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Lens space surgeries and a conjecture of Goda and Teragaito

Jacob Rasmussen

Department of Mathematics, Princeton University
Princeton, NJ 08544, USA

Email: jrasmus@math.princeton.edu

Abstract

Using work of Ozsváth and Szabó, we show that if a nontrivial knot in S^3 admits a lens space surgery with slope p, then $p \leq 4g+3$, where g is the genus of the knot. This is a close approximation to a bound conjectured by Goda and Teragaito.

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1 Introduction

Let K be a knot in S^3, and denote by K_r the three-manifold obtained by performing Dehn surgery on K with slope $r = p/q$. If K_r is homeomorphic to a lens space we say that K admits a lens space surgery with slope r. In recent years, Kronheimer, Mrowka, Ozsváth, and Szabó have used Floer homology for three-manifolds to give constraints on such knots [10], [12], [7]. Generally speaking, these constraints are derived from the fact that lens spaces belong to a larger class of spaces, known as *L–spaces*, for which the reduced Floer homology groups HF^{red} vanish.

On the other hand there are many L–spaces which are not lens spaces. In particular, if K admits a single L–space surgery with positive slope, then K_p is an L–space for every integer $p \geq 2g(K) - 1$, where $g(K)$ denotes the genus of K [7]. In contrast, when K is hyperbolic, the cyclic surgery theorem of [2] tells us that at most two of these surgeries are actually lens spaces. In this note, we show that Floer homology can be used to distinguish at least some of these L–space surgeries from lens spaces. In particular, we prove the following result.

Theorem 1 *Suppose K is a nontrivial knot which admits a lens space surgery of slope r. Then*

$$|r| \leq 4g(K) + 3.$$

The inequality is sharp — equality holds for the case of $4k + 3$ surgery on the right-handed $(2, 2k + 1)$ torus knot, which gives the lens space $L(4k + 3, 4)^*$.

This result closely approximates a bound conjectured by Goda and Teragaito in [4]. More specifically, they showed that if K is a *hyperbolic* knot which admits a lens space surgery of slope p, then $|p| \leq 12g(K) - 7$, and conjectured that in fact

$$2g(K) + 8 \leq |p| \leq 4g(K) - 1.$$

Something close to the first inequality was proved in Corollary 8.5 of [7], where it was shown that if K admits an L–space surgery of slope p, then $2g(K)-1 \leq |p|$. Theorem 1 seems to be a natural (and only minimally weaker) reformulation of the second inequality which applies to all knots.

The proof of the theorem is based on work of Ozsváth and Szabó in [10]. In addition, we use an analog of an inequality of Frøyshov [3] and some elementary facts about Dedekind sums. The paper is arranged as follows: in section 2, we review the results of [10] and outline the proof of Theorem 1. Section 3 is devoted to the proof of Frøyshov's inequality, and section 4 contains the necessary results on Dedekind sums.

Throughout this note, we work in the category of oriented manifolds. All maps, homeomorphisms, *etc.* are assumed to be orientation preserving unless specified otherwise. For lens spaces, our orientation convention is the one used in [5] and [16], namely, that $-p$ surgery on the unknot produces the oriented lens space $L(p,1)$. (Note that this is the opposite of the convention used in [10] and [7]).

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2 Outline of proof

Suppose that K is a nontrivial knot and that K_r is a lens space. Without loss of generality, we may assume that r is an integer. Indeed, the cyclic surgery theorem implies that this must be case unless K is a torus knot. On the other hand, if K is the right-handed (a,b) torus knot, it is well known [8] that $K_{p/q}$ is a lens space if and only if $qab - p = \pm 1$. In particular, the slope attains its largest value when $p/q = ab+1$ is an integer.

We now review the results of [10] on knots admitting integral lens space surgeries. For technical reasons, it is convenient to assume that the slope of these surgeries is negative. By considering the mirror image of K, if necessary, we may arrange that this is the case. From this point on, then, we will assume that K_{-p} is a lens space, where p is a positive integer.

2.1 The exact triangle

Let W_1 be the surgery cobordism from K_0 to S^3, and let x be a generator of $H^2(W_1) \cong \mathbf{Z}$. We use the notation $\mathfrak{s}_i$ to refer either to the Spin^c structure on W_1 with $c_1(\mathfrak{s}_i) = 2ix$ or its restriction to K_0. (It should be clear from context which manifold is being considered.) Likewise, if W_2 is the surgery cobordism from S^3 to K_{-p} and $y \in H^2(W_2)$ is a generator, we let $\mathfrak{t}_i$ be the Spin^c structure on W_2 with $c_1(\mathfrak{t}_i) = (-p+2i)y$, and $\mathfrak{t}'_i$ be its restriction to K_{-p}. Note that $\mathfrak{t}'_i$ only depends on the value of i mod p.

The exact triangle with twisted coefficients [9] gives a long exact sequence

$$\oplus_{i\equiv k\,(p)}\underline{HF}^+(K_0,\mathfrak{s}_i) \xrightarrow{\oplus F_{W_1,\mathfrak{s}_i}} HF^+(S^3)[T,T^{-1}] \xrightarrow{F_{W_2,k}} HF^+(K_{-p},\mathfrak{t}'_k)[T,T^{-1}]$$

where

$$F_{W_2,k}(x) = \sum_{n \in \mathbf{Z}} T^n \cdot F_{W_2,\mathfrak{t}_{k+pn}}(x).$$

Let h_i be the rank of $F_{W_1,\mathfrak{s}_i}\colon \underline{HF}^+(K_0,\mathfrak{s}_i) \to HF^+(S^3)[T,T^{-1}]$, viewed as a map of $\mathbf{Z}[T,T^{-1}]$ modules. Combining results from [10] and [13] gives the following:

