Existence of ruled wrappings in hyperbolic 3–manifolds

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We present a short elementary proof of an existence theorem of certain CAT(−1)–surfaces in open hyperbolic 3–manifolds. The main construction lemma in Calegari and Gabai's proof of Marden's Tameness Conjecture can be replaced by an applicable version of our theorem. Finally, we will give a short proof of the conjecture along their ideas.

Agol [1] and Calegari and Gabai [5] proved independently that any hyperbolic 3–manifold M with finitely generated fundamental group is homeomorphic to the interior of a compact 3–manifold. This is the affirmative answer to Marden's Tameness Conjecture in [8]. Subsequently, Choi [7] gave another proof of the conjecture, similar to Agol's, in the case when M has no parabolic cusps.

We are here interested in arguments in [5], where the notion of "shrinkwrapping" was introduced. Shrinkwrappings play an important role in their proof. For the proof of the existence of shrinkwrappings and that of their CAT(−1)–property, Calegari and Gabai used very deep and rarefied arguments, which some readers, including the author, may find difficult to approach. This paper is intended to give an elementary proof of part of their proof by using ruled wrappings instead of shrinkwrappings.

For simplicity, we only consider the case when a hyperbolic 3–manifold has no parabolic cusps and will prove the following theorem.

Theorem 0.1 *Let N be an orientable hyperbolic 3–manifold without parabolic cusps, Δ a disjoint union of finitely many simple closed geodesics in N, and $f\colon \Sigma \longrightarrow N$ a 2–incompressible map rel. Δ from a closed orientable surface Σ of genus greater than 1 to $N \setminus \Delta$. Then there exists a homotopy $F\colon \Sigma \times [0, 1] \longrightarrow N$ satisfying the following conditions.*

(i) *$F(x, 0) = f(x)$ for any $x \in \Sigma$.*

(ii) *$F(\Sigma \times [0, 1)) \cap \Delta = \varnothing$.*

(iii) *The map $g\colon \Sigma \longrightarrow N$ defined by $g(x) = F(x, 1)$ is a CAT(−1)–piecewise ruled map.*

Here, a continuous map $f\colon \Sigma \longrightarrow N$ is said to be 2–*incompressible* in N rel. Δ if $f(\Sigma) \cap \Delta = \varnothing$, $f_*\colon \pi_1(\Sigma) \longrightarrow \pi_1(N \setminus \Delta)$ is injective, and for any simple non-contractible loop l in Σ the restriction $f|_l$ is not freely homotopic in $N \setminus \Delta$ to a (multiplied) meridian of any component of Δ. See Definition 1.4 for the definition of piecewise ruled maps. We say that a map g satisfying properties (i)–(iii) as above or its image $g(\Sigma)$ is a CAT(-1)–*ruled wrapping* of Δ in N homotopic to f. In fact, Theorem 0.1 is a special case of Proposition 2.1, which corresponds to the main construction lemma in [5].

In Section 4, we will give a short proof of Marden's Conjecture along ideas in [5]. Our proof is self-contained in the sense that it does not rely on published partial solutions to the conjecture. It is not hard to see that all arguments and proofs in this paper work also in the case when the ambient manifold N has pinched negative curvature and hyperbolic cusps by invoking results in Canary [6, Section 4].

1 Completion of certain hyperbolic metrics

For a closed subset A in a metric space (X, d), the r–neighborhood of A in X, $\{y \in X;\ d(y, A) \leq r\}$, is denoted by $\mathcal{N}_r(A)$ (or more strictly by $\mathcal{N}_r(A, X)$). In the case when A is a single point set $\{x\}$, we also set $\mathcal{N}_r(\{x\}) = \mathcal{B}_r(x)$. The link of x in X with radius r, $\{y \in X;\ d(y, x) = r\}$, is denoted by $\mathcal{S}_r(x)$.

Let U be a simply connected incomplete hyperbolic 3–manifold with metric completion $\overline{U}$ such that each component l of $L = \overline{U} \setminus U$ is a geodesic line, and there exists a constant $c > 0$ with $\mathrm{dist}(x, y) \geq 3c$ for any points x, y contained in distinct components of L. Moreover, we suppose that for any component l of L there exists an infinite cyclic branched covering $p_l\colon \mathcal{N}_c(l, \overline{U}) \longrightarrow \mathcal{N}_c(j, \mathbf{H}^3)$ branched over a geodesic line j of $\mathbf{H}^3$, such that the restriction $p_l|_{\mathcal{N}_c(l,\overline{U})\setminus l}$ is a locally isometric covering. From the definition, $\mathcal{N}_c(l, \overline{U})$ is homeomorphic to the quotient space of $\mathbf{R}^2 \times [0, 1]$ by the identification map $a\colon \mathbf{R}^2 \times \{0\} \longrightarrow \mathbf{R}$ defined by $a(x, y, 0) = x$.

