Quadrisecants give new lower bounds for the ropelength of a knot

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Using the existence of a special quadrisecant line, we show the ropelength of any nontrivial knot is at least 15.66. This improves the previously known lower bound of 12. Numerical experiments have found a trefoil with ropelength less than 16.372, so our new bounds are quite sharp.

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1 Introduction

The ropelength problem seeks to minimize the length of a knotted curve subject to maintaining an embedded tube of fixed radius around the curve; this is a mathematical model of tying the knot tight in rope of fixed thickness.

More technically, the thickness $\tau(K)$ of a space curve K is defined by Gonzalez and Maddocks [11] to be twice the infimal radius $r(a,b,c)$ of circles through any three distinct points of K. It is known from the work of Cantarella, Kusner and Sullivan [4] that $\tau(K) = 0$ unless K is $C^{1,1}$, meaning that its tangent direction is a Lipschitz function of arclength. When K is C^1, we can define normal tubes around K, and then indeed $\tau(K)$ is the supremal diameter of such a tube that remains embedded. We note that in the existing literature thickness is sometimes defined to be the radius rather than diameter of this thick tube.

We define ropelength to be the (scale-invariant) quotient of length over thickness. Because this is semi-continuous even in the C^0 topology on closed curves, it is not hard to show [4] that any (tame) knot or link type has a ropelength minimizer.

Cantarella, Kusner and Sullivan gave certain lower bounds for the ropelength of links; these are sharp in certain simple cases where each component of the link is planar [4]. However, these examples are still the only known ropelength minimizers. Recent work by Cantarella, Fu, Kusner, Sullivan and Wrinkle [2] describes a much more complicated

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tight (ropelength-critical) configuration B_0 of the Borromean rings. (Although the somewhat different Gehring notion of thickness is used there, B_0 should still be tight, and presumably minimizing, for the ordinary ropelength we consider here.) Each component of B_0 is still planar, and it seems significantly more difficult to describe explicitly the shape of any tight knot.

Numerical experiments by Pierański [17], Sullivan [19] and Rawdon [18] suggest that the minimum ropelength for a trefoil is slightly less than 16.372, and that there is another tight trefoil with different symmetry and ropelength about 18.7. For comparison, numerical simulations of the tight figure-eight knot show ropelength just over 21. The best lower bound in [4] was 10.726; this was improved by Diao [8], who showed that any knot has ropelength more than 12 (meaning that "no knot can be tied in one foot of one-inch rope").

Here, we use the idea of quadrisecants, lines that intersect a knot in four distinct places, to get better lower bounds for ropelength. Almost 75 years ago, Pannwitz [16] showed, using polygonal knots, that a generic representative of any nontrivial knot type must have a quadrisecant. Kuperberg [13] extended this result to all knots by showing that generic knots have essential (or topologically nontrivial) quadrisecants. (See also the article by Morton and Mond [15].) We will define this precisely below, as essential quadrisecants are exactly what we need for our improved ropelength bounds. (Note that a curve arbitrarily close to a round circle, with ropelength thus near π, can have a nonessential quadrisecant.)

By comparing the orderings of the four points along the knot and along the quadrisecant, we distinguish three types of quadrisecants. For each of these types we use geometric arguments to obtain a lower bound for the ropelength of the knot having a quadrisecant of that type. The worst of these three bounds is 13.936.

In her doctoral dissertation [5], Denne shows that nontrivial knots have essential quadrisecants of alternating type. This result, combined with our Theorem 9.4, shows that any nontrivial knot has ropelength at least 15.66.

Nontrivial links also necessarily have quadrisecants. We briefly consider ropelength bounds obtained for links with different types of quadrisecants. These provide another interpretation of the argument showing that the tight Hopf link has ropelength 4π, as in the Gehring link problem. But we have not found any way to improve the known ropelength estimates for other links.

2 Definitions and lemmas

Definition A *knot* is an oriented simple closed curve K in $\mathbb{R}^3$. Any two points a and b on a knot K divide it into two complementary subarcs γ_{ab} and γ_{ba}. Here γ_{ab} is the arc from a to b following the given orientation on K. If $p \in \gamma_{ab}$, we will sometimes write $\gamma_{apb} = \gamma_{ab}$ to emphasize the order of points along K. We will use ℓ_{ab} to denote the length of γ_{ab}; by comparison, $|a-b|$ denotes the distance from a to b in space, the length of the segment $\overline{ab}$.

Definition An *n–secant line* for a knot K is an oriented line in $\mathbb{R}^3$ whose intersection with K has at least n components. An *n–secant* is an ordered n–tuple of points in K (no two of which lie in a common straight subarc of K) which lie in order along an n–secant line. We will use *secant*, *trisecant* and *quadrisecant* to mean 2–secant, 3–secant and 4–secant, respectively. The *midsegment* of a quadrisecant $abcd$ is the segment $\overline{bc}$.

The orientation of a trisecant either agrees or disagrees with that of the knot. In detail, the three points of a trisecant abc occur in that linear order along the trisecant line, but may occur in either cyclic order along the oriented knot. (Cyclic orders are cosets of the cyclic group C_3 in the symmetric group S_3.) These could be labeled by their lexicographically least elements (abc and acb), but we choose to call them *direct* and *reversed* trisecants, respectively, as in Figure 1. Changing the orientation of either the knot or the trisecant changes its class. Note that abc is direct if and only if $b \in \gamma_{ac}$.

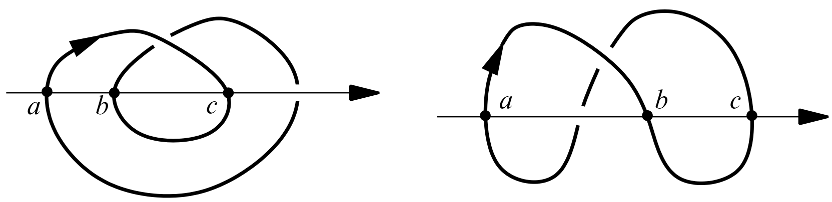

Figure 1: These trisecants abc are reversed (left) and direct (right) because the cyclic order of the points along K is acb and abc, respectively. Flipping the orientation of the knot or the trisecant would change its type.

Similarly, the four points of a quadrisecant $abcd$ occur in that order along the quadrisecant line, but may occur in any order along the knot K. Of course, the order along K is only a cyclic order, and if we ignore the orientation on K it is really just a dihedral order, meaning one of the three cosets of the dihedral group D_4 in S_4. Picking the lexicographically least element in each, we could label these cosets $abcd$, $abdc$

and $acbd$. We will call the corresponding classes of quadrisecants *simple*, *flipped* and *alternating*, respectively, as in Figure 2. Note that this definition ignores the orientation of K, and switching the orientation of the quadrisecant does not change its type.

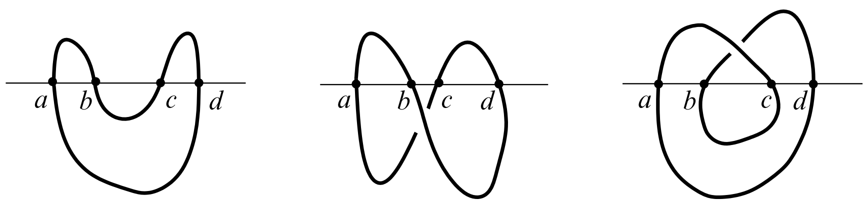

Figure 2: Here we see quadrisecants $abcd$ on each of three knots. From left to right, these are simple, flipped and alternating, because the dihedral order of the points along K is $abcd$, $abdc$ and $acbd$, respectively.

When discussing a quadrisecant $abcd$, we will usually orient K so that $b \in \gamma_{ad}$. That means the cyclic order of points along K will be $abcd$, $abdc$ or $acbd$, depending on the type of the quadrisecant.

