FOURIER INTEGRAL OPERATORS FOR SYSTEMS

by

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INTRODUCTION

This talk is a report on some joint work with J. J. Duistermaat. Last summer at Stanford we announced some results concerning the spectrum of a positive mth order self-adjoint elliptic pseudodifferential operator P. $C^\infty(|\Lambda|^{\frac{1}{2}} X) \to C^\infty(|\Lambda|^{\frac{1}{2}} X)$ X being a compact manifold and $|\Lambda|^{\frac{1}{2}}$ the half-density bundle on X. These results concerned the assymptotic behavior of the mth roots of the eigenvalues of P, $\sqrt[m]{\lambda_0}$, $\sqrt[m]{\lambda_1}$, ..., and the singularities of the spectral function $\Sigma\, e^{i \sqrt[m]{\lambda_k} t}$

It seems unlikely that these results can be extended to systems except in special cases. We have decided to describe one such special case here, the case when the symbol of P admits a smooth diagonal form, since it includes a number of examples of interest to differential topologists (such as the Laplace operator on forms).

TRIVIALITIES

Let $E \to X$ be a vector bundle and let

$$P: \; C^{\infty}(E \otimes |\Lambda|^{\frac{1}{2}}) \longrightarrow C^{\infty}(E \otimes |\Lambda|^{\frac{1}{2}})$$

be an mth order pseudodifferential operator whose leading symbol is a scalar multiple $\wp$ of the identity. Let $\pi: T^{*}X \to X$ be projection and let $\pi^{*}E$ be the pull-back of E. We will show that there is intrinsically associated to P a first order differential operator

$$L_P: \; C^{\infty}(\pi^{*}E) \longrightarrow C^{\infty}(\pi^{*}E)$$

with symbol equal to $H_{\wp}$ (i.e. $L_P f \Delta - f L_P \Delta = (H_{\wp} f)\Delta$), $H_{\wp}$ being the vector field $\Sigma \frac{\partial \wp}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial \wp}{\partial x_i}\frac{\partial}{\partial \xi_i}$. Let $P_{\circ}$ be a pseudodifferential operator operating on half-densities with $\wp(x,\xi)$ as leading symbol. Let $F: \pi^{*}E \to \pi^{*}E_{\circ}$ be a homogeneous trivialization of $\pi^{*}E$, and let

$$\mathfrak{F}: \; C^{\infty}(E \otimes |\Lambda|^{\frac{1}{2}}) \longrightarrow C^{\infty}(E_{\circ} \otimes |\Lambda|^{\frac{1}{2}})$$

be a zeroth order pseudodifferential operator with F as its leading symbol. Notice that $\mathfrak{F}P - P_{\circ}\mathfrak{F}$ is of order m-1. Choose $\mathcal{Q}$ of order m-1 so that

$$\text{(1.1)} \qquad \mathfrak{F}\mathcal{Q} - (\mathfrak{F}P - P_{\circ}\mathfrak{F})$$

is of order m-2. (Note that the symbol of $\mathcal{Q}$ depends only on F not $\mathfrak{F}$.) Now set

$$\text{(1.2)} \qquad L_P = F^{-1}\left(\frac{1}{\sqrt{-1}}\,\mathcal{L}_{H_{\wp}} + \mathrm{sub}(P_{\circ})\right) F + \sigma\,(\mathcal{Q})$$

PROPOSITION: The definition 1.2 is independent of the choice of P_o and F.

