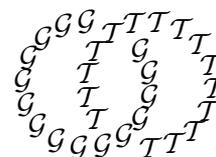


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## Splitting the concordance group of algebraically slice knots

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### Abstract

As a corollary of work of Ozsváth and Szabó [8], it is shown that the classical concordance group of algebraically slice knots has an infinite cyclic summand and in particular is not a divisible group.

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Let  $\mathcal{A}$  denote the concordance group of algebraically slice knots, the kernel of Levine's homomorphism  $\phi: \mathcal{C} \rightarrow \mathcal{G}$ , where  $\mathcal{C}$  is the classical knot concordance group and  $\mathcal{G}$  is Levine's algebraic concordance group [6]. Little is known about the algebraic structure of  $\mathcal{A}$ : it is countable and abelian, Casson and Gordon [2] proved that  $\mathcal{A}$  is nontrivial, Jiang [5] showed it contains a subgroup isomorphic to  $\mathbf{Z}^\infty$ , and the author [7] proved that it contains a subgroup isomorphic to  $\mathbf{Z}_2^\infty$ . We add the following theorem, a quick corollary of recent work of Ozsváth and Szabó [8].

**Theorem 1** *The group  $\mathcal{A}$  contains a summand isomorphic to  $\mathbf{Z}$  and in particular  $\mathcal{A}$  is not divisible.*

**Proof** In [8] a homomorphism  $\tau: \mathcal{C} \rightarrow \mathbf{Z}$  is constructed. We prove that  $\tau$  is nontrivial on  $\mathcal{A}$ . The theorem follows since, because  $\text{Im}(\tau)$  is free, there is the induced splitting,  $\mathcal{A} \cong \text{Im}(\tau) \oplus \text{Ker}(\tau)$ . No element representing a generator of  $\text{Im}(\tau)$  is divisible.

According to [8],  $|\tau(K)| \leq g_4(K)$ , where  $g_4$  is the 4-ball genus of a knot, and there is the example of the  $(4, 5)$ -torus knot  $T$  for which  $\tau(T) = 6$ . We will show that there is a knot  $T^*$  algebraically concordant to  $T$  with  $g_4(T^*) < 6$ . Hence,  $T \# -T^*$  is an algebraically slice knot with nontrivial  $\tau$ , as desired.

Recall that  $T$  is a fibered knot with fiber  $F$  of genus  $(4-1)(5-1)/2 = 6$ . Let  $V$  be the  $12 \times 12$  Seifert matrix for  $T$  with respect to some basis for  $H_1(F)$ . The quadratic form  $q(x) = xVx^t$  on  $\mathbf{Z}^{12}$  is equal to the form given by  $(V + V^t)/2$ . Using [3] the signature of this symmetric bilinear form can be computed to be 8, so  $q$  is indefinite, and thus by Meyer's theorem [4] there is a nontrivial primitive element  $z$  with  $q(z) = 0$ . Since  $z$  is primitive, it is a member of a symplectic basis for  $H_1(F)$ . Let  $V^*$  be the Seifert matrix for  $T$  with respect to that basis. The canonical construction of a Seifert surface with Seifert matrix  $V^*$  ([9], or see [1]) yields a surface  $F^*$  such that  $z$  is represented by a simple closed curve on  $F^*$  that is unknotted in  $S^3$ . Hence,  $F^*$  can be surgered in the 4-ball to show that its boundary  $T^*$  satisfies  $g_4(T^*) < 6$ . Since  $T^*$  and  $T$  have the same Seifert form, they are algebraically concordant.  $\square$

**Addendum** An alternative proof of Theorem 1 follows from the construction of knots with trivial Alexander polynomial for which  $\tau$  is nontrivial, to appear in a forthcoming paper.

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