Suppose that σ is a shortest arc in $\overline{U}$ connecting two given points and consisting of two hyperbolic segments σ_1, σ_2 with $\sigma \cap L = \sigma_1 \cap \sigma_2 = \{x\}$. Let x_i $(i = 1, 2)$ be the point in σ_i with $\mathrm{dist}(x_1, l) = \mathrm{dist}(x_2, l) = s > 0$, where l is the component of L containing x; see Figure 1 (a). There exist totally geodesic half planes P_i in $\overline{U}$ with $\sigma_i \subset P_i$ and $\partial P_i = l$. Since the subsegment τ of σ with $\partial \tau = \{x_1, x_2\}$ is the shortest
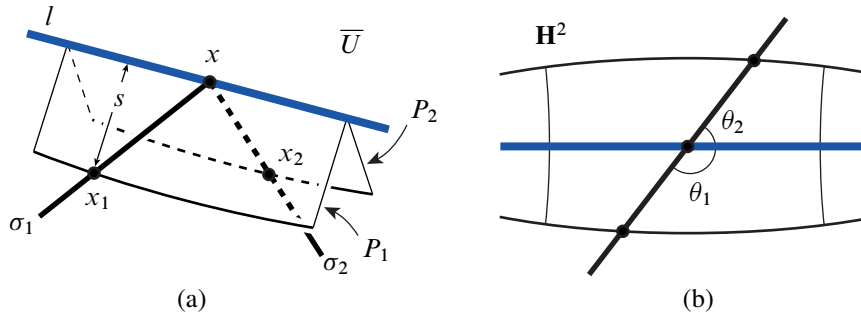
(a)   (b)

Figure 1

arc in $\overline{U}$ connecting x_1 with x_2, we have

(1–1) $$\theta_1 + \theta_2 = \pi,$$

where θ_i is the angle made by σ_i and a fixed ray in l emanating from x. This fact is easily seen by considering the developing of $P_1 \cup P_2$ on $\mathbf{H}^2$; see Figure 1 (b).

For any d with $0 < d \leq c$, the set $\mathcal{B}_d(x, \overline{U})$ is homeomorphic to the subset of $\mathbf{R}^3$

$$\{(u, v, w) \in \mathbf{R}^3; \; u^2 + v^2 + w^2 \leq 1, w > 0\} \cup \{(u, 0, 0) \in \mathbf{R}^3; \; -1 \leq u \leq 1\}.$$

In particular, $\mathcal{B}_d(x, \overline{U})$ is simply connected. The image $p_l(\mathcal{B}_d(x, \overline{U}))$ coincides with the hyperbolic ball $\mathcal{B}_d(\hat{x}, \mathbf{H}^3)$, where $\hat{x} = p_l(x)$. Rescaling the metric on the boundary $S = \mathcal{S}_d(\hat{x}, \mathbf{H}^3)$ of the ball, we have the spherical metric ν on S isometric to the unit sphere in the Euclidean 3–space. Consider the metric on $\widetilde{S} = \mathcal{S}_d(x, \overline{U})$, still denoted by ν, so that the infinite cyclic branched covering $p_l|\widetilde{S}: \widetilde{S} \longrightarrow S$ is locally pathwise isometric. Here, $p_l|\widetilde{S}$ being *locally pathwise isometric* means that $\text{length}_\nu(\alpha) = \text{length}_\nu(p_l(\alpha))$ for any rectifiable arc α in $\widetilde{S}$. One can take $d > 0$ so that $\sigma' = \sigma \cap \mathcal{B}_d(x)$ is a geodesic segment in $\mathcal{B}_d(x)$ with $\partial\sigma' \subset \widetilde{S}$. Let γ be any rectifiable arc in $\widetilde{S}$ with $\partial\gamma = \partial\sigma'$. Since $\mathcal{B}_d(x)$ is simply connected, γ is homotopic rel. $\partial\gamma$ to σ' in $\mathcal{B}_d(x)$. Then the following lemma is proved immediately from the equality (1–1) and by checking the situation of $\hat{\gamma} = p_l(\gamma)$ in S; see Figure 2.

Lemma 1.1 $\text{length}_\nu(\gamma) \geq \pi$.

