Geometry & Topology 10 (2006) 335–357 **335**

Infinitely many universally tight contact manifolds with trivial Ozsváth–Szabó contact invariants

PAOLO GHIGGINI

In this article we present infinitely many 3–manifolds admitting infinitely many universally tight contact structures each with trivial Ozsváth–Szabó contact invariants. By known properties of these invariants the contact structures constructed here are non weakly symplectically fillable.

57R17; 57R57

1 Introduction

Recently Ozsváth and Szabó introduced a new isotopy invariant $c(\xi)$ for contact 3–manifolds (Y, ξ) belonging to the Heegaard Floer homology group $\widehat{HF}(-Y)$. They proved [27] that $c(\xi) = 0$ if ξ is an overtwisted contact structure, and that $c(\xi) \neq 0$ if ξ is Stein fillable. Later, they introduced also a refined version of the contact invariant denoted by $\underline{c}(\xi)$ taking values in the so-called Heegaard Floer homology group with twisted coefficients. They proved [24, Theorem 4.2] that $\underline{c}(\xi) \neq 0$ if (Y, ξ) is weakly fillable.

The Ozsváth–Szabó contact invariants have been successfully used to prove tightness for several manifolds which had resisted to any previously known technique: see the papers [19; 17; 18] by Lisca and Stipsicz. This fact raised the hope that these invariants could be non trivial for any tight contact structure. However in [6] we showed that the untwisted contact invariant reduced modulo 2 can vanish even for weakly symplectically fillable contact structures. Those examples however left open the question whether the twisted invariants were non trivial for every tight contact structures. In this article we give the following negative answer.

Theorem 1.1 *For any choice of coefficients $r_i \in (0,1) \cap \mathbb{Q}$ the Seifert manifold $M(r_1, r_2, r_3, r_4)$ defined by the surgery diagram in Figure 1 admits infinitely many pairwise non isomorphic universally tight contact structures with trivial Ozsváth–Szabó contact invariants.*

Published: 2 April 2006 DOI: 10.2140/gt.2006.10.335

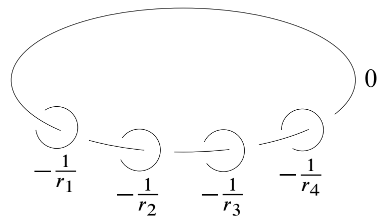


Figure 1: The surgery diagram of $M(r_1, r_2, r_3, r_4)$

Although Theorem 1.1 will be proved for the untwisted invariant, it holds for the twisted ones as well because $M(r_1, r_2, r_3, r_4)$ is a rational homology sphere for our choice of Seifert coefficients, and for rational homology spheres the twisted and the untwisted contact invariants coincide. The following corollary is therefore a consequence of the non triviality of the twisted invariant for weakly symplectically fillable contact structures [24, Theorem 4.2], and of the fact that $H_1(M(r_1, r_2, r_3, r_4), \mathbb{Q}) = 0$.

Corollary 1.2 *For any choice of coefficients $r_i \in (0,1) \cap \mathbb{Q}$ the Seifert manifold $M(r_1, r_2, r_3, r_4)$ admits infinitely many pairwise non isomorphic universally tight contact structures which are not weakly symplectically fillable.*

This corollary provides the first universally tight contact structures which are known to be non weakly symplectically fillable, therefore it answers negatively to a question [5, Question 4] of Etnyre and Ng, asking whether any universally tight contact 3–manifold has to be weakly fillable.

Remark 1.3 The non fillable contact structures constructed here are all homotopic to Stein fillable contact structures. Our construction contrasts with all previous non fillability results, whose proofs relied on homotopic properties of the contact structures.

Acknowledgements

This work was initiated while the author was visiting Princeton University supported by the NSF Focused Research Grant FRG-024466. We would like to thank András Stipsicz for several useful discussions about Heegaard Floer homology, and Steve Boyer for his help with the diffeomorphisms of Seifert manifolds. The author was partially supported by a CIRGET fellowship and by the Chaire de Recherche du Canada en algèbre, combinatoire et informatique mathématique de l'UQAM.

2 Construction of the contact structures

In this section for any natural number n we will construct a universally tight contact structure ξ_n on $M(r_1, r_2, r_3, r_4)$.

We denote by $M'(r_1, \dots, r_k)$ the Seifert manifold over D^2 with k singular fibres with Seifert coefficients $r_1, \dots, r_k$. We can decompose the manifold $M(r_1, r_2, r_3, r_4)$ as

$$M(r_1, r_2, r_3, r_4) = M'(r_1, r_2) \cup T^2 \times [-1, 1] \cup M'(r_3, r_4).$$

The orientation on $T^2 \times \{-1\}$ is given by the inward normal convention, while the orientation on $T^2 \times \{1\}$ is given by the outward normal convention, therefore $\partial M'(r_1, r_2)$ is identified to $T^2 \times \{-1\}$, and $T^2 \times \{1\}$ is identified to $-\partial M'(r_3, r_4)$.

Since our construction will not use the Seifert coefficients of the fibres in any specific way, except for the fact $r_i \in (0, 1)$, we will suppress them from the notation, and will call $M = M(r_1, r_2, r_3, r_4)$, $M_1' = M'(r_1, r_2)$, and $M_2' = M'(r_3, r_4)$.

By Hatcher [11, Proposition 2.2 and Section 1.2], M_i' is a surface bundle over S^1. Let Σ_i be the fibre, and ϕ_i be the monodromy of this bundle. The Seifert fibration on M_i' restricted to Σ_i is a branched cover $\Sigma_i \to D^2$ with finite fibre, and ϕ_i is a deck transformation, therefore ϕ_i has finite order n_i.

Call $\mathcal{C}_i$ the set of all 1–forms β over Σ_i such that

(1) $d\beta$ is a volume form on Σ_i,

(2) $\beta|_{\partial \Sigma_i}$ is a volume form on $\partial \Sigma_i$.

By Thurston–Winkelnkemper [29] the sets $\mathcal{C}_i$ are nonempty and convex. We define a ϕ_i–invariant 1–form $\overline{\beta}_i$ on Σ_i by averaging as follows: we pick any 1–form $\beta_i \in \mathcal{C}_i$ and define

$$\overline{\beta}_i = \frac{1}{n_i} \sum_{k=0}^{n_i - 1} (\phi_i^k)^* \beta.$$

By the convexity of $\mathcal{C}_i$ we have $\overline{\beta}_i \in \mathcal{C}_i$. If t is the coordinate of [0, 1], for any $K > 0$ the 1–form $dt + K\overline{\beta}_i$ is a contact form on $\Sigma_i \times [0, 1]$ which gives a well defined contact form on M_i'. By Gray's Theorem the contact structures on M_i' obtained from different choices of β_i are isotopic, while the actual value of K has no relevance for our construction. We denote by ξ_+ the kernel of $dt + K\overline{\beta}_i$, and by ξ_- the kernel of $-(dt + K\overline{\beta}_i)$. The following lemma is a straightforward computation on the contact forms.

Lemma 2.1 *$\partial M_i'$ is a prelagrangian torus with slope s_i with respect to ξ_+ and ξ_-. We can chose K so that $-2 < s_1 < -(r_1 + r_2)$ and $-2 < s_2 < -(r_3 + r_4)$. The Reeb vector fields of the contact forms $\pm(dt + K\overline{\beta}_i)$ are tangent to the fibres of the Seifert fibration of M_i', and ξ_+ and ξ_- are transverse to the Seifert fibration.*

In the following we will always assume that K has been chosen so that the inequalities in Lemma 2.1 hold.

On $T^2 \times [-1, 1]$ we consider the contact structures

$$\alpha_n(s_1, s_2) = \ker\big(\cos(\varphi_n(t))dx + \sin(\varphi_n(t))dy\big)$$

for a smooth function $\varphi_n \colon [-1, 1] \to \mathbb{R}$ such that

(1) $\varphi_n'(t) > 0$ for any $t \in [-1, 1]$,

(2) $[\frac{\varphi_n(1) - \varphi_n(-1)}{\pi}] = n$,

(3) $T^2 \times \{-1\}$ has slope s_1 and $T^2 \times \{1\}$ has slope $-s_2$,

(4) $\varphi_n(-1) \in (0, \frac{\pi}{2})$.

Condition (1) implies that $\alpha_n(s_1, s_2)$ is a contact structure, condition (2) implies that it has twisting $n\pi$ in the sense of Honda [13, Section 2.2.1], and condition (4) is simply a normalisation condition. The set of functions satisfying these conditions is convex, therefore by Gray's Theorem the isotopy type of $\alpha_n(s_1, s_2)$ depends only on n, s_1 and s_2.