In some sense, alternating quadrisecants are the most interesting. These have also been called NSNS quadrisecants, because if we view $\overrightarrow{cd}$ and $\overrightarrow{ba}$ as the North and South ends of the midsegment $\overline{bc}$, then when $abcd$ is an alternating quadrisecant, K visits these ends alternately NSNS as it goes through the points $acbd$. It was noted by Cantarella, Kuperberg, Kusner and Sullivan [3] that the midsegment of an alternating quadrisecant for K is automatically in the *second hull* of K. Denne shows [5] that nontrivial knots have alternating quadrisecants. Budney, Conant, Scannell and Sinha [1] have shown that the finite-type (Vassiliev) knot invariant of type 2 can be computed by counting alternating quadrisecants with appropriate multiplicity.

3 Knots with unit thickness

Because the ropelength problem is scale invariant, we find it most convenient to rescale any knot K to have thickness (at least) 1. This implies that K is a $C^{1,1}$ curve with curvature bounded above by 2.

For any point $a \in \mathbb{R}^3$, let $B(a)$ denote the open unit ball centered at a. Our first lemma, about the local structure of a thick knot, is by now standard. (Compare [8, Lemma 4] and [4, Lemma 5].)

Lemma 3.1 *Let K be a knot of unit thickness. If $a \in K$, then $B(a)$ contains a single unknotted arc of K; this arc has length at most π and is transverse to the nested spheres* *centered at* a. *If* ab *is a secant of* K *with* $|a-b|<1$, *then the ball of diameter* $\overline{ab}$ *intersects* K *in a single unknotted arc (either* γ_{ab} *or* γ_{ba}*) whose length is at most* $\arcsin|a-b|$.

Proof If there were an arc of K tangent at some point c to one of the spheres around a within $B(a)$, then triples near (a,c,c) would have radius less than $\frac{1}{2}$. For the second statement, if K had a third intersection point c with the sphere of diameter $\overline{ab}$, then we would have $r(a,b,c)<\frac{1}{2}$. The length bounds come from Schur's lemma. □

An immediate corollary is:

Corollary 3.2 *If* K *has unit thickness,* $a,b\in K$ *and* $p\in\gamma_{ab}$ *with* $a,b\notin B(p)$ *then the complementary arc* γ_{ba} *lies outside* $B(p)$.

The following lemma should be compared to [8, Lemma 5] and [4, Lemma 4], but here we give a slightly stronger version with a new proof.

Lemma 3.3 *Let* K *be a knot of unit thickness. If* $a\in K$, *then the radial projection of* $K\smallsetminus\{a\}$ *to the unit sphere* $\partial B(a)$ *is* 1*–Lipschitz, that is, it does not increase length.*

Proof Consider what this projection does infinitesimally near a point $b\in K$. Let $d=|a-b|$ and let θ be the angle at b between K and the segment $\overline{ab}$. The projection stretches by a factor $1/d$ near b, but does not see the radial part of the tangent vector to K. Thus the local Lipschitz constant on K is $(\sin\theta)/d$. Now consider the circle through a and tangent to K at b. Plane geometry (see Figure 3) shows that its radius is $r=d/(2\sin\theta)$, but it is a limit of circles through three points of K, so by the three-point characterization of thickness, r is at least $\frac{1}{2}$; that is, the Lipschitz constant $1/2r$ is at most 1. □

Corollary 3.4 *Suppose* K *has unit thickness, and* $p,a,b\in K$ *with* $p\notin\gamma_{ab}$. *Let* $\angle apb$ *be the angle between the vectors* $a-p$ *and* $b-p$. *Then* $\ell_{ab}\geq\angle apb$. *In particular, if* apb *is a reversed trisecant in* K, *then* $\ell_{ab}\geq\pi$.

Proof By Lemma 3.3, ℓ_{ab} is at least the length of the projected curve on $B(p)$, which is at least the spherical distance $\angle apb$ between its endpoints. For a trisecant apb we have $\angle apb=\pi$, and $p\notin\gamma_{ab}$ exactly when the trisecant is reversed. □

Observe that a quadrisecant $abcd$ includes four trisecants: abc, abd, acd and bcd. Simple, flipped and alternating quadrisecants have different numbers of reversed trisecants. We can apply Corollary 3.4 to these trisecants to get simple lower bounds on ropelength for any curve with a quadrisecant.

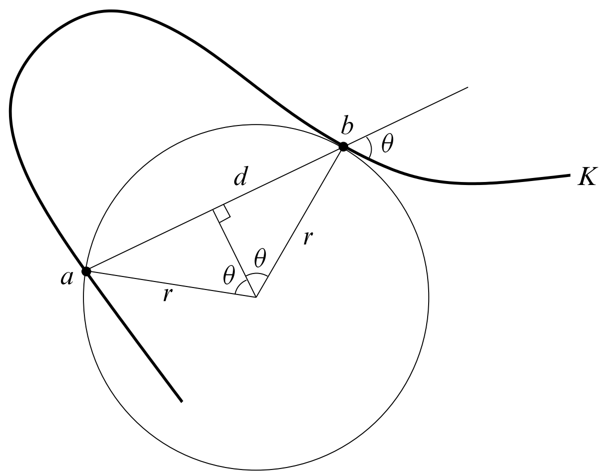

Figure 3: In the proof of Lemma 3.3, the circle tangent to K at b and passing through $a \in K$ has radius $r = d/(2\sin\theta)$ where $d = |a-b|$ and θ is the angle at b between K and $\overrightarrow{ab}$.

Theorem 3.5 *The ropelength of a knot with a simple, flipped or alternating quadrisecant is at least π, 2π or 3π, respectively.*

Proof Rescale K to have unit thickness, so that its ropelength equals its length ℓ. Let $abcd$ be the quadrisecant, and orient K in the usual way, so that $b \in \gamma_{ad}$. In the case of a simple quadrisecant, the trisecant dba is reversed, so $\ell \geq \ell_{da} \geq \pi$, using Corollary 3.4. In the case of a flipped quadrisecant, the trisecants cba and bcd are reversed, so $\ell \geq \ell_{ca} + \ell_{bd} \geq 2\pi$. In the case of an alternating quadrisecant, the trisecants abc, bcd and dca are reversed, so $\ell \geq \ell_{ac} + \ell_{bd} + \ell_{da} \geq 3\pi$. □

Of course, any closed curve has ropelength at least π, independent of whether it is knotted or has any quadrisecants, because its total curvature is at least 2π. But curves arbitrarily close (in the C^1 or even C^∞ sense) to a round circle can have simple quadrisecants, so at least the first bound in the theorem above is sharp.

Although Kuperberg has shown that any nontrivial (tame) knot has a quadrisecant, and Denne shows that in fact it has an alternating quadrisecant, the bounds from Theorem 3.5 are not as good as the previously known bounds of [8] or even [4]. To improve our bounds, in Section 5 we will consider Kuperberg's notion of an essential quadrisecant.

4 Length bounds in terms of segment lengths

Given a thick knot K with quadrisecant $abcd$, we can bound its ropelength in terms of the distances along the quadrisecant line. Whenever we discuss such a quadrisecant, we will abbreviate these three distances as $r := |a-b|$, $s := |b-c|$ and $t := |c-d|$. We start with some lower bounds for r, s and t for quadrisecants of certain types.