For any $k \leq 0$, let $\mathbf{F}(k)$ be a complete Riemannian plane of constant curvature k. A simply connected geodesic metric space X is called a CAT(k)–*space* if any geodesic triangle Δ in X is *not thicker* than a comparison triangle $\overline{\Delta}$ in $\mathbf{F}(k)$, that is, for any two points s and t in the edges of Δ and their comparison points $\overline{s}$ and $\overline{t}$ in $\overline{\Delta}$, $\text{dist}_X(s, t) \leq \text{dist}_{\mathbf{F}(k)}(\overline{s}, \overline{t})$. See Bridson–Haefliger [3] for fundamental properties of such spaces.
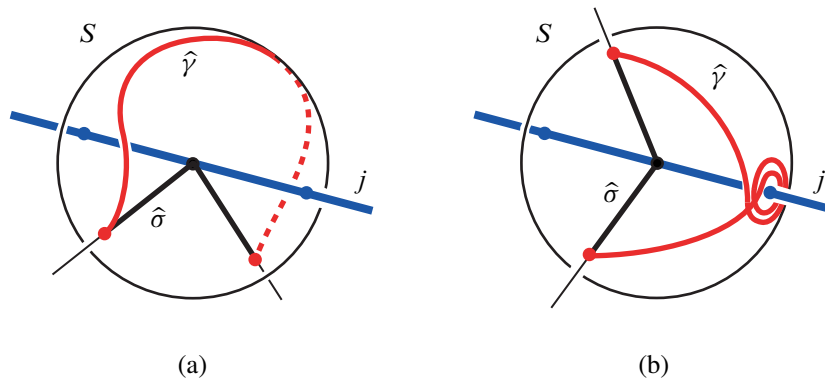
Figure 2: $j = p_l(l)$, $\hat{\sigma} = p_l(\sigma')$. (b) is the case when $\hat{\gamma}$ winds around j more than once, but $\text{length}_\nu(\gamma) = \text{length}_\nu(\hat{\gamma})$ does not exceed π very much.

Lemma 1.2 $\overline{U}$ *is a* CAT(-1)*–space.*

Proof By the generalized Cartan–Hadamard Theorem [3, Chapter II.4, Theorem 4.1(2)], it suffices to show that, for any point $x \in \overline{U}$, there exists $r > 0$ such that $B_r(x)$ is a CAT(-1)–space. If $x \in U$ then $B_r(x)$ is obviously a CAT(-1)–space if we take r so that $B_r(x) \cap L = \varnothing$. So suppose that $x \in L$. Let Δ be any triangle in $B_c(x)$ with geodesic edges $\gamma_1, \gamma_2, \gamma_3$. Each Int γ_i either meets L at most one point p_i or is contained in L. From this, we know that the union $\gamma_1 \cup \gamma_2 \cup \gamma_2$ bounds a triangle Δ_0 in $B_c(x)$ consisting of at most three totally geodesic hyperbolic subtriangles. In fact, when p_1, p_2, p_3 exist, the totally geodesic triangle with vertices p_1, p_2, p_3 is degenerated to a geodesic segment in L. By Lemma 1.1, the internal angle of Δ_0 at any p_i is at least π. This shows that Δ_0 is not thicker than a comparison triangle in $\mathbf{H}^2$. It follows that $B_c(x)$ is a CAT(-1)–space. $\square$

It is not hard to see that any geodesic segment in $\overline{U}$ is a broken line consisting of finitely many hyperbolic segments, and any vertex of the broken line other than its end points lies in L. By Lemma 1.2, a geodesic segment σ in $\overline{U}$ connecting two given points is uniquely determined. Moreover, σ varies continuously with its end points.

Let Z be an incomplete hyperbolic 3–manifold such that the total space of the universal covering $q: U \longrightarrow Z$ has the induced metric as above. We suppose moreover that for the metric completion $\overline{Z}$ each component l of $\overline{Z} \setminus Z$ is either a geodesic line or a geodesic loop. That is, no component of $\overline{Z} \setminus Z$ is a one point set. Then $q: U \longrightarrow Z$ is extended to a locally pathwise isometric map $\overline{q}: \overline{U} \longrightarrow \overline{Z}$. Note that the frontier $\{x \in \overline{Z}; \text{dist}(x, l) = c\}$ of $\mathcal{N}_c(l, \overline{Z})$ in $\overline{Z}$ is homeomorphic to either $\mathbf{R}^2$ or an open annulus or a torus.

Remark 1.3 Even in the case when $\mathcal{N}_c(l, \overline{Z})$ is homeomorphic to a solid torus, we always suppose that *homotopies in* $\overline{Z}$ starting from a continuous map $f \colon A \longrightarrow Z$ never cross l (possibly they touch l), where A is a manifold of dimension less than three. In other words, we only consider homotopies $F \colon A \times [0, 1] \longrightarrow \overline{Z}$ which can be covered by a map $\widetilde{F} \colon \widetilde{A} \times [0, 1] \longrightarrow \overline{U}$, where $\widetilde{A}$ is the universal covering space of A.

