Classification of continuously transitive circle groups

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Let $G$ be a closed transitive subgroup of Homeo($S^1$) which contains a non-constant continuous path $f : [0, 1] \to G$. We show that up to conjugation $G$ is one of the following groups: $\text{SO}(2, \mathbb{R})$, $\text{PSL}(2, \mathbb{R})$, $\text{PSL}_k(2, \mathbb{R})$, $\text{Homeo}_k(S^1)$, $\text{Homeo}(S^1)$. This verifies the classification suggested by Ghys in [5]. As a corollary we show that the group $\text{PSL}(2, \mathbb{R})$ is a maximal closed subgroup of Homeo($S^1$) (we understand this is a conjecture of de la Harpe). We also show that if such a group $G < \text{Homeo}(S^1)$ acts continuously transitively on $k$–tuples of points, $k > 3$, then the closure of $G$ is $\text{Homeo}(S^1)$ (cf [1]).

1 Introduction

Let $\text{Homeo}(S^1)$ denote the group of orientation preserving homeomorphisms of $S^1$ which we endow with the uniform topology. Let $G$ be a subgroup of Homeo($S^1$) with the topology induced from Homeo($S^1$). We say that $G$ is transitive if for every two points $x, y \in S^1$, there exists a map $f \in G$, such that $f(x) = y$. We say that a group $G$ is closed if it is closed in the topology of Homeo($S^1$). A continuous path in $G$ is a continuous map $f : [0, 1] \to G$.

Let $\text{SO}(2, \mathbb{R})$ denote the group of rotations of $S^1$ and $\text{PSL}(2, \mathbb{R})$ the group of Möbius transformations. The first main result we prove describes transitive subgroups of Homeo($S^1$) that contain a non constant continuous path.

**Theorem 1.1** Let $G$ be a transitive subgroup of Homeo($S^1$) which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds:

1. $G$ is conjugate to $\text{SO}(2, \mathbb{R})$ in Homeo($S^1$).
2. $G$ is conjugate to $\text{PSL}(2, \mathbb{R})$ in Homeo($S^1$).
3. For every $f \in \text{Homeo}(S^1)$ and each finite set of points $x_1, \ldots, x_n \in S^1$ there exists $g \in G$ such that $g(x_i) = f(x_i)$ for each $i$.

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(4) $G$ is a cyclic cover of a conjugate of $\text{PSL}(2, \mathbb{R})$ in $\text{Homeo}(\mathbb{S}^1)$ and hence conjugate to $\text{PSL}_k(2, \mathbb{R})$ for some $k > 1$.

(5) $G$ is a cyclic cover of a group satisfying condition 3 above.

Here we write $\text{PSL}_k(2, \mathbb{R})$ and $\text{Homeo}_k(\mathbb{S}^1)$ to denote the cyclic covers of the groups $\text{PSL}(2, \mathbb{R})$ and $\text{Homeo}(\mathbb{S}^1)$ respectively, for some $k \in \mathbb{N}$.

The proof begins by showing that the assumptions of the theorem imply that $G$ is continuously 1–transitive. This means that if we vary points $x, y \in \mathbb{S}^1$ in a continuous fashion, then we can choose corresponding elements of $G$ which map $x$ to $y$ that also vary in a continuous fashion. In Theorems 3.8 and 3.10 we show that this leads us to two possibilities, either $G$ is conjugate to $\text{SO}(2, \mathbb{R})$, or $G$ is a cyclic cover of a group which is continuously 2–transitive.

We then analyse groups which are continuously 2–transitive and show that they are in fact all continuously 3–transitive. Furthermore, if such a group is not continuously 4–transitive, we show that it is a convergence group and hence conjugate to $\text{PSL}(2, \mathbb{R})$. On the other hand if it is continuously 4–transitive, then we use an induction argument to show that it is continuously $n$–transitive for all $n \geq 4$. This implies that for every $f \in \text{Homeo}(\mathbb{S}^1)$ and each finite set of points $x_1, \ldots, x_n \in \mathbb{S}^1$ there exists a group element $g$ such that $g(x_i) = f(x_i)$ for each $i$.

The remaining possibilities, namely cases 2 and 3, arise when the aforementioned cyclic cover is trivial.

In the case where the group $G$ is also closed we can use Theorem 1.1 to make the following classification.

**Theorem 1.2** Let $G$ be a closed transitive subgroup of $\text{Homeo}(\mathbb{S}^1)$ which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds:

1. $G$ is conjugate to $\text{SO}(2, \mathbb{R})$ in $\text{Homeo}(\mathbb{S}^1)$.
2. $G$ is conjugate to $\text{PSL}_k(2, \mathbb{R})$ in $\text{Homeo}(\mathbb{S}^1)$ for some $k \geq 1$.
3. $G$ is conjugate to $\text{Homeo}_k(\mathbb{S}^1)$ in $\text{Homeo}(\mathbb{S}^1)$ for some $k \geq 1$.

