiv c\varpi^n \bmod \mathfrak{p}^{n+1}\}.$$

**Remarks.** (1) Since $P_\gamma$ is irreducible over $F$, there exists $r$ such that $P_\gamma(X) \equiv 0 \bmod \mathfrak{p}^r$ has no solution, and hence $\mathcal{N}_\gamma(c, n) = 0$ for all pairs $(c, n)$ with $n \geq r$. So the series is actually a finite sum.

(2) When $n = 1$ the sum over $y$ is a Kloosterman sum. When $n > 1$,

$$\sum_y \psi\left(\frac{yc}{z}\right) = \begin{cases} q - 1 & \text{if } c \equiv 0 \bmod \mathfrak{p}, \\ -1 & \text{otherwise.} \end{cases}$$

(3) When $F = \mathbb{Q}_p$, the integer $\mathcal{N}_\gamma(c, n)$ is given explicitly in [26, Lemma 9.6], and presumably there is a version of that lemma for an arbitrary $p$-adic field. In particular, $N_\gamma(c, n) = 0$ unless $n \leq \operatorname{ord}_p(\Delta_\gamma) - 1$, and for such $n$,

$$N_\gamma(c, n) \leq p^{\lfloor (n+1)/2 \rfloor}$$

assuming $\gamma$ is elliptic in $G(\mathbb{Q}_p)$ and satisfies (6-17). This gives the following bound for the orbital integral: setting $\delta = \operatorname{ord}_p(\Delta_\gamma)$,

$$\begin{aligned} |\Phi(\gamma, f_p)| \leq \sum_{n=1}^{\delta-1} (p-1)(p^{1/2})^{n+1} &= p(p-1) \sum_{n=0}^{\delta-2} (p^{1/2})^n \\ &= p(p-1) \frac{(p^{1/2})^{\delta-1} - 1}{p^{1/2} - 1} \\ &= p(p^{1/2} + 1)(p^{-1/2} p^{\delta/2} - 1) \leq 2p|\Delta_\gamma|_p^{-1/2}. \end{aligned}$$

This illustrates the general bound given in [21, (1.8) and Theorem 3.11], according to which

$$|\Phi(\gamma, f_{\mathfrak{p}})| \le C \cdot (d_{\sigma_{\mathfrak{p}}})^{\eta} |\Delta_\gamma|_{\mathfrak{p}}^{-1/2},$$

where $C > 0$ and $\eta < 1$ depend only on $G(F)$.

*Proof of Proposition 6.5.* The first statement was proven in Proposition 5.6. Suppose $\Phi(\gamma, f_{\mathfrak{p}}) \neq 0$ for $\gamma$ as in (6-16). We will compute each term of (6-7). It is not hard to check that $f_{\mathfrak{p}}(\gamma^y) = 0$ and $K_\gamma\left(\left(\begin{smallmatrix} y & \\ & \varpi^n \end{smallmatrix}\right)\right) = 0$, since the matrices involved do not intersect the support of $f_{\mathfrak{p}}$. Therefore

$$\Phi(\gamma, f_{\mathfrak{p}}) = 2 \sum_{n=1}^{\infty} \frac{q^n}{q^2-1} \sum_{y \in (\mathcal{O}/\mathfrak{p})^*} K_\gamma\left(\left(\begin{smallmatrix} \varpi^n & \\ & y \end{smallmatrix}\right)\right). \tag{6-19}$$

Now fix $n \ge 1$ and $y \in \mathcal{O}^*$ and let $x = \left(\begin{smallmatrix} \varpi^n & \\ & y \end{smallmatrix}\right)$. As in (6-13), we have

$$K_\gamma(x) = \int_{\mathcal{O}} f_{\mathfrak{p}}\left(x^{-1} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} x\right) db.$$

By a quick calculation (see (6-14) with $u, v$ in place of $u\varpi, v\varpi$),

$$x^{-1} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} x = \begin{pmatrix} b & -y P_\gamma(b)/\varpi^n \\ \varpi^n/y & v-b \end{pmatrix}. \tag{6-20}$$

Since the determinant is $u \in \mathcal{O}^*$, this belongs to the support of $f_{\mathfrak{p}}$ only if it belongs to $\mathcal{O}^* K'$. In particular, $b \in \mathcal{O}^*$ and $P_\gamma(b) \equiv 0 \bmod \mathfrak{p}^n$. Therefore $b \in z + \mathfrak{p}$ for $z$ as in (6-17). From (6-17) we see that $v \equiv 2z \bmod \mathfrak{p}$ so $v - b \in z + \mathfrak{p}$ as well. Therefore, pulling out a factor of $z$ from the above matrix,

$$K_\gamma\left(\left(\begin{smallmatrix} \varpi^n & \\ & y \end{smallmatrix}\right)\right) = \overline{\omega_{\mathfrak{p}}(z)} \int_{z+\mathfrak{p}} f_{\mathfrak{p}}\left(\left(\begin{smallmatrix} 1 & -yP_\gamma(b)/z\varpi^n \\ \varpi^n/yz & 1 \end{smallmatrix}\right)\right) db.$$

Writing $P_\gamma(b) \equiv c\varpi^n \bmod \mathfrak{p}^{n+1}$ for some $c \in \mathcal{O}/\mathfrak{p}$, by (5-18) the integrand becomes

$$f_{\mathfrak{p}}\left(\left(\begin{smallmatrix} 1 & -yc/z \\ \varpi^n/yz & 1 \end{smallmatrix}\right)\right) = \tfrac{1}{2}(q^2-1)\, \overline{\psi\left(-\frac{yc}{z} + \frac{t\varpi^{n-1}}{yz}\right)}.$$

This depends (via $c$) only on the coset $b + \mathfrak{p}^{n+1}$ (in fact it depends only on $b + \mathfrak{p}^n$ but we will not use this). Each such coset has measure $q^{-(n+1)}$. Therefore if we let

$$\mathcal{N}_\gamma(c, n) = \#\{b \bmod \mathfrak{p}^{n+1} \mid P_\gamma(b) \equiv c\varpi^n \bmod \mathfrak{p}^{n+1}\}$$

for $c \in \mathcal{O}/\mathfrak{p}$, we find that

$$K_\gamma(x) = \overline{\omega_{\mathfrak{p}}(z)} \frac{q^2-1}{2q^{n+1}} \sum_{c \bmod \mathfrak{p}} \psi\left(\frac{yc}{z}\right) \psi\left(-\frac{t}{yz}\right)^{\delta(n=1)} \mathcal{N}_\gamma(c, n).$$

Inserting this into (6-19) gives the result. □

**Example 6.6.** For $M \in \mathbb{Z}_2^*$ and $f_2$ as in (5-18),

$$\Phi\left(\begin{pmatrix} & -M \\ 1 & \end{pmatrix}, f_2\right) = \begin{cases} 1 & \text{if } M \equiv 1 \bmod 4, \\ -3 & \text{if } M \equiv 3 \bmod 8, \\ 0 & \text{if } M \equiv 7 \bmod 8. \end{cases}$$

*Proof.* First, $\gamma = \left(\begin{smallmatrix} & -M \\ 1 & \end{smallmatrix}\right)$ is hyperbolic in $G(\mathbb{Q}_2)$ if and only if $-M$ is a square in $\mathbb{Q}_2$, which holds if and only if $M \equiv 7 \bmod 8$. In this case, $\Phi(\gamma, f_2) = 0$ by (3-4).

Assuming $M \not\equiv 7 \bmod 8$, we may apply Proposition 6.5. We need to determine

$$\mathcal{N}_\gamma(0, n) = \text{number of solutions to } x^2 \equiv -M \bmod 2^{n+1}$$

and

$$\mathcal{N}_\gamma(1, n) = \text{ number of solutions to } x^2 \equiv 2^n - M \bmod 2^{n+1}.$$

Given any odd integer $D$, the number of solutions to $x^2 \equiv D \bmod 2^j$ is

$$\begin{cases} 1, & j = 1, \\ 2, & j = 2,\ D \equiv 1 \bmod 4, \\ 0, & j = 2,\ D \equiv 3 \bmod 4, \\ 4, & j > 2,\ D \equiv 1 \bmod 8, \\ 0, & j > 2,\ D \not\equiv 1 \bmod 8 \end{cases}$$

[32, Theorem 87]. Therefore

$$\mathcal{N}_\gamma(0, n) = \begin{cases} 2 & \text{if } n = 1 \text{ and } M \equiv 3 \bmod 4, \\ 0 & \text{if } n = 1 \text{ and } M \equiv 1 \bmod 4, \\ 4 & \text{if } n \geq 2 \text{ and } M \equiv 7 \bmod 8, \\ 0 & \text{if } n \geq 2 \text{ and } M \not\equiv 7 \bmod 8, \end{cases}$$

$$\mathcal{N}_\gamma(1, n) = \begin{cases} 0 & \text{if } n = 1 \text{ and } M \equiv 3 \bmod 4, \\ 2 & \text{if } n = 1 \text{ and } M \equiv 1 \bmod 4, \\ 4 & \text{if } n = 2 \text{ and } M \equiv 3 \bmod 8, \\ 0 & \text{if } n = 2 \text{ and } M \not\equiv 3 \bmod 8, \\ 4 & \text{if } n \geq 3 \text{ and } M \equiv 7 \bmod 8, \\ 0 & \text{if } n \geq 3 \text{ and } M \not\equiv 7 \bmod 8. \end{cases}$$

By definition, $\psi_2(x) = (-1)^x$ for $x \in \mathbb{Z}$, and $\omega_2$ is trivial on $1 + 2\mathbb{Z}_2 = \mathbb{Z}_2^*$. So by (6-18) and the above,

$$\begin{aligned} \Phi(\gamma, f_2) &= \tfrac{1}{2}[\mathcal{N}_\gamma(0,1)\,\psi_2(0)\,\psi_2(1) + \mathcal{N}_\gamma(1,1)\,\psi_2(1)^2] \\ &\quad + \tfrac{1}{2}[\mathcal{N}_\gamma(0,2)\,\psi_2(0) + \mathcal{N}_\gamma(1,2)\,\psi_2(1)] \\ &= \begin{cases} \tfrac{1}{2}[0+2] + \tfrac{1}{2}[0+0] = 1 & \text{if } M \equiv 1 \bmod 4, \\ \tfrac{1}{2}[-2+0] + \tfrac{1}{2}[0-4] = -3 & \text{if } M \equiv 3 \bmod 8. \end{cases} \end{aligned}$$

□

**Example 6.7.** For $f_3$ as in (5-18) and any $m \in \mathbb{Z}_3^*$,

$$\Phi\left(\begin{pmatrix} 0 & -m^2 \\ 1 & m \end{pmatrix}, f_3\right) = \omega_3(-m)\, t \cdot 2_{t=1},$$

where $2_{t=1}$ is a factor of 2 which is present only when $t=1$. Here, $t \in \{\pm 1\} = (\mathbb{Z}/3\mathbb{Z})^*$ is the parameter of the fixed simple supercuspidal representation $\sigma_t^\zeta$ of $G(\mathbb{Q}_3)$.

**Remark.** When $N = 3$, we have $\omega_3(-1) = \omega'(-1) = (-1)^k$, so

$$\omega_3(-m) = \begin{cases} (-1)^k & \text{if } m \in 1 + 3\mathbb{Z}_3, \\ 1 & \text{if } m \in -1 + 3\mathbb{Z}_3. \end{cases}$$

*Proof.* We will apply Proposition 6.5. First note that

$$P_\gamma(X) = X^2 - mX + m^2 \equiv (X+m)^2 \bmod 3,$$

so we can take $z = -m$ in (6-17). We need to find

$$\mathcal{N}_\gamma(c, n) = \#\{b \mid b^2 - mb + m^2 \equiv c3^n \bmod 3^{n+1}\}.$$

If $b \in \mathbb{Z}_3^*$ is a double root of $P_\gamma$ modulo 3, then we may write $b = -m + 3d$, so

$$P_\gamma(b) = (-m+3d)^2 - m(-m+3d) + m^2 = 3m^2 + 9(d^2 - md) \in 3\mathbb{Z}_3^*.$$

Thus, $\operatorname{ord}_3(P_\gamma(b)) = 1$, which means that $\mathcal{N}_\gamma(c, n) = 0$ for all $n \geq 2$, and also $\mathcal{N}_\gamma(0, 1) = 0$. Some elementary calculations show that independently of $m$, $\mathcal{N}_\gamma(-1, 1) = 0$ and $\mathcal{N}_\gamma(1, 1) = 3$. In view of (6-18), this means

$$\begin{aligned} \Phi(\gamma, f_3) &= \tfrac{1}{3}(\overline{\omega_3(-m)})\, \mathcal{N}_\gamma(1,1)\left(\psi_3\left(\frac{1}{-m}\right)\psi_3\left(\frac{-t}{-m}\right) + \psi_3\left(\frac{1}{m}\right)\psi_3\left(\frac{-t}{m}\right)\right) \\ &= \overline{\omega_3(-m)}\left(e\left(\frac{1-t}{-3m}\right) + e\left(\frac{1-t}{3m}\right)\right) \\ &= \omega_3(-m)\left[e\left(\frac{t-1}{3}\right) + e\left(\frac{1-t}{3}\right)\right]. \end{aligned}$$

When $t = 1$ (resp. $t = -1$), the expression in the brackets equals 2 (resp. $-1$). □

**6.4.** ***The case where $p \mid S$.*** When $p \mid S$, the support of $f_p$ is contained in $Z_p K_p$, so the orbital integral vanishes unless $\gamma$ is unramified. We again work over a $p$-adic field $F$, with the usual notation, and fix a depth zero supercuspidal representation $\sigma_\nu$ of $G = G(F)$ for $\nu$ a primitive character of $\mathbb{F}_{q^2}^*$.

**Proposition 6.8.** *Let $f_\mathfrak{p}$ be the test function defined in (5-4), and let $\gamma = \left(\begin{smallmatrix} & -u \\ 1 & v \end{smallmatrix}\right)$ be an elliptic element of $G(F)$, where $u \in \mathcal{O}^*$ and $v \in \mathcal{O}$. If there exists $z \in (\mathcal{O}/\mathfrak{p})^*$ such that*

$$P_\gamma(X) \equiv (X - z)^2 \bmod \mathfrak{p},$$

*then*

$$\Phi(\gamma, f_{\mathfrak{p}}) = -\overline{\omega_{\mathfrak{p}}(z)} + \frac{\overline{\omega_{\mathfrak{p}}(z)}}{q} \sum_{n=1}^{\infty} \left[ (q-1)\, \mathcal{N}_\gamma(0,n) - \sum_{c \in (\mathcal{O}/\mathfrak{p})^*} \mathcal{N}_\gamma(c,n) \right], \quad (6\text{-}21)$$

*where* $\mathcal{N}_\gamma(c,n) = \#\{b \bmod \mathfrak{p}^{n+1} \mid P_\gamma(b) \equiv c\varpi^n \bmod \mathfrak{p}^{n+1}\}$.

*If* $P_\gamma(X)$ *is irreducible modulo* $\mathfrak{p}$, *then*

$$\Phi(\gamma, f_{\mathfrak{p}}) = -\overline{\nu(\gamma)} - \overline{\nu^q(\gamma)}, \quad (6\text{-}22)$$

*where we interpret the above to mean* $-\overline{\nu(x)} - \overline{\nu^q(x)}$ *if* $x \in \mathbb{F}_{q^2}^*$ *has the same minimum polynomial over* $\mathbb{F}_q$ *as the reduction of* $\gamma$ *mod* $\mathfrak{p}$, *i.e.*, $\gamma$ *is conjugate to* $x \in \mathbb{T}$.

**Remarks.** (1) The remaining possibility where $P_\gamma(X)$ has two distinct roots mod $\mathfrak{p}$ cannot occur due to Hensel's Lemma, since we are assuming that $\gamma$ is elliptic in $G(F)$.