Lemma 4.1 *If $abcd$ is a flipped quadrisecant for a knot of unit thickness, then the midsegment has length $s \geq 1$. Furthermore if $r \geq 1$ then the whole arc γ_{ca} lies outside $B(b)$; similarly if $t \geq 1$ then γ_{bd} lies outside $B(c)$.*

Proof Orient the knot in the usual way. If $s = |b-c| < 1$, then by Lemma 3.1 either $\ell_{cab} < \pi/2$ or $\ell_{bdc} < \pi/2$. But since cba and bcd are reversed trisecants, we have $\ell_{ca} \geq \pi$ and $\ell_{bd} \geq \pi$. This is a contradiction because $\ell_{cab} = \ell_{ca} + \ell_{ab}$ and $\ell_{bdc} = \ell_{bd} + \ell_{dc}$. The second statement follows immediately from Corollary 3.2. □

Lemma 4.2 *If $abcd$ is an alternating quadrisecant for a knot of unit thickness, then $r \geq 1$ and $t \geq 1$. With the usual orientation, the entire arc γ_{da} thus lies outside $B(b) \cup B(c)$. If $s \geq 1$ as well, then γ_{ac} lies outside $B(b)$ and γ_{bd} lies outside $B(c)$.*

Proof If $r = |a-b| < 1$, then by Lemma 3.1 either $\ell_{acb} < \pi/2$ or $\ell_{bda} < \pi/2$. Similarly, if $t = |c-d| < 1$, then either $\ell_{cbd} < \pi/2$ or $\ell_{dac} < \pi/2$. But as in the proof of Theorem 3.5, we have $\ell_{ac} \geq \pi$ and $\ell_{bd} \geq \pi$, contradicting any choice of the inequalities above. Thus we have $r, t \geq 1$. Because a and d are outside $B(b)$ and $B(c)$, the remaining statements follow from Corollary 3.2. □

As suggested by the discussion above, we will often find ourselves in the situation where we have an arc of a knot known to stay outside a unit ball. We can compute exactly the minimum length of such an arc in terms of the following functions.

Definition For $r \geq 1$, let $f(r) := \sqrt{r^2-1} + \arcsin(1/r)$. For $r, s \geq 1$ and $\theta \in [0, \pi]$, the minimum length function is defined by

$$m(r,s,\theta) := \begin{cases} \sqrt{r^2+s^2-2rs\cos\theta} & \text{if } \theta \leq \arccos(1/r) + \arccos(1/s), \\ f(r) + f(s) + (\theta - \pi) & \text{if } \theta \geq \arccos(1/r) + \arccos(1/s). \end{cases}$$

The function $f(r)$ will arise again in other situations. The function m was defined exactly to make the following bound sharp:

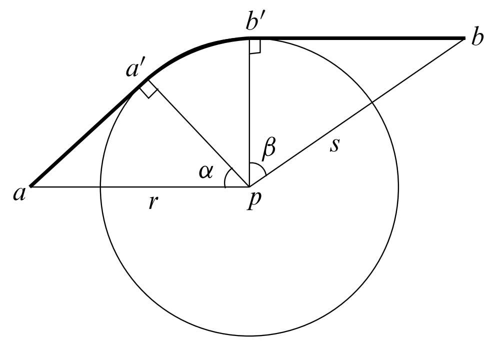

Figure 4: If points a, b are at distances $r, s \geq 1$ (respectively) from p, then the shortest curve from a to b avoiding $B(p)$ is planar. Either it is a straight segment or (in the case illustrated) it includes an arc of $\partial B(p)$. In either case, its length is $m(r, s, \angle apb)$.

Lemma 4.3 *Any arc γ from a to b, staying outside $B(p)$, has length at least $m(|a-p|, |b-p|, \angle apb)$.*

Proof Let $r := |a-p|$ and $s := |b-p|$ be the distances to p (with $r, s \geq 1$) and let $\theta := \angle apb$ be the angle between $a-p$ and $b-p$. The shortest path from a to b staying outside $B := B(p)$ either is the straight segment or is the C^1 join of a straight segment from a to ∂B, a great-circle arc in ∂B, and a straight segment from ∂B to b. In either case, we see that the path lies in the plane through a, p, b (shown in Figure 4). In this plane, draw the lines from a and b tangent to ∂B. Let $\alpha := \angle apa'$ and $\beta := \angle bpb'$, where a' and b' are the points of tangency. Then $\cos\alpha = 1/r$ and $\cos\beta = 1/s$. Clearly if $\alpha + \beta \geq \theta$ then the shortest path is the straight segment from a to b, with length $\sqrt{r^2 + s^2 - 2rs\cos\theta}$. If $\alpha + \beta \leq \theta$ then the shortest path consists of the C^1 join described above, with length

$$\sqrt{r^2 - 1} + \big(\theta - (\alpha + \beta)\big) + \sqrt{s^2 - 1} = f(r) + f(s) + (\theta - \pi). \qquad \square$$

An important special case is when $\theta = \pi$. Here we are always in the case $\alpha + \beta \leq \theta$, so we get the following corollary.

Corollary 4.4 *If a and b lie at distances r and s along opposite rays from p (so that $\angle apb = \pi$) then the length of any arc from a to b avoiding $B(p)$ is at least*

$$f(r) + f(s) = \sqrt{r^2 - 1} + \arcsin(1/r) + \sqrt{s^2 - 1} + \arcsin(1/s).$$

We note that the special case of this formula when $r = s$ also appears in recent papers by Dumitrescu, Ebbers-Baumann, Grüne, Klein and Rote [9; 10] investigating the geometric dilation (or distortion) of planar graphs. Gromov had given a lower bound for the distortion of a closed curve (see the paper by Kusner and Sullivan [14]); in [9; 10] sharper bounds in terms of the diameter and width of the curve are derived using this minimum-length arc avoiding a ball. (Although the bounds are stated there only for plane curves they apply equally well to space curves.) See also the article by Denne and Sullivan [6] for further development of these ideas and a proof that knotted curves have distortion more than 4.

Lemma 4.5 *Let $abcd$ be an alternating quadrisecant for a knot of unit thickness (oriented in the usual way). Let $r := |a-b|$, $s := |b-c|$ and $t := |c-d|$ be the lengths of the segments along $abcd$. Then $\ell_{ad} \geq f(r) + s + f(t)$. The same holds if $abcd$ is a simple quadrisecant as long as $r, t \geq 1$.*

Proof In either case (and as we already noted in Lemma 4.2 for the alternating case) we find, using Corollary 3.2, that γ_{da} lies outside $B(b) \cup B(c)$. As in the proof of Lemma 4.3, the shortest arc from d to a outside these balls will be the C^1 join of various pieces: these alternate between straight segments in space and great-circle arcs in the boundaries of the balls. Here, the straight segment in the middle always has length exactly $s := |b-c|$. As in Corollary 4.4, the overall length is then at least $f(r) + s + f(t)$ as desired. □

5 Essential secants

We have seen that the existence of a quadrisecant for K is not enough to get good lower bounds on ropelength, because some quadrisecants do not capture the knottedness of K. Kuperberg [13] introduced the notion of *essential* secants and quadrisecants (which he called "topologically nontrivial"). We will see below that these give us much better ropelength bounds.

We extend Kuperberg's definition to say when an arc γ_{ab} of a knot K is essential, capturing part of the knottedness of K. Generically, the knot K together with the segment $S = \overline{ab}$ forms a knotted Θ–graph in space (that is, a graph with three edges connecting the same two vertices). To adapt Kuperberg's definition, we consider such knotted Θ–graphs.

Definition Suppose α, β and γ are three disjoint simple arcs from p to q, forming a knotted Θ–graph. Then we say that the ordered triple (α, β, γ) is *inessential* if there is a disk D bounded by the knot $\alpha \cup \beta$ having no interior intersections with the knot $\alpha \cup \gamma$. (We allow self-intersections of D, and interior intersections with β; the latter are certainly necessary if $\alpha \cup \beta$ is knotted.)