The above theorem provides the classification of closed, transitive subgroups of $\text{Homeo}(\mathbb{S}^1)$ that contain a non-trivial continuous path. This classification was suggested by Ghys for all transitive and closed subgroups of $\text{Homeo}(\mathbb{S}^1)$ (See [5]).

One well known problem in the theory of circle groups is to prove that the group of M"obius transformations is a maximal closed subgroup of $\text{Homeo}(\mathbb{S}^1)$. We understand that this is a conjecture of de la Harpe (see [1]). The following theorem follows directly from our work and answers this question.
Theorem 1.3  PSL(2, \mathbb{R}) is a maximal closed subgroup of Homeo(\mathbb{S}^1).

In the following five sections we develop the techniques needed to prove our results. Here we prove the results about the transitivity on \(k\)--tuples of points. In Section 7 we give the proofs of all the main results stated above.

2 Continuous Transitivity

Let \(G < \text{Homeo}(\mathbb{S}^1)\) be a transitive group of orientation preserving homeomorphisms of \(\mathbb{S}^1\). We begin with some definitions which generalize the notion of transitivity.

Set,

\[ P_n = \{ (x_1, \ldots, x_n) : x_i \in \mathbb{S}^1, x_i = x_j \iff i = j \} \]

to be the set of distinct \(n\)--tuples of points in \(\mathbb{S}^1\). Two \(n\)--tuples

\( (x_1, \ldots, x_n), (y_1, \ldots, y_n) \in P_n \)

have matching orientations if there exists \(f \in \text{Homeo}(\mathbb{S}^1)\) such that \(f(x_i) = y_i\) for each \(i\).

**Definition 2.1**  \(G\) is \(n\)--transitive if for every pair \((x_1, \ldots, x_n), (y_1, \ldots, y_n) \in P_n\) with matching orientations there exists \(g \in G\) such that \(g(x_i) = y_i\) for each \(i\).

**Definition 2.2**  \(G\) is uniquely \(n\)--transitive if it is \(n\)--transitive and for each pair \((x_1, \ldots, x_n), (y_1, \ldots, y_n) \in P_n\) with matching orientations there is exactly one element \(g \in G\) such that \(g(x_i) = y_i\). Equivalently, the only element of \(G\) fixing \(n\) distinct points is the identity.

Endow \(\mathbb{S}^1\) with the standard topology and \(P_n\) with the topology it inherits as a subspace of the \(n\)--fold Cartesian product \(\mathbb{S}^1 \times \cdots \times \mathbb{S}^1\). These are metric topologies. With the topology on \(P_n\) being induced by the distance function

\[ d_{P_n}( (x_1, \ldots, x_n), (y_1, \ldots, y_n) ) = \max\{ d_{\mathbb{S}^1}(x_i, y_i) : i = 1, \ldots, n \}, \]

where \(d_{\mathbb{S}^1}\) is the standard Euclidean distance function on \(\mathbb{S}^1\).

Endow \(G\) with the uniform topology. This is also a metric topology, induced by the distance function,

\[ d_G(g_1, g_2) = \sup\{ \max\{ d_{\mathbb{S}^1}(g_1(x), g_2(x)), d_{\mathbb{S}^1}(g_1^{-1}(x), g_2^{-1}(x)) \} : x \in \mathbb{S}^1 \} \]

A path in a topological space \(X\) is a continuous map \(\gamma : [0, 1] \to X\). If \(\mathcal{X} : [0, 1] \to P_n\) is a path in \(P_n\) we will write \(x_i(t) = \pi_i \circ \mathcal{X}(t)\), where \(\pi_i\) is projection onto the \(i\)--th
We have the following lemma.

We start by showing

\textbf{Definition 2.3} \quad G \text{ is continuously } n\text{--transitive if for every compatible pair of paths} \ X, Y : [0,1] \to P_n \text{ there exists a path } g : [0,1] \to \text{Homeo}(\mathbb{S}^1) \text{ with } g(t)(x_i(t)) = y_i(t) \text{ for each } i \text{ and } t .

\textbf{Definition 2.4} \quad A \text{ continuous deformation of the identity in } G \text{ is a non constant path of homeomorphisms } f_t \in G \text{ for } t \in [0,1] \text{ with } f_0 = \text{id}.

We have the following lemma.

