ISSN 1364-0380 (on line) 1465-3060 (printed) **1953**

Geometry & Topology
Volume 9 (2005) 1953–1976
Published: 17 October 2005

On finite subgroups of groups of type VF

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Abstract

For any finite group Q not of prime power order, we construct a group G that is virtually of type F, contains infinitely many conjugacy classes of subgroups isomorphic to Q, and contains only finitely many conjugacy classes of other finite subgroups.

AMS Classification numbers Primary: 20F65

Secondary: 19A31, 20E45, 20J05, 57M07

Keywords: Conjugacy classes, finite subgroups, groups of type F

Proposed: Martin Bridson | Received: 11 June 2005
Seconded: Wolfgang Lück, Haynes Miller | Accepted: 10 October 2005

1 Introduction

A group H is said to be of type F if there is a finite classifying space for H, ie, if there exists a finite simplicial complex whose fundamental group is isomorphic to H and whose universal cover is contractible. A group of type F is necessarily torsion-free. It is easily seen that any finite-index subgroup of a group of type F is also of type F.

A group G is said to be of type VF if G contains a finite-index subgroup H which is of type F, ie, if G is virtually of type F. If H has index n in G, then the kernel of the action of G on the cosets of H has index at most $n!$. Hence any group of type VF contains a finite-index normal subgroup of type F, and so for any group G of type VF there is a bound on the orders of finite subgroups of G.

K S Brown's book 'Cohomology of Groups' contains a result that implies that a group of type VF can contain only finitely many conjugacy classes of subgroups of prime power order [4, IX.13.2]. The question of whether a group of type VF could ever contain infinitely many conjugacy classes of finite subgroups was posed in [11, 8], and remained unanswered until Brita Nucinkis and the author constructed examples in [7]. These examples may be summarized as follows:

Theorem 1 *Let Q be a finite group admitting a simplicial action on a finite contractible simplicial complex L such that the fixed point set L^Q is empty. Then there is a group H_L of type F (depending only on L) and an action of Q on H_L such that the semi-direct product $H_L{:}Q$ contains infinitely many conjugacy classes of subgroup isomorphic to Q.*

R Oliver has shown that a finite group Q admits an action on a finite contractible L without a global fixed point if and only if Q is not expressible as p–group-by-cyclic-by-q–group for any primes p and q [9]. (Oliver's main result is the construction of actions: the proof that actions do not exist in the other cases is far simpler and we include it in Section 3.)

The purpose of this paper is to close the gap between Brown's result and the construction of Theorem 1. For any finite group Q that is not of prime power order, we construct a group H of type F with an action of Q so that the semi-direct product $H{:}Q$ contains infinitely many conjugacy classes of subgroup isomorphic to Q, and finitely many conjugacy classes of other finite subgroups. As a corollary we obtain the following apparently stronger result.

Theorem 2 *Let $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$ be a finite list of isomorphism types of finite group, such that no Q_i is a group of prime power order. There exists a group $G = G(\mathcal{Q})$ of type VF such that G contains infinitely many conjugacy classes of subgroup isomorphic to a finite group Q if and only if $Q \in \mathcal{Q}$.*

In particular, it follows that a group of type VF may contain infinitely many conjugacy classes of *elements* of finite order, although any such group can only contain finitely many conjugacy classes of elements of prime power order.

Our techniques also apply to other weaker finiteness conditions. Recall that a group G is of type FP over a ring R if the trivial module R for the group ring RG admits a finite resolution by finitely generated projective RG–modules, ie, if and only if there is an integer n and an exact sequence of RG–modules

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to R \to 0$$

in which each P_i is a finitely generated projective. If there exists such a sequence in which each P_i is a finitely generated free module, then G is said to be of type FL over R.

In [7] Brita Nucinkis and the author proved the following.

Theorem 3 *Let Q be a finite group admitting a simplicial action on a finite $\mathbb{Q}$–acyclic simplicial complex L such that the fixed point set L^Q is empty. Then there is a virtually torsion-free group $G = H_L{:}Q$ of type FP over $\mathbb{Q}$ containing infinitely many conjugacy classes of subgroup isomorphic to Q.*

R Oliver has shown that a finite group Q admits such an action if and only if Q is not of the form cyclic-by-p–group for some prime p [9]. In particular, the above construction did not give rise to any groups of type FP over $\mathbb{Q}$ containing infinitely many conjugacy classes of *elements* of finite order. The question of whether such groups can exist was posed by H Bass in [1, 11]. One reason why this question is of interest is that if G contains infinitely many conjugacy classes of elements of finite order, then the Grothendieck group $K_0(\mathbb{Q}G)$ may be shown to have infinite rank. (We give a proof of this fact below in Theorem 22.)

Any group of type F is of type FP over any ring R, and a group G of type VF is of type FP over any ring R in which the orders of all finite subgroups of G are units. In particular, every group of type VF is of type FP over $\mathbb{Q}$. It follows that examples coming from Theorem 2 may be used to answer Bass's question. By Brown's result, groups of type VF necessarily contain only finitely many conjugacy classes of elements of prime power order. This is not the case for groups of type FP over $\mathbb{Q}$, and in fact for any non-trivial finite group Q we construct a group of type FP over $\mathbb{Q}$ containing infinitely many conjugacy classes of subgroup isomorphic to Q, and finitely many conjugacy classes of other finite subgroup. The following is a corollary of our result.

Theorem 4 *Let $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$ be a finite list of isomorphism types of non-trivial finite groups. There exists a virtually torsion-free group $G = G(\mathcal{Q})$ of type FP over $\mathbb{Q}$ such that G contains infinitely many conjugacy classes of subgroup isomorphic to a finite group Q if and only if $Q \in \mathcal{Q}$.*

The groups H_L appearing in the statements of Theorems 1 and 3 are the groups introduced by M Bestvina and N Brady, who used them to solve a number of open problems concerning homological finiteness conditions [2]. In particular, in the case when L is a finite acyclic complex that is not contractible, they showed that the group H_L is of type FL over $\mathbb{Z}$ but is not finitely presented. The main idea in [7] was to allow a finite group Q to act on the complex L, and hence on the group H_L.

The main idea in this paper is to consider Bestvina–Brady groups H_L for *infinite* complexes L. If Q is any finite group not of prime power order, then there exists a complex L with a $\mathbb{Z} \times Q$–action such that

(1) L is contractible;

(2) $\mathbb{Z} \times Q$ acts cocompactly on L;

(3) all cell stabilizers are finite;

(4) $\{0\} \times Q$ fixes no point of L.

The first three properties together imply that the semi-direct product $H_L{:}\mathbb{Z}$ is of type F, and the fourth property implies that the semi-direct product $H_L{:}(\mathbb{Z} \times Q)$ contains infinitely many conjugacy classes of subgroup isomorphic to Q. A construction for L as above in the case when Q is cyclic was given by Conner and Floyd [5]. In Section 3 we give a construction for arbitrary finite Q which was shown to us by Bob Oliver.

A similar (but simpler) construction involving an infinite $\mathbb{Q}$–acyclic complex L is used in proving our theorem concerning groups of type FP over $\mathbb{Q}$.

