

Pro- p groups and towers of rational homology spheres

NIGEL BOSTON
JORDAN S ELLENBERG

In the preceding paper, Calegari and Dunfield exhibit a sequence of hyperbolic 3-manifolds which have increasing injectivity radius, and which, subject to some conjectures in number theory, are rational homology spheres. We prove unconditionally that these manifolds are rational homology spheres, and give a sufficient condition for a tower of hyperbolic 3-manifolds to have first Betti number 0 at each level. The methods involved are purely pro- p group theoretical.

20E18; 22E40

In [1], Calegari and Dunfield give a conditional answer to a question of Cooper [3, Problem 3.58] by exhibiting a series of hyperbolic 3-manifolds M_1, M_2, \dots , such that

- the injectivity radius of M_n is unbounded;
- subject to the Generalized Riemann Hypothesis and Langlands-type conjectures about the existence of Galois representations attached to automorphic forms, $H^1(M_n, \mathbb{Q}) = 0$ for all n .

These 3-manifolds are constructed as quotients of hyperbolic 3-space by certain arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$. In the following note, we explain how to prove unconditionally that $H^1(M_n, \mathbb{Q}) = 0$ for all n , without use of automorphic forms. The argument uses only the theory of pro- p groups (see Dixon–du Sautoy–Mann–Segal [2]). More specifically we prove that the pro- p completion of $\pi_1(M_n)$ is p -adic analytic. This should generalize to some other lattices in $\mathrm{SL}_2(\mathbb{C})$. We emphasize, however, that the present argument is not in *general* a replacement for the argument of Calegari and Dunfield; we expect there will be many hyperbolic manifolds to which the method of Galois representations might be applicable, but whose fundamental groups do not have analytic pro- p completion. In particular, it follows from results of Lubotzky [5, Theorem 1.2, Remark 1.4] that when Γ is a lattice with $\dim H^1(\Gamma, \mathbb{F}_p) \geq 4$, the pro- p completion of Γ is never analytic. On the other hand, the argument here does apply to some non-arithmetic lattices [1, Section 6.7].

Note We use number theorists' notation throughout, in which \mathbb{Z}_3 denotes the ring of 3-adic integers, not the field with 3 elements.

We recall some basic facts about cocompact lattices in $SL_2(\mathbb{C})$. Let Γ be a torsion-free cocompact lattice. Then there is a number field K (which can be taken to be the *trace field* of Γ) and a quaternion algebra A admitting an injection $\Gamma \hookrightarrow A^\times$. (See Maclachlan–Reid [7, 3.2].) For each prime \mathfrak{p} of K , let $A_{\mathfrak{p}}/K_{\mathfrak{p}}$ be the completion of A at the prime \mathfrak{p} , and write $A_{\mathfrak{p}}^1$ for the subgroup of elements of norm 1. If U is a uniformly powerful subgroup of $A_{\mathfrak{p}}^\times$, the lower p -central series $U = U_0, U_1, U_2, \dots$ is defined by $U_{i+1} = U_i^p[U, U_i]$. Write \mathcal{H} for hyperbolic 3-space; then \mathcal{H}/Γ is a compact hyperbolic 3-manifold, which is a rational homology sphere just when $H^1(\Gamma, \mathbb{Q}) = 0$.

Proposition 1 *Let Γ be a cocompact lattice of $SL_2(\mathbb{C})$ and let \mathfrak{p} be a prime of K such that*

- *the norm of \mathfrak{p} is an odd rational prime p ;*
- *the closure of the image of $\pi: \Gamma \hookrightarrow A_{\mathfrak{p}}^\times$ contains an open pro- p subgroup U of $A_{\mathfrak{p}}^1$ such that if $\Gamma_0 = \pi^{-1}(U)$, then Γ_0/Γ_0^p is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$.*

Then every normal subgroup H of Γ_0 with p -group quotient has $H^1(H, \mathbb{Q}) = 0$. In particular, taking Γ_i to be $\pi^{-1}(U_i)$, the tower of compact 3-manifolds \mathcal{H}/Γ_i ($i = 0, 1, \dots$) has unbounded injectivity radius, and each \mathcal{H}/Γ_i is a rational homology 3-sphere.

Proof The unboundedness of the injectivity radii of \mathcal{H}/Γ_i follows immediately from the fact that the Γ_i have trivial intersection.

Write T for the pro- p completion of Γ_0 . By the *dimension* of a pro- p group T , we mean the \mathbb{F}_p -dimension of $H^1(T_0, \mathbb{F}_p)$ for any uniformly powerful open subgroup T_0 of T as in Dixon–du Sautoy–Mann–Segal [2, Definition 4.7]. The fact that $T/T^p \cong \Gamma_0/\Gamma_0^p \cong (\mathbb{Z}/p\mathbb{Z})^3$ implies that T is powerful [2, Definition 3.1(i)] and has dimension at most 3 [2, Theorem 3.8]. Since U is torsion-free and has $U/U^p \cong (\mathbb{Z}/p\mathbb{Z})^3$, it is uniformly powerful [2, Theorem 4.5] and has dimension 3. Since dimension is additive in exact sequences of pro- p groups [2, Theorem 4.8]) we have that the surjection $T \rightarrow U$ has finite kernel. It is clear that every open subgroup of U has finite abelianization; the same now follows for T . This completes the proof. \square

We now explain how to show that the tower of manifolds studied in the preceding article in this volume, by Calegari and Dunfield [1] satisfies the conditions of Proposition 1. We recall some definitions and notation from [1]. Let D be the quaternion algebra over $\mathbb{Q}(\sqrt{-2})$ which is ramified precisely at the two primes π and $\bar{\pi}$ dividing 3, let B be

a maximal order of D , and let B^\times be the group of units of B . Calegari and Dunfield consider a manifold M_0 whose fundamental group is isomorphic to $B^\times / \pm 1$.