PROOF: It is clear that it is independent of the choice of P_o because of the sub(P_o) term. To prove its independent of F we can assume to begin with that E is trivial i.e. that F is a function on $T^*X/0$ with values in GL(N,C). With $P = P_o + \mathcal{Q}_o$ we have to show

$$\frac{1}{\sqrt{-1}} \mathcal{L}_{H_p} + \text{sub}(P_o) + \sigma(\mathcal{Q}_o) \quad = \quad F^{-1}\left(\frac{1}{\sqrt{-1}} \mathcal{L}_{H_p} + \text{sub}(P_o)\right) F + \sigma(\mathcal{Q}),$$

where $\sigma(\mathcal{Q})$ is given by (1.1). After some obvious cancellations this equation becomes

$$(1.3) \qquad \sigma(\mathcal{Q}_o) = \frac{1}{\sqrt{-1}} F^{-1} \{p, F\} + \sigma(\mathcal{Q}).$$

On the other hand, using local coordinates, we can write (1.1) in the form

$$F\sigma(\mathcal{Q}) = \frac{1}{\sqrt{-1}} \Sigma \left(\frac{\partial F}{\partial \xi_i} \frac{\partial p}{\partial x_i} - \frac{\partial p}{\partial \xi_i} \frac{\partial F}{\partial x_i}\right) + F\sigma(\mathcal{Q}_o),$$

which is the same as (1.3). Q. E. D.

Now suppose the symbol of P is not a scalar multiple of the identity but is smoothly diagonalizable in the following sense: There exists a vector bundle splitting

$$(1.4) \qquad \pi^* E = E_1 \oplus \dots \oplus E_N$$

such that $\sigma(P)$ preserves this splitting and is equal to a scalar multiple p_i of the identity on E_i with $p_i(x,\xi) \neq p_j(x,\xi)$ for all

(x,ξ). Then there exists an intrinsically defined first order differential operator

$$L^i_P: \ C^\infty(E_i) \to C^\infty(E_i)$$

with its symbol equal to H_{p_i}. This is defined just as in (1.2) except that $F: \ \pi^*E \to \pi^*E_0$ is now a simultaneous trivialization of all the E_i's.

Let γ be a closed H_{p_i} solution curve of period T. Let $(x,\xi)\varepsilon\gamma$. For each $s \,\varepsilon\, (E_i)_{(x,\xi)}$ we can find a unique solution $\tilde{s}$ of $L^i_P\tilde{s} = 0$ on the interval [0, T] with $\tilde{s}(0) = s$. The map of $(E_i)_{(x,\xi)}$ onto itself given by $s \to \tilde{s}(T)$ will be called the <u>holonomy map</u> associated to γ and denoted H_γ. Up to conjugacy it is independent of the choice of (x,ξ).

<u>EXAMPLE</u>: Let X be an oriented Riemannian manifold and let P be the square root of the Laplace operator, $-\delta d + d\delta = \Delta$, on k forms. Then H_γ is the usual holonomy map along closed geodesics. Before proving this we will first prove a general fact.

<u>LEMMA</u>: Let P_1 and P_2 be pseudodifferential operators on $C^\infty(E \otimes |\Lambda|^{\frac{1}{2}})$ with scalar top symbols. Then

$$(1.5) \qquad L_{P_1P_2} = \sigma(P_1)L_{P_2} + \sigma(P_2)L_{P_1} + \frac{1}{2i}\{\sigma(P_1),\ \sigma(P_2)\}$$

<u>PROOF</u>: Let p_1 and p_2 be the symbols of P_1 and P_2. Fix a trivialization of E, and let P_{10} and P_{20} be scalar operators with p_1 and p_2 as leading symbols. Let $P_1 = P_{10} + Q_1$ and $P_2 = P_{20} + Q_2$. Then

(1.6) $$P_1P_2 = P_{10}P_{20} + P_{10}\mathcal{Q}_2 + P_{20}\mathcal{Q}_1 + \dots$$

The principal symbol of $P_{10}P_{20}$ is $p_1 p_2$, and the subprincipal symbol is $p_1 \mathrm{sub}(P_{20}) + p_2 \mathrm{sub}(P_{10}) + \frac{1}{2i}\{P_1, P_2\}$. (see (2)). Therefore, from the right hand side of (1.6), we get (1.5) as asserted. Q. E. D.