\textbf{Lemma 2.5} \quad For } n \geq 2 \text{ the following are equivalent:

\begin{enumerate}
  \item \quad G \text{ is continuously } n\text{--transitive.}
  \item \quad G \text{ is continuously } n-1\text{--transitive and the following holds. For every } n-1\text{--tuple} \quad (a_1, \ldots , a_{n-1}) \in P_{n-1} \text{ and } x \in \mathbb{S}^1 \setminus \{a_1, \ldots , a_{n-1}\} \text{ there exists a continuous map } F_x : I_x \to G \text{ satisfying the following conditions,}
  \begin{enumerate}
  \item \quad F_x(y) \text{ fixes } a_1, \ldots , a_{n-1} \text{ for all } y \in I_x
  \item \quad (F_x(y))(x) = y \text{ for all } y \in I_x
  \item \quad F_x(x) = \text{id}
  \end{enumerate}
  \text{where } I_x \text{ is the component of } \mathbb{S}^1 \setminus \{a_1, \ldots , a_{n-1}\} \text{ containing } x.
  \item \quad G \text{ is continuously } n-1\text{--transitive and there exists } (a_1, \ldots , a_{n-1}) \in P_{n-1} \text{ with the following property. There is a component } I \text{ of } \mathbb{S}^1 \setminus \{a_1, \ldots , a_{n-1}\} \text{ a point } x \in I \text{ and a continuous map } F_x : I \to G \text{ satisfying the following conditions,}
  \begin{enumerate}
  \item \quad F_x(y) \text{ fixes } a_1, \ldots , a_{n-1} \text{ for all } y \in I
  \item \quad (F_x(y))(x) = y \text{ for all } y \in I
  \item \quad F_x(x) = \text{id}
  \end{enumerate}
  \item \quad G \text{ is continuously } n-1\text{--transitive and there exists } (a_1, \ldots , a_{n-1}) \in P_{n-1} \text{ with the following property. There is a component } I \text{ of } \mathbb{S}^1 \setminus \{a_1, \ldots , a_{n-1}\} \text{ such that for each } x \in I \text{ there exists a continuous deformation of the identity } f_t, \text{ satisfying } f_t(a_i) = a_i \text{ for each } t \text{ and } i \text{ and } f_t(x) \neq x \text{ for some } t .
\end{enumerate}

\textbf{Proof} \quad \text{We start by showing } [1 \Rightarrow 4]. \text{ As } G \text{ is continuously } n\text{--transitive, it will automatically be continuously } n-1\text{--transitive. Take } (a_1, \ldots , a_{n-1}) \in P_{n-1} \text{ and } x \in \mathbb{S}^1 \setminus \{a_1, \ldots , a_{n-1}\} . \text{ Let } I_x \text{ be the component of } \mathbb{S}^1 \setminus \{a_1, \ldots , a_{n-1}\} \text{ which contains } x . \text{ Take } y \in I_x \setminus \{x\} \text{ and let } x_t \text{ be an injective path in } I_x \text{ with } x_0 = x \text{ and } x_1 = y .

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Let $\mathcal{X} : [0, 1] \to P_n$ be the constant path defined by $\mathcal{X}(t) = (a_1, \ldots, a_{n-1}, x_0)$ and let $\mathcal{Y} : [0, 1] \to P_n$ be the path defined by $\mathcal{Y}(t) = (a_1, \ldots, a_{n-1}, x_t)$. Then since $x_t \in I_x$ for every time $t$ these form a compatible pair of paths. Consequently, there exists a path $g_t \in G$ which fixes each $a_i$ and such that $g_t(x) = (x_t)$. Defining $f_t = g_t \circ (g_0^{-1})$ gives us the required continuous deformation of the identity.

We now show that $[4 \Rightarrow 3]$. For $\vec{x} \in I$ set $K_{\vec{x}}$ to be the set of points $x \in I$ for which there is a path of homeomorphisms $f_t \in G$ satisfying,

1. $f_0 = \text{id}$
2. $f_t(a_i) = a_i$ for each $i$ and $t$
3. $f_1(\vec{x}) = x$.

Obviously, $K_{\vec{x}}$ will be a connected subset of $I$ and hence an interval for each $\vec{x} \in I$.

Choose $\vec{x} \in I$ and take $x \in K_{\vec{x}}$. Let $f_t$ and $g_t$ be continuous deformations of the identity which fix the $a_i$ for all $t$ and such that $f_{t_0}(x) \neq x$ for some $t_0 \in (0, 1]$ and $g_1(\vec{x}) = x$. $f_t$ exists by the assumptions of condition 4. and $g_t$ exists because $x \in K_{\vec{x}}$. The following paths show that the interval between $f_{t_0}(x)$ and $(f_{t_0})^{-1}(x)$ is contained in $K_{\vec{x}}$:

$$h_1(t) = \begin{cases} g_{2t} & t \in [0, 1/2] \\ f_{t_0(2t-1)} \circ g_1 & t \in [1/2, 1] \end{cases}$$

$$h_2(t) = \begin{cases} g_{2t} & t \in [0, 1/2] \\ (f_{t_0(2t-1)})^{-1} \circ g_1 & t \in [1/2, 1] \end{cases}$$

As $x$ is contained in this interval and cannot be equal to either of its endpoints we see that $K_{\vec{x}}$ is open for every $\vec{x} \in I$. On the other hand, $\vec{x} \in K_{\vec{x}}$ for each $\vec{x} \in I$ and if $x_1 \in K_{\vec{x}_1}$ then $K_{\vec{x}_1} = K_{\vec{x}_2}$. Consequently, the sets $\{K_{\vec{x}} : \vec{x} \in I\}$ form a partition of $I$ and hence $K_{\vec{x}} = I$ for every $\vec{x} \in I$.

