Geometry & Topology 10 (2006) 715–736

Dynamics of the mapping class group action on the variety of $\mathrm{PSL}_2\mathbb{C}$ characters

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We study the action of the mapping class group $\mathrm{Mod}(S)$ on the boundary $\partial\mathcal{Q}$ of quasifuchsian space $\mathcal{Q}$. Among other results, $\mathrm{Mod}(S)$ is shown to be topologically transitive on the subset $\mathcal{C} \subset \partial\mathcal{Q}$ of manifolds without a conformally compact end. We also prove that any open subset of the character variety $\mathcal{X}(\pi_1(S), \mathrm{SL}_2\,\mathbb{C})$ intersecting $\partial\mathcal{Q}$ does not admit a nonconstant $\mathrm{Mod}(S)$–invariant meromorphic function. This is related to a question of Goldman.

57M50; 58D27

1 Introduction

Let S be a closed oriented surface of genus $g \geq 2$ and let Γ be its fundamental group. The mapping class group

$$\mathrm{Mod}(S) = \mathrm{Diff}(S)/\,\mathrm{Diff}_0(S) = \mathrm{Out}(\Gamma)$$

of S acts on the character variety

$$\mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C}) = \mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbb{C}) /\!\!/ \mathrm{PSL}_2\,\mathbb{C}$$

by precomposition. Quasifuchsian space $\mathcal{Q}$ is the open cell in the character variety $\mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ formed by the conjugacy classes of faithful representations with convex cocompact image. It is invariant under the mapping class group, and the action of $\mathrm{Mod}(S)$ on $\mathcal{Q}$ is properly discontinuous. On the other hand, our first result shows that the action of $\mathrm{Mod}(S)$ on $\partial\mathcal{Q}$ has very complicated dynamics.

Theorem 1.1 *Let $\mathcal{C} \subset \overline{\mathcal{Q}}$ denote the set of representations whose quotient manifold has no conformally compact end and let $\overline{\mathcal{C}}$ denote the closure of $\mathcal{C}$. Then:*

1. *$\overline{\mathcal{C}}$ is a $\mathrm{Mod}(S)$–invariant nowhere dense topologically perfect set.*
2. *The action of $\mathrm{Mod}(S)$ on $\overline{\mathcal{C}}$ is topologically transitive.*
3. *The points $\rho \in \partial\mathcal{Q}$ satisfying $\overline{\mathcal{C}} \subset \overline{\mathrm{Mod}(S) \cdot \rho}$ form a dense G_δ–set.*

Published: 11 July 2006 DOI: 10.2140/gt.2006.10.715

In particular, Theorem 1.1 implies that any continuous $\mathrm{Mod}(S)$–invariant function on $\partial\mathcal{Q}$ is constant. Recall that the action of a group on a locally compact Hausdorff separable topological space is topologically transitive if the translates of any two open sets intersect, or equivalently, if there is a dense orbit.

The group $\mathrm{PSL}_2\,\mathbb{C}$ is the group of orientation preserving isometries of hyperbolic 3-space $\mathbb{H}^3$. It is well known that ρ is faithful and that the action of $\rho(\Gamma)$ on $\mathbb{H}^3$ is free and properly discontinuous for each $\rho \in \overline{\mathcal{Q}}$ (see Kapovich [14, Theorem 9.1.4]). In particular $M_\rho = \mathbb{H}^3/\rho(\Gamma)$ is an orientable hyperbolic manifold homotopy equivalent to S. From this point of view the set $\mathcal{C}$ of Theorem 1.1 is the set of all $\rho \in \overline{\mathcal{Q}}$ such that the boundary of the convex core of the associated hyperbolic manifold M_ρ does not have a compact component.

A special role is played by geometrically finite representations $\rho \in \mathcal{C}$ where the boundary of the convex core of M_ρ is a collection of thrice-punctured spheres. We call these representations full maximal cusps. The proof of Theorem 1.1 involves studying the dynamics of the $\mathrm{Mod}(S)$–action near these points. For example, the techniques used to prove Theorem 1.1 are also used to prove that for any open neighborhood $U \subset \partial\mathcal{Q}$ of a full maximal cusp, the orbit $\mathrm{Mod}(S)\cdot U$ is dense in $\partial\mathcal{Q}$ (see Corollary 4.3).

It is well known that two representations $\rho, \rho' \in \mathcal{Q}$ are close if and only if the associated hyperbolic manifolds M_ρ and $M_{\rho'}$ are bi-Lipschitz with a small bi-Lipschitz constant. This is why geometric invariants of the manifold M_ρ, for example the volume of the convex core, the injectivity radius, the lowest eigenvalue of the Laplacian, or the Hausdorff dimension of the limit set, are continuous functions on quasifuchsian space. However, in the larger set $\overline{\mathcal{Q}}$ the picture is more complex. Two representations $\rho, \rho' \in \overline{\mathcal{Q}}$ may be close without there being any bi-Lipschitz homeomorphism from M_ρ to $M_{\rho'}$. With this motivation one may ask which geometric invariants remain continuous on $\overline{\mathcal{Q}}$. In each case it was previously known via different methods that these quantities are no longer continuous on $\overline{\mathcal{Q}}$. We derive from Theorem 1.1 a unified proof of this fact:

Theorem 1.2 *The volume of the convex core, the injectivity radius, the lowest eigenvalue of the Laplacian and the Hausdorff dimension of the limit set do not vary continuously on $\overline{\mathcal{Q}}$.*

We also apply Theorem 1.1 to study $\mathrm{Mod}(S)$–invariant meromorphic functions defined on subsets of the character variety $\mathcal{X}(\Gamma, \mathrm{SL}_2\,\mathbb{C})$. In [12], Goldman proved that every $\mathrm{Mod}(S)$–invariant meromorphic function defined on the whole of $\mathcal{X}(\Gamma, \mathrm{SL}_2\,\mathbb{C})$ must be constant. This result motivates the question of which connected open subsets U of $\mathcal{X}(\Gamma, \mathrm{SL}_2\,\mathbb{C})$ admit nonconstant $\mathrm{Mod}(S)$–invariant meromorphic functions. (A weaker form of this question can be found in [12, Section 1.4].) Goldman [11] deduced the nonexistence of invariant nonconstant meromorphic functions from the ergodicity of the $\mathrm{Mod}(S)$–action on the (real) subvariety $\mathcal{X}(\Gamma, \mathrm{SU}_2)$. In particular his result applies to every connected open set $U \subset \mathcal{X}(\Gamma, \mathrm{SL}_2\,\mathbb{C})$ containing unitary representations. We obtain the following analogue of Goldman's result:

Theorem 1.3 *Let $U \subset \mathcal{X}(\Gamma, \mathrm{SL}_2\,\mathbb{C})$ be a $\mathrm{Mod}(S)$–invariant connected open set. If U contains both (faithful) convex cocompact representations and indiscrete representations then any $\mathrm{Mod}(S)$–invariant meromorphic function on U is a constant function.*

After some preliminaries in Section 2 we present in Section 3 a concrete construction of hyperbolic 3-manifolds which will be one of the key ingredients in the proof of Theorem 1.1. Full maximal cusps are the other key ingredient. In particular, in Section 4 we deduce that full maximal cusps are dense in $\mathcal{C}$. The proof uses techniques developed by McMullen [16], Canary, Culler, Hersonsky, and Shalen [6; 7], who studied the ubiquity of maximally cusped representations on the boundaries of various deformation spaces. Theorem 1.2 is proved in Section 5 and Theorem 1.3 is proved in Section 6.

