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Refined Kirby calculus for integral homology spheres

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A theorem of Kirby states that two framed links in the 3–sphere produce orientation-preserving homeomorphic results of surgery if they are related by a sequence of stabilization and handle-slide moves. The purpose of the present paper is twofold: First, we give a sufficient condition for a sequence of handle-slides on framed links to be able to be replaced with a sequences of algebraically canceling pairs of handle-slides. Then, using the first result, we obtain a refinement of Kirby's calculus for integral homology spheres which involves only ± 1–framed links with zero linking numbers.

57M25; 57M27

This paper is dedicated to Professor Yukio Matsumoto on the occasion of his sixtieth birthday.

1 Introduction

Every closed, connected, oriented 3–manifold is realized as the result of surgery along a framed link in the 3–sphere, Lickorish [16], Wallace [25]. Kirby's calculus of framed links [10] states that two framed links in the 3–sphere have orientation-preserving homeomorphic results of surgeries if and only if these two links are related by a sequence of two kinds of moves: *stabilizations* and *handle-slides*. Thus Kirby's calculus provides a method to study closed 3–manifolds through a study of framed links. One of the most successful applications of Kirby's calculus is Reshetikhin and Turaev's definition of quantum 3–manifold invariants [22], which is considered to give a mathematical definition of Witten's Chern–Simons path integral [26].

Kirby's calculus involves all the framed links in the 3–sphere, which represent all the closed, connected, oriented 3–manifolds. However, one is sometimes interested in a more special class of 3–manifolds, eg, integral homology spheres. It is natural to expect that, by restricting our attention to a special class of framed links which can represent all the 3–manifolds under consideration, we would be able to obtain a *refinement* of Kirby's calculus of special framed links involving some special types of

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moves, and consequently we would be able to obtain better results than what we would obtain by using Kirby's calculus directly.

The present paper is intended as the first of a series of papers in which we study such refinements of Kirby's calculus. The purpose of the present paper is twofold: First, we establish a general result about sequences of handle-slides on framed links, which will be used as a "main lemma" in the series of papers. Second, we use the main lemma to obtain a refinement of Kirby's calculus for integral homology spheres.

Let us give a rough description of the main lemma (Theorem 2.1). Let M be a connected, oriented 3–manifold, and let $n \geq 0$ be an integer. We consider a category $\mathcal{S}_{M,n}$ whose objects are the isotopy classes of n–component, oriented, ordered, framed links in M, and whose morphisms between two framed links L and L' are sequences from L to L' of handle-slides, orientation reversals and permutations. To each such sequence S, we associate in a functorial way an element $\varphi(S)$ of $\mathrm{GL}(n;\mathbb{Z})$, the group of integral $n \times n$ matrix of determinant ± 1. Then the main lemma states that if the matrix $\varphi(S)$ for $S\colon L \to L'$ is the identity matrix I_n, then there is a sequence from L to L' of *band-slides*. A band-slide on a framed link is an algebraically canceling pair of handle-slides of one component over another, see Figure 1. Note that if the link is null-homologous in M, then a band-slide preserves the linking matrix.

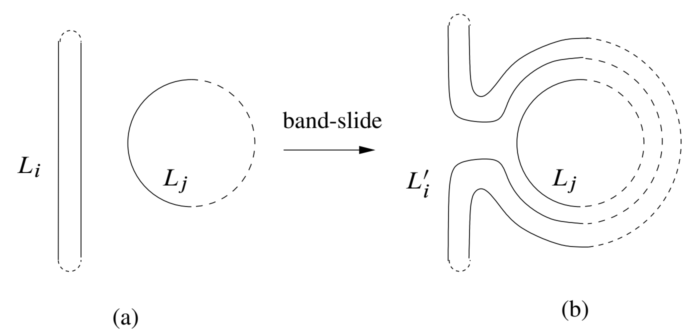

Figure 1: (a) Two components L_i and L_j of a framed link. (b) The result of a band-slide of L_i over L_j.

It is well known that every integral homology sphere can be expressed as the result from S^3 of surgery along a framed link of diagonal linking matrix with diagonal entries ± 1. We call such a framed link *admissible*. (In the literature, it is also called *algebraically split, unit-framed*.) Using the main lemma, we can prove the following refined version of Kirby's calculus for integral homology spheres.

Theorem 1.1 *Two admissible framed links in S^3 have orientation-preserving homeomorphic results of surgery if and only if they are related by a sequence of stabilizations, band-slides and isotopies.*

Hoste [9] conjectures that if two rationally-framed links in S^3 with zero linking numbers and with framings in $\{1/m \mid m \in \mathbb{Z}\}$ have orientation-preserving homeomorphic results of surgery, then they are related by a sequence of Rolfsen's moves [23] through such rationally-framed links. This conjecture follows as a corollary to Theorem 1.1, see Corollary 5.2. We also prove a similar variant of Theorem 1.1 for Fenn and Rourke's theorem [2], see Corollary 5.1. Theorem 1.1 can also be extended to pairs of integral homology spheres and knots, see Corollary 6.2, which is a refined version of a result by Garoufalidis and Kricker [3].

Now we make some comments on applications of the results in the present paper.

Remark 1.2 Hoste [9] proves a surgery formula for the Casson invariant of integral homology 3–spheres and shows that if Corollary 5.2 is true, then his surgery formula provides a simple existence proof of the Casson invariant. This approach to the Casson invariant is perhaps the simplest known one if one admits Corollary 5.2.

Remark 1.3 Recall that Ohtsuki's finite type invariants of integral homology 3–spheres [20], which are generalizations of the Casson invariant, are defined in terms of admissible framed links. Thus it is expected that one can use Theorem 1.1 in the study of Ohtsuki finite type invariants of integral homology spheres. Though this theory of finite type invariants over $\mathbb{Q}$ has been understood to a great extent using the Le–Murakami–Ohtsuki invariant [14; 15], this is not the case for arbitrary coefficient ring. It is expected that, using Theorem 1.1, one can construct a universal Ohtsuki finite invariants of integral homology spheres over $\mathbb{Z}$, and perhaps over arbitrary coefficient ring.

Remark 1.4 In papers in preparation partially joint with T T Q Le [7; 8], we will use Corollary 5.1 to define, for each simple Lie algebra $\mathfrak{g}$, an invariant $J^{\mathfrak{g}}_M$ of an integral homology sphere M which unifies the Witten–Reshetikhin–Turaev invariants of M at all roots of unity (for which the invariant is defined), which is announced in [6], [21, Conjecture 7.29]. Existence of this invariant implies strong integrality properties of the Witten–Reshetikhin–Turaev invariants. Corollary 5.1 enables us to prove the well-definedness of $J^{\mathfrak{g}}_M$ without using any previously known definitions of the Witten–Reshetikhin–Turaev 3–manifold invariant. Thus the definition of $J^{\mathfrak{g}}_M$ provides a new, unified definition of the Witten–Reshetikhin–Turaev invariants of integral homology spheres.

We organize the rest of the paper as follows. In Section 2, we state the main lemma, which is proved in Section 3. In Section 4, we prove Theorem 1.1. In Section 5, we prove Hoste's conjectures. In Section 6, we generalize Theorem 1.1 to pairs of integral homology spheres and knots. In Section 7, we give a short description of several applications of the main lemma, which we will study in future papers.

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2 Definitions and the statement of Main Lemma

In the rest of the paper, all the 3–manifolds are connected and oriented. All homeomorphisms of 3–manifolds are orientation-preserving.

In this and the next sections, we fix a connected, oriented 3–manifold M and an integer $n \geq 0$. Let $\mathcal{L} = \mathcal{L}_{M,n}$ denote the set of isotopy classes of n–component, oriented, ordered, framed links in M. We will systematically confuse a framed link and its isotopy class. We set $\mathsf{I} = \{1, \ldots, n\}$. For $i \in \mathsf{I}$, the ith component of a framed link $L \in \mathcal{L}$ will be denoted by L_i.

2.1 The category $\mathcal{S}$ of framed links and elementary moves

Definition 1 Let $\mathcal{E} = \mathcal{E}_n$ denote the set of symbols

$$\begin{aligned} &\mathsf{P}_{i,j} && \text{for } i,j \in \mathsf{I},\ i \neq j, \\ &\mathsf{Q}_i && \text{for } i \in \mathsf{I}, \\ &\mathsf{W}^{\epsilon}_{i,j} && \text{for } i,j \in \mathsf{I},\ i \neq j \text{ and } \epsilon = \pm 1. \end{aligned}$$

For $e \in \mathcal{E}$, an e–move on $L \in \mathcal{L}$ is defined as follows.

- A $\mathsf{P}_{i,j}$*–move* on L exchanges the order of L_i and L_j.
- A Q_i*–move* on L reverses the orientation of L_i.
- A $\mathsf{W}^{\epsilon}_{i,j}$*–move* on L is a handle-slide of L_i over L_j. If $\epsilon = +1$ (resp. $\epsilon = -1$), then L_i is added to (resp. subtracted from) L_j, see Figure 2.

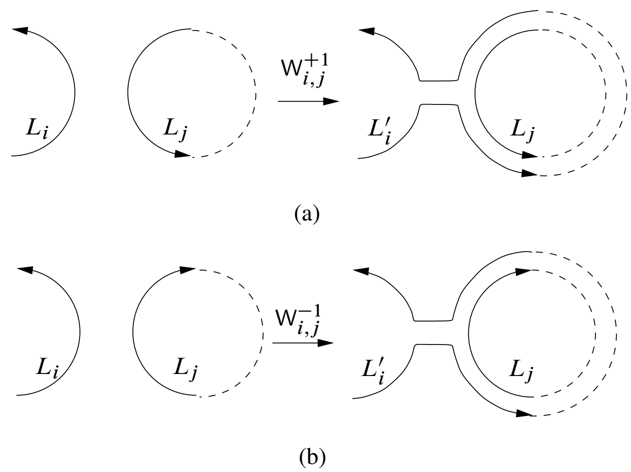

Figure 2: (a) A $\mathsf{W}_{i,j}^{+1}$–move. (b) A $\mathsf{W}_{i,j}^{-1}$–move.

These moves are called *elementary moves*. For $L, L' \in \mathcal{L}$, $e \in \mathcal{E}$, by $L \xrightarrow{e} L'$ we mean that L' is obtained from L by an e–move.

If $e = \mathsf{P}_{i,j}$ or Q_i, then the result from L of an e–move is unique. In this case, we denote the result by $e(L)$. For $e = \mathsf{W}_{i,j}^{\epsilon}$, however, there are in general infinitely many distinct L' satisfying

$$L \xrightarrow{\mathsf{W}_{i,j}^{\epsilon}} L'.$$

Definition 2 Let $\mathcal{S} = \mathcal{S}_{M,n}$ be the free category generated by a graph (in the sense of category theory) whose set of vertices are $\mathcal{L}$, and whose edges are elementary moves. In other words, $\mathcal{S}$ is the category with $\mathrm{Ob}(\mathcal{S}) = \mathcal{L}$, and, for $L, L' \in \mathcal{L}$, the set $\mathcal{S}(L, L')$ of morphisms from L to L' consists of the sequences $S = (L^0, e_1, L^1, e_2, L^2, \dots, e_p, L^p)$ such that $p \geq 0$, $L^0, L^1, \dots, L^p \in \mathcal{L}$, $L^0 = L$, $L^p = L'$, $e_1, \dots, e_p \in \mathcal{E}$, and for $s = 1, \dots, p$ we have $L^{s-1} \xrightarrow{e_s} L^s$. It is convenient to express the sequence S as

$$S\colon L^0 \xrightarrow{e_1} L^1 \xrightarrow{e_2} \cdots \xrightarrow{e_p} L^p.$$

The identity morphism $1_L \in \mathcal{S}(L, L)$ of $L \in \mathcal{L}$ is given by

$$1_L = (L)\colon L \to L.$$

The composite $S'S$ of $S\colon L^0 \xrightarrow{e_1} \cdots \xrightarrow{e_p} L^p$ and $S'\colon K^0 \xrightarrow{e'_1} \cdots \xrightarrow{e'_p} K^p$ with $L^p = K^0$ is given by

$$S'S\colon L^0 \xrightarrow{e_1} \cdots \xrightarrow{e_p} L^p = K^0 \xrightarrow{e'_1} \cdots \xrightarrow{e'_p} K^p.$$

2.2 The functor $\varphi\colon \mathcal{S} \to \mathrm{GL}(n;\mathbb{Z})$ and the statement of Main Lemma

For $i,j \in \mathsf{I}$, let $E_{i,j}$ denote the $n \times n$ matrix such that the (i,j)–entry is 1 and the other entries are 0. Let $I_n = \sum_{i=1}^n E_{i,i}$ be the identity matrix of size n. Define matrices $P_{i,j}, Q_i, W_{i,j} \in \mathrm{GL}(n;\mathbb{Z})$ by

$$\begin{aligned} P_{i,j} &= I_n - E_{i,i} - E_{j,j} + E_{i,j} + E_{j,i}, \\ Q_i &= I_n - 2E_{i,i}, \\ W_{i,j} &= I_n + E_{i,j} \end{aligned}$$

for $i,j \in \mathsf{I},\ i \neq j$. It is well known that these elements generate $\mathrm{GL}(n;\mathbb{Z})$. Note that

$$W_{i,j}^{-1} = I_n - E_{i,j}.$$

We regard the group $\mathrm{GL}(n;\mathbb{Z})$ as a category with one object $*$ in the standard way. Define a functor $\varphi\colon \mathcal{S} \to \mathrm{GL}(n;\mathbb{Z})$ by $\varphi(L) = *$ for $L \in \mathcal{L}$ and

$$\varphi(L \xrightarrow{\mathsf{P}_{i,j}} L') = P_{i,j}, \quad \varphi(L \xrightarrow{\mathsf{Q}_i} L') = Q_i, \quad \varphi(L \xrightarrow{\mathsf{W}_{i,j}^{\pm 1}} L') = W_{i,j}^{\pm 1}.$$

For a morphism $S\colon L^0 \xrightarrow{e_1} L^1 \xrightarrow{e_2} \cdots \xrightarrow{e_p} L^p$, we have

$$\varphi(S) = \varphi(L^{p-1} \xrightarrow{e_p} L^p) \cdots \varphi(L^1 \xrightarrow{e_2} L^2)\varphi(L^0 \xrightarrow{e_1} L^1).$$

Now we state the main lemma.