Definition 2.2 We define the contact structures ξ_n on $M(r_1, r_2, r_3, r_4)$ so that they coincide with ξ_+ on M_1', with ξ_- on M_2', and with $\alpha_{2n}(s_1, s_2)$ on $T^2 \times [-1, 1]$, where s_i is the boundary slope of M_i'.

Following Colin and Honda [2] we say that a contact structure is *hypertight* if it can be defined by a contact form whose associated Reeb vector field has no contractible periodic orbits. By Hofer [12, Theorem 1] hypertight contact structures are tight.

Theorem 2.3 *The contact structures ξ_n on $M(r_1, r_2, r_3, r_4)$ are hypertight for any $n \geq 0$.*

Proof By [2, Lemma 9.1] we can isotope ξ_+ and ξ_- relative to the boundary, so that they are defined by contact 1–forms which glue to the contact form of $\alpha_{2n}(s_1, s_2)$ to give a globally defined contact form for ξ_n. Moreover, the isotopy can be chosen so that the Reeb vector fields of ξ_+ and ξ_- remain transverse to the fibrations over S^1 defined on M_1' and M_2'.

Any periodic orbit of the Reeb flow must be completely contained in one of the three pieces M_1', M_2' and $T^2\times[-1,1]$ in which M has been decomposed, because the Reeb vector field is tangent to $\partial M_1'$ and to $\partial M_2'$. Moreover, the inclusions $M_i' \hookrightarrow M$ and $T^2\times[-1,1]\hookrightarrow M$ induce injective maps between the fundamental groups, therefore a periodic orbit of the Reeb flow is contractible in M if and only it is contractible in the piece it is contained in. This implies that there are no contractible periodic orbits of the Reeb flow in M, because the Reeb vector field is transverse to the S^1–fibrations in M_i', and in $T^2\times[-1,1]$ its closed orbits are homotopically non trivial in the incompressible tori $T^2\times\{c\}$. □

In order to prove universal tightness for ξ_n we need the following lemma about the coverings of Seifert manifolds.

Lemma 2.4 *Let M be a Seifert manifold with base B, and denote its universal covering by $\widetilde{M}$. If K is a compact set of $\widetilde{M}$, then there exists a finite covering $\overline{M}$ of M for which the projection $\widetilde{M}\to\overline{M}$ is injective on K.*

Proof We have to consider only the case when the universal cover is not already a finite cover itself. Consider the orbifold structure induced on B by the Seifert fibration on M. Either M has at least three singular fibres, or the genus of B (as a surface) is $g>0$ because we have assumed that the universal cover of M is infinite. In these cases B is a good orbifold, therefore there is a finite orbifold covering $B'\to B$ such that B' has no singular points: see Scott [28, Theorem 2.3 and Theorem 2.5].

We pull back the Seifert fibration of M to B' in order to obtain a Seifert manifold M' which fibres over B' and a finite covering $M'\to M$. The Seifert fibration on M' has no singular fibres because B' has no singular points, therefore M' is a circle bundle over B'. This concludes the proof because the lemma holds trivially for circle bundles over surfaces, and the composition of finite coverings is still a finite covering. □

Corollary 2.5 *The contact structures ξ_n on $M(r_1,r_2,r_3,r_4)$ are universally tight for all $n\geq 0$.*

Proof Suppose by contradiction that the universal cover of (M,ξ_n) contains an overtwisted disc. Since the overtwisted disc is compact, by Lemma 2.4 (M,ξ_n) has an overtwisted finite cover. This is a contradiction because any finite cover of a hypertight contact manifold is hypertight again, and therefore it is tight. □

3 Decomposition of the tight contact structures

Let M be a Seifert manifold with base B and k singular fibres $F_1, \dots, F_k$, and let U_i be a standard neighbourhood of F_i, $i = 1, \dots, k$. Then $M \setminus \bigcup_{i=1}^k U_i$ can be identified with $S \times S^1$ where S is the surface obtained by removing k disjoint discs centred at the images of the singular fibres from the base B. This diffeomorphism determines identifications of $-\partial(M \setminus U_i)$ with $\mathbb{R}^2/\mathbb{Z}^2$ so that $\binom{1}{0}$ is the direction of the section $S \times \{1\}$ and $\binom{0}{1}$ is the direction of the regular fibres. In order to fix one among the infinitely many product structures on $M \setminus \bigcup_{i=1}^k U_i$ we also require the meridian of each U_i to have slope $-\frac{\beta_i}{\alpha_i}$ in $-\partial(M \setminus V_i)$ with $\frac{\beta_i}{\alpha_1} = r_1$.

We also choose an identification between ∂U_i and $\mathbb{R}^2/\mathbb{Z}^2$ so that $\binom{1}{0}$ is the direction of the meridian of U_i and $\binom{0}{1}$ is the direction of a longitude. Notice that ∂U_i and $-\partial(M \setminus U_i)$ coincide as sets, but are identified with $\mathbb{R}^2/\mathbb{Z}^2$ in different ways. We can choose the longitude on U_i so that these two identifications are related by gluing matrices $A_i \colon \partial U_i \to -\partial(M \setminus U_i)$ given by

$$A_i = \begin{pmatrix} \alpha_i & \alpha_i' \\ -\beta_i & -\beta_i' \end{pmatrix}$$

with $\beta_i \alpha_i' - \alpha_i \beta_i' = 1$ and $0 < \alpha_i' < \alpha_i$.

Lemma 3.1 *Let ξ be a tight contact structure on M, and assume that (M, ξ) has a Legendrian regular fibre L with twisting number 0 (possibly after isotopy), which means that its contact framing coincides with the framing determined by the fibration. Then for $i = 1, \dots, k$ there exist tubular neighbourhoods V_i of the singular fibres F_i so that $\partial(M \setminus V_i)$ is a convex torus in standard form with infinite slope.*

Proof Make the singular fibres F_i Legendrian with very low twisting numbers $n_i < 0$, and consider their standard neighbourhoods U_i for $i = 1, \dots, k$. Without loss of generality we can assume $L \cap U_i = \varnothing$ for $i = 1, \dots, k$. Let A_i be a convex vertical annulus between L and a Legendrian ruling curve of $\partial(M \setminus U_i)$. By the Imbalance Principle, Honda [13, Proposition 3.17], A_i produces a bypass attached to $-\partial(M \setminus U_i)$ along a vertical Legendrian ruling curve. Then using the bypass attachment Lemma [13, Lemma 3.15] we can thicken U_i until we obtain a convex solid torus V_i such that $-\partial(M \setminus V_i)$ has infinite slope. □

We call $(M \setminus \bigcup V_i, \xi|_{M \setminus \bigcup V_i})$ the *background* of (M, ξ). Since the background of (M, ξ) has infinite boundary slopes, by Honda [14, Section 4.3] it is isomorphic to an S^1–invariant tight contact structure $(S \times S^1, \xi_{\Gamma_{S_0}})$. Here $\xi_{\Gamma_{S_0}}$ denotes the S^1–invariant contact structure on $S \times S^1$ inducing the dividing set Γ_{S_0} on a convex $\#\Gamma$–minimising section $S_0 \subset S \times S^1$ with Legendrian boundary. By [14, Proposition 4.4] the isotopy class of $\xi_{\Gamma_{S_0}}$ is completely determined by Γ_{S_0}.

Let c_1 be the smallest number for which the torus $T^2 \times \{c_1\}$ in the contact manifold $(T^2 \times [-1,1], \alpha_{2n}(s_1,s_2))$ has infinite slope, and let c_2 be the biggest one. Also, let c_1' be the first number for which $T^2 \times \{c_1'\}$ has slope -2 and let c_2' be the last number for which $T^2 \times \{c_2'\}$ has slope 2. c_1' and c_2' exist because of Lemma 2.1. Call

$$\begin{aligned}
(M_{c_1'}, \widetilde{\xi}_+') &= (M_1', \xi_+) \cup (T^2 \times [-1, c_1'], \alpha_{2n}(s_1,s_2)|_{T^2 \times [-1,c_1']}) \\
(M_{c_2'}, \widetilde{\xi}_-') &= (M_2', \xi_-) \cup (T^2 \times [c_2', 1], \alpha_{2n}(s_1,s_2)|_{T^2 \times [c_2',1]}) \\
(M_{c_1}, \widetilde{\xi}_+) &= (M_1', \xi_+) \cup (T^2 \times [-1, c_1], \alpha_{2n}(s_1,s_2)|_{T^2 \times [-1,c_1]}) \\
(M_{c_2}, \widetilde{\xi}_-) &= (M_2', \xi_-) \cup (T^2 \times [c_2, 1], \alpha_{2n}(s_1,s_2)|_{T^2 \times [c_2,1]}).
\end{aligned}$$

Lemma 3.2 *$(M_{c_1'}, \widetilde{\xi}_+')$ and $(M_{c_2'}, \widetilde{\xi}_-')$ contain no Legendrian curves with twisting number 0 isotopic to regular fibres.*

Proof We prove the lemma only for $(M_{c_1'}, \widetilde{\xi}_+')$ because the proof for $(M_{c_2'}, \widetilde{\xi}_-')$ is the same.