COROLLARY: $$L_{P^m} = m\sigma(P)^{m-1}L_P$$

In particular the holonomy map associated with the Laplacian is the same as the holonomy map for the square root of the Laplacian; so we only have to check the assertion above for the Laplacian. Let $x_\circ \varepsilon X$. Trivialize the k form bundle in a neighborhood of $x_\circ$ by means of the geodesic coordinate system centered at $x_\circ$. Then by the Weitzenbock theorem (See [5]) $\Delta = \Delta_\circ + \mathcal{Q}$ where $\Delta_\circ$ is the scalar Laplacian and $\mathcal{Q}$ is a first order operator whose coefficients are linear combinations of the Christoffel symbols, hence equal to zero at $x_\circ$. Therefore, at any point $(x_\circ, \xi_\circ)$, $L_\Delta = \mathcal{L}_\Xi$, where Ξ is the vector field defining geodesic flow. The trivialization of the k form bundle is such that along the geodesic through $(x_\circ, \xi_\circ)$ the trivialization is by means of parallel transport. So $\mathcal{L}_\Xi$ is just covariant differentiation in the direction of Ξ at $(x_\circ, \xi_\circ)$ Q. E. D.

Finally, let Λ be a Lagrangian manifold on which $p_i = 0$. Then there exists an intrinsic first order differential operator on the tensor product:

$$(E_i \ 1 \ \Lambda) \otimes |\Lambda|^{\frac{1}{2}} \otimes (\text{Maslov})(\Lambda)$$

defined by

$$L^i_P(\mathfrak{s} \otimes \mu \otimes m) = L^i_P\mathfrak{s} \otimes \mu \otimes m + \mathfrak{s} \otimes \mathcal{L}_{H_{p_i}} \mu \otimes m$$

where $\mathfrak{s}$ is a section of $E_i 1\Lambda$, μ a half-density, and m a constant section of the Maslov bundle. To check that L^i_P is well defined we must show that

$$(1.7) \qquad L^i_P(\mathfrak{s} \otimes \mu) = L^i_P(\mathfrak{s}' \otimes \mu')$$

when $\mathfrak{s} = f\mathfrak{s}'$ and $\mu' = f\mu$, $f \neq 0$. Since the symbol of L^i_P is H_{p_i} $L^i_P f\mathfrak{s}' = fL^i_P\mathfrak{s}' + (H_{p_i}f)\mathfrak{s}'$ so

$$\begin{aligned} L^i_P(\mathfrak{s} \otimes \mu) &= L^i_P\mathfrak{s}' \otimes \mu' + (H_{p_i}f/f)\mathfrak{s}' \otimes \mu' \\ &\quad - (H_{p_i}f/f)\mathfrak{s}' \otimes \mu' + \mathfrak{s}' \otimes \mathcal{L}_{H_{p_i}} \mu' \\ &= L^i_P(\mathfrak{s}' \otimes \mu') \end{aligned}$$

Q. E. D.

SOME RESULTS ON THE SPECTRUM OF P

Now assume that X is compact and P self adjoint, elliptic, and positive definite, with spectrum: $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \dots$. By a bicharacteristic of P we will mean an H_{p_i} integral curve, γ, normalized so that $p_i = 1$ on γ.

THEOREM I: $e(t) = \Sigma\, e^{i\sqrt{\lambda_k}t}$ is well defined as a generalized function, and if $T \in$ sing. supp. e, there exists a periodic bicharacteristic of period T.