Definition 1.4 Let $f \colon \Sigma \longrightarrow \overline{Z}$ be a continuous map from a closed orientable surface Σ. Suppose that Σ admits a cell decomposition K consisting of finitely many triangular 2–cells. We say that f is a *piecewise ruled map* with respect to K if (i) for each edge of e of K, $f(e)$ is a broken line consisting of finitely many hyperbolic segments, and (ii) for each 2–cell F of K, $f(F)$ is a ruled triangle based on a single vertex. Since the Gaussian curvature at any smooth point of a ruled surface is nonpositive, the (intrinsic) curvature of Σ at any smooth point is at most -1. This map f is called a CAT(-1)–*piecewise ruled map* if the cone-angle of Σ at any singular point is at least 2π.

2 Applicable version of Theorem 0.1

Throughout this section, we assume that all hyperbolic 3–manifolds and surfaces are orientable.

Let N be a complete hyperbolic 3–manifold N without parabolic cusps, and W a 3–dimensional compact connected submanifold of N. Consider a link Δ in Int W consisting of finitely many disjoint simple closed geodesics in N. Let $p \colon X \longrightarrow W$ be the covering of W associated to a finitely generated subgroup of $\pi_1(W)$. Here we make the following assumptions, which correspond to those in the main construction lemma [5, Lemma 2.3].

(i) ∂W is incompressible in $N \setminus \Delta$.

(ii) There exists a union $\widehat{\Delta}$ of components of $p^{-1}(\Delta)$ such that the restriction $p|\widehat{\Delta} \colon \widehat{\Delta} \longrightarrow \Delta$ is a homeomorphism.

(iii) There exists a continuous map $f \colon \Sigma \longrightarrow X$ from a closed surface Σ of genus $m > 1$ which is 2–incompressible in X rel. $\widehat{\Delta}$.

Set $X^\circ = X \setminus p^{-1}(\Delta)$. The fundamental group $\pi_1(X^\circ)$ may be infinitely generated. By (i), the restriction $p^\circ = p|X^\circ \colon X^\circ \longrightarrow W \setminus \Delta \subset N \setminus \Delta$ is π_1–injective. Thus, we may assume that X° is a subset of the total space of the covering $q \colon Y^\circ \longrightarrow N \setminus \Delta$ associated to the subgroup $p^\circ_*(\pi_1(X^\circ))$ of $\pi_1(N \setminus \Delta)$ and the inclusion $i \colon X^\circ \longrightarrow Y^\circ$ is a homotopy equivalence. Since $\partial X^\circ = \partial X = p^{-1}(\partial W)$ is a deformation retract of

$Y^{\circ} \setminus \operatorname{Int} X^{\circ}$, condition (iii) implies that $f: \Sigma \longrightarrow Y$ is 2–incompressible in Y rel. $\widehat{\Delta}$, where $Y = X \cup (Y^{\circ} \setminus \operatorname{Int} X^{\circ}) = X \cup Y^{\circ}$. The complement $Z = Y \setminus \widehat{\Delta}$ has the induced incomplete metric as was studied in Section 1. Let $\overline{Z}$ be the metric completion of Z.

With the notation and assumptions as above, we will prove the following proposition.

Proposition 2.1 *There exists a homotopy* $F: \Sigma \times [0, 1] \longrightarrow \overline{Z}$ *which never crosses* $\widehat{\Delta}$ *and connects* f *with a* CAT(-1)*–ruled wrapping* $g: \Sigma \longrightarrow \overline{Z}$ *of* $\widehat{\Delta}$.

Proof Let $c_1, \ldots, c_{3m-3}$ be mutually disjoint simple loops in Σ which define a pants decomposition of Σ. Consider a cell decomposition K of Σ consisting of triangular 2–cells and such that each vertex of K is contained in $c_1 \cup \cdots \cup c_{3m-3}$. If necessary by deforming f by a homotopy in the sense of Remark 1.3 we may assume that each $f(c_i)$ is a closed geodesic in $\overline{Z}$, and $f(e)$ is a geodesic segment in $\overline{Z}$ for any edge e of K not contained in $c_1 \cup \cdots \cup c_{3m-3}$. In fact, $f(c_i)$ is the image of an axis of a hyperbolic transformation on the metric completion $\overline{U}$ of the universal covering space of Z. See for example [3, Theorem 6.8 (1)]. For any 2–cell F of K, take a vertex v_0 and the opposite edge e_0. Then, $f|F$ can be homotoped rel. ∂F to a map $g|F$ such that $g(F)$ is a ruled triangle consisting of all geodesic segments connecting $f(v_0)$ with points of $f(e_0)$. These $g|F$ define a piecewise ruled map $g: \Sigma \longrightarrow \overline{Z}$ homotopic to f. From our construction of g, for any singular point $g(v)$ of $g(\Sigma)$, there exists an arc α in Σ with $\operatorname{Int} \alpha \ni v$ and such that $g(\alpha)$ is a geodesic segment in $\overline{Z}$. If $g(v)$ is not an element of $\overline{Z} \setminus Z$, then it is easily seen that the cone-angle of Σ at v is at least 2π. So we may assume that $g(v)$ is contained in a component l of $\overline{Z} \setminus Z$. For any sufficiently small $d > 0$, α divides the circle $\mathcal{S}_d(v, \Sigma)$ into two arcs γ_1, γ_2. By Lemma 1.1, the ν–length of $g(\gamma_i)$ $(i = 1, 2)$ in $\mathcal{S}_d(g(v), \overline{Z})$ is at least π. Thus, the cone-angle of Σ at v is at least 2π. This shows that g is a CAT(-1)–ruled wrapping of $\widehat{\Delta}$ in $\overline{Z}$. $\qquad\square$

Note that Theorem 0.1 is proved quite similarly to Proposition 2.1 by considering (N, Δ) instead of $(\overline{Z}, \widehat{\Delta})$.