In the final section of the paper we discuss some further finiteness properties of the groups that we construct. We show that the groups are residually finite, although we are unable to decide whether they are linear. We also show that each of the groups used in the proofs of Theorems 2 and 4 occurs as the kernel of a map to $\mathbb{Z}$ from a group that acts cocompactly with finite stabilizers on a CAT(0) cube complex.

The work in this paper builds on the author's joint work with Brita Nucinkis and uses theorems concerning actions of finite groups which the author learned from Bob Oliver. The author gratefully acknowledges their contributions to this work. Some of this work was done at Paris 13 and at the ETH, Zürich. The author thanks these institutions for their hospitality.

The author was partially supported by NSF grant DMS-0505471.

2 Bestvina–Brady groups

In this section we define the Bestvina–Brady group H_L associated to a flag complex L, and we check that some of the results in [2, 7] extend to the case when L is an infinite flag complex.

A flag complex, L, is a simplicial complex which contains as many higher dimensional simplices as possible, given its 1–skeleton. In other words, whenever the complete graph on a finite subset of the vertex set of L is contained in the 1–skeleton of L, then there is a simplex of L with that set of vertices. The realisation of any poset is a flag complex (since a subset is totally ordered if any two of its members are comparable). In particular, the barycentric subdivision of any simplicial complex is a flag complex.

Given a flag complex L, the associated right-angled Artin group G_L is the group with generators the vertices of L subject only to the relations that the ends of each edge commute. There is a model for the classifying space BG_L with one n–dimensional cubical cell for each $(n-1)$–simplex of L (including one vertex corresponding to the empty simplex in L). Let X_L denote the universal cover of this space. Cells of X_L are n–cubes of the form $(g, v_1, \ldots, v_n)$ where $(v_1, \ldots, v_n)$ is an $n-1$–simplex of L and g is an element of G. The ith pair of opposite faces of this n–cube consists of the cubes $(g, v_1, \ldots, \hat{v}_i, \ldots, v_n)$ and $(gv_i, v_1, \ldots, \hat{v}_i, \ldots, v_n)$, where gv_i is the product of two elements of G_L, and as usual $\hat{v}_i$ means 'omit v_i'. The action of G_L is given by $g'(g, v_1, \ldots, v_n) = (g'g, v_1, \ldots, v_n)$. If $\sigma = (v_1, \ldots, v_n)$ is a simplex of L, we will write (g, σ) in place of $(g, v_1 \ldots, v_n)$. In particular, we will write (g) for a vertex of X_L.

The space X_L admits the structure of a CAT(0) cubical complex: there is a geodesic CAT(0) metric on X_L in which each cubical cell is isometric to a standard Euclidean unit cube, and the action of G_L is by isometries of this metric. In the case when L is infinite, X_L is not locally finite, and the metric topology on X_L is not the same as the CW–topology, but this will not cause any difficulties.

Suppose now that $f\colon L \to L'$ is a simplicial map. Then f defines a group homomorphism $f_*\colon G_L \to G_{L'}$, which takes the generator v to the generator $f(v)$, and f defines a piecewise-linear continuous map $f_!\colon X_L \to X_{L'}$, which takes the vertex (g) to the vertex $(f(g))$, and extends linearly across each cube. The map $f_!$ is G_L–equivariant, where f_* is used to define the G_L–action on $X_{L'}$, and so induces a map from X_L/G_L to $X_{L'}/G_{L'}$, which is an explicit construction for the map $B(f_*)\colon BG_L \to BG_{L'}$. If f is an injective simplicial map, then f_* is an injective group homomorphism and $f_!$ is an isometric embedding.

Two special cases of this construction are of interest to us. Firstly, any group Γ of automorphisms of L gives rise to a group of automorphisms of G_L and to a group of cellular automorphisms of X_L/G_L. Since the unique vertex in X_L/G_L is fixed by Γ, the group of all lifts of elements of Γ to the covering space $X_L \to X_L/G_L$ is the semi-direct product $G_L{:}\Gamma$, where Γ acts on G_L via the action described above.

Secondly, let $*$ denote a 1–point simplicial complex. For this choice of simplicial complex, G_* is infinite cyclic, and X_* is the real line triangulated with one orbit of vertices and one orbit of edges. For any L, there is a unique map $f_L\colon L \to *$, and the Bestvina–Brady group H_L is defined to be the kernel of $f_{L*}\colon G_L \to G_*$. The map $f_{L!}\colon X_L \to X_* \cong \mathbb{R}$ may be considered as defining a 'height function' on X_*. Identifying the integers $\mathbb{Z} \subseteq \mathbb{R}$ with the vertex set in X_*, one sees that $f_{L!}$ sends each vertex of X_L to an integer, and that each cube of X_L has a unique minimal and maximal vertex for this height function. For the cube C, we shall write $\min(C)$ and $\max(C)$ respectively for its minimal and maximal vertices. Any simplicial map $f\colon L \to L'$ fits in to a commutative triangle with $f_L\colon L \to *$ and $f_{L'}\colon L' \to *$, and hence one obtains an induced map $f_*\colon H_L \to H_{L'}$. In particular, if Γ is a group of simplicial automorphisms of L, then the semi-direct product $H_L{:}\Gamma$ is defined and is equal to the kernel of the composite $G_L{:}\Gamma \to G_* \times \Gamma \to G_*$.

The work of Bestvina and Brady [2] relies on a study of the height function $f\colon X_L \to X_* = \mathbb{R}$. We recall part of this, and check that it applies to the case when L is infinite (which was not considered in [2]).

Pick a point c in the interior of an edge of X_*, and define $Y = Y_L = f^{-1}(c) \subseteq X_L$. (The point c will remain fixed for the remainder of this section, but will be suppressed from the notation.) Give Y the structure of a polyhedral CW–complex by taking as cells the sets $C \cap Y$ where C is a cube of X_L. Note that the CW–structure on Y gives rise to the same topology as the subspace topology coming from the CW–topology on X.

Now let C be a cube in X_L whose highest vertex is v_1. Define a subset C_c of C by

$$C_c = C \cap f^{-1}([c,\infty)) = C \cap f^{-1}([c, f(v_1)]).$$

Similarly, if the lowest vertex of C is v_0, define a subset C^c by

$$C^c = C \cap f^{-1}((-\infty, c]) = C \cap f^{-1}([f(v_0), c]).$$

If $C = (g,\sigma)$ for some simplex $\sigma \in L$, then the link of v_1 in C is homeomorphic to σ. It follows that if $f(v_1) > c$, then C_c is homeomorphic to the cone on L with vertex v_1. If $f(v_1) < c$, then C_c is empty. Similarly, if $f(v_0) < c$ then C^c is empty, and otherwise C^c is homeomorphic to the cone on σ. Now for v a vertex of X_L, define $F(v)$ to be either

$$F(v) = \begin{cases} \bigcup_{v=\max(C)} C_c & f(v) > c \\ \bigcup_{v=\min(C)} C^c & f(v) < c \end{cases}$$

For each v, one may show that $F(v)$ is homeomorphic to the cone on L with vertex v. (Here, as usual, we are using the CW–topology on both $F(v)$ and the cone on L.) Now for $a, b \in X_* = \mathbb{R}$ with $a < c < b$, define a subspace $Y_{[a,b]}$ of X_L by

$$Y_{[a,b]} = Y \cup \bigcup_{a \le f(v) \le b} F(v).$$

Each $Y_{[a,b]}$ is a CW–complex, with cells the truncated cubes C_c, C^c and $C \cap Y$ for each cube C of X_L, and if $\alpha \le a < c < b \le \beta$, then $Y_{[a,b]}$ is a subcomplex of $Y_{[\alpha,\beta]}$. As a decreases (resp. b increases) the complex $Y_{[a,b]}$ only changes as a (resp. b) passes through an integer. For each $a < c < b$, one has that $Y_{[a-1,b+1]}$ is homeomorphic to $Y_{[a,b]}$ with a family of subspaces homeomorphic to L coned off. (There is one such cone for each vertex in $Y_{[a-1,b+1]} - Y_{[a,b]}$.) Thus one obtains the following lemma and corollary due to Bestvina–Brady [2], for any simplicial complex L.