Theorem 2.1 (Main Lemma) *If a morphism $S\colon L \to L'$ in $\mathcal{S}$ satisfies $\varphi(S) = I_n$, then L and L' are related by a sequence of band-slides.*

2.3 Linking matrices

If a framed link $L \in \mathcal{L}_{M,n}$ is null-homologous in M (ie, each component of L is null-homologous in M), then let A_L denote the linking matrix of L, which is a symmetric matrix with integer entries of size n. Note that if moreover $S \in \mathcal{S}(L,L')$, $L' \in \mathcal{L}_{M,n}$, then L' also is null-homologous.

For a matrix T, let T^t denote the transpose of T.

Lemma 2.2 *If $L, L' \in \mathcal{L}_{M,n}$ are null-homologous and $S \in \mathcal{S}(L, L')$, then we have*

$$A_{L'} = \varphi(S) A_L \varphi(S)^t.$$

Proof The proof is reduced to the case where S consists of only one elementary move, which is well known (see eg Kirby [11]) and can be verified easily. □

2.4 Explanation using 4–manifolds

The following observation is not necessary in the rest of the paper, but explains some ideas of the above definitions.

The functor $\varphi\colon \mathcal{S} \to \mathrm{GL}(n;\mathbb{Z})$ has the following natural topological meaning. For simplicity, we assume $M = S^3$. Recall that for $L \in \mathcal{L}$, we have a 4–manifold X_L obtained from the 4–ball B^4 by attaching 2–handles $h_1, \ldots, h_n$ along the components $L_1, \ldots, L_n \subset S^3 = \partial B^4$ of L, see Kirby [11]. The boundary of X_L is the result of surgery $(S^3)_L$. There is a natural basis $u_1, \ldots, u_n \in H_2(X_L;\mathbb{Z})$, where u_i is represented by the union of the core of the 2–handle h_i and the cone of L_i in B^4.

Suppose $L \xrightarrow{e} L'$ with $e \in \mathcal{E}$. Then we can define a canonical (up to isotopy) diffeomorphism $\widetilde{e}\colon X_L \cong X_{L'}$ as follows. For $e = \mathsf{P}_{i,j}$ or $e = \mathsf{Q}_i$, $\widetilde{e}$ is the obvious one. For $e = \mathsf{W}_{i,j}^{\pm 1}$, $\widetilde{e}$ is the diffeomorphism given by sliding h_i along h_j. Let $u'_1, \ldots, u'_n$ be the basis of $H_2(X_{L'};\mathbb{Z})$. Then we have

$$\widetilde{e}_* = \varphi(e)\colon H_2(X_L;\mathbb{Z}) \to H_2(X_{L'};\mathbb{Z}).$$

Here we regard the matrix $\varphi(e)$ as a $\mathbb{Z}$–linear map using the bases of $H_2(X_L;\mathbb{Z})$ and $H_2(X_{L'};\mathbb{Z})$. More precisely, we have

$$\widetilde{e}_*(u_i) = \sum_{j=1}^{n} \varphi(e)_{i,j} u'_j.$$

For a sequence $S\colon L \xrightarrow{e_1} \cdots \xrightarrow{e_p} L'$ of elementary moves, the matrix $\varphi(S)$ corresponds to the isomorphism $H_2(X_L;\mathbb{Z}) \to H_2(X_{L'};\mathbb{Z})$ obtained as the composite of the isomorphisms corresponding to the elementary moves e_s, $s = 1, \ldots, p$.

3 Proof of Theorem 2.1

3.1 Bands and annuli for handle-slides

In the following, it is sometimes convenient to use bands and annuli in order to keep track of handle-slides.

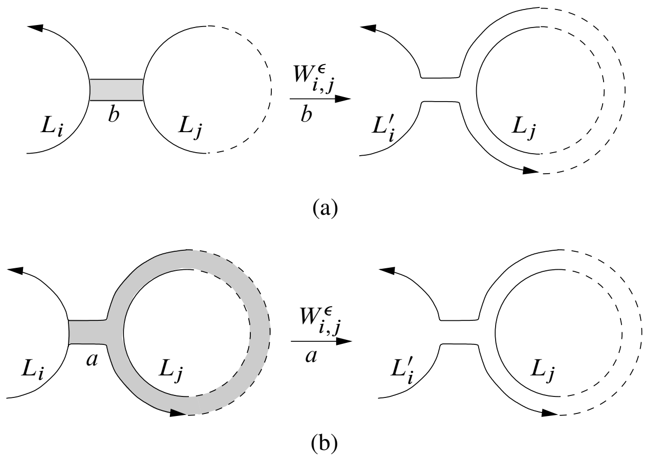

Figure 3

Definition 3 Suppose that $L \xrightarrow{\mathsf{W}^{\epsilon}_{i,j}} L'$.

By a *band for* $L \xrightarrow{\mathsf{W}^{\epsilon}_{i,j}} L'$, we mean a band b in M joining L_i and L_j such that sliding L_i over L_j along b (i.e., replacing L_i with a band sum of L_i and a parallel copy of L_j along b) is a $\mathsf{W}^{\epsilon}_{i,j}$–move, see Figure 3 (a). In this case, we write

$$L \xrightarrow[b]{\mathsf{W}^{\epsilon}_{i,j}} L'.$$

By an *annulus for* $L \xrightarrow{\mathsf{W}^{\epsilon}_{i,j}} L'$, we mean an annulus a in M which looks as depicted in Figure 3 (b), such that "handle-slide of L_i over L_j along a" (ie, replacing L_i with $L'_i = (L_i \cup \partial a) \setminus \mathrm{int}(L_i \cap \partial a))$ is a $\mathsf{W}^{\epsilon}_{i,j}$–move. In this case, we write

$$L \xrightarrow[a]{\mathsf{W}^{\epsilon}_{i,j}} L'.$$

3.2 Reverse moves and reverse sequences

The *reverse* $\overline{e}$ of $e \in \mathcal{E}$ is defined by

$$\overline{\mathsf{P}}_{i,j} = \mathsf{P}_{i,j}, \quad \overline{\mathsf{Q}}_i = \mathsf{Q}_i, \quad \overline{\mathsf{W}}^{\epsilon}_{i,j} = \mathsf{W}^{-\epsilon}_{i,j}.$$

Lemma 3.1 *If* $e \in \mathcal{E}$, $L, L' \in \mathcal{L}$ *and* $L \xrightarrow{e} L'$, *then we have* $L' \xrightarrow{\overline{e}} L$.

Proof If $e = \mathsf{P}_{i,j}$ or $e = \mathsf{Q}_i$, then the assertion is obvious.

Let $e = \mathsf{W}^{\epsilon}_{i,j}$. Choose an annulus a such that $L \xrightarrow[a]{\mathsf{W}^{\epsilon}_{i,j}} L'$. Then we have $L' \xrightarrow[a]{\mathsf{W}^{-\epsilon}_{i,j}} L$. □

For $S\colon L^0 \xrightarrow{e_1} L^1 \xrightarrow{e_2} \cdots \xrightarrow{e_p} L^p$, the *reverse* $\overline{S} \in \mathcal{L}(L^p, L^0)$ of S is defined by

$$\overline{S}\colon L^p \xrightarrow{\overline{e_p}} \cdots \xrightarrow{\overline{e_2}} L^1 \xrightarrow{\overline{e_1}} L^0.$$

We have

$$\varphi(\overline{S}) = \varphi(S)^{-1}.$$

3.3 Decomposition of φ

Let $\mathcal{M}$ denote the free monoid generated by the set $\mathcal{E}$, which is regarded as a category with one object $*$. Define a functor

$$\alpha\colon \mathcal{S} \to \mathcal{M}$$

by $\alpha(L) = *$ for $L \in \mathcal{L}$ and $\alpha(L \xrightarrow{e} L') = e$ for $e \in \mathcal{E}$ with $L \xrightarrow{e} L'$. For a morphism $S\colon L^0 \xrightarrow{e_1} L^1 \xrightarrow{e_2} \cdots \xrightarrow{e_p} L^p$, we have

$$\alpha(S) = e_p \cdots e_2 e_1.$$

For each element $x = e_p \cdots e_2 e_1 \in \mathcal{M}$, the *reverse* of x is defined by

$$\overline{x} = \overline{e_1}\,\overline{e_2} \cdots \overline{e_p}.$$

Clearly, we have $\alpha(\overline{S}) = \overline{\alpha(S)}$ for any morphism S in $\mathcal{S}$.

Define $\mathcal{E}^+$ to be the set of symbols

$$\mathcal{E}^+ = \{\mathsf{p}_{i,j}, \mathsf{q}_i, \mathsf{w}_{i,j} \mid i, j \in \mathsf{I},\ i \neq j\}.$$

Let $\mathcal{G}$ denote the free group generated by the set $\mathcal{E}^+$. Define a homomorphism

$$\beta\colon \mathcal{M} \to \mathcal{G}$$

by

$$\beta(\mathsf{P}_{i,j}) = \mathsf{p}_{i,j}, \quad \beta(\mathsf{Q}_i) = \mathsf{q}_i, \quad \beta(\mathsf{W}_{i,j}^{\pm 1}) = \mathsf{w}_{i,j}^{\pm 1}.$$

For $x \in \mathcal{M}$, we have $\beta(\overline{x}) = \beta(x)^{-1}$.

Define a homomorphism

$$\gamma\colon \mathcal{G} \to \mathrm{GL}(n;\mathbb{Z})$$

by

$$\gamma(\mathsf{p}_{i,j}) = P_{i,j}, \quad \gamma(\mathsf{q}_i) = Q_i, \quad \gamma(\mathsf{w}_{i,j}) = W_{i,j}.$$

Clearly, we have

$$\varphi = \gamma\beta\alpha\colon \mathcal{S} \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{G} \xrightarrow{\gamma} \mathrm{GL}(n;\mathbb{Z}).$$

3.4 Realization lemma

Lemma 3.2 *(1) If $L \in \mathcal{L}$ and $x \in \mathcal{M}$, then there is $S \in \mathcal{S}(L, L')$, $L' \in \mathcal{L}$, such that $\alpha(S) = x$.*

(2) If $L \in \mathcal{L}$ and $x \in \mathcal{M}$, then there is $S \in \mathcal{S}(L', L)$, $L' \in \mathcal{L}$, such that $\alpha(S) = x$.

Proof We prove only (1), since (2) can be similarly proved. Let l be the length of x.

If $l = 0$, then the result is obvious.

If $l = 1$, then $x \in \mathcal{E}$. If $x = \mathsf{P}_{i,j}$ or $x = \mathsf{Q}_i$, then set $S\colon L \xrightarrow{x} x(L)$. If $x = \mathsf{W}^{\epsilon}_{i,j}$, then choose a band b connecting L_i and L_j, and set

$$S\colon L \xrightarrow[b]{\mathsf{W}^{\epsilon}_{i,j}} L',$$

where L' is the result of $\mathsf{W}^{\epsilon}_{i,j}$–move.

The case $l \geq 2$ reduces to the case $l = 1$ by induction. □

3.5 A preorder on $\mathcal{M}$

Recall that a *preorder* on a set X is a binary relation $\Rightarrow$ such that

(1) $x \Rightarrow x$ for all $x \in X$,

(2) $x \Rightarrow y$ and $y \Rightarrow z$ implies $x \Rightarrow z$ for all $x, y, z \in X$.