Fix a rational number $r_3' < -2$, and consider the matrix with integral entries

$$A(r_3') = \begin{pmatrix} \alpha & \alpha' \\ -\beta & -\beta' \end{pmatrix}$$

such that $r_3' = \frac{\beta}{\alpha}$, $\alpha'\beta - \alpha\beta' = 1$, and $0 < \alpha' < \alpha$. Applying $A(r_3')^{-1}$ to $-\partial M_{c_1'}$ we obtain a prelagrangian torus with slope

$$-\frac{\beta + 2\alpha}{\beta' + 2\alpha'} = -\frac{\alpha}{\alpha' - \frac{1}{\beta + 2\alpha}}$$

which is greater than the slope of the Seifert fibration $-\frac{\alpha}{\alpha'}$ because $\frac{\beta}{\alpha} < -2$.

Put polar coordinates (ρ, θ) on $\mathbb{R}^2$, and call

$$D(\rho_0) = \{(\rho, \theta) \in \mathbb{R}^2 : \rho \leq \rho_0\}.$$

We can choose ρ_0 so that $\big(D(\rho_0) \times S^1, \ker(dz - \rho^2 d\theta)\big)$ has prelagrangian boundary and boundary slope

$$-\frac{\beta + 2\alpha}{\beta' + 2\alpha'}.$$

If we glue the tight solid torus

$$\big(D(\rho_0)\times S^1,\ \ker(dz-\rho^2 d\theta)\big)$$

to $-\partial M_{c_1'}$ by the map $A(r_3')$, we obtain a contact structure τ on $M(r_1,r_2,r_3')$. This contact structure is transverse to the Seifert fibration because the contact planes do not twist enough to become tangent to the fibres.

Transverse contact structures in Seifert manifolds are tight by Lisca–Matić [16, Corollary 2.2], therefore Wu [30, Theorem 1.4] implies that $(M(r_1,r_2,r_3'),\ \tau)$ contains no Legendrian curve with twisting number 0 isotopic to a regular fibre. Consequently, $(M_{c_1'},\ \widetilde{\xi}'_+)$ contains no such a curve either. □

Lemma 3.3 *For any $i=1,2,3,4$ there exists a tubular neighbourhood V_i' of the singular fibre F_i such that $V_1', V_2'\subset M_{c_1'}$, $V_3', V_4'\subset M_{c_2'}$, and $-\partial(M\setminus V_i')$ has slope -1. Moreover there exist collars C_1 of ∂M_{c_1} in $M_{c_1}\setminus(V_1'\cup V_2')$ and C_2 of ∂M_{c_2} in $M_{c_2}\setminus(V_3'\cup V_4')$ such that $\partial(M_{c_1}\setminus C_1)$ and $\partial(M_{c_2}\setminus C_2)$ are convex tori with slope -1.*

Proof Again, we prove the lemma only for $M_{c_1'}$. We perturb $\partial M_{c_1'}$ so that it becomes a convex torus in standard form with vertical ruling, then we make the singular fibres F_1 and F_2 Legendrian with very low twisting numbers k_1 and k_2, and take standard neighbourhoods U_i of F_i. The slopes of $-\partial(M_{c_1'}\setminus(U_1\cup U_2))$ are

$$\begin{cases} -\frac{k_1\beta_1+\beta_1'}{k_1\alpha_1+\alpha_1'} & \text{on the component corresponding to } \partial U_1,\\ -\frac{k_2\beta_2+\beta_2'}{k_2\alpha_2+\alpha_2'} & \text{on the component corresponding to } \partial U_2,\\ \quad 2 & \text{on } -\partial M_{c_1'}.\end{cases}$$

We can make $-\frac{k_i\beta_i+\beta_i'}{k_i\alpha_i+\alpha_i'}$ arbitrarily close to $-r_i\in(-1,0)$ by making k_i very large in absolute value.

Take convex vertical annuli A_i between a vertical Legendrian ruling curve of $\partial M_{c_1'}$ and a vertical Legendrian ruling curve of the component of $\partial(M_{c_1'}\setminus(U_1\cup U_2))$ corresponding to ∂U_i for $i=1,2$. By the imbalance principle of Honda [13, Proposition 3.17] A_i carries a bypass on the side of ∂U_i. Attaching this bypass we can thicken U_i and reduce the slope of the component of $-\partial(M_{c_1'}\setminus(U_1\cup U_2))$ corresponding to ∂U_i. We can repeat this procedure until we get neighbourhoods V_1' and V_2' of F_1 and F_2 such that the slopes of $-\partial(M_{c_1'}\setminus(V_1'\cup V_2'))$ are -1, -1, and 2. Denote by T_1 and T_2 the components of $-\partial(M_{c_1'}\setminus(V_1'\cup V_2'))$ corresponding to V_1' and V_2' respectively. Let B be a convex vertical annulus between vertical Legendrian ruling curves of T_1 and T_2.

The dividing set of B contains no boundary parallel dividing curves, otherwise there would be a bypass attached vertically to T_1 or T_2 by [13, Proposition 3.18], and we could use this bypass to produce a torus with infinite slope inside $M_{c'_1}$. This would be a contradiction because there are no Legendrian curves with twisting number 0 isotopic to regular fibres in $M_{c'_1}$ by Lemma 3.2. After cutting $M_{c'_1} \setminus (V'_1 \cup V'_2)$ along B and rounding the edges, by the Edge rounding lemma [13, Lemma 3.11] we get a convex torus T_3 with slope -1 isotopic to ∂M_{c_1}. The collar C_1 is bounded by T_3 and ∂M_{c_1}. □

Now we determine the isotopy type of the contact structures $\xi_n|_{V_i}$. We recall that $D^2 \times S^1$ admits exactly two universally tight contact structures with $\#\Gamma_{\partial D^2 \times S^1} = 2$ if its boundary slope is lesser than -1, and exactly one its boundary slope is -1; see Honda [13, Proposition 5.1(2)]. It follows from the computation of the relative Euler class of the universally tight contact structures on $T^2 \times [0,1]$ [13, Proposition 5.1] and from the correspondence between tight contact structures on $T^2 \times [0,1]$ and on $D^2 \times S^1$ [13, Proposition 4.15] that all the basic slices in the basic slices decomposition of a universally tight contact structures have the same sign. We define the sign of a universally tight contact structure on $D^2 \times S^1$ as the sign of any basic slices in its decomposition.

Proposition 3.4 *The contact structures $\xi_n|_{V_i}$ are universally tight for $i = 1, 2, 3, 4$. They are positive for $i = 1, 2$ and negative for $i = 3, 4$.*

Proof $(V_i, \xi_n|_{V_i})$ is universally tight because (M, ξ_n) is universally tight and the inclusions $\iota_i \colon V_i \to M$ induce injective maps $(\iota_i)_* \colon \pi_1(V_i) \to \pi_1(M)$. We can assume without loss of generality that V'_i is contained in V_i for $i = 1, 2, 3, 4$, then $V_i \setminus V'_i$ is the outermost basic slice in the decomposition of V_i. Since all basic slice in the decomposition of a universally tight contact structure have the same sign, the sign of $\xi_n|_{V_i \setminus V'_i}$ determines the sign of $\xi_n|_{V_i}$. By Ghiggini–Lisca–Stipsicz [7, Lemma 2.7] the signs of $V_1 \setminus V'_1$ and of $V_2 \setminus V'_2$ are the same as the sign of C_1, and the signs of $V_3 \setminus V'_3$ and of $V_4 \setminus V'_4$ are the same as the sign of C_2. In applying [7, Lemma 2.7] we must notice that here the sign of C_i is computed after orienting ∂M_{c_1} by the outward normal convention, while in [7, Lemma 2.7] all boundaries are oriented by the inward normal. By a direct check on $\alpha_{2n}(s_1, s_2)$ it easy to see that C_1 is positive and C_2 is negative. □

Lemma 3.5 *For $i = 1, \dots, 4$, let V_i the neighbourhood of the singular fibre F_1 obtained by applying Lemma 3.1 to $(M_{c_1}, \widetilde{\xi}_+)$ and to $(M_{c_2}, \widetilde{\xi}_-)$. Then the dividing set on a convex $\#\Gamma$–minimising section of the backgrounds of $(M_{c_1}, \widetilde{\xi}_+)$ and $(M_{c_2}, \widetilde{\xi}_-)$ has no boundary parallel dividing curves.*

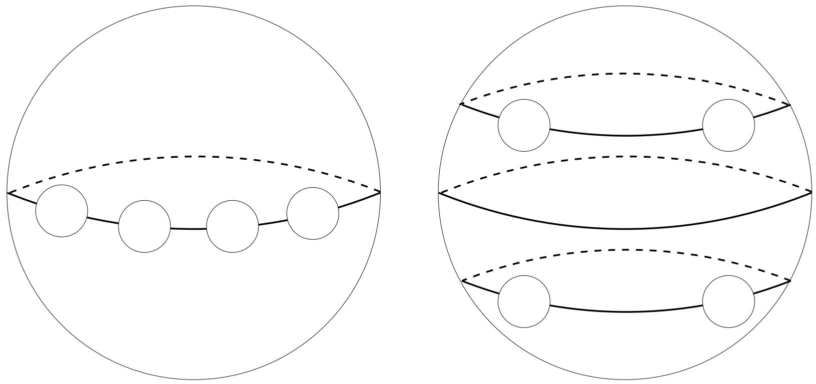


Figure 2: On the left the dividing set on a $\#\Gamma$–minimising section of the background of $(M,\, \xi_0)$. On the right the dividing set on a $\#\Gamma$–minimising section of the background of $(M,\, \xi_1)$.