3 Compact cores and end reductions

A 3–manifold X is *topologically tame* if there exists an embedding $f: X \longrightarrow Y$ into a compact manifold Y with $f(X) \supset \operatorname{Int} Y$. Throughout this section, we suppose that M is an orientable, open, irreducible and connected 3–manifold with finitely generated fundamental group. An end $\mathcal{E}$ of M is said to be *topologically tame* if there exits a closed neighborhood of $\mathcal{E}$ in M homeomorphic to $S \times [0, \infty)$ for some closed

connected surface S. It is easily seen that the open 3–manifold M is topologically tame if and only if each end of M is so.

Scott [10] proved that M contains a 3–dimensional submanifold C, called a *compact core* of M, such that the inclusion $i: C \longrightarrow M$ is a homotopy equivalence. Let S be the component of ∂C facing an end $\mathcal{E}$ of M, and $p: \widetilde{M} \longrightarrow M$ the covering associated with the image of $\pi_1(S)$ in $\pi_1(M)$. There exists a compact core $\widetilde{C}$ of $\widetilde{M}$ such that $\partial \widetilde{C}$ has a component $\widetilde{S}$ mapped onto S homeomorphically by p. The manifold $\widetilde{C}$ is a *compression body*, ie, it is homeomorphic to $E \cup h_1 \cup \cdots \cup h_m$ where E is either a 3–ball or $F \times [0, 1]$ for some closed surface F consisting of nonspherical components and the h_i's are 1–handles attached to one side of E. In particular, when E is a 3–ball, the compression body $\widetilde{C}$ is a handlebody. Note that the end $\widetilde{\mathcal{E}}$ of $\widetilde{M}$ faced by $\widetilde{S}$ is topologically tame if and only if $\mathcal{E}$ is also.

Let $\Delta = \delta_1 \cup \cdots \cup \delta_{i_0}$ be an i_0–component link in the compression body $\widetilde{C}$ such that $[\delta_k]$ $(k = 1, \cdots, i_0 - 1)$ form a basis for the free abelian group $H_1(\widetilde{C}, \mathbf{Z})$ and $[\delta_{i_0}] = [\delta_1] + \cdots + [\delta_{i_0-1}]$. An advantage of considering compression bodies is that any nontrivial free decomposition of $\pi_1(\widetilde{C})$ induces a nontrivial decomposition of $H_1(\widetilde{C}; \mathbf{Z})$. In particular, this implies that the link Δ is *algebraically disk-busting*, that is, for any nontrivial free decomposition $A * B$ of $\pi_1(\widetilde{C})$, there exists a component δ_k of Δ such that the element of $\pi_1(\widetilde{C})$ represented by δ_k is neither conjugate into A nor B.

Some results in Myers [9] concerning end reductions play an important role in the proof of Theorem 4.1. The paper is useful also as an expository article on end reductions. A compact, connected, 3–dimensional submanifold R of M is *regular* if $M \setminus R$ is irreducible and the closure of any component of $M \setminus R$ in M is not compact. Let Δ be a link in M each component of which is noncontractible in M. An open submanifold V of M containing Δ is called an *end reduction* of M at Δ if it satisfies the following conditions.

 (i) No component of $M \setminus V$ is compact.
 (ii) There exists a sequence $\{R_n\}$ of regular submanifolds of M with $\Delta \subset R_1$, $R_n \subset \text{Int } R_{n+1}$, $V = \bigcup_n R_n$ and such that ∂R_n is incompressible in $M \setminus \Delta$.
 (iii) V satisfies the *engulfing property* at Δ, that is, for any regular submanifold N of M with $\Delta \subset \text{Int } N$ such that ∂N is incompressible in $M \setminus \Delta$, V is ambient isotopic rel. Δ to a manifold containing N.

We refer to Brin–Thickstun [4] for the existence and uniqueness up to isotopy of end reductions. According to Myers [9, Theorem 9.2], if the link Δ is algebraically disk-busting, then an end reduction V of M at Δ is connected and the homomorphism $i_*: \pi_1(V) \longrightarrow \pi_1(M)$ induced from the inclusion is an isomorphism.