Lemma 5 *If L is contractible, then for any $a < c < b$, the inclusion of Y in $Y_{[a,b]}$ is a homotopy equivalence. If L is R–acyclic for some ring R, then for any $a < c < b$, the inclusion of Y in $Y_{[a,b]}$ induces an isomorphism of R–homology.*

Corollary 6 *If L is contractible, then Y is contractible. If L is R–acyclic, then Y is R–acyclic.*

Proof We know that X_L is contractible, and the lemma implies that the inclusion $Y \to X_L$ is a homotopy equivalence if L is contractible and is an R–homology isomorphism if L is R–acyclic. □

Theorem 7 *Suppose that Γ acts freely cocompactly on a simplicial complex L. If L is contractible, then $H_L{:}\Gamma$ is type F. If L is R–acyclic, then $H_L{:}\Gamma$ is type FL over R.*

Proof It follows from Corollary 6 that Y is contractible or R–acyclic whenever L is. Thus it suffices to show that $H_L{:}\Gamma$ acts freely cocompactly on Y. To see this, first note that $G_L{:}\Gamma$ has only finitely many orbits of cells in its action on X_L. If C is an n–cube of X_L with top vertex v, then $C \cap Y$ is non-empty if and only if $c < f(v) < c + n$. It follows that each $G_L{:}\Gamma$–orbit of n–cubes in X_L gives rise to exactly n $H_L{:}\Gamma$–orbits of $(n-1)$–cells in Y. □

It remains to study the conjugacy classes of finite subgroups of groups of the form $H_L{:}\Gamma$ and $G_L{:}\Gamma$. In fact it is no more difficult to study conjugacy classes of subgroups Q' such that $Q' \cap G_L = \{1\}$. Consider first the collection of subgroups Γ' of $G_L{:}\Gamma$ which map isomorphically to $G_L{:}\Gamma/G_L \cong \Gamma$. The action of Γ on X_L/G_L fixes the unique vertex. It follows that each such Γ' fixes some vertex v of X_L. Since the vertices form a single orbit, it follows that all such Γ' are conjugate in $G_L{:}\Gamma$.

Proposition 8 *Let Γ act on L, let $Q \leq \Gamma$, and let Q' be any subgroup of $G_L{:}\Gamma$ that maps isomorphically to $Q \leq \Gamma = G_L{:}\Gamma/G_L$. If $L^Q = \emptyset$, then Q' fixes a unique vertex in X_L. If L^Q contains the barycentre of an m–simplex, and Q' fixes a vertex $(g) \in X_L$ of height $f(g) = a$, then Q' also fixes a vertex of height $a + (m+1)n$ for each integer n.*

Remark Since we are not assuming that the action of Γ on L makes L into a Γ–CW–complex, it is not necessarily the case that L^Q is a subcomplex of L. However there can be a point of L^Q in the interior of the simplex σ only if $q\sigma = \sigma$ for all $q \in Q$. In this case the barycentre of σ is a point fixed by Q.

Proof For the first time, we shall make use of the CAT(0) metric on X_L. Suppose that Q' fixes two distinct vertices $(g), (h)$ of X_L. Since geodesics in a CAT(0) metric space are unique, it follows that the geodesic arc from (g) to (h) is also fixed by Q'. The start of this arc is a straight line passing from (g) into the interior of C, an n–cube of X_L for some $n > 0$, which must be preserved (setwise) by Q'. If $C = (g', v_1, \ldots, v_n)$, then it follows that the $(n-1)$–simplex $(v_1, \ldots, v_n)$ in L is (setwise) preserved by Q, and hence that $L^Q \neq \emptyset$.

For the second statement, suppose that (g) is fixed by Q', and that the m–simplex $(v_0, \ldots, v_m)$ in L is setwise fixed by Q. Then the long diagonal from

(g) to $(gv_0v_1\cdots v_m)$ in the $(m+1)$–cube $(g, v_0, \ldots, v_m)$ is an arc fixed by Q', which connects two vertices whose heights differ by $m+1$. It follows that for any n, the vertex $g(v_0v_1\cdots v_m)^n$ is fixed by Q'. □

Theorem 9 *Let Γ act on L, and let $Q \le \Gamma$. If $L^Q = \emptyset$, then there are infinitely many conjugacy classes of subgroups Q' of $H_L{:}\Gamma$ whose members map isomorphically to conjugates of Q in Γ. If L^Q contains the barycentre of an m–simplex, then there are at most $m+1$ conjugacy classes of such Q' in $H_L{:}\Gamma$. In particular, if L^Q contains a vertex of L, then any two such subgroups are conjugate.*

Proof We know that any such Q' fixes a vertex of X_L and that every vertex is fixed by some such Q'. In the case when $L^Q = \emptyset$, each Q' fixes exactly one vertex of X_L. Since vertices of different heights are in different orbits for the action of $H_L{:}\Gamma$, it follows that in this case there are infinitely many conjugacy classes of such Q'.

In general, H_L acts transitively on the vertices of fixed height. If L^Q contains the barycentre of an m–simplex, and Q' fixes a vertex of height a, then Q' also fixes a vertex of height $a+(m+1)n$ for each n. Hence given any $(m+2)$ subgroups of $H_L{:}\Gamma$ which map isomorphically to Q or one of its conjugates, some pair Q', Q'' of these subgroups must fix vertices of the same height. Let $\Gamma' \ge Q'$ and $\Gamma'' \ge Q''$ be the stabilizers of these vertices, which map isomorphically to Γ. The subgroups Γ' and Γ'' are conjugate by an element of H_L. Hence it follows that Q' and Q'' are conjugate by some element of $H_L{:}\Gamma$. □

3 Group actions

Here we construct the actions of finite groups Q and direct products of the form $\mathbb{Z} \times Q$ on finite-dimensional simplicial complex that are needed in order to apply the constructions of the previous section. The first two propositions are included to show why actions of finite groups alone cannot give all the examples that we need.

Proposition 10 *Suppose that Q is a finite group with normal subgroups $P \le P'$, so that P and Q/P' are groups of prime power order and P'/P is cyclic. For any action of Q on a finite contractible complex L, the fixed point set L^Q is non-empty.*

Proof Let p and q be the primes (not necessarily distinct) so that $|P|$ is a power of p and $|Q:P|$ is a power of q. Let C denote P'/P, and let Q' denote Q/P'.