(2) See the remarks after Proposition 6.5 regarding $\mathcal{N}_\gamma(c,n)$. In particular, the sum in (6-21) is finite, and when $F = \mathbb{Q}_p$ we find $|\Phi(\gamma, f_p)| \le 1 + 4|\Delta_\gamma|_p^{-1/2}$.

*Proof.* In this proof we write $\overline{G}$ for $\overline{G}(F)$, $Z$ for $Z(F)$, and $K$ for $G(\mathcal{O})$. By Lemma 6.1,

$$\begin{aligned}\Phi(\gamma, f_{\mathfrak{p}}) &= \int_{\overline{G}} f_{\mathfrak{p}}(g^{-1}\gamma g)\, dg \\ &= \sum_{x \in \overline{K}\backslash\overline{G}/\overline{K}} \mathrm{meas}_{\overline{G}}(\overline{K}x\overline{K}) \int_{\overline{K}} \int_{\overline{K}} f_{\mathfrak{p}}(h_2^{-1}x^{-1}h_1^{-1}\gamma h_1 x h_2)\, dh_1\, dh_2,\end{aligned}$$

with $dh_1$ and $dh_2$ each having total measure 1. The integrand is nonzero only if $x^{-1}h_1^{-1}\gamma h_1 x \in ZK$. Therefore, since $f_{\mathfrak{p}}$ is a trace, $h_2$ has no effect, so

$$\Phi(\gamma, f_{\mathfrak{p}}) = \sum_{x \in \overline{K}\backslash\overline{G}/\overline{K}} \mathrm{meas}_{\overline{G}}(\overline{K}x\overline{K}) \int_{\overline{K}} f_{\mathfrak{p}}(x^{-1}h\gamma h^{-1}x)\, dh.$$

(For convenience in what follows, we have set $h = h_1^{-1}$.)

By the Cartan decomposition of $G$, a set of representatives for $\overline{K}\backslash\overline{G}/\overline{K}$ is given by

$$\left\{ \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} \,\middle|\, n \ge 0 \right\}.$$

Arguing as in [36, Lemma 4.5.6(2)], for $x = \left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right)$,

$$|K\backslash KxK| = \begin{cases} q^{n-1}(q+1) & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

Therefore $\mathrm{meas}_{\overline{G}}(\overline{K}x\overline{K}) = q^{n-1}(q+1)$ when $n > 0$, so

$$\Phi(\gamma, f_{\mathfrak{p}}) = f_{\mathfrak{p}}(\gamma) + \sum_{n=1}^{\infty} q^{n-1}(q+1)K_{\gamma}(n), \tag{6-23}$$

where

$$K_{\gamma}(n) = \int_K f_{\mathfrak{p}}\left(\begin{pmatrix} \varpi^{-n} & \\ & 1 \end{pmatrix} h\gamma h^{-1} \begin{pmatrix} \varpi^{n} & \\ & 1 \end{pmatrix}\right) dh \quad (n > 0).$$

(We may integrate over $K$ since both $K$ and $\overline{K}$ have measure 1.) Write $h\gamma h^{-1} = \left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right) \in K$. Then

$$\begin{pmatrix} \varpi^{-n} & \\ & 1 \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} \varpi^{n} & \\ & 1 \end{pmatrix} = \begin{pmatrix} w & x/\varpi^{n} \\ y\varpi^{n} & z \end{pmatrix}. \tag{6-24}$$

This belongs to the support of $f_{\mathfrak{p}}$ only if $x \in \mathfrak{p}^n$.

In the integrand above, we can freely multiply $h$ by a diagonal element of $K$ since such an element commutes with $x$ and can be eliminated since $f_{\mathfrak{p}}$ is a trace. In particular, we can assume $\det h = 1$. Write $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\det h = 1$. Then

$$h\gamma h^{-1} = \begin{pmatrix} * & -b^2 + abv - a^2u \\ * & * \end{pmatrix}.$$

If $a \in \mathfrak{p}$ then we must have $b \in \mathcal{O}^*$ since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K$. But then the upper-right entry above cannot belong to $\mathfrak{p}^n$, so the integrand vanishes by (6-24). Therefore we may assume $a \in \mathcal{O}^*$, i.e., $h \in A$, where

$$A = \begin{pmatrix} \mathcal{O}^* & * \\ * & * \end{pmatrix} \cap K.$$

Let's find the measure of $A$. Let $K(\mathfrak{p}) = 1 + M_2(\mathfrak{p}) \subseteq A$. This is the kernel of the reduction mod $\mathfrak{p}$ map $K \to G(\mathcal{O}/\mathfrak{p})$. Since $|G(\mathcal{O}/\mathfrak{p})| = (q^2-1)(q^2-q)$, we see that $\mathrm{meas}(K(\mathfrak{p})) = 1/((q^2-1)(q^2-q))$. Let $\overline{A} = A \bmod K(\mathfrak{p})$. Thinking of $\overline{A}$ as a set of matrices in $G(\mathcal{O}/\mathfrak{p})$, we see that

$$|\overline{A}| = (q-1)\,q(q^2-q).$$

(There are $(q-1)\,q$ possible top rows, and then $q^2-q$ remaining choices for the bottom row.) Hence

$$\mathrm{meas}(A) = \frac{(q-1)\,q(q^2-q)}{(q^2-1)(q^2-q)} = \frac{q}{q+1}.$$

It is not hard to show that

$$A = \begin{pmatrix} \mathcal{O}^* & \\ & \mathcal{O}^* \end{pmatrix}\begin{pmatrix} 1 & \\ \mathcal{O} & 1 \end{pmatrix}\begin{pmatrix} 1 & \mathcal{O} \\ & 1 \end{pmatrix},$$

and that the corresponding decomposition of any element of $A$ is unique. Therefore by Lemma 6.3 we can use the above as a coordinate system for integration over $A$. Since, as noted above, the diagonal component has no effect on the value of the integral, we have

$$K_\gamma(n) = \frac{q}{q+1}\int_{\mathcal{O}}\int_{\mathcal{O}} f_{\mathfrak{p}}\left(x^{-1}\begin{pmatrix}1 & \\ c & 1\end{pmatrix}\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\gamma\begin{pmatrix}1 & -b\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ -c & 1\end{pmatrix}x\right) db\, dc,$$

where $x = \left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right)$, and $db$ and $dc$ each have total measure 1. The integral over $c$ can be eliminated, since $\left(\begin{smallmatrix}1 & \\ -c & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \\ -\varpi^n c & 1\end{smallmatrix}\right)$, and the rightmost matrix belongs to $K$. Therefore

$$\begin{aligned} K_\gamma(n) &= \frac{q}{q+1}\int_{\mathcal{O}} f_{\mathfrak{p}}\left(\begin{pmatrix}\varpi^{-n} & \\ & 1\end{pmatrix}\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\begin{pmatrix} & -u\\ 1 & v\end{pmatrix}\begin{pmatrix}1 & -b\\ & 1\end{pmatrix}\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}\right) db \\ &= \frac{q}{q+1}\int_{\mathcal{O}^*} f_{\mathfrak{p}}\left(\begin{pmatrix} b & -P_\gamma(b)/\varpi^n\\ 0 & v-b\end{pmatrix}\right) db \end{aligned}$$

as in (6-20). (As a reminder, $db$ is additive measure.) We have replaced the lower-left entry by 0, using the fact that by definition (see (5-4)), $f_{\mathfrak{p}}$ is sensitive only to the reduction of its argument mod $\mathfrak{p}$. Further, the integrand is nonzero only if $P_\gamma(b) \in \mathfrak{p}^n$. Under this condition, given that the characteristic polynomial of the matrix in the integrand is $P_\gamma(X)$, and this cannot have distinct roots mod $\mathfrak{p}$ as $\gamma$ is elliptic in $G(\mathbb{Q}_p)$, there exists $z \in (\mathcal{O}/\mathfrak{p})^*$ such that $b \equiv v - b \equiv z \bmod \mathfrak{p}$. In particular, the matrix (viewed modulo $\mathfrak{p}$) belongs to $ZU$, with notation as in (5-2). Write $P_\gamma(b) \equiv c\varpi^n \bmod \mathfrak{p}^{n+1}$, for $c \in \mathcal{O}/\mathfrak{p}$. The integrand becomes

$$\overline{\omega_{\mathfrak{p}}(z)}\, f_{\mathfrak{p}}\left(\begin{pmatrix}1 & -c/z\\ 0 & 1\end{pmatrix}\right) = \begin{cases}\overline{\omega_{\mathfrak{p}}(z)}(q-1) & \text{if } c = 0,\\ -\overline{\omega_{\mathfrak{p}}(z)} & \text{if } c \in (\mathcal{O}/\mathfrak{p})^*.\end{cases}$$

This depends (via $c$) only on the coset $b + \mathfrak{p}^{n+1}$, which has measure $q^{-(n+1)}$. Therefore

$$K_\gamma(n) = \overline{\omega_{\mathfrak{p}}(z)} \cdot \frac{q}{q+1} \cdot \frac{1}{q^{n+1}}\left[(q-1)\,\mathcal{N}_\gamma(0,n) - \sum_{c\in(\mathcal{O}/\mathfrak{p})^*}\mathcal{N}_\gamma(c,n)\right].$$

Plugging the above into (6-23), equation (6-21) follows. (Note that $f_{\mathfrak{p}}(\gamma) = -\overline{\omega_{\mathfrak{p}}(z)}$ in this case, since $\gamma - z = \left(\begin{smallmatrix}-z & -u\\ 1 & v-z\end{smallmatrix}\right) \not\equiv 0 \bmod \mathfrak{p}$, so $\gamma$ is conjugate mod $\mathfrak{p}$ to $zu$ for some $1 \neq u \in U$).

By the above discussion $K_\gamma(n) = 0$ for all $n > 0$ if $P_\gamma(X)$ is irreducible mod $\mathfrak{p}$. So in this case (6-23) gives $\Phi(\gamma, f_{\mathfrak{p}}) = f_{\mathfrak{p}}(\gamma) = -\overline{\nu(\gamma)} - \overline{\nu^q(\gamma)}$ by (5-2) and (5-4). □

## 7. General dimension formula, and examples with $N = S^2T^3$

When $\mathtt{n} = 1$, the list of relevant $\gamma$ in Theorem 4.2 can be simplified. The result is the following general dimension formula.

**Theorem 7.1.** *Let $N = \prod_{p|N} p^{N_p} > 1$ with $N_p \geq 2$ for all $p|N$. Fix $k > 2$ and a tuple $\hat{\sigma} = (\sigma_p)_{p|N}$ of supercuspidal representations so that $S_k(\hat{\sigma}) \subseteq S_k^{\mathrm{new}}(N, \omega')$ for a Dirichlet character $\omega'$, as detailed at the beginning of Section 4. Let $T$ be the product of all primes $p|N$ with $N_p$ odd. Let $f = f^1$ be the test function defined in (4-10) with $\mathtt{n} = 1$ but with $f_p$ chosen as in (7-10) below for all $p|T$. Then*

$$\begin{aligned}\dim S_k(\hat{\sigma}) = \frac{1}{12}(k-1)\prod_{p|N} d_{\sigma_p} + \frac{1}{2}\Phi\left(\begin{pmatrix} & -T \\ 1 & \end{pmatrix}, f\right) \\ + \frac{1}{2}\delta_{T\in 2\mathbb{Z}^+}\,\Phi\left(\begin{pmatrix} & -T/2 \\ 1 & \end{pmatrix}, f\right) + \delta_{T=2}\,\Phi\left(\begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}, f\right) \\ + \delta_{T=3}\,\Phi\left(\begin{pmatrix} 0 & -3 \\ 1 & 3 \end{pmatrix}, f\right) + \delta_{T\in\{1,3\}}\,\Phi\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, f\right).\end{aligned}$$

*Here, $d_{\sigma_p}$ is the formal degree of $\sigma_p$ relative to the Haar measure fixed in Section 2, and the orbital integrals $\Phi(\gamma, f)$ are given as in Theorem 4.2.*

*Proof.* The case where $T = 1$ is already contained in Theorem 4.2 by taking $\mathtt{n} = 1$. The simplifications when $T > 1$ are proven in Proposition 7.9 below. □

As with Theorem 1.1, using the results of Section 6 we can compute the above explicitly in any case of interest when $N = S^2T^3$ with $S$ and $T$ square-free relatively prime positive integers. Although there is not a particularly nice formula for all such levels, as an illustration we will work everything out in the two special cases where $N = S^2$ and $N = T^3$. These results are stated in Sections 7.1 and 7.4 respectively. In Section 7.5 we give some examples to illustrate Theorem 1.1 with $\mathtt{n} > 1$.

First, we highlight the following consequence of Theorem 7.1.

**Corollary 7.2.** *In the setting of Theorem 7.1 above, suppose that the weight $k$ is odd, so $\omega'(-1) = -1$. For $T$ as in Theorem 7.1, if $T > 3$ the elliptic terms vanish, so*

$$\dim S_k(\hat{\sigma}) = \tfrac{1}{12}(k-1)\prod_{p|N} d_{\sigma_p} \quad (k > 2 \text{ odd}, T > 3).$$

**Remark.** If $N = 2^2$ or $N = 2^3$, then $S_k(\hat{\sigma})$ is undefined when $k$ is odd since by Proposition 5.4 there is no appropriate nebentypus.

*Proof.* If $\gamma = \left(\begin{smallmatrix} & -T \\ 1 & \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} & -T/2 \\ 1 & \end{smallmatrix}\right)$, then $\Phi(\gamma, f_\infty) = 0$ when $k$ is odd, by (4-13). □

### 7.1. *Dimension formula and root number bias when $N = S^2$.*

When we set $T = 1$ and take $N = S^2$, the formula in Theorem 7.1 gives the following.

**Theorem 7.3.** *Let $N = S^2$ for $S$ square-free, $k > 2$, $\omega'$ a Dirichlet character of modulus $N$ and conductor dividing $S$, and let $\hat{\sigma} = (\sigma_p)_{p \mid S}$ be a tuple of depth zero supercuspidal representations chosen compatibly with $\omega'$ as in Section 5.3, with $T = 1$. Then the subspace $S_k(\hat{\sigma}) \subseteq S_k^{\mathrm{new}}(S^2, \omega')$ has dimension*

$$\dim S_k(\hat{\sigma}) = \tfrac{1}{12}(k-1)\prod_{p \mid N}(p-1) + A_1 + A_2,$$

*where*

$$A_1 = \tfrac{1}{4}(-1)^{S+1+(k/2)}\,\delta_{k \in 2\mathbb{Z}} \prod_{\text{odd } p \mid N} (-\overline{\nu_p(\alpha)} - \overline{\nu_p^p(\alpha)})\,\delta_{p \equiv 3 \bmod 4}, \tag{7-1}$$

*where $\nu_p$ is the primitive character of $\mathbb{F}_{p^2}^*$ defining $\sigma_p$ and $\alpha \in \mathbb{F}_{p^2}^*$ satisfies $\alpha^2 = -1$, and*

$$A_2 = \tfrac{1}{3}(\delta_{k \equiv 0,2 \bmod 3})\,(-1)^{\delta_{k \equiv 2,3 \bmod 6}}(-\omega_3(-1))^{\delta(3 \mid N)} \prod_{p \mid N,\, p \neq 3} (-\overline{\nu_p(\beta)} - \overline{\nu_p^p(\beta)})\,\delta_{p \equiv 2 \bmod 3}, \tag{7-2}$$

*where $\beta \in \mathbb{F}_{p^2}^*$ satisfies $\beta^2 - \beta + 1 = 0$.*

**Remarks.** (1) Note that $A_1 = A_2 = 0$ in each of the following situations: (i) $k \equiv 1 \bmod 6$, (ii) there exist primes $p, q \mid N$ (which could be equal) such that $p \equiv 1 \bmod 4$ and $q \equiv 1 \bmod 3$, (iii) $k$ is odd and $p \equiv 1 \bmod 3$ for some $p \mid N$, (iv) $k \equiv 1 \bmod 3$ and $p \equiv 1 \bmod 4$ for some $p \mid N$.