Proof We prove the lemma only for $(M_{c_1},\, \widetilde{\xi}_+)$ because the proof for $(M_{c_2},\, \widetilde{\xi}_-)$ is the same.

Let $C_0 = M_{c_1} \setminus (V_1' \cup V_2' \cup C_1)$. Since $\xi_n|_{C_0} = \widetilde{\xi}_+|_{C_0}$ is a tight contact structure with boundary slopes -1 and without Legendrian curves with twisting number 0 isotopic to fibres, and $(C_1,\, \widetilde{\xi}_+)$ is a basic slice with boundary slopes -1 and ∞, we apply [7, Lemma 2.7] and conclude that the dividing set on a convex $\#\Gamma$–minimising section of $M_{c_1} \setminus (V_1 \cup V_2)$ has no boundary parallel dividing arcs. □

Proposition 3.6 *Let S be a four-punctured sphere, and let $S_0 \subset S \times S^1$ be a convex $\#\Gamma$–minimising section in the background $(S \times S^1, \xi_{\Gamma_{S_0}})$ of (M, ξ_n). Then, if we choose the neighbourhoods V_i so that V_1, V_2 are contained in M_{c_1} and V_3, V_4 are contained in M_{c_2}, the dividing set of S_0 is isotopic to one of the following:*

(1) *four dividing arcs joining the components of ∂S_0 in cyclic order, as in a necklace, if $n = 0$ (see Figure 2(a)), or*

(2) *two dividing arcs joining V_1 to V_2, two dividing arcs joining V_3 to V_4, and $2n-1$ homotopically non trivial parallel dividing curves in-between (see Figure 2(b)).*

Proof We divide S_0 into three pieces $S_0^{(1)} \subset M_{c_1}$, $S_0^{(2)} \subset T^2 \times [c_1, c_2]$, and $S_0^{(3)} \subset M_{c_2}$, so that each piece is convex with Legendrian boundary and $\#\Gamma$–minimising in its relative homology class. Here we assume that the product structure on $T^2 \times I$ has been deformed in small neighbourhoods of c_1 and c_2 so that $T^2 \times \{c_1\}$ and $T^2 \times \{c_2\}$

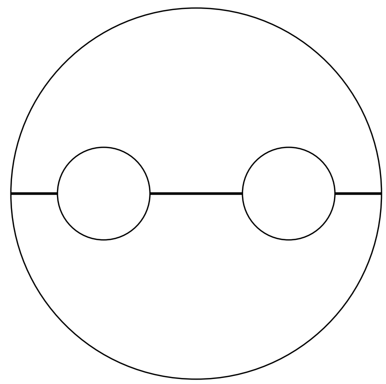


Figure 3: The dividing set on $S_0^{(1)}$ and $S_0^{(3)}$

has become convex tori with two dividing curves. If $n = 0$ $c_1 = c_2$, then we assume further that c_1 and c_2 have been replaced by $c_1 - \epsilon$ and $c_2 + \epsilon$, so that $T^2 \times [c_1, c_2]$ has become the invariant neighbourhood of a convex torus.

By Lemma 3.5 the dividing set on $S_0^{(1)}$ and $S_0^{(3)}$ consists of three arcs connecting the boundary components in pairs, as in Figure 3. If $n = 0$ $T^2 \times [c_1, c_2]$ is an invariant neighbourhood of a convex torus, therefore the dividing set on $S_0^{(2)}$ consists of two arcs connecting the two components of $\partial S_0^{(2)}$. If $n > 0$, on the other hand, by Honda–Kazez–Matić [15, Proposition 2.2] $S_0^{(2)}$ consists of a boundary parallel dividing arc for each component of $\partial S_0^{(2)}$ and of $2n-1$ closed homotopically trivial curves. To see that the number of closed dividing curves is odd we glue the boundary components of $T^2 \times [c_1, c_2]$ together by the identity map, and observe from the equation defining $\alpha_{2n}(s_1, s_2)$ that we obtain a tight contact structure on T^3. This is possible only if the boundary parallel arcs of $\Gamma_{S_0^{(2)}}$ glue to give a homotopically non trivial closed curve, and this happens only if the number of closed dividing curves on $S_0^{(2)}$ is odd. Gluing $S_0^{(1)}$, $S_0^{(2)}$, and $S_0^{(3)}$ together we obtain the dividing set on S_0. □

4 Distinguishing the contact structures

Our next goal is to prove that the contact manifolds (M, ξ_n) are pairwise non isomorphic. We will follow the line of Honda–Kazez–Matić [15, Section 4.3]. For $i = 1, 2$ the fibre bundle $M_i' \to S^1$ with fibre Σ_i defined in Section 2 extends to a fibre bundle $M_{c_i} \to S^1$ with fibre $\overline{\Sigma}_i$ containing Σ_i. We define a surface with boundary $\widehat{\Sigma} \subset M$ as follows. Identify $T^2 \times \{c_i\}$ with T^2 for $i = 1, 2$, and regard $\partial \overline{\Sigma}_i$ as a curve in T^2, then isotope $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ so that $\partial\overline{\Sigma}_1$ and $\partial\overline{\Sigma}_2$ minimise their geometric intersection and call $x_1, \dots, x_m$ the intersection points between $\partial\overline{\Sigma}_1$ and $\partial\overline{\Sigma}_2$ in T^2. Then for any intersection point x_i join $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ by a small band thickening the segment $\{x_i\} \times [c_1, c_2]$ so that the band intersects $T^2 \times \{t\}$ in a linear arc whose slope is never vertical. Call the resulting surface $\widehat{\Sigma}$.

Now take two boundary-incompressible arcs $\gamma_1 \subset \overline{\Sigma}_1$ and $\gamma_2 \subset \overline{\Sigma}_2$ with endpoints on the same intersection points, and extend $\gamma_1 \cup \gamma_2$ over the bands to get a simple closed curve $\gamma \subset \widehat{\Sigma}$. We define the framing of γ to be the one coming from $\widehat{\Sigma}$. Let $\mathcal{L}_n$ be the set of the Legendrian curves in (M, ξ_n) which are smoothly isotopic to γ, and define the *maximal twisting* $t(\mathcal{L}_n)$ to be the maximum attained by the twisting number of the curves in $\mathcal{L}_n$.

Proposition 4.1 $t(\mathcal{L}_n) = -2n + 1$.

Proof This is a corollary of [15, Proposition 4.9] once we have proved that the contact structures ξ_n are isomorphic to the contact structures ζ_{2k} defined in that article. To prove this we need to find a convex decomposition of M_{c_1} and M_{c_2} such that the dividing sets induced by $\widetilde{\xi}_+$ and $\widetilde{\xi}_-$ on the cutting surfaces are isotopic to the dividing sets induced by the contact structures constructed by Honda, Kazez and Matić [15]. This means that we want the dividing set induced by $\widetilde{\xi}_+$ and $\widetilde{\xi}_-$ on any cutting surface to be boundary parallel. We will work out the details only for $(M_{c_1}, \widetilde{\xi}_+)$, because the proof for $(M_{c_2}, \widetilde{\xi}_-)$ is the same.

We take $\overline{\Sigma}_1$ as the first cutting surface; $M_{c_1} \setminus \overline{\Sigma}_1$ is diffeomorphic to $\overline{\Sigma}_1 \times [0, 1]$, therefore we can further decompose it by cutting along discs of the form $\alpha_i \times I$, where $\{\alpha_1, \dots, \alpha_{1-\chi(\overline{\Sigma}_1)}\}$ is a set of properly embedded and pairwise disjoint arcs with boundary on $\partial\overline{\Sigma}_1$, such that $\overline{\Sigma}_1 \setminus (\alpha_1 \cup \dots \cup \alpha_{1-\chi(\overline{\Sigma}_1)})$ is disc.