Let B_π be the maximal order in the completion of D at π ; then B_π^\times is a profinite group with a finite-index pro-3 subgroup, and the natural map $B^\times \rightarrow B_\pi^\times$ is an inclusion whose image contains a dense subgroup of the group B_π^1 of elements of reduced norm 1.

Let Q be the unique maximal two-sided ideal of B_π ; then $(1 + Q^n) \cap B_\pi^1$ is an open subgroup of B_π^1 for all $n \geq 0$, and is a pro-3 group for $n \geq 1$. Let Γ_n be the preimage of $(1 + Q^n) \cap B_\pi^1$ under $B^\times \rightarrow B_\pi^\times$. Then the content of [1, Theorem 1.4] is that Γ_n has finite abelianization for all sufficiently large n . In a slight discord of notation, the group denoted Γ_2 by Calegari and Dunfield plays the role of Γ_0 in Proposition 1. It remains only to check that Γ_2 / Γ_2^3 is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$.

A presentation of Γ_1 is obtained by using Magma to calculate the normal subgroups of index 4 in $\pi_1(M_0)$, for which a Wirtinger presentation was given in [1].

$$\begin{aligned} \text{Gamma1} := \text{Group} \langle a, b, c, d \mid & a*b^{-1}*c^{-1}*b*a^{-1}*d*c*d^{-1}, \\ & a*b*a^{-1}*d*c*d*a^{-2}*b*c, \\ & a*d*c*d*a^{-1}*b*d^{-2}*c^{-1}*b^{-1}, \\ & c*d^2*c*d^2*c*d^2, c^3, a*c*b*c*a*b*d^{-2} \rangle \end{aligned}$$

Then Γ_2 is the kernel of the map from Γ_1 to its maximal elementary abelian 3-quotient. One can easily compute a presentation of Γ_2 (too long to be worth including here) and from there it is a simple matter to compute Γ_2 / Γ_2^3 . We have thus shown that the manifolds appearing in [1] are all rational homology spheres.

Remark 2 The group Γ_2 does not have the congruence subgroup property (see Lubotzky [5]); however, one might think of Proposition 1 as asserting a kind of “pro-3 congruence subgroup property”: every finite-index normal subgroup of Γ_2 whose quotient is a 3-group is indeed congruence. It would be interesting to understand which lattices in $SL_2(\mathbb{C})$ are residual p -groups with the pro- p congruence subgroup property for some p . This property certainly cannot hold for all lattices, since there exist lattices with infinite abelianization (see Labesse–Schwermer [4] and Lubotzky [6]). For such lattices, $\dim H^1(\Gamma, \mathbb{F}_p) \leq 3$ by Lubotzky [5].

Acknowledgments

The second author was partially supported by NSF-CAREER Grant DMS-0448750 and a Sloan Research Fellowship. The authors thank the anonymous referee for his helpful comments and simplification of the proof of their main proposition.

References

- [1] **F Calegari, NM Dunfield**, *Automorphic forms and rational homology 3–spheres*, *Geom. Topol.* 10 (2006) 295–330
- [2] **JD Dixon, MPF du Sautoy, A Mann, D Segal**, *Analytic pro- p groups*, Cambridge Studies in Advanced Mathematics 61, Cambridge University Press, Cambridge (1999) MR1720368
- [3] **RC Kirby**, *Problems in low-dimensional topology*, from: “Geometric topology (Athens, GA, 1993)”, AMS/IP Stud. Adv. Math. 2, Amer. Math. Soc., Providence, RI (1997) 35–473 MR1470751
- [4] **J-P Labesse, J Schwermer**, *On liftings and cusp cohomology of arithmetic groups*, *Invent. Math.* 83 (1986) 383–401 MR818358
- [5] **A Lubotzky**, *Group presentation, p -adic analytic groups and lattices in $SL_2(\mathbb{C})$* , *Ann. of Math. (2)* 118 (1983) 115–130 MR707163
- [6] **A Lubotzky**, *Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem*, *Ann. of Math. (2)* 144 (1996) 441–452 MR1418904
- [7] **C Maclachlan, A W Reid**, *The arithmetic of hyperbolic 3–manifolds*, Graduate Texts in Mathematics 219, Springer, New York (2003) MR1937957

Department of Mathematics, University of Wisconsin
Van Vleck Hall, 480 Lincoln Drive, Madison WI 53706, USA

boston@math.wisc.edu, ellenber@math.wisc.edu

Proposed: Walter Neumann

Received: 22 November 2005

Seconded: David Gabai, Tomasz Mrowka

Revised: 11 December 2005