(2) By summing the above formula over all tuples $\hat{\sigma}$, one obtains a formula for the dimension of the space $S_k^{\min}(S^2, \omega')$ of twist-minimal newforms. See Proposition 7.7.

(3) Theorem 1.3 from the introduction follows from the above by taking $\omega'$ trivial. We will prove this after first proving the above result.

*Proof.* Taking $T = 1$ in Theorem 7.1, we have

$$\dim S_k(\hat{\sigma}) = \frac{1}{12}(k-1)\prod_{p \mid N}(p-1) + \frac{1}{2}\Phi\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, f\right) + \Phi\left(\begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix}, f\right).$$

Consider $\gamma = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$. Its discriminant is $\Delta_\gamma = -4$, and we adopt the shorthand

$$\Phi(\gamma) = m\Phi_\infty\Phi_2 \prod_{\text{odd } p \mid N} \Phi_p$$

for (1-4), where $m = 2h(E)/(w(E)2^{\omega(d_E)})$ for $E = \mathbb{Q}[\gamma]$. We find that $m = \frac{1}{4}$ and $\Phi_\infty = (-1)^{k/2}\delta_{k \in 2\mathbb{Z}}$. If $S$ is odd, then $\Phi_2 = 2$ by Example 4.10. If $S$ is even, $\Phi_2$ is given by (6-21). Here, $\mathcal{N}_\gamma(c, n) = 0$ for all $n \geq 2$, $\mathcal{N}_\gamma(0, 1) = 0$ and $\mathcal{N}_\gamma(1, 1) = 2$. So $\Phi_2 = -1 + \frac{1}{2}(-2) = -2$. Thus in both cases, $\Phi_2 = 2(-1)^{S+1}$. Finally, for odd $p \mid S$, $\gamma$ is elliptic in $G(\mathbb{Q}_p)$ if and only if $-1$ is not a square in $\mathbb{Q}_p$, i.e., $p \equiv 3 \bmod 4$.

In such cases, $P_\gamma(X)$ is irreducible modulo $p$, so by (6-22), $\Phi_p = -\overline{\nu_p(\gamma)} - \overline{\nu_p^p(\gamma)}$. Multiplying everything together, we see that $\frac{1}{2}\Phi\left(\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right), f\right)$ gives (7-1).

Now consider $\gamma = \left(\begin{smallmatrix} & -1 \\ 1 & 1 \end{smallmatrix}\right)$. Then $\Delta_\gamma = -3$, so

$$\Phi(\gamma) = m\Phi_\infty\Phi_3 \prod_{p\,|\,N,\, p\neq 3} \Phi_p.$$

We find that $m = \frac{1}{6}$, and

$$\Phi_\infty = -\frac{\sin((k-1)\,\pi/3)}{\sin(\pi/3)} = \begin{cases} 0 & \text{if } k \equiv 1 \bmod 3, \\ 1 & \text{if } k \equiv 0, 5 \bmod 6, \\ -1 & \text{if } k \equiv 2, 3 \bmod 6. \end{cases}$$

If $3 \nmid N$, then by Proposition 4.8, $\Phi_3 = 2$ since 3 is ramified in $\mathbb{Q}_3[\gamma] = \mathbb{Q}_3[\sqrt{-3}]$ and $\mathcal{O}_\gamma = \mathbb{Z}_3\left[\frac{1}{2}(1+\sqrt{-3})\right] = \mathbb{Z}_3[\sqrt{-3}]$ is the full ring of integers. If $3\,|\,N$, then $\Phi_3$ is given by (6-21) with $z = -1$. We find that $\mathcal{N}_\gamma(c, n) = 0$ for all $n \geq 2$, $\mathcal{N}_\gamma(1, 1) = 3$, and $\mathcal{N}_\gamma(0, 1) = \mathcal{N}_\gamma(2, 1) = 0$. So

$$\Phi_3 = -\overline{\omega_3(-1)} + \tfrac{1}{3}\overline{\omega_3(-1)}(-3) = -2\omega_3(-1).$$

For $p\,|\,N$ with $p \neq 3$, $P_\gamma(X) = X^2 - X + 1$ is irreducible in $\mathbb{Q}_p$ if and only if $-3$ is not a square in $\mathbb{Q}_p$, or equivalently, $p \equiv 2 \bmod 3$ (see [23, Lemma 27.4]). For such $p$, $\Phi_p$ is given by (6-22). Multiplying these factors together gives (7-2), and the theorem follows. □

Now suppose $\omega'$ is trivial, so $k > 2$ is even. In this case we can simplify the expressions for $A_1$ and $A_2$ to obtain Theorem 1.3, as follows.

*Proof of Theorem 1.3.* Recall that by (5-6), when $p \equiv 3 \bmod 4$ and $\omega_p$ is trivial, $-\nu_p(\alpha) = -\nu_p^p(\alpha) = \epsilon_p$ is the root number of $\sigma_p$. Likewise, by (5-5), $(-1)^{S+1} = \epsilon_2$ when $S$ is even (and 1 otherwise). So in this case, we simply have

$$A_1 = \tfrac{1}{4}(\epsilon(k, \hat{\sigma}))\, D_4(S) \prod_{\text{odd } p\,|\,S} 2, \tag{7-3}$$

where $\epsilon(k, \hat{\sigma}) = (-1)^{k/2}\prod_{p\,|\,S}\epsilon_p$ is the common global root number of the newforms in $S_k(\hat{\sigma})$, and $D_4(S) \in \{0, 1\}$ vanishes exactly when $S$ is divisible by a prime $p \equiv 1 \bmod 4$.

Turning to (7-2), if $p \equiv 2 \bmod 3$, the polynomial $X^2 - X + 1$ is irreducible over $\mathbb{F}_p$. So $L = \mathbb{F}_{p^2}$ has a root $\beta \in L^* - \mathbb{F}_p^*$. Let $t$ be a generator of the cyclic group $L^*$. The dual group of $L^*$ is the set $\{\nu_m \mid 1 \leq m \leq p^2 - 1\}$, where $\nu_m = \nu_{p,m}$ is defined by

$$\nu_m(t) = \nu_{p,m}(t) = e\left(\frac{m}{p^2-1}\right).$$

Suppose $p$ is odd. As shown in the proof of Corollary 5.2, the list of depth zero supercuspidal representations of $G(\mathbb{Q}_p)$ with trivial central character is

$$\{\sigma_{\nu_{p-1}}, \sigma_{\nu_{2(p-1)}}, \dots \sigma_{\nu_{\frac{p-1}{2}(p-1)}}\}.$$

So there exists $m = k(p-1)$ such that the primitive character $\nu_p$ of $\mathbb{F}_{p^2}^*$ fixed in Theorem 1.3 is given by

$$\nu_p = \nu_{p,m} = \nu_m.$$

Hopefully this conflict of notation ($\nu_p = \nu_m$) will cause no confusion, since $m$ cannot equal $p$.

Noting that $\beta^3 = -1$, we can take $\beta = t^{(p^2-1)/6}$. Then for $m = k(p-1)$,

$$\nu_m(\beta) = e\left(\frac{k(p-1)(p^2-1)}{6(p^2-1)}\right) = e\left(\frac{k(p-1)}{6}\right) = \begin{cases} 1 & \text{if } 3|k, \\ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} & \text{otherwise.} \end{cases}$$

Therefore

$$B(\nu_p) := -\overline{\nu_p(\beta)} - \overline{\nu_p(\beta^p)} = -2\,\mathrm{Re}(\nu_m(\beta)) = \begin{cases} -2 & \text{if } 3|k, \\ 1 & \text{if } 3\nmid k. \end{cases} \tag{7-4}$$

When $p = 2$, there is only one supercuspidal, corresponding to $m = k = 1$, we can take $t = \beta$, and (7-4) holds as well. By (5-7), $3|k$ if and only if the order of $\nu_m$ divides $\frac{1}{3}(p+1)$. So the above coincides with $B(\nu_p)$ defined in Theorem 1.3, and the theorem follows from (7-3) and (7-4). □

Next we will use Theorem 1.3 to count the locally supercuspidal newforms of level $S^2$ with a given global root number. (What we will actually compute is the bias in global root number, but the count for each sign could be determined easily by following the proof of Proposition 7.6.)

To understand the impact of the local root numbers on the product of $B(\nu_p)$ in (1-11), the primes of interest are equivalent to 2 mod 3, so aside from $p = 2$, we have $p \equiv 5 \bmod 6$. It is helpful to look at two typical examples:

| | $\nu$ | $\nu_{10}$ | $\nu_{20}$ | $\nu_{30}$ | $\nu_{40}$ | $\nu_{50}$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| $p = 11$ | AL | + | − | + | − | + | | | |
| | $B(\nu)$ | 1 | 1 | −2 | 1 | 1 | | | |
| | $\nu$ | $\nu_{16}$ | $\nu_{2\cdot16}$ | $\nu_{3\cdot16}$ | $\nu_{4\cdot16}$ | $\nu_{5\cdot16}$ | $\nu_{6\cdot16}$ | $\nu_{7\cdot16}$ | $\nu_{8\cdot16}$ |
| $p = 17$ | AL | + | − | + | − | + | − | + | − |
| | $B(\nu)$ | 1 | 1 | −2 | 1 | 1 | −2 | 1 | 1 |

(7-5)

The Atkin–Lehner sign (AL) in the second row comes from (5-8), and the third row is from (7-4).

**Lemma 7.4.** *Given $S > 1$ square-free, let $H_S^+$ (resp. $H_S^-$) denote the set of tuples $\hat{\sigma} = (\sigma_p)_{p|S}$ satisfying*:

- *For each $p \mid S$, $\sigma_p$ has trivial central character and conductor $p^2$.*
- *$\prod_{p\mid S} \epsilon_p = 1$ (resp. $-1$), where $\epsilon_p$ is the root number of $\sigma_p$.*

*For $\nu_p$ the primitive character of $\mathbb{F}_{p^2}^*$ attached to $\sigma_p$, and $B(\nu_p)$ defined in (7-4), define*

$$\mathcal{B}(S)^{\pm} = \sum_{\hat{\sigma}\in H_S^{\pm}} \prod_{p\mid S,\, p\neq 3} B(\nu_p).$$

*Suppose $D_3(S) = 1$ (in the notation of Theorem 1.3), and let $\omega(S)$ denote the number of prime factors of $S$. Then if* $\gcd(S, 6) = 1$,

$$\mathcal{B}(S)^{+} = \begin{cases} 2^{\omega(S)-1} & \text{if there exists } p\mid S \text{ with } p \equiv 5 \bmod 12, \\ 2^{\omega(S)} & \text{if } \omega(S) \text{ is even and } p \equiv 11 \bmod 12 \text{ for all } p\mid S, \\ 0 & \text{if } \omega(S) \text{ is odd and } p \equiv 11 \bmod 12 \text{ for all } p\mid S, \end{cases} \tag{7-6}$$

*and $\mathcal{B}(S)^{-}$ is the same but with "even" and "odd" interchanged, i.e., $\mathcal{B}(S)^{-} = 2^{\omega(S)} - \mathcal{B}(S)^{+}$.*

*If $S$ is odd and $3\mid S$, then $\mathcal{B}(S)^{\pm} = \mathcal{B}\big(\frac{S}{3}\big)^{\pm}$ if $S > 3$, and $\mathcal{B}(3)^{+} = 1$, $\mathcal{B}(3)^{-} = 0$.*

*If $S$ is even, then $\mathcal{B}(S)^{\pm} = \mathcal{B}\big(\frac{S}{2}\big)^{\mp}$ if $S > 2$, and $\mathcal{B}(2)^{+} = 0$, $\mathcal{B}(2)^{-} = 1$.*

*Proof.* Suppose $\gcd(S, 6) = 1$. We prove (7-6) by induction on $\omega(S)$. For the base case, we take $S = p$ for a prime $p \equiv 5 \bmod 6$. As in (7-5), there are $\frac{1}{3}(p+1)$ representations with $B(\nu_p) = 1$, of which $\frac{1}{6}(p+1)$ have $\epsilon_p = 1$ and $\frac{1}{6}(p+1)$ have $\epsilon_p = -1$. There are $\frac{1}{6}(p-5)$ representations with $B(\nu_p) = -2$, of which $\lceil \frac{1}{12}(p-5)\rceil$ have $\epsilon_p = 1$, and $\lfloor \frac{1}{12}(p-5)\rfloor$ have $\epsilon_p = -1$. Therefore

$$\mathcal{B}(p)^{+} = \sum_{\sigma_p\in H_p^{+}} B(\nu_p) = \tfrac{1}{6}(p+1) - 2\lceil \tfrac{1}{12}(p-5)\rceil = \begin{cases} 1 & \text{if } p \equiv 5 \bmod 12, \\ 0 & \text{if } p \equiv 11 \bmod 12. \end{cases}$$

Likewise

$$\mathcal{B}(p)^{-} = \sum_{\sigma_p\in H_p^{-}} B(\nu_p) = \tfrac{1}{6}(p+1) - 2\lfloor \tfrac{1}{12}(p-5)\rfloor = \begin{cases} 1 & \text{if } p \equiv 5 \bmod 12, \\ 2 & \text{if } p \equiv 11 \bmod 12. \end{cases}$$

This proves the base case. Suppose (7-6) holds for some $S > 1$ with $\gcd(S, 6) = 1$, and $\ell \equiv 5 \bmod 6$ is a prime not dividing $S$. Then the result follows, by considering cases, from the fact that

$$\mathcal{B}(S\ell)^{+} = \mathcal{B}(S)^{+}\cdot\mathcal{B}(\ell)^{+} + \mathcal{B}(S)^{-}\cdot\mathcal{B}(\ell)^{-}$$

and

$$\mathcal{B}(S\ell)^{-} = \mathcal{B}(S)^{+}\cdot\mathcal{B}(\ell)^{-} + \mathcal{B}(S)^{-}\cdot\mathcal{B}(\ell)^{+}.$$

When $3\mid S$, the claim follows from the fact that there is a unique depth zero supercuspidal representation of $\mathrm{PGL}_2(\mathbb{Q}_3)$, and it has root number $+1$ (see Corollary 5.2). When $2\mid S$, the claim follows from the fact that there is a unique depth zero supercuspidal representation $\sigma_\nu$ of $\mathrm{PGL}(\mathbb{Q}_2)$, and it has $B(\nu) = 1$ and root number $-1$. □

**Lemma 7.5.** *Let $H_S^+$ and $H_S^-$ be defined as in Lemma 7.4 above. As in (1-11), define $D_4(S) \in \{0,1\}$ to be 0 if and only if $S$ is divisible by a prime $p \equiv 1 \bmod 4$. Then*

$$|H_S^{\pm}| = \begin{cases} \frac{1}{2}\prod_{\text{odd } p\,|\,S} \frac{1}{2}(p-1) & \text{if } D_4(S)=0, \\ \frac{1}{2}\prod_{\text{odd } p\,|\,S} \frac{1}{2}(p-1) \pm \frac{1}{2}(-1)^{\delta(2\,|\,S)} & \text{if } D_4(S)=1. \end{cases}$$

*Proof.* For each odd prime $p$, there are $\frac{1}{2}(p-1)$ depth zero supercuspidals with trivial central character (see Section 5.1). For $p=2$, there is only one. Therefore for all square-free $S>1$, the total number of tuples $\hat{\sigma} = (\sigma_p)_{p\,|\,S}$ with each $\sigma_p$ having depth zero and trivial central character is

$$|H_S^+| + |H_S^-| = \prod_{\text{odd } p\,|\,S} \tfrac{1}{2}(p-1). \tag{7-7}$$

Now suppose $S$ is divisible by a prime $p_0 \equiv 1 \bmod 4$. Fix $\epsilon_{\text{fin}} = \pm 1$. By the above, the number of tuples $(\sigma_p)_{p\,|\,(S/p_0)}$ is $\prod_{\text{odd } p\,|\,(S/p_0)} \frac{1}{2}(p-1)$. Having fixed one such tuple, by Corollary 5.2 there are then $\frac{1}{4}(p_0-1)$ choices for $\sigma_{p_0}$ subject to $\prod_{p\,|\,S} \epsilon_{\sigma_p} = \epsilon_{\text{fin}}$. This proves the result when $D_4(S)=0$.