Define a binary relation $\Rightarrow$ on $\mathcal{M}$ such that for $x, x' \in \mathcal{M}$ we have $x \Rightarrow x'$ if and only if, for any $L, L' \in \mathcal{L}$ and for any $S \in \mathcal{S}(L, L')$ with $\alpha(S) = x$, there is $S' \in \mathcal{S}(L, L')$ satisfying $\alpha(S') = x'$. (Note that the definition of $\Rightarrow$ depends on n and M.) It is obvious that $\Rightarrow$ is a preorder. By $x \Leftrightarrow y$, we mean "$x \Rightarrow y$ and $y \Rightarrow x$", which is an equivalence relation.

Lemma 3.3 *For all $x, x', y, y', z \in \mathcal{M}$, we have the following.*

(1) *$y \Rightarrow y'$ implies $zyx \Rightarrow zy'x$.*

(2) *$x \Rightarrow y$ implies $\bar{x} \Rightarrow \bar{y}$.*

(3) *$yx \Rightarrow z$ implies $x \Rightarrow \bar{y}z$ and $y \Rightarrow z\bar{x}$.*

(4) *$1 \Rightarrow \bar{x}x$.*

Proof (1) Suppose $y \Rightarrow y'$, and $S\colon L \to L'$ with $\alpha(S) = zyx$. S can be decomposed into $S = S^3 S^2 S^1$ with $S^1\colon L \to L^1$, $S^2\colon L^1 \to L^2$, $S^3\colon L^2 \to L'$, $L^1, L^2 \in \mathcal{L}$, such that $\alpha(S^1) = x$, $\alpha(S^2) = y$, $\alpha(S^3) = z$. Since $y \Rightarrow y'$, there is $(S^2)'\colon L^1 \to L^2$ such that $\alpha((S^2)') = y'$. Then we have $S^3 (S^2)' S^1\colon L \to L'$ and $\alpha(S^3 (S^2)' S^1) = zy'x$. Hence $y \Rightarrow y'$ implies $zyx \Rightarrow zy'x$.

(2) Suppose $x \Rightarrow y$, and $S\colon L \to L'$ with $\alpha(S) = \bar{x}$. Then $\bar{S}\colon L' \to L$ satisfies $\alpha(\bar{S}) = x$. Since $x \Rightarrow y$, there is a sequence $S'\colon L' \to L$ such that $\alpha(S') = y$. Then we have $\bar{S}'\colon L \to L'$ and $\alpha(\bar{S}') = \bar{y}$. Hence $x \Rightarrow y$ implies $\bar{x} \Rightarrow \bar{y}$.

(3) Suppose $yx \Rightarrow z$, and $S\colon L \to L'$ with $\alpha(S) = x$. By Lemma 3.2, there is a sequence $S'\colon L' \to L''$ with $\alpha(S') = y$. Since $yx \Rightarrow z$, there is a sequence $S''\colon L \to L''$ with $\alpha(S'') = z$. Then we have $\bar{S}' S''\colon L \to L'$ and $\alpha(\bar{S}' S'') = \bar{y}z$. Hence $yx \Rightarrow z$ implies $x \Rightarrow \bar{y}z$. Similarly, we can prove that $yx \Rightarrow z$ implies $y \Rightarrow z\bar{x}$.

(4) Since $x1 = x \Rightarrow x$, it follows from (3) that $1 \Rightarrow \bar{x}x$. □

Lemma 3.4 *We have the following.*

(A) $\mathsf{P}_{i,k} \Leftrightarrow \mathsf{P}_{k,i}$, $\mathsf{P}_{i,k}^2 \Leftrightarrow 1$, $\mathsf{P}_{i,k}\mathsf{P}_{r,s} \Leftrightarrow \mathsf{P}_{r,s}\mathsf{P}_{i,k}$, $\mathsf{P}_{i,k}\mathsf{P}_{k,r} \Leftrightarrow \mathsf{P}_{i,r}\mathsf{P}_{i,k} \Leftrightarrow \mathsf{P}_{k,r}\mathsf{P}_{i,r}$,

(B) $\mathsf{Q}_i^2 \Leftrightarrow 1$, $\mathsf{Q}_i\mathsf{Q}_k \Leftrightarrow \mathsf{Q}_k\mathsf{Q}_i$, $\mathsf{Q}_j\mathsf{P}_{i,k} \Leftrightarrow \mathsf{P}_{i,k}\mathsf{Q}_j$, $\mathsf{P}_{i,k}\mathsf{Q}_i\mathsf{P}_{i,k} \Leftrightarrow \mathsf{Q}_k$,

(C) $\mathsf{P}_{r,s}\mathsf{W}_{i,k}^{\epsilon} \Leftrightarrow \mathsf{W}_{i,k}^{\epsilon}\mathsf{P}_{r,s}$, $\mathsf{P}_{i,k}\mathsf{W}_{i,k}^{\epsilon} \Leftrightarrow \mathsf{W}_{k,i}^{\epsilon}\mathsf{P}_{i,k}$, $\mathsf{P}_{i,j}\mathsf{W}_{i,k}^{\epsilon} \Leftrightarrow \mathsf{W}_{j,k}^{\epsilon}\mathsf{P}_{i,j}$,
$\mathsf{P}_{i,j}\mathsf{W}_{k,i}^{\epsilon} \Leftrightarrow \mathsf{W}_{k,j}^{\epsilon}\mathsf{P}_{i,j}$,

(D) $\mathsf{Q}_r\mathsf{W}_{i,k}^{\epsilon} \Leftrightarrow \mathsf{W}_{i,k}^{\epsilon}\mathsf{Q}_r$, $\mathsf{Q}_k\mathsf{W}_{i,k}^{\epsilon} \Leftrightarrow \mathsf{W}_{i,k}^{-\epsilon}\mathsf{Q}_k$, $\mathsf{Q}_i\mathsf{W}_{i,k}^{\epsilon} \Leftrightarrow \mathsf{W}_{i,k}^{-\epsilon}\mathsf{Q}_i$,

(E) $\mathsf{W}_{i,k}^{\epsilon}\mathsf{W}_{l,m}^{\xi} \Leftrightarrow \mathsf{W}_{l,m}^{\xi}\mathsf{W}_{i,k}^{\epsilon}$, $\mathsf{W}_{i,k}^{\epsilon}\mathsf{W}_{l,k}^{\xi} \Leftrightarrow \mathsf{W}_{l,k}^{\xi}\mathsf{W}_{i,k}^{\epsilon}$, $\mathsf{W}_{i,k}^{\epsilon}\mathsf{W}_{i,l}^{\xi} \Leftrightarrow \mathsf{W}_{i,l}^{\xi}\mathsf{W}_{i,k}^{\epsilon}$,
$\mathsf{W}_{i,k}^{\epsilon}\mathsf{W}_{i,k}^{\xi} \Leftrightarrow \mathsf{W}_{i,k}^{\xi}\mathsf{W}_{i,k}^{\epsilon}$,

(F) $\mathsf{W}_{i,k}^{+1} \Rightarrow \mathsf{W}_{k,i}^{+1}\mathsf{W}_{i,k}^{-1}\mathsf{P}_{i,k}\mathsf{Q}_i$,

(G) $\mathsf{W}_{i,k}^{\xi}\mathsf{W}_{k,l}^{\epsilon} \Rightarrow \mathsf{W}_{k,l}^{\epsilon}\mathsf{W}_{i,l}^{\epsilon\xi}\mathsf{W}_{i,k}^{\xi}$, $\mathsf{W}_{k,l}^{\epsilon}\mathsf{W}_{i,k}^{\xi} \Rightarrow \mathsf{W}_{i,k}^{\xi}\mathsf{W}_{i,l}^{-\epsilon\xi}\mathsf{W}_{k,l}^{\epsilon}$,

where i, k, etc, are distinct elements in I, and $\epsilon, \xi \in \{\pm 1\}$.

Lemma 3.4 is related to Nielsen's presentation of $\mathrm{GL}(n;\mathbb{Z})$. Taking a look at the statement of Lemma 3.6 below before proceeding may be useful.

Proof (A)–(D) are easy and straightforward.

We will prove (E). They mean that $\mathsf{W}_{i,k}^{\epsilon}$ and $\mathsf{W}_{p,q}^{\xi}$ commute up to $\Leftrightarrow$, if $p \neq k$ and $q \neq i$. Suppose that

$$L \xrightarrow[a]{\mathsf{W}_{i,k}^{\epsilon}} L'' \xrightarrow[a']{\mathsf{W}_{p,q}^{\xi}} L',$$

where a and a' are annuli. We can move a' by an isotopy of M fixing L'' as a subset of M so that

(1) if $q = k$ (ie, $(p,q) = (l,k), (i,k)$), then $a \cap a' = L''_k$, where L''_k denotes the kth component of L'',

(2) if $q \neq k$ (i.e., $(p,q) = (l,m), (i,l)$), then $a \cap a' = \varnothing$.

This can be shown as follows. All the isotopies below fix L'' as a subset. First, if $p = i$, then we isotop a' so that $a' \cap L''_i$ is disjoint from a. Second, if $q = k$, then we isotop a' so that in a small neighborhood of L''_k, a and a' meet only along L''_k. Choose a properly embedded arc c in a' from a point in L''_q to a point $L''_p \cap a'$. Then we isotop a' into a small regular neighborhood of $c \cup L''_q$ in a'. Then we can sweep $(a \cap a') \setminus L''_q$ out of a by an isotopy.

We may regard a' as an annulus for a $\mathsf{W}^{\xi}_{p,q}$–move

$$L \xrightarrow[a']{\mathsf{W}^{\xi}_{p,q}} L''',$$

where $L''' \in \mathcal{L}$. Since $p \neq k$ and $q \neq i$, we have

$$L \xrightarrow[a']{\mathsf{W}^{\xi}_{p,q}} L''' \xrightarrow[a]{\mathsf{W}^{\epsilon}_{i,j}} L'.$$

This shows the direction $\Rightarrow$. The other direction is similar.

We prove (F). Suppose

$$L \xrightarrow[b]{\mathsf{W}^{+1}_{i,k}} L'.$$

Let V be a small regular neighborhood of $L_i \cup L_k \cup b$, which is a handlebody of genus 2. The inside of V looks as depicted in the upper left corner of Figure 4. The

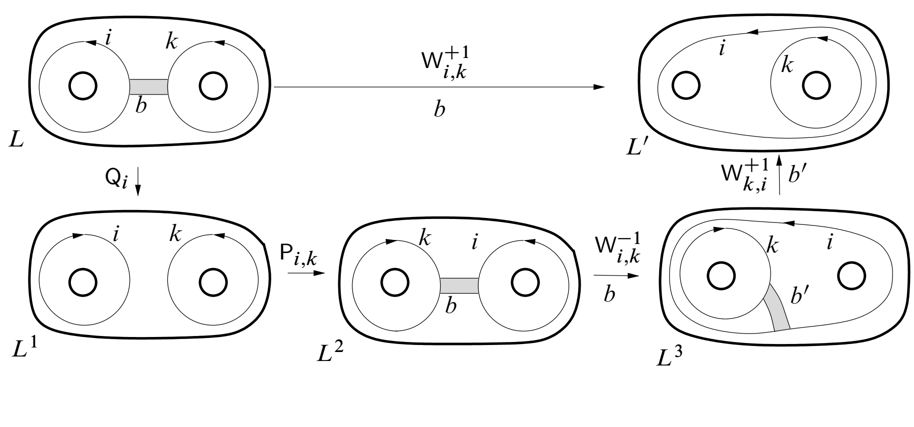

Figure 4

result of $\mathsf{W}^{+1}_{i,k}$–move along b is depicted in the upper right corner. There is a sequence

$$L \xrightarrow{\mathsf{Q}_i} L^1 \xrightarrow{\mathsf{P}_{i,k}} L^2 \xrightarrow[b]{\mathsf{W}^{-1}_{i,k}} L^3 \xrightarrow[b']{\mathsf{W}^{+1}_{k,i}} L'$$

as depicted in Figure 4, which implies (F).

We prove the first formula in (G) for $\epsilon = \xi = 1$. The second formula can be proved similarly. Suppose

$$L \xrightarrow[a]{\mathsf{W}^{+1}_{k,l}} L' \xrightarrow[b]{\mathsf{W}^{+1}_{i,k}} L'', \tag{1}$$

where a is an annulus and b is a band. By moving b with an isotopy of M fixing L' as a subset of M, we may assume that a and b are disjoint. Let V be a small regular neighborhood of $L_i \cup L_k \cup L_l \cup a \cup b$ in M, which is a handlebody of genus 3. The inside of V looks as depicted in the upper left corner of Figure 5, where the sequence (1) is depicted in the top row.