By P A Smith theory [4, VII.10.5], the fixed point set $L' = L^P$ has the same mod-p homology as a point, and hence has Euler characteristic equal to 1. By character theory, it follows that the Euler characteristic of $L'' = L^{P'} = L'^C$ is equal to 1. By counting lengths of orbits of cells, one sees that $L^Q = L''^{Q'}$ has Euler characteristic congruent to 1 modulo q. This implies that L^Q is not empty. □

The above proof also gives:

Proposition 11 *Let Q be a finite group with a normal cyclic subgroup P' so that Q/P' is a group of prime power order. For any action of Q on a finite complex L' with Euler characteristic $\chi(L') = 1$, the fixed point set L'^Q is non-empty.*

The actions on $\mathbb{Q}$–acyclic spaces that we shall need will all come from Theorem 13. In the proof of we shall need Lemma 12 concerning Wall's finiteness obstruction.

Suppose that G is a group of type FP over the ring R, and that

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to R \to 0$$

is a resolution of R over RG by finitely generated projectives. As usual, let $K_0(RG)$ denote the Grothendieck group of finitely-generated projective RG–modules. The Wall obstruction or Euler characteristic of G over R is the element of $K_0(RG)$ given by the alternating sum

$$w(R,G) = \sum_i (-1)^i [P_i]$$

and is independent of the choice of resolution [10, I.7].

Lemma 12 *Let Q be a finite group. The group $G = \mathbb{Z} \times Q$ is FP over $\mathbb{Q}$, and the Wall obstruction for this group is zero.*

Proof Let the group $G = \mathbb{Z} \times Q$ act on the real line via the projection $G \to \mathbb{Z}$. There is a G–equivariant triangulation of the line with one orbit of 0–cells of type G/Q and one orbit of 1–cells, also of type G/Q. The cellular chain complex for this space gives a projective resolution for $\mathbb{Q}$ over $\mathbb{Q}G$ of length one:

$$0 \to \mathbb{Q}G/Q \to \mathbb{Q}G/Q \to \mathbb{Q} \to 0,$$

in which the modules in degrees 0 and 1 are isomorphic to each other. □

Theorem 13 *Let Q be a finite group, and let $\mathcal{F}$ be a non-empty family of subgroups of Q which is closed under conjugation and inclusion. There is a 3–dimensional $\mathbb{Q}$–acyclic simplicial complex L admitting a cocompact action of $\Gamma = \mathbb{Z} \times Q$ so that all cell stabilizers are finite and so that $P \leq Q$ fixes some point of L if and only if $P \in \mathcal{F}$.*

Proof Let Δ be a finite set with a Q–action, such that $\Delta^P \neq \emptyset$ if and only if $P \in \mathcal{F}$, and let $Z = Q * Q * \Delta$ be the join of two copies of Q and one copy of Δ, with the diagonal action of Q. This Z is a 2–dimensional simply-connected Q–space, with the property that the Q–action is free except on the 0–skeleton. Let $\mathbb{Z}$ act on $\mathbb{R}$ in the usual way, and let L_0 be the product $\mathbb{R} \times Z$ with the product action of $\Gamma = \mathbb{Z} \times Q$. Now let L_1 be the 2–skeleton of L_0. The cells of L_1 in non-free orbits form a copy of $\mathbb{R} \times \Delta$ with the product action of Γ. Let C_* be the rational chain complex for L_1. Since L_1 is 1–connected, C_* forms the start of a projective resolution for $\mathbb{Q}$ over $\mathbb{Q}\Gamma$. As $\mathbb{Q}\Gamma$–modules, C_2 is free and each of C_1 and C_0 is the direct sum of a free module and a copy of $\mathbb{Q}[\mathbb{Z} \times \Delta]$. Hence the element of $K_0(\mathbb{Q}\Gamma)$ defined by the alternating sum $[C_2] - [C_1] + [C_0]$ is in the subgroup of $K_0(\mathbb{Q}\Gamma)$ generated by the free module. Since we know by Lemma 12 that the Wall obstruction for Γ over $\mathbb{Q}$ is zero, it follows that $H_2(C_*)$ is a stably-free $\mathbb{Q}\Gamma$–module. Make L_2 by attaching finitely many free Γ–orbits of 2–spheres to L_1 in such a way that $H_2(L_2; \mathbb{Q})$ is a free $\mathbb{Q}\Gamma$–module. Let $c_1, \ldots, c_k$ be cycles in $C_2(L_2, \mathbb{Q})$ representing a $\mathbb{Q}\Gamma$–basis for $H_2(L_2; \mathbb{Q})$, and pick a large integer M so that each $M.c_i$ is an integral cycle. Since L_2 is 1–connected, each $M.c_i$ is realized by the image of the fundamental class for S^2 under some map $f_i \colon S^2 \to L_2$. Now define L_3 by attaching free Γ–orbits of 3–balls to L_2, using the f_i as attaching maps for orbit representatives. This L_3 has all of the required properties, except that it has been constructed as a Γ–CW–complex rather than as a Γ–simplicial complex. By the simplicial approximation theorem, we can construct a 3–dimensional Γ–simplicial complex L together with an equivariant homotopy equivalence $L \to L_3$. □

Before stating and proving Theorem 15, which will provide all the actions on contractible spaces that we shall need, we begin by establishing some notation, and proving a lemma concerning equivariant self-maps of spheres in linear representations. Lemma 14 and Theorem 15 were shown to the author by Bob Oliver.

Let S denote the unit sphere in $\mathbb{C}^n$, so that S is a sphere of odd dimension. For $x \in S$, let T_xS be the tangent space to S at x, and let B_x be the closed unit ball in T_xS, with boundary ∂B_x. For $\epsilon > 0$, let $e_{\epsilon,x} \colon B_x \to S$ denote the scalar multiple of the exponential map such that the image of B_x is a ball of radius ϵ in S. In the case when $\epsilon = \pi$, this map sends the whole of ∂B_x to the point $-x$. The cases of interest to us include the case $\epsilon = \pi$ and the case when ϵ is small. Suppose that a finite group P acts linearly on S, fixing the point x. This induces a P–action on T_xS, and the exponential map $e_{\epsilon,x}$ is P–equivariant in the sense that $e_{\epsilon,x}(gv) = ge_{\epsilon,x}(v)$ for all $v \in B_x$ and all $g \in P$.

Each of the self-maps of spheres that we shall construct will have the property that it is equal to the identity except on a number of small balls. For such a map $f\colon S \to S$, its support, $\mathrm{supp}(f)$, is defined to be the closure of the set of points $x \in S$ so that $f(x) \neq x$. Given another such map $f'\colon S \to S$ with $\mathrm{supp}(f) \cap \mathrm{supp}(f') = \emptyset$, the map $f \coprod f'$ is defined by

$$f{\coprod}f'(x) = \begin{cases} f(x) & x \in \mathrm{supp}(f) \\ f'(x) & x \in \mathrm{supp}(f') \\ x & x \notin \mathrm{supp}(f) \cup \mathrm{supp}(f'). \end{cases}$$

Suppose that a group Q acts on S. For $f\colon S \to S$ a self-map of S and $g \in Q$, define $g * f(s) = g(f(g^{-1}(s)))$. The support of $g * f$ is equal to $g.\mathrm{supp}(f)$.