Proposition 2.1 *Suppose K_{-p} is a lens space. Then $p \geq 2g(K)-1$, and h_i is nonzero if and only if $-g(K)+1 \leq i \leq g(K)-1$. Moreover, for $-p/2 \leq k \leq p/2$*

$$\text{rank } \ker F_{W_2,k} = \text{rank } \underline{HF}^+(K_0,\mathfrak{s}_k) = h_k.$$

Proof Since K_{-p} is a lens space, $HF^+(K_{-p},\mathfrak{t}'_k) \cong HF^+(S^3) \cong \mathbf{Z}[u^{-1}]$. Now $\underline{HF}^+(K_0,\mathfrak{s}_i)$ has finite rank as a $\mathbf{Z}[T,T^{-1}]$ module, and $F_{W_2,k}$ is a u–equivariant map. From this, it follows that $\ker F_{W_2,k}$ must be of the form $\langle 1, u^{-1}, \ldots, u^{-n}\rangle \otimes \mathbf{Z}[T,T^{-1}]$ for some value of n, and that coker $F_{W_2,k}$ must have zero rank. Thus the maps $F_{W_1,\mathfrak{s}_i}$ must all be of full rank, and the sum

$$\bigoplus_{i \equiv k\,(p)} \underline{HF}^+(K_0,\mathfrak{s}_i) \tag{1}$$

can contain at most one term of nonzero rank. In this case, we have

$$\text{rank } \ker F_{W_2,k} = \text{rank } \underline{HF}^+(K_0,\mathfrak{s}_i) = h_i.$$

where i is the index of the nontrivial summand. Since $h_i \leq h_j$ whenever $|i| > |j|$ (Proposition 7.6 of [15]), it follows that i must be the representative of k mod p with smallest absolute value, so $-p/2 \leq i \leq p/2$.

According to [13], the largest value of i for which $HF^+(K_0,\mathfrak{s}_i)$ is nontrivial is $g(K)-1$. It follows that h_i is nonzero if and only if

$$-g(K)+1 \leq i \leq g(K)-1.$$

Thus there are $2g(K)-1$ values of i for which h_i is nontrivial, and the condition that the sum in equation (1) contains at most one nontrivial term for all values of k is equivalent to the statement that $p \geq 2g(K)-1$. □

2.2 The d–invariant

Let Y be a rational homology three-sphere, and let $\mathfrak{s}$ be a Spin^c structure on it. Then $HF^+(Y,\mathfrak{s}) \otimes \mathbf{Q}$ is absolutely graded and contains a unique summand isomorphic to $\mathbf{Q}[u^{-1}]$. The invariant $d(Y,\mathfrak{s})$ is defined [10] to be the absolute grading of $1 \in \mathbf{Q}[u^{-1}]$. (This is the analog of Frøyshov's h invariant in Seiberg–Witten theory.)

The d–invariants of K_{-p} are easily expressed in terms of the h_k's. Recall that the map $F_{W_2,k}$ is a sum of maps $F_{W_2,\mathfrak{t}_i}\colon HF^+(S^3) \to HF^+(K_{-p}, \mathfrak{t}'_k)$. Since the intersection form on W_2 is negative-definite, each $F_{W_2,\mathfrak{t}_i}$ is a surjection. Moreover, $F_{W_2,\mathfrak{t}_i}$ is a graded map; it shifts the absolute grading by a fixed rational number $\epsilon(p,i)$ which depends only on homological data associated to the cobordism W_2. In particular, this number is independent of K.

The kernel of $F_{W_2,k}$ is generated by $\langle 1, u^{-1}, \ldots, u^{-h_k+1}\rangle$, so $u^{-h_k} \in HF^+(S^3)$ must map to a nontrivial multiple of $1 \in HF^+(K_{-p}, \mathfrak{t}_k)[T, T^{-1}]$. Thus if we let

$$E(p,k) = \max\, \{\epsilon(p,i) \,|\, i \equiv k\ (p)\}$$

it follows that for $-p/2 \leq k \leq p/2$,

$$d(K_{-p}, \mathfrak{t}'_k) = E(p,k) + 2h_k. \tag{2}$$

Specializing to the case where K is the unknot, we see that the set of d–invariants of the lens space $L(p,1)$ is $\{E(p,k) \,|\, k \in \mathbf{Z}/p\}$.

2.3 The Casson–Walker invariant

If Y is an integral homology three-sphere, then its Casson invariant $\lambda(Y)$ is determined by $d(Y)$ and the group $HF^{\mathrm{red}}(Y)$ (Theorem 5.1 of [10].) More generally, if Y is a rational homology three-sphere, it is expected that its Casson–Walker invariant $\lambda(Y)$ will be related to $HF^{\mathrm{red}}(Y)$ and the d–invariants of Y. This relation is particularly simple when Y is a lens space:

Lemma 2.2

$$\sum_{\mathfrak{s}} d(L(p,q), \mathfrak{s}) = p\lambda(L(p,q)))$$

where the sum runs over all Spin^c *structures on* $L(p,q)$.

The proof will be given in section 4. Combining with equation (2), we get

$$p(\lambda(K_{-p}) - \lambda(L(p,1))) = 2 \sum_{i=-g+1}^{g-1} h_i.$$

2.4 Frøyshov's inequality

The new geometric input in the proof of Theorem 1 is the following fact, which is analogous to a theorem of Frøyshov in instanton Floer homology [3]. Its proof is the subject of section 3.