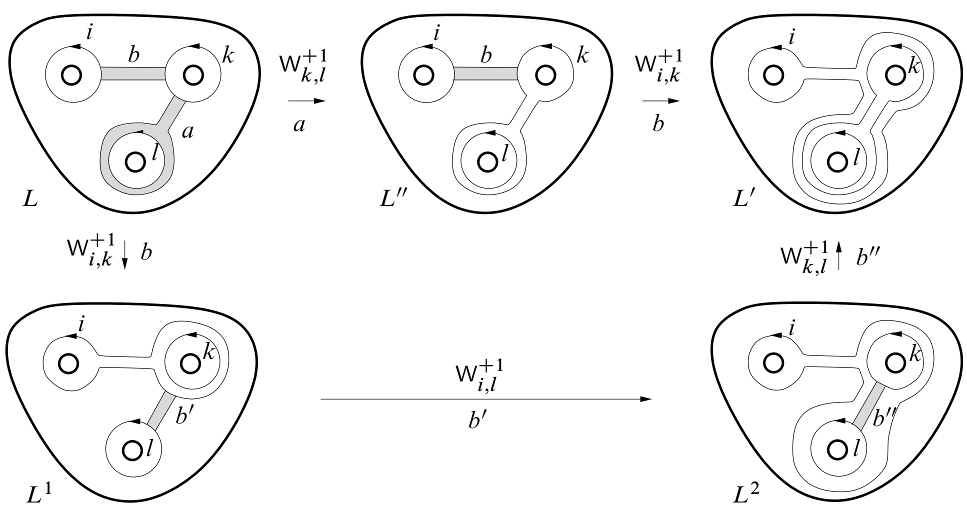

Figure 5

There is a sequence

$$L \xrightarrow[b]{\mathsf{W}^{+1}_{i,k}} L^1 \xrightarrow[b']{\mathsf{W}^{+1}_{i,l}} L^2 \xrightarrow[b'']{\mathsf{W}^{+1}_{k,l}} L'$$

as depicted in Figure 5. Hence we have the first formula for $\epsilon = \xi = 1$. The general case of the first formula can be obtained by conjugating the formula by $\mathsf{Q}_i^{\xi}\mathsf{Q}_k^{\epsilon}$. □

Lemma 3.5 *If $S\colon L \to L'$ with $\alpha(S) = \mathsf{W}^{\mp 1}_{i,k}\mathsf{W}^{\pm 1}_{i,k}$, then L and L' are related by a band-slide.*

Proof Consider the case $(p,q) = (i,k)$ in the proof of (E) of Lemma 3.4. We may assume

$$L \xrightarrow[a]{\mathsf{W}^{\pm 1}_{i,k}} L'' \xrightarrow[a']{\mathsf{W}^{\mp 1}_{i,k}} L',$$

where $a \cap a' = L''_k$. Thus a, a', L_i, L_k look as depicted in Figure 6 (a). By isotopy, we obtain Figure 6 (b). By a band-slide of L_i over L_k as indicated in the figure, we obtain a framed link, which is isotopic to L'. □

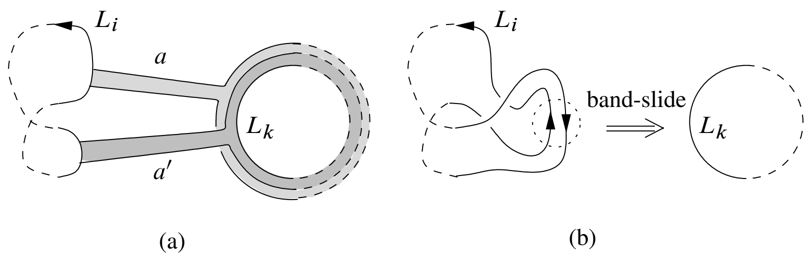

Figure 6

3.6 Realization of relators in GL(n; $\mathbb{Z}$)

The following lemma follows from a result of Nielsen [19], see Magnus–Karrass–Solitar [17, Section 3.5].

Lemma 3.6 (Nielsen) *The group* GL(n; $\mathbb{Z}$) *has a presentation such that the generators are the elements of* $\mathcal{E}^+$ *and the relators are as follows.*

(a) $\mathsf{p}_{i,k} = \mathsf{p}_{k,i}$, $\mathsf{p}_{i,k}^2 = 1$, $\mathsf{p}_{i,k}\mathsf{p}_{r,s} = \mathsf{p}_{r,s}\mathsf{p}_{i,k}$, $\mathsf{p}_{i,k}\mathsf{p}_{k,r} = \mathsf{p}_{i,r}\mathsf{p}_{i,k} = \mathsf{p}_{k,r}\mathsf{p}_{i,r}$,

(b) $\mathsf{q}_i^2 = 1$, $\mathsf{q}_i\mathsf{q}_k = \mathsf{q}_k\mathsf{q}_i$, $\mathsf{q}_j\mathsf{p}_{i,k} = \mathsf{p}_{i,k}\mathsf{q}_j$, $\mathsf{p}_{i,k}\mathsf{q}_i\mathsf{p}_{i,k} = \mathsf{q}_k$,

(c) $\mathsf{p}_{r,s}\mathsf{w}_{i,k} = \mathsf{w}_{i,k}\mathsf{p}_{r,s}$, $\mathsf{p}_{i,k}\mathsf{w}_{i,k} = \mathsf{w}_{k,i}\mathsf{p}_{i,k}$, $\mathsf{p}_{i,j}\mathsf{w}_{i,k} = \mathsf{w}_{j,k}\mathsf{p}_{i,j}$,
$\mathsf{p}_{i,j}\mathsf{w}_{k,i} = \mathsf{w}_{k,j}\mathsf{p}_{i,j}$,

(d) $\mathsf{q}_r\mathsf{w}_{i,k} = \mathsf{w}_{i,k}\mathsf{q}_r$, $\mathsf{q}_k\mathsf{w}_{i,k} = \mathsf{w}_{i,k}^{-1}\mathsf{q}_k$, $\mathsf{q}_i\mathsf{w}_{i,k} = \mathsf{w}_{i,k}^{-1}\mathsf{q}_i$,

(e) $\mathsf{w}_{i,k}\mathsf{w}_{l,m} = \mathsf{w}_{l,m}\mathsf{w}_{i,k}$, $\mathsf{w}_{i,k}\mathsf{w}_{l,k} = \mathsf{w}_{l,k}\mathsf{w}_{i,k}$, $\mathsf{w}_{i,k}\mathsf{w}_{i,l} = \mathsf{w}_{i,l}\mathsf{w}_{i,k}$,

(f) $\mathsf{w}_{i,k}^{-1}\mathsf{w}_{k,i}\mathsf{w}_{i,k}^{-1} = \mathsf{q}_i\mathsf{p}_{i,k}$,

(g) $\mathsf{w}_{i,k}\mathsf{w}_{k,l}\mathsf{w}_{i,k}^{-1}\mathsf{w}_{k,l}^{-1} = \mathsf{w}_{i,l}$.

Here i, k*, etc, denote distinct elements in* I.

Proof Magnus–Karrass–Solitar [17, Section 3.5, Theorem N1] gives a presentation of the automorphism group Φ_n of a free group of rank n, with generators $P_{i,j}, \sigma_i, U_{i,j}, V_{i,j}$ (in the notation of [17]) for $i, j \in \mathsf{I}$, $i \neq j$. This presentation of Φ_n yields a presentation of GL(n; $\mathbb{Z}$) (denoted by Λ_n in [17]) by setting $U_{i,j} = V_{i,j}$. In our notations, $P_{i,j}$ and σ_i are denoted by $\mathsf{p}_{i,j}$ and q_i, respectively, and $U_{i,j} = V_{i,j}$ are denoted by $\mathsf{w}_{i,j}$.

Then we easily obtain from the presentation given in [17] a presentation of $\mathrm{GL}(n;\mathbb{Z})$ with a set of generators $\mathcal{E}^+$ and a set of relations consisting of (a)–(f) above and the following.

(g1) $\mathsf{w}_{i,k}\mathsf{w}_{k,l}^{\epsilon}\mathsf{w}_{i,k}^{-1}\mathsf{w}_{k,l}^{-\epsilon} = \mathsf{w}_{i,l}^{\epsilon} = \mathsf{w}_{k,l}^{-\epsilon}\mathsf{w}_{i,k}\mathsf{w}_{k,l}^{\epsilon}\mathsf{w}_{i,k}^{-1}$,

(g2) $\mathsf{w}_{i,k}^{-1}\mathsf{w}_{k,l}^{\epsilon}\mathsf{w}_{i,k}\mathsf{w}_{k,l}^{-\epsilon} = \mathsf{w}_{i,l}^{-\epsilon} = \mathsf{w}_{k,l}^{-\epsilon}\mathsf{w}_{i,k}^{-1}\mathsf{w}_{k,l}^{\epsilon}\mathsf{w}_{i,k}$.

It is easy to see that (g1) and (g2) reduces to (g) modulo the other relations. □

For each relation of the form $x = y$ in Lemma 3.6, the element $x^{-1}y \in \mathcal{G}$ will be called a *relator* of $\mathrm{GL}(n;\mathbb{Z})$.

Definition 4 Define a map (not a homomorphism) $\lambda\colon \mathcal{G} \to \mathcal{M}$ as follows. For $x \in \mathcal{G}$, let $y_1 \cdots y_p$ be the shortest word representing x such that for $k = 1, \dots, p$ we have either $y_k \in \mathcal{E}^+$ or $y_k^{-1} \in \mathcal{E}^+$. Then we set $\lambda(x) = \lambda(y_1)\cdots\lambda(y_p)$, where

$$\lambda(\mathsf{p}_{i,j}^{\pm 1}) = \mathsf{P}_{i,j}, \quad \lambda(\mathsf{q}_i^{\pm 1}) = \mathsf{Q}_i, \quad \lambda(\mathsf{w}_{i,j}^{\pm 1}) = \mathsf{W}_{i,j}^{\pm 1}.$$

Clearly, we have $\beta\lambda = \mathrm{id}_{\mathcal{G}}$.

Lemma 3.7 *If r is a relator of $\mathrm{GL}(n;\mathbb{Z})$, then we have $1 \Rightarrow \lambda(r)$ and $1 \Rightarrow \lambda(r^{-1})$.*

Proof By Lemma 3.3 (2) and $\lambda(r^{-1}) = \overline{\lambda(r)}$, it suffices to prove $1 \Rightarrow \lambda(r)$.

The assertion follows easily from Lemma 3.4 and Lemma 3.3, where the relations (A)–(G) in Lemma 3.4 corresponds to (a)–(g) in Lemma 3.6. For example, we show the case of the relation (g). In this case, $r = \mathsf{w}_{k,l}\mathsf{w}_{i,k}\mathsf{w}_{k,l}^{-1}\mathsf{w}_{i,k}^{-1}\mathsf{w}_{i,l}$. Hence

$$\lambda(r) = \mathsf{W}_{k,l}^{+1}\mathsf{W}_{i,k}^{+1}\mathsf{W}_{k,l}^{-1}\mathsf{W}_{i,k}^{-1}\mathsf{W}_{i,l}^{+1}.$$

By Lemma 3.4 (G), we have

$$\mathsf{W}_{i,k}^{+1}\mathsf{W}_{k,l}^{+1} \Rightarrow \mathsf{W}_{k,l}^{+1}\mathsf{W}_{i,l}^{+1}\mathsf{W}_{i,k}^{+1}.$$

Hence by Lemma 3.3 (3), we have

$$1 \Rightarrow \mathsf{W}_{k,l}^{+1}\mathsf{W}_{i,l}^{+1}\mathsf{W}_{i,k}^{+1}\mathsf{W}_{k,l}^{-1}\mathsf{W}_{i,k}^{-1}.$$

Since $\mathsf{W}_{i,l}^{+1}$ commutes up to $\Leftrightarrow$ with $\mathsf{W}_{i,k}^{+1}$, $\mathsf{W}_{k,l}^{-1}$, $\mathsf{W}_{i,k}^{-1}$ by Lemma 3.4 (E), we obtain $1 \Rightarrow \lambda(r)$ using Lemma 3.3 (1). □

Lemma 3.8 *For each $x \in \mathcal{M}$ with $\gamma\beta(x) = I_n$, there is $x' \in \mathcal{M}$ such that $\beta(x') = \beta(x)$ and $1 \Rightarrow x'$.*