For $x \in S$, let $r\colon (B_x, \partial B_x) \to (B_x, \partial B_x)$ be any map of degree -1, for example a reflection in a hyperplane through 0 in B_x. Define $\tilde\phi_x, \tilde\psi_x : B_x \to S$ by

$$\tilde\phi_x(v) = \begin{cases} -e_{\pi,x}(2v) & |v| \leq 1/2 \\ (|v| - 1/2)v & |v| \geq 1/2 \end{cases}$$
$$\tilde\psi_x(v) = \begin{cases} -e_{\pi,x}(r(2v)) & |v| \leq 1/2 \\ (|v| - 1/2)v & |v| \geq 1/2 \end{cases}$$

and define self-maps $\phi_{\epsilon,x}$ and $\psi_{\epsilon,x}$ to be the identity outside of the image of $e_{\epsilon,x}$ and equal to $\tilde\phi_x \circ e_{\epsilon,x}^{-1}$ and $\tilde\psi_x \circ e_{\epsilon,x}^{-1}$ respectively on their supports. If f is a self-map of S of degree n whose support is disjoint from the ϵ–ball around x, then $f \coprod \phi_{\epsilon,x}$ is a self-map of degree $n+1$ and $f \coprod \psi_{\epsilon,x}$ is a self-map of degree $n-1$.

Suppose that a finite group Q acts linearly on S, so that the distance between any two points of the orbit $Q.x$ is greater than 2ϵ. If $g \in Q$, then $g * \phi_{\epsilon,x}$ and $\phi_{\epsilon,g.x}$ are equal. In particular, if g is an element of Q_x, the stabilizer of the point x, then the equation $g * \phi_{\epsilon,x} = \phi_{\epsilon,x}$ holds. Since the definition of ψ involved an arbitrary choice of function r, there is no corresponding equivariance property for the ψ self-maps. However, the map $g * \psi_{\epsilon,x}$ is a self-map whose support is the ϵ–ball in S centred at $g.x$, and if f is any self-map of S whose support is disjoint from this ball, the coproduct $f \coprod g * \psi_{\epsilon,x}$ is a self-map whose degree is one less than that of f.

For any $x \in S$, define

$$Q.\phi_{\epsilon,x} = \coprod_{g \in Q/Q_x} g * \phi_{\epsilon,x},$$

for any sufficiently small ϵ, where the sum runs over a transversal to Q_x in Q. For x in a free Q–orbit, define

$$Q.\psi_{\epsilon,x} = \coprod_{g \in Q} g * \psi_{\epsilon,x},$$

for small ϵ. Each of these maps is Q–equivariant.

Lemma 14 *Let S be the unit sphere in a complex representation of the finite group Q, and suppose that S contains points in Q–orbits of coprime lengths. Then S admits a Q–equivariant self-map of degree zero.*

Proof Without loss of generality, we may suppose that Q acts faithfully on S. The action of the unit circle in $\mathbb{C}$ on S commutes with the Q–action, and so whenever S contains a Q–orbit of a given length, S contains infinitely many Q–orbits of that length. Pick points $x_1, \ldots, x_m$ in distinct Q–orbits, such that the sum of the lengths of the orbits is congruent to -1 modulo $|Q|$, ie, so that there exists n with

$$|Q|n = 1 + \sum_{i=1}^{m} |Q.x_i|.$$

Now pick $y_1, \ldots, y_n$ in distinct free Q–orbits. Choose ϵ sufficiently small that any two points in any of these orbits are separated by more than 2ϵ. The coproduct

$$f = \coprod_{i=1}^{m} Q.\phi_{\epsilon,x_i} \coprod \coprod_{j=1}^{n} Q.\psi_{\epsilon,y_j}$$

is the required degree zero map. □

Theorem 15 *Let Q be a finite group not of prime power order. Then there exists a finite-dimensional contractible simplicial complex L with a cocompact action of $\mathbb{Z} \times Q$ such that all stabilizers are finite and such that $L^Q = \emptyset$. Furthermore, L may be chosen in such a way that $L^P \neq \emptyset$ for P any proper subgroup of Q.*

Proof Let S be the unit sphere in the 'reduced regular representation of Q', ie, the regular representation $\mathbb{C}Q$ minus the trivial representation. This S has the property that $S^Q = \emptyset$ but $S^P \neq \emptyset$ for any proper subgroup $P < Q$. Since Q is not of prime power order, S satisfies the hypotheses of Lemma 14, and so there exists a Q–equivariant map $f\colon\ S \to S$ of degree zero.

Take a Q–equivariant triangulation of the space $I \times S$, where Q acts trivially on the interval I. By the simplicial approximation theorem, there is an integer $n \geq 0$ and a simplicial map $f'\colon\ \{1\} \times S^{(n)} \to \{0\} \times S$ which is equivariantly homotopic to f. Now let M be the nth barycentric subdivision of $I \times S$ relative to $\{0\} \times S$. This is a copy of $I \times S$, with the original triangulation on the subspace $\{0\} \times S$ and the nth barycentric subdivision of this triangulation on $\{1\} \times S$. Construct L from the direct product $\mathbb{Z} \times M$ by identifying $(m, 1, s)$ with $(m+1, 0, f'(s))$ for each $s \in S$ and $m \in \mathbb{Z}$. This space L is a triangulation of the doubly infinite mapping telescope of the map $f'\colon\ S \to S$. The fact that f' has degree zero implies that L is contractible. □

One difference between Theorem 13 and Theorem 15 is that the the dimension of the space constructed in Theorem 15 varies with Q. The final results in this section show that this difference cannot be avoided.

Lemma 16 *Let Q be the special linear group $SL_n(\mathbb{F}_p)$ over the field of p elements. Let $e_1, \dots, e_n$ be the standard basis for the vector space $\mathbb{F}_p^n$. Define elements $\tau_1, \dots, \tau_n \in Q$ by*

$$\tau_i(e_j) = \begin{cases} e_j & i \neq j \\ e_i + e_{i+1} & i = j < n \\ e_n + e_1 & i = j = n. \end{cases}$$

The elements $\tau_1, \dots, \tau_n$ generate Q, and any proper subset of them generates a subgroup of order a power of p.

Proof Let θ be the cyclic permutation of the n standard basis elements, so that $\theta(e_i) = e_{i+1}$ for $i < n$ and $\theta(e_n) = \theta_1$. The action of θ on Q by conjugation induces a cyclic permutation of the τ_i.

The elements $\tau_1, \dots, \tau_{n-1}$ generate the upper triangular matrices, which form a Sylow p–subgroup of Q. This group contains each of the elementary matrices $E_{i,j}$ for $i < j$, defined by

$$E_{i,j}(e_k) = \begin{cases} e_k & k \neq i \\ e_k + e_j & k = i. \end{cases}$$

Conjugation by powers of θ induces a transitive permutation of the size $n-1$ subsets of $\tau_1, \dots, \tau_n$. Hence one sees that each such set generates a Sylow p–subgroup of Q.