Theorem 2.3 *Let K be a knot in S^3, and let $g_*(K)$ be its slice genus. Then $h_i(K) = 0$ for $|i| \geq g_*(K)$, while for $|i| < g_*(K)$*

$$h_i(K) \leq \left\lceil \frac{g_*(K) - |i|}{2} \right\rceil.$$

If K admits a lens space surgery, the results of [12] and [13] show that $g_*(K) = g(K)$, so

$$p(\lambda(K_{-p}) - \lambda(L(p,1))) \leq 2 \sum_{i=-g+1}^{g-1} \left\lceil \frac{g_*(K) - |i|}{2} \right\rceil = g(K)(g(K)+1).$$

2.5 Proof of the theorem

Suppose that the inequality is false. Then $p \geq 4g(K) + 4$, so $g(K) + 1 \leq \frac{p}{4}$. Substituting into the previous inequality, we find that

$$\lambda(K_{-p}) - \lambda(L(p,1)) \leq \frac{1}{4}(\frac{p}{4} - 1).$$

The value of $\lambda(L(p,q))$ is given by a certain arithmetic function of p and q, known as a Dedekind sum. The following purely arithmetic result is proved in section 4:

Proposition 2.4 *Suppose that Y is a lens space with $|H_1(Y)| = p$, and that*

$$\lambda(Y) - \lambda(L(p,1)) \leq \frac{1}{4}(\frac{p}{4} - 1).$$

Then Y is homeomorphic to one of $L(p,1)$, $L(p,2)$, or $L(p,3)$.

The possibility that K_{-p} is homeomorphic to $L(p,1)$ is ruled out by the main theorem of [7]. To eliminate the other two cases, we use the following proposition, which is also proved in section 4.

Proposition 2.5 *If K_{-p} is homeomorphic to $L(p,2)$, then $p = 7$. If it is homeomorphic to $L(p,3)$, then either $p = 11$ or $p = 13$.*

From the table at the end of [10], we see that if $L(7,2)$, $L(11,3)$, or $L(13,3)$ is realized by integer surgery on a knot K, then K must have genus 1, 2, or 3, respectively, and the inequality of Theorem 1 is satisfied. (In fact, these lens spaces are realized by surgery on the torus knots $T(2,3)$, $T(2,5)$, and $T(3,4)$, respectively.) This concludes the proof of the theorem. □

3 Frøyshov's inequality

We now turn to the proof of Theorem 2.3. The argument we give is essentially that of [3], but adapted along the lines of [10] to fit the Heegaard Floer homology. We begin by reformulating the problem slightly. Let K be a knot in S^3, and choose $n \gg 0$. Then for $-n/2 \leq k \leq n/2$, we have $d(K_{-n}, \mathfrak{t}'_k) = E(n,k) + 2h_k(K)$. To prove the theorem, we will estimate the size of $d(K_{-n}, \mathfrak{t}'_k)$.

To this end, we consider the surgery cobordism ${W_2}^*$ from K_{-n} to S^3. (This is the cobordism W_2 of section 2.1 with its orientation reversed.) We fill in the S^3 boundary component of ${W_2}^*$ with a four-ball to get a four-manifold W'. Then $H_2(W') \cong \mathbf{Z}$, and the generator of this group can be represented by an embedded surface Σ_g with genus $g = g_*(K)$ and self-intersection n. Finally, let W be the four-manifold obtained by removing a tubular neighborhood of Σ_g from W'. W is a cobordism from K_{-n} to the circle bundle over Σ_g with Euler number $-n$, which we denote by B_{-n}. This choice of name is a natural one, since B_{-n} can be obtained by doing $-n$ surgery on the "Borromean knot" $B \subset \#^{2g}(S^1 \times S^2)$.

We now consider the topology of the cobordism W. An easy computation shows that $H^2(W) \cong H^2(B_{-n}) \cong \mathbf{Z}^{2g} \oplus \mathbf{Z}/n$, and the restriction map to $H^2(K^*_{-n})$ is projection onto the second factor. It follows that there is a unique torsion Spin^c structure on W which restricts to $\mathfrak{t}'_k$ on K^*_{-n}. We denote this Spin^c structure and its restriction to B_{-n} by $\mathfrak{t}'_k$ as well.

Note that there is another natural way to label the torsion Spin^c structures on B_{-n}. Namely, we can view B_{-n} as $-n$ surgery on the knot B and use the labeling convention of section 2.1. To be precise, let X_2 be the surgery cobordism from $\#^{2g}(S^1 \times S^2)$ to B_{-n}. Then the restriction map $H^2(X_2) \to H^2(\#^{2g}(S^1 \times S^2))$ has kernel isomorphic to $\mathbf{Z}$. If x is a generator of this group, we let $\mathfrak{u}_k$ be the Spin^c structure on X_2 with $c_1(\mathfrak{u}_k) = (-n+2k)x$, and $\mathfrak{u}'_k$ be its restriction to B_{-n}.

Lemma 3.1 *For an appropriate choice of the generator x, we have $\mathfrak{t}'_k = \mathfrak{u}'_k$.*

Proof Let X' be the double of W', and let $S \subset X'$ be the embedded sphere which is obtained by gluing together the cocore of the two-handle in W' (which is an embedded disk generating $H_2(W', \partial W')$) and its mirror image in $(W')^*$. S intersects Σ_g geometrically once. If we remove a tubular neighborhood of Σ_g from X, we get a four-manifold X which is the union of $(W')^*$ and W. $S \cap X$ is an embedded disk D whose boundary is a fiber of the circle bundle B_{-n}. Let D_1 be the disk $D \cap (W')^*$. Then D_1 is Poincare dual to the generator of $H^2((W')^*)$.