Proof Since $\gamma\beta(x) = I_n$, $\beta(x)$ is contained in the normal subgroup of $\mathcal{G}$ generated by the relators of $\mathrm{GL}(n;\mathbb{Z})$. ie, we can express $\beta(x)$ as

$$\beta(x) = (u_1^{-1} r_1^{\epsilon_1} u_1) \cdots (u_p^{-1} r_p^{\epsilon_p} u_p),$$

where $p \geq 0$, and for $s = 1, \ldots, p$, r_s is a relator of $\mathrm{GL}(n;\mathbb{Z})$, $\epsilon_s = \pm 1$, and $u_s \in \mathcal{G}$. Set

$$\begin{aligned} x' &= (\lambda(u_1^{-1})\lambda(r_1^{\epsilon_1})\lambda(u_1)) \cdots (\lambda(u_p^{-1})\lambda(r_p^{\epsilon_p})\lambda(u_p)) \\ &= (\overline{\lambda(u_1)}\lambda(r_1^{\epsilon_1})\lambda(u_1)) \cdots (\overline{\lambda(u_p)}\lambda(r_p^{\epsilon_p})\lambda(u_p)). \end{aligned}$$

Clearly, we have $\beta(x') = \beta(x)$. For each $s = 1, \ldots, p$, we have

$$\begin{aligned} 1 &\Rightarrow \overline{\lambda(u_s)}\lambda(u_s) \quad \text{by Lemma 3.3 (4)} \\ &= \overline{\lambda(u_s)}1\lambda(u_s) \\ &\Rightarrow \overline{\lambda(u_s)}\lambda(r_s^{\epsilon_s})\lambda(u_s) \quad \text{by Lemma 3.3 (1) and } 1 \Rightarrow \lambda(r_s^{\epsilon_s}), \end{aligned} \tag{2}$$

where $1 \Rightarrow \lambda(r_s^{\epsilon_s})$ follows from Lemma 3.7. By Lemma 3.3 (1) and (2) for $s = 1, \ldots, p$, it follows that $1 \Rightarrow x'$. □

Lemma 3.9 *For each $x \in \mathcal{M}$ with $\gamma\beta(x) = I_n$, there is $x'' \in \mathcal{M}$ such that $\beta(x'') = 1$ and $x \Rightarrow x''$.*

Proof Set $x'' = \overline{x}'x$, where $x' \in \mathcal{M}$ is as in Lemma 3.8. We have $\beta(x'') = \beta(\overline{x}'x) = \beta(x')^{-1}\beta(x) = 1$. Moreover, using Lemma 3.3, we have $x = 1x \Rightarrow \overline{x}'x = x''$. □

3.7 Submonoids $\mathcal{M}^0$ and $\tilde{\mathcal{M}}^0$ of $\mathcal{M}$

Let $\mathcal{M}^0$ be the submonoid of $\mathcal{M}$ generated by the elements $\mathsf{W}_{i,j}^{-1}\mathsf{W}_{i,j}^{+1}$ for all $i, j \in \mathsf{I}$, $i \neq j$. Let $\tilde{\mathcal{M}}^0$ denote the submonoid of $\mathcal{M}$ consisting of the elements $x \in \mathcal{M}$ such that $x \Rightarrow y$ for some $y \in \mathcal{M}^0$.

Lemma 3.4 (A), (B) and (E) imply that if $e \in \mathcal{E}$, then we have $\overline{e}e \in \tilde{\mathcal{M}}^0$. We will freely use this fact in the proof of the following lemma.

Lemma 3.10 *$\tilde{\mathcal{M}}^0$ is invariant under "conjugation" in $\mathcal{M}$. Ie, if $x \in \tilde{\mathcal{M}}^0$ and $y \in \mathcal{M}$, then we have $\overline{y}xy \in \tilde{\mathcal{M}}_0$.*

Proof Clearly, we may assume that the length of y is 1, ie, $y \in \mathcal{E}$. Since $x \in \widetilde{\mathcal{M}}^0$, we have

$$x \Rightarrow \mathsf{W}_{i_1,j_1}^{-1}\mathsf{W}_{i_1,j_1}^{+1}\mathsf{W}_{i_2,j_2}^{-1}\mathsf{W}_{i_2,j_2}^{+1}\cdots\mathsf{W}_{i_t,j_t}^{-1}\mathsf{W}_{i_t,j_t}^{+1}$$

where $t \geq 0$ and $i_s, j_s \in \mathsf{I}, i_s \neq j_s$ for $s = 1, \ldots, t$. Hence, using Lemma 3.3, we have

$$\begin{aligned}\bar{y}xy &\Rightarrow \bar{y}\mathsf{W}_{i_1,j_1}^{-1}\mathsf{W}_{i_1,j_1}^{+1}\mathsf{W}_{i_2,j_2}^{-1}\mathsf{W}_{i_2,j_2}^{+1}\cdots\mathsf{W}_{i_t,j_t}^{-1}\mathsf{W}_{i_t,j_t}^{+1}y\\ &\Rightarrow \bar{y}\mathsf{W}_{i_1,j_1}^{-1}\mathsf{W}_{i_1,j_1}^{+1}y\bar{y}\mathsf{W}_{i_2,j_2}^{-1}\mathsf{W}_{i_2,j_2}^{+1}y\bar{y}\cdots y\bar{y}\mathsf{W}_{i_t,j_t}^{-1}\mathsf{W}_{i_t,j_t}^{+1}y\\ &= (\bar{y}\mathsf{W}_{i_1,j_1}^{-1}\mathsf{W}_{i_1,j_1}^{+1}y)(\bar{y}\mathsf{W}_{i_2,j_2}^{-1}\mathsf{W}_{i_2,j_2}^{+1}y)\cdots(\bar{y}\mathsf{W}_{i_t,j_t}^{-1}\mathsf{W}_{i_t,j_t}^{+1}y).\end{aligned}$$

It suffices to show that $\bar{y}\mathsf{W}_{i_s,j_s}^{-1}\mathsf{W}_{i_s,j_s}^{+1}y \in \widetilde{\mathcal{M}}^0$ for $s = 1, \ldots, t$. Ie, we may assume that $t = 1$, and what we have to show is the following:

Claim If $i, j \in \mathsf{I}$, $i \neq j$, and $y \in \mathcal{E}$, then we have $\bar{y}\mathsf{W}_{i,j}^{-1}\mathsf{W}_{i,j}^{+1}y \in \widetilde{\mathcal{M}}^0$.

First consider the case where we have $\mathsf{W}_{i,j}^{\pm 1}y \Leftrightarrow y\mathsf{W}_{i',j'}^{\pm\epsilon}$ for some $i', j' \in \mathsf{I}, i' \neq j'$ and $\epsilon = \pm 1$. We have

$$\bar{y}\mathsf{W}_{i,j}^{-1}\mathsf{W}_{i,j}^{+1}y \Leftrightarrow \bar{y}\mathsf{W}_{i,j}^{-1}y\mathsf{W}_{i',j'}^{\epsilon} \Leftrightarrow \bar{y}y\mathsf{W}_{i',j'}^{-\epsilon}\mathsf{W}_{i',j'}^{\epsilon}.$$

If $y = \mathsf{P}_{p,q}$ or Q_p, then the claim follows from $\bar{y}y \Rightarrow 1$. If $y = \mathsf{W}_{p,q}^{+1}$, then the claim immediately holds. If $y = \mathsf{W}_{p,q}^{-1}$, then Lemma 3.4 (E) implies the claim.

Now consider the other cases. We have $y = \mathsf{W}_{p,q}^{\epsilon}$ with either $p = j$ or $q = i$ or both, and $\epsilon = \pm 1$. It suffices to consider the following three cases:

Case 1 $(p,q) = (j,k)$, $k \neq i, j$. We have

$$\begin{aligned}\mathsf{W}_{j,k}^{-\xi}\mathsf{W}_{i,j}^{-\epsilon}\mathsf{W}_{i,j}^{\epsilon}\mathsf{W}_{j,k}^{\xi} &\underset{(G)}{\Rightarrow} \mathsf{W}_{j,k}^{-\xi}\mathsf{W}_{i,j}^{-\epsilon}\mathsf{W}_{j,k}^{\xi}\mathsf{W}_{i,k}^{\epsilon\xi}\mathsf{W}_{i,j}^{\epsilon} \underset{(G)}{\Rightarrow} \mathsf{W}_{j,k}^{-\xi}\mathsf{W}_{j,k}^{\xi}\mathsf{W}_{i,k}^{-\epsilon\xi}\mathsf{W}_{i,j}^{-\epsilon}\mathsf{W}_{i,k}^{\epsilon\xi}\mathsf{W}_{i,j}^{\epsilon}\\ &\underset{(E)}{\Rightarrow} \mathsf{W}_{j,k}^{-\xi}\mathsf{W}_{j,k}^{\xi}\mathsf{W}_{i,k}^{-\epsilon\xi}\mathsf{W}_{i,k}^{\epsilon\xi}\mathsf{W}_{i,j}^{-\epsilon}\mathsf{W}_{i,j}^{\epsilon} \in \widetilde{\mathcal{M}}^0.\end{aligned}$$

Case 2 $(p,q) = (k,i)$, $k \neq i, j$. We have

$$\begin{aligned}\mathsf{W}_{k,i}^{-\xi}\mathsf{W}_{i,j}^{-\epsilon}\mathsf{W}_{i,j}^{\epsilon}\mathsf{W}_{k,i}^{\xi} &\underset{(G)}{\Rightarrow} \mathsf{W}_{k,i}^{-\xi}\mathsf{W}_{i,j}^{-\epsilon}\mathsf{W}_{k,i}^{\xi}\mathsf{W}_{k,j}^{-\epsilon\xi}\mathsf{W}_{i,j}^{\epsilon} \underset{(G)}{\Rightarrow} \mathsf{W}_{k,i}^{-\xi}\mathsf{W}_{k,i}^{\xi}\mathsf{W}_{k,j}^{\epsilon\xi}\mathsf{W}_{i,j}^{-\epsilon}\mathsf{W}_{k,j}^{-\epsilon\xi}\mathsf{W}_{i,j}^{\epsilon}\\ &\underset{(E)}{\Rightarrow} \mathsf{W}_{k,i}^{-\xi}\mathsf{W}_{k,i}^{\xi}\mathsf{W}_{k,j}^{\epsilon\xi}\mathsf{W}_{k,j}^{-\epsilon\xi}\mathsf{W}_{i,j}^{-\epsilon}\mathsf{W}_{i,j}^{\epsilon} \in \widetilde{\mathcal{M}}^0.\end{aligned}$$

Case 3 $(p,q) = (j,i)$. We have

$$\mathsf{W}_{j,i}^{-\xi}\mathsf{W}_{i,j}^{-\epsilon}\mathsf{W}_{i,j}^{\epsilon}\mathsf{W}_{j,i}^{\xi} \underset{(E)}{\Rightarrow} \mathsf{W}_{j,i}^{-\xi}\mathsf{W}_{i,j}^{\xi}\mathsf{W}_{i,j}^{-\xi}\mathsf{W}_{j,i}^{\xi}.$$

If $\xi=+1$, then

$$\mathsf{W}_{j,i}^{-1}\mathsf{W}_{i,j}^{+1}\mathsf{W}_{i,j}^{-1}\mathsf{W}_{j,i}^{+1} \underset{\text{(F)}}{\Rightarrow} \mathsf{W}_{j,i}^{-1}(\mathsf{W}_{j,i}^{+1}\mathsf{W}_{i,j}^{-1}\mathsf{P}_{i,j}\mathsf{Q}_i)(\mathsf{Q}_i\mathsf{P}_{i,j}\mathsf{W}_{i,j}^{+1}\mathsf{W}_{j,i}^{-1})\mathsf{W}_{j,i}^{+1}$$
$$\underset{\text{(A),(B)}}{\Rightarrow} \mathsf{W}_{j,i}^{-1}\mathsf{W}_{j,i}^{+1}\mathsf{W}_{i,j}^{-1}\mathsf{W}_{i,j}^{+1}\mathsf{W}_{j,i}^{-1}\mathsf{W}_{j,i}^{+1} \in \mathcal{M}^0.$$

If $\xi=-1$, then

$$\mathsf{W}_{j,i}^{+1}\mathsf{W}_{i,j}^{-1}\mathsf{W}_{i,j}^{+1}\mathsf{W}_{j,i}^{-1} \underset{\text{(F)}}{\Rightarrow} \mathsf{W}_{j,i}^{+1}(\mathsf{Q}_i\mathsf{P}_{i,j}\mathsf{W}_{i,j}^{+1}\mathsf{W}_{j,i}^{-1})(\mathsf{W}_{j,i}^{+1}\mathsf{W}_{i,j}^{-1}\mathsf{P}_{i,j}\mathsf{Q}_i)\mathsf{W}_{j,i}^{-1}$$
$$\underset{\text{(A),(B),(C),(D)}}{\Longrightarrow} \mathsf{W}_{j,i}^{+1}\mathsf{W}_{j,i}^{-1}\mathsf{W}_{i,j}^{+1}\mathsf{W}_{i,j}^{-1}\mathsf{W}_{j,i}^{+1}\mathsf{W}_{j,i}^{-1} \in \widetilde{\mathcal{M}}^0.$$