Now suppose $p \equiv 3 \bmod 4$ for all odd $p\,|\,S$. For this case, in view of (7-7), the given formula is equivalent to $|H_S^+| - |H_S^-| = (-1)^{\delta(2\,|\,S)}$. We will prove the latter by induction on the number $\omega(S)$ of primes dividing $S$. If $S=2$, the given formula holds since there is just one representation $\sigma_2$, and it has $\epsilon_{\sigma_2} = -1$. If $S = p \equiv 3 \bmod 4$, the given formula holds by Corollary 5.2. Having established the base case, suppose now that the given formula holds for some $S$ satisfying $D_4(S)=1$, and that $p_0 \nmid S$ is a prime satisfying $p_0 \equiv 3 \bmod 4$. We construct a tuple $\hat{\sigma} = (\sigma_p)_{p\,|\,Sp_0}$ by first choosing the components at $p\,|\,S$, and then at $p_0$. Let $P = |H_S^+|$ and $Q = |H_{p_0}^+|$, so $|H_S^-| = P - (-1)^{\delta(2\,|\,S)}$ and $|H_{p_0}^-| = Q - 1$ by the inductive hypothesis. Then

$$|H_{Sp_0}^+| = PQ + (P - (-1)^{\delta(2\,|\,S)})(Q-1) = 2PQ - P - (-1)^{\delta(2\,|\,S)}Q + (-1)^{\delta(2\,|\,S)},$$

and

$$|H_{Sp_0}^-| = P(Q-1) + (P - (-1)^{\delta(2\,|\,S)})Q = 2PQ - P - (-1)^{\delta(2\,|\,S)}Q.$$

Subtracting,

$$|H_{Sp_0}^+| - |H_{Sp_0}^-| = (-1)^{\delta(2\,|\,Sp_0)},$$

as needed. □

**Proposition 7.6.** *For $S>1$ square-free, let*

$$\Delta(S^2,k)^{\min} = \dim S_k^{\min}(S^2)^+ - \dim S_k^{\min}(S^2)^-.$$

*Then for $k>2$ even,*

$$\Delta(S^2,k)^{\min} = \Delta_M + \Delta_{A_1} + \Delta_{A_2}, \tag{7-8}$$

*where*

$$\Delta_M = D_4(S)(-1)^{k/2+\delta(2\mid S)}\tfrac{1}{12}(k-1)\prod_{p\mid S}(p-1), \quad \Delta_{A_1} = \tfrac{1}{4}(D_4(S))\prod_{p\mid S}(p-1)$$

*for* $D_4(S)$ *as in Lemma 7.5 above, and*

$$\Delta_{A_2} = \delta(k\equiv 0,2 \bmod 6)\tfrac{1}{3}(D_3(S))(-1)^{\delta(k\equiv 6,8 \bmod 12)}\mu(S)\,\Omega_0(S'),$$

*where* $D_3(S)\in\{0,1\}$ *is* 0 *if and only if* $p\equiv 1 \bmod 3$ *for some* $p\mid S$, $\mu(S)=\prod_{p\mid S}(-1)$ *is the Möbius function, and for* $S' = S/\gcd(S,6)$,

$$\Omega_0(S') = \begin{cases} 0 & \text{if there exists } p\mid S' \text{ such that } p\equiv 5 \bmod 12,\\ 2^{\omega(S')} & \text{if } p\equiv 11 \bmod 12 \text{ for all } p\mid S', \end{cases}$$

*where* $\omega(S') = \sum_{p\mid S'} 1$. *(Note that* $\Omega_0(1)=1$.)

**Remark.** Proposition 1.4, which summarizes the conditions under which $\Delta(S^2,k)^{\min}$ vanishes, is positive, or is negative, follows easily. The claim in the third paragraph of Proposition 1.4 is due to the fact that when $D_4(S)=1$,

$$\begin{aligned}|\Delta_{A_1}+\Delta_{A_2}| &\le \frac{1}{4}\prod_{p\mid S}(p-1)+\frac{1}{3}\prod_{p\mid S'}2\\ &= \Big[\frac{1}{4}+\frac{1}{3}\prod_{p\mid S'}\frac{2}{p-1}\prod_{p\mid \gcd(S,6)}\frac{1}{p-1}\Big]\prod_{p\mid S}(p-1) \le \frac{7}{12}\prod_{p\mid S}(p-1),\end{aligned}$$

where the last inequality is strict if $S>2$. So if $k\ge 10$, or $k=8$ and $S>2$, it follows that $|\Delta_{A_1}+\Delta_{A_2}|<|\Delta_M|$, and hence the sign of $\Delta_M$ is the sign of the bias. One checks by hand (or LMFDB) that $S_8^{\min}(2^2)=0$. The case $k=6$ follows similarly, replacing the rightmost inequality by $<\frac{5}{12}\prod_{p\mid S}(p-1)$ when $S>6$ and $D_4(S)=1$, and checking the $S\mid 6$ cases by hand.

*Proof of Proposition 7.6.* We have

$$\Delta(S^2,k)^{\min} = \sum_{\hat\sigma:\epsilon(k,\hat\sigma)=1}\dim S_k(\hat\sigma) - \sum_{\hat\sigma:\epsilon(k,\hat\sigma)=-1}\dim S_k(\hat\sigma). \tag{7-9}$$

Applying Theorem 1.3 to each summand, we get a sum of three terms as in (7-8). Since the archimedean factor of the global root number is $(-1)^{k/2}$ (see [20, Theorem 14.17] and [9]), the set of tuples $\hat\sigma$ with global root number $\epsilon$ is $H_S^{(-1)^{k/2}\epsilon}$, with notation as in Lemma 7.4. Therefore the contribution of the main term is

$$\Delta_M = \tfrac{1}{12}(k-1)\prod_{p\mid S}(p-1)(|H_S^{(-1)^{k/2}}| - |H_S^{-(-1)^{k/2}}|),$$

and using Lemma 7.5 we obtain the formula given for $\Delta_M$.

Likewise, the contribution of the $A_1$ term of Theorem 1.3 to (7-9) is

$$\begin{aligned}\Delta_{A_1} &= |H_S^{(-1)^{k/2}}|\tfrac{1}{4}D_4(S)\cdot 1\prod_{\text{odd } p\mid S}2 - |H_S^{-(-1)^{k/2}}|\tfrac{1}{4}D_4(S)\cdot(-1)\prod_{\text{odd } p\mid S}2\\ &= \tfrac{1}{4}D_4(S)(|H_S^+|+|H_S^-|)\prod_{\text{odd } p\mid S}2,\end{aligned}$$

and the given formula follows from (7-7).

In the notation of Theorem 1.3 and Lemma 7.4, the contribution of $A_2$ to (7-9) is

$$\begin{aligned}\Delta_{A_2} &= \tfrac{1}{3}(D_3(S)\,b(k)(-1)^{\delta(3\mid S)})(\mathcal{B}(S)^{(-1)^{k/2}} - \mathcal{B}(S)^{-(-1)^{k/2}})\\ &= \tfrac{1}{3}(D_3(S)\,b(k)(-1)^{\delta(3\mid S)+k/2})(\mathcal{B}(S)^{+} - \mathcal{B}(S)^{-}).\end{aligned}$$

By considering possibilities for $\gcd(6, S)$, it is easy to check using Lemma 7.4 that

$$\mathcal{B}(S)^{+} - \mathcal{B}(S)^{-} = (-1)^{\delta(2\mid S)}\mu(S')\Omega_0(S').$$

The result then follows from $(-1)^{\delta(2\mid S)+\delta(3\mid S)}\mu(S') = \mu(S)$ and the fact that

$$(-1)^{k/2}b(k) = \begin{cases} 1 & \text{if } k\equiv 0, 2 \bmod 12,\\ -1 & \text{if } k\equiv 6, 8 \bmod 12,\\ 0 & \text{if } k\equiv 4 \bmod 6.\end{cases}$$

□

By similar arguments, we obtain the dimension of the space of twist-minimal forms of level $S^2$.

**Proposition 7.7.** *For $S > 1$ square-free and $k > 2$ even,*

$$\begin{aligned}\dim S_k^{\min}(S^2) = \tfrac{1}{12}(k-1)\prod_{\text{odd } p\mid S}\tfrac{1}{2}(p-1)^2 + \tfrac{1}{4}(D_4(S))(-1)^{\delta(2\mid S)+k/2}\prod_{\text{odd } p\mid S}2\\ + \tfrac{1}{3}(D_3(S)\,b(k))(-1)^{\delta(3\mid S)}\prod_{p\mid (S/\gcd(6,S))}2\end{aligned}$$

*for*

$$b(k) = \begin{cases} 1 & \text{if } 6\mid k,\\ -1 & \text{if } k\equiv 2 \bmod 6,\\ 0 & \text{otherwise.}\end{cases}$$

**Remark.** Although we have assumed $k > 2$, the above formula is valid when $k = 2$ as well. More generally, the dimension of $S_k^{\min}(N, \chi)$ has been computed by Child [8, Section 5.1].

*Proof.* We have

$$\dim S_k(S^2)^{\min} = d_M + d_{A_1} + d_{A_2},$$

where

$$d_M = \tfrac{1}{12}(k-1)\prod_{p\mid S}(p-1)(|H_S^{(-1)^{k/2}}|+|H_S^{-(-1)^{k/2}}|),$$
$$d_{A_1} = \tfrac{1}{4}(D_4(S))(-1)^{k/2}(|H_S^+|-|H_S^-|)\prod_{\text{odd } p\mid S}2,$$

and

$$d_{A_2} = \tfrac{1}{3}(D_3(S)\,b(k)(-1)^{\delta(3\mid S)})(\mathcal{B}(S)^+ + \mathcal{B}(S)^-).$$

The result follows upon applying (7-7) to $d_M$, Lemma 7.5 to $d_{A_1}$, and the fact that $\mathcal{B}(S)^+ + \mathcal{B}(S)^- = 2^{\omega(S')} = \prod_{p\mid S'}2$, for $S' = S/\gcd(6, S)$. □

**7.2.** ***Simplification when*** $\mathtt{n} = 1$ ***and*** $T > 1$**.** We return to the general setting of Theorem 4.2 with no constraint on the conductor exponents of the $\sigma_p$. Our aim here is to cull the list of matrices that appear in Theorem 4.2 when $\mathtt{n} = 1$ and $T > 1$. The result is Proposition 7.9, from which Theorem 7.1 follows.

Recall that for $p\mid T$, $\sigma_p$ is a supercuspidal representation whose conductor is of the form $p^n$ with $n \geq 3$ odd. It is well known (see, e.g., [6, Section A.3.8]) that there is a ramified quadratic extension $E/\mathbb{Q}_p$ with $E^*$ embedded in $G(\mathbb{Q}_p)$ such that $\sigma_p$ is compactly induced from a character $\chi$ of $J_n = E^*U^{(n-1)/2}$, where $U^r = 1+\left(\begin{smallmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p \end{smallmatrix}\right)^r$ is an open compact subgroup of $G(\mathbb{Q}_p)$ and $\chi|_{F^*} = \omega_p$. In the notation of Section 5.2, $U^1$ coincides with $K'$, $J_3$ with $H'$, and in general $J_n \subseteq H'$. We use the local test function defined for $g \in G(\mathbb{Q}_p)$ by

$$f_p(g) = \begin{cases} d_{\sigma_p}\overline{\chi(g)} & \text{if } g \in J_n, \\ 0 & \text{otherwise,} \end{cases} \tag{7-10}$$

where $d_{\sigma_p}$ is the formal degree (depending only on the conductor). This coincides with (5-18) when $n = 3$.

If $p\mid T$, the support of $f_p$ is the disjoint union of its unramified and ramified elements:

$$\mathrm{Supp}(f_p) = J_n = (J_n \cap ZK') \cup (J_n \cap \pi_E ZK'), \tag{7-11}$$

where $\pi_E$ is a prime element of $E$ whose square is a prime element of $\mathbb{Q}_p$. We may decompose $f_p$ as $f_p = f_u + f_r$, a sum of two functions supported on the unramified and ramified elements of $J_n$ respectively. In the paper of Gross [17, p. 1240], discussed in Section 1.3, $n = 3$ and the local test function used is a multiple of $f_u$. The following is largely contained in [17, Proposition 5.1].

**Proposition 7.8.** *Let* $f^1 = f^{\mathtt{n}}$ *for* $\mathtt{n} = 1$. *Suppose* $\gamma$ *is elliptic in* $G(\mathbb{Q})$ *and unramified at some prime* $p\mid T$. *Then either* $\gamma$ *has* $p$*-torsion in* $\overline{G}(\mathbb{Q})$ *and* $p \in \{2, 3\}$, *or* $\Phi(\gamma, f^1) = 0$. *As a result,* $\Phi(\gamma, f^1) = 0$ *in each of the following situations*:

(1) $\gamma$ *is unramified at some prime* $p\mid T$ *with* $p > 3$.

(2) $\gamma$ *is unramified at* $3\mid T$ *and* $T \neq 3$.

*Proof.* Write $f = f^1$. Suppose $\Phi(\gamma, f) \neq 0$. By Proposition 4.3, $\gamma$ is elliptic in $G(\mathbb{R})$ and $\det \gamma > 0$. Hence it belongs to a compact-mod-center subgroup $U_\infty$ of $G(\mathbb{R})$ ($U_\infty$ being some conjugate of $\mathbb{R}^* \cdot \mathrm{SO}(2)$). Likewise, at every finite place $v$, the support of $f_v$ is a compact-mod-center subgroup $J_v$ of $G_v$ (here is where we use $\mathtt{n} = 1$), and $\gamma$ belongs to some conjugate $U_v$ of $J_v$. (In fact since $\gamma \in K_v$ a.e., we can take $U_v = K_v$ a.e.) Hence $\gamma$ belongs to a compact-mod-center subgroup $\prod_v U_v$ of $G(\mathbb{A})$. Identifying $\gamma$ with its image modulo the center, we have

$$\gamma \in \overline{G}(\mathbb{Q}) \cap \prod_v \overline{U}_v.$$

This is a *finite* group since $\overline{G}(\mathbb{Q})$ is discrete in $\overline{G}(\mathbb{A})$ [23, Section 7.11]. In particular, $\gamma$ is a torsion element of $\overline{G}(\mathbb{Q})$, i.e., some power of $\gamma$ lies in the center $Z(\mathbb{Q})$.

Since $\gamma$ is unramified at $p \,|\, T$, some conjugate of $\gamma$ belongs to the unramified part of the support of $f_p$, which is a subset of the pro-$p$ group $\overline{K}'$. (Recall that $K'$ is the pro-$p$-Sylow subgroup of the Iwahori subgroup of $G(\mathbb{Q}_p)$). It follows that the order of $\gamma$ in $\overline{G}(\mathbb{Q})$ is a power of $p$. However, it is known that any torsion element of $\overline{G}(\mathbb{Q})$ has order 1, 2, 3, 4, or 6 [11, Lemma 1]. Since $\gamma \neq 1$, we conclude that $p \leq 3$. This proves (1).

The 3-torsion elements of $\overline{G}(\mathbb{Q})$ comprise a single conjugacy class containing $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ [11, Lemma 1]. Therefore, if $p = 3$, $\gamma$ is conjugate in $G(\mathbb{Q})$ to a matrix of the form $\left(\begin{smallmatrix} 0 & -z \\ z & z \end{smallmatrix}\right)$ and is hence everywhere unramified. By the above, this means $T$ is not divisible by any prime $p > 3$. It is also odd, because otherwise $\gamma$ would somehow simultaneously have 3-torsion and 2-power torsion. Hence $T = 3$, which proves (2).