It is well-known that the elementary matrices $E_{i,j}$ for all $i \neq j$ form a generating set for Q. Each elementary matrix may be expressed as the conjugate of an upper triangular elementary matrix by some power of θ. It follows that the subgroup generated by $\tau_1, \dots, \tau_n$ contains all elementary matrices and so is equal to Q. □

Theorem 17 *As in the previous lemma, let $Q = SL_n(\mathbb{F}_p)$. Suppose that L is contractible, or that L is mod-p acyclic, and that Q acts on L so that $L^Q = \emptyset$. Then the dimension of L is at least $n-1$.*

Proof We may assume that L is finite-dimensional, or there is nothing to prove. Let L_i be the fixed point subspace for the action of τ_i. By P. A. Smith theory, the fixed point set for the action of a p–group on a finite-dimensional mod-p acyclic space is itself mod-p acyclic. From Lemma 16 it follows that each intersection of at most $n-1$ of the L_i is mod-p acyclic, and that the intersection $L_1 \cap \ldots \cap L_n$ is empty. Let X be the union of the L_i. The Mayer–Vietoris spectral sequence for the covering of X by the L_i with mod-p coefficients is isomorphic to the spectral sequence for the covering of the boundary of an $(n-1)$–simplex by its faces. It follows that the mod-p homology of X is isomorphic to the mod-p homology of an $(n-2)$–sphere. Hence X cannot be a subspace of a mod-p acyclic space of dimension strictly less than $n-1$. □

Remark For a discrete group G, the minimal dimension of any contractible simplicial complex admitting a G–action without a global fixed point is an interesting invariant of G. The above theorem shows that this invariant can take arbitrarily large finite values. When G is a finite group of prime power order, the invariant takes the value infinity. Peter Kropholler has asked whether there are any other finitely generated groups G for which the invariant takes the value infinity.

4 Examples

Here we combine the results of Sections 2 and 3 to construct groups with strong homological finiteness properties that contain infinitely many conjugacy classes of certain finite subgroups.

Theorem 18 *Let Q be a finite group not of prime power order. There is a group H of type F and a group $G = H{:}Q$ such that G contains infinitely many conjugacy classes of subgroup isomorphic to Q and finitely many conjugacy classes of other finite subgroups.*

Proof By Theorem 15, there is a contractible finite-dimensional simplicial complex L with a cocompact action of $\mathbb{Z}\times Q$ such that all stabilizers are finite, $L^Q = \emptyset$ and $L^P \neq \emptyset$ if $P < Q$. Take a flag triangulation of L, and consider the Bestvina–Brady group H_L. By Theorem 7, the semi-direct product $H = H_L{:}\mathbb{Z}$ is of type F. By Theorem 9 the group $G = H_L{:}(\mathbb{Z}\times Q)$ contains infinitely many conjugacy classes of subgroups isomorphic to Q and finitely many conjugacy classes of other finite subgroups. □

We can now prove Theorem 2 as stated in the introduction. We first give a lemma concerning free products.

Lemma 19 *Let $G = G_1 * \cdots * G_n$ be a free product of groups, and let H_i be a finite-index normal subgroup of G_i. There is a bijection between conjugacy classes of non-trivial finite subgroups of G and the disjoint union of the sets of conjugacy classes of non-trivial finite subgroups of the G_i. The kernel of the map from G to $\prod_i G_i/H_i$ is isomorphic to the free product of finitely many copies of the H_i and a finitely-generated free group.*

Proof Take a classifying space BG_i for each G_i, take a star-shaped tree with n edges whose central vertex has valency n, and make a classifying space BG for G by attaching the given BG_i to the ith boundary vertex of the tree. Now consider the regular covering of this space BG corresponding to the kernel of the homomorphism $G \to \prod_i G/H_i$. This is a finite covering. The subspace of this covering lying above each BG_i is a finite disjoint union of copies of BH_i, and the subspace lying above the tree is a finite disjoint union of trees. Hence the whole space, which is a classifying space for the kernel, consists of a finite number of copies of the BH_i's, connected together by a finite number of trees. The fundamental group of such a space is the free product of finitely many copies of the H_i and a finitely-generated free group.

For the claimed result concerning conjugacy classes of finite subgroups, we consider the tree obtained from the given expression for G as a free product. One way to construct this tree is by considering the universal covering space of the model for BG given above. This consists of copies of the EG_i's, connected together by trees. Now contract each copy of EG_i to a point. The resulting

G–space is contractible (since replacing EG_i by a single point does not change its homotopy type) and is 1–dimensional. It is therefore a G–tree, with $n+1$ orbits of vertices and n orbits of edges. Each edge orbit is free, one of the vertex orbits is free, and there is one vertex orbit of type G/G_i for each $1 \leq i \leq n$. Whenever a finite group acts on a tree, it has a fixed point. (To see this, take the finite subtree spanning an orbit, and peel off orbits of 'leaves' until the remainder is fixed.) Since the stabilizer of each edge is trivial, it follows that each non-trivial finite subgroup of G must fix exactly one vertex of the tree. This implies that each non-trivial finite subgroup of G is conjugate to a subgroup of exactly one of the G_i, and that two finite subgroups of G_i are conjugate in G if and only if they were already conjugate in G_i. □

Proof of Theorem 2 Let $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$ be a finite list of isomorphism types of finite groups not of prime power order. For each Q_i, let $G_i = H_i{:}Q_i$ be a group as in Theorem 18. Let G be $G_1 * \cdots * G_n$, the free product of the G_i. By Lemma 19, the group G is of type VF, contains infinitely many conjugacy classes of subgroup isomorphic to each Q_i, and contains finitely many conjugacy classes of finite subgroups of all other isomorphism types. □

Theorem 20 *Let Q be a non-trivial finite group. There exists a group $G = H{:}Q$ of type FP over $\mathbb{Q}$, containing infinitely many conjugacy classes of subgroups isomorphic to Q and finitely many conjugacy classes of other finite subgroups. Furthermore, H is torsion-free, has rational cohomological dimension at most 4 and has integral cohomological dimension at most 5.*

Proof By Theorem 13 there is a 3–dimensional $\mathbb{Q}$–acyclic simplicial complex L with a cocompact $\mathbb{Z}\times Q$–action such that all stabilizers are finite, $L^Q = \emptyset$ and $L^P \neq \emptyset$ if $P < Q$. Take a flag triangulation of L, and consider the Bestvina–Brady group H_L. By Theorem 7, the semi-direct product $H = H_L{:}\mathbb{Z}$ is FP over $\mathbb{Q}$. By Theorem 9 the group $G = H_L{:}(\mathbb{Z} \times Q)$ contains infinitely many conjugacy classes of subgroups isomorphic to Q and finitely many conjugacy classes of other finite subgroups. The rational cohomological dimension of H_L is at most the dimension of the $\mathbb{Q}$–acyclic space Y appearing in Section 2, which is equal to the dimension of L, and the integral cohomological dimension of H_L is at most the dimension of the space X_L, which is one more than the dimension of L. The cohomological dimension of $H_L{:}\mathbb{Z}$ over any ring is at most one more than the cohomological dimension of H_L over the same ring. □

Proof of Theorem 4 For each $Q_i \in Q$, Theorem 20 gives a group G_i of type FP over $\mathbb{Q}$ containing infinitely many conjugacy classes of subgroup isomorphic to Q and only finitely many conjugacy classes of other finite subgroups. By Lemma 19, the free product $G = G_1 * \cdots * G_n$ is FP over $\mathbb{Q}$, contains infinitely many conjugacy classes of subgroups isomorphic to each $Q_i \in \mathcal{Q}$, and contains finitely many conjugacy classes of all other finite subgroups. □

Remark One difference between Theorems 2 and 4 is that each of the groups constructed in Theorem 4 has virtual cohomological dimension at most 5, whereas the virtual cohomological dimensions of the groups constructed in Theorem 2 seem to depend on the list $\mathcal{Q}$. We do not know whether this necessarily happens, but the following proposition may be relevant.