We can find a convex surface with Legendrian boundary isotopic to $\overline{\Sigma}_1$ by "patching meridional discs" of V_1 and V_2 as described in Etnyre–Honda [4, proof of Proposition 3.5] or Ghiggini–Schönenberger [9, Section 4.1]. Since the contact structures on V_1 and V_2 are both universally tight and positive, the dividing sets on the meridional discs of V_1 and V_2 consist of boundary-parallel arcs cutting out regions all with the same signs, therefore when we patch the meridional discs their dividing arcs join to give boundary-parallel dividing arcs in $\overline{\Sigma}_1$.

If we choose the arc α_i disjoint from the dividing set of $\overline{\Sigma}_i$ for all i, then $\partial(\alpha_i \times [0, 1])$ intersects the dividing set of $\partial(\overline{\Sigma}_1 \times [0, 1])$ in exactly two points. When we make $\alpha_i \times [0, 1]$ convex with Legendrian boundary, its dividing set consists of exactly one arc, therefore we have obtained a convex decomposition of $(M_{c_1}, \widetilde{\xi}_+)$ of the required form. □

Corollary 4.2 *ξ_n is not isotopic to ξ_m if $n \neq m$*

Proof $t(\mathcal{L}_n)$ is clearly an isotopy invariant for ξ_n. □

Theorem 4.3 *ξ_n is not isomorphic to ξ_m if $n \neq m$*

Proof Let $\phi\colon M \to M$ be a diffeomorphism of M such that $\phi_*(\xi_n) = \xi_m$. Denote by T_0 a fibred torus in M isotopic to $\partial M_1'$ and $\partial M_2'$. From the equation defining $\alpha_{2n}(s_1, s_2)$ and $\alpha_{2m}(s_1, s_2)$ it is immediate to check that for both contact structures the isotopy class of T_0 contains a prelagrangian torus. Any other incompressible torus in M non isotopic to T_0 intersect T_0 persistently, therefore by Colin [1, Theorem 1.6] any prelagrangian torus in M is isotopic to T_0. This implies that $\phi(T_0)$ is isotopic to T_0. By Orlik [21, Theorem 8.1.7] ϕ is isotopic to a fibre-preserving diffeomorphism, therefore it defines an element $\overline{\phi}$ in the mapping class group of the base orbifold B of M. Call C_0 the projection of T_0 to B. Since $\overline{\phi}$ fixes C_0, it must be a product of Dehn twists around C_0. Let C_1 be an essential curve in B which intersects C_0 in exactly two points, and let T_1 be the pre-image of C_1 in M. T_1 is a fibred torus which splits M in two submanifolds M_l and M_r, and we may also assume without loss of generality that $F_1, F_3 \subset M_l$ and $F_2, F_4 \subset M_r$. Let ϕ' be a composition of ϕ with Dehn twists around T_0 and isotopies so that $\phi'(T_0) = T_0$ and $\phi'(T_1) = T_1$. Then we can assume that T_0 and T_1 are fixed not only as sets, but also pointwise because the action of ϕ' on the homology of T_0 and T_1 must preserve the kernels of the maps

$$H_1(T_0) \to H_1(M_1'), \quad H_1(T_0) \to H_1(M_2')$$

and

$$H_1(T_1) \to H_1(M_l), \quad H_1(T_1) \to H_1(M_r).$$

Since these kernels are linearly independent, ϕ' acts trivial on $H_1(T_0)$ and $H_1(T_1)$, therefore it is isotopic to the identity.

Since $M \setminus (T_0 \cup T_1)$ is a disjoint union of four solid tori and ϕ' fixes their boundaries, ϕ' is isotopic relative to the boundary to the identity in any of the components of $M \setminus (T_0 \cup T_1)$, therefore it is isotopic to the identity on M. This implies that ϕ is isotopic to a product of Dehn twists around T_0. We may further assume that ϕ is supported in $M \setminus (M_1' \cup M_2') \cong T^2 \times [-1, 1]$. The contact structure $\alpha_{2n}(s_1, s_2)$ is invariant up to isotopy under Dehn twists around T_0, therefore $\xi_m = \phi_*(\xi_n)$ is isotopic to ξ_n. This proves that $m = n$, because ξ_m is not isotopic to ξ_n if $m \neq n$ by Corollary 4.2. □

5 Ozsváth–Szabó contact invariants

In this section we give a brief overview of those properties of Heegaard Floer homology and of the related Ozsváth–Szabó contact invariant which will be used in this article.

Heegaard Floer homology is a family of functors introduced by Ozsváth and Szabó in [25; 26; 22] which, in their simplest form, associate finitely generated Abelian groups $\widehat{HF}(Y,\mathfrak{t})$ to any closed connected[1] oriented Spin^c 3–manifold $(Y,\mathfrak{t})$, and homomorphisms

$$\widehat{F}_{W,\mathfrak{s}}\colon \widehat{HF}(Y_1,\mathfrak{t}_1)\to \widehat{HF}(Y_2,\mathfrak{t}_2)$$

to any oriented Spin^c–cobordism $(W,\mathfrak{s})$ between two Spin^c–manifolds $(Y_1,\mathfrak{t}_1)$ and $(Y_2,\mathfrak{t}_2)$ such that $\mathfrak{s}|_{Y_i}=\mathfrak{t}_i$. If we do not need to specify the Spin^c–structure on W we write

$$\widehat{F}_W=\sum_{\substack{\mathfrak{s}\in \mathrm{Spin}^c(W)\\ \mathfrak{s}|_{Y_1}=\mathfrak{t}_1,\ \mathfrak{s}|_{Y_2}=\mathfrak{t}_2}}\widehat{F}_{W,\mathfrak{s}}.$$

This notation makes sense because $\widehat{F}_{W,\mathfrak{s}}\neq 0$ only for finitely many Spin^c–structures $\mathfrak{s}$.

A feature of Heegaard Floer homology is that, when Y is a rational homology sphere, $\chi(\widehat{HF}(Y,\mathfrak{t}))=1$ for any $\mathfrak{t}\in\mathrm{Spin}^c(Y)$, where the Euler characteristic is computed using a suitably defined $\mathbb{Z}/2\mathbb{Z}$–grading (see Ozsváth–Szabó [25, Proposition 5.1]). This implies that $\mathrm{rk}\,\widehat{HF}(Y,\mathfrak{t})\geq 1$ for any Spin^c–structure $\mathfrak{t}$.

Definition 5.1 A rational homology sphere Y is an *L–space* if $\widehat{HF}(Y,\mathfrak{t})\cong\mathbb{Z}$ for any $\mathfrak{t}\in\mathrm{Spin}^c(Y)$.

A contact structure ξ on a 3–manifold Y determines a Spin^c–structure $\mathfrak{t}_\xi$ on Y such that $c_1(\mathfrak{t}_\xi)=c_1(\xi)$. To any contact manifold (Y,ξ) we can associate an element $c(\xi)\in\widehat{HF}(-Y,\mathfrak{t}_\xi)/\pm 1$ which is an isotopy invariant of ξ, see [27]. In the following we will always abuse the notation and consider $c(\xi)$ as an element of $\widehat{HF}(-Y,\mathfrak{t}_\xi)$, although it is, strictly speaking, defined only up to sign. This abuse does not lead to mistakes as long as we do not use the additive structure on $\widehat{HF}(-Y,\mathfrak{t}_\xi)$.

Theorem 5.2 (Ozsváth–Szabó [27, Theorem 1.4 and Theorem 1.5]) *If (Y,ξ) is overtwisted, then $c(\xi)=0$. If (Y,ξ) is Stein fillable, then $c(\xi)$ is a primitive element of $\widehat{HF}(-Y,\mathfrak{t}_\xi)$.*

[1] This is the main deviation of the properties of Heegaard Floer homology from the axioms of a topological quantum field theory. We thank Tom Mrowka for pointing out this issue.

Theorem 5.3 (Ozsváth–Szabó [27, Theorem 4.2], Lisca–Stipsicz [20, Theorem 2.3]) *Suppose that (Y', ξ') is obtained from (Y, ξ) by Legendrian surgery along a Legendrian link. Then we have*

$$\widehat{F}_W(c(\xi')) = c(\xi)$$

where W is the cobordism induced by the surgery viewed as a cobordism from $-Y'$ to $-Y$.