Acknowledgements The first author is supported by a postdoctoral fellowship of the Deutsche Forschungs Gemeinschaft. The second author is supported by a National Science Foundation Postdoctoral Fellowship. The first author would like to thank the members of the University of Chicago Department of Mathematics for their hospitality. The second author would like to thank Richard Canary and Misha Kapovich for helpful conversations on these topics.

2 Preliminaries

We refer to Heusener and Porti [13] for basic facts about the character variety and to Anderson [1] for a survey about the deformation theory of discrete subgroups of $\mathrm{PSL}_2\,\mathbb{C}$. If H is a finitely generated non-virtually abelian torsion free group then $\mathrm{Hom}(H, \mathrm{PSL}_2\,\mathbb{C})$ is a complex algebraic variety on which $\mathrm{PSL}_2\,\mathbb{C}$ acts by conjugacy. The character variety

$$\mathcal{X}(H, \mathrm{PSL}_2\,\mathbb{C}) = \mathrm{Hom}(H, \mathrm{PSL}_2\,\mathbb{C}) /\!\!/ \mathrm{PSL}_2\,\mathbb{C}$$

is the quotient of $\mathrm{Hom}(H, \mathrm{PSL}_2\,\mathbb{C})$ under this action in the sense of invariant theory. We remind the reader that $\mathcal{X}(H, \mathrm{PSL}_2\,\mathbb{C})$ does not coincide with the set theoretic quotient $\mathrm{Hom}(H, \mathrm{PSL}_2\,\mathbb{C}) / \mathrm{PSL}_2\,\mathbb{C}$. However, the set of conjugacy classes of discrete faithful representations is contained in a smooth open manifold in $\mathcal{X}(H, \mathrm{PSL}_2\,\mathbb{C})$ [13, Section 4]. This paper is concerned only with discrete faithful representations, so the machinery of invariant theory will not be needed.

Notation The Greek letters ρ and σ (possibly with decoration) will be used to indicate *conjugacy classes* of representations. Thus ρ and σ will be elements of the appropriate character variety. The notation $\rho(H)$ will indicate the image of H under any fixed homomorphism of the conjugacy class ρ.

We identify the group $\mathrm{PSL}_2\,\mathbb{C}$ with the group of orientation preserving isometries of hyperbolic 3-space $\mathbb{H}^3$. We will denote the convex core of a hyperbolic manifold M by $CC(M)$. The convergence of sequences in $\mathcal{X}(H, \mathrm{PSL}_2\,\mathbb{C})$ is said to be *algebraic convergence*. Let $\rho_i \to \rho$ be an algebraically convergent sequence. If the representations ρ_i are discrete and faithful for all i, then it is well known that ρ is discrete and faithful as well (see Kapovich [14, Theorem 9.1.4]). Moreover, up to a choice of a subsequence, the groups $\rho_i(H)$ converge in the Chabauty topology to a discrete subgroup H_G of $\mathrm{PSL}_2\,\mathbb{C}$ which contains the image of ρ. H_G is the *geometric limit* of the sequence $\{\rho_i(H)\}$.

In this paper we will mainly consider discrete and faithful representations of the fundamental group Γ of a closed surface S. A faithful and discrete representation $\rho \in \mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ induces a homotopy class of homotopy equivalences $S \longrightarrow M_\rho$. (Recall that M_ρ denotes the hyperbolic 3–manifold $\mathbb{H}^3/\rho(\Gamma)$.) A theorem of Bonahon [3] ensures that M_ρ is homeomorphic to a trivial interval bundle over S. Moreover, the homotopy equivalence $S \longrightarrow M_\rho$ determines a unique isotopy class of orientation preserving homeomorphisms $S \times (-1,1) \longrightarrow M_\rho$. In other words, M_ρ has a positive end (the top end) and a negative end (the bottom end). Observe that orientation reversing elements in $\mathrm{Mod}(S)$ extend in a canonical way to orientation preserving homeomorphisms of $S \times (-1,1)$ which interchange the top and bottom ends.

A component of $\partial CC(M_\rho)$ is said to face the top (resp. bottom) end of M_ρ if it is isotopic in $M_\rho - CC(M_\rho)$ out the top (resp. bottom) end of M_ρ. With this terminology, the top (resp. bottom) end of M_ρ is *conformally compact* if there is a single compact component of $\partial CC(M_\rho)$ facing the top (resp. bottom) end of M_ρ, or equivalently if there are compact embedded convex surfaces exiting the top (resp. bottom) end of M_ρ. (This terminology comes from the fact that a conformally compact end limits onto a compact boundary component of the conformal manifold $\left(\mathbb{H}^3 \cup \Omega_\rho\right)/\rho(\Gamma)$, where $\Omega_\rho \subset \mathbb{S}^2_\infty$ denotes the domain of discontinuity of $\rho(\Gamma)$.)

Quasifuchsian space $\mathcal{Q} \subset \mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ is the open set of conjugacy classes of faithful convex cocompact representations. The closure $\overline{\mathcal{Q}}$ of quasifuchsian space $\mathcal{Q}$ consists of faithful representations with discrete image and hence it is contained in an open submanifold of $\mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$. Let $\partial\mathcal{Q}$ denote the boundary of quasifuchsian space $\overline{\mathcal{Q}} - \mathcal{Q} \subset \mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$. Sullivan [21] proved that $\partial\mathcal{Q}$ is also the set

$$\left(\overline{\mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C}) - \overline{\mathcal{Q}}}\right) - \left(\mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C}) - \overline{\mathcal{Q}}\right).$$

In other words, $\partial\mathcal{Q}$ is the frontier of $\overline{\mathcal{Q}}$. Note that $\rho \in \overline{\mathcal{Q}}$ is quasifuchsian if and only if it has two conformally compact ends.

The following $\mathrm{Mod}(S)$–invariant subset of $\partial\mathcal{Q}$ will play a central role:

$$\mathcal{C} = \{\rho \in \partial\mathcal{Q} \mid M_\rho \text{ has no conformally compact end}\}.$$

For example, if $CC(M_\rho) = M_\rho$ then $\rho \in \mathcal{C}$. In particular, a cyclic cover of a fibered hyperbolic 3–manifold lies in $\mathcal{C}$. On the other hand, the closure of any Bers slice is disjoint from $\mathcal{C}$. Finally, any representation in $\overline{\mathcal{Q}}$ without parabolic elements has a neighborhood disjoint from $\mathcal{C}$ (see the proof of Theorem 1.1). This gives a decomposition of $\overline{\mathcal{Q}}$ into three pieces: the manifolds with two conformally compact ends $\mathcal{Q}$, the manifolds with exactly one conformally compact end $\partial\mathcal{Q} - \mathcal{C}$, and finally $\mathcal{C}$, which contains several types of manifolds. Roughly speaking, the $\mathrm{Mod}(S)$–action becomes increasingly chaotic on these pieces. Surprisingly, $\mathcal{C}$ is not closed (see Section 7).

