

## Pro- $p$ groups and towers of rational homology spheres

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In the preceding paper, Calegari and Dunfield exhibit a sequence of hyperbolic 3-manifolds which have increasing injectivity radius, and which, subject to some conjectures in number theory, are rational homology spheres. We prove unconditionally that these manifolds are rational homology spheres, and give a sufficient condition for a tower of hyperbolic 3-manifolds to have first Betti number 0 at each level. The methods involved are purely pro- $p$  group theoretical.

[20E18](#); [22E40](#)

In [1], Calegari and Dunfield give a conditional answer to a question of Cooper [3, Problem 3.58] by exhibiting a series of hyperbolic 3-manifolds  $M_1, M_2, \dots$ , such that

- the injectivity radius of  $M_n$  is unbounded;
- subject to the Generalized Riemann Hypothesis and Langlands-type conjectures about the existence of Galois representations attached to automorphic forms,  $H^1(M_n, \mathbb{Q}) = 0$  for all  $n$ .

These 3-manifolds are constructed as quotients of hyperbolic 3-space by certain arithmetic lattices in  $\mathrm{SL}_2(\mathbb{C})$ . In the following note, we explain how to prove unconditionally that  $H^1(M_n, \mathbb{Q}) = 0$  for all  $n$ , without use of automorphic forms. The argument uses only the theory of pro- $p$  groups (see Dixon–du Sautoy–Mann–Segal [2]). More specifically we prove that the pro- $p$  completion of  $\pi_1(M_n)$  is  $p$ -adic analytic. This should generalize to some other lattices in  $\mathrm{SL}_2(\mathbb{C})$ . We emphasize, however, that the present argument is not in *general* a replacement for the argument of Calegari and Dunfield; we expect there will be many hyperbolic manifolds to which the method of Galois representations might be applicable, but whose fundamental groups do not have analytic pro- $p$  completion. In particular, it follows from results of Lubotzky [5, Theorem 1.2, Remark 1.4] that when  $\Gamma$  is a lattice with  $\dim H^1(\Gamma, \mathbb{F}_p) \geq 4$ , the pro- $p$  completion of  $\Gamma$  is never analytic. On the other hand, the argument here does apply to some non-arithmetic lattices [1, Section 6.7].

**Note** We use number theorists’ notation throughout, in which  $\mathbb{Z}_3$  denotes the ring of 3-adic integers, not the field with 3 elements.

We recall some basic facts about cocompact lattices in  $SL_2(\mathbb{C})$ . Let  $\Gamma$  be a torsion-free cocompact lattice. Then there is a number field  $K$  (which can be taken to be the trace field of  $\Gamma$ ) and a quaternion algebra  $A$  admitting an injection  $\Gamma \hookrightarrow A^\times$ . (See Maclachlan–Reid [7, 3.2].) For each prime  $\mathfrak{p}$  of  $K$ , let  $A_{\mathfrak{p}}/K_{\mathfrak{p}}$  be the completion of  $A$  at the prime  $\mathfrak{p}$ , and write  $A_{\mathfrak{p}}^1$  for the subgroup of elements of norm 1. If  $U$  is a uniformly powerful subgroup of  $A_{\mathfrak{p}}^\times$ , the lower  $p$ -central series  $U = U_0, U_1, U_2, \dots$  is defined by  $U_{i+1} = U_i^p[U, U_i]$ . Write  $\mathcal{H}$  for hyperbolic 3-space; then  $\mathcal{H}/\Gamma$  is a compact hyperbolic 3-manifold, which is a rational homology sphere just when  $H^1(\Gamma, \mathbb{Q}) = 0$ .

**Proposition 1** *Let  $\Gamma$  be a cocompact lattice of  $SL_2(\mathbb{C})$  and let  $\mathfrak{p}$  be a prime of  $K$  such that*

- *the norm of  $\mathfrak{p}$  is an odd rational prime  $p$ ;*
- *the closure of the image of  $\pi: \Gamma \hookrightarrow A_{\mathfrak{p}}^\times$  contains an open pro- $p$  subgroup  $U$  of  $A_{\mathfrak{p}}^1$  such that if  $\Gamma_0 = \pi^{-1}(U)$ , then  $\Gamma_0/\Gamma_0^p$  is isomorphic to  $(\mathbb{Z}/p\mathbb{Z})^3$ .*

*Then every normal subgroup  $H$  of  $\Gamma_0$  with  $p$ -group quotient has  $H^1(H, \mathbb{Q}) = 0$ . In particular, taking  $\Gamma_i$  to be  $\pi^{-1}(U_i)$ , the tower of compact 3-manifolds  $\mathcal{H}/\Gamma_i$  ( $i = 0, 1, \dots$ ) has unbounded injectivity radius, and each  $\mathcal{H}/\Gamma_i$  is a rational homology 3-sphere.*

**Proof** The unboundedness of the injectivity radii of  $\mathcal{H}/\Gamma_i$  follows immediately from the fact that the  $\Gamma_i$  have trivial intersection.