An equivalent definition, illustrated in Figure 5, is as follows: Let $X := \mathbb{R}^3 \smallsetminus (\alpha \cup \gamma)$, and consider a parallel curve δ to $\alpha \cup \beta$ in X. Here by *parallel* we mean that $\alpha \cup \beta$ and δ cobound an annulus embedded in X. We choose the parallel to be homologically trivial in X. (Since the homology of the knot complement X is $\mathbb{Z}$, this simply means we take δ to have linking number zero with $\alpha \cup \gamma$. This determines δ uniquely up to homotopy.) Let $x_0 \in \delta$ near p be a basepoint for X, and let $h = h(\alpha, \beta, \gamma) \in \pi_1(X, x_0)$ be the homotopy class of δ. Then (α, β, γ) is *inessential* if h is trivial.

We say that (α, β, γ) is *essential* if it is not inessential, meaning that $h(\alpha, \beta, \gamma)$ is nontrivial.

Now let λ be a meridian loop (linking $\alpha \cup \gamma$ near x_0) in the knot complement X. If the commutator $[h(\alpha, \beta, \gamma), \lambda]$ is nontrivial then we say (α, β, γ) is *strongly essential*.

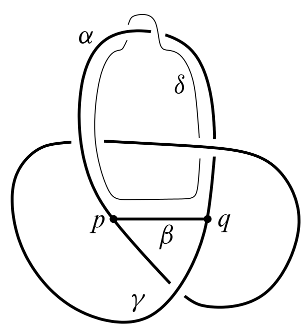

Figure 5: In this knotted Θ–graph $\alpha \cup \beta \cup \gamma$, the ordered triple (α, β, γ) is essential. To see this, we find the parallel $\delta = h(\alpha, \beta, \gamma)$ to $\alpha \cup \beta$ which has linking number zero with $\alpha \cup \gamma$, and note that it is homotopically nontrivial in the knot complement $\mathbb{R}^3 \smallsetminus (\alpha \cup \gamma)$. In this illustration, β is the straight segment $\overline{pq}$, so we equally say that the arc α of the knot $\alpha \cup \gamma$ is essential.

This notion is clearly a topological invariant of the (ambient isotopy) class of the knotted Θ–graph. For an introduction to the theory of knotted graphs, see the article by Kauffman [12]; note that since the vertices of the Θ–graph have degree three, in our situation there is no distinction between what Kauffman calls topological and rigid vertices. The three arcs approaching one vertex can be braided arbitrarily without affecting the topological type of the knotted Θ–graph.

Lemma 5.1 *In a knotted Θ–graph $\alpha \cup \beta \cup \gamma$, the triple (α, β, γ) is strongly essential if and only if (γ, β, α) is.*

Proof The homotopy classes $h = h(\alpha, \beta, \gamma)$ and $h' = h(\gamma, \beta, \alpha)$ proceed outwards from x_0 along β and then return backwards along α or γ. Note that the product $h^{-1}h'$

is homotopic to a parallel of the knot $\alpha \cup \gamma$. Since a torus has abelian fundamental group, this parallel commutes with the meridian λ. It follows that $[\lambda, h] = [\lambda, h']$. $\square$

This commutator $[\lambda, h(\alpha, \beta, \gamma)]$ that comes up in the definition of strongly essential will later be referred to as the *loop* l_β *along* β; it can be represented by a curve which follows a parallel β' of β, then loops around $\alpha \cup \gamma$ along a meridian near q, then follows β'^{-1}, then loops backwards along a meridian near p, as in Figure 6.

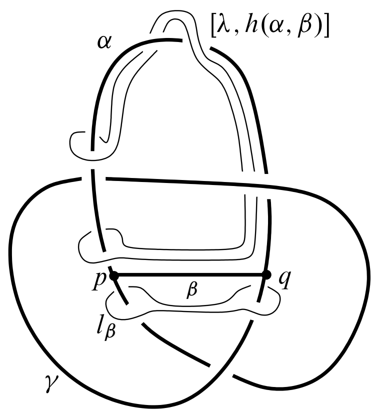

Figure 6: If λ is a meridian curve linking $\alpha \cup \gamma$, then the commutator $[\lambda, h(\alpha, \beta)]$ is homotopic to the loop l_β along β.

We apply the definition of essential to arcs of a knot as follows.

Definition If K is a knot and $a, b \in K$, let $S = \overline{ab}$. We say γ_{ab} is *(strongly) essential* in K if for every $\varepsilon > 0$ there exists some ε–perturbation of S (with endpoints fixed) to a curve S' such that $K \cup S'$ is an embedded Θ–graph in which $(\gamma_{ab}, S', \gamma_{ba})$ is (strongly) essential.

Remark 5.2 Allowing the ε–perturbation ensures that the set of essential secants is closed in the set of all secants of K, and lets us handle the case when S intersects K. We could allow the perturbation only in that case of intersection; the combing arguments of [7] show the resulting definition is equivalent. We require only that S' be ε–close to S in the C^0 sense; it thus could be locally knotted, but in the end we care only about the homotopy class h, and not an isotopy class.

In [3] it was shown that if K is an unknot, then any arc γ_{ab} is inessential. In our context, this follows immediately, because the homology and homotopy groups of $X := \mathbb{R}^3 \setminus K$ are equal for an unknot, so any curve δ having linking number zero with K is homotopically trivial in X. We can use Dehn's lemma to prove a converse statement:

Theorem 5.3 *If $a, b \in K$ and both γ_{ab} and γ_{ba} are inessential, then K is unknotted.*

Proof Let S be the secant $\overline{ab}$, perturbed if necessary to avoid interior intersections with K. We know that $\gamma_{ab} \cup S$ and $\gamma_{ba} \cup S$ bound disks whose interiors are disjoint from K. Glue these two disks together along S to form a disk D spanning K. This disk may have self intersections, but these occur away from K, which is the boundary of D. By Dehn's lemma, we can replace D by an embedded disk, hence K is unknotted. □

Definition A secant ab of K is *essential* if both subarcs γ_{ab} and γ_{ba} are essential. A secant ab is *strongly essential* if γ_{ab} (or, equivalently, γ_{ba}) is strongly essential.

To call a quadrisecant $abcd$ essential, we could follow Kuperberg and require that the secants ab, bc and cd all be essential. But instead, depending on the order type of the quadrisecant, we require this only of those secants whose length could not already be bounded as in Theorem 3.5, namely those secants whose endpoints are consecutive along the knot. That is, for simple quadrisecants, all three secants must be essential; for flipped quadrisecants the end secants ab and cd must be essential; for alternating quadrisecants, the middle secant bc must be essential. Figure 7 shows a knot with essential and inessential quadrisecants.

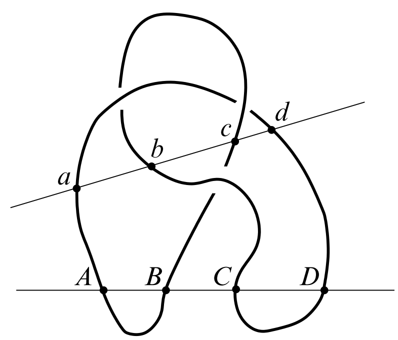

Figure 7: This trefoil knot has two quadrisecants. Quadrisecant $abcd$ is alternating and essential (meaning that bc is essential, although here in fact also ab and cd are essential). Quadrisecant $ABCD$ is simple and inessential, since AB and CD are inessential (although BC is essential).

More formally, we can give the following definition for any n–secant.

Definition An n–secant $a_1 a_2 \dots a_n$ is *essential* if we have $a_i a_{i+1}$ essential for each i such that one of the arcs $\gamma_{a_i a_{i+1}}$ and $\gamma_{a_{i+1} a_i}$ includes no other a_j.

Kuperberg introduced the notion of essential secants and showed the following result.