3 The main construction

In this section we present a construction which is the core of the proof of Theorem 1.1. The main building pieces in our construction are so called maximal cusps. Fix a compact hyperbolic surface S with fundamental group Γ.

We will say that a discrete finitely generated subgroup of $\mathrm{PSL}_2\,\mathbb{C}$ is a *full maximal cusp* if it is geometrically finite and every component of the boundary of the convex core of the associated hyperbolic manifold is a thrice punctured sphere. Observe that $\rho \in \mathcal{C}$ if $\rho(\Gamma)$ is a full maximal cusp. We will say a representation $\rho \in \partial\mathcal{Q}$ is a *one sided maximal cusp* if it is geometrically finite, has one conformally compact end, and each component of $\partial CC(M_\rho)$ facing the end of M_ρ which is not conformally compact is a thrice punctured sphere. (A one sided maximal cusp is often simply called a maximal cusp (see McMullen [16]). Our modified terminology has been chosen for clarity.) The set of full maximal cusps in $\partial\mathcal{Q}$ is countable, and intuitively forms a set of "rational points" on the boundary. This intuition can be made precise in the punctured torus case.

Maximal cusps are very convenient when making concrete constructions because any pair of totally geodesic thrice punctured spheres in any pair of hyperbolic 3-manifolds are isometric. In particular, if M and M' are hyperbolic manifolds whose convex cores boundaries $\partial CC(M)$ and $\partial CC(M')$ contain thrice punctured spheres X and X', and $\phi\colon X \longrightarrow X'$ is a homeomorphism, then ϕ is isotopic to an isometry which we denote again by ϕ. Hence there is a hyperbolic manifold N which is covered by M and M', whose convex core is isometric to $CC(M) \cup_\phi CC(M')$.

The second main ingredient in our constructions is the following lemma, which is a consequence of Thurston's Dehn-filling Theorem (see Thurston [22, Chapter 4], Bonahon and Otal [4], and Comar [8]).

Lemma 3.1 *Let $H < \mathrm{PSL}_2\,\mathbb{C}$ be a geometrically finite group such that $\mathbb{H}^3/H$ is homeomorphic to $S \times (0,1) - \mathcal{P}$ where $\mathcal{P}$ is an unlinked collection of disjoint simple closed curves. Then the group H is the geometric limit of geometrically finite groups H_n isomorphic to $\pi_1(S)$. Moreover, if H is a full (resp. one sided) maximal cusp, then the H_n can be chosen to be full (resp. one sided) maximal cusps.*

Recall that a collection of disjoint simple closed curves in $S \times [0,1]$ is unlinked if every curve is contained in an embedded boundary parallel surface which is disjoint from all the other curves. The groups H_n in the statement of Lemma 3.1 are obtained by performing a hyperbolic $(1,n)$–Dehn surgery on a neighborhood of each of the curves in $\mathcal{P}$.

Lemma 3.2 *Let $H, H_n < \mathrm{PSL}_2\,\mathbb{C}$ be as in the statement of Lemma 3.1, and let $\rho_n\colon \Gamma \longrightarrow H_n$ be isomorphisms. If $\rho\colon \Gamma \longrightarrow \mathrm{PSL}_2\,\mathbb{C}$ is the representation induced by an embedded level surface*

$$S \times \{t\} \subset (S \times (0,1) - \mathcal{P}) \cong \mathbb{H}^3/H$$

then there are automorphisms α_n of Γ such that $\rho_n \circ \alpha_n$ converges algebraically to ρ.

Proof Since $H_n \to H$ geometrically, for any a compact submanifold $K \subset \mathbb{H}^3/H$ there is a sequence of smooth embeddings $\phi_n\colon K \longrightarrow \mathbb{H}^3/H_n$ which converge in the $\mathcal{C}^\infty$–topology to isometric embeddings (see McMullen [17, Section 2.2]). Since $\mathbb{H}^3/H_n$ is obtained by $(1,n)$–Dehn surgery on $\mathbb{H}^3/H$, it is clear that by choosing a sufficiently large compact submanifold K the restricted homomorphism

$$(\phi_{n*})|_{\rho(\Gamma)}\colon \rho(\Gamma) \longrightarrow H_n < \mathrm{PSL}_2\,\mathbb{C}$$

will be well defined and injective, and thus an isomorphism. Since the maps ϕ_n are converging to isometries, the sequence of homomorphisms $(\phi_{n*})|_{\rho(\Gamma)}$ is converging to the identity map. Therefore

$$\phi_{n*} \circ \rho\colon \Gamma \longrightarrow H_n$$

is a sequence of discrete faithful representations converging to ρ. Define α_n to be $\rho_n^{-1} \circ \phi_{n*} \circ \rho$. This proves the lemma. □

The following is the main result of the present section. For clarity it has been split into three similar pieces.

Proposition 3.3 *Let $\rho, \bar\rho \in \partial\mathcal{Q}$ be one sided maximal cusps. There exists a sequence of representations $\{\rho_i\}$ in quasifuchsian space and a sequence of mapping classes $\{\alpha_i\}$ such that*

$$\rho_i \to \rho \quad \text{and} \quad (\alpha_i \cdot \rho_i) \to \bar\rho.$$

Proposition 3.4 *Let $\rho \in \partial\mathcal{Q}$ be a one sided maximal cusp. Let $\rho^{\mathcal{C}}$ be a full maximal cusp. There exists a sequence of one sided maximal cusps $\{\rho_i\}$ and a sequence of mapping classes $\{\alpha_i\}$ such that*

$$\rho_i \to \rho \quad \text{and} \quad (\alpha_i \cdot \rho_i) \to \rho^{\mathcal{C}}.$$

Proposition 3.5 *Let $\rho^{\mathcal{C}}, \bar\rho^{\mathcal{C}} \in \partial\mathcal{Q}$ be full maximal cusps. There exists a sequence of full maximal cusps $\{\rho_i\}$ and a sequence of mapping classes $\{\alpha_i\}$ such that*

$$\rho_i \to \rho^{\mathcal{C}} \quad \text{and} \quad (\alpha_i \cdot \rho_i) \to \bar\rho^{\mathcal{C}}.$$

The proofs of Propositions 3.3, 3.4, and 3.5 are very similar, so we will prove carefully only the first one.

Proof Up to composition with an orientation reversing element in $\mathrm{Mod}(S)$, we may assume that the top end of M_ρ and the bottom end of $M_{\bar\rho}$ are conformally compact.

By Lemma 3.2 it suffices to construct a hyperbolic manifold N homeomorphic to the complement in $S \times (0,1)$ of an unlinked collection of simple curves, which is (locally isometrically) covered by both M_ρ and $M_{\bar\rho}$.