4 Proof of Marden's Conjecture

Our proof of Marden's Conjecture is based on that of Calegari–Gabai [5], but the importance of the disk-busting property is suggested by Agol [1]. For simplicity, we only consider hyperbolic 3–manifolds without parabolic cusps. It is not hard to modify our argument for the case of manifolds with parabolic cusps.

Theorem 4.1 (Marden's Tameness Conjecture) *Let N be an orientable hyperbolic 3–manifold without parabolic cusps. If $\pi_1(N)$ is finitely generated, then N is topologically tame.*

First of all, we fix the setting for the proof. It suffices to show that each end $\mathcal{E}$ of N is topologically tame. As was seen in Section 3, we may assume that a compact core C of N is a compression body. Let S be the component of ∂C facing $\mathcal{E}$. If $\mathcal{E}$ is *geometrically finite*, that is, C is locally convex in S, then it is well known that $\mathcal{E}$ is topologically tame, for example see Marden [8]. So we may assume that $\mathcal{E}$ is not geometrically finite. Then Bonahon [2] shows that there exists a sequence $\{\delta_i\}$ of closed geodesics in N exiting $\mathcal{E}$. If necessary, by adding finitely many closed geodesics to $\{\delta_i\}$ one can suppose that $\Delta_i = \delta_1 \cup \cdots \cup \delta_i$ is algebraically disk-busting for all i not less than some fixed integer $i_0 > 0$. If necessarily, by slightly deforming the hyperbolic metric in a small neighborhood of $\bigcup_i \delta_i$ in N we may assume that the closed geodesics δ_i are simple and mutually disjoint, ie, each Δ_i is a link in N. In fact, the resulting metric is no longer hyperbolic but pinched negatively curved. However, all the results concerting hyperbolic manifolds which we need, eg Proposition 2.1 in Section 2, still hold under this metric if we replace CAT(-1) with CAT$(-a^2)$, where $-a^2$ is the supremum of sectional curvatures of N with respect to the new metric. We refer to Canary [6, Sections 4 and 5] for standard arguments on such a metric deformation.

For any $i \geq i_0$, let V_i be an end reduction of N at Δ_i. By [9], the π_1–homomorphism induced from the inclusion $V_i \longrightarrow N$ is an isomorphism. It follows that a compact core C_i of V_i is also a compact core of N, and each δ_k $(k = 1, \ldots, i)$ is freely homotopic in V_i to a loop of C_i. By property (ii) of the end reduction V_i, there exists a regular submanifold W_i of V_i containing both C_i and the traces of these free homotopies in C_i and such that ∂W_i is incompressible in $N \setminus \Delta_i$. In fact, R_n in Section 3 with sufficiently large n satisfies the properties of W_i. Since the inclusion $C_i \longrightarrow W_i \longrightarrow N$ is π_1–isomorphic, $\pi_1(C_i)$ can be regarded as a subgroup of $\pi_1(W_i)$. Consider the covering $p_i \colon X_i \longrightarrow W_i$ associated to $\pi_1(C_i) \subset \pi_1(W_i)$. Let $\hat{\delta}_k$ be a component of $p_i^{-1}(\delta_k)$ such that $p_i|\hat{\delta}_k \colon \hat{\delta}_k \longrightarrow \delta_k$ is homeomorphic, and let $\hat{\Delta}_i = \hat{\delta}_1 \cup \cdots \cup \hat{\delta}_i$. The

component S_i of ∂C_i facing $\mathcal{E}$ has a lift $S_i^{\bullet}$ to X_i which is also a boundary component of a compact core of X_i.

Lemma 4.2 X_i *is topologically tame.*

Proof The claim in [5, page 426] shows that W_i is an atoroidal manifold such that ∂W_i has a component of genus greater than 1. This fact together with Canary [6] proves that X_i is topologically tame.

Here, we will give an another proof without invoking the atoroidality of W_i. First we outline the proof. Divide the covering $p_i \colon X_i \longrightarrow W_i$ to restricted coverings associated with a torus decomposition of W_i. By finite generation, all but finitely many restrictions are universal coverings. By a result in Waldhausen [14], the total spaces of such universal coverings are topologically tame. Other total spaces cover atoroidal Haken manifolds with non-torus boundary component, and hence they are topologically tame. Since X_i is obtained by attaching topologically tame manifolds to each other along simply connected boundary components, Simon's Combination Theorem [11] shows that X_i is also topologically tame.