Theorem 5.4 *If K is a nontrivial knot parameterized by a generic polynomial, then K has an essential quadrisecant.*

As mentioned above, Kuperberg did not distinguish the different order types, and so he actually obtained a quadrisecant $abcd$ where all three segments ab, bc and cd are essential. Since this is more than we will need here, we have given our weaker definition of essential for the flipped and alternating cases, making these easier to produce.

Kuperberg used the fact that a limit of essential quadrisecants must still be a quadrisecant in order to show that every knot has a quadrisecant. Since we want an essential quadrisecant for every knot, we next need to show that being essential is preserved in such limits.

6 Limits of essential secants

Being essential is a topological property of a knotted Θ–graph. One approach to show that a limit of essential secants remains essential is to show that nearby knotted Θs are isotopic. In fact, we have pursued this approach in another paper [7] where we show that given any knotted graph of finite total curvature, any other graph which is sufficiently close (in a C^1 sense) is isotopic. (Our definition makes essential a closed condition, so the case where a secant has interior intersections with the knot, and is thus not a knotted theta, causes no trouble.)

Here, however, our knots are thick, hence $C^{1,1}$, and we give a simpler direct argument for the limit of essential secants.

Lemma 6.1 *Let K be a knot of thicknes $\tau > 0$, and K' be a C^1 knot which is close to K in the following sense: corresponding points p and p' are within distance $\varepsilon < \tau/4$ and their tangent vectors are within angle $\pi/6$. Then K and K' are ambient isotopic; the isotopy can be chosen to move each point by a distance less than ε.*

Note that the constants here are sharp within a factor of two or three: if the distance from K to K' exceeds $\tau/2$ we can have strand passage, while if the angle between tangent vectors exceeds $\pi/2$ we can have local knotting in K'.

Proof Rescale so K has unit thickness and $\varepsilon < 1/4$. Clearly K' lies within the thick tube around K. Each point $p' \in K'$ corresponds to some point $p \in K$, but also has a unique nearest point $p_0 \in K$, which is within distance $|p' - p| < \varepsilon$ of p', hence within 2ε of p. By Lemma 3.1, the arclength from p_0 to p is at most $\arcsin 2\varepsilon$, so the angle between the tangent vectors there is at most $2\arcsin 2\varepsilon < \pi/3$. The point p' is thus in the normal disk at p_0 and has tangent vector within $\pi/2$ of that at p_0. In other words, K' is transverse to the foliation of the thick tube by normal disks. Construct the isotopy from K' to K as the union of isotopies in these disks: On each disk, we move p' to p_0, coning this outwards to the fixed boundary. No other point moves further than p' which moves at most distance ε. □

Proposition 6.2 *If the $C^{1,1}$ knots K_i have essential arcs $\gamma_{a_i b_i}$, and if the K_i converge in C^1 to some thick limit knot K, with $a_i \to a$ and $b_i \to b$, then the arc γ_{ab} is essential for K.*

Proof We can reduce to the case $a_i = a$, $b_i = b$ by applying euclidean similarities (approaching the identity) to the K_i.

Given any $\varepsilon > 0$, we prove there is a 2ε–perturbation of $\overline{ab}$ making an essential knotted Θ. Then by definition, γ_{ab} is essential.

For large enough i, the knot K_i is within ε of K. Let I_i be the ambient isotopy described in Lemma 6.1 with $K = I_i(K_i)$. Since K_i is essential, by definition, we can find an ε–perturbation S_i' of S_i such that $\Theta_i := K_i \cup S_i'$ is an embedded essential Θ–graph. Setting $S_i'' := I_i(S_i')$, this is the desired 2ε perturbation of $\overline{ab}$. By definition $K_i \cup S_i'$ is isotopic to $K \cup S_i''$, so the latter is also essential. □

Corollary 6.3 *Every nontrivial $C^{1,1}$ knot has an essential quadrisecant.*

Proof Any $C^{1,1}$ knot K is a C^1–limit of polynomial knots, which can be taken to be generic in the sense of [13]. Then by Theorem 5.4 they have essential quadrisecants. Some subsequence of these quadrisecants converges to a quadrisecant for K, which is essential by Proposition 6.2. □

7 Arcs becoming essential

We showed in Corollary 6.3 that every nontrivial $C^{1,1}$ knot has an essential quadrisecant. Our aim is to find the least length of an essential arc γ_{pq} for a thick knot K, and use this to get better lower bounds on the ropelength of knots. This leads us to consider what happens when arcs change from inessential to essential.

Theorem 7.1 *Suppose γ_{ac} is in the boundary of the set of essential arcs for a knot K. (That is, γ_{ac} is essential, but there are inessential arcs of K with endpoints arbitrarily close to a and c.) Then K must intersect the interior of segment $\overline{ac}$, and in fact there is some essential trisecant abc.*

Proof If K did not intersect the interior of segment $\overline{ac}$ (in a component separate from a and c) then all nearby secant segments would form ambient isotopic Θ–graphs, and thus would all be essential. Therefore a and c are the first and third points of some trisecant abc. Let S and S' be two perturbations of $\overline{ac}$, forming Θ–graphs with K, such that $(\gamma_{ac}, S, \gamma_{ca})$ is essential but $(\gamma_{ac}, S', \gamma_{ca})$ is not. (The first exists because γ_{ac} is essential, and the second because it is near inessential arcs.)

For clarity, we first assume that b is the only point of K in the interior of $\overline{ac}$ and that S and S' differ merely by going to the two different sides of K near b, as in Figure 8. (We will return to the more general case at the end of the proof.) We will show that ab is a strongly essential secant; by symmetry the same is true of bc, and thus abc is an essential trisecant, as desired.

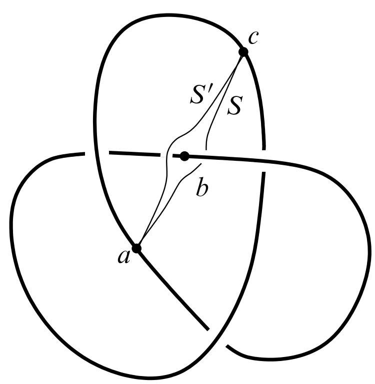

Figure 8: Secant ac is essential but some nearby secants are not. By Theorem 7.1 there must be an essential trisecant abc, because there are perturbations S and S' of $\overline{ac}$ which are essential and inessential, respectively.

By the definition of essential, the homotopy class $h := h(\gamma_{ac}, S, \gamma_{ca})$ is nontrivial in $\pi_1(\mathbb{R}^3 \smallsetminus K)$, but $h' := h(\gamma_{ac}, S', \gamma_{ca})$ is trivial. Since both have linking number zero with K, they differ not only by the meridian loop around K near b (seen in the change from S to S') but also by a meridian loop around K somewhere along the arc γ_{ac}, say near a. Let δ and δ' be the standard loops representing these homotopy classes (as in the definition of essential), and consider the subarc of δ which follows along $\overline{bc}$ and then back along γ_{ac}. The fact that δ' is null-homotopic means that this subarc is homotopic to a parallel to $\overline{ba}$. This means δ is homotopic to the loop $l_{\overline{ab}}$ along $\overline{ab}$, as in Figure 9. Thus $l_{\overline{ab}}$ represents the nontrivial homotopy class h, so by definition $\overline{ab}$ is strongly essential.