We observe that it makes sense to write $\widehat{F}_W(c(\xi'))$ because $\widehat{F}_W$ descends to a well defined map

$$\widehat{F}_W\colon \widehat{HF}(-Y', \mathfrak{t}_{\xi'})/\pm 1 \to \widehat{HF}(-Y, \mathfrak{t}_{\xi})/\pm 1.$$

There is a also a more refined version of Heegaard Floer homology called *Heegaard Floer homology with twisted coefficients*; see Ozsváth–Szabó [25, Section 8]. Denote by $\mathbb{Z}[H^1(Y,\mathbb{Z})]$ the group ring of the first cohomology group of Y with integer coefficients. For any $\mathbb{Z}[H^1(Y,\mathbb{Z})]$–module M and for any compact connected oriented 3–manifold Y endowed with a Spin^c–structure $\mathfrak{t}$ one can define a Heegaard Floer homology group $\underline{\widehat{HF}}(Y, \mathfrak{t}; M)$. On this Heegaard Floer homology group there is a natural structure of module over $\mathbb{Z}[H^1(Y,\mathbb{Z})]$ inherited by the coefficient group M.

The contact invariant in the context of Heegaard Floer homology with twisted coefficients is denoted by $\underline{c}(\xi; M)$. It is well defined only up to multiplication by invertibles elements of $\mathbb{Z}[H^1(Y,\mathbb{Z})]$, therefore it is, properly speaking, an element of the quotient $\underline{\widehat{HF}}(Y, \mathfrak{t}; \mathbb{Z})/\mathbb{Z}[H^1(Y,\mathbb{Z})]^*$. If we consider $\mathbb{Z}[H^1(Y,\mathbb{Z})]$ as a module over itself, we write $\underline{\widehat{HF}}(Y, \mathfrak{t})$ for $\underline{\widehat{HF}}(Y, \mathfrak{t}; \mathbb{Z}[H^1(Y,\mathbb{Z})])$, and $\underline{c}(\xi)$ for $\underline{c}(\xi; \mathbb{Z}[H^1(Y,\mathbb{Z})])$. The untwisted Heegaard Floer homology group $\widehat{HF}(Y, \mathfrak{t})$ can be seen in the theory with twisted coefficients as $\underline{\widehat{HF}}(Y, \mathfrak{t}; \mathbb{Z})$, where $\mathbb{Z}$ is considered as a trivial $\mathbb{Z}[H^1(Y,\mathbb{Z})]$–module, and $c(\xi) = \underline{c}(\xi; \mathbb{Z})$.

Remark 5.4 If Y is a rational homology sphere, then $\mathbb{Z}[H^1(Y,\mathbb{Z})] = \mathbb{Z}$, therefore $\underline{\widehat{HF}}(Y, \mathfrak{t}) = \widehat{HF}(Y, \mathfrak{t})$ and $\underline{c}(\xi) = c(\xi)$.

We say that a contact manifold (Y, ξ) is *weakly symplectically fillable* if there is a symplectic manifold (X, Ω) such that $Y = \partial X$ and $\Omega|_{\xi} > 0$. The Ozsváth–Szabó contact invariant with twisted coefficients is non trivial for weakly fillable contact manifolds. More precisely, consider the $\mathbb{Z}[H^1(Y,\mathbb{Z})]$–module $\mathbb{Z}[\mathbb{R}]$ generated over $\mathbb{Z}$ by the elements T^r, where T is a formal variable and r is any real number. A closed 2–form ω on Y endows $\mathbb{Z}[\mathbb{R}]$ with the $H^1(Y,\mathbb{Z})$–action $\gamma \cdot T = T^{\langle \gamma \cup [\omega], Y \rangle}$. Denote by $\underline{\widehat{HF}}(Y, \mathfrak{t}; [\omega])$ the Heegaard Floer homology group of $(Y, \mathfrak{t})$ with twisted coefficients in $\mathbb{Z}[\mathbb{R}]$ with the module structure defined by ω.

Theorem 5.5 (Ozsváth–Szabó [24, Theorem 4.2]) *Let (Y,ξ) be a weakly fillable contact 3–manifold, and let (X,Ω) be one of its weak fillings. Call $\omega = \Omega|_Y$, then $\underline{c}(\xi,[\omega])$ is a primitive element of $\underline{\widehat{HF}}(-Y,\mathfrak{t}_\xi;[\omega])$.*

Remark 5.6 If ω is exact (in particular, if Y is a rational homology sphere), then $\underline{\widehat{HF}}(Y,\mathfrak{t}_\xi;[\omega]) \cong \widehat{HF}(Y,\mathfrak{t}_\xi)\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{R}]$ and $\underline{c}(\xi,[\omega]) = c(\xi)\otimes 1$.

In the following we will use only the untwisted invariant because we are concerned only with rational homology spheres. We have made this excursion into Heegaard Floer homology with twisted coefficients only to show that we are not losing any generality by considering only the untwisted invariant in this article.

6 Computation of the Ozsváth–Szabó invariants

We construct contact structures ζ_n on $M(r_1,r_2,r_3)$ for $n\geq 0$ so that we obtain $(M(r_1,r_2,r_3,r_4),\ \xi_n)$ from $(M(r_1,r_2,r_3),\ \zeta_n)$ by a sequence of negative Legendrian surgeries.

The contact manifold $(M\setminus V_4,\ \xi_n|_{M\setminus V_4})$ has infinite boundary slope for any $n\geq 0$. Let λ be the unique tight contact structure with infinite boundary slope on the solid torus $D^2\times S^1$. Then $(M(r_1,r_2,r_3),\ \zeta_n)$ is obtained by gluing $(D^2\times S^1,\lambda)$ to $(M\setminus V_4,\ \xi_n|_{M\setminus V_4})$ by a map $\partial D^2\times S^1\to -\partial(M\setminus V_4)$ represented by the identity matrix in the bases described in Section 3.

$\partial D^2\times S^1\to -\partial(M\setminus V_4)$

Proposition 6.1 *All the contact structures ξ_n are homotopic to ξ_0. All the contact structures ζ_n are homotopic to ζ_0.*

Proof All the contact structures ξ_n coincide with $\widetilde{\xi}_+$ on M_{c_1}, and with $\widetilde{\xi}_-$ on M_{c_2}, therefore we need to show that they are homotopic relative to the boundary on $M\setminus(M_{c_1}\cup M_{c_2})$. We recall that $\xi_n|_{M\setminus(M_{c_1}\cup M_{c_2})}$ is a perturbation of $\alpha_{2n}(s_1,s_2)|_{T^2\times[c_1,c_2]}$ near to the boundary in order to make it convex. If we choose the functions φ_{2n} so that they coincide on $[-1,c_1+\epsilon]$ and on $[c_2-\epsilon,1]$, and do the perturbations in $[-1,c_1+\frac{\epsilon}{2}]\cup[c_2-\frac{\epsilon}{2},1]$, we only need to show that all α_{2n} are homotopic through a homotopy which is constant in $[-1,c_1+\epsilon]\cup[c_2-\epsilon,1]$.

In order to construct such a homotopy, take a cut-off function β which is 0 on the union $[-1,c_1+\frac{\epsilon}{2}]\cup[c_2-\frac{\epsilon}{2},1]$ and 1 on $[c_1+\epsilon,c_2-\epsilon]$. For any n_0 and n_1 we can take a number $K \gg 0$ which is big enough so that, for any $\lambda \in [0,3]$, the kernel of the 1–form H_λ defined by

$$H_\lambda = \begin{cases} \lambda K\beta(t)dt + \big(\cos(\varphi_{2n_0}(t))dx + \sin(\varphi_{2n_0}(t))dy\big) \quad \text{if} \quad \lambda \in [0,1] \\ \\ K\beta(t)dt + (2-\lambda)\big(\cos(\varphi_{2n_0}(t))dx + \sin(\varphi_{2n_0}(t))dy\big) + \\ (\lambda-1)\big(\cos(\varphi_{2n_1}(t))dx + \sin(\varphi_{2n_1}(t))dy\big) \quad \text{if} \quad \lambda \in [1,2] \\ \\ (3-\lambda)K\beta(t)dt + \big(\cos(\varphi_{2n_1}(t))dx + \sin(\varphi_{2n_1}(t))dy\big) \;\text{ if }\; \lambda \in [2,3] \end{cases}$$

is a tangent 2–plane field. A smoothing of H_λ provides the wanted homotopy. The homotopy between ζ_{n_0} and ζ_{n_1} follows at once because (N, ζ_{n_0}) and (N, ζ_{n_1}) are obtained by modifying (M, ξ_{n_0}) and (M, ξ_{n_1}) in M_{c_2} where the homotopy H_λ is constant. □

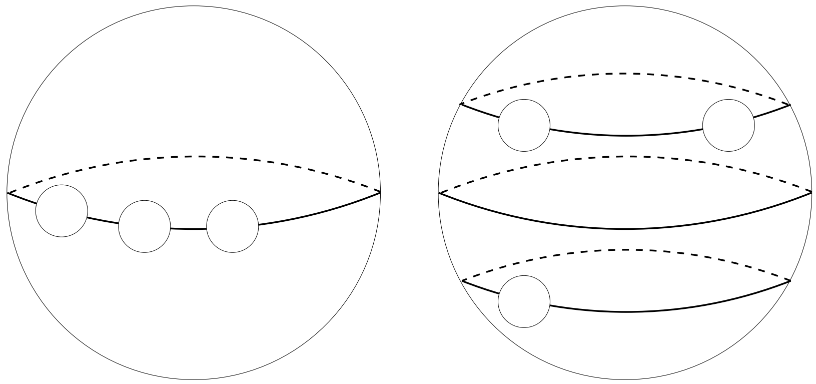
Proposition 6.2 *$(M(r_1,r_2,r_3,r_4), \xi_n)$ is obtained from $(M(r_1,r_2,r_3), \zeta_n)$ by a sequence of Legendrian surgeries.*