Recall that the Spin^c structure $\mathfrak{t}_k$ on W_2 was defined by $c_1(\mathfrak{t}_k) = (-n+2k)y$, where y was a generator of $H^2(W_2)$. $(W')^* = W_2 \cup D^4$, so $\mathfrak{t}_k$ extends uniquely to a Spin^c structure $\mathfrak{t}_k$ on $(W')^*$ with $c_1(\mathfrak{t}_k) = (-n+2k)\mathrm{PD}(D_1)$. Now let $\mathfrak{v}_k$ be the Spin^c structure on X with $c_1(\mathfrak{v}_k) = (-n+2k)\mathrm{PD}(D)$. Then $\mathfrak{v}_k|_{(W')^*} = \mathfrak{t}_k$, and $\mathfrak{v}_k|_{B_{-n}}$ is torsion, so $\mathfrak{v}_k|_W = \mathfrak{t}'_k$. On the other hand, if $X'' \subset X$ is a regular neighborhood of $B_{-n} \cup D$, it is not difficult to see that $X'' \cong X_2$, and that the kernel of the restriction map $H^2(X'') \to H^2(\#^{2g}(S^1 \times S^2))$ is generated by $\mathrm{PD}(D)$. Thus $\mathfrak{v}_k|_{X''} = \mathfrak{u}_k$, and the claim follows. □

Returning to the topology of W, we further calculate that $b_1(W) = b_2^+(W) = 0$. Now if W were a cobordism between two rational homology spheres, the fact that $b_2^+(W) = 0$ would imply that the induced map $F^\infty_{W,\mathfrak{s}}$ is an isomorphism for any Spin^c structure $\mathfrak{s}$ on W. In our case, B_{-n} is not a homology sphere, so the situation is somewhat more complicated. Nevertheless, it is still true that $F^\infty_{W,\mathfrak{t}'_k}$ is an injection:

Lemma 3.2 *Suppose W is a cobordism from Y_1 to Y_2 and that*

$$b_1(Y_1) = b_1(W) = b_2^+(W) = 0.$$

Let $\mathfrak{s}$ be a Spin^c *structure on W whose restriction $\mathfrak{s}_i$ to Y_i is torsion, and suppose moreover that $HF^\infty(Y_2, \mathfrak{s}_2)$ is "standard," in the sense that its rank as a $\mathbf{Z}[u, u^{-1}]$ module is $2^{b_1(Y_2)}$. Then*

$$F^\infty_{W,\mathfrak{s}} : HF^\infty(Y_1, \mathfrak{s}_1) \to HF^\infty(Y_2, \mathfrak{s}_2)$$

maps $HF^\infty(Y_1, \mathfrak{s}_1)$ isomorphically onto $A_{\mathfrak{s}_2} \subset HF^\infty(Y_2, \mathfrak{s}_2)$, where

$$A_{\mathfrak{s}_2} = \{x \in HF^\infty(Y_2, \mathfrak{s}_2) \,|\, \gamma \cdot x = 0 \;\; \forall\, \gamma \in H_1(Y_2)/\mathrm{Tors}\}.$$

Remark If we wish to avoid the use of twisted coefficients, the condition that $HF^\infty(Y_2, \mathfrak{s}_2)$ be standard is clearly necessary. For example, let W be the cobordism from S^3 to T^3 obtained by removing a ball and a neighborhood of a regular fiber from the rational elliptic surface $E(1)$. Then W satisfies the homological conditions of the lemma, but $F^\infty_{W,\mathfrak{s}}$ is the zero map.

Proof This is essentially contained in the proof of Theorem 9.1 in [10]. The argument may be summarized as follows. The cobordism W can be broken into a composition of three cobordisms W_i $(1 \leq i \leq 3)$, each corresponding to the addition of handles of index i. The hypothesis that $b_1(W) = b_2^+(W) = 0$ implies that W_2 can be further decomposed into a composition of cobordisms W_2^-, W_2^0 and W_2^+, where each two-handle addition in W_2^- decreases b_1 of the terminal end by 1, each two-handle addition in W_2^0 does not change b_1 of the terminal end, and each two-handle in W_2^+ increases b_1 of the terminal end by 1.

We further subdivide W_2^+ into a sequence of elementary cobordisms, each corresponding to the addition of a two-handle. Let Y^i be the terminal end of the i-th such cobordism, and let Y^0 be the initial end of the first one. Then the hypothesis that $b_1(W) = 0$ implies that Y^0 is a rational homology sphere, so $b_1(Y^i) = i$. Likewise, the fact that $b_2^+(W) = 0$ implies that the restriction of $\mathfrak{s}$ to Y^i (which we continue to denote by $\mathfrak{s}$) must be torsion. Finally, the exact triangle shows that

$$\text{rank } HF^\infty(Y^{i+1}, \mathfrak{s}) \leq 2 \text{ rank } HF^\infty(Y^i, \mathfrak{s}).$$

It follows that if $HF^\infty(Y^i, \mathfrak{s})$ is standard, then $HF^\infty(Y^j, \mathfrak{s})$ is standard for all $j \leq i$. Since W_3 is composed entirely of three-handles, the terminal end of W_2^+ is homeomorphic to $Y_2 \#^n (S^1 \times S^2)$. Now $HF^\infty(Y_2, \mathfrak{s}_2)$ is standard by hypothesis, and this implies that $HF^\infty(Y_2 \#^n (S^1 \times S^2), \mathfrak{s})$ is standard as well. Thus $HF^\infty(Y^i, \mathfrak{s})$ is standard for all i.

One can now check directly, using Proposition 9.3 of [10], that $F^\infty_{W_1 \cup W_2^-, \mathfrak{s}}$ and $F^\infty_{W_2^0, \mathfrak{s}}$ are isomorphisms, that $F^\infty_{W_2^+, \mathfrak{s}}$ is injective and maps onto $A_{\mathfrak{s}}$, and that $F^\infty_{W_3, \mathfrak{s}}$ preserves this property. □

In order to apply the lemma, we must check that $HF^\infty(B_{-n}, \mathfrak{t}'_k)$ is standard. From the exact triangle for the knot $B \subset \#^{2g}(S^1 \times S^2)$, we see that

$$HF^\infty(B_{-n}, \mathfrak{t}'_k) \cong HF^\infty(\#^{2g}(S^1 \times S^2), \mathfrak{t})$$

where $\mathfrak{t}$ is the unique torsion Spin^c structure on $\#^{2g}(S^1 \times S^2)$. The latter group is standard, so $HF^\infty(B_{-n}, \mathfrak{t}'_k)$ must be standard as well.