By the same reference, the 4-torsion elements of $\overline{G}(\mathbb{Q})$ are all conjugate to $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. But such an element is ramified at 2. Hence $\gamma$ has 2-torsion if $p = 2$. □

**Proposition 7.9.** *With notation as in Section 4.1, let $T$ be the product of the primes $p$ for which $\mathrm{ord}_p(N)$ is odd, and for $p \,|\, T$ take $f_p$ as in (7-10). Then for $\gamma \in \overline{G}(\mathbb{Q})$, $\Phi(\gamma, f^1) = 0$ unless either $\gamma = 1$ or the conjugacy class of $\gamma$ has a representative in $G(\mathbb{Q})$ of one of the forms given in the table below*:

| *form of $T$* | *list of relevant elliptic $\gamma$ for $\mathtt{n} = 1$* |
|---|---|
| *even* $T \neq 2$ | $\left(\begin{smallmatrix} 1 & -T \\ & \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -T/2 \\ & \end{smallmatrix}\right)$ |
| $T = 2$ | $\left(\begin{smallmatrix} 1 & -2 \\ & \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -1 \\ & \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -2 \\ 1 & 2 \end{smallmatrix}\right)$ |
| *odd* $T > 3$ | $\left(\begin{smallmatrix} 1 & -T \\ & \end{smallmatrix}\right)$ |
| $T = 3$ | $\left(\begin{smallmatrix} 1 & -3 \\ & \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -3 \\ 1 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ |
| $T = 1$ | $\left(\begin{smallmatrix} 1 & -1 \\ & \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ |

**Remark.** When $T/2 \equiv 7 \bmod 8$, the matrix $\left(\begin{smallmatrix} 1 & -T/2 \\ & \end{smallmatrix}\right)$ is hyperbolic (rather than elliptic) in $G(\mathbb{Q}_2)$, so its orbital integral vanishes. All other entries in the above

table are elliptic in $G(\mathbb{Q}_p)$ for each $p \mid T$, but for $p \mid S$ this needs to be checked on a case-by-case basis.

*Proof.* The case where $T = 1$ is already contained in Theorem 4.2, taking $\mathtt{n} = 1$. So suppose $T > 1$ and $\Phi(\gamma, f) \neq 0$. By Proposition 4.13, we may take $\gamma = \left(\begin{smallmatrix} 0 & -M \\ 1 & rM \end{smallmatrix}\right)$ for some $M \mid T$ and $0 \leq r < \sqrt{4/M}$. Notice that if $M > 3$ then $r = 0$. Suppose first that $T \neq 3$. By Proposition 7.8, $\gamma$ must be ramified at all odd primes dividing $T$, so $M = T$ or $M = T/2$. If $T$ is odd, this means $M = T$ and we obtain the third row of the above table. Suppose $T$ is even and $M = T/2$. By Proposition 7.8, $\gamma$ has 2-torsion in $\overline{G}(\mathbb{Q})$. Note that

$$\gamma^2 = \begin{pmatrix} -M & -rM^2 \\ rM & r^2M^2 - M \end{pmatrix}$$

is a scalar matrix if and only if $r = 0$. Therefore $\gamma = \left(\begin{smallmatrix} 0 & -M \\ 1 & 0 \end{smallmatrix}\right)$. This establishes the top two rows of the table. (When $M = T = 2$, $r = 1$ is admissible, and for $\gamma = \left(\begin{smallmatrix} 0 & -2 \\ 1 & 2 \end{smallmatrix}\right)$, $P_\gamma(X) = X^2 - 2X + 2$ is an Eisenstein polynomial for the prime 2, which is indeed irreducible in $\mathbb{Q}_2[X]$ [50, p. 19].)

Now suppose $T = 3$. Then $M = 1$ or $M = 3$. In the latter case, $\gamma = \left(\begin{smallmatrix} 0 & -3 \\ 1 & 3r \end{smallmatrix}\right)$ for $r = 0, 1$. If $M = 1$, then $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & r \end{smallmatrix}\right)$ for $r = 0, 1$, and $\gamma$ is unramified at 3. If $r = 0$, this matrix has 2-torsion, in violation of Proposition 7.8. Hence $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. (In this case, $P_\gamma(X) = X^2 - X + 1$ has discriminant $-3$, which is not a square in $\mathbb{Q}_3$, and hence $\gamma$ is indeed elliptic in $G(\mathbb{Q}_3)$.) □

**7.3.** ***Global orbital integrals for*** $\mathtt{n} = 1$, $N = T^3$**.** Here we will evaluate the global elliptic orbital integrals of Theorem 7.1 explicitly when $N = T^3 > 1$ for $T$ square-free. We must consider

$$\gamma = \begin{pmatrix} & -T \\ 1 & \end{pmatrix}, \underbrace{\begin{pmatrix} & -T/2 \\ 1 & \end{pmatrix}}_{(T \text{ even})}, \underbrace{\begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}}_{(T=2)}, \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}}_{(T=3)}, \underbrace{\begin{pmatrix} 0 & -3 \\ 1 & 3 \end{pmatrix}}_{(T=3)}$$

as appearing in Proposition 7.9.

We introduce some notation before stating the global results. Given our tuple $\hat{\sigma} = (\sigma_{t_p}^{\zeta_p})_{p \mid T}$ of simple supercuspidal representations, for $k > 2$ define

$$\epsilon(k, \hat{\sigma}) = i^k \prod_{p \mid N} \zeta_p. \tag{7-12}$$

This is the common global root number of the cusp forms comprising $H_k(\hat{\sigma})$ (see Proposition 5.3 and [9; 20, Theorem 14.17]). Throughout this section $f = f^1$ as in (5-21).

**Proposition 7.10.** *For $N = T^3$, with notation as above, suppose that for each odd prime factor $p$ of the square-free integer $T > 1$, $-pt_p/T$ is a square modulo $p$.*

*Then for $k \geq 4$ even,*

$$\Phi\left(\begin{pmatrix} & -T \\ 1 & \end{pmatrix}, f\right) = \frac{\overline{\epsilon(k,\hat{\sigma})}\, 2_7\, 4_3\, h(-T)}{3_{T=3}\, 2^{\omega(T)}} \sum_y \omega'(y),$$

*where numbers with subscripts are present only when $T$ falls into the subscript's equivalence class modulo 8, $3_{T=3}$ is a factor of 3 which is present only when $T = 3$, and $y$ ranges over all integers modulo $T$ that satisfy $y^2 \equiv -pt_p/T \bmod p$ for all $p \mid T$. If the central character is trivial, the above simplifies to*

$$\Phi\left(\begin{pmatrix} & -T \\ 1 & \end{pmatrix}, f\right) = \frac{\epsilon(k,\hat{\sigma})\, h(-T)\, w_T}{3_{T=3}}, \tag{7-13}$$

*where*

$$w_T = \begin{cases} \frac{1}{2} & \text{if } T \text{ is even}, \\ 1 & \text{if } T \equiv 1 \bmod 4, \\ 2 & \text{if } T \equiv 7 \bmod 8, \\ 4 & \text{if } T \equiv 3 \bmod 8. \end{cases}$$

**Remark.** If the first hypothesis is not satisfied or $k$ is odd, then $\Phi(\gamma, f) = 0$; see Proposition 5.6.

*Proof.* Take $\gamma = \left(\begin{smallmatrix} & -T \\ 1 & \end{smallmatrix}\right)$, $\Delta_\gamma = -4T$, and let $M$ be the odd part of $T$, so that $T = 2^a M$ for some $a \in \{0, 1\}$. Corresponding to (1-4), write

$$\Phi(\gamma, f) = m\Phi_\infty \Phi_2 \prod_{p \mid M} \Phi_p = m(-1)^{k/2} \Phi_2 \prod_{p \mid M} \overline{\zeta_p} \sum_{y_p} \omega_p(y_p),$$

where we have applied (4-13) and Proposition 6.4, with $y_p$ running over the two (since $p$ is odd) solutions to $y_p^2 \equiv -pt_p/T \bmod p$. We can exchange the sum and product. To each of the $2^{\omega(M)}$ tuples $(y_p)_{p \mid M}$, the Chinese remainder theorem assigns a unique integer $y$ modulo $T$ satisfying $y \equiv y_p \bmod p$ for all $p \mid T$, where we take $y_2 = 1$ if $T$ is even. Further,

$$\omega'(y) = \prod_{p \mid T} \omega_p(y) = \prod_{p \mid T} \omega_p(y_p) = \prod_{p \mid M} \omega_p(y_p).$$

The first equality holds because $\gcd(y, T) = 1$ (see [23, (12.4)]); the second holds since each $\omega_p$ is trivial on $1 + p\mathbb{Z}_p$. By Example 4.10 (for $T$ odd) or Proposition 6.4 (for $T$ even),

$$\Phi_2 = \begin{cases} \overline{\zeta_2} & \text{if } T \text{ is even}, \\ 2 & \text{if } T \equiv 1, 5, 7 \bmod 8, \\ 4 & \text{if } T \equiv 3 \bmod 8. \end{cases}$$

It follows that

$$\Phi(\gamma, f) = \frac{2h(E)}{w_E 2^{\omega(d_E)}} \overline{\epsilon(k,\hat{\sigma})}\, a_T \sum_y \omega'(y)$$

for $y$ as in the statement of the proposition,

$$a_T = \begin{cases} 1 & \text{if } T \text{ is even,} \\ 2 & \text{if } T \equiv 1, 5, 7 \bmod 8, \\ 4 & \text{if } T \equiv 3 \bmod 8, \end{cases}$$

and $E = \mathbb{Q}(\sqrt{-T})$. Since $T > 1$, we know that

$$w_E = |\mathcal{O}_E^*| = \begin{cases} 6 & \text{if } T = 3, \\ 2 & \text{otherwise.} \end{cases}$$

So $\frac{1}{2} w_E = 3_{T=3}$ and $2h(E)/w_E = h(-T)/3_{T=3}$. Recall that

$$d_E = \begin{cases} -4T, & -T \equiv 2, 3 \bmod 4, \\ -T, & -T \equiv 1 \bmod 4. \end{cases}$$

Therefore, placing the congruence condition on $T$ rather than $-T$,

$$2^{\omega(d_E)} = \begin{cases} 2 \cdot 2^{\omega(T)} & \text{if } T \equiv 1 \bmod 4, \\ 2^{\omega(T)} & \text{if } T \equiv 2, 3 \bmod 4. \end{cases}$$

Hence using the definition of $a_T$ in the following numerator,

$$\Phi(\gamma, f) = \overline{\epsilon(k, \hat{\sigma})}\, h(-T) \frac{2_{1,5,7} \cdot 4_3}{3_{T=3} \cdot 2_{1,5} \cdot 2^{\omega(T)}} \sum_y \omega'(y),$$

where numbers with subscripts are only present when $T$ falls into one of the subscript equivalence classes modulo 8. The general result now follows.

If $\omega'$ is trivial, the sum over $y$ equals the number of terms, namely $2^{\omega(M)}$. Then (7-13) follows from

$$\frac{2^{\omega(M)}}{2^{\omega(T)}} = \begin{cases} 1 & \text{if } T \text{ is odd,} \\ \frac{1}{2} & \text{if } T \text{ is even} \end{cases}$$

and the fact that $\epsilon(k, \hat{\sigma}) \in \{\pm 1\}$ is real in this case. □

**Proposition 7.11.** *For $N = T^3$, suppose that the square-free integer $T = 2M$ is even, and that for each prime factor $p$ of $T$, $-pt_p/M$ is a square modulo $p$. Then for even $k \geq 4$,*

$$\Phi\left(\begin{pmatrix} & -M \\ 1 & \end{pmatrix}, f\right) = h(-M) \frac{\overline{\epsilon(k, \hat{\sigma})}}{\zeta_2} \cdot \frac{z_M}{2_{M=1}\, 3_{M=3}\, 2^{\omega(M)}} \sum_y \omega'(y),$$

*where $2_{M=1}$ is a factor of 2 which is present only when $M = 1$, $3_{M=3}$ is defined similarly,*

$$z_M = \begin{cases} \frac{1}{2} & \text{if } M \equiv 1 \bmod 4, \\ -3 & \text{if } M \equiv 3 \bmod 8, \\ 0 & \text{if } M \equiv 7 \bmod 8, \end{cases}$$

*and $y$ ranges over all elements modulo $M$ that satisfy $y^2 \equiv -pt_p/M \bmod p$ for each $p \mid M$. If $\omega'$ is trivial, the sum over $y$ simply cancels with the factor of $2^{\omega(M)}$. (Again, if the condition on the $t_p$ fails to hold or $k$ is odd, the orbital integral vanishes.)*

*Proof.* We use the same proof as for the previous proposition, with minor modifications. First, by Example 6.6,

$$\Phi\left(\begin{pmatrix} & -M \\ 1 & \end{pmatrix}, f_2\right) = \begin{cases} 1 & \text{if } M \equiv 1 \bmod 4, \\ -3 & \text{if } M \equiv 3 \bmod 8, \\ 0 & \text{if } M \equiv 7 \bmod 8. \end{cases}$$

Taking $E = \mathbb{Q}[\sqrt{-M}]$ we have

$$2^{\omega(d_E)} = \begin{cases} 2 \cdot 2^{\omega(M)} & \text{if } M \equiv 1 \bmod 4, \\ 2^{\omega(M)} & \text{if } M \equiv 3 \bmod 4 \end{cases}$$

as in the previous proof, and $2h(E)/w_E = h(-M)/(3_{M=3}\, 2_{M=1})$ since $\mathbb{Q}[\sqrt{-1}]$ has unit group of order 4 when $M = 1$. Hence (assuming $M \not\equiv 7 \bmod 8$)

$$\Phi\left(\begin{pmatrix} & -M \\ 1 & \end{pmatrix}, f\right) = \frac{h(-M)(-3)_3}{3_{M=3}\, 2_{M=1}\, 2_{1,5}\, 2^{\omega(M)}} \frac{\overline{\epsilon(k, \hat{\sigma})}}{\zeta_2} \sum_y \omega'(y),$$

where numerical subscripts refer to the congruence class of $M$ modulo 8. □

**Proposition 7.12.** *Suppose $N = 2^3$, $\zeta \in \{\pm 1\}$ and $\sigma = \sigma^\zeta$ is our fixed simple supercuspidal representation of $G(\mathbb{Q}_2)$ (the parameter $t$ must equal 1 when $p = 2$). Then*

$$\Phi\left(\begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}, f\right) = \tfrac{1}{4}\epsilon(k, \sigma)\, g_8(k),$$

*where $g_8(k) = -1$ if $k \equiv 0, 2 \bmod 8$, and $g_8(k) = 1$ if $k \equiv 4, 6 \bmod 8$.*

**Remark.** In view of Proposition 5.4, we assume that $k$ is even.

*Proof.* Given that $\gamma$ has characteristic polynomial $X^2 - 2X + 2$ with discriminant $\Delta_\gamma = -4$, we find $E = \mathbb{Q}[\gamma] = \mathbb{Q}[i]$. Hence $h(E) = 1$, $w_E = |\mathcal{O}_E^*| = 4$, and $d_E = -4$. By (1-4),

$$\Phi(\gamma, f) = m\Phi_\infty\Phi_2 = \tfrac{1}{4}\Phi_\infty\Phi_2.$$

Applying Proposition 6.4 with $p = 2$ and $v = 1$, we have $\Phi_2 = -\zeta$. So

$$\Phi(\gamma, f) = -\tfrac{1}{4}\Phi_\infty\, \zeta. \tag{7-14}$$

The complex eigenvalues of $\gamma$ are $1 \pm i$, so we apply (4-12) with $\theta = \frac{\pi}{4}$ to get

$$\Phi_\infty = -\sqrt{2}\sin\left(\tfrac{1}{4}(k-1)\,\pi\right) = \begin{cases} 1 & \text{if } k \equiv 0, 6 \bmod 8, \\ -1 & \text{if } k \equiv 2, 4 \bmod 8. \end{cases}$$