In full generality, the secant $\overline{ac}$ may intersect K in many points (even infinitely many). But still, the two fundamental group elements h and h' differ by some finite word:

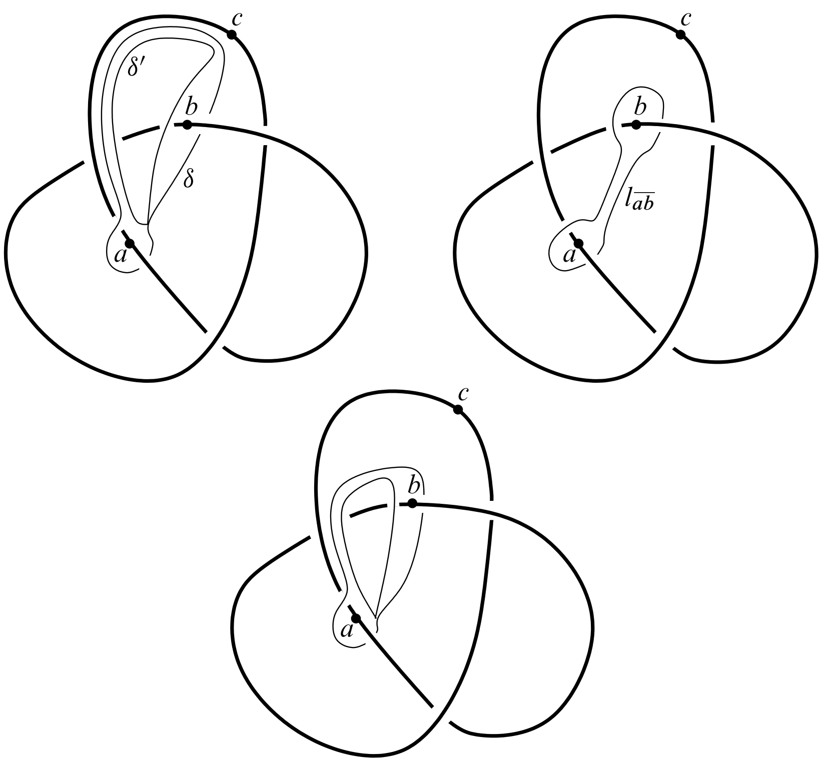

Figure 9: At the top left, we see the loops δ and δ', representing the homotopy classes h and h' arising from the essential and inessential perturbations S and S' of $\overline{ac}$ as in Figure 8. Because δ' is null-homotopic, applying the same motion to parts of δ shows it is homotopic to the loop $l_{\overline{ab}}$ (top right) along $\overline{ab}$. At the bottom we see an intermediate stage of the homotopy.

the difference between S and S' is captured by wrapping a different number of times around K at some finite number of intersection points $b_1, \dots, b_k$, as in Figure 10. In particular, for some integers n_i, we have that S wraps n_i more times around b_i than S' does. We can change from S to S' in $\sum n_i$ steps, at each step making just the simple kind of change shown in Figure 8. At (at least) one of these steps, we see a change from essential to inessential. As in the simple case above, the homotopy class h just before such a step can be represented by a loop $l_{\overline{ab_i}}$ along some segment $\overline{ab_i}$.

Note that because of the intersections (including $b_1, \dots, b_{i-1}$) of this segment with K, the loop $l_{\overline{ab_i}}$ is not *a priori* uniquely defined; it should be interpreted as wrapping around those previous intersection points the same way the current S does. With this convention, however, we see that $l_{\overline{ab_i}}$ represents the nontrivial homotopy class h. The definition of strongly essential allows arbitrary small perturbations, so again it follows

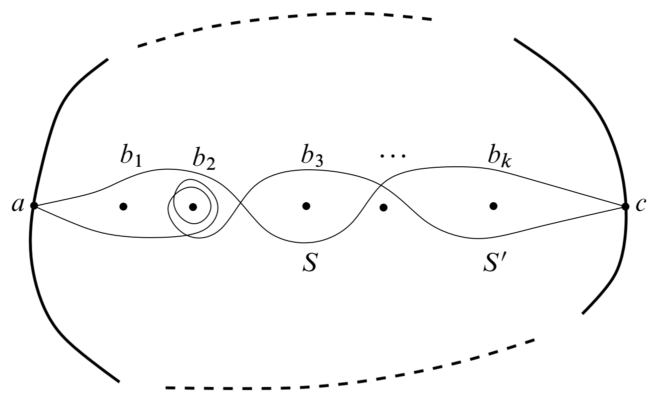

Figure 10: In case $\overline{ac}$ intersects K at many points, it still has essential and inessential perturbations S and S', and these differ by finitely many loops around intersection points b_i.

immediately that $\overline{ab_i}$ is strongly essential. By symmetry, $\overline{b_i c}$ is also strongly essential. Thus $b := b_i$ is the desired intersection point for which abc is essential. □

8 Minimum arclength for essential subarcs of a knot

We will improve our previous ropelength bounds by getting bounds on the length of an essential arc. A first bound is very easy:

Lemma 8.1 *If secant ab is essential in a knot of unit thickness then $|a-b| \geq 1$, and if arc γ_{ab} is essential then $\ell_{ab} \geq \pi$.*

Proof If $|a-b| < 1$ then by Lemma 3.1 the ball B of diameter $\overline{ab}$ contains a single unknotted arc (say γ_{ab}) of K. Now for any perturbation S of $\overline{ab}$ which is disjoint from γ_{ab}, we can span $\gamma_{ab} \cup S$ by an embedded disk within B, whose interior is then disjoint from K. This means that γ_{ab} (and thus ab) is inessential.

Knowing that sufficiently short arcs starting at any given point a are inessential, consider now the shortest arc γ_{aq} which is essential. From Theorem 7.1 there must be a trisecant apq with both secants ap and pq essential, implying by the first part that a and q are outside $B(p)$. Since ap is essential, by the definition of q we have $p \notin \gamma_{aq}$, meaning that apq is reversed. From Corollary 3.4 we get $\ell_{ab} \geq \ell_{aq} \geq \pi$. □

Intuitively, we expect an essential arc γ_{ab} of a knot to "wrap at least halfway around" some point on the complementary arc γ_{ba}. Although when $|a-b| = 2$ we can have $\ell_{ab}=\pi$, when $|a-b|<2$ we expect a better lower bound for ℓ_{ab}. Even though in fact an essential γ_{ab} might instead "wrap around" some point on itself, we can still derive the desired bound.

Lemma 8.2 *If γ_{ab} is an essential arc in a unit-thickness knot and $|a-b|<2$, then $\ell_{ab}\ge 2\pi-2\arcsin(|a-b|/2)$.*

Proof Note that $|a-b|\in[1,2]$, so $2\pi-2\arcsin(|a-b|/2)\le 5\pi/3$. As in the previous proof, let γ_{aq} be the shortest essential arc from a, and find a reversed trisecant apq. We have $b\notin\gamma_{aq}$ and $\ell_{aq}\ge\pi$, so we may assume $\ell_{qb}<2\pi/3$ or the bound is trivially satisfied.

Since γ_{qp} is essential, $\ell_{qp}\ge\pi>\ell_{qb}$, so $b\in\gamma_{qp}$. If $b\notin B(p)$ then the whole arc γ_{aqb} stays outside $B(p)$. Let Π denote the radial projection to $\partial B(p)$ as in Figure 11. From

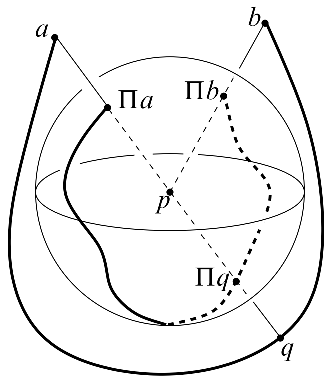

Figure 11: In the proof of Lemma 8.2, projecting γ_{ab} to the unit ball around p increases neither its length nor the distance between its endpoints. The projected curve includes antipodal points Πa and Πq, which bounds its length from below.