Let $A_{\mathfrak{t}'_k} \subset HF^\infty(B_{-n}, \mathfrak{t}'_k)$ be as in the lemma. Then $A_{\mathfrak{t}'_k} \otimes \mathbf{Q} \cong \mathbf{Q}[u, u^{-1}]$, and its image under the map $\pi\colon HF^\infty(B_{-n}, \mathfrak{t}'_k) \otimes \mathbf{Q} \to HF^+(B_{-n}, \mathfrak{t}'_k) \otimes \mathbf{Q}$ will be isomorphic to $\mathbf{Q}[u^{-1}]$. In analogy with the d–invariant for rational homology spheres, we define $d(B_{-n}, \mathfrak{t}'_k)$ to be the absolute grading of $1 \in \pi(A_{\mathfrak{t}'_k}) \otimes \mathbf{Q} \cong \mathbf{Q}[u^{-1}]$.

Lemma 3.3 $d(K_{-n}, \mathfrak{t}'_k) \leq d(B_{-n}, \mathfrak{t}'_k) + g$.

Proof Consider the map $F^+_{W,\mathfrak{t}'_k}: HF^+(K_{-n}, \mathfrak{t}'_k) \to HF^+(B_{-n}, \mathfrak{t}'_k)$. This map is u–equivariant and agrees with $F^\infty_{W,\mathfrak{t}'_k}$ in high degrees, which implies that it takes $\pi(HF^\infty(K_{-n}, \mathfrak{t}'_k))$ onto $\pi(A_{\mathfrak{t}'_k})$. Thus if $1 \in \pi(HF^\infty(K_{-n}, \mathfrak{t}'_k)) \otimes \mathbf{Q} \cong \mathbf{Q}[u^{-1}]$ is the element with the lowest absolute grading, we must have

$$\mathrm{gr}(F^+_{W,\mathfrak{t}'_k}(1)) \leq d(B_{-n}, \mathfrak{t}'_k).$$

But $\mathrm{gr}(1) = d(K_{-n}, \mathfrak{t}'_k)$ and $F^+_{W,\mathfrak{t}'_k}$ shifts the absolute grading by

$$\frac{c_1^2(\mathfrak{t}'_k) - 2\chi(W) - 3\sigma(W)}{4} = \frac{0 - 2 \cdot 2g - 3 \cdot 0}{4} = -g. \qquad \square$$

Since $d(K_{-n}, \mathfrak{t}'_k) = E(n,k) + 2h_k$, the proof of Theorem 2.3 reduces to the following computation:

Proposition 3.4 *For $n \gg 0$, $d(B_{-n}, \mathfrak{t}'_k) = E(n,k) - g + 2\left\lceil \frac{g - |k|}{2} \right\rceil$.*

Proof The Floer homology of B_{-n} was computed by Ozsváth and Szabó in section 9 of [11]. More specifically, they show that the knot Floer homology of the Borromean knot $B \subset \#^{2g}(S^1 \times S^2)$ is given by

$$\widehat{HFK}(B, i) \cong \Lambda^{g+i}(H^1(\Sigma_g)).$$

This complex is *perfect*, in the sense that the homological grading is equal to the Alexander grading, and there are no differentials, even in the larger complex $HFK^\infty(B) \cong \widehat{HFK}(B) \otimes \mathbf{Z}[u, u^{-1}]$. Ozsváth and Szabó also compute the action of

$$H_1(\#^{2g}(S^1 \times S^2)) \cong H_1(B_{-n})/\mathrm{Tors} \cong H_1(\Sigma_g)$$

on $HFK^\infty(B)$; it is given by

$$\gamma \cdot (\omega \otimes u^n) = \iota_\gamma \omega \otimes u^n + \mathrm{PD}(\gamma) \wedge \omega \otimes u^{n+1},$$

where $\mathrm{PD}(\gamma)$ denotes the Poincare dual of γ viewed as an element of $H_1(\Sigma_g)$.

Let $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be a symplectic basis of $H^1(\Sigma_g)$. We can write any $\omega \in HFK^\infty(B)$ in the form $\omega = \omega_1 + b_1 \wedge \omega_2$, where b_1 does not appear in the expressions for ω_1 and ω_2. Then

$$\mathrm{PD}(a_1) \cdot (\omega_1 + b_1 \wedge \omega_2) = \omega_2 + a_1 \wedge \omega_1 \otimes u + a_1 \wedge b_1 \wedge \omega_2 \otimes u.$$

For this expression to vanish, we must have $\omega_2 = -a_1 \wedge \omega_1 \otimes u$. Thus $\omega = (1+a_1\wedge b_1\otimes u)\omega_1$, where b_1 does not appear in the expression for ω_1. Applying the same argument to the action of the other generators of $H_1(\Sigma_g)$, we find that

$$\Omega = \prod_{i=1}^{g}(1 + a_i \wedge b_i \otimes u)$$

generates $A = \{x \in HFK^\infty(B) \,|\, \gamma \cdot x = 0 \quad \forall\ \gamma \in H_1(\Sigma_g)\}$ as a $\mathbf{Z}[u, u^{-1}]$ module. For future reference we note that the Alexander grading of Ω has the same parity as g.

When $n \gg 0$, the knot Floer homology tell us that $HF^+(B_{-n}, \mathfrak{t}'_k) \cong H_*(C_k)$, where C_k is the quotient complex of $HFK^\infty(B)$ spanned by

$$\{\omega \otimes u^n \,|\, \omega \in \Lambda^{g+i}(H^1(\Sigma)),\, n \geq \max\{k-i, 0\}\}.$$

Moreover, this isomorphism respects the H_1 action. There are no differentials in this complex, so $H_*(C_k) \cong C_k$.