This completes the proof of the claim, and hence the lemma. □

Lemma 3.11 *If $x\in\mathcal{M}$ and $\gamma\beta(x)=I_n$, then we have $x\in\widetilde{\mathcal{M}}^0$.*

Proof By Lemma 3.9, we may assume without loss of generality that $\beta(x)=1$. This implies that there is a sequence $x_0=1,x_1,\dots,x_p=x\in\mathcal{M}$ such that for each $s=1,\dots,p$, x_s is obtained from s by inserting $\overline{e}e$ with $e\in\mathcal{E}$, ie, we can write $x_{s-1}=y_{s-1}z_{s-1}$ and $x_s=y_{s-1}\overline{e}ez_{s-1}$. Hence, by inserting $\overline{e}e$, $e\in\mathcal{E}$, finitely many times into x, we obtain $x'\in\mathcal{M}$ with $x\Rightarrow x'$ and

$$x'=(\overline{u_1}u_1)(\overline{u_2}u_2)\cdots(\overline{u_q}u_q),$$

where $q\geq 0$, $u_1,\dots,u_q\in\mathcal{M}$. By Lemma 3.10, it follows that $\overline{u_t}u_t\in\widetilde{\mathcal{M}}^0$ for $t=1,\dots,q$ (using induction on the length of u_t). Hence we have $x'\in\widetilde{\mathcal{M}}^0$. This and $x\Rightarrow x'$ imply $x\in\widetilde{\mathcal{M}}^0$. □

3.8 Proof of Theorem 2.1

By assumption, we have $\gamma\beta\alpha(S)=\varphi(S)=I_n$ for $S\colon L\to L'$ in $\mathcal{S}$. By Lemma 3.11, we have $\alpha(S)\in\widetilde{\mathcal{M}}^0$. Hence there is $y\in\mathcal{M}^0$ such that $\alpha(S)\Rightarrow y$. Hence there is $S'\colon L\to L'$ with $\alpha(S')=y\in\mathcal{M}^0$. By Lemma 3.5, it follows that there is a sequence of band-slides from L to L'.

4 Proof of Theorem 1.1

4.1 Definitions and notations

In this section, we consider null-homotopic framed links in a fixed oriented 3–manifold M. Here a framed link L is said to be *null-homotopic* if every component of L is null-homotopic. For $p, q \geq 0$, set $I_{p,q} = I_p \oplus (-I_q)$, where $\oplus$ denotes block sum. Set

$$\mathcal{L}^0_{M;\, p,q} = \{L \in \mathcal{L}_{M,\, p+q} \mid A_L = I_{p,q},\ L \text{ is null-homotopic in } M\},$$

where A_L denotes the linking matrix of L. Let $\mathcal{S}^0_{M;\, p,q}$ denote the full subcategory of $\mathcal{S}_{M,\, p+q}$ such that $\mathrm{Ob}(\mathcal{S}^0_{M;\, p,q}) = \mathcal{L}^0_{M;\, p,q}$.

A component L_i of a framed link L is said to be *trivial* if it bounds a disc which is disjoint from the other components of L. Here the framing of L_i may be arbitrary.

For $0 \leq p \leq p'$ and $0 \leq q \leq q'$, we define a *stabilization map*

$$\iota_{p',q'}\colon \mathcal{L}^0_{M;\, p,q} \to \mathcal{L}^0_{M;\, p',q'}$$

as follows. For $L \in \mathcal{L}^0_{M;\, p,q}$, let

$$\widehat{L} = \iota_{p',q'}(L) \in \mathcal{L}^0_{M;\, p',q'}$$

denote the framed link obtained from L by adjoining $p' - p$ trivial, $+1$–framed components $O^+_1, \dots, O^+_{p'-p}$, and $q' - q$ trivial, -1–framed components $O^-_1, \dots, O^-_{q'-q}$, so that

$$\widehat{L} = (L_1, \dots, L_p, O^+_1, \dots, O^+_{p'-p}, L_{p+1}, \dots, L_{p+q}, O^-_1, \dots, O^-_{q'-q}),$$

where we express the ordered link $\widehat{L}$ as a sequence of components.

By abuse of notation, we extend this ι notation for elementary move sequences and matrices. For $0 \leq p \leq p'$ and $0 \leq q \leq q'$, define a map

$$\iota_{p',q'}\colon \mathcal{S}^0_{M;\, p,q}(L, L') \to \mathcal{S}^0_{M;\, p',q'}(\iota_{p',q'}(L), \iota_{p',q'}(L'))$$

such that for $S \in \mathcal{S}^0_{M;\, p,q}(L, L')$, $\iota_{p',q'}(S)$ is defined to be the obvious sequence of moves from L to L' obtained from S by adjoining trivial components which are not involved in the sequence of moves. The $\iota_{p',q'}$ defines a functor

$$\iota_{p',q'}\colon \mathcal{S}^0_{M;\, p,q} \to \mathcal{S}^0_{M;\, p',q'}.$$

We also define a homomorphism

$$\iota_{p',q'}\colon \mathrm{GL}(p+q; \mathbb{Z}) \to \mathrm{GL}(p'+q'; \mathbb{Z})$$

as follows. For a matrix $T = \begin{pmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{pmatrix} \in \mathrm{GL}(p+q;\mathbb{Z})$ with $\mathrm{size}(T_{++}) = p$, $\mathrm{size}(T_{--}) = q$, set

$$\iota_{p',q'}(T) = \begin{pmatrix} T_{++} & 0 & T_{+-} & 0 \\ 0 & I_{p'-p} & 0 & 0 \\ T_{-+} & 0 & T_{--} & 0 \\ 0 & 0 & 0 & I_{q'-q} \end{pmatrix}.$$

Note that if $S \in \mathcal{S}^0_{M;\,p,q}(L, L')$, $L, L' \in \mathcal{L}^0_{M;\,p,q}$, then we have

$$\varphi(\iota_{p',q'}(S)) = \iota_{p',q'}(\varphi(S)).$$

For $L, L' \in \mathcal{L}^0_{M;\,p,q}$, by $L \sim_b L'$ we mean that there is a sequence from L to L' of isotopies and band-slides.

4.2 Proof of Theorem 1.1

Theorem 1.1 follows from the case $M = S^3$ of Theorem 4.1 below, which will be proved in the following subsections.

Theorem 4.1 *Let M be a connected, oriented 3–manifold. Let $L, L' \in \mathcal{L}^0_{M;\,p,q}$ and suppose that $\mathcal{S}^0_{M;\,p,q}(L, L') \neq \varnothing$. Then for some $p' \geq p$, $q' \geq q$, we have $\iota_{p',q'}(L) \sim_b \iota_{p',q'}(L')$.*

To prove Theorem 1.1, we need only the case $M = S^3$ of Theorem 4.1. It is for later convenience that we state Theorem 4.1 in a general form.

Proof of Theorem 1.1 assuming Theorem 4.1 The "if" part is obvious. We prove the "only if" part below.

Suppose that two admissible, unoriented, unordered framed links $\widetilde{L}$ and $\widetilde{L}'$ in S^3 have homeomorphic results of surgery. By Kirby's theorem, $\widetilde{L}$ and $\widetilde{L}'$ are related by a sequence of handle-slides after adjoining some trivial ± 1–framed components. Thus we may assume without loss of generality that $\widetilde{L}$ and $\widetilde{L}'$ are related by a sequence of handle-slides.

We choose orientations and orderings of components to $\widetilde{L}$ and $\widetilde{L}'$, obtaining an oriented, ordered framed links $L, L' \in \mathcal{L}_{S^3,n}$, where n is the number of components of L and L'. Here L and L' are chosen so that the linking matrix A_L of L is $I_{p,q}$ with $p, q \geq 0$, and the linking matrix $A_{L'}$ of L' is $I_{p',q'}$ with $p', q' \geq 0$. Since $\widetilde{L}$ and $\widetilde{L}'$ are related by a sequence of handle-slides, the signatures of A_L and $A_{L'}$ are the same. Hence we have $p = p'$, $q = q'$, and $A_L = A_{L'} = I_{p,q}$.

Since there is a sequence from $\widetilde{L}$ to $\widetilde{L}'$ of handle-slides, there is a sequence from L to L' of handle-slides, orientation change, and permutation of components. In other words, $\mathcal{S}^0_{S^3;\,p,q}(L, L') \neq \varnothing$. By Theorem 4.1, there are $p'' \geq p$ and $q'' \geq q$ such that $\iota_{p'',q''}(L) \sim_b \iota_{p'',q''}(L')$ are related by a sequence of band-slides. Hence we have the assertion. □

4.3 Realizing a matrix as a sequence between unlinks

The rest of this section is devoted to the proof of Theorem 4.1. Fix a connected, oriented 3–manifold M. For $p, q \geq 0$, we write $\mathcal{L}_{p,q} = \mathcal{L}_{M;\,p,q}$ and $\mathcal{S}_{p,q} = \mathcal{S}_{M;\,p,q}$.

For $p, q \geq 0$, set

$$O(p, q; \mathbb{Z}) = \{T \in \mathrm{GL}(p + q; \mathbb{Z}) \mid T I_{p,q} T^t = I_{p,q}\},$$

which is a subgroup of $\mathrm{GL}(p + q; \mathbb{Z})$.

In this subsection, we will prove the following lemma.

Lemma 4.2 *Let $p, q \geq 2$ and let $U \in \mathcal{L}^0_{B^3;\,p,q}$ be an unlink. If $T \in O(p, q; \mathbb{Z})$ with $p, q \geq 2$, then there is $S \in \mathcal{S}^0_{B^3;\,p,q}(U, U)$ such that $\varphi(S) = T$.*

Lemma 4.2 holds for any connected, oriented 3–manifold M instead of a 3–ball B^3, but we need only the case of B^3.

To prove Lemma 4.2, we need a set of generators of $O(p, q; \mathbb{Z})$.

Lemma 4.3 (Wall [24, 1.8]) *If $p, q \geq 2$, then $O(p, q; \mathbb{Z})$ is generated by the matrices*

$$\begin{aligned} &P_{i,j} \quad \text{for } 1 \leq i < j \leq p \text{ and for } p + 1 \leq i < j \leq p + q, \\ &Q_i \quad \text{for } 1 \leq i \leq p + q, \end{aligned} \tag{3}$$

and the matrix $D_{p,q} = \iota_{p,q}(D) \in O(p, q; \mathbb{Z})$, where we set

$$D = \begin{pmatrix} 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \in O(2, 2; \mathbb{Z}).$$

Proof of Lemma 4.2 It suffices to prove Lemma 4.2 when T is each of the generators of $O(p,q;\mathbb{Z})$ given in Lemma 4.3. If $T = P_{i,j}$ or $T = Q_i$, then the assertion follows since U is an unlink.

Let us consider the case $T = D_{p,q}$. Without loss of generality we may assume that $p=q=2$, since the case $p=q=2$ implies the general case via the stabilization map $\iota_{p,q}$.

The upper left corner of Figure 7 depicts U. By performing four handle-slides as indicated in the first row in the figure, we obtain $L \in \mathcal{L}_{B^3,4}$. These four handle-slides are realized as $S' \in \mathcal{S}_{B^3,4}(U,L)$ such that

$$\alpha(S') = \mathsf{W}_{4,3}^{-1}\mathsf{W}_{1,3}^{-1}\mathsf{W}_{4,2}^{+1}\mathsf{W}_{1,2}^{+1}.$$

Similarly, as depicted in the second row in Figure 7, there is $S'' \in \mathcal{S}_{B^3,4}(U,L)$ such that

$$\alpha(S'') = \mathsf{W}_{3,4}^{-1}\mathsf{W}_{2,4}^{-1}\mathsf{W}_{3,1}^{+1}\mathsf{W}_{2,1}^{+1}.$$

(Note that S'' is obtained from S' by a permutation of indices $1 \leftrightarrow 2,\ 3 \leftrightarrow 4$.) Set $S = \bar{S}''S' \in \mathcal{S}^0_{2,2}(U,U)$. We have

$$\alpha(S) = \overline{\alpha(S'')}\alpha(S') = \mathsf{W}_{2,1}^{-1}\mathsf{W}_{3,1}^{-1}\mathsf{W}_{2,4}^{+1}\mathsf{W}_{3,4}^{+1}\mathsf{W}_{4,3}^{-1}\mathsf{W}_{1,3}^{-1}\mathsf{W}_{4,2}^{+1}\mathsf{W}_{1,2}^{+1},$$

and hence

$$\varphi(S) = W_{2,1}^{-1}W_{3,1}^{-1}W_{2,4}W_{3,4}W_{4,3}^{-1}W_{1,3}^{-1}W_{4,2}W_{1,2} = D. \qquad \square$$

4.4 Reordering components

Let $L \in \mathcal{L}^0_{p,q}$, and $p' \geq 2p$, $q' \geq 2q$. Let $L^+ = (L_1,\dots,L_p)$ (resp. $L^- = (L_{p+1},\dots,L_{p+q})$) be the sublinks of L consisting of the $+1$–framed (resp. -1–framed) components of L. Set

$$\hat{L} = \iota_{p',q'}(L) = (L^+, O^{+,p}, O^{+,p'-2p}, L^-, O^{-,q}, O^{-,q'-2q}) \in \mathcal{L}^0_{p',q'},$$

where $O^{\pm,k}$ denotes k trivial components of framings ± 1. (Here, by abuse of notation, the sequence of sublinks in the right hand side means a sequence of components.) We also set

$$\hat{L}^\sharp = (O^{+,p}, L^+, O^{+,p'-2p}, O^{-,q}, L^-, O^{-,q'-2q}) \in \mathcal{L}^0_{p',q'},$$

which is obtained from $\hat{L}$ by interchanging L^+ and $O^{+,p}$, and interchanging L^- and $O^{-,q}$.