More precisely, let $\mathcal{T}_i = T_1 \cup \cdots \cup T_m$ be a maximal union of mutually disjoint and nonparallel incompressible tori in $\mathrm{Int}\, W_i$. Since W_i is regular and N is atoroidal and irreducible, each T_j bounds a compact manifold A_j which is contained in a 3–ball in N and homeomorphic to the exterior of a nontrivial knot in S^3. Either any two A_j are mutually disjoint or one of them contains the other. Set $\mathcal{A} = A_1 \cup \cdots \cup A_m$. Note that, for any component $\widehat{A}$ of $p_i^{-1}(\mathcal{A})$, the image $\mathrm{inc}_i \circ p_i(\widehat{A})$ is contained in a 3–ball in N, where $\mathrm{inc}_i \colon W_i \longrightarrow N$ is the inclusion. Since $\mathrm{inc}_i \circ p_i \colon X_i \longrightarrow N$ is π_1–isomorphic, it follows that $\widehat{A}$ is simply connected. Then $\widehat{A}$ is topologically tame by [14, Theorem 8.1]. Since each component of $\partial \widehat{A}$ is simply connected, $p_i^{-1}(\partial \mathcal{A})$ induces a free decomposition of $\pi_1(X_i)$. The classical Grushko Theorem implies that the fundamental group of any component $\widehat{B}$ of $p_i^{-1}(W_i \setminus \mathrm{Int}\, \mathcal{A})$ is finitely generated. Since $W_i \setminus \mathrm{Int}\, \mathcal{A}$ is an atoroidal Haken manifold such that one of the boundary components has genus greater than 1, $\mathrm{Int}\, \widehat{B}$ is topologically tame by [6, Proposition 3.2]. In the present case, it is not hard to show that $\widehat{B}$ is also topologically tame. See Soma [12] for a more general case. Finally, Theorem 3.1 in [11] implies that X_i is topologically tame. $\quad\square$

Proof of Theorem 4.1 By Lemma 4.2, X_i is realized as the interior of a compact manifold $\overline{X}_i$. Let $\overline{S}_i$ be the component of $\partial \overline{X}_i$ facing $S_i^{\bullet}$ in X_i, and $\widehat{S}_i$ a closed surface in $\mathrm{Int}\, X_i$ obtained by a small isotopy of $\overline{S}_i$ in $\overline{X}_i$. We show that $\widehat{S}_i$ is 2–incompressible in X_i rel. $\widehat{\Delta}_i$. If not, there would exist a compressing disk D for $\overline{S}_i$ in $\overline{X}_i$ such that the intersection $D \cap \widehat{\Delta}_i$ consists of at most one point. If D separates

X_i, then $\pi_1(X_i)$ has a nontrivial free decomposition $A * B$ each factor of which is isomorphic to the fundamental group of a component of $X_i \setminus D$. Otherwise, $\pi_1(X_i)$ is isomorphic to $A * \mathbf{Z}$ with $A = \pi_1(X_i \setminus D)$. When $D \cap \hat{\Delta}_i = \varnothing$, any element of $\pi_1(X_i)$ represented by a component of $\hat{\Delta}_i$ is conjugate into one of the factors. When $D \cap \hat{\delta}_j \neq \varnothing$ for some component $\hat{\delta}_j$ of $\hat{\Delta}_i$, one can suppose that the cyclic factor $\mathbf{Z}$ is generated by the element represented by $\hat{\delta}_j$. Any element represented by a component of $\hat{\Delta}_i \setminus \hat{\delta}_j$ is conjugate into A. It follows that $\hat{\Delta}_i$ is not algebraically disk-busting in X_i. Since $(\mathrm{inc}_i \circ p_i)_* \colon \pi_1(X_i) \longrightarrow \pi_1(N)$ is isomorphic and $\mathrm{inc}_i \circ p_i(\hat{\Delta}_i) = \Delta_i$, the link Δ_i would not be algebraically disk-busting in N, a contradiction. One can show similarly that $\hat{S}_i$ is 2–incompressible in X_i also rel. $\hat{\Delta}_{i;i_0} = \hat{\Delta}_i \cap p_i^{-1}(\Delta_{i_0})$.

For any $i \geq i_0$, let $\overline{q}_i \colon \overline{Z}_i \longrightarrow N$ be the locally pathwise isometric map extending the covering $q_i \colon Y_i^\circ \longrightarrow N \setminus \Delta_i$ given in Section 2 which satisfies $q_i = p_i$ on $X_i \setminus p_i^{-1}(\Delta_i)$. Note that $\overline{Z}_i$ is the metric completion of $Z_i = X_i \cup Y_i^\circ \setminus \hat{\Delta}_i$. By Proposition 2.1, $\hat{S}_i$ is homotopic in $\overline{Z}_i$ to a CAT$(-a^2)$–ruled wrapping $\hat{\Sigma}_i$ without crossing $\hat{\Delta}_i$. The image $\Sigma_i = \overline{q}_i(\hat{\Sigma}_i)$ is also a CAT$(-a^2)$–surface homotopic in N to S_i.