Let $\pi(A)$ be the image of A in C_k. We claim that for $k \geq -g$, the minimum Alexander grading of a nonzero element of $\pi(A)$ is

$$m_k = -g + 2\left\lceil \frac{g+k}{2} \right\rceil.$$

Indeed, it is not difficult to see that m_k is the minimum Alexander grading of *any* element in C_k with the same parity as g, and that this grading is realized by any element in $\Lambda^{g+m_k}(H^1(\Sigma_g))$. The expansion of Ω certainly contains terms of the form $\omega \otimes u^n$, $\omega \in \Lambda^{g+m_k}(H^1(\Sigma_g))$, so the element of A with Alexander grading m_k has a nontrivial image in C_k. This proves the claim.

Since B is perfect, the absolute grading on $HF^+(B_{-n}, \mathfrak{s}_k)$ coincides with the Alexander grading on C_k up to an overall shift. We claim that for $k \leq 0$, this shift is $E(n,k)$. Indeed, for $k \leq 0$, the map

$$F^+_{X_2,\mathfrak{t}_k}\colon\ HF^+(\#^{2g}(S^1 \times S^2), \mathfrak{t}) \to HF^+(B_{-n}, \mathfrak{t}'_k)$$

is induced by the quotient map $HFK^+(B) \to C_k$. Now the Alexander grading on $HFK^+(B)$ is equal to the absolute grading on $HF^+(\#^{2g}(S^1 \times S^2))$, and $F^+_{X_2,\mathfrak{t}_k}$ shifts the absolute grading by $E(n,k)$. This proves the claim, and thus the proposition, when $k \leq 0$. Finally, when $k > 0$, the result follows from the conjugation symmetry of HF^+. ◻

4 Invariants of lens spaces

In this section, we establish the various properties of the Casson–Walker and d–invariants of lens spaces which were used in section 2. For the most part, the proofs involve little more than elementary arithmetic. Our starting point is the recursive formula for the d–invariants of a lens space. In [10] Ozsváth and Szabó introduce natural maps from $\mathbf{Z}$ to the set of Spin^c structures on any $L(p,q)$, which send an integer i to a Spin^c structure $\mathfrak{s}_i$. These maps have the property that $\mathfrak{s}_i = \mathfrak{s}_j$ whenever $i \equiv j\ (p)$. With this labeling, they prove

Proposition 4.1 (Proposition 4.8 of [10]) *Suppose $p > q > 0$ are relatively prime integers, and $0 \leq i < p+q$. Then*

$$d(L(p,q),\mathfrak{s}_i) = \frac{1}{4} - \frac{(2i+1-p-q)^2}{4pq} - d(L(q,p),\mathfrak{s}_i).$$

(Note that our orientation convention for lens spaces is the opposite of the one in [10].)

Together with the fact that $d(L(1,1),\mathfrak{s}_0) = d(S^3) = 0$, this relation clearly determines $d(L(p,q),\mathfrak{s}_i)$ for any values of p, q, and i such that p and q are relatively prime. For the remainder of this section, we adopt the shorthand notation $d(p,q,i)$ to stand for $d(L(p,q),\mathfrak{s}_i)$.

The Casson–Walker invariant also satisfies a recursive formula. To be specific, $\lambda(L(p,q))$ is given by a classical arithmetic function, known as a *Dedekind sum* [17], and it is well known that this function satisfies a recursion relation [14]. For our purposes, this relation can be stated as follows:

Proposition 4.2 *Suppose $p > q > 0$ are relatively prime. Then*

$$\lambda(L(p,q)) = \frac{1}{4} - \frac{p^2+q^2+1}{12pq} - \lambda(L(q,p)).$$

Again, it is clear that this formula, together with the condition $\lambda(L(1,1)) = \lambda(S^3) = 0$ is sufficient to determine $\lambda(L(p,q))$ for all relatively prime p and q. As with d, we adopt the shorthand notation $\lambda(p,q)$ for $\lambda(L(p,q))$.

4.1 Preliminaries

Our first order of business is to show that d and λ are related:

Proof of Lemma 2.2 Let

$$\tilde{\lambda}(p,q) = \frac{1}{p}\sum_{i=0}^{p-1} d(p,q,i).$$

We will show that $\tilde{\lambda}$ satisfies the same recursion relation as λ. We write

$$\tilde{\lambda}(p,q) = \frac{1}{pq}\sum_{j=0}^{q-1}\sum_{i=0}^{p-1} d(p,q,i+j).$$

Applying the recursion formula and switching the order of summation, we get

$$\begin{aligned}\tilde{\lambda}(p,q) &= \frac{1}{pq}\sum_{j=0}^{q-1}\sum_{i=0}^{p-1}\left[\frac{1}{4} - \frac{(2(i+j)+1-p-q)^2}{4pq}\right] - \frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} d(q,p,i+j)\\ &= \frac{1}{4} - \frac{1}{pq}\sum_{j=0}^{q-1}\sum_{i=0}^{p-1}\frac{(2(i+j)+1-p-q)^2}{4pq} - \tilde{\lambda}(q,p).\end{aligned}$$

Using standard identities (or simply asking Mathematica) one finds that

$$\sum_{j=0}^{q-1}\sum_{i=0}^{p-1}\frac{(2(i+j)+1-p-q)^2}{4pq} = \frac{p^2+q^2+1}{12}.$$

This proves the claim. □

To estimate the size of $\lambda(p,q)$, we will express it using continued fractions. To be precise, we consider the Hirzebruch–Jung continued fraction expansion

$$p/q = [a_1, a_2, \ldots, a_n] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ldots - \cfrac{1}{a_n}}} \qquad (a_i \geq 2)$$

common in the theory of lens spaces. The a_i may be found recursively using the division algorithm:

$$p_i/q_i = a_i - q_{i+1}/p_{i+1}$$

where $p/q = p_1/q_1$, $p_{i+1} = q_i$ and $0 < q_{i+1} < p_{i+1}$. Then we have

Lemma 4.3

$$\lambda(p,q) = -\frac{1}{12}\Big(\frac{q}{p} + \frac{q'}{p} + \sum_{i=1}^{n}(a_i - 3)\Big)$$

where $1 \leq q' < p$ *and* $qq' \equiv 1\,(p)$.