We now construct the map $F_{\vec{x}}$. To do this, take a nested sequence of intervals $[x_n, y_n]$ containing $\vec{x}$ for each $n$ and such that $x_n, y_n$ converge to the endpoints of $I$ as $n \to \infty$. We define $F_{\vec{x}}$ inductively on these intervals. Since $K_{\vec{x}} = I$ we can find a path of homeomorphisms $f_t \in G$ satisfying,

1. $f_0 = \text{id}$
2. $f_t(a_i) = a_i$ for each $i$ and $t$
3. $f_1(\vec{x}) = x_1$.

We now show that there exists a path $\tilde{f}_t \in G$, which also satisfies the above, but with the additional condition that the path $\tilde{f}_t(\vec{x})$ is simple.

To see this, let $[x^*, \vec{x}]$ be the largest subinterval of $[x_1, \vec{x}]$ for which there exists a path $\tilde{f}_t \in G$ which satisfies,
We can use the path 

\[ f_t(\alpha_i) = a_i \text{ for each } i \text{ and } t \]

(3) \[ f_t(\tilde{x}) = x^* \]

(4) \[ f_t(\tilde{x}) \text{ is simple.} \]

We want to show that \( x^* = x_1 \). Assume for contradiction that \( x^* \neq x_1 \). Then since \( x^* \in [x_1, \tilde{x}] \) there exists \( s \in [0, 1] \) such that \( f_s(\tilde{x}) = x^* \) and for small \( \epsilon > 0 \), we have that \( f_{s+\epsilon}(\tilde{x}) \not\in [x^*, \tilde{x}] \). Then if we concatenate the path \( f_t \) with \( f_{s+\epsilon} \circ f_s^{-1} \circ f_t \) for small \( \epsilon \) we can construct a simple path satisfying the same conditions as \( f_t \) but on an interval strictly bigger than \([x^*, \tilde{x}]\), this contradicts the maximality of \( x^* \) and we deduce that \( x^* = x_1 \).

We can use the path \( f_t \) to define a map \( F^1_{\tilde{x}}: [x_1, y_1] \to G \) satisfying,

(1) \( F^1_{\tilde{x}}(y) \) fixes each \( a_i \) for each \( y \in I \)

(2) \( (F^1_{\tilde{x}}(y))(\tilde{x}) = y \) for all \( y \in I \)

(3) \( F^1_{\tilde{x}}(\tilde{x}) = \text{id} \)

by taking paths of homeomorphisms that move \( \tilde{x} \) to \( x_1 \) and \( y_1 \) along simple paths in \( \Sigma^1 \).

Now assume we have defined a map \( F^k_{\tilde{x}}: [x_k, y_k] \to G \) satisfying,

(1) \( F^k_{\tilde{x}}(y) \) fixes each \( a_i \) for each \( y \in I \)

(2) \( (F^k_{\tilde{x}}(y))(\tilde{x}) = y \) for all \( y \in I \)

(3) \( F^k_{\tilde{x}}(\tilde{x}) = \text{id} \)

We can use the same argument used to produce \( F^1_{\tilde{x}} \) to show that there exists a map \( \tilde{\sigma}_{x_k}: [x_{k+1}, x_k] \to G \) such that \( \tilde{\sigma}_{x_k}(x) \) fixes the \( a_i \) for each \( x \), \( \tilde{\sigma}_{x_k}(x_k) = \text{id} \) and \( (\tilde{\sigma}_{x_k}(x))(x_k) = x \). Similarly there exists a map \( \tilde{\sigma}_{y_k}: [y_k, y_{k+1}] \to G \) such that \( \tilde{\sigma}_{y_k}(x) \) fixes the \( a_i \) for each \( x \), \( \tilde{\sigma}_{y_k}(y_k) = \text{id} \) and \( (\tilde{\sigma}_{y_k}(x))(y_k) = x \).

This allows us to define, \( F^{k+1}_{\tilde{x}}: [x_{k+1}, y_{k+1}] \to G \) by:

\[
F^{k+1}_{\tilde{x}}(x) = \begin{cases} 
F^k_{\tilde{x}}(x) & x \in [x_k, y_k] \\
(\tilde{\sigma}_{x_k}(x))(x_k) & x \in [x_{k+1}, x_k] \\
(\tilde{\sigma}_{y_k}(x))(y_k) & x \in [y_k, y_{k+1}] 
\end{cases}
\]

Inductively, we can now define the full map \( F_{\tilde{x}}: I \to G \).