Proposition 21 *Suppose that G contains infinitely many conjugacy classes of subgroup isomorphic to $SL_n(\mathbb{F}_p)$, and that G acts cocompactly with finite stabilizers on a mod-p–acyclic simplicial complex X. Then X must have dimension at least $n-1$.*

Proof There are only finitely many orbits in X, and hence only finitely many conjugacy classes of subgroup of G can fix some point of X. It follows that there is a subgroup isomorphic to $SL_n(\mathbb{F}_p)$ that has no fixed point, and we may apply Theorem 17 to deduce the required result. □

Remark If G is virtually torsion-free and acts cocompactly with finite stablizers on a contractible simplicial complex X, then G is of type VF. It seems to be unknown whether every group of type VF admits such an action. It also seems to be unknown whether every group of type FL over a prime field F admits a free cocompact action on an F–acyclic simplicial complex X. If F is not assumed to be a prime field, then there are counterexamples. In [6] we exhibited a group which is FL over $\mathbb{C}$ but which is not FL over $\mathbb{R}$. This group cannot admit a cocompact free action on any $\mathbb{C}$–acyclic simplicial complex X.

We conclude this section with a brief discussion of the Grothendieck group $K_0(\mathbb{Q}G)$ of finitely generated projective modules for $\mathbb{Q}G$ and its connection with conjugacy classes of elements of finite order in G. First, we recall the definition of the Hattori–Stallings trace [1].

For any ring R, let $T(R)$ denote the quotient of R by the additive subgroup generated by commutators of the form $rs - sr$ for $r, s \in R$. For a square matrix A with coefficients in R, the Hattori–Stallings trace $\mathrm{tr}(A)$ is the element of $T(R)$ defined as the equivalence class containing the sum of the diagonal entries of A. As an element of $T(R)$, this satisfies the usual trace condition $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any matrices A and B.

Now suppose that P is a finitely generated projective R module, and that P is isomorphic to a summand of R^n. Pick an idempotent $n \times n$ matrix e_P whose image is isomorphic to P. The Hattori–Stallings rank of P is defined to be $\mathrm{tr}(e_P)$. It may be shown that this is independent of the choice of n and e_P. The Hattori–Stallings rank defines a group homomorphism from $K_0(R)$ to $T(R)$.

Theorem 22 *For any group G, there is a subgroup of $K_0(\mathbb{Q}G)$ which is free abelian of rank equal to the number of conjugacy classes of finite cyclic subgroups of G.*

Proof For the group algebra $\mathbb{Q}G$, the group $T(\mathbb{Q}G)$ is the $\mathbb{Q}$–vector space with basis the conjugacy classes of elements of G. For any finite cyclic subgroup $C \leq G$, define an element $e_C \in \mathbb{Q}G$ by

$$e_C = \frac{1}{|C|} \sum_{g \in C} g.$$

The element e_C is an idempotent, and the $\mathbb{Q}G$–module P_C defined by $P_C = \mathbb{Q}Ge_C$ is a projective $\mathbb{Q}G$–module. With respect to the basis for $T(\mathbb{Q}G)$ given by the conjugacy classes of elements of G, the non-zero coefficients in the Hattori–Stallings trace for e_C are those corresponding to elements of C. If $C_1, \ldots, C_n$ are pairwise non-conjugate finite cyclic subgroups of G, it follows that the Hattori–Stallings traces $e_{C_1}, \ldots, e_{C_n}$ are linearly independent. It follows that the projectives of the form P_C generate a subgroup of $K_0(\mathbb{Q}G)$ which is free abelian of rank equal to the number of conjugacy classes of finite cyclic subgroups of G. □

Corollary 23 *There are groups G of type VF for which $K_0(\mathbb{Q}G)$ is not finitely generated.*

Proof Apply Theorem 22 to the groups with infinitely many conjugacy classes of finite cyclic subgroups constructed in Theorem 18. □

5 Other properties of the groups

Suppose that Q is a group of automorphisms of a finite flag complex L with n vertices. It is shown in [7] that in this case the group $G_L{:}Q$ is isomorphic to a subgroup of the special linear group $SL_{2n}(\mathbb{Z})$. We do not know whether the groups $G_L{:}\Gamma$, for infinite L, are linear. Residual finiteness however is easier to establish.

Lemma 24 *Suppose that Γ is residually finite and that Γ acts cocompactly and with finite stabilizers on a flag complex L. There is a finite-index normal subgroup Γ' such that for any $\Gamma'' \leq \Gamma'$, the quotient $L' = L/\Gamma''$ is a flag simplicial complex.*

Proof There are finitely many conjugacy classes of simplex stabilizer in L, and each simplex stabilizer is finite. It follows that there is a finite-index normal subgroup Γ_1 of Γ that acts freely on L. Since L is locally finite, there are only finitely many Γ_1–orbits of paths of length 1, 2 and 3 in the 1–skeleton of L. Hence we may pick Γ_2 of finite-index in Γ_1, such that no two points in the same Γ_2–orbit are joined by an edge path of length less than four. We claim that we may take $\Gamma' = \Gamma_2$.

If Γ'' is any subgroup of Γ_2, then there is no edge path of length less than four between any two vertices in the same Γ''–orbit. In particular, there can be no loops in L/Γ'. Hence every simplex of L maps injectively to a subspace of L/Γ''. There can be no double edges in L/Γ'', since that would give rise to an edge path of length two between vertices in the same Γ''–orbit. Thus the 1–skeleton of L/Γ'' is a simplicial complex.

Now suppose that $\bar{v}_0, \ldots, \bar{v}_n$ are a mutually adjacent set of vertices of L/Γ'', and let v_0 be a lift of $\bar{v}_0$. There exists a unique lift v_i of each $\bar{v}_i$ that is adjacent to v_0. For each $i \neq j$, there exists a unique $g \in \Gamma_2$ so that v_i is adjacent to gv_j. But if $g \neq e$, then the path (v_j, v_0, v_i, gv_j) gives rise to a contradiction. Thus the v_i are all adjacent to each other, and so there is a simplex σ of L with vertex set $v_0, \ldots, v_n$. It follows that the quotient L/Γ'' contains a simplex $\bar{\sigma}$ spanning each complete subgraph of its 1–skeleton. Suppose that $\bar{\sigma}'$ is any simplex of L/Γ'' spanning the same complete subgraph as $\bar{\sigma}$. There is a unique lift σ' of $\bar{\sigma}'$ containing v_0. If $\sigma' \neq \sigma$, then there exists i and $g \neq e$ so that gv_i is a vertex of σ'. But then there is an edge path of length 2 from v_i to gv_i. Hence any finite full subgraph of the 1–skeleton of L/Γ'' is spanned by a unique simplex, and so L/Γ'' is a flag complex. □

Theorem 25 *Let Γ be residually finite and let Γ act cocompactly and with finite stabilizers on a flag complex L. Then the group $G_L{:}\Gamma$ is also residually finite.*

Proof Let g be a non-identity element of $G_L{:}\Gamma$. Since $(G_L{:}\Gamma)/G_L$ is isomorphic to Γ, it suffices to consider the case when $g \in G_L$. Let K be a finite full subcomplex of L (ie, a subcomplex containing as many simplices as possible given its 0–skeleton) such that g is in the subgroup generated by the vertices of K, and let J be a finite full subcomplex of L containing K and every vertex adjacent to a vertex of K. Let Γ' be a finite-index subgroup of Γ as in Lemma 24, and let Γ'' be a finite-index normal subgroup of Γ contained in Γ' such that any two vertices of J lie in distinct Γ''–orbits. Now $M = L/\Gamma''$ is a finite flag complex, and K maps to a full subcomplex of L/Γ''.