The existance of such formulae is well known (see for example [1], [17], [6].) For the reader's convenience, we sketch an elementary proof here.

Proof We induct on the length n of the continued fraction expansion. If $n = 1$, then $q = 1$ and $p = a_1$, and we have

$$\begin{aligned}\lambda(p,1) &= \frac{1}{4} - \frac{p^2+2}{12p} \\ &= -\frac{1}{12}\Big(\frac{1}{p} + \frac{1}{p} + p - 3\Big)\end{aligned}$$

which agrees with the stated form.

In general, we note that $q_2 \equiv -p_1\,(q_1)$ and that

$$\lambda(p,-q) = \lambda(L(p,-q)) = \lambda(L(p,q)^*) = -\lambda(p,q)$$

so the recursion relation becomes

$$\begin{aligned}\lambda(p_1,q_1) &= \frac{1}{4} - \frac{p_1^2+q_1^2+1}{12p_1q_1} - \lambda(q_1,p_1) \\ &= -\frac{1}{12}\Big(\frac{p_1}{q_1} + \frac{q_1}{p_1} + \frac{1}{p_1q_1} - 3\Big) + \lambda(p_2,q_2).\end{aligned}$$

Applying the induction hypothesis to $\lambda(p_2,q_2)$, we get

$$\lambda(p_1,q_1) = -\frac{1}{12}\Big(\frac{p_1}{q_1} + \frac{q_1}{p_1} + \frac{1}{p_1q_1} - 3 + \frac{q_2}{p_2} + \frac{q_2'}{p_2} + \sum_{i=2}^{n}(a_i-3)\Big).$$

Now by definition

$$\frac{p_1}{q_1} + \frac{q_2}{p_2} = a_1,$$

and it is not difficult to see that

$$\frac{1}{p_1q_1} + \frac{q_2'}{p_2} = \frac{q_1'}{p_1}.$$

Substituting these relations into the equation above, we obtain the desired formula. □

4.2 Proof of Proposition 2.4

Suppose that

$$\lambda(p,q) \leq \frac{1}{4}(\frac{p}{4} - 1) + \lambda(p,1).$$

Substituting the formula of Lemma 4.3 and simplifying, we find that

$$\frac{q}{p} + \frac{q'}{p} + \sum_{i=1}^{n}(a_i - 3) \geq \frac{p}{4} + \frac{2}{p}.$$

Since the two fractions on the left-hand side are both < 1, this implies that

$$\sum_{i=1}^{n}(a_i - 3) > \frac{p}{4} - 2.$$

Thus for $p > 100$, we see that

$$S = \sum_{a_i > 3}(a_i - 3) \geq \sum_{i=1}^{n}(a_i - 3) > \frac{2p}{9}.$$

We investigate the conditions which this inequality puts on the continued fraction expansion. Our first step is to estimate the size of p in terms of the a_i.

Lemma 4.4 $p_i \geq p_{i+1}(a_i - 1)$.

Proof We have

$$p_i = p_{i+1}(a_i - \frac{q_{i+1}}{p_{i+1}}) \geq p_{i+1}(a_i - 1). \qquad \square$$

Corollary 4.5 $\prod_{i=1}^{n}(a_i - 1) \leq p$. *Moreover, the inequality still holds if one (but not both) of the factors* $a_1 - 1$ *and* $a_n - 1$ *is replaced by* a_1 *or* a_n*, respectively.*

Proof An obvious induction. The case where $a_1 - 1$ is replaced by a_1 follows from the fact that the continued fractions $[a_1, a_2, \ldots, a_n]$ and $[a_n, a_{n-1}, \ldots, a_1]$ have the same numerator. □

Lemma 4.6 *If* $S > 2p/9$*, then at most two* a_i *are greater than* 3.

Proof Suppose more than one of the a_i is > 3. If there are m such terms, it is clear that they must all be less than $p/3^{m-1}$. Then

$$\sum_{i=1}^{n}(a_i - 3) \leq m \cdot \frac{p}{3^{m-1}} \leq \frac{4p}{27}$$

if $m \geq 4$. Now supposing that $m = 3$, we try to maximize $a_1 + a_2 + a_3 - 9$ subject to the constraints $(a_1 - 1)(a_2 - 1)(a_3 - 1) \leq p$, $a_i \geq 4$. It is not difficult to see that the maximum is $\frac{p}{9} - 1$ (attained when two of the three are equal to 4), so this case is ruled out as well. □

Lemma 4.7 *Let $A = \{a_i \,|\, a_i > 2\}$, and let x be the largest of the a_i. If $S > 2p/9$ then A is equal to one of $\{x\}$, $\{x, 3\}$, or $\{x, 4\}$.*

Proof Clearly $x > 3$, or $S = 0$. Suppose that two of the a_i, say x and y, are > 3. If $y > 5$, then the same sort of maximization argument used in the previous lemma shows that $S \leq p/5$. If $y = 4$ or 5, and one of the other $a_i = 3$ in addition, then $(x-1)(y-1) \leq p/2$, and it follows that $S \leq p/6$. Finally, suppose that x is the only value of $a_i > 3$. Then if three or more of the other a_i equal 3, we have $S \leq p/8$. It follows that A must be one of $\{x\}$, $\{x, 3\}$, $\{x, 4\}$, $\{x, 5\}$, or $\{x, 3, 3\}$.

To elimate the last two possibilities, we use the sharper version of Corollary 4.5. For example, if $A = \{x, 3, 3\}$, then one of a_1 or a_n is equal to 2 or 3. If it is 2, we must have $x \leq p/8$, whence $S \leq p/8$ as well. If it is 3, we get that $S \leq p/6$. A similar argument takes care of the case $A = \{x, 5\}$. □

In all remaining cases, we have $S < x$. To analyze these cases, suppose $x = a_k$, and call the continued fractions $[a_1, a_2, \ldots, a_{k-1}]$ and $[a_{k+1}, a_{k+2}, \ldots, a_n]$ the *head* and *tail*, respectively.