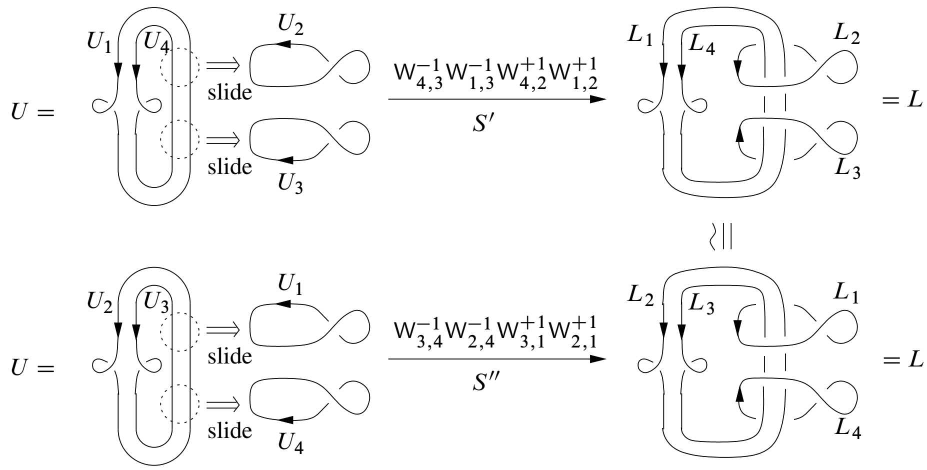

Figure 7

Lemma 4.4 *Let $L \in \mathcal{L}^0_{p,q}$. Then there are integers $p' \geq 2p$ and $q' \geq 2q$ such that in the above notations we have $\widehat{L} \sim_b \widehat{L}^\sharp$.*

Proof We may assume $p, q \geq 2$ without loss of generality.

Let n be a sufficiently large integer which will be determined later. Set $p' = 2p + n$ and $q' = 2q + n$. Define $\widehat{L}^k \in \mathcal{L}^0_{p',q'}$ for $k = 0, \dots, p + q$ inductively by

$$\widehat{L}^k = \begin{cases} \widehat{L} & \text{if } k = 0, \\ \mathsf{P}_{k,p+k}(\widehat{L}^{k-1}) & \text{if } 1 \leq k \leq p, \\ \mathsf{P}_{p'+k-p,p'+q+k-p}(\widehat{L}^{k-1}) & \text{if } p+1 \leq k \leq p+q. \end{cases}$$

We have $\widehat{L}^\sharp = \widehat{L}^{p+q}$.

It suffices to prove the following:

Claim For each $k = 1, \dots, p + q$, we have $\widehat{L}^{k-1} \sim_b \widehat{L}^k$.

Note that each permutation move involved in the definition of $\widehat{L}^k$ permutes a component in L^+ or L^- and a trivial component in $O^{+,p}$ or $O^{-,q}$, respectively. We have only to show that such a permutation can be realized as a sequence of band-slides.

For simplicity, we assume $k = 1$; the other cases are similar. Since L_1 is null-homotopic in M, L_1 can be unknotted after performing finitely many crossing changes of strings of L_1. Since one can take n to be sufficiently large, we can perform each of these crossing changes by a band-slide of L_1 over a trivial component distinct from L_{p+1}, see Figure 8. Let K^1 denote the framed link obtained from $\widehat{L}$ by applying such

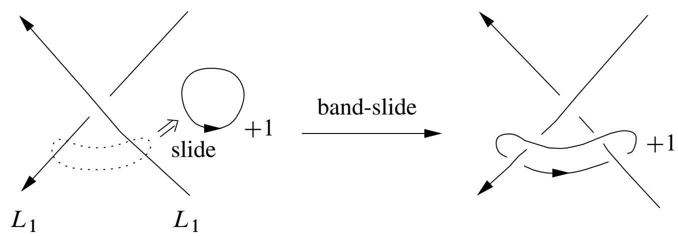

Figure 8: Self crossing change realized as a band-slide over $+1$–framed trivial component. The case of the other sign is similar.

band-slides at $c_1, \dots, c_r$. Note that the first component K^1_1 of K^1 is unknotted and of framing $+1$. Let K^2 denote the result from K^1 by band-sliding all the strands linking with K^1_1 over K^1_1 so that the first component K^2_1 of K^2 is trivial in K^2.

Now we have $K^2 = \mathsf{P}_{1,p+1}(K^2)$ since both the two components K^2_1 and K^2_{p+1} are trivial in K^2. Hence we have

$$\widehat{L}^0 \sim_b K^1 \sim_b K^2 = \mathsf{P}_{1,p+1}(K^2) \sim_b \mathsf{P}_{1,p+1}(K^1) \sim_b \mathsf{P}_{1,p+1}(\widehat{L}^0) = \widehat{L}^1.$$

Here the last two $\sim_b$ can be proved similarly to the first two. This completes the proof of the claim, and hence the lemma. Note that it is sufficient to take n as the maximum of the unknotting numbers of the components of L. □

4.5 Realizing a matrix as a sequence from one link to itself

Lemma 4.5 *Let $L \in \mathcal{L}^0_{p,q}$. There are integers $p' \geq p$, $q' \geq q$ (depending on L) such that for each $T \in O(p,q;\mathbb{Z})$ there is $S \in \mathcal{S}^0_{p',q'}(\widehat{L}, \widehat{L})$, $\widehat{L} = \iota_{p',q'}(L)$, satisfying $\varphi(S) = \iota_{p',q'}(T)$.*

Proof Let $p' \geq p, q' \geq q$, $\widehat{L}^\sharp \in \mathcal{L}^0_{p',q'}$ be as in Lemma 4.4. By Lemma 4.4, there is $S' \in \mathcal{S}^0_{p',q'}(\widehat{L}, \widehat{L}^\sharp)$ with $\varphi(S') = I_{p'+q'}$. Note that the sublink

$$L' = \widehat{L}^\sharp_1 \cup \cdots \cup \widehat{L}^\sharp_p \cup \widehat{L}^\sharp_{p'+1} \cup \cdots \cup \widehat{L}^\sharp_{p'+q}$$

of $\widehat{L}^\sharp$ is an unlink separated from the other components of $\widehat{L}^\sharp$ by a sphere. We can apply Lemma 4.2 to the sublink L' to obtain a sequence $S'' \in \mathcal{S}^0_{p',q'}(\widehat{L}^\sharp, \widehat{L}^\sharp)$ such that

$\varphi(S'') = \iota_{p',q'}(T)$. Set $S = \bar{S}' S'' S' \in \mathcal{S}^0_{p',q'}(\hat{L}, \hat{L})$. Then we have

$$\varphi(S) = \varphi(S')^{-1}\varphi(S'')\varphi(S') = \iota_{p',q'}(T).$$

This completes the proof. □

4.6 Proof of Theorem 4.1

Let $S \in \mathcal{S}^0_{p,q}(L, L')$. By Lemma 4.5, there are $p' \geq p$, $q' \geq q$, $S' \in \mathcal{S}^0_{p',q'}(\hat{L}, \hat{L})$, $\hat{L} = \iota_{p',q'}(L)$ such that $\varphi(S') = \iota_{p',q'}(\varphi(S))$. Set $S'' = \iota_{p',q'}(S)\overline{S'} \in \mathcal{S}^0_{p',q'}(\hat{L}, \hat{L}')$, $\hat{L}' = \iota_{p',q'}(L')$. Then we have $\varphi(S'') = I_{p'+q'}$. Hence it follows from Theorem 2.1 that $\hat{L} \sim_b \hat{L}'$. This completes the proof of Theorem 4.1.

5 Hoste's conjecture

Fenn and Rourke [2] prove that Kirby's moves can be generated by local twisting moves. Rolfsen [23] extends it to framed links with rational framings.

The purpose of this section is to state and prove "Fenn–Rourke version" and "Rolfsen version" of Theorem 1.1, conjectured by Hoste [9].

In this section, framed links are unoriented and unordered for simplicity.

A *Hoste move* is defined to be a Fenn–Rourke move between two admissible framed links, see Figure 9. In (a), the component L_i of L is unknotted and of framing ± 1.

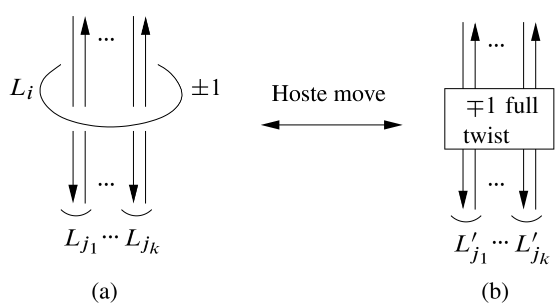

Figure 9: A Hoste move.

Each of the other components of L links with L_i algebraically 0 times. Thus the strands linking with L_i can be paired as depicted, where $i_1, \dots, i_k \neq i$. (Here $i_1, \dots, i_k$ may not be distinct.) The result L' from L of a Hoste move on L_i is shown in (b), which is obtained from L by performing surgery along L_i, i.e., by discarding L_i and giving a ∓ 1 full twist to the bunch of strands linking with L_i.

Corollary 5.1 (Essentially conjectured by Hoste [9]) *Two admissible framed links in S^3 have orientation-preserving homeomorphic results of surgery if and only if they are related by a sequence of Hoste moves.*

A rationally-framed link in S^3 is said to be *admissible* if the linking numbers of any pairs of distinct components are 0, and if the framings are in $\{1/m \mid m \in \mathbb{Z}\}$. Surgery along an admissible rationally-framed link yields an integral homology sphere. A *rational Hoste move* is defined to be a Rolfsen move between two admissible rationally-framed links, see Figure 10.

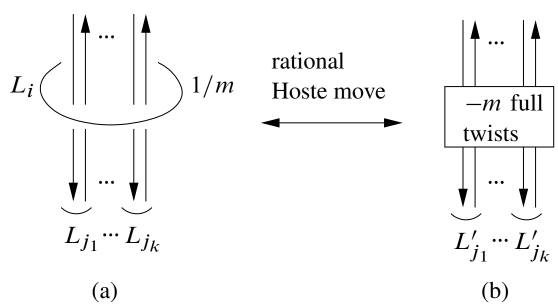

Figure 10: A rational Hoste move. Here m is any integer.

Corollary 5.2 (Conjectured by Hoste [9]) *Two admissible rationally-framed links in S^3 have orientation-preserving homeomorphic results of surgery if and only if they are related by a sequence of rational Hoste moves.*

Corollaries 5.1 and 5.2 can be proved by adapting the proofs by Fenn and Rourke [2] and by Rolfsen [23] of the equivalence of their calculi and Kirby's.

Proof of Corollary 5.1 It is easy to see that a Hoste move can be replaced with a sequence of stabilizations and band-slides. Hence it suffices to prove that a band-slide can be replaced with a sequence of Hoste moves.

Suppose that we are going to perform a band-slide of a component L_i of a framed link L over another component L_j of L. By finitely many crossing changes for strands in L_j we can unknot L_j. This unknotting process can be realized as a sequence of finitely many Hoste moves, see Figure 11. Let L'_j denote the unknotted component obtained from L_j by this process, and let $K_1, \dots, K_r$ be the newly created unknotted components. A band-slide of L_i over L'_j can then be realized by two Hoste moves, see Figure 12. Then we perform Hoste moves for the unknotted component $K_1, \dots, K_r$. The result is isotopic to the result from L by the band-slide of L_i over L_j. □

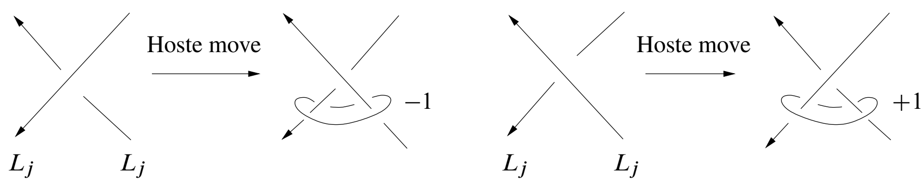

Figure 11: A realization of crossing change of two strands of L_j as a Hoste move.