Multiplying this by $-1$ as in (7-14) yields $(-1)^{k/2} g_8(k)$ with $g_8$ as given. □

**Proposition 7.13.** *Suppose $T = 3$ so $N = 3^3$, and let $\sigma = \sigma_t^\zeta$ be our fixed simple supercuspidal representation of $G(\mathbb{Q}_3)$, for $t = \pm 1$. Then*

$$\Phi\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, f\right) = \frac{t}{2_{t=-1}} c_3(k),$$

*where $c_3(k) = \frac{1}{3} + \left\lfloor \frac{k}{3} \right\rfloor - \frac{k}{3}$.*

*Proof.* Let $E = \mathbb{Q}[\gamma] = \mathbb{Q}[\sqrt{-3}]$. Then $h(E) = 1$, $d_E = -3$, and $w_E = |\mathcal{O}_E^*| = 6$. By (1-4) and taking $m = 1$ in Example 6.7 and its remark,

$$\Phi(\gamma, f) = \Phi_\infty \cdot \tfrac{1}{6}((-1)^k t \cdot 2_{t=1}) = \tfrac{1}{3}((-1)^k \Phi_\infty) \frac{t}{2_{t=-1}}.$$

By (4-12), we find that

$$(-1)^k \Phi(\gamma, f_\infty) = (-1)^{k+1} \frac{\sin\left(\frac{(k-1)\pi}{3}\right)}{\sin(\pi/3)} = \begin{cases} 1 & \text{if } k \equiv 0 \bmod 3, \\ 0 & \text{if } k \equiv 1 \bmod 3, \\ -1 & \text{if } k \equiv 2 \bmod 3. \end{cases} \tag{7-15}$$

Using the above, we see that $\frac{1}{3}((-1)^k \Phi(\gamma, f_\infty)) = \frac{1}{3} + \left\lfloor \frac{k}{3} \right\rfloor - \frac{k}{3}$. □

**Proposition 7.14.** *Suppose $N = 3^3$, and let $\sigma = \sigma_t^\zeta$ be a fixed simple supercuspidal representation of $G(\mathbb{Q}_3)$. Then*

$$\Phi\left(\begin{pmatrix} 0 & -3 \\ 1 & 3 \end{pmatrix}, f\right) = \begin{cases} 0 & \text{if } t = 1, \\ \epsilon(k, \sigma)\, g_6(k) & \text{if } t = -1, \end{cases}$$

*where*

$$g_6(k) = \begin{cases} 0 & \text{if } k \equiv 1 \bmod 6, \\ -\frac{1}{6} & \text{if } k \equiv 0, 2 \bmod 6, \\ \frac{1}{2} & \text{if } k \equiv 3 \bmod 6, \\ \frac{1}{3} & \text{if } k \equiv 4 \bmod 6, \\ -\frac{1}{2} & \text{if } k \equiv 5 \bmod 6. \end{cases}$$

*Proof.* Let $\gamma = \left(\begin{smallmatrix} 0 & -3 \\ 1 & 3 \end{smallmatrix}\right)$, so $\Delta_\gamma = -3$. We have $E = \mathbb{Q}[\gamma] = \mathbb{Q}[\sqrt{-3}]$, so the measure factor is $\frac{1}{6}$ as in the previous proof. Therefore as in (1-4), we may write

$$\Phi(\gamma, f) = \tfrac{1}{6} \Phi_\infty \Phi_3. \tag{7-16}$$

By Proposition 6.4, $\Phi_3 = 0$ unless $-t$ is a square modulo 3, i.e., unless $t = -1$. Assuming this holds, we have

$$\begin{aligned} \Phi_3 &= \bar{\zeta} \cdot \left(\overline{\psi(1)}\, \omega_3(1) + \overline{\psi(-1)}\, \omega_3(-1)\right) \\ &= \bar{\zeta} \cdot (e^{-2\pi i/3} + (-1)^k e^{2\pi i/3}) = -\bar{\zeta}\, [i\sqrt{3}]_{k \text{ odd}}, \end{aligned}$$

where the factor of $i\sqrt{3}$ is present only when $k$ is odd.

By (4-4) with $N=3$, $\omega_3(3)=1$. So $\zeta^2=\omega_3(t)=\omega_3(-1)=(-1)^k$, so $\zeta=\pm(i^k)$. In particular, the global root number $\varepsilon(\sigma,\zeta)=i^k\zeta$ is real and $\bar{\zeta}=(-1)^k\zeta$.

The complex roots of $P_\gamma(X)=X^2-3X+3$ are $\frac{1}{2}(3\pm i\sqrt{3})=\sqrt{3}\big(\frac{1}{2}(\sqrt{3}\pm i)\big)$, so in (4-12) we can take $\theta=\frac{\pi}{6}$ and $\Phi(\gamma,f_\infty)=-2\sin\big(\frac{1}{6}(k-1)\pi\big)$. Hence (7-16) becomes

$$\Phi(\gamma,f)=\tfrac{1}{3}(-1)^k\zeta\sin\big(\tfrac{1}{6}(k-1)\pi\big)[i\sqrt{3}]_{k\text{ odd}}=\begin{cases}\zeta/3 & \text{if } k\equiv 4 \bmod 12,\\ -i\zeta/2 & \text{if } k\equiv 3,5 \bmod 12,\\ \zeta/6 & \text{if } k\equiv 2,6 \bmod 12,\\ 0 & \text{if } k\equiv 1,7 \bmod 12,\\ -\zeta/6 & \text{if } k\equiv 0,8 \bmod 12,\\ i\zeta/2 & \text{if } k\equiv 9,11 \bmod 12,\\ -\zeta/3 & \text{if } k\equiv 10 \bmod 12.\end{cases}$$

Upon factoring out $\epsilon(k,\sigma)=i^k\zeta$, we obtain $g_6(k)$ as given. □

**7.4.** ***Dimension formulas when $N=T^3$.*** Here we put everything together to compute $|H_k(\hat{\sigma})|=\dim S_k(\hat{\sigma})$ for $\hat{\sigma}=(\sigma_p)_{p|N}$ a tuple of simple supercuspidal representations of $G(\mathbb{Q}_p)$ as in Theorem 7.1 with $S=1$.

We begin with the case $N=2^3$, where the central character is necessarily trivial due to (4-2) and Proposition 5.4.

**Theorem 7.15.** *Let $N=2^3$, fix $\zeta\in\{\pm1\}$, and let $\sigma=\sigma_\zeta$ be the associated simple supercuspidal representation of $G(\mathbb{Q}_2)$ with trivial central character. Then*

$$|H_k(\sigma)|=\begin{cases}0 & \text{if } k \text{ is odd},\\ \lfloor\frac{k}{8}\rfloor & \text{if } k\equiv 0,2 \bmod 8,\\ \lfloor\frac{k}{8}\rfloor+\frac{1}{2}(1+\epsilon(k,\sigma)) & \text{if } k\equiv 4,6 \bmod 8,\end{cases}$$

*where $\epsilon(k,\sigma)=(-1)^{k/2}\zeta$ is the global root number.*

*Proof.* When $k$ is odd, the assertion follows from Proposition 5.4. Suppose $k$ is even. By Theorem 7.1,

$$|H_k(\sigma)|=\tfrac{1}{12}(k-1)\cdot\tfrac{3}{2}+\tfrac{1}{2}\Phi\left(\begin{pmatrix}&-2\\1&\end{pmatrix},f\right)+\tfrac{1}{2}\Phi\left(\begin{pmatrix}&-1\\1&\end{pmatrix},f\right)+\Phi\left(\begin{pmatrix}0&-2\\1&2\end{pmatrix},f\right).$$

Applying the results of Section 7.3 using $h(-2)=h(-1)=1$, we find

$$|H_k(\sigma)|=\tfrac{1}{8}(k-1)+\tfrac{1}{4}((-1)^{k/2}\zeta)+\tfrac{1}{8}((-1)^{k/2})+\tfrac{1}{4}((-1)^{k/2}\zeta)\,g_8(k)$$

for

$$g_8(k)=\begin{cases}-1 & \text{if } k\equiv 0,2 \bmod 8,\\ 1 & \text{if } k\equiv 4,6 \bmod 8.\end{cases}$$

The result follows upon simplifying each of the cases. □

**Theorem 7.16.** *Let $N = 3^3$, fix $t \in \{\pm 1\}$, a character $\omega_3$ of $\mathbb{Q}_3^*$ trivial on $1+3\mathbb{Z}_3$, $\zeta \in \mathbb{C}$ with $\zeta^2 = \omega_3(t)$ (see (4-2)), and let $\sigma = \sigma_t^\zeta$ be the associated simple supercuspidal representation of $G(\mathbb{Q}_3)$ with central character $\omega_3$. Then for $k > 2$, setting $\epsilon = i^k\zeta$, we have*

$$|H_k(\sigma)| = \begin{cases} \lfloor \frac{k}{3} \rfloor + \frac{1}{2}(\epsilon - 1) & \text{if } k \equiv 0 \bmod 3 \text{ and } t = -1, \\ \lfloor \frac{k}{3} \rfloor & \text{if } k \equiv 1 \bmod 6 \text{ or } t = 1, \\ \lfloor \frac{k}{3} \rfloor + \frac{1}{2}(\epsilon + 1) & \text{if } k \equiv 2 \bmod 6 \text{ and } t = -1, \\ \lfloor \frac{k}{3} \rfloor + \epsilon & \text{if } k \equiv 4 \bmod 6 \text{ and } t = -1, \\ \lfloor \frac{k}{3} \rfloor + \frac{1}{2}(1 - \epsilon) & \text{if } k \equiv 5 \bmod 6 \text{ and } t = -1. \end{cases}$$

**Remarks.** (1) If $t = -1$, then $\zeta^2 = \omega_3(-1) = (-1)^k$, so $\zeta = \pm i^k$, as noted earlier. Therefore $\epsilon \in \{\pm 1\}$ when $t = -1$. When $t = 1$ and $k$ is odd, $\epsilon = \pm i$.

(2) There is one more newform with $\epsilon = -1$ than with $\epsilon = 1, i$, or $-i$ when $k \equiv 5 \bmod 6$, i.e., the root number has a slight bias toward $-1$ in this case. For example, when $k = 5$ and $\omega'$ is the Dirichlet character of conductor 3, there are five newforms of level 27, with respective root numbers $1, -1, -1, i, -i$. These newforms are discussed further in Section 7.5.

*Proof.* By Theorem 7.1,

$$\begin{aligned} |H_k(\sigma)| &= \frac{1}{12}(k-1)\cdot\frac{8}{2} + \frac{1}{2}\Phi\left(\begin{pmatrix} & -3 \\ 1 & \end{pmatrix}, f\right) + \Phi\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, f\right) + \Phi\left(\begin{pmatrix} 0 & -3 \\ 1 & 3 \end{pmatrix}, f\right) \\ &= \frac{1}{3}(k-1) + \frac{2\epsilon}{3}\delta_{t=-1}\cdot\delta_{k\in 2\mathbb{Z}} + \frac{t}{2_{t=-1}}c_3(k) + \epsilon g_6(k)\,\delta_{t=-1}, \end{aligned}$$

where we have applied Propositions 7.10, 7.13, and 7.14, and $c_3(k)$, $g_6(k)$ are recalled below. (For nonvanishing of $\Phi\left(\left(\begin{smallmatrix} & -3 \\ 1 & \end{smallmatrix}\right), f\right)$, the hypothesis in Proposition 7.10 requires that $-t$ be a square modulo 3, i.e., $t = -1$, and $k$ even. Then $\bar{\epsilon} = \epsilon$ and the sum over $y$ in that result is $1 + (-1)^k = 2$.)

If $t = 1$, then because $c_3(k) = \frac{1}{3}(1-k) + \lfloor \frac{k}{3} \rfloor$, the above simplifies to $\lfloor \frac{k}{3} \rfloor$, as needed.

Now suppose $t = -1$, and write $k = a + 6c$ for some $0 \le a \le 5$. If $k$ is odd, then

$$|H_k(\sigma)| = \tfrac{1}{3}(k-1) - \tfrac{1}{2}\left(\tfrac{1}{3}(1-k) + \lfloor \tfrac{k}{3} \rfloor\right) + \epsilon g_6(k) = \tfrac{1}{2}(k-1) - \tfrac{1}{2}\lfloor \tfrac{k}{3} \rfloor + \epsilon g_6(k).$$

Using the fact that $g_6(k) = 0, \frac{1}{2}, -\frac{1}{2}$ when $a = 1, 3, 5$ respectively, we get

$$|H_k(\sigma)| = \begin{cases} 2c = \lfloor \frac{k}{3} \rfloor & \text{if } a = 1, \\ 2c + 1 + \frac{1}{2}(\epsilon - 1) = \lfloor \frac{k}{3} \rfloor + \frac{1}{2}(\epsilon - 1) & \text{if } a = 3, \\ 2c + 1 + \frac{1}{2}(1 - \epsilon) = \lfloor \frac{k}{3} \rfloor + \frac{1}{2}(1 - \epsilon) & \text{if } a = 5. \end{cases}$$

If $k$ is even, then there is one extra term, namely $\frac{2\epsilon}{3}$, so

$$|H_k(\sigma)| = \tfrac{1}{2}(k-1) - \tfrac{1}{2}\lfloor \tfrac{k}{3} \rfloor + \epsilon\big(\tfrac{2}{3} + g_6(k)\big).$$

Here, $g_6(k) = -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}$ when $a = 0, 2, 4$ respectively. Upon simplifying,

$$|H_k(\sigma)| = \begin{cases} 2c + \frac{1}{2}(\epsilon - 1) = \lfloor \frac{k}{3} \rfloor + \frac{1}{2}(\epsilon - 1) & \text{if } a = 0, \\ 2c + \frac{1}{2}(1 + \epsilon) = \lfloor \frac{k}{3} \rfloor + \frac{1}{2}(1 + \epsilon) & \text{if } a = 2, \\ 2c + 1 + \epsilon = \lfloor \frac{k}{3} \rfloor + \varepsilon & \text{if } a = 4. \end{cases}$$ □

**Theorem 7.17.** *Suppose $N = T^3$ with $T > 3$ square-free, $M = \frac{T}{2}$, $k \geq 4$ is even, and $\hat{\sigma} = (\sigma_{t_p}^{\zeta_p})_{p|N}$ is a tuple of simple supercuspidal representations with trivial central characters. Then*

$$|H_k(\hat{\sigma})| = \tfrac{1}{12}(k-1) \prod_{p|T} \tfrac{1}{2}(p^2 - 1) + \Delta_1(\hat{t})\, \epsilon(k, \hat{\sigma})\, b_T\, h(-T) + \Delta_2(\hat{t}) \frac{\epsilon(k, \hat{\sigma})\, j_M\, h(-M)}{\zeta_2\, 3_{M=3}}, \quad (7\text{-}17)$$

*where $\epsilon(k, \hat{\sigma}) \in \{\pm 1\}$ is the common global root number of the newforms in $H_k(\hat{\sigma})$ given in (7-12),*

$$b_T = \begin{cases} \frac{1}{4} & \text{if } T \text{ is even}, \\ \frac{1}{2} & \text{if } T \equiv 1 \bmod 4, \\ 1 & \text{if } T \equiv 7 \bmod 8, \\ 2 & \text{if } T \equiv 3 \bmod 8, \end{cases} \qquad j_M = \begin{cases} \frac{1}{4} & \text{if } M \equiv 1 \bmod 4, \\ -\frac{3}{2} & \text{if } M \equiv 3 \bmod 8, \\ 0 & \text{if } M \equiv 7 \bmod 8, \end{cases}$$

*$h(d)$ is the class number of $\mathbb{Q}[\sqrt{-d}]$, and $\Delta_i(\hat{t}) \in \{0, 1\}$ is nonzero if and only if* (i) *$T$ is even in the case $i = 2$, and* (ii) *$-2^{i-1} p t_p / T$ is a square modulo $p$ for each odd $p \,|\, T$.*

**Remarks.** To keep the formula simple, we have restricted ourselves to the case of trivial central character; the general case is obtained similarly. Even in the general case, one may restrict to $k$ even because by Corollary 7.2,

$$|H_k(\hat{\sigma})| = \tfrac{1}{12}(k-1) \prod_{p|N} \tfrac{1}{2}(p^2 - 1) \quad (T > 3,\ k \text{ odd}). \quad (7\text{-}18)$$

*Proof.* This follows from Theorem 7.1 and Propositions 7.10 and 7.11. □

As a corollary, we recover the following dimension formulas of [37].