The group Γ/Γ'' acts on M, and g has non-trivial image under the homomorphism $G_L{:}\Gamma \to G_M{:}(\Gamma/\Gamma'')$. Since this group is isomorphic to a subgroup of $SL_{2n}(\mathbb{Z})$, where n is the number of vertices of M (see corollary 8 of [7]), it follows that there is a finite quotient of $G_L{:}\Gamma$ in which the image of g is non-zero. □

In the special case when $\Gamma = \mathbb{Z}$ (which is the main case used earlier in the paper), we shall show how to describe the group $G_L{:}\Gamma$ as the fundamental group of a finite locally CAT(0) cube complex. First we present two lemmas concerning right-angled Artin groups.

Lemma 26 *Let N be a full subcomplex of a flag complex M. The inclusion $i\colon N \to M$ induces a split injection $G_N \to G_M$.*

Proof The quotient of G_M by the subgroup generated by the vertices of $M - N$ is naturally isomorphic to G_N. □

Lemma 27 *Let the flag complex K be expressed as $K = L \cup M$, where L and M are full subcomplexes with $N = L \cap M$. Then the group homomorphisms induced by the inclusion of each subcomplex in K induce an isomorphism $G_L *_{G_N} G_M \to G_K$.*

Proof Immediate from the presentations of the groups, given the result of Lemma 26. □

Theorem 28 *Let Γ be an infinite cyclic group generated by γ, let Γ act on the flag complex L, and let M be a 'fundamental domain' for Γ in the sense that $L = \bigcup_i \gamma^i M$. Define subcomplexes N_0 and N_1 by*

$$N_0 = \gamma^{-1} M \cap M, \qquad N_1 = M \cap \gamma M.$$

Then $G_L{:}\Gamma$ is isomorphic to the HNN–extension $G_M *_{G_{N_0} = G_{N_1}}$. (In this HNN–extension, the base group is G_M, and the stable letter conjugates the subgroup G_{N_0} to the subgroup G_{N_1} by the map induced by $\gamma\colon N_0 \to N_1$.)

Proof Let t denote the stable letter in the HNN–extension, and consider the homomorphism ϕ from the HNN–extension to $\mathbb{Z}$ that sends t to 1 and sends each element of G_M to 0. The kernel of ϕ is an infinite free product with amalgamation:

$$\cdots * G_{-2} *_{H_{-1}} G_{-1} *_{H_0} G_0 *_{H_1} G_1 *_{H_2} G_2 * \cdots ,$$

where G_i denotes $t^i G_M t^{-i}$, and H_i denotes $t^i G_{N_0} t^{-i}$. If we define $M_i = \gamma^i M$ and $N_i = \gamma^i N_0$, there is an isomorphism $\psi_i \colon G_i \to G_{M_i}$ defined as the composite

$$G_i \xrightarrow{c(t^{-i})} G_0 \xrightarrow{1} G_M \xrightarrow{c(\gamma^i)} G_{M_i}$$

of conjugation by t^{-i} followed by the identification of G_0 and G_M, followed by conjugation by γ^i. Each of ψ_i and ψ_{i-1} induces an isomorphism from H_i to H_{N_i}, and these two are the same isomorphism. The ψ_i therefore fit together to make an isomorphism from $\ker(\phi)$, described as an infinite free product with amalgamation, to the following infinite free product with amalgamation:

$$\cdots * G_{M_{-2}} *_{H_{N_{-1}}} G_{M_{-1}} *_{H_{N_0}} G_{M_0} *_{H_{N_1}} G_{M_1} *_{H_{N_2}} G_{M_2} * \cdots .$$

Furthermore, this isomorphism is equivariant for the $\mathbb{Z}$–actions given by conjugation by powers of t and γ. By Lemma 27, the inclusions of the G_{M_i} in G_L induce a Γ–equivariant isomorphism between the second free product with amalgamation and G_L. Hence we obtain an isomorphism $G_M *_{G_{N_0}=G_{N_1}} \to G_L{:}\Gamma$ as required. □

Corollary 29 *Under the hypotheses of Theorem 28, the group $G_L{:}\Gamma$ is the fundamental group of a finite locally* CAT(0) *cube complex.*

Proof For any flag complex K, let Y_K denote the explicit model for the classifying space BG_K described in Section 2, so that $Y_K = X_K/G_K$. The naturality properties of this construction are such that Y_{N_0} and Y_{N_1} are subcomplexes of Y_M. We construct a model Z for $B(G_L{:}\Gamma)$ from Y_M and $Y_{N_0} \times I$ by identifying $\{0\} \times Y_{N_0}$ with $Y_{N_0} \subseteq Y_M$ via the identity map and identifying $\{1\} \times Y_{N_0}$ with $Y_{N_1} \subseteq Y_M$ via the action of γ which gives an isomorphism from N_0 to N_1.

The space Z as above is a model for $B(G_L{:}\Gamma)$. To see that Z has the structure of a locally CAT(0) cube complex, one may either quote a gluing lemma (such as in [3], proposition II.11.13), or one may show that the link of the unique vertex in Z is a flag complex, which suffices by Gromov's lemma ([3], theorem II.5.18). For any flag complex K, the link of the unique vertex in Y_K is a flag complex $S(K)$, which is a sort of 'double' of K: each vertex v of K corresponds to two vertices v', v'' of $S(K)$, and a set of vertices of $S(K)$ is the vertex set of an

n–simplex in $S(K)$ if and only if its image in the vertex set of K is the vertex set of an n–simplex. (For example, in the case when K is a 2–simplex, $S(K)$ is the boundary of an octahedron.) The link of the vertex in Z is isomorphic to $S(M)$ with a cone attached to each of the subspaces $S(N_0)$ and $S(N_1)$, and hence it is a flag complex. □

Corollary 30 *Each of the groups $G_L{:}(\mathbb{Z} \times Q)$ constructed in Section 4 acts cocompactly with finite stabilizers on some* CAT(0) *cube complex. In particular, there is a model for the universal space for proper actions of $G_L{:}(\mathbb{Z} \times Q)$ which has finitely many orbits of cells.*

Proof Take a finite 'fundamental domain', M', for the action of $\mathbb{Z}$ on L (as in the statement of Theorem 28). In case M' is not Q–invariant, replace M' by $M = \bigcup_{q \in Q} qM'$. For this choice of M, there is a base-point preserving cellular Q–action on Z, the model for $B(G_L{:}\mathbb{Z})$ constructed in Corollary 29. This induces the required action of $G_L{:}(\mathbb{Z} \times Q)$ on the universal cover of Z. Whenever a group H acts with finite stabilizers on a CAT(0) cube complex, that space is a model for the universal space for proper actions of H [7]. □

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