Lemma 4.8 *If $x > 2p/9$, the numerator of the head and tail must both be less than 5.*

Proof In the case of the tail, this follows immediately from Lemma 4.4. To get the same result for the head, use the fact that a continued fraction and its inverse have the same numerator. □

Thus there are only six possibilities for the head and tail: $[\ ], [2], [3], [2, 2], [4]$, and $[2, 2, 2]$ (corresponding to the fractions $1/1, 2/1, 3/1, 3/2, 4/1$, and $4/3$, respectively.) Since a continued fraction and its inverse correspond to the same lens space, we need only consider one element of each such pair. Thus there are 21 possible head–tail combinations. It is not difficult to check that 14 of these 21 have $S \leq p/6$. The remaining 7 possiblities are listed in table 1, which shows the continued fraction expansion, the associated fraction p/q, and the difference $\Delta = 12(\lambda(p, q) - \lambda(p, 1))$.

The only expansions which have $\Delta \leq 3(\frac{p}{4} - 1)$ are those corresponding to $L(p, 1)$, $L(p, 2)$, and $L(p, 3)$. It follows that the proposition holds for all values of $p > 100$. Using a computer, it is elementary to check that it holds for all values of $p \leq 100$ as well. This concludes the proof of the proposition. □

$[a_1, a_2, \ldots, a_n]$	p	q	Δ
$[x]$	x	1	0
$[x, 2]$	$2x-1$	2	$x - x/p$
$[x, 3]$	$3x-1$	3	$2x-1-(x+1)/p$
$[x, 4]$	$4x-1$	4	$3x-2-(x+2)/p$
$[x, 2, 2]$	$3x-2$	3	$2x-2x/p$
$[x, 2, 2, 2]$	$4x-3$	4	$3x-3x/p$
$[2, x, 2]$	$4x-4$	$2x-1$	$3x$

Table 1

4.3 Proof of Proposition 2.5

To rule out the exceptional cases $L(p,2)$ and $L(p,3)$, we return to considering the d–invariants. Suppose that $L(p,2)$ is K_{-p} for some knot K. Then

$$\{d(p,2,i)\,|\,i \in \mathbf{Z}/p\} = \{d(p,1,i) + 2n_i\,|\,i \in \mathbf{Z}/p\}$$

where the n_i are non-negative integers. Since the continued fraction expansions of $p/1$ and $p/2$ are short, it is easy to use the formula of Proposition 4.1 to work out $d(p,1,i)$ and $d(p,2,i)$. We get

$$d(p,1,i) = \frac{1}{4}\Big(1 - \frac{(2i-p)^2}{p}\Big)$$

and

$$d(p,2,i) = \begin{cases} \frac{1}{4}\Big(2 - \frac{(2i-p-1)^2}{2p}\Big) & \text{if } i \text{ is even,} \\ \frac{1}{4}\Big(-\frac{(2i-p-1)^2}{2p}\Big) & \text{if } i \text{ is odd.} \end{cases}$$

We consider the largest values attained by $L(p,1,i)$. Since we are working with $L(p,2)$, p is odd, and the largest possible value of $L(p,1,i)$ is $(1-\frac{1}{p})/4$. By hypothesis, this is equal to $d(p,2,i) - 2n_i$ for some value of i. The only way this can happen is if i is even and $n_i = 0$. In this case, we get

$$1 - \frac{1}{p} = 2 - \frac{(2i-p-1)^2}{2p}.$$

After some simplification, this becomes

$$(2i-p-1)^2 = 2p + 2$$

so $2p+2$ is a perfect square. We now apply the same argument to the second largest value of $d(p,1,i)$, which is $(1-\frac{9}{p})/4$. Assuming $p > 9$, we again see that this must be equal to $d(p,2,i')$, where i' is even. Thus we have

$$1-\frac{9}{p}=2-\frac{(2i'-p-1)^2}{2p}$$

which reduces to

$$(2i'-p-1)^2=2p+18.$$

So $2p+2$ and $2p+18$ are even perfect squares differing by 16. But this is impossible, since the only pair of squares with this property is 0 and 16. Thus we need only consider those values of p which are ≤ 9. Consulting the list at the end of [10], we see that $L(7,2)=L(7,4)$ is the only case in which $L(p,2)$ can be realized as $-p$ surgery on a knot.

The proof for $L(p,3)$ is similar in spirit, but involves a larger number of cases. We have

$$d(L(p,3,i))=\frac{1}{4}\Big(1-\frac{(2i-p-2)^2}{3p}\Big)-d(3,p,i).$$

Suppose $p\equiv 1\,(6)$. Then $d(3,p,i)=-1/2$ if $i\equiv 0\,(3)$ and $1/6$ otherwise. We need

$$\frac{1}{4}(1-\frac{1}{p})=d(3,p,i)+2n_i \qquad \text{and} \qquad \frac{1}{4}(1-\frac{9}{p})=d(p,3,i')+2n_{i'}.$$

If $p>14$, this can only occur if i and i' are divisible by 3 and $n_i=n_i'=0$. In this case, we find

$$(2i-p-2)^2=6p+3 \qquad\qquad (2i'-p-2)^2=6p+27$$

so we are looking for a pair of perfect squares which differ by 24. Again, it is easy to see that there is no such pair with the right values $\bmod 6$. Among the possible values of $p\equiv 1\,(6)$ less than 14, the table in [10] shows that $L(13,3)\cong L(13,9)$ can be a lens space surgery, while $L(7,3)$ cannot.

If $p\equiv 5\,(6)$, a similar analysis leads to the equations

$$(2i-p-2)^2=2p+3 \qquad\qquad (2i'-p-2)^2=2p+27$$

which actually has a solution when $p=11$. Finally, the remaining cases $p\equiv 2,4\,(6)$ do not admit any solutions. $\square$

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