**Corollary 7.18.** *For $T = 2, 3$ and $k \geq 4$ even,*

$$\dim S_k^{\text{new}}(8) = \lfloor \tfrac{k}{4} \rfloor, \quad \dim S_k^{\text{new}}(27) = k - 1 + \lfloor \tfrac{k}{3} \rfloor.$$

*For $T > 3$ square-free, and $k \geq 4$ even,*

$$\dim S_k^{\text{new}}(T^3) = \tfrac{1}{12}(k-1) \prod_{p|T} (p-1)^2(p+1). \quad (7\text{-}19)$$

**Remarks.** As shown in [37], the formula is also valid for $k = 2$. When $k$ is odd and $\omega'$ has conductor dividing $T$, $\dim S_k^{\mathrm{new}}(T^3, \omega')$ is also equal to (7-19). This follows from (7-18).

*Proof.* For $T = 2$, by Theorem 7.15,

$$|H_k(2^3)| = |H_k(\sigma^+)| + |H_k(\sigma^-)| = \begin{cases} 2\lfloor \frac{k}{8} \rfloor & \text{if } k \equiv 0, 2 \bmod 8, \\ 2\lfloor \frac{k}{8} \rfloor + 1 & \text{if } k \equiv 4, 6 \bmod 8. \end{cases}$$

This is easily seen to be the same as $\lfloor \frac{k}{4} \rfloor$.

For $T = 3$, for fixed $k$ we add the formula in Theorem 7.16 over all $t, \zeta \in \{\pm 1\}$, or equivalently, $t, \epsilon \in \{\pm 1\}$. Writing the $t = 1$ contribution first, we obtain

$$|H_k(3^3)| = 2\lfloor \tfrac{k}{3} \rfloor + \begin{cases} 2\lfloor \frac{k}{3} \rfloor - 1 & \text{if } k \equiv 0 \bmod 3, \\ 2\lfloor \frac{k}{3} \rfloor & \text{if } k \equiv 1 \bmod 3, \\ 2\lfloor \frac{k}{3} \rfloor + 1 & \text{if } k \equiv 2 \bmod 3. \end{cases}$$

The above is easily seen to equal $k - 1 + \lfloor \frac{k}{3} \rfloor$, as required.

For $T > 3$ we have

$$\dim S_k^{\mathrm{new}}(T^3) = |H_k(T^3)| = \sum_{\hat{\sigma}} |H_k(\hat{\sigma})|,$$

where $\hat{\sigma}$ ranges over the $\prod_{p|T} 2(p-1)$ tuples $(t_p, \zeta_p)$, with trivial central character. By (7-17), this equals

$$\tfrac{1}{12}(k-1) \prod_{p|T} \tfrac{1}{2}(p^2-1)\, 2(p-1) + \sum_{\hat{\sigma}} \Delta_1(\hat{t})\, \epsilon(k, \hat{\sigma})\, b_T\, h(-T) + \sum_{\hat{\sigma}} \Delta_2(\hat{t}) \frac{\epsilon(k, \hat{\sigma})\, j_M\, h(-M)}{\zeta_2\, 3_{M=3}}.$$

Clearly, from (7-12), exactly half of the $\hat{\sigma}$ satisfying $\Delta_1(\hat{t}) = 1$ have $\epsilon(k, \hat{\sigma}) = +1$, and half have $\epsilon(k, \hat{\sigma}) = -1$. So the first sum over $\hat{\sigma}$ vanishes. Likewise if $T$ is even, $\epsilon(k, \hat{\sigma})/\zeta_2 = +1$ (resp. $-1$) exactly half of the time since $T$ is divisible by at least one prime different from 2, so the second sum also vanishes. □

Next, we compute the dimension of the subspace of forms with a given root number, which recovers the main result (1-9) of [46].

**Corollary 7.19** [46]. *For $T > 3$ square-free and $k \geq 4$ even, the subspace of $S_k^{\mathrm{new}}(T^3)$ with root number $\pm 1$ has dimension*

$$|H_k^{\pm}(T^3)| = \tfrac{1}{24}(k-1) \prod_{p|T} (p-1)^2(p+1) \pm \tfrac{1}{2}(c_T\, h(-T)) \prod_{p|T} (p-1),$$

*where $c_T = b_T$ if $T$ is odd, and $c_T = 2b_T$ if $T$ is even, i.e.,*

$$c_T = \begin{cases} \frac{1}{2} & \text{if } T \equiv 1, 2 \bmod 4, \\ 1 & \text{if } T \equiv 7 \bmod 8, \\ 2 & \text{if } T \equiv 3 \bmod 8. \end{cases} \tag{7-20}$$

*Proof.* Given $\hat{\sigma} = (\sigma_{t_p}^{\zeta_p})_{p|T}$, let $\hat{t} = (t_p)_{p|T}$ and $\hat{\zeta} = (\zeta_p)_{p|T}$. The root number is determined by $\hat{\zeta}$ and $k$. Let $A_k^{\pm}$ be the set of all tuples $\hat{\zeta}$ for which $(-1)^{k/2} \prod_{p|T} \zeta_p = \pm 1$. Then

$$|A_k^+| = |A_k^-| = \tfrac{1}{2} \prod_{p|T} 2. \tag{7-21}$$

By (7-17), we see that

$$\begin{aligned} &|H_k^{\pm}(T^3)| \\ &\quad = \sum_{\hat{\zeta} \in A_k^{\pm}} \sum_{\hat{t}} |H_k(\hat{\sigma})| \\ &\quad = \sum_{\hat{\zeta} \in A_k^{\pm}} \sum_{\hat{t}} \left( \tfrac{1}{12}(k-1) \prod_{p|T} \tfrac{1}{2}(p^2-1) \pm b_T\, h(-T) \Delta_1(\hat{t}) \pm \zeta_2 \frac{j_M\, h(-M)}{3_{M=3}} \Delta_2(\hat{t}) \right), \end{aligned}$$

where $M$ is the odd part of $T$. Recall that $\Delta_2(\hat{t}) = 0$ if $T$ odd. If $T$ is even, upon summing over $\zeta_2 = \pm 1$ the last term will be eliminated, so we can ignore it henceforth. For any given odd prime $p$, exactly half of the elements $t_p \in (\mathbb{Z}/p\mathbb{Z})^*$ have the property that $-pt_p/T$ is a square. Therefore, the number of tuples $\hat{t}$ for which $\Delta_1(\hat{t}) \neq 0$ is $\prod_{p|M} \frac{1}{2}(p-1)$. The total number of tuples $\hat{t}$ is $\prod_{p|T}(p-1) = \prod_{p|M}(p-1)$. It follows that

$$|H_k^{\pm}(T^3)| = \sum_{\hat{\zeta} \in A_k^{\pm}} \left( \tfrac{1}{12}(k-1) \prod_{p|T} \tfrac{1}{2}(p^2-1)(p-1) \pm b_T\, h(-T) \prod_{p|M} \tfrac{1}{2}(p-1) \right).$$

By (7-21), we obtain

$$|H_k^{\pm}(T^3)| = \tfrac{1}{24}(k-1) \prod_{p|T} (p-1)^2(p+1) \pm \tfrac{1}{2}(2_T b_T\, h(-T)) \prod_{p|T} (p-1),$$

where $2_T$ is a factor of 2 which is only present when $T$ is even. We see immediately that $2_T b_T = c_T$ as given. □

**7.5.** ***Some examples with*** $\mathtt{n} > 1$**.** In this section we illustrate Theorem 1.1 with some examples. (A different set of examples is given in the earliest version of this paper posted on the arXiv.) We will compare with the Galois orbits of newforms tabulated in LMFDB [34]. Though $S_k(\hat{\sigma})$ occasionally forms a Galois orbit, typically the orbit is a direct sum of more than one such space. It also happens that a space $S_k(\hat{\sigma})$ decomposes as a direct sum of more than one Galois orbit.

Examples of these phenomena can be found in $S_4^{\min}(23^2)$, where Theorem 1.3 gives $\dim S_4(\hat{\sigma}) = \frac{1}{2}(11+\epsilon) \in \{5, 6\}$, but the twist-minimal Galois orbits can have dimensions 1, 2, 5, 6, 12 or 24.

**7.5.1.** We first consider an example with odd weight. Take $N = 3^3$, $k = 5$, and $\omega'$ the Dirichlet character of modulus 27 and conductor 3. We consider simple supercuspidal representations $\sigma_t^\zeta$, where $t \in \{\pm 1\}$ and $\zeta^2 = \omega'(t)$. In LMFDB [34] we find the following data for the space $S_5(27, \omega')$:

| LMFDB label | $\epsilon$ | dim | $\operatorname{tr} T_4$ | $\operatorname{tr} T_7$ | $(\zeta, t)$ |
|---|---|---|---|---|---|
| 27.5.b.a | 1 | 1 | 16 | 71 | $(-i, -1)$ |
| 27.5.b.b | $-1$ | 2 | $-76$ | 34 | $(i, -1)$ |
| 27.5.b.c | $\pm i$ | 2 | 14 | $-38$ | $(1, 1) \oplus (-1, 1)$ |

The final column, using the shorthand $(\zeta, t) = S_5(\sigma_t^\zeta)$, is immediate upon comparing Theorem 7.16 with the $\epsilon$ and dim columns. Using Theorem 7.17 we find the following, which refines the above.

**Example 7.20.** With notation as above,

$$\operatorname{tr}(T_4 \mid S_5(\sigma_t^\zeta)) = \tfrac{1}{2}(37t - 23) + 46i\zeta \cdot \delta_{t=-1},$$
$$\operatorname{tr}(T_7 \mid S_5(\sigma_t^\zeta)) = \tfrac{1}{4}(67 - 143t) + \tfrac{1}{2}(37i\zeta)\delta_{t=-1}.$$

We will give an indication of the proof of the above formulas. The calculations for $\mathfrak{n} = 7$ are a little bit more interesting, so we start with this case. By Theorem 1.1,

$$\operatorname{tr}(T_7 \mid S_5(\sigma_t^\zeta)) = 7^{3/2}\left[\Phi\left(\begin{pmatrix} & -21 \\ 1 & 3 \end{pmatrix}\right) + \Phi\left(\begin{pmatrix} & -21 \\ 1 & 6 \end{pmatrix}\right) + \Phi\left(\begin{pmatrix} & -21 \\ 1 & 9 \end{pmatrix}\right) + \sum_{r=1}^{5} \Phi\left(\begin{pmatrix} & -7 \\ 1 & r \end{pmatrix}\right)\right].$$

We have used (4-13) to eliminate the trace zero matrices, since $k$ is odd. The matrix $\left(\begin{smallmatrix} & -7 \\ 1 & 3 \end{smallmatrix}\right)$ is unramified at $p = 3$ but has no double characteristic root mod 3. So its orbital integral vanishes by Proposition 5.6. The first three integrals vanish unless

$$y^2 \equiv -\frac{t}{7} \equiv -t \mod 3$$

has a solution, i.e., $t = -1$. In this case, applying Proposition 6.4 to $\gamma = \left(\begin{smallmatrix} & -21 \\ 1 & 9 \end{smallmatrix}\right)$ and $p = 3$, we see that $v = 3$ so the local integral has the value $\overline{\zeta_5}(\omega_3(1) + \omega_3(-1)) = 0$. Hence this $\gamma$ can be discarded. We compute the remaining orbital integrals locally as summarized in the following table, where $m = 2h(E)/(w(E)2^{\omega(d_E)})$ is the global measure factor for $E = \mathbb{Q}[\gamma]$, and $\ell$ denotes a prime factor of the discriminant $\Delta_\gamma$ other than 3 (if such exists). The global orbital integral is then $\Phi = m\Phi_\infty\Phi_3\Phi_\ell$. The factor

$$\Phi_\infty = -\frac{\sin(4\arctan(\sqrt{|\Delta_\gamma|}/\operatorname{tr}\gamma))}{\sin(\arctan(\sqrt{|\Delta_\gamma|}/\operatorname{tr}\gamma))}$$

was computed using software.

| $\gamma$ | $\Delta_\gamma$ | $\ell$ | $m$ | $\Phi_\infty$ | $\Phi_3$ | $\Phi_\ell$ |
|---|---|---|---|---|---|---|
| $\left(\begin{smallmatrix}1 & -21\\ & 3\end{smallmatrix}\right)$ | $-3\cdot 5^2$ | 5 | $\frac{1}{6}$ | $11\sqrt{3}\cdot 7^{-3/2}$ | $-i\bar{\zeta}\sqrt{3}\cdot\delta_{t=-1}$ | 7 |
| $\left(\begin{smallmatrix}1 & -21\\ & 6\end{smallmatrix}\right)$ | $-2^4\cdot 3$ | 2 | $\frac{1}{6}$ | $4\sqrt{3}\cdot 7^{-3/2}$ | $i\bar{\zeta}\sqrt{3}\cdot\delta_{t=-1}$ | 10 |
| $\left(\begin{smallmatrix}1 & -7\\ & 1\end{smallmatrix}\right)$ | $-3^3$ | | $\frac{1}{6}$ | $13\cdot 7^{-3/2}$ | 4 | |
| $\left(\begin{smallmatrix}1 & -7\\ & 2\end{smallmatrix}\right)$ | $-2^3\cdot 3$ | 2 | $\frac{1}{2}$ | $20\cdot 7^{-3/2}$ | $\frac{1}{2}(1-3t)$ | 2 |
| $\left(\begin{smallmatrix}1 & -7\\ & 4\end{smallmatrix}\right)$ | $-2^2\cdot 3$ | 2 | $\frac{1}{6}$ | $-8\cdot 7^{-3/2}$ | $-\frac{1}{2}(3t+1)$ | 4 |
| $\left(\begin{smallmatrix}1 & -7\\ & 5\end{smallmatrix}\right)$ | $-3$ | | $\frac{1}{6}$ | $-55\cdot 7^{-3/2}$ | $\frac{1}{2}(3t+1)$ | |

The formula for $\operatorname{tr} T_7$ in Example 7.20 follows upon simplifying. Most of the entries in the above table are straightforward, but we highlight a few. For example, $\gamma = \left(\begin{smallmatrix}1 & -21\\ & 6\end{smallmatrix}\right)$ is elliptic in $G(\mathbb{Q}_2)$, and by the quadratic formula,

$$\mathbb{Z}_2[\gamma] = \mathbb{Z}_2\big[\tfrac{1}{2}(6+2^2\sqrt{-3})\big] = \mathbb{Z}_2 + \mathbb{Z}_2 2^2\varepsilon,$$

where $\varepsilon = \frac{1}{2}(1+\sqrt{-3})$. So $n_\gamma = 2$ and $\Phi_2(\gamma) = 1+(2+1)+(4+2) = 10$ by Proposition 4.8 and (4-20).