We now show that \( [3 \Rightarrow 2] \). So take \( x' \in I \) with \( x' \neq \tilde{x} \) and define \( F_{x'}: I \to G \) by:

(1) \( F_{x'}(y) = F_{\tilde{x}}(y) \circ (F_{\tilde{x}}(x'))^{-1} \)

Then \( F_{x'} \) satisfies,
(1) \( F_{x'}(y) \) fixes \( a_1, \ldots, a_{n-1} \) for all \( y \in I \)
(2) \( (F_{x'}(y))(x') = y \) for all \( y \in I \)
(3) \( F_{x'}(x') = \text{id} \).

Moreover, we can use (1) to define a map \( F: I \times I \to G \) which is continuous in each variable and satisfies,

(1) \( F(x, y) \) fixes \( a_1, \ldots, a_{n-1} \) for all \( x, y \in I \)
(2) \( F(x, y)(x) = y \) for all \( x, y \in I \)
(3) \( F(x, x) = \text{id} \) for all \( x \in I \).

Now take \( x' \) to be a point in \( S^1 \setminus I \cup \{a_1, \ldots, a_{n-1}\} \) and let \( I' \) be the component of \( S^1 \setminus \{a_1, \ldots, a_{n-1}\} \) which contains \( x' \). Then since \( G \) is continuously \( n-1 \)-transitive there exists \( g \in G \) which permutes the \( a_i \) so that \( g(I) = I' \). Define \( F_{x'}: I' \to G \) by

\[
F_{x'}(y) = g \circ F_{g^{-1}(x')}(g^{-1}(y)) \circ g^{-1}
\]

for \( y \in I' \). Then \( F_{x'} \) satisfies,

(1) \( F_{x'}(y) \) fixes \( a_1, \ldots, a_{n-1} \) for all \( y \in I \)
(2) \( (F_{x'}(y))(x') = y \) for all \( y \in I' \)
(3) \( F_{x'}(x') = \text{id} \).

Now let \( (b_1, \ldots, b_{n-1}) \in P_{n-1} \) have the same orientation as \( (a_1, \ldots, a_{n-1}) \) then since \( G \) is continuously \( n-1 \)-transitive there exists \( g \in G \) so that \( g(a_i) = b_i \) for each \( i \).

Let \( x' \in S^1 \setminus \{b_1, \ldots, b_{n-1}\} \) and let \( I' \) be the component of \( S^1 \setminus \{b_1, \ldots, b_{n-1}\} \) in which it lies. Define \( F_{x'}: I' \to G \) by

\[
F_{x'}(y) = g \circ F_{g^{-1}(x')}(g^{-1}(y)) \circ g^{-1}
\]

for \( y \in I' \). Then \( F_{x'} \) satisfies,

(1) \( F_{x'}(y) \) fixes \( b_1, \ldots, b_{n-1} \) for all \( y \in I \)
(2) \( (F_{x'}(y))(x') = y \) for all \( y \in I' \)
(3) \( F_{x'}(x') = \text{id} \)

and we have that \([3 \Rightarrow 2]\)

Finally we have to show that \([2 \Rightarrow 1]\). Let \( \mathcal{X}, \mathcal{Y}: [0, 1] \to P_n \) be an compatible pair of paths. We define \( \mathcal{X}': [0, 1] \to P_{n-1} \) by

\[
\mathcal{X}'(t) = (x_1(t), \ldots, x_{n-1}(t))
\]
and $Y': [0, 1] \to P_{n-1}$ by

$$Y'(t) = (y_1(t), \ldots, y_{n-1}(t)).$$

Notice that $X'$ and $Y'$ will also be a compatible pair of paths. Furthermore, as $G$ is continuously $n-1$–transitive there will exist a path $g': [0, 1] \to G$ such that $g'(t)(x_i(t)) = y_i(t)$ for $1 \leq i \leq n-1$.

The paths $X', Y': [0, 1] \to P_{n-1}$ will also be compatible with the constant paths,

$X'_0: [0, 1] \to P_{n-1}$

$X'_0(t) = X'(0)$

and

$Y'_0: [0, 1] \to P_{n-1}$

$Y'_0(t) = Y'(0)$

respectively. So that there exist paths $g'_x, g'_y: [0, 1] \to G$ with $g'_x(x_i(0)) = x_i(t)$ and $g'_y(y_i(0)) = y_i(t)$ for $1 \leq i \leq n-1$. Furthermore, by pre composing with $(g'_x(0))^{-1}$ and $(g'_y(0))^{-1}$ if necessary, we can assume that $g'_x(0) = g'_y(0) = \text{id}$.