Proof $(D^2 \times S^1, \lambda)$ is diffeomorphic to a standard neighbourhood of a Legendrian curve with twisting number 0. Moreover the core of $D^2 \times S^1$ is isotopic in $M(r_1,r_2,r_3)$ to a regular fibre, therefore $(M \setminus V_4, \xi_n|_{M\setminus V_4})$ is the complement of the standard neighbourhood of a regular fibre with twisting number 0 in $M(r_1,r_2,r_3)$. This implies that $(M(r_1,r_2,r_3,r_4), \xi_n)$ is obtained from $(M(r_1,r_2,r_3), \zeta_n)$ by rational contact surgery. Since we have performed the surgery on a Legendrian curve with twisting number 0, the contact surgery coefficient is equal to the smooth surgery coefficient, which is $-\frac{1}{r_4} < 0$. By Ding–Geiges [3, Proposition 3] any contact surgery with negative coefficient can be expanded into a sequence of Legendrian surgeries. □

Proposition 6.3 *$(M(r_1,r_2,r_3), \zeta_0)$ is Stein fillable. $(M(r_1,r_2,r_3), \zeta_n)$ is overtwisted for $n \geq 1$.*

Proof The background of (N, ζ_n) is isomorphic to a S^1–invariant contact structure in $S' \times S^1$ where S' is a three-punctured sphere. A convex $\#\Gamma$–minimising section S_0' of the background of (N, ζ_n) is obtained by gluing a meridional disc of $(D^2 \times S^1, \lambda)$ to S_0 along the component of ∂S_0 corresponding to V_4. The dividing set of S_0' consists of

(1) three arcs joining the boundary components of $\partial S_0'$ in pairs when $n = 0$ (Figure 4(a)), or



(a) (b)

Figure 4: On the left the dividing set on a $\#\Gamma$–minimising section of the background of $(N,\ \zeta_0)$. On the right the dividing set on a $\#\Gamma$–minimising section of the background of $(N,\ \zeta_1)$.

(2) two arcs joining two boundary components, one arc with both endpoints on the third boundary component, and $2n-1$ closed curves parallel to the third boundary component when $n \geq 1$ (Figure 4(b)).

We consider ζ_0 first. It has been proved by Ghiggini, Lisca and Stipsicz [8] that any tight contact structure on $N = M(r_1, r_2, r_3)$ which is isomorphic to the background of (N, ζ_0) in the complement of tubular neighbourhoods of the singular fibres is Stein fillable independently of its restrictions to the neighbourhoods of the singular fibres, provided that the restrictions are tight.

Now we consider ζ_n for $n \geq 1$. The dividing arc with both endpoints on $-\partial(N \setminus V_3)$ produces a singular bypass on S_0' by Honda [13, Proposition 3.18]. By [13, Lemma 3.15] attaching this bypass to $-\partial(N \setminus V_3)$ we thicken V_3 to V_3' so that $-\partial(M \setminus V_3')$ has slope 0.

Take a point p belonging to another dividing curve of S_0', then $\{p\}\times S^1$ is a Legendrian fibre with twisting number 0 because $\zeta_n|_{N\setminus(V_1\cup V_2\cup V_3)}$ is S^1–invariant by Honda [14, Section 4.3]. Applying the Imbalance principle [13, Proposition 3.17] we use this curve to find a vertical bypass attached to $\partial(N \setminus V_3')$. The attachment of this bypass gives a further thickening of V_3' to V_3'' so that $-\partial(N \setminus V_3'')$ has infinite boundary slope again. By [13, Proposition 4.16] there is a standard torus with slope $-r_3$ in $V_3'' \setminus V_3$. This torus produces an overtwisted disc in (N, ζ_n). □

Corollary 6.4 $(M(r_1, r_2, r_3, r_4),\ \xi_0)$ *is Stein fillable.*

Theorem 6.5 $c(\xi_n) = 0$ *for* $n \geq 1$.

Proof The contact structures ξ_n are homotopic for $n \geq 0$ by Proposition 6.1, in particular they determine the same Spin^c–structure on M. The same is true for the contact structures ζ_n on N. Let $\mathfrak{t}_\xi$ denote the Spin^c–structure on M determined by the ξ_n's, and let $\mathfrak{t}_\zeta$ denote the Spin^c–structure on N determined by the ζ_n's.

The surgery links from Proposition 6.2 are all smoothly isotopic independently of the contact structure, therefore they determine the same smooth cobordism W from $-M$ to $-N$. This implies that for any $n \geq 0$ we have $\widehat{F}_W(c(\xi_n)) = c(\zeta_n)$. Both $-M$ and $-N$ are L–spaces by Ozsváth–Szabó [23, Lemma 2.6], hence $\widehat{HF}(-M, \mathfrak{t}_\xi)$ is generated by $c(\xi_0)$ and $\widehat{HF}(-N, \mathfrak{t}_\zeta)$ is generated by $c(\zeta_0)$ because ξ_0 and ζ_0 are Stein fillable by Corollary 6.4 and Proposition 6.3. This implies that $\widehat{F}_W$ is injective. $c(\zeta_n) = 0$ when $n \geq 1$ because ζ_n is overtwisted by Proposition 6.3, therefore the injectivity of $\widehat{F}_W$ implies that $c(\xi_n) = 0$. □

$c(\zeta_n) = 0$ when $n \geq 1$ because ζ_n is overtwisted by 6.3, therefore

Corollary 6.6 $(M(r_1, r_2, r_3, r_4), \xi_n)$ *is not weakly symplectically fillable for* $n \geq 1$.

7 Further examples

More examples of tight contact manifolds with trivial contact invariants can be constructed by Legendrian surgery on $(M(r_1, r_2, r_3), \xi_n))$ with the help of the following tightness criterion, which was implicitly proved in Hofer's work on the Weinstein conjecture for overtwisted contact structures [12].

Proposition 7.1 *Let* (Y, ξ) *be a hypertight contact manifold. Then any contact manifold* (Y', ξ') *obtained by Legendrian surgery on* (Y, ξ) *is tight.*

Sketch of the proof Pick a contact form α on Y whose Reeb flow has no contractible periodic orbits. The Legendrian surgery defines an exact symplectic cobordism $(W, d\lambda)$ from (Y, ξ) to (Y', ξ') such that $\lambda|_Y = \alpha$. Extend W to a non compact exact symplectic manifold $(\widehat{W}, d\widehat{\lambda})$ by gluing $(Y \times (-\infty, 0], d(e^t\alpha))$ to $Y \subset W$.

Suppose that (Y', ξ') is overtwisted, then there is a 2–sphere S embedded in Y' whose characteristic foliation contains a closed periodic orbit and has an elliptic point in each connected component of the complement of the periodic orbit as unique singularities. Now we proceed as in [12]: we start filling S by a Bishop family of holomorphic discs originating from an elliptic singularity of the characteristic foliation of S. Since S cannot be filled completely because of the periodic orbit in the characteristic foliation, we end up with a contractible periodic orbit of the Reeb flow associated to α exactly like in the case of the symplectisation, contradicting our hypothesis. □

Isotope the singular fibres F_i to Legendrian curves with twisting number -1 in $(M(r_1, r_2, r_3, r_4), \xi_n)$. Denote by $(M(r_1'', r_2'', r_3'', r_4''), \xi_n'')$ a contact manifold obtained by contact surgery on the singular fibres F_i with smooth surgery coefficient r_i'. From well known properties of rational surgery and Seifert invariants we have $-\frac{1}{r_i''} = -\frac{1}{r_1} - \frac{1}{r_1'}$, therefore $r_i'' = \frac{r_i r_i'}{r_i + r_i'}$. Observe that, in general, ξ_n'' is not uniquely determined by the surgery coefficients, however we will not care about this fact because our argument is independent of the choices determining ξ_n''.

If $r_i' < -1$ by Ding–Geiges [3, Proposition 3] any contact surgery on F_i with smooth surgery coefficient r_i' can be realised as a Legendrian surgery on a Legendrian link in $(M(r_1, r_2, r_3, r_4), \xi_n)$, and ξ_n'' is determined by the rotation numbers of the components of the Legendrian surgery link. By Proposition 7.1 the resulting contact manifold $(M(r_1'', r_2'', r_3'', r_4''), \xi_n'')$ is tight.