The convex cores $CC(M_\rho)$ and $CC(M_{\bar\rho})$ are by definition homeomorphic to

$$(S \times [-1,1]) - (\mathcal{P}_+ \times \{-1\}) \text{ and } (S \times [-1,1]) - (\mathcal{P}_- \times \{1\}),$$

where $\mathcal{P}_+$ and $\mathcal{P}_-$ are pants decompositions of S. Choose sufficiently complicated homeomorphisms $\phi_+ \colon S \longrightarrow S$ and $\phi_- \colon S \longrightarrow S$ such that the two collections $\phi_+(\mathcal{P}_+)$ and $\phi_-(\mathcal{P}_-)$ bind the surface S. Thurston's hyperbolization theorem (see Otal [19]) implies that there is a geometrically finite hyperbolic 3-manifold $S \longrightarrow M$ whose convex core is homeomorphic to

$$(S \times [-1,1]) - (\phi_-(\mathcal{P}_-) \times \{-1\} \cup \phi_+(\mathcal{P}_+) \times \{1\}).$$

The middle manifold M is a full maximal cusp. M_ρ and $M_{\bar\rho}$ are one sided maximal cusps. Therefore the convex cores of M_ρ, $M_{\bar\rho}$ and M can be glued together to obtain the convex core $CC(N)$ of a hyperbolic 3-manifold N covered by M_ρ and $M_{\bar\rho}$ and homeomorphic to

$$(S \times [-3,3]) - (\phi_-(\mathcal{P}_-) \times \{-1\} \cup \phi_+(\mathcal{P}_+) \times \{1\}).$$

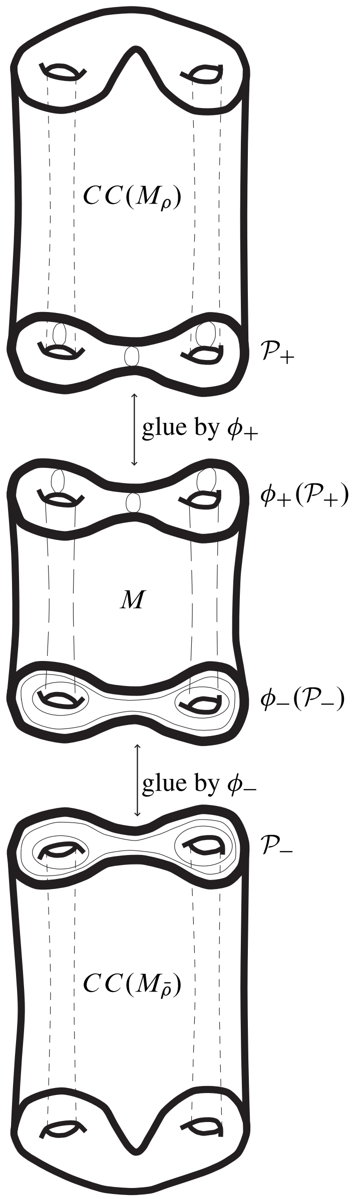

Figure 1: Constructing the manifold N

(See Figure 1 for a picture of this construction. In the figure the homeomorphisms $\phi_\pm$ are the identity map, because the pants decompositions $\mathcal{P}_\pm$ already bind S.)

We are now in the situation of Lemma 3.1. The holonomy representation of N corresponds to $H < \mathrm{PSL}_2\,\mathbb{C}$, and

$$\phi_-(\mathcal{P}_-)\times\{-1\} \;\cup\; \phi_+(\mathcal{P}_+)\times\{1\}$$

corresponds to the unlinked collection of curves $\mathcal{P}$. Two applications of Lemma 3.2 conclude the proof of Proposition 3.3. First apply Lemma 3.2 to the level surface $S\times\{2\}$. Then apply it to the level surface $S\times\{-2\}$. □

For the proof of Proposition 3.4, replace $\bar\rho$ with $\rho^{\mathcal{C}}$ and follow the above argument. This will require adding a third pants decomposition to the above notation, corresponding to the bottom end of $M_{\rho^{\mathcal{C}}}$. Similarly, for the proof of Proposition 3.5, replace ρ with $\rho^{\mathcal{C}}$, $\bar\rho$ with $\bar\rho^{\mathcal{C}}$, and follow the above argument. In this case it will be necessary to add a fourth pants decomposition to the above notation. Otherwise the proof goes through unchanged in both cases.

4 Ubiquity of maximal cusps

In this section we study the ubiquity of maximal cusps in $\partial\mathcal{Q}$. In particular, full maximal cusps are shown to be dense in the subset $\mathcal{C}\subset\partial\mathcal{Q}$ given by manifolds without a conformally compact end (see Proposition 4.5). This result will be used in the proof of Theorem 1.1. We will use techniques developed by McMullen [16], Canary, Culler, Hersonsky and Shalen [6], and Canary and Hersonsky [7]. We begin by combining results from [16] and [6] with work of Brock, Bromberg, Evans and Souto [5] to prove the following statement.

Theorem 4.1 *One sided maximal cusps form a dense subset of $\partial\mathcal{Q}$.*

Proof A standard Baire category argument shows that the set of representations $\rho\in\partial\mathcal{Q}$ without parabolic elements forms a dense subset of $\partial\mathcal{Q}$ (see McMullen [16, Corollary 1.5] or Canary, Culler, Hersonsky and Shalen [6, Lemma 15.2]). So it suffices to approximate only representations without parabolic elements by one sided maximal cusps. Pick such a representation $\rho\in\partial\mathcal{Q}$. There are only two possibilities: either the limit set of ρ is the entire sphere at infinity, or M_ρ has exactly one conformally compact end and one geometrically infinite end.

If the limit set of such a ρ is the entire sphere at infinity, then Canary, Culler, Hersonsky and Shalen have proven [6, Theorem 6.1] that ρ is an algebraic limit of full maximal cusps. Proposition 3.4 shows that a full maximal cusp is always an algebraic limit of one sided maximal cusps in $\partial\mathcal{Q}-\mathcal{C}$. Therefore it only remains to consider the case where M_ρ has one conformally compact end.

Let us assume without loss of generality that the top end of M_ρ is conformally compact. Applying [5], we know the representation $\rho\in\partial\mathcal{Q}$ is a strong limit of quasifuchsian representations ρ_i. By Kerckhoff and Thurston [15], the conformal structure at infinity of the top end of M_ρ must be the limit in Teichmüller space of the conformal structures at infinity of the top ends of the M_{ρ_i}. Therefore altering the manifolds M_{ρ_i} by a K_i–bi-Lipschitz deformation, where $K_i\to 1$, produces a sequence of quasifuchsian manifolds in a Bers slice converging strongly to M_ρ. We may now apply [16] to conclude that ρ is an algebraic limit of one sided maximal cusps. □

From Proposition 3.4 and Theorem 4.1 we derive the following ostensibly stronger result:

Proposition 4.2 *In every open subset of $\partial\mathcal{Q}$ we find a $\bar\rho$ with the following property: For every full maximal cusp $\rho^{\mathcal{C}}$ there is a sequence $\{\alpha_i\}$ in $\mathrm{Mod}(S)$ with*

$$\rho^{\mathcal{C}}=\lim_i \alpha_i\cdot\bar\rho$$

Proof Recall that the set of all full maximal cusps is countable. In particular there is a sequence of full maximal cusps $\{\rho_j\}\subset\mathcal{C}$ such that every full maximal cusp appears infinitely often in the sequence.