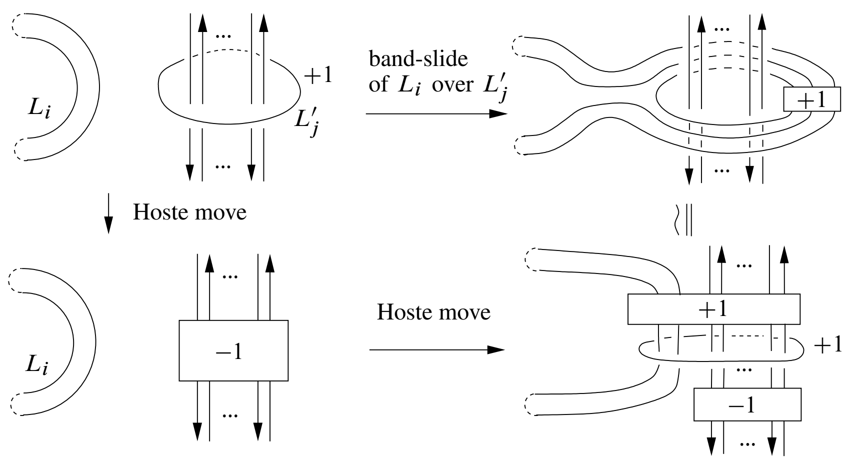

Figure 12: The top row depicts a band-slide of L_i over a $+1$–framed unknotted component L'_j. This can be replaced with two Hoste moves. The case where L'_j is -1–framed is similar.

Proof of Corollary 5.2 It suffices to prove that an admissible rationally-framed link L is related by a sequence of rational Hoste moves to an admissible (integrally) framed link. The proof is by induction on the number of components in L with non-integral framings. Suppose there is a component, say L_1, of non-integral framing $1/m$, $m \in \mathbb{Z}$, $m \neq \pm 1$. We can unknot L_1 by some self-crossing changes of L_1, which can be realized as a sequence of rational Hoste moves introducing ± 1–framed components. Let $L' = L'_1 \cup \cdots$ be the result of these moves, where L'_1 is unknotted and of framing $1/m$. Then we perform a rational Hoste move at L'_1. The resulting framed link and L have homeomorphic result of surgery. Moreover, the number of components of non-integral framing is reduced by one. Hence the assertion follows. □

6 Knots in integral homology spheres

In this section, framed links are unoriented and unordered for simplicity.

Let M be a connected, oriented 3–manifold. If a framed link L in M is null-homotopic and if a framed link L' is related to L by a sequence of Kirby moves, then L' is also null-homotopic.

A (unoriented, unordered) framed link is said to be π_1*–admissible* if it is null-homotopic and has diagonal linking matrix with diagonal entries ± 1. If L is a null-homotopic framed link in L with linking matrix of determinant ± 1, then L is related by a sequence of handle-slides to a π_1–admissible framed link.

As before, a *Hoste move* will mean a Fenn–Rourke move between two π_1–admissible framed links.

Proposition 6.1 *For two π_1–admissible framed links L and L' in M, the following conditions are equivalent.*

(1) *L and L' are related by a sequence of Kirby moves (ie, stabilizations and handle-slides).*
(2) *L and L' are related by a sequence of stabilizations and band-slides.*
(3) *L and L' are related by a sequence of Hoste moves.*

Proof Obviously, (3) implies (2), and (2) implies (1). That (1) implies (2) follows easily from Theorem 4.1. That (2) implies (3) follows from the proof of Corollary 5.1. (Note that we need the fact that L and L' are null-homotopic, in order to unknot some components by Hoste moves in the proofs of Corollary 5.1.) □

Note that any pair (M, K) of an integral homology sphere M and an oriented knot K in M can be realized as a result from the pair (S^3, U) of S^3 and an unknot U of surgery along a π_1–admissible framed link in $S^3 \setminus U$. Using Proposition 6.1, we have the following refined version of a theorem by Garoufalidis and Kricker [3, Theorem 1] on surgery presentations of pairs of integral homology spheres and knots.

Corollary 6.2 *Let L and L' be two π_1–admissible framed links in $S^3 \setminus U$. Then the following conditions are equivalent.*

(1) *The results of surgeries, $(S^3, U)_L$ and $(S^3, U)_{L'}$, are homeomorphic.*
(2) *L and L' are related by a sequence of stabilizations and band-slides.*
(3) *L and L' are related by a sequence of Hoste moves.*

Proof Garoufalidis and Kricker [3, Theorem 1] prove that two null-homotopic framed links in $S^3 \setminus U$ with linking matrices of determinants ± 1 are related by a sequence of Kirby moves if and only if they have homeomorphic results of surgeries. Hence the corollary follows immediately from Proposition 6.1. □

7 Applications

In this section we describe some applications of Theorems 2.1 and 4.1, which we plan to prove in future papers.

7.1 Splitting the degenerate part

A framed link L, or the linking matrix A_L, is *degenerate-split* if A_L is of the form $O_m \oplus A$, where $m \geq 0$ and $\det A \neq 0$. Note that, for any closed 3–manifold M, there is a degenerate-split framed link L such that $M \cong S^3_L$. The first m components of L, which is 0–framed and has 0 linking number with the other components, are called the *degenerate components* or D*–components*. The other components of L are called *nondegenerate components* or N*–components*. Note that handle-slide of a component over a D–component preserves the linking matrix.

Theorem 7.1 *Let L and L' be two (unoriented, unordered) degenerate-split framed links in S^3. Then $(S^3_L) \cong (S^3)_{L'}$ if and only if L and L' are related by a sequence of the following types of moves:*

- *stabilization, ie, adding or removing a ± 1–framed, trivial* N*–component,*
- *handle-slide of a (*D*– or* N*–)component over a* D*–component,*
- *handle-slide of an* N*–component over an* N*–component,*
- *band-slide of a* D*–component over an* N*–component.*

Remark 7.2 A remarkable application of Theorem 7.1 is a refinement of the Le–Murakami–Ohtsuki invariant [15] of closed, connected, oriented 3–manifolds which is universal for all the rational-valued finite type invariants in the sense of Goussarov and the author [4; 5].

We can also prove the following, which is a generalization of Theorem 1.1.

A framed link L in S^3 is *split-admissible* if it is degenerate-split and diagonal with diagonal entries $0, \pm 1$.

Theorem 7.3 *Let L and L' be split-admissible framed links in S^3. Then $(S^3_L) \cong (S^3)_{L'}$ if and only if L and L' are related by a sequence of the following types of moves:*

- *stabilization,*
- *sliding a (*D*– or* N*–)component over a* D*–component,*
- *band-sliding a (*D*– or* N*–)component over an* N*–component.*

We can also give variants of Theorems 7.1 and 7.3 involving only local moves, like Fenn and Rourke's theorem or Theorems 5.1 and 5.2.

It is natural to ask what happens if we drop some of the moves listed in Theorems 7.1 and 7.3. In other words, what kind of topological structure does the equivalence classes of framed links correspond to? For example, we have the following variant of Theorem 7.3.

Theorem 7.4 *Let $m \geq 0$. Let $\mathcal{L}_m$ denote the set of isotopy classes of framed links in S^3 with linking matrices of the form $O_m \oplus I_{p,q}$, $p, q \geq 0$. Let $\bar{\mathcal{L}}_m$ denote the quotient of $\mathcal{L}_m$ by the equivalence relation generated by stabilization and band-sliding. Let $\mathcal{M}_m$ denote the set of equivalence classes of pairs (M, f) of closed, oriented, spin 3–manifolds M and an isomorphism $f\colon \mathbb{Z}^m \to H_1(M;\mathbb{Z})$, where two such pairs (M, f) and (M', f') are equivalent if there is a spin-structure-preserving homeomorphism $\phi\colon M \cong M'$ such that $f' = \phi_* f$. Then there is a natural bijection*

$$\bar{\mathcal{L}}_m \xrightarrow{\simeq} \mathcal{M}_m,$$

which maps a framed link L to the pair (S^3_L, f_L). Here the spin structure of S^3_L is such that the meridian (with 0–framing in S^3) to each D*–component of L represents an "even-framed" curve in S^3_L, and the map f_L maps the i th basis element of $\mathbb{Z}^n$ to the elements represented by the meridian (with 0–framing in S^3) to the i th* D*–component of L.*

7.2 Double-slides

A *double-slide* on a framed link L is defined to be handle-slides of two strands from one component L_i over another component L_j, see Figure 13. Thus a double-slide is either a band-slide or a *parallel double-slide*, where the two strands are parallel. It is easy to see that a band-slide can be realized as a sequence of two parallel double-slides.

A framed link L, or its linking matrix A_L, is 2*–diagonal* if all the non-diagonal entries of A_L are even. For any symmetric integer matrix A of size n there is $B \in \mathrm{GL}(n;\mathbb{Z})$

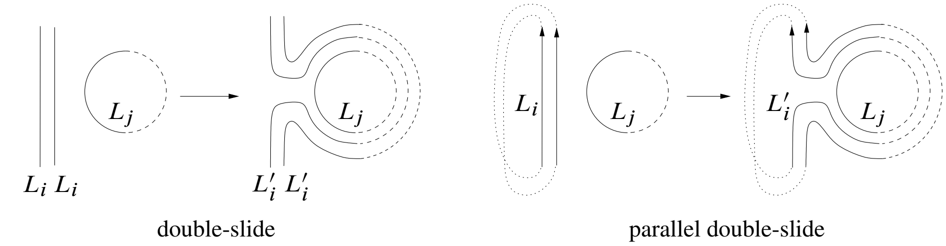

Figure 13: A double-slide and a parallel double-slide.

such that BAB^t is 2–diagonal. Hence any closed, connected, oriented 3–manifold can be obtained from S^3 by surgery along a 2–diagonal framed link. A double-slide on a 2–diagonal framed link L transforms into another 2–diagonal framed link, and preserves the diagonal entries of the linking matrix modulo 4.

For a 2–diagonal framed link L with linking matrix A_L with no diagonal entries congruent to 2 modulo 4, define the *Brown number* $b(L) \in \mathbb{Z}$ of L by

$$b(L) = \sigma_4(A_L) - \sigma(A_L).$$

Here $\sigma_4(A_L) = n_1 - n_{-1}$, where $n_{\pm 1}$ is the number of diagonal entries congruent modulo 4 to ± 1. $\sigma(A_L)$ denotes the signature of A_L. If A_L has at least one diagonal entry $\equiv 2 \pmod 4$, then we formally set $\sigma_4(A_L) = b(L) = \infty$. $\sigma_4(A_L) \bmod 8$ is known as the *Brown invariant* [1] (see also Matsumoto [18], Kirby–Melvin [13]) of the $\mathbb{Z}_4$–valued quadratic form associated to A_L. Moreover, $b(L) \bmod 8$ is known to be an invariant of the 3–manifold S^3_L, called the *Brown invariant* of S^3_L, see Kirby–Melvin [12]. (Here we formally set $\infty \bmod 8 = \infty$.) One can prove that the integer $b(L)$ is invariant under stabilization and double-slides. For each $k \in \mathbb{Z}$, there is a framed link L^k such that $S^3_{L^k} \cong S^3$ and $b(L^k) = 8k$.

A component of a 2–diagonal framed link is *even* (resp. *odd*) if its framing is even (resp. odd).

We have the following $\mathbb{Z}_2$–version of Theorem 7.3.

Theorem 7.5 *Let L and L' be two 2–diagonal framed link of the same Brown number $n \in \mathbb{Z} \cup \{\infty\}$. Then $S^3_L \cong S^3_{L'}$ if and only if L and L' are related by a sequence of the following types of moves:*

- *stabilizations,*
- *double-slides,*
- *handle-slides of (even or odd) components over even components.*

We can modify Theorem 7.5 as follows, which may be regarded as the $\mathbb{Z}_2$–version of Theorem 7.4.

Theorem 7.6 *Let $n \in \mathbb{Z} \cup \{\infty\}$. There is a natural bijection between the set of 2–diagonal, oriented, ordered framed links of Brown number n modulo stabilization and double-slides, and the set of the closed 3–manifolds M of Brown invariant n mod 8, equipped with spin structure and parameterization of $H_1(M;\mathbb{Z}_2)$. The bijection is defined similarly as in Theorem 7.4.*

One can also derive "local move versions" of Theorems 7.5 and 7.6.

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