Compared with [1], theorem 1

THEOREM II: Suppose that there are only a finite number of bicharacteristics of period T: $\gamma_1, \gamma_2, \dots, \gamma_N$ and that for each γ_i the Poincare map P_{γ_i} satisfies the Lefschetz condition, $\det(I-P_\gamma) \neq 0$. Then e(t) is smooth in an interval $0 < |t-T| < a$ and $\lim_{t\to T} (t-T)\, e\,(t)$ exists and is equal to

$$\Sigma(\text{trace } H_{\gamma_i}) \frac{|T|}{2\pi} (\sqrt{-1})^{\sigma_i} \,|\det(I-P_\gamma)|^{-\frac{1}{2}}$$

where σ_i is the Maslov index of γ_i

Compare with [1], theorem 2

THEOREM III: Let $f(\lambda)$ be the number of eigenvalues of P less than λ. Then

$$f(\lambda) = \Sigma \dim E_i \frac{\text{vol}(B_i)}{(2\pi)^n} \lambda^n + 0(\lambda^{n-1})$$

where $\mathrm{vol}(B_i)$ is the volume, in T^*X, of the ball $\{(x,\xi), p_i(x,\xi) \leq 1\}$ (with respect to the symplectic volume form).

Compare with Hormander [3]

In [1] we studied the relationship between periodic bicharacteristic flow and "clustering" of the eigenvalues of $\sqrt[m]{P}$ when P is a scalar operator. By "clustering" we mean there exist numbers α and B such that most of the eigenvalues lie near the lattice points $\alpha n + B$, n = 0, 1, 2, etc.. More precisely given $\varepsilon > 0$ let $f_\varepsilon(\lambda)$ be the number of eigenvalues less than λ lying in one of the intervals

$$(2.1) \qquad |\lambda - \alpha n + B| < \frac{\varepsilon}{\sqrt{n}}$$

Then clustering means that for every ε, $f_\varepsilon(\lambda)/f(\lambda) \to 1$ as $\lambda \to \infty$. We showed this phenomenon occurs if and only if the bicharacteristic flow is periodic, (and, if it occurs, α and B are related in a simple way to the period and Maslov index of the flow). We now state a generalization of this result for systems satisfying (1.4).

<u>THEOREM</u> <u>IV</u>: The spectrum of $\sqrt[m]{P}$ clusters if and only if there exists a fixed T and σ such that the bicharacteristic flow for all the p_i's is periodic of period T and index σ and, in addition, the holonomy map, H_γ, around every periodic bicharacteristic γ of period T is the identity.

These results can be refined to allow for clustering with respect to the eigenvalues associated with certain modes (i.e. with certain E_i) and not with respect to others. We won't bother to discuss these refinements here.

SOME REMARKS ON THE PROOFS OF THEOREMS I - IV

Let $E \to X$ be a vector bundle, and $\Lambda \subset T^*X\setminus 0$ a homogeneous Lagrangian manifold. We will denote by $I^k(E,\Lambda)$ the space of all generalized sections of $E \otimes |\Lambda|^{\frac{1}{2}}X$ which can be written in the form

$$(3.1) \qquad e_1 \otimes \mu_1 + \dots + e_N \otimes \mu_N$$

where the e_i's are ordinary sections of E and the μ_i's are elements of $I^k(X,\Lambda)$. (For the definition of these spaces see Hormander [4].) The symbol of (3.1) is defined to be

$$e_1 \otimes \sigma(\mu_1) + \dots + e_N \otimes \sigma(\mu_N)$$

and is to be viewed as a section of the vector bundle:

$$(\pi^* E l \Lambda) \quad \otimes \quad |\Lambda|^{\frac{1}{2}}\Lambda \quad \otimes \quad \mathrm{Maslov}(\Lambda).$$

If $P: C^\infty(E \otimes |\Lambda|^{\frac{1}{2}}) \to C^\infty(E \otimes |\Lambda|^{\frac{1}{2}})$ is a pseudodifferential operator of order m then P maps $I^k(E,\Lambda)$ into $I^{k+m}(E,\Lambda)$, and

$$(3.2) \qquad \sigma(P\mu) \quad = \quad \sigma(P)\sigma(\mu)$$

If $\sigma(P)$ is a scalar multiple of the identity and $p = 0$ on Λ then by (3.2) $P\mu \in I^{k+m-1}(E,\Lambda)$. In this case the symbol of $P\mu$ is given by the transport equation

$$(3.3) \qquad \sigma(P\mu) \quad = \quad L_p\sigma(\mu)$$

More generally, suppose $\sigma(P)$ is smoothly diagonalizable in the sense of (1.4) and $p_i = 0$ on Λ. Then if the E_j components of $\sigma(\mu)$, $j \neq i$, are zero the E_i component of $\sigma(P\mu)$ is $L^i{}_P\sigma(\mu)$. (Compare with Duistermaat - Hormander, [2], (5.3.1).)