We now construct a path $g_x: [0, 1] \to G$ which satisfies,

$$g_x(t)(x_i(0)) = x_i(t)$$

for $1 \leq i \leq n$. To do this let $I$ be the component of $S^1 \setminus \{x_1(0), \ldots, x_{n-1}(0)\}$ containing $x_n(0)$. By assumption we have a continuous map $F_{x_n(0)}: I \to G$ satisfying

1. $F_{x_n(0)}(y)$ fixes $x_1(0), \ldots, x_{n-1}(0)$ for all $y \in I$
2. $(F_{x_n(0)}(y))(x) = y$ for all $y \in I$
3. $F_{x_n(0)}(x) = \text{id}$.

Define $g_x: [0, 1] \to G$ by

$$g_x(t) = g'_x(t) \circ (F_{x_n(0)}((g'_x(0))^{-1}(x_n(t))))^{-1}.$$
This is a path in $G$ which satisfies $g_t(x_i(t)) = y_i(t)$ for each $i$ and $t$. Since we can do this for any two compatible paths, $G$ is continuously $n$–transitive and we have shown that $[2 \Rightarrow 1]$. \[
\]

**Proposition 2.6** If $G$ is 1–transitive and there exists a continuous deformation of the identity $f_t : [0, 1] \to G$ in $G$, then $G$ is continuously 1–transitive.

**Proof** Let $x_0 \in \mathbb{S}^1$ be such that $f_{t_0}(x_0) \neq x_0$ for some $t_0 \in [0, 1]$. Take $x \in \mathbb{S}^1$ then there exists $g \in G$ such that $g(x) = x_0$. Consequently, $g^{-1} \circ f_t \circ g$ is a continuous deformation of the identity which doesn’t fix $x$ for some $t$. Since these deformations exist for each $x \in \mathbb{S}^1$ the proof follows in exactly the same way as $[4 \Rightarrow 1]$ from the proof of Lemma 2.5. \[
\]
From now on we will assume that $G$ contains a continuous deformation of the identity, and hence is continuously 1–transitive.

### 3 The set $J_x$

**Definition 3.1** For $x \in \mathbb{S}^1$ we define $J_x$ to be the set of points $y \in \mathbb{S}^1$ which satisfy the following condition. There exists a continuous deformation of the identity $f_t \in G$ which fixes $x$ for all $t$ and such that $f_{t_0}(y) \neq y$ for some $t_0 \in [0, 1]$.

It follows directly from this definition that $x \notin J_x$.

**Lemma 3.2** $J_{f(x)} = f(J_x)$ for every $f \in G$ and $x \in \mathbb{S}^1$.

**Proof** Let $y \in J_{f(x)}$ and let $f_t$ be the corresponding continuous deformation of the identity with $f_{t_0}(y) \neq y$. Then $f^{-1} \circ f_t \circ f$ is also a continuous deformation of the identity which now fixes $x$, and for which $f_{t_0}(f^{-1}(y)) \neq f^{-1}(y)$. This means that $f^{-1}(y) \in J_x$ and hence $y \in f(J_x)$ so that $J_{f(x)} \subseteq f(J_x)$. The other inclusion is an identical argument. \[
\]

**Lemma 3.3** $J_x$ is open for every $x \in \mathbb{S}^1$.

**Proof** Let $y \in J_x$ and take $f_t$ to be the corresponding continuous deformation of the identity with $f_{t_0}(y) \neq y$ for some $t_0 \in [0, 1]$. Then since $f_{t_0}$ is continuous there exists a neighborhood $U$ of $y$ such that $f_{t_0}(z) \neq z$ for all $z \in U$. This implies that $U \subseteq J_x$ and hence that $J_x$ is open. \[
\]
Lemma 3.4  \( J_x = \emptyset \) for every \( x \in \mathbb{S}^1 \) or \( J_x \) has a finite complement for every \( x \in \mathbb{S}^1 \).

To prove this lemma we will use the Hausdorff maximality Theorem which we now recall.

Definition 3.5  A set \( P \) is partially ordered by a binary relation \( \leq \) if,

1. \( a \leq b \) and \( b \leq c \) implies \( a \leq c \)
2. \( a \leq a \) for every \( a \in P \)
3. \( a \leq b \) and \( b \leq a \) implies that \( a = b \).

Definition 3.6  A subset \( Q \) of a partially ordered set \( P \) is totally ordered if for every pair \( a, b \in Q \) either \( a \leq b \) or \( b \leq a \). A totally ordered subset \( Q \subset P \) is maximal if for any member \( a \in P \setminus Q \), \( Q \cup \{a\} \) is not totally ordered.

Theorem 3.7  (Hausdorff Maximality Theorem) Every nonempty partially ordered set contains a maximal totally ordered subset.

We now prove Lemma 3.4.

Proof  Assume that there exists \( x \in \mathbb{S}^1 \) for which \( J_x = \emptyset \). Then for every \( y \in \mathbb{S}^1 \) there exists a map \( g \in G \) such that \( g(x) = y \). Consequently,

\[
J_y = J_{g(x)} = g(J_x) = g(\emptyset) = \emptyset
\]

for every \( y \in \mathbb{S}^1 \).