Lemma 3.3, this projection does not increase length. Because $|\Pi a-\Pi b|\le|a-b|$, we have $2\pi-2\arcsin(|\Pi a-\Pi b|/2)\ge 2\pi-2\arcsin(|a-b|/2)$. It therefore suffices to consider the case $\gamma_{ab}\subset\partial B(p)$. For any two points $x,y\in\partial B(p)$, the spherical distance between them is $2\arcsin(|x-y|/2)$. Thus

$$\begin{aligned}\ell_{ab}=\ell_{aq}+\ell_{qb}&\ge\pi+2\arcsin(|q-b|/2)\\&=\pi+2\arccos(|a-b|/2)=2\pi-2\arcsin(|a-b|/2).\end{aligned}$$

So we now assume that $|b-p|<1$. Let γ_{qy} be the shortest essential arc starting from q, and note $|q-y|\ge 2$. Since $\ell_{qy}\ge\pi>\ell_{qb}$ we have $b\in\gamma_{qy}$. Let $h:=|p-y|\le\ell_{yp}$

and note that $h \in [0,1]$ since $b \in B(p)$. (See Figure 12.) Since $|q-y| \geq 2$, we have $|p-q| \geq 2-h$, so $\ell_{aq} \geq \pi/2 + f(2-h)$ by Corollary 4.4. On the other hand, since $\ell_{bp} \leq \pi/2$ (by Lemma 3.1) and $\ell_{qy} \geq \pi$, we have $\ell_{qb} \geq \pi/2 + \ell_{yp} \geq \pi/2 + h$. Thus $\ell_{ab} \geq \pi + f(2-h) + h$. An elementary calculation shows that the right-hand side is an increasing function of $h \in [0,1]$, minimized at $h = 0$, where its value is $\pi + f(2) = 7\pi/6 + \sqrt{3} > 5\pi/3$. That is, we have as desired

$$\ell_{ab} \geq 5\pi/3 \geq 2\pi - 2\arcsin(|a-b|/2). \qquad \square$$

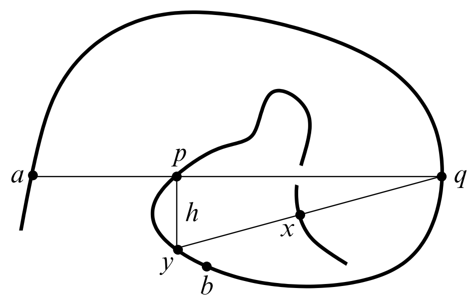

Figure 12: In the most intricate case in the proof of Lemma 8.2, we let γ_{aq} be the first essential arc from a, giving an essential trisecant apq. We then let γ_{qy} be the first essential arc from q, giving an essential trisecant qxy. Since $|x-y| \geq 1$ and $|x-q| \geq 1$, setting $h = |p-y|$ we have $|p-q| \geq 2-h$.

If we define the continuous function

$$g(r) := \begin{cases} 2\pi - 2\arcsin(r/2) & \text{if } 0 \leq r \leq 2, \\ \pi & \text{if } r \geq 2. \end{cases}$$

then we can collect the results of the previous two lemmas as:

Corollary 8.3 *If γ_{ab} is an essential arc in a knot K of unit thickness, then*

$$\ell_{ab} \geq g(|a-b|).$$

9 Main results

We now prove ropelength bounds for knots with different types of quadrisecants. The following lemma will be used repeatedly.

Lemma 9.1 *Recall that*

$$f(r) := \sqrt{r^2-1} + \arcsin(1/r),$$
$$g(r) := \begin{cases} 2\pi - 2\arcsin(r/2) & \text{for } r \le 2, \\ \pi & \text{for } r \ge 2. \end{cases}$$

Then, for $r \ge 1$,

(1) *the minimum of $f(r)$ is $\pi/2$ and occurs at $r = 1$,*

(2) *the minimum of $f(r) + g(r)$ is $7\pi/6 + \sqrt{3} > 5.397$ and occurs at $r = 2$,*

(3) *the minimum of $g(r) + r$ is $\pi + 2 > 5.141$ and also occurs at $r = 2$, and*

(4) *the minimum of $2f(r)+g(r)+r$ is just over 9.3774 and occurs for $r \approx 1.00305$.*

Proof Note that f is increasing, and g is constant for $r \ge 2$. Thus the minima will occur in the range $r \in [1,2]$, where $f' = \frac{1}{r}\sqrt{r^2-1}$ and $g' = -2/\sqrt{4-r^2}$. Elementary calculations then give the results we want, where $r \approx 1.00305$ is a polynomial root expressible in radicals. See also Figure 13. □

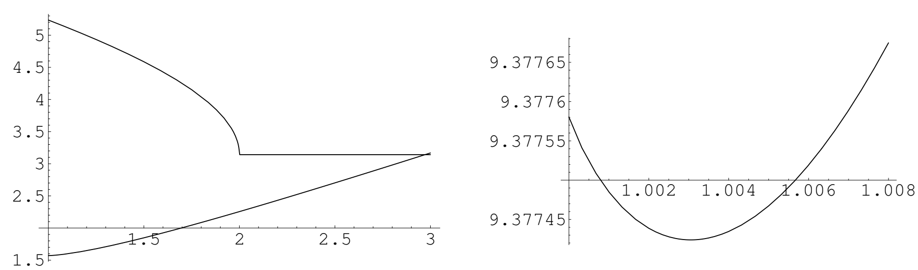

Figure 13: Left, a plot of $f(r)$ and $g(r)$ for $r \in [1,3]$, and right, a plot of $2f(r)+g(r)+r$ for $r \in [1, 1.008]$.

Theorem 9.2 *A knot with an essential simple quadrisecant has ropelength at least $10\pi/3 + 2\sqrt{3} + 2 > 15.936$.*

Proof Rescale the knot K to have unit thickness, let $abcd$ be the quadrisecant and orient K in the usual way. Then the length of K is $\ell_{ab} + \ell_{bc} + \ell_{cd} + \ell_{da}$. As before, let $r = |a-b|$, $s = |b-c|$ and $t = |c-d|$.

Corollary 8.3 bounds γ_{ab}, γ_{bc} and γ_{cd}. The quadrisecant is essential, so from Lemma 8.1 we have $r, s, t \ge 1$, and Lemma 4.5 may be applied to bound ℓ_{da}. Thus the length of K is at least

$$\begin{aligned} & g(r) + g(s) + g(t) + \big(f(r) + s + f(t)\big) \\ = \quad & \big(g(r) + f(r)\big) + \big(g(s) + s\big) + \big(g(t) + f(t)\big). \end{aligned}$$

Since this is a sum of functions in the individual variables, we can minimize each term separately. These are the functions considered in Lemma 9.1, so the minima are achieved at $r = s = t = 2$. Adding the three values together, we find the ropelength of K is at least $10\pi/3 + 2\sqrt{3} + 2 > 15.936$. □

Theorem 9.3 *A knot with an essential flipped quadrisecant has ropelength at least* $10\pi/3 + 2\sqrt{3} > 13.936$.

Proof Rescale K to have unit thickness, let $abcd$ be the quadrisecant. With the usual orientation, the length of K is $\ell_{ab} + \ell_{bd} + \ell_{dc} + \ell_{ca}$. Since the quadrisecant is essential, from Lemma 8.1 and Lemma 4.1 we have $r, s, t \geq 1$. We apply Corollary 8.3 to γ_{ab} and γ_{dc} and Corollary 4.4 to γ_{bd} and γ_{ca}.

Thus the length of K is at least

$$\begin{aligned} & g(r) + \big(f(r) + f(s)\big) + g(t) + \big(f(s) + f(t)\big) \\ = \quad & \big(g(r) + f(r)\big) + 2f(s) + \big(g(t) + f(t)\big). \end{aligned}$$

Again we minimize the terms separately using Lemma 9.1. We find the ropelength of K is at least $10\pi/3 + 2\sqrt{3} > 13.936$. □

Theorem 9.4 *A knot with an essential alternating quadrisecant has ropelength at least* 15.66.