In the case of a single equation the proofs of theorems I-IV depended on finding a fairly explicit analytical description of the operator $\mu(t) = \exp\sqrt{-1}\, t\, P$. More precisely we needed the

THEOREM: Let $\mu(x, y, t)$ be the Schwartz kernel of the operator $\mu(t)$ viewed as a generalized half-density on $X \times X \times \mathbb{R}$. Then $\mu \in I^{-\frac{1}{4}}(X\times X\times\mathbb{R}, C)$ where

$$C = \{(x, \xi, y, \eta, t, \tau),\ \tau = p(x,\xi),\ (y,\eta) = (\exp t\, H_p)\ (x, -\xi)\}$$

moreover the symbol of μ, ignoring Maslov factors, is equal to $\sqrt{\pi^* dx\wedge d\xi}$ where $\pi: C \to T^*X$ is the map $(x, \xi, y, \eta, t, \tau) \to (x, \xi)$.

The proof of this theorem, which can be found in Duistermaat-Hormander, (§5.3) involves an iterated solution of the transport equation with initial data at $t = 0$ prescribed by $\mu(0) = \mathrm{Id}$. Using the transport equation described in the previous paragraph one can prove a similar result for $\exp\sqrt{-1}tP$, P being an operator on a vector bundle, providing the symbol of P satisfies (1.4). Note that $\mu(x, y, t)$ is now a generalized section of

$$\mathrm{Hom}(\rho_1{}^*E,\ \rho_2{}^*E) \quad \otimes \quad |\Lambda|^{\frac{1}{2}}(X\times Y)$$

where $\rho_1: X \times X \times \mathbb{R} \to X$ and $\rho_2: X \times X \times \mathbb{R} \to X$ are the projections $(x, y, t) \to x$ and $(x, y, t) \to y$.

THEOREM: $\mu(x, y, t) = \sum_{i=1}^{N} \mu_i(x, y, t)$ with $\mu_i \in I^{-\frac{1}{4}}(\mathrm{Hom}(\rho_1{}^*E, \rho_2{}^*E), C_i)$ where $C_i = \{(x,\xi,y,\eta,t,\tau),\ \tau = p_i(x,\xi),\ (y,\eta) = \exp tH_{p_i}(x,-\xi)\}$

Moreover the symbol of μ_i is equal, modulo Maslov factors, to the tensor product of the half-density $\sqrt{\pi^* dx \wedge d\xi}$ and the following section, H^i, of $\mathrm{Hom}(\rho_1{}^*E, \rho_2{}^*E) \mid C_i$: Let $(x, -\xi, y, \eta, t, \tau)$ be a point of C_i. Define H^i at $(x, -\xi, y, \eta, t, \tau)$ to be the map which maps $E^j_{(x,\xi)}$ onto zero for $j \neq i$ and maps $E^i{}_{(x,\xi)}$ onto $E^i{}_{(y,\eta)}$ by the map $\mathfrak{s} \in E^i{}_{(x,\xi)} \to \mathfrak{s}' \in E^i{}_{(y,\eta)}$ where $\mathfrak{s}'$ is obtained from $\mathfrak{s}$ by solving $L^i{}_P \tilde{\mathfrak{s}} = 0$ along the bicharacteristic joining (x,ξ) to (y,η) with $\tilde{\mathfrak{s}}(0) = \mathfrak{s}$.

With this description of $\exp \sqrt{-1}tP$, the proofs of theorems I - IV proceed along the same general lines as for the analogous theorems discussed in [1].

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