Moreover, we can fix open neighborhoods V_j of ρ_j in $\partial\mathcal{Q}$ such that when a subsequence ρ_{j_i} converges then for every choice of $\rho'_{j_i}\in V_{j_i}$ the sequence ρ'_{j_i} also converges to the same limit.

Given an open neighborhood U_1 in $\partial\mathcal{Q}$ we obtain from Proposition 3.4 and Theorem 4.1 a one sided maximal cusp $\sigma_1\in U_1\subset\partial\mathcal{Q}$ and $\alpha_1\in\mathrm{Mod}(S)$ with $\alpha_1\cdot\sigma_1\in V_1$. Let U_2 be an open and relatively compact neighborhood of σ_1 in $U\cap\alpha_1^{-1}(V_1)$. Applying again the same argument we find a one sided maximal cusp $\sigma_2\in U_2$ and $\alpha_2\in\mathrm{Mod}(S)$ with $\alpha_2\cdot\sigma_2\in V_2$. Let U_3 be an open and relatively compact neighborhood of σ_2 in $U_2\cap\alpha_2^{-1}(V_2)$. Inductively we find a sequence of one sided maximal cusps $\{\sigma_i\}$, a sequence $\{\alpha_i\}$ in $\mathrm{Mod}(S)$, and a sequence of open sets $\{U_i\}$ such that: U_{i+1} is relatively compact in U_i, σ_i is in U_i, and $\alpha_i(U_i)\subset V_i$. By our choice of the neighborhoods V_i we can conclude that every $\bar\rho$ in $\cap_i U_i$ has the desired property. □

By reversing the logic of Proposition 4.2 we obtain:

Corollary 4.3 *If $W \subset \partial\mathcal{Q}$ is an open set containing a full maximal cusp then* $\mathrm{Mod}(S)\cdot W$ *is a dense open subset of* $\partial\mathcal{Q}$*. Moreover, the set*

$$\{\rho \in \partial\mathcal{Q} \mid \overline{\mathrm{Mod}(S)\cdot\rho} \text{ contains every full maximal cusp}\}$$

is a dense G_δ*–set.*

Proof The first sentence of the corollary follows immediately from Proposition 4.2. To prove the second, let us reuse the notation $\{\rho_j\}$ and $\{V_j\}$ from the proof of Proposition 4.2. Then

$$\bigcap_j \mathrm{Mod}(S)\cdot V_j = \{\rho \in \partial\mathcal{Q} \mid \overline{\mathrm{Mod}(S)\cdot\rho} \text{ contains every full maximal cusp}\}.$$

From this the corollary follows. □

The following theorem paraphrases the results of Canary and Hersonsky [7, Section 10]. It will be used in the proof of Proposition 4.5.

Theorem 4.4 [7, Theorem 10.1] *Let M be a hyperbolic 3–manifold with a holonomy representation ρ. Assume there is a sequence of geometrically finite representations ρ_i converging algebraically to ρ such that the homomorphisms $\rho(\pi_1(M)) \longrightarrow \rho_i(\pi_1(M))$ are induced by homeomorphisms*

$$(M, \text{cusps of } M) \longrightarrow (M_{\rho_i}, \text{cusps of } M_{\rho_i}).$$

Let $\mathcal{E}$ denote the geometrically infinite ends of $M - \{\text{cusps}\}$. Then there exists a sequence of geometrically finite representations $\widehat{\rho}_j$ converging algebraically to ρ satisfying:

1. *There exists a homeomorphism $\phi_j\colon M \longrightarrow M_{\widehat{\rho}_j}$ taking the cusps of M into the cusps of $M_{\widehat{\rho}_j}$.*
2. *If Σ is a component of $\partial CC(M_{\widehat{\rho}_j})$ isotopic in*
$$M_{\widehat{\rho}_j} - int(CC(M_{\widehat{\rho}_j}))$$
outside every bounded subset of $\phi_j(\mathcal{E})$, then Σ is a thrice punctured sphere.

Using some standard terminology we have not defined here (see Anderson [1]), this theorem can be restated informally as follows. If M lies on the boundary of the deformation space of a pared manifold, then it can be approximated by geometrically finite manifolds on the boundary of the same deformation space, where each geometrically infinite end of $M - \{\text{cusps}\}$ has been replaced by a maximally cusped geometrically finite end. The statement of Theorem 4.4 differs slightly from the statement of [7, Theorem 10.1]. However, the proof of [7, Theorem 10.1] proves Theorem 4.4.

Recall that $\mathcal{C} \subset \partial\mathcal{Q}$ is the set of manifolds without a conformally compact end. We are now ready to prove the main result of this section, which will be used in the proof of Theorem 1.1.

Proposition 4.5 *Full maximal cusps are dense in* $\mathcal{C}$.

Proof Pick a representation $\rho \in \mathcal{C}$. The goal is to approximate ρ by a sequence of full maximal cusps. We divide the proof into four cases, depending on what type of representation ρ is.

Case 1 Assume ρ is geometrically finite.

We will produce a hyperbolic manifold covered by M_ρ which is the geometric limit of a sequence of full maximal cusps in $\mathcal{C}$. Then we will apply Lemma 3.2 to complete Case 1.

The convex core $CC(M_\rho)$ of M_ρ is homeomorphic to

$$N_0 := (S \times [-1,1]) - (\mathcal{P}_- \times \{-1\} \cup \mathcal{P}_+ \times \{1\}),$$

where $\mathcal{P}_\pm$ are nonempty collections of disjoint, nonparallel simple closed curves. Define

$$N := (S \times [-1,2]) - (\mathcal{P}_- \times \{-1\} \cup \mathcal{P}_+ \times \{1\}).$$

We consider the embedding $N_0 \longrightarrow N$ and the induced morphism

$$\Pi\colon \mathcal{X}(\pi_1(N), \mathrm{PSL}_2\,\mathbb{C}) \longrightarrow \mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C}).$$

By a result of Brock, Bromberg, Evans and Souto [5, Lemma 5.1] there is an open set in $\Pi^{-1}(\rho)$ consisting of discrete, faithful, geometrically finite, and minimally parabolic representations. (Here, minimally parabolic means that every parabolic conjugacy class of $\pi_1(N)$ corresponds to a component of one of the pants decompositions $\mathcal{P}_-$ or $\mathcal{P}_+$.) Moreover, the marked conformal structure at infinity corresponding to the bottom end of any manifold in the fiber $\Pi^{-1}(\rho)$ is equivalent to the marked conformal structure at infinity corresponding to the bottom end of M_ρ.