Proof Rescale K to have unit thickness, let $abcd$ be the quadrisecant and orient K in the usual way. Then the ropelength of K is $\ell_{ac} + \ell_{cb} + \ell_{bd} + \ell_{da}$. Again, let $r = |a-b|$, $s = |b-c|$ and $t = |c-d|$.

The quadrisecant is essential, so from Lemma 8.1 and Lemma 4.2 we see $r, s, t \geq 1$. Thus Lemma 4.5 may be applied to γ_{da}. We apply Corollary 4.4 to γ_{ac} and γ_{bd}, and Corollary 8.3 to γ_{cb}.

We find that the length of K is at least

$$\begin{aligned} & \big(f(r) + f(s)\big) + \big(f(s) + f(t)\big) + g(s) + \big(f(r) + s + f(t)\big) \\ = \quad & 2f(r) + \big(2f(s) + g(s) + s\big) + 2f(t). \end{aligned}$$

Again, we can minimize in each variable separately, using Lemma 9.1. Hence the ropelength of K is at least $2\pi + 9.377 > 15.66$. □

Theorem 9.5 *Any nontrivial knot has ropelength at least* 13.936.

Proof Any knot of finite ropelength is $C^{1,1}$, so by Corollary 6.3 it has an essential quadrisecant. This must be either simple, alternating, or flipped, so one of the theorems above applies; we inherit the worst of the three bounds. □

In her doctoral dissertation [5], Denne shows:

Theorem 9.6 *Any nontrivial* $C^{1,1}$ *knot has an essential alternating quadrisecant.*

Combining this with Theorem 9.4 gives:

Corollary 9.7 *Any nontrivial knot has ropelength at least* 15.66.

We note that this bound is better even than the conjectured bound of 15.25 from [4, Conjecture 26]. We also note that our bound cannot be sharp, for a curve which is C^1 at b cannot simultaneously achieve the bounds for ℓ_{cb} and ℓ_{bd} when $s \approx 1.003$. Probably a careful analysis based on the tangent directions at b and c could yield a slightly better bound. However, we note again that numerical simulations have found trefoil knots with ropelength no more than 5% greater than our bound, so there is not much further room for improvement.

10 Quadrisecants and links

Quadrisecants may be used in a similar fashion to give lower bounds for the ropelength of a nontrivial link. For a link, n–secant lines and n–secants are defined as before. They can be classified in terms of the order in which they intersect the different components of the link. We begin our considerations with a simple lemma bounding the length of any nonsplit component of a link.

Lemma 10.1 *Suppose* L *is a link, and* A *is a component of* L *not split from the rest of the link. Then for any point* $a \in A$, *there is a trisecant* aba' *where* $a' \in A$ *but* b *lies on some other component. If* L *has unit thickness, then* A *has length at least* 2π.

Proof For the given a, the union of all secants aa' is a disk spanning A. Because A is not split from $L \smallsetminus A$, this disk must be cut by $L \smallsetminus A$, at some point b. This gives the desired trisecant. If L has unit thickness, then A stays outside $B(b)$, so as in Corollary 3.4, we have $\ell_{aa'} \geq \pi$ and $\ell_{a'a} \geq \pi$. □

This construction of a trisecant is adapted from Ortel's original solution of the Gehring link problem. (See [2].) The length bound can also be viewed as a special case of [4, Theorem 10], and immediately implies the following corollary.

Corollary 10.2 *A nonsplit link with k components has ropelength at least $2\pi k$.*

This bound is sharp in the case of the tight Hopf link, where each component has length exactly 2π.

We know that nontrivial links have many trisecants, and want to consider when they have quadrisecants. Pannwitz [16] was the first to show the existence of quadrisecants for certain links:

Proposition 10.3 *If A and B are disjoint generic polygonal knots, linked in the sense that neither one is homotopically trivial in the complement of the other, then $A \cup B$ has a quadrisecant line intersecting them in the order $ABAB$.*

Note that, when A or B is unknotted, Pannwitz's hypothesis is equivalent to having nonzero linking number. The theorem says nothing, for instance, about the Whitehead link.

Kuperberg [13] extended this result to apply to all nontrivial link types (although with no information about the order type), and for generic links he again guaranteed an essential quadrisecant:

Proposition 10.4 *A generic nontrivial link has an essential quadrisecant.*

Here, a secant of a link L is automatically essential if its endpoints lie on different components of L. If its endpoints are on the same component K, we apply our previous definition of inessential secants, but require the disk to avoid all of L, not just the component K.

Combining Kuperberg's result with Proposition 6.2 (which extends easily to links) immediately gives us:

Theorem 10.5 *Every nontrivial $C^{1,1}$ link has an essential quadrisecant.*

Depending on the order in which the quadrisecant visits the different components of the link, we can hope to get bounds on the ropelength. Sometimes, as for $ABAB$ quadrisecants, we cannot improve the bound of Corollary 10.2. Here of course, the example of the tight Hopf link (which does have an $ABAB$ quadrisecant) shows that this bound has no room for improvement.

Note that Lemma 8.1 extends immediately to links, since if a and b are on different components of a unit-thickness link we automatically have $|a-b| \geq 1$.

Theorem 10.6 *Let L be a link of unit thickness with an essential quadrisecant Q. If Q has type $AAAB$, $AABA$, $ABBA$ or $ABCA$, then the length of component A is at least $7\pi/3+2\sqrt{3}$, $8\pi/3+1+\sqrt{3}$, $2\pi+2$ or $2\pi+2$, respectively.*

Proof Suppose $a_1a_2a_3b$ is an essential quadrisecant of type $AAAB$, and set

$$r := |a_2-a_1|, \qquad s := |a_3-a_2|.$$

The quadrisecant is essential, so from Lemma 8.1 we have $r,s\geq 1$ and we may apply Corollary 4.4 to $\gamma_{a_1a_3}$ and Corollary 8.3 to $\gamma_{a_1a_2}$ and $\gamma_{a_2a_3}$. Thus the length of A is at least $f(r)+g(r)+f(s)+g(s)$. As before, we can minimize in each variable separately, using Lemma 9.1. Hence the length of A is at least $7\pi/3+2\sqrt{3}$.

Now suppose that $a_1a_2ba_3$ is an essential quadrisecant of type $AABA$, and set

$$r := |a_1-a_2|, \qquad s := |a_2-b|, \qquad t := |b-a_3|.$$

Since the quadrisecant is essential, we have $r,s,t\geq 1$ and we may apply Corollary 8.3 to $\gamma_{a_1a_2}$, Corollary 4.4 to $\gamma_{a_2a_3}$ and Lemma 4.5 to $\gamma_{a_1a_3}$. We find that the length of A is at least

$$\begin{aligned} g(r) + \big(f(s)+f(t)\big) + \big(f(r)+s+f(t)\big) \\ = \quad \big(f(r)+g(r)\big) + \big(f(s)+s\big) + 2f(t). \end{aligned}$$

Again, minimizing in each variable separately using Lemma 9.1, we find the length of A is at least $8\pi/3+1+\sqrt{3}$.

Finally suppose a_1bca_2 is an essential quadrisecant of type $ABBA$ or type $ABCA$. Because the quadrisecant is essential we must have $|a_1-b|\geq 1$, $|b-c|\geq 1$ and $|c-a_2|\geq 1$. Just as in the proof of Lemma 4.5, we find that $\ell_{a_1a_2}\geq\pi+1$ and that $\gamma_{a_2a_1}\geq\pi+1$, showing that the length of A is at least $2\pi+2$ as desired. □

Unfortunately, we do not know any link classes which would be guaranteed to have one of these types of quadrisecants, so we know no way to apply this theorem.

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