Write  $T$  for the pro- $p$  completion of  $\Gamma_0$ . By the *dimension* of a pro- $p$  group  $T$ , we mean the  $\mathbb{F}_p$ -dimension of  $H^1(T_0, \mathbb{F}_p)$  for any uniformly powerful open subgroup  $T_0$  of  $T$  as in Dixon–du Sautoy–Mann–Segal [2, Definition 4.7]. The fact that  $T/T^p \cong \Gamma_0/\Gamma_0^p \cong (\mathbb{Z}/p\mathbb{Z})^3$  implies that  $T$  is powerful [2, Definition 3.1(i)] and has dimension at most 3 [2, Theorem 3.8]. Since  $U$  is torsion-free and has  $U/U^p \cong (\mathbb{Z}/p\mathbb{Z})^3$ , it is uniformly powerful [2, Theorem 4.5] and has dimension 3. Since dimension is additive in exact sequences of pro- $p$  groups [2, Theorem 4.8]) we have that the surjection  $T \rightarrow U$  has finite kernel. It is clear that every open subgroup of  $U$  has finite abelianization; the same now follows for  $T$ . This completes the proof.  $\square$

We now explain how to show that the tower of manifolds studied in the preceding article in this volume, by Calegari and Dunfield [1] satisfies the conditions of Proposition 1. We recall some definitions and notation from [1]. Let  $D$  be the quaternion algebra over  $\mathbb{Q}(\sqrt{-2})$  which is ramified precisely at the two primes  $\pi$  and  $\bar{\pi}$  dividing 3, let  $B$  be

a maximal order of  $D$ , and let  $B^\times$  be the group of units of  $B$ . Calegari and Dunfield consider a manifold  $M_0$  whose fundamental group is isomorphic to  $B^\times / \pm 1$ .

Let  $B_\pi$  be the maximal order in the completion of  $D$  at  $\pi$ ; then  $B_\pi^\times$  is a profinite group with a finite-index pro-3 subgroup, and the natural map  $B^\times \rightarrow B_\pi^\times$  is an inclusion whose image contains a dense subgroup of the group  $B_\pi^1$  of elements of reduced norm 1.

Let  $Q$  be the unique maximal two-sided ideal of  $B_\pi$ ; then  $(1 + Q^n) \cap B_\pi^1$  is an open subgroup of  $B_\pi^1$  for all  $n \geq 0$ , and is a pro-3 group for  $n \geq 1$ . Let  $\Gamma_n$  be the preimage of  $(1 + Q^n) \cap B_\pi^1$  under  $B^\times \rightarrow B_\pi^\times$ . Then the content of [1, Theorem 1.4] is that  $\Gamma_n$  has finite abelianization for all sufficiently large  $n$ . In a slight discord of notation, the group denoted  $\Gamma_2$  by Calegari and Dunfield plays the role of  $\Gamma_0$  in Proposition 1. It remains only to check that  $\Gamma_2 / \Gamma_2^3$  is isomorphic to  $(\mathbb{Z}/3\mathbb{Z})^3$ .

A presentation of  $\Gamma_1$  is obtained by using Magma to calculate the normal subgroups of index 4 in  $\pi_1(M_0)$ , for which a Wirtinger presentation was given in [1].

$$\begin{aligned} \text{Gamma1} := \text{Group} \langle a, b, c, d \mid & a*b^{-1}*c^{-1}*b*a^{-1}*d*c*d^{-1}, \\ & a*b*a^{-1}*d*c*d*a^{-2}*b*c, \\ & a*d*c*d*a^{-1}*b*d^{-2}*c^{-1}*b^{-1}, \\ & c*d^2*c*d^2*c*d^2, c^3, a*c*b*c*a*b*d^{-2} \rangle \end{aligned}$$

Then  $\Gamma_2$  is the kernel of the map from  $\Gamma_1$  to its maximal elementary abelian 3-quotient. One can easily compute a presentation of  $\Gamma_2$  (too long to be worth including here) and from there it is a simple matter to compute  $\Gamma_2 / \Gamma_2^3$ . We have thus shown that the manifolds appearing in [1] are all rational homology spheres.

**Remark 2** The group  $\Gamma_2$  does not have the congruence subgroup property (see Lubotzky [5]); however, one might think of Proposition 1 as asserting a kind of “pro-3 congruence subgroup property”: every finite-index normal subgroup of  $\Gamma_2$  whose quotient is a 3-group is indeed congruence. It would be interesting to understand which lattices in  $SL_2(\mathbb{C})$  are residual  $p$ -groups with the pro- $p$  congruence subgroup property for some  $p$ . This property certainly cannot hold for all lattices, since there exist lattices with infinite abelianization (see Labesse–Schwermer [4] and Lubotzky [6]). For such lattices,  $\dim H^1(\Gamma, \mathbb{F}_p) \leq 3$  by Lubotzky [5].

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## References

- [1] **F Calegari, NM Dunfield**, *Automorphic forms and rational homology 3–spheres*, *Geom. Topol.* 10 (2006) 295–330
- [2] **J D Dixon, MP F du Sautoy, A Mann, D Segal**, *Analytic pro- $p$  groups*, Cambridge Studies in Advanced Mathematics 61, Cambridge University Press, Cambridge (1999) [MR1720368](#)
- [3] **R C Kirby**, *Problems in low-dimensional topology*, from: “Geometric topology (Athens, GA, 1993)”, AMS/IP Stud. Adv. Math. 2, Amer. Math. Soc., Providence, RI (1997) 35–473 [MR1470751](#)
- [4] **J-P Labesse, J Schwermer**, *On liftings and cusp cohomology of arithmetic groups*, *Invent. Math.* 83 (1986) 383–401 [MR818358](#)
- [5] **A Lubotzky**, *Group presentation,  $p$ -adic analytic groups and lattices in  $SL_2(\mathbb{C})$* , *Ann. of Math. (2)* 118 (1983) 115–130 [MR707163](#)
- [6] **A Lubotzky**, *Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem*, *Ann. of Math. (2)* 144 (1996) 441–452 [MR1418904](#)
- [7] **C Maclachlan, A W Reid**, *The arithmetic of hyperbolic 3–manifolds*, Graduate Texts in Mathematics 219, Springer, New York (2003) [MR1937957](#)

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