Assume that \( J_x \neq \emptyset \) for every \( x \in \mathbb{S}^1 \) and let \( S_x = \mathbb{S}^1 \setminus J_x \) denote the complement of \( J_x \). This means that \( S_x \) consists of the points \( y \in \mathbb{S}^1 \) such every continuous deformation of the identity which fixes \( x \) also fixes \( y \). The set \( P = \{S_x : x \in \mathbb{S}^1\} \) is partially ordered by inclusion so that by Theorem 3.7 there exists a maximal totally ordered subset, \( Q = \{S_x : x \in A\} \), where \( A \) is the appropriate subset of \( \mathbb{S}^1 \).

If we set \( S = \bigcap_{x \in A} S_x \) then we have the following:

1. \( S \neq \emptyset \)
2. if \( x \in S \) then \( S_x = S \).
Classification of continuously transitive circle groups

(1) follows from the fact that $S$ is the intersection of a descending family of compact sets, and hence is nonempty.

To see that (2) is also true, fix $x \in S$. Then from the definition of $S$, we will have $x \in S_a$ for each $a \in A$. In other words, if we take $a \in A$, then every continuous deformation of the identity which fixes $a$ will also fix $x$. Furthermore, if $y \in S_x$ then every continuous deformation of the identity which fixes $a$ not only fixes $x$ but $y$ too, so that $S_x \subset S_a$. This is true for every $a \in A$ so that $S_x \subset S$. On the other hand, by the maximality of $Q$, it must contain $S_x$. Consequently, if $x \in S$ then $S_x = S$.

Fix $x_0 \in S$ and assume for contradiction that $S_{x_0}$ is infinite. Take a sequence $x_n \in S_{x_0}$ and let $x_{n_k}$ be a convergent subsequence with limit $x'$. This limit will also be in $S_{x_0}$ as it is closed. As $J_{x_0}$ is a nonempty open subset of $S^1$ it will contain an interval $(a, b)$ with $a, b \in S_{x_0}$. Take maps $g_a, g_b \in G$ so that $g_a(x') = a$ and $g_b(x') = b$. Since $x', a \in S_{x_0}$ we have that,

$g_a(S_{x_0}) = g_a(S_{x'}) = S_{g_a(x')} = S_a = S_{x_0}$

and similarly for $g_b$. As a result $g_a(x_n), g_b(x_n) \in S_{x_0}$ for each $n$, but $g_a, g_b$ are orientation preserving homeomorphisms so that at least one of these points will lie in $(a, b)$, a contradiction.

We have shown that $S_{x_0}$ is finite. If we now take any other point $x \in S^1$ then there exists a map $g \in G$ such that $g(x_0) = x$. This means that the set $S_x = S_{g(x_0)} = g(S_{x_0})$ will also be finite and we are done.

**Theorem 3.8** If $J_x = \emptyset$ for all $x \in S^1$ then $G$ is conjugate in $\text{Homeo}(S^1)$ to the group of rotations $SO(2, \mathbb{R})$.

We require the following lemma for the proof of this Theorem.

**Lemma 3.9** If $f: \mathbb{R} \to \mathbb{R}$ is a homeomorphism which conjugates translations to translations, then it is an affine map.

**Proof** Let $f$ be a homeomorphism which conjugates translations to translations and set $f_1 = T \circ f$ where $T$ is the translation that sends $f(0)$ to 0. Then $f_1$ fixes 0 and also conjugates translations to translations. In particular there exists $\alpha$ such that $f_1$ conjugates $x \mapsto x + 1$ to the map $x \mapsto x + \alpha$. Notice that $\alpha \neq 0$ since the identity is only conjugate to itself.

Now define $f_2 = f_1 \circ M_\alpha$ where $M_\alpha(x) = \alpha x$. A simple calculation shows that $f_2$ conjugates $x \mapsto x + 1$ to itself and conjugates translations to translations. Since $f_2$
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fixes 0 and conjugates \( x \mapsto x + 1 \) to itself, we deduce that it must fix all the integer points.

Now, for \( n \in \mathbb{N} \) let \( \gamma \in \mathbb{R} \) be such that \((f_2)^{-1} \circ T_{1/n} \circ f_2 = T_{\gamma}\) where \( T_{\alpha}(x) = x + \alpha \). It follows that,

\[
T_1 = (f_2)^{-1} \circ (T_{1/n})^n \circ f_2 = ((f_2)^{-1} \circ T_{1/n} \circ f_2)^n = (T_{\gamma})^n
\]

so that \( \gamma = 1/n \) and \((f_2)^{-1} \circ T_{1/n} \circ f_2 = T_{1/n} \) for every \( n \in \mathbb{N} \). Combining this with the fact that \( f_2 \) fixes 0, we see that \( f_2 \) must fix all the rational points and hence is the identity. This implies that \( f_1 \) and hence \( f \) are affine.