On the other hand, by deforming the added generator in any $\mathbb{Z} \times \mathbb{Z}$ subgroup of $\pi_1(N)$, we see that every connected component of $\Pi^{-1}(\rho)$ contains indiscrete representations. So there must be a nonempty separating set of discrete faithful representations in $\Pi^{-1}(\rho)$ which are either geometrically infinite or not minimally parabolic. A generic $\sigma \in \Pi^{-1}(\rho)$ is minimally parabolic. (This is again a Baire category argument; compare with McMullen [16, Corollary 1.5] or Canary, Culler, Hersonsky and Shalen [6, Lemma 15.2].) We deduce that there is a discrete, faithful, and minimally parabolic $\sigma \in \Pi^{-1}(\rho)$ such that the top end of M_σ is geometrically infinite.

One should imagine that M_σ has been obtained by adding a geometrically infinite cap onto the top end of M_ρ. With its geometrically infinite end, M_σ is more easily approximated by one sided maximal cusps. Indeed, we may now apply Theorem 4.4 to conclude that σ is the algebraic limit of a sequence σ_i in $\mathcal{X}(\pi_1(N), \mathrm{PSL}_2\,\mathbb{C})$ such σ_i is discrete, faithful, geometrically finite, and the top end of M_{σ_i} is maximally cusped.

Since M_σ is minimally parabolic, we may apply a theorem of Evans [10] to conclude that the manifolds M_{σ_i} also converge geometrically to M_σ. Lemma 3.1 implies now that for all i the manifold M_{σ_i} is the geometric limit of a sequence M_i^j of geometrically finite hyperbolic manifolds homeomorphic to $S \times (0,1)$ such that the top end of each M_i^j is maximally cusped. Taking a diagonal sequence we deduce that M_σ is the geometric limit of a sequence of geometrically finite manifolds whose top ends are maximally cusped. Since M_σ is covered by M_ρ, we may apply Lemma 3.2 to conclude that

M_ρ is an algebraic limit of geometrically finite manifolds in $\mathcal{C}$ whose top ends are maximally cusped.

Therefore, we may without loss of generality assume the the top end of M_ρ is maximally cusped. Now apply the above argument again to M_ρ, swapping the roles of the bottom end and the top end. This yields a sequence of full maximal cusps converging algebraically to M_ρ. The completes the proof of Case 1.

Case 2 M_ρ has at least one rank one cusp in each end.

By Case 1 it suffices to approximate M_ρ by geometrically finite manifolds in $\mathcal{C}$. Let P_+ (resp. P_-) be a nonempty collection of disjoint essential annuli on S such that there is a relative homeomophism

$$(S \times (0,1)\ ,\ (P_+ \times [.8,1)) \bigcup (P_- \times (0,.2])) \longrightarrow (M_\rho, \text{cusps of } M_\rho).$$

By a result of Brock, Bromberg, Evans and Souto [5, Corollary 3.2], M_ρ is the algebraic and geometric limit of a sequence of geometrically finite manifolds M_{ρ_i} such that for each i there is a relative *homotopy equivalence*

$$(S \times (0,1)\ ,\ (P_+ \times [.8,1)) \bigcup (P_- \times (0,.2])) \longrightarrow (M_{\rho_i}, \text{cusps of } M_{\rho_i}).$$

The concurrence of algebraic and geometric convergence implies that, after passing to a subsequence, these relative homotopy equivalences are rel homotopic to relative homeomorphisms. This implies that the manifolds M_{ρ_i} are in $\mathcal{C}$.

Case 3 M_ρ has all of its rank one cusps in exactly one end.

Assume without loss of generality that M_ρ has rank one cusps in its bottom end. Then the top end of M_ρ is necessarily geometrically infinite. The proof begins as in Case 3. Let P_- be a nonempty collection of disjoint essential annuli on S such that there is a relative homeomophism

$$(S \times (0,1), P_- \times (0,.2]) \longrightarrow \big(M_\rho, \text{cusps of } M_\rho\big)\,.$$

As above, applying [5, Corollary 3.2] yields that M_ρ is the algebraic and geometric limit of a sequence of geometrically finite manifolds M_{ρ_i} such that for each i there is a relative homotopy equivalence

$$(S \times (0,1), P_- \times (0,.2]) \longrightarrow \big(M_{\rho_i}, \text{cusps of } M_{\rho_i}\big)\,.$$

Again using the fact that M_{ρ_i} converges both algebraically and geometrically to M_ρ, after possibly passing to a subsequence we may assume these relative homotopy equivalences are homotopic to relative homeomorphisms.

Having verified its hypotheses, we may now apply Theorem 4.4 to find an M_ρ convergent sequence of geometrically finite manifolds in $\mathcal{C}$ whose top ends are maximally cusped. This reduces the proof to Case 1.

Case 4 M_ρ has no cusps.

In this case M_ρ must have an empty domain of discontinuity. We may then directly apply [7, Theorem 10.1] to conclude that M_ρ is the algebraic limit of a sequence of full maximal cusps. □

5 The main theorem

In this section we prove the main result of this paper, Theorem 1.1. As an application we also prove Theorem 1.2.

Recall that $\mathcal{C}$ denotes the set of representations whose quotient manifolds have no conformally compact end. A goal of Theorem 1.1 is to prove that the closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ is a small set, but up to this point we have not found even a single discrete faithful representation outside of $\overline{\mathcal{C}}$. This we now do, using a theorem of Evans [10].

Lemma 5.1 *If $\sigma \in \partial\mathcal{Q}$ is a representation with no parabolic elements and exactly one conformally compact end (i.e. a singly degenerate representation without parabolics), then σ is not contained in $\overline{\mathcal{C}}$.*

Proof Pick a representation $\sigma \in \partial\mathcal{Q}$ with no parabolic elements and exactly one conformally compact end. Suppose that $\sigma \in \overline{\mathcal{C}}$. Then by Proposition 4.5 there is a sequence of full maximal cusps $\tau_i \to \sigma$. Since σ has no parabolic elements, it follows from a theorem of Evans [10] that the manifolds M_{τ_i} converge geometrically to M_σ. Since the top end of M_σ is conformally compact, there is a strictly convex surface Σ embedded in the top end of M_σ. Use the almost isometric embeddings provided by geometric convergence to push the surface Σ into M_{τ_i}. For sufficiently large i this yields a strictly convex embedded surface in M_{τ_i}, showing that the manifolds M_{τ_i} eventually have a conformally compact end. Since they are all full maximal cusps, this is a contradiction. Therefore σ is not in $\overline{\mathcal{C}}$. □

Theorem 1.1 Let $\mathcal{C} \subset \mathcal{Q}$ denote the set of representations whose quotient manifold has no conformally compact end. Then:

(1) The closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ is a $\mathrm{Mod}(S)$–invariant nowhere dense topologically perfect set.

(2) The action of $\mathrm{Mod}(S)$ on $\overline{\mathcal{C}}$ is topologically transitive.