Since the components S_i of ∂C_i are homeomorphic to each other, all $\hat{\Sigma}_i$ are closed surfaces of the same genus. Fix an $\varepsilon > 0$ less than the Margulis constant for N and so that $\mathcal{N}_{2\varepsilon}(\Delta_{i_0}, N)$ is a tubular neighborhood of Δ_{i_0} in N. Let l be any simple noncontractible loop in $\hat{\Sigma}_i$ of length less than ε. If l were contractible in $\overline{Z}_i$, then l would either bound a disk in $\overline{Z}_i$ disjoint from $\hat{\Delta}_{i;i_0}$ or be contained in $\mathcal{N}_{2\varepsilon}(\hat{\Delta}_{i;i_0}, \overline{Z}_i) \setminus \hat{\Delta}_{i;i_0}$. In either case, this contradicts that $\hat{\Sigma}_i$ is 2–incompressible rel. $\hat{\Delta}_{i;i_0}$. Thus l is not contractible in $\overline{Z}_i$, and hence $\overline{q}_i(l)$ is contained in the ε–thin part $N_{\mathrm{thin}(\varepsilon)}$ of N. From this fact together with Bounded Diameter Lemma [5, Lemma 1.15] for CAT$(-a^2)$–surfaces, we know that the diameter of any component of $\Sigma_i \setminus N_{\mathrm{thin}(\varepsilon)}$ is less than a constant independent of i. Let $\hat{\alpha}_i$ be a ray in $\overline{Z}_i$ beginning at $\hat{\delta}_i$ and covering a proper ray α_i in N such that the sequence $\{\alpha_i\}$ exits $\mathcal{E}$. Since the algebraic intersection number of $\hat{\alpha}_i$ with $\hat{S}_i$ is one, $\hat{\Sigma}_i \cap \hat{\alpha}_i$ and hence $\Sigma_i \cap \alpha_i$ are not empty. This shows that $\{\Sigma_i\}$ exits $\mathcal{E}$. Under the present situation, the tameness of $\mathcal{E}$ is proved by standard arguments in hyperbolic geometry, for example see [2; 6; 13] or Tameness Criteria in [5, Section 6]. However, the case when C_i is a handlebody is exceptional. As is pointed out in the paragraph following the statement of Theorem 3 in Souto [13], we need furthermore to show that $[\Sigma_i] \neq 0$ in $H_2(N \setminus \delta_1; \mathbf{Z})$ for guaranteeing that any surface homologous to Σ_i in $N \setminus C$ excises the end $\mathcal{E}$ from N, where C is a compact core of N with $\mathrm{Int}\, C \supset \delta_1$ and $C \cap \Sigma_i = \varnothing$.

Suppose that a compact core of N and hence any $\overline{X}_i$ are handlebodies. Let J be a tubular neighborhood of δ_1 in N and $\hat{J}$ a lift of J to X_i containing $\hat{\delta}_1$. For i sufficiently large, $J \cap \Sigma_i$ is empty and hence $\overline{q}_i^{-1}(J) \cap \hat{\Sigma}_i \subset \overline{q}_i^{-1}(J \cap \Sigma_i) = \varnothing$.

The closure K of the union of bounded components of $\overline{Z}_i \setminus (\widehat{\Sigma}_i \cup \widehat{J})$ is compact. As $\overline{q}_i^{-1}(\delta_1)$ is disjoint from $\widehat{\Sigma}_i \cup \partial \widehat{J} \supset \partial K$, the preimage $\overline{q}_i^{-1}(\delta_1)$ contains no line components meeting K nontrivially. Since $\overline{q}_i^{-1}(\delta_1) \setminus p_i^{-1}(\delta_1) \subset \partial \overline{Z}_i$, if $\overline{q}_i^{-1}(\delta_1) \cap K$ were nonempty, then each component $\widehat{\delta}'_1$ of the intersection would be a loop component of $p_i^{-1}(\delta_1)$. Since $(\text{inc}_i \circ p_i)_*\colon \pi_1(X_i) \longrightarrow \pi_1(N)$ is an isomorphism, $\widehat{\delta}'_1$ is freely homotopic in $X_i \subset \overline{Z}_i$ to $\widehat{\delta}_1$ up to multiplicity. Then there would exist a CAT($-a^2$)–piecewise ruled annulus in $\overline{Z}_i$ with the geodesic boundary $\widehat{\delta}_1 \cup \widehat{\delta}'_1$, which is a contradiction by the Gauss–Bonnet Theorem. It follows that $\overline{q}_i^{-1}(\delta_1) \cap K = \varnothing$, and so $[\partial \widehat{J}] + [\widehat{\Sigma}_i] = 0$ in $H_2(\overline{Z}_i \setminus \overline{q}_i^{-1}(\delta_1); \mathbf{Z})$. This shows that $[\Sigma_i] = -[\partial J] \neq 0$ in $H_2(N \setminus \delta_1; \mathbf{Z})$. $\qquad\qquad\square$

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