We can now prove Theorem 3.8.

Proof Let \( \widehat{G} \subset G \) denote the path component of the identity in \( G \). We are going to show that \( \widehat{G} \) is a compact group. Proposition 4.1 in [5] will then imply that it is conjugate in \( \text{Homeo}(\mathbb{S}^1) \) to a subgroup of \( \text{SO}(2, \mathbb{R}) \). Moreover, as \( \widehat{G} \) is 1–transitive it will be equal to the whole of \( \text{SO}(2, \mathbb{R}) \).

For \( x \in \mathbb{S}^1 \) let \( \pi_x : \mathbb{R} \to \mathbb{S}^1 \) be the usual projection map which sends each integer to \( x \) and for each integer translation \( T : \mathbb{R} \to \mathbb{R} \) satisfies \( \pi_x \circ T = \pi_x \).

If we fix \( x \in \mathbb{S}^1 \) then since \( G \) is continuously 1–transitive we can choose a continuous path \( g : [0, 1] \to G \) such that \( g(t)(x) = \pi_x(t) \) and \( g(0) = \text{id} \). Notice that this path is contained in \( \widehat{G} \) and \( g(1) \) is not necessarily the identity even though it fixes \( x \).

For \( x \in \mathbb{S}^1 \) we define a continuous map \( F_x \) by

\[
F_x(t) = g(t - \lfloor t \rfloor) \circ g(1)^{\lfloor t \rfloor} \quad (*)
\]

where \( \lfloor t \rfloor \) is the greatest integer less than or equal to \( t \). Set \( f = F_x(1) \). Note that \( F_x(n) = f^n \) for every \( n \in \mathbb{Z} \).

We claim that \( F_x \) has the following properties,

1. \( F_x(t)(x) = \pi_x(t) \) for every \( t \in \mathbb{R} \)
2. \( F_x(0) = \text{id} \)
3. The map \( F_x \) is a surjection, that is \( F_x(\mathbb{R}) = \widehat{G} \)
4. If the map \( f = F_x(1) \) is not equal to the identity map then \( F_x \) is a bijection

The first two properties follow directly from the definition. To see that the third property holds, let \( h_s \) be a path in \( \widehat{G} \), \( s \geq 0 \), \( h_0 = \text{id} \). Let \( \alpha(s) = h_s(x) \). We have that \( \alpha \) is a continuous map from the non-negative reals \( \mathbb{R}^+ \) into the circle. Since the set \( \mathbb{R}^+ \)
is contractible we can lift the map \( \alpha \) into the universal cover of the circle. That is, there is a map \( \beta: \mathbb{R}^+ \to \mathbb{R} \) such that \( \pi_x \circ \beta = \alpha \). We have \( F_x(\beta(s))(x) = h_x(x) \).

Then \((h_x^{-1} \circ F_x(\beta(s)))(x) = x\). It follows from the assumption of the theorem that \( F_x(\beta(s)) = h_x \) and \( F_x \) is surjective. The map \( F_x \) is injective for \( 0 \leq t < 1 \), because \( F_x(t)(x) = \pi_x(t) \). If \( F_x(1) \) is not the identity, and since \( F_x(1)(x) = x \) we have that \( F_x(m) = F_x(n) \) if and only if \( m = n \), for every two integers \( m, n \). This implies the fourth property.

It follows from \((\ast)\), and the surjectivity of \( F_x \), that \( \hat{G} \) is a compact group if and only if the cyclic group generated by \( F_x(1) = f \) is a compact group. We will prove that \( f = \text{id} \).

Assume that \( f \) is not the identity map. Since \( F_x \) is a bijection for each \( t \in \mathbb{R} \) there exists a unique \( s_n(t) \in \mathbb{R} \) such that,

\[
f^n \circ F_x(t) \circ f^{-n} = F_x(s_n(t)). \tag{**}
\]

This defines a function \( s_n: \mathbb{R} \to \mathbb{R} \) which we claim is continuous for each \( n \). To see this, fix \( n \) and let \( t_m \in \mathbb{R} \) be a convergent sequence with limit \( t' \). Since \( F_x \) is continuous,

\[
f^n \circ F_x(t_m) \circ f^{-n} \to f^n \circ F_x(t') \circ f^{-n}
\]

and so \( F_x(s_n(t_m)) \to F_x(s_n(t')) \) as \( m \to \infty \).

Now, if \( s_n(t_{m_k}) \) is a convergent subsequence, with limit \( t_0 \), then using continuity \( F_x(s_n(t_{m_k})) \) will converge to \( F_x(t_0) \). Since \( F_x \) is a bijection this gives us that \( t_0 = s_n(t') \). