Elliptic and Parabolic Methods in Geometry

Estimates for the Biharmonic Energy on Unbounded Planar Domains, and the Existence of Surfaces of Every Genus That Minimize the Squared-Mean-Curvature Integral

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ABSTRACT. We outline our approach to proving there is a surface of each genus that minimizes the integral W of the squared mean curvature.

1. Introduction to the Willmore problem and the main result

The Willmore problem seeks the compact surface M embedded or immersed in $\mathbb{R}^3$ that minimizes the squared-mean-curvature integral

$$W(M) = \int_M H^2 \, dA$$

among surfaces of fixed topological type, such as prescribed genus or regular homotopy class. It is easy to show that $W \geq 4\pi$, with the round sphere S^2 minimizing W among all surfaces, and in particular among those of genus zero.

A well-known conjecture of Willmore states that the circular torus of revolution with a radius ratio of $\sqrt{2}$ is the minimizer for W (with value $2\pi^2$) among embedded surfaces of genus one, but only recently has Simon [1993] proved that a W-minimizer of genus one actually exists. The difficulty is that W is invariant not only under Euclidean motions, but also under dilation and inversion. Hence the noncompact group of Möbius tranformations preserves W. This can interfere with proving that a W-minimizing sequence converges to a surface of the same topological type. For example, one could always apply a sequence of Möbius transformations to any sequence of surfaces so that the result converges to a round sphere. Analytically, this Möbius invariance is reflected in the fact that W is Sobolev-borderline.

In this note we outline our approach [Kusner a] to proving the following existence result for W-minimizers of each genus:

Suppose that $M_i \subset \mathbb{R}^3$ is a sequence of embedded surfaces of genus g, with $W(M_i)$ converging to the infimum W_g of W among surfaces of genus g. Then there is a subsequence M_j, and a sequence of Möbius tranformations G_j, such that $G_j(M_j)$ converge smoothly to an embedded surface $M \subset \mathbb{R}^3$ of genus g, with $W(M) = W_g$.

In fact, extensive computer experiments [Hsu et al. 1992] using Brakke's Evolver [Brakke 1992] suggest that the minimizers are the stereographic images of the Lawson [1970] minimal surfaces in S^3, as the author had originally conjectured [Kusner 1989]. (Since the Lawson minimal surface of genus one is simply the Clifford torus, one of whose stereographic images is the torus of revolution mentioned above, this conjecture includes Willmore's conjecture.)

2. An easy estimate for the Willmore energy

To get a better idea of what can go wrong with a high-genus W-minimizing sequence, and to introduce a way to overcome this, we first sketch why the connected sum $M\#N$ of two embedded surfaces M and N satisfies the weak inequality

$$\inf_{P\in[M\#N]} W^*(P) \le \inf_{M\in[M]} W^*(M) + \inf_{N\in[N]} W^*(N),$$

where we use the notation $W^* = W - 4\pi$, and where $[S]$ means the regular homotopy class [Kusner 1989] of an immersed surface S.

Consider deforming small neighborhoods of points on M and N to make them umbilic: this will change W by an arbitrarily small amount, and allow us to invert through the respective points to get a pair of noncompact surfaces M^* and N^*, each with one planar end, and with $W(M^*) = W^*(M)$ and $W(N^*) = W^*(N)$. Clearly we can weld togther M^* and N^* along their planar ends to obtain a one-ended surface $P^* \in [(M\#N)^*]$, and invert back to get a compact conformal connected sum surface $P \in [M\#N]$ with

$$W^*(P) = W(P^*) = W(M^*) + W(N^*) = W^*(M) + W^*(N).$$

Taking infima yields the desired inequality.

3. Induction on the genus

Now suppose that equality holds in the above inequality; then we can easily get a (divergent) W-minimizing sequence simply by welding the two summands at greater and greater distances apart: even if we applied a sequence of Möbius transformations to this, we would at most recover a surface regularly homotopic to either M or N in the limit, but not one regularly homotopic to $M\#N$.

On the other hand, at least in the case of embedded surfaces, suppose we can show that the *strict* inequality holds for all nontrivial connected sums—that is, for both M and N of genus at least one. Then there is an inductive argument to prove that there is a W-minimizer of each genus g, as follows. Since there is genus-one minimizer [Simon 1993], we may assume (by induction) that we have minimizers for every genus $< g$, and consider what can happen to a minimizing sequence of genus-g surfaces. If (after Möbius transformation) the sequence fails to converge to a minimizer of genus g, we can find two subsequences, together with two sequences of Möbius transformations, so that the transformed sequences of surfaces converge to W-minimizing surfaces of positive genus h and k, with $h + k = g$. But then one can use an argument like that of the preceding section to see that if the strict inequality held, the original sequence could not have been minimizing.

4. Reducing the Willmore energy to the biharmonic energy

The key new idea in proving for embedded surfaces the strict inequality for the infima,

$$W^*_{h+k} < W^*_h + W^*_k,$$

is to analyze the W-energy saved when we weld together a pair of surfaces at nonumbilic points (which always exist on surfaces of positive genus). Indeed, if we invert through a nonumbilic point on a W-critical surface M, the end of M^* (after rotation, translation and dilation) is asymptotic to the graph of the biharmonic function $\cos 2\theta = (x^2 - y^2)/r^2$, with higher-order (biharmonic) terms decaying at least as fast as $1/r$.

Thus, if one welds together the ends of suitably scaled surfaces, the W-energy saved can be computed in terms of the savings in the "linearized" biharmonic energy

$$B(u; \Omega) = \frac{1}{4} \int_\Omega (u_{xx} + u_{yy})^2 \, dx \, dy$$

for a biharmonic function u over a noncompact planar domain Ω whose graph represents the welded region on the surface. Here we use the fact that on this region

$$H^2 \, dA = \tfrac{1}{4}(u_{xx} + u_{yy})^2 \, dx \, dy + \mathcal{O}(\ni),$$

where the second term on the right means that only things cubic in u and its gradient occur.

5. The tricky estimate for the biharmonic energy

The relevant biharmonic energy estimate is that, when Ω is the exterior of a pair of unit disks centered on the x-axis at $-\frac{1}{2}s$ and $\frac{1}{2}s$, there is a biharmonic function u with boundary values (approximately) $\cos 2(\theta-\theta_+)$ and $\cos 2(\theta-\theta_-)$ on the respective circles, satisfying

$$B(u;\Omega) \le 2B_0 - \frac{c(\theta_+,\, \theta_-)}{s^2}\,,$$

where $c(\theta_+,\, \theta_-)$ is a nonzero function of the phase angles θ_+ and θ_-, and where B_0 is the biharmonic energy of $\cos 2\theta$ on the exterior of the unit disk centered at the origin. The function $c(\theta_+,\, \theta_-)$ changes sign as the relative phase rotates through 90 degrees. Thus, by suitably aligning the phase angles, we have a C/s^2 biharmonic energy savings when we weld together the ends of a pair of handles, and a corresponding savings in the W-energy for sufficiently large s, for some positive constant C. This inequality can be proved by considering the associated bilinear form

$$B(v,w;\Omega) = \frac{1}{4}\int_\Omega (v_{xx}+v_{yy})(w_{xx}+w_{yy})\,dx\,dy,$$

where v and w are x-translates by $\pm\frac{1}{2}s$ of the restriction to Ω of $\cos 2(\theta-\theta_+)$ and $\cos 2(\theta-\theta_-)$, respectively, and where u is approximated by a linear combination of v and w.

6. Long-range elastodynamics of fluid membranes

In physical terms, the above means that the force fields associated to the gradient of both W and B can be attractive, with an s^{-3} power law. For example, if one thinks of a genus-two surface as a plane with a pair of handles welded into it, there is a relative orientation of the handles such that the handles are attracted to one another when they align; however, if one rotates one handle by 90 degrees, they will repel. Thus the interaction is like that between a pair of quadrupoles.

These dynamical features of are of considerable interest to physicists studying elastic fluid membranes (see [Hsu et al. 1992] and the references therein), whose energy is governed by W. In fact, though it will be a "many-body" problem, the behavior of several widely separated handles with various orientations could be treated in a similar way as we do here for a pair of handles. Perhaps the most tractable case is the limit where the handles are infinitely separated, or (equivalently) shrunk to zero size. This leads to a finite-dimensional variational problem concerning a collection of points on S^2 with a quadrupole interaction, which (since W is Möbius invariant) is invariant under the group $\mathrm{PSL}(2,\mathbb{C})$ of fractional linear transformations, and which asymptotically models the W-interaction of the handles [Kusner b].

7. Some questions about the W-minimizers and higher codimension

To actually determine the W-minimizer of a particular genus is a much more difficult problem. For low genus, it may turn out that integrable systems methods (see [Ferus et al. 1992], for example) will shed some light on this, but it may also be of interest to consider the limit as $g \to \infty$. In this case we can show (see [Hsu et al. 1992] for a brief discussion) that the minimal surface in S^3 of smallest area among those of genus g must converge to a union of two (orthogonal) great two-spheres. It would be interesting to prove that (the stereographic projection of) these smallest-area minimal surfaces are in fact the W-minimizers, even if we cannot identify the former as the Lawson surfaces.

We expect that an analysis of $\mathbb{R}^k$-valued biharmonic functions should yield a similar inequality for W on connected sums of surfaces in $\mathbb{R}^{k+2}$, but at the moment it is unclear whether all our results extend directly to higher codimension. As observed in [Li and Yau 1982], the Veronese embedding of $\mathbb{R}P^2$ into S^4 realizes the absolute minimum value (6π) for W among all immersed projective planes. Since any nonorientable surface is diffeomorphic to the connected sum of projective planes, the following questions are quite tempting:

- Is there a compact nonorientable embedded surface of each topological type in $\mathbb{R}^4$ with energy $W < 8\pi$?
- Must a compact surface in $\mathbb{R}^{k+2}$ with $W < 6\pi$ be a sphere?

Of course, affirmation of the latter question would provide a direct demonstration of the strict connected sum inequality for W on orientable surfaces [Kusner 1989], while if both are affirmed, the strict connected sum inequality would hold for nonorientable surfaces as well.

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