The proof of Theorem 6.5 goes through for this more general family of contact manifolds, therefore we have the following theorem.

Theorem 7.2 *If $r_i' < -1$ and $n > 0$, for any choice of the rotation numbers of the components of the Legendrian surgery link defining ξ_n'' we have $c(\xi_n'') = 0$.*

Most of the contact structures ξ_n'' are virtually overtwisted. To prove this we apply Lemma 3.1 to $(M(r_1'', r_2'', r_3'', r_4''), \xi_n'')$ in order to find tubular neighbourhoods V_i of the singular fibres, and observe that the restriction of ξ_n'' to V_i may contain basic slices with different signs, depending on the choices made in the contact surgery. In those cases $\xi_n''|_{V_i}$, and therefore ξ_n'', is virtually overtwisted.

An alternative way to prove that most contact structures ξ_n'' are virtually overtwisted is by applying Gompf's trick [10, Proposition 5.1] to the Legendrian surgery diagrams for $(M(r_1'', r_2'', r_3'', r_4''), \xi_n'')$ having components with non maximal rotation number in order to find an overtwisted disc in a finite cover.

The proof of Proposition 4.1 does not extend immediately to ξ_n'' because the contact structures considered by Honda–Kazez–Matić [15] are universally tight, therefore we are not able to prove that they are all distinct. However we believe that ξ_n'' and ξ_m'' are non isomorphic if $m \neq n$, and that two contact structures on $M(r_1'', r_2'', r_3'', r_4'')$ obtained by Legendrian surgery on $(M(r_1, r_2, r_3, r_4), \xi_n)$ are non isotopic if they are obtained by surgery on Legendrian links with different rotation numbers.

8 Final considerations

Giroux has introduced the a topological invariant for 3–dimensional contact manifolds defined as follows.

Definition 8.1 Let ξ be a contact structure on a 3–manifold Y. The *torsion* of (Y, ξ) is the supremum of the integers $n \geq 1$ for which there exists a contact embedding

$$\big(T^2 \times [0,1], \ker(\cos(2n\pi z)dx - \sin(2n\pi z)dy)\big) \hookrightarrow (Y, \xi).$$

We declare the torsion of (Y, ξ) to be 0 if no such an embedding exists.

We denote the torsion of (Y, ξ) by $\mathrm{Tor}(Y, \xi)$. One can deduce from Theorem 4.3 that $\mathrm{Tor}(M, \xi_n) = n$, therefore Corollary 6.6 adds further evidence to the following conjecture, which the author learnt from Eliashberg.

Conjecture 8.2 If $\mathrm{Tor}(Y, \xi) > 0$ then (Y, ξ) is not strongly symplectically fillable.

It seems also plausible that the following stronger statement holds.

Conjecture 8.3 If $\mathrm{Tor}(Y, \xi) > 0$ then $c(\xi) = 0$.

Conjecture 8.3 implies Conjecture 8.2 because a strongly symplectically fillable contact manifold has non trivial contact invariant by Remark 5.6. On the other hand there are contact manifolds with positive torsion which are weakly fillable, therefore they have non trivial twisted contact invariant.

We are able to define another family of universally tight contact structures η_n on $M(r_1, r_2, r_3, r_4)$ for $n \geq 0$ coinciding with ξ_+ on $M'(r_1, r_2)$ and on $M'(r_3, r_4)$, and with $\alpha_{2n+1}(s_1, s_2)$ on $T^2 \times [-1, 1]$. We observe that $\mathrm{Tor}(M, \eta_n) = n$. Since all the example of tight contact structures with trivial Ozsváth–Szabó invariants we know at present have positive torsion, it would be interesting to compute $c(\eta_0)$. Unfortunately the strategy adopted in this article fails for computing $c(\eta_n)$, because we cannot show that the η_n's are homotopic to Stein fillable contact structures.

References

[1] **V Colin**, *Sur la torsion des structures de contact tendues*, Ann. Sci. École Norm. Sup. (4) 34 (2001) 267–286 MR1841879

[2] **V Colin**, **K Honda**, *Constructions contrôlées de champs de Reeb et applications* arXiv:math.GT/0411640

[3] **F Ding**, **H Geiges**, *A Legendrian surgery presentation of contact 3–manifolds*, Math. Proc. Cambridge Philos. Soc. 136 (2004) 583–598 MR2055048

[4] **J B Etnyre**, **K Honda**, *Tight contact structures with no symplectic fillings*, Invent. Math. 148 (2002) 609–626 MR1908061

[5] **J B Etnyre**, **L L Ng**, *Problems in low dimensional contact topology*, from: "Topology and geometry of manifolds (Athens, GA, 2001)", Proc. Sympos. Pure Math. 71, Amer. Math. Soc., Providence, RI (2003) 337–357 MR2024641

[6] **P Ghiggini**, *Ozsváth–Szabó invariants and fillability of contact structures*, Math. Z. 253 (2006) 159–175

[7] **P Ghiggini**, **P Lisca**, **A I Stipsicz**, *Classification of tight contact structures on some small Seifert 3-manifolds* arXiv:math.SG/0509714

[8] **P Ghiggini**, **P Lisca**, **A I Stipsicz**, *Classification of tight contact structures on small Seifert 3–manifolds with $e_0 \geq 0$*, Proc. Amer. Math. Soc. 134 (2006) 909–916 MR2180909

[9] **P Ghiggini**, **S Schönenberger**, *On the classification of tight contact structures*, from: "Topology and geometry of manifolds (Athens, GA, 2001)", Proc. Sympos. Pure Math. 71, Amer. Math. Soc., Providence, RI (2003) 121–151 MR2024633

[10] **R E Gompf**, *Handlebody construction of Stein surfaces*, Ann. of Math. (2) 148 (1998) 619–693 MR1668563

[11] **A Hatcher**, *Notes on Basic 3–Manifold Topology*

[12] **H Hofer**, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Invent. Math. 114 (1993) 515–563 MR1244912

[13] **K Honda**, *On the classification of tight contact structures. I*, Geom. Topol. 4 (2000) 309–368 MR1786111

[14] **K Honda**, *On the classification of tight contact structures. II*, J. Differential Geom. 55 (2000) 83–143 MR1849027

[15] **K Honda**, **W H Kazez**, **G Matić**, *Convex decomposition theory*, Int. Math. Res. Not. (2002) 55–88 MR1874319

[16] **P Lisca**, **G Matić**, *Transverse contact structures on Seifert 3–manifolds*, Algebr. Geom. Topol. 4 (2004) 1125–1144 MR2113899

[17] **P Lisca**, **A I Stipsicz**, *Ozsváth–Szabó invariants and tight contact three-manifolds II* arXiv:math.SG/0404136

[18] **P Lisca**, **A I Stipsicz**, *Ozsváth–Szabó invariants and tight contact three-manifolds III* arXiv:math.SG/0505493

[19] **P Lisca**, **A I Stipsicz**, *Ozsváth–Szabó invariants and tight contact three-manifolds I*, Geom. Topol. 8 (2004) 925–945 MR2087073

[20] **P Lisca**, **A I Stipsicz**, *Seifert fibered contact three-manifolds via surgery*, Algebr. Geom. Topol. 4 (2004) 199–217 MR2059189

[21] **P Orlik**, *Seifert manifolds*, Springer, Berlin (1972) MR0426001

[22] **P Ozsváth**, **Z Szabó**, *Holomorphic triangles and invariants for smooth four-manifolds* arXiv:math.SG/0110169

[23] **P Ozsváth**, **Z Szabó**, *On the Floer homology of plumbed three-manifolds*, Geom. Topol. 7 (2003) 185–224 MR1988284

[24] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and genus bounds*, Geom. Topol. 8 (2004) 311–334 MR2023281

[25] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2) 159 (2004) 1159–1245 MR2113020

[26] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) 159 (2004) 1027–1158 MR2113019

[27] **P Ozsváth**, **Z Szabó**, *Heegaard Floer homology and contact structures*, Duke Math. J. 129 (2005) 39–61 MR2153455

[28] **P Scott**, *The geometries of 3–manifolds*, Bull. London Math. Soc. 15 (1983) 401–487 MR705527

[29] **W P Thurston**, **H E Winkelnkemper**, *On the existence of contact forms*, Proc. Amer. Math. Soc. 52 (1975) 345–347 MR0375366

[30] **H Wu**, *Legendrian vertical circles in small Seifert spaces* arXiv:math.GT/0310034

CIRGET, Université du Québec à Montréal, Case Postale 8888, succursale Centre-Ville
Montréal (Québec) H3C 3P8, Canada

ghiggini@math.uqam.ca

Proposed: Peter Ozsváth Received: 4 November 2005
Seconded: Tomasz Mrowka, Ronald Stern Accepted: 26 December 2005