(3) The points $\rho \in \partial\mathcal{Q}$ satisfying $\overline{\mathcal{C}} \subset \overline{\mathrm{Mod}(S)\cdot\rho}$ form a dense G_δ–set.

Proof The set $\mathcal{C}$ is $\mathrm{Mod}(S)$–invariant by definition, implying the same for its closure $\overline{\mathcal{C}}$. The set $\mathcal{C}$ is topologically perfect since full maximal cusps are dense in $\mathcal{C}$ and Proposition 3.5 implies full maximal cusps are not isolated.

To finish claim (1), it remains to prove that $\overline{\mathcal{C}}$ is nowhere dense. Following Lemma 5.1, let $\sigma \notin \overline{\mathcal{C}}$ be a representation without parabolics whose quotient manifold has exactly one conformally compact end. Let U be a neighborhood of σ in the complement of $\overline{\mathcal{C}}$. By Proposition 4.2 there is a representation in U whose orbit limits onto all of $\mathcal{C}$. As $\mathrm{Mod}(S)\cdot U \cap \overline{\mathcal{C}} = \varnothing$, this proves that $\overline{\mathcal{C}}$ is nowhere dense.

We now prove claim (2). Let U^1 and U^2 be open sets in $\mathcal{C}$. By Proposition 4.5 there are full maximal cusps $\rho^1 \in U^1$ and $\rho^2 \in U^2$. By Proposition 3.5 there exist sequences $\{\rho_i\}$ in $\mathcal{C}$ and $\{\alpha_i\}$ in $\mathrm{Mod}(S)$ such that

$$\rho^1 = \lim_i \rho_i \quad \text{and} \quad \rho^2 = \lim_i \alpha_i\cdot\rho_i$$

This shows that for sufficiently large i, $(\alpha_i\cdot U^2)\cap U^1$ is not empty. Hence the action of $\mathrm{Mod}(S)$ on $\mathcal{C}$ is topologically transitive. This implies topological transitivity on $\overline{\mathcal{C}}$.

Finally, claim (3) follows immediately from Proposition 4.5 and Corollary 4.3. □

As an application we prove the following theorem.

Theorem 1.2 The volume of the convex core, the injectivity radius, the lowest eigenvalue of the Laplacian and the Hausdorff dimension of the limit set do not vary continuously on $\overline{\mathcal{Q}}$.

Each part of Theorem 1.2 was known previously. We present merely a unified proof.

Proof All these invariants are $\mathrm{Mod}(S)$–invariant functions

$$\partial\mathcal{Q} \longrightarrow \mathbb{R} \cup \{\infty\}.$$

In particular we derive from Theorem 1.1 (3) and (4) that they either are constant or fail to be continuous. In particular, it suffices to find $\rho, \rho' \in \partial\mathcal{Q}$ for which they do not take the same value. Let ρ be a full maximal cusp and ρ' the cyclic cover of the mapping torus of a pseudo-Anosov $\phi \in \mathrm{Mod}(S)$; ρ' is in the boundary of quasifuchsian space by the work of Thurston [23]. We know that

$$\mathrm{vol}(CC(M_\rho)) < \infty \text{ and } \mathrm{vol}(CC(M_{\rho'})) = \infty$$

because ρ is geometrically finite and ρ' is not. The manifold $M_{\rho'}$ covers a compact manifold and hence has positive injectivity radius while M_ρ has cusps and hence its injectivity radius vanishes. Geometric finiteness of ρ implies that Λ_ρ has Hausdorff dimension less than 2 (see Tukia [24] and Sullivan [20]) while $\Lambda_{\rho'} = \mathbb{C}P^1$ has dimension 2.

Sullivan's theorem tells us that the lowest eigenvalue of the Laplacian on M_ρ is

$$\dim(\Lambda_\rho)\,(2-\dim(\Lambda_\rho)) > 0.$$

Finally, the lowest eigenvalue of the Laplacian on $M_{\rho'}$ equals the infimum

$$\inf_{f\in C_0^\infty(M_{\rho'})} \frac{\int |\nabla f|^2}{\int |f|^2}.$$

(This is true for any Riemannian manifold.) Since $M_{\rho'}$ is the cyclic cover of the mapping torus of ϕ, it is easy to show without any machinery that this infimum is zero. □

From this we see that it is in fact impossible to define a nontrivial purely geometric invariant which varies continuously on $\overline{\mathcal{Q}}$.

6 Map(S)–invariant meromorphic functions

In this section we prove Theorem 1.3. As a warm up to the $\mathrm{SL}_2\,\mathbb{C}$ Case, we first consider meromorphic functions on open subsets of $\mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$.

Theorem 6.1 *Let $U \subset \mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ be a $\mathrm{Mod}(S)$–invariant connected open set. If U contains both (faithful) convex cocompact representations and indiscrete representations then any $\mathrm{Mod}(S)$–invariant meromorphic function on U is a constant function.*

Note that indiscrete representations are dense in the complement of quasifuchsian space (see Sullivan [21]).

Proof Without loss of generality we may assume that U is a manifold. The set U intersects the interior and the exterior of $\overline{\mathcal{Q}}$ and hence $\partial\mathcal{Q} \cap U \neq \varnothing$. We claim that every continuous $\mathrm{Mod}(S)$–invariant function $f\colon U \longrightarrow \mathbb{C}$ is constant on $\partial\mathcal{Q} \cap U$. Given one sided maximal cusps $\rho, \bar\rho \in \partial\mathcal{Q} \cap U$ we obtain from Proposition 3.3 a sequence $\{\rho_i\} \subset \mathcal{Q}$ and a sequence of mapping classes $\{\alpha_i\} \subset \mathrm{Mod}(S)$ with

$$\lim_i \rho_i = \rho \text{ and } \lim_i \alpha_i \cdot \rho_i = \bar\rho.$$

We have $\rho_i \in U$ for sufficiently large i. The continuity and the $\mathrm{Mod}(S)$–invariance of f imply that $f(\rho) = f(\bar\rho)$. Since maximal cusps are dense in $\partial\mathcal{Q} \cap U$, the claim follows.

Assume now that $f\colon U \longrightarrow \mathbb{C}$ is meromorphic and $\mathrm{Mod}(S)$–invariant. The divisor D of poles of f is either empty or has complex codimension 1. The open set $U - D$ is open, connected, $\mathrm{Mod}(S)$–invariant and also intersects $\partial\mathcal{Q}$. In particular we may assume without loss of generality that f is holomorphic. We proved above that f is constant on the separating set $\partial\mathcal{Q} \cap U$. By holomorphicity, this implies that f is everywhere constant. □

We now discuss the relationship between the character varieties $\mathcal{X}(\Gamma, \mathrm{SL}_2\,\mathbb{C})$ and $\mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ and prove Theorem 1.3. The homomorphism $\mathrm{SL}_2\,\mathbb{C} \longrightarrow \mathrm{PSL}_2\,\mathbb{C}$ induces a map

$$p\colon \mathcal{X}(\Gamma, \mathrm{SL}_2\,\mathbb{C}) \longrightarrow \mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$$

which maps onto the connected component of $\mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ containing $\overline{\mathcal{Q}}$ (see Heusener and Porti [13, Remark 4.3] and Culler [9]). Recall that the closure of quasifuchsian space $\overline{\mathcal{Q}} \subset \mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ is contained in a smooth open $\mathrm{Mod}(S)$–invariant manifold $\mathcal{O} \subset \mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ (see [13, Section 4] and Goldman [12]). After possibly shrinking $\mathcal{O}$ slightly we may assume that

$$p|_{p^{-1}(\mathcal{O})}\colon p^{-1}(\mathcal{O}) \longrightarrow \mathcal{O}$$

is a Galois cover with covering transformation group $H_1(S, \mathbb{Z}/2\mathbb{Z})$ (see [13]). See Goldman [12] for properties of $\mathcal{X}(\Gamma, \mathrm{SL}_2\,\mathbb{C})$.

Theorem 1.3 Let $U \subset \mathcal{X}(\Gamma, \mathrm{SL}_2\,\mathbb{C})$ be a $\mathrm{Mod}(S)$–invariant connected open set. If U contains both (faithful) convex cocompact representations and indiscrete representations then any $\mathrm{Mod}(S)$–invariant meromorphic function on U is a constant function.

Again note that indiscrete representations are dense in the complement of the set of convex cocompact representations.

Proof As above we may assume without loss of generality that U is a manifold contained in $\mathcal{O}$. Moreover, it suffices to show that a holomorphic $\mathrm{Mod}(S)$–invariant function f on U is constant.

The conditions on U imply that there is some $\sigma \in \partial\mathcal{Q} \cap p(U)$ since the image under p of a faithful convex cocompact representation lies in $\mathcal{Q}$ and the image of an indiscrete representation is again indiscrete. Choose a neighborhood V' of σ and $V \subset U$ open such that $p|_V\colon V \longrightarrow V'$ is a homeomorphism. Since the actions of $\mathrm{Mod}(S)$ on $\mathcal{X}(\Gamma, \mathrm{SL}_2\,\mathbb{C})$ and $\mathcal{X}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ are compatible we obtain that the restriction of p to the open $\mathrm{Mod}(S)$–invariant set

$$W = \bigcup_{\alpha \in \mathrm{Mod}(S)} \alpha \cdot V$$

is a covering. It suffices to show that the restriction of f to W is constant. Given $n \in \mathbb{N}$ we consider the holomorphic function on $p(W)$ given by

$$S_n(z) = \sum_{w \in p^{-1}(z)} f^n(w).$$

We claim that S_n is constant for all n. The function S_n is holomorphic and $\mathrm{Mod}(S)$–invariant, but we cannot directly apply Theorem 6.1 since $p(W)$ may not be connected. However, every connected component of $p(W)$ intersects $\partial\mathcal{Q}$ and therefore the same proof as in Theorem 6.1 applies to show that S_n is locally constant.

The following lemma shows that S_n being locally constant for all n implies that f is itself constant on W.

Lemma 6.2 *Let $\Omega \subset \mathbb{C}^r$ be a connected open set and $f_1, \ldots, f_k\colon \Omega \longrightarrow \mathbb{C}$ holomorphic such that for all n the function $S_n(z) = \sum f_i(z)^n$ is constant. Then all the functions f_i are constant.*

Proof Before going further observe that it suffices to consider the case where Ω is a unit disk in $\mathbb{C}$, because a nonconstant holomorphic function will have a nonconstant restriction to some holomorphic disk. Seeking a contradiction assume that the lemma is false. One can assume without loss of generality that none of the functions f_i are constant and that no two of the functions f_i and f_j are proportional. We may further assume, up to reducing Ω, that for all i and z we have $f_i(z) \neq 0$ and $f_i'(z) \neq 0$. Moreover, the assumption that no two of the functions are proportional implies that, up to relabeling, there is some z_0 satisfying $|f_1(z_0)| > |f_2(z_0)| \geq \cdots \geq |f_k(z_0)|$. Multiplying by a suitable scalar we may assume that $f_1(z_0) = 1$. Computing the derivative of S_n at the point 0 we obtain the following identity for all n:

$$0 = S_n'(z_0) = n(f_1'(z_0) + f_2(z_0)^{n-1} f_2'(z_0) + \cdots + f_k(z_0)^{n-1} f_k'(z_0))$$

Dividing by n and taking a limit $n \to \infty$ we derive that $f_1'(z_0) = 0$. This is a contradiction. □

As mentioned above this concludes the proof of Theorem 1.3. □

7 $\mathcal{C} \subset \partial \mathcal{Q}$ is not closed

This section will show that the set

$$\mathcal{C} := \{\rho \in \partial\mathcal{Q} \mid M_\rho \text{ has no conformally compact end}\}$$

(see Section 2) is not a closed subset of $\partial\mathcal{Q}$. This fact surprised the authors. This section is logically independent of Sections 3–6.

The proof is a slight elaboration of a construction due to McMullen [18, Lemma A.4] (which was in turn based on results of Kerckhoff and Thurston [15] and Anderson and Canary [2]). As Lemma A.4 is very clearly written, we will not attempt to reproduce its construction here. We will assume that the reader has read Lemma A.4 and the example which follows it. (These total only two and half pages.) Our notation will be chosen to conform to McMullen's.

Choose two pants decompositions $P_\pm$ which bind the surface S. Choose an embedded essential closed curve $C \subset S$ such that: C and P_+ bind S, and C and P_- bind S. By Thurston's hyperbolization theorem (see Otal [19]) there is an infinite volume hyperbolic 3–manifold N whose convex core is a finite volume manifold with totally geodesic boundary homeomorphic to

$$(S\times[0,1])-(P_+\times\{1\}\cup P_-\times\{0\}\cup C\times\{1/2\}).$$

With this manifold N, perform McMullen's construction in [18, Lemma A.4]. Following his notation, let N_n be a sequence of $(1,n)$–Dehn surgeries on $C\times\{1/2\}\subset N$ together with maps $F_n\colon N\longrightarrow N_n$ converging in the compact–$\mathcal{C}^\infty$ topology to an isometric embedding. Let $f\colon S\longrightarrow N$ be an immersed essential surface which wraps around C.

Mark the manifolds N_n by the composition $F_n\circ f$. Equipped with these markings the sequence N_n is contained in $\mathcal{C}\subset\partial\mathcal{Q}$ and converges algebraically to the covering space of N given by $f_*(\pi(S))$. Recall that the curve C binds S with either pants decomposition P_+ or P_-. From this it follows that the only parabolics of $f_*(\pi_1(S))$ correspond to the curve C. Therefore the algebraic limit must have one conformally compact end, and does not lie in $\mathcal{C}\subset\partial\mathcal{Q}$. This shows that $\mathcal{C}$ is not closed and concludes the construction.

It would be interesting to find a geometric characterization of the manifolds in the closure $\overline{\mathcal{C}}$. This appears to be difficult.

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Received: 14 January 2006
Revised: 16 May 2006