The matrix $\gamma = \left(\begin{smallmatrix}1 & -7\\ & 1\end{smallmatrix}\right)$ is unramified at $p=3$, so $\Phi_3(\gamma)$ is computed using Proposition 6.5. We find (using software) that $\mathcal{N}_\gamma(0,1) = \mathcal{N}_\gamma(0,2) = 3$, $\mathcal{N}_\gamma(1,2) = 6$, $\mathcal{N}_\gamma(1,3) = 9$, and $\mathcal{N}_\gamma(c,n) = 0$ for all other pairs $(c,n)$. Since $P_\gamma(X) \equiv (X+1)^2 \bmod 3$, we take $z=-1$, so, using the third remark after Proposition 6.5, for $t=\pm 1$ we have

$$\Phi_3\left(\begin{pmatrix}1 & -7\\ & 1\end{pmatrix}\right) = \frac{-1}{3}\Big[3\Big(e\Big(\frac{t}{3}\Big)+e\Big(\frac{-t}{3}\Big)\Big)+3(2)+6(-1)+9(-1)\Big] = 4.$$

Finally, $\gamma = \left(\begin{smallmatrix}1 & -7\\ & 2\end{smallmatrix}\right)$ is unramified at $p=3$ and $\mathcal{N}_\gamma(-1,1) = 3$ is the only nonzero value of $\mathcal{N}_\gamma(c,n)$. We take $z=1$ in Proposition 6.5 to get

$$\Phi_3\left(\begin{pmatrix}1 & -7\\ & 2\end{pmatrix}\right) = \frac{1}{3}\cdot 3\Big[e\Big(\frac{-1-t}{3}\Big)+e\Big(\frac{1+t}{3}\Big)\Big] = 2\cos\frac{2\pi(1+t)}{3} = \begin{cases}-1 & \text{if } t=1,\\ 2 & \text{if } t=-1.\end{cases}$$

This equals $\frac{1}{2}(1-3t)$ for $t=\pm 1$. The remaining entries in the above $T_7$ table are found in a similar fashion.

For $\operatorname{tr} T_4$, in the identity term we have $\omega'(\sqrt{4}) = -1$. So

$$\operatorname{tr}(T_4 \,|\, S_5(\sigma_t^\zeta)) = 8\Big[-\frac{4}{3}+\Phi\left(\begin{pmatrix}1 & -12\\ & 3\end{pmatrix}\right)+\Phi\left(\begin{pmatrix}1 & -12\\ & 6\end{pmatrix}\right) + \Phi\left(\begin{pmatrix}1 & -4\\ & 1\end{pmatrix}\right)+\Phi\left(\begin{pmatrix}1 & -4\\ & 2\end{pmatrix}\right)+\Phi\left(\begin{pmatrix}1 & -4\\ & 3\end{pmatrix}\right)\Big].$$

The last term can be eliminated since it is unramified at $p = 3$ and it has no characteristic root modulo 3. The remaining orbital integrals are computed locally as follows, and the formula for $\operatorname{tr} T_4$ in Example 7.20 follows upon simplification.

| $\gamma$ | $\Delta_\gamma$ | $\ell$ | $m$ | $\Phi_\infty$ | $\Phi_3$ | $\Phi_\ell$ |
|---|---|---|---|---|---|---|
| $\left(\begin{smallmatrix}1 & -12\\ 3\end{smallmatrix}\right)$ | $-3\cdot 13$ | 13 | 1 | $5\sqrt{3}\cdot 8^{-1}$ | $-i\bar{\zeta}\sqrt{3}\cdot\delta_{t=-1}$ | 2 |
| $\left(\begin{smallmatrix}1 & -12\\ 6\end{smallmatrix}\right)$ | $-2^2\cdot 3$ | 2 | $\frac{1}{6}$ | $-\sqrt{3}$ | $i\bar{\zeta}\sqrt{3}\cdot\delta_{t=-1}$ | 4 |
| $\left(\begin{smallmatrix}1 & -4\\ 1\end{smallmatrix}\right)$ | $-3\cdot 5$ | 5 | $\frac{1}{2}$ | $7\cdot 8^{-1}$ | $\frac{1}{2}(3t-1)$ | 2 |
| $\left(\begin{smallmatrix}1 & -4\\ 2\end{smallmatrix}\right)$ | $-2^2\cdot 3$ | 2 | $\frac{1}{6}$ | 1 | $\frac{1}{2}(3t+1)$ | 4 |

**7.5.2.** Let $N = 2^3 11^2$ and $k = 6$, and let $\sigma^\zeta$ be a simple supercuspidal representation of $\mathrm{PGL}_2(\mathbb{Q}_2)$ and $\sigma_\nu$ a depth zero supercuspidal representation of $\mathrm{PGL}_2(\mathbb{Q}_{11})$. Here, $\zeta \in \{\pm 1\}$, and $\nu$ is one of the five primitive characters of $L^*$ listed in (7-5), where $L = \mathbb{F}_{11^2}$ and we take the generator $t$ of $L^*$ to be a root of the polynomial $X^2 + 7X + 2 \in \mathbb{F}_{11}[X]$. Let $\hat{\sigma}$ be the associated tuple. Then by Theorem 7.1,

$$\dim S_6(\hat{\sigma}) = \frac{25}{4} + \frac{1}{2}\Phi\left(\left(\begin{smallmatrix}1 & -2\\ 1\end{smallmatrix}\right), f^1\right) + \frac{1}{2}\Phi\left(\left(\begin{smallmatrix}1 & -1\\ 1\end{smallmatrix}\right), f^1\right) + \Phi\left(\left(\begin{smallmatrix}1 & -2\\ 1 & 2\end{smallmatrix}\right), f^1\right).$$

Over $\mathbb{F}_{11}$, $X^2 + 2 = (x+3)(x-3)$, so $\left(\begin{smallmatrix}1 & -2\\ 1\end{smallmatrix}\right)$ is hyperbolic in $G(\mathbb{Q}_{11})$ by Hensel's lemma, and therefore its orbital integral vanishes. Using Example 6.6 and the argument at (7-3),

$$\frac{1}{2}\Phi\left(\left(\begin{smallmatrix}1 & -1\\ 1\end{smallmatrix}\right)\right) = \frac{1}{2}m\Phi_\infty\Phi_2\Phi_{11} = \frac{1}{2}\cdot\frac{1}{4}\cdot(-1)^{6/2}\cdot 1\cdot 2\epsilon_{11} = -\frac{1}{4}\epsilon_{11}.$$

Taking $\gamma = \left(\begin{smallmatrix}1 & -2\\ 2\end{smallmatrix}\right)$, $P_\gamma(X) = X^2 - 2X + 2$ is irreducible over $\mathbb{F}_{11}$, so by (6-22),

$$\Phi_{11} = -\overline{\nu(\gamma)} - \overline{\nu^{11}(\gamma)}.$$

For $L^* = \langle t\rangle$ as above, we find (using software) that $t^{51}$ has minimum polynomial $P_\gamma(X)$. Therefore, if $\nu = \nu_m$ for $m = 10w \in \{10, 20, 30, 40, 50\}$ as in (7-5) where $\mu_m(t) = e\left(\frac{m}{120}\right)$, we have

$$\nu_m(\gamma) = e\left(\frac{51m}{120}\right) = e\left(\frac{17w}{4}\right) = e\left(\frac{w}{4}\right) = i^w.$$

Using this, $\Phi_{11}(\gamma)$ is given by

| $\nu$ | $\nu_{10}$ | $\nu_{20}$ | $\nu_{30}$ | $\nu_{40}$ | $\nu_{50}$ |
|---|---|---|---|---|---|
| $\epsilon_{11}$ | + | − | + | − | + |
| $\Phi_{11}$ | 0 | 2 | 0 | −2 | 0 |

(7-22)

As in the proof of Proposition 7.12, $m = \frac{1}{4}$, $\Phi_\infty = 1$ (since $k = 6$), and $\Phi_2 = -\zeta$. Hence $\Phi(\gamma) = -\frac{1}{4}\zeta\Phi_{11}$ for $\Phi_{11}$ as above. Thus

$$\dim S_6(\hat{\sigma}) = \tfrac{25}{4} - \tfrac{1}{4}\epsilon_{11} - \tfrac{1}{4}\zeta\Phi_{11} = \begin{cases} 6 & \text{if } \epsilon_{11} = 1, \text{ or } \zeta = 1 \text{ and } \nu = \nu_{20}, \\ & \text{or } \zeta = -1 \text{ and } \nu = \nu_{40}, \\ 7 & \text{if } \zeta = 1 \text{ and } \nu = \nu_{40}, \\ & \text{or } \zeta = -1 \text{ and } \nu = \nu_{20}. \end{cases} \tag{7-23}$$

We would like to match the above spaces to Galois orbits of twist-minimal newforms in $S_6^{\mathrm{new}}(2^3 11^2)$. In the table below, the first five columns show LMFDB [34] data, with AL entries corresponding to the Atkin–Lehner signs at $p = 2, 11$. These are equal to $\zeta$ and $\epsilon_{11}$ respectively. The dim column gives the size of the orbit.

| LMFDB label | dim | tr $T_7$ | AL 2 | AL 11 | $(\zeta, \nu)$ |
|---|---|---|---|---|---|
| 968.6.a.f | 6 | $-124$ | $-$ | $-$ | $(-1, \nu_{40})$ |
| 968.6.a.g | 6 | 124 | $+$ | $-$ | $(1, \nu_{20})$ |
| 968.6.a.h | 6 | $-88$ | $+$ | $+$ | $(1, \nu_{30})$ |
| 968.6.a.i | 6 | 88 | $-$ | $+$ | $(-1, \nu_{30})$ |
| 968.6.a.j | 7 | $-62$ | $-$ | $-$ | $(-1, \nu_{20})$ |
| 968.6.a.k | 7 | 62 | $+$ | $-$ | $(1, \nu_{40})$ |
| 968.6.a.l | 6 | $-206$ | $+$ | $+$ | $(1, \nu_{10}) \oplus (1, \nu_{50})$ |
| 968.6.a.m | 6 | 206 | $-$ | $+$ | $(-1, \nu_{10}) \oplus (-1, \nu_{50})$ |

In the final column we have adopted the notation $S_6(\hat{\sigma}) = (\zeta, \nu)$. This column was obtained as follows. Comparing (7-22) and (7-23) with the AL and dim columns, we immediately infer the entries with $\epsilon_{11} = -1$, i.e., with $\nu_{20}$ and $\nu_{40}$. We can distinguish the remaining entries by looking at Hecke eigenvalues. For this we apply Theorem 1.1 to compute $\mathrm{tr}(T_7 \,|\, S_6(\hat{\sigma}))$. The result is the following.

**Example 7.21.** Let $N = 2^3 11^2$ and $\hat{\sigma} = (\sigma^\zeta, \sigma_\nu)$ be a tuple of supercuspidal representations of conductors $2^3$ and $11^2$ respectively, as above. Then

$$\mathrm{tr}(T_7 \,|\, S_6(\hat{\sigma})) = -98\zeta\epsilon_{11} - 5\zeta X_{11} - 31 Y_{11},$$

where $\epsilon_{11}$, $X_{11}$ and $Y_{11}$ are given as follows:

| $\nu$ | $\nu_{10}$ | $\nu_{20}$ | $\nu_{30}$ | $\nu_{40}$ | $\nu_{50}$ |
|---|---|---|---|---|---|
| $\epsilon_{11}$ | $+$ | $-$ | $+$ | $-$ | $+$ |
| $X_{11}$ | 1 | 1 | $-2$ | 1 | 1 |
| $Y_{11}$ | $\sqrt{3}$ | $-1$ | 0 | 1 | $-\sqrt{3}$ |

For example, in the notation used above,

$$\mathrm{tr}(T_7 \,|\, (1, \nu_{10})) = -103 - 31\sqrt{3}, \quad \mathrm{tr}(T_7 \,|\, (1, \nu_{50})) = -103 + 31\sqrt{3}.$$

We sketch the proof as follows. By Theorem 1.1,

$$\begin{aligned}&\operatorname{tr}(T_7 \mid S_6(\hat{\sigma}))\\ &= 7^2\left[\frac{1}{2}\Phi\left(\begin{pmatrix} & -7\\ 1 & \end{pmatrix}\right)+\frac{1}{2}\Phi\left(\begin{pmatrix} & -14\\ 1 & \end{pmatrix}\right)+\sum_{r=1}^{5}\Phi\left(\begin{pmatrix} & -7\\ 1 & r\end{pmatrix}\right)+\sum_{r=1}^{3}\Phi\left(\begin{pmatrix} & -14\\ 1 & 2r\end{pmatrix}\right)\right].\end{aligned}$$

All but three of the orbital integrals vanish for simple reasons. The matrices $\left(\begin{smallmatrix} & -7\\ 1 & \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} & -7\\ 1 & 2\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} & -7\\ 1 & 3\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} & -14\\ 1 & 1\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} & -14\\ 1 & 2\end{smallmatrix}\right)$ are hyperbolic in $G(\mathbb{Q}_{11})$, since their characteristic polynomials have two distinct roots modulo 11. The matrices $\left(\begin{smallmatrix} & -7\\ 1 & 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} & -7\\ 1 & 5\end{smallmatrix}\right)$ are unramified at $p=2$ but do not have characteristic roots modulo 2. So the associated orbital integrals vanish by Proposition 5.6, and

$$\operatorname{tr}(T_7 \mid S_6(\hat{\sigma})) = 7^2\left[\frac{1}{2}\Phi\left(\begin{pmatrix} & -14\\ 1 & \end{pmatrix}\right)+\Phi\left(\begin{pmatrix} & -7\\ 1 & 4\end{pmatrix}\right)+\Phi\left(\begin{pmatrix} & -14\\ 1 & 6\end{pmatrix}\right)\right].$$

The formula in Example 7.21 follows upon computing each of these terms locally. The local results are shown in the following table, with notation as in the previous $N=27$ example. The global orbital integral for a given row is $\Phi = m\Phi_\infty\Phi_2\Phi_{11}\Phi_\ell$.

| $\gamma$ | $\Delta_\gamma$ | $\ell$ | $m$ | $\Phi_\infty$ | $\Phi_2$ | $\Phi_{11}$ | $\Phi_\ell$ |
|---|---|---|---|---|---|---|---|
| $\left(\begin{smallmatrix} & -14\\ 1 & \end{smallmatrix}\right)$ | $-2^3\cdot 7$ | 7 | 1 | $-1$ | $\zeta$ | $2\epsilon_{11}$ | 2 |
| $\left(\begin{smallmatrix} & -14\\ 1 & 6\end{smallmatrix}\right)$ | $-2^2\cdot 5$ | 5 | $\frac{1}{2}$ | $\frac{5}{7^2}$ | $-\zeta$ | $X_{11}$ | 2 |
| $\left(\begin{smallmatrix} & -7\\ 1 & 4\end{smallmatrix}\right)$ | $-2^2\cdot 3$ | 3 | $\frac{1}{6}$ | $\frac{31}{7^2}$ | $-3$ | $Y_{11}$ | 2 |

The $\Phi_{11}$ column was determined as follows. As described earlier, $\mathbb{F}_{11^2}^* = \langle t\rangle$ where $t^2+7t+2=0$. For each $\gamma$ as above, there is a power $t^j$ whose minimum polynomial over $\mathbb{F}_{11}$ is $P_\gamma(X)$. The power $j$ was found with software, and is given as follows:

| $\gamma$ | $\left(\begin{smallmatrix} & -14\\ 1 & \end{smallmatrix}\right)$ | $\left(\begin{smallmatrix} & -14\\ 1 & 6\end{smallmatrix}\right)$ | $\left(\begin{smallmatrix} & -7\\ 1 & 4\end{smallmatrix}\right)$ |
|---|---|---|---|
| $j$ | 18 | 8 | 17 |

In each case, (6-22) implies that

$$\Phi_{11} = -\overline{\nu(\gamma)} - \overline{\nu^{11}(\gamma)} = -\overline{\nu(t^j)} - \overline{\nu(t^{11j})}.$$

By definition, $\nu_m(t) = e\left(\frac{m}{120}\right)$, so if $\nu = \nu_m$ for $m = 10w$,

$$\Phi_{11}(\gamma) = -e\left(-\frac{jw}{12}\right) - e\left(-\frac{11jw}{12}\right),$$

which can be evaluated by hand or using software to obtain the $\Phi_{11}$ column in the above table.

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Most drawing and graphing packages — Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc. — allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with as many details as you can about how your graphics were generated.

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**White Space.** Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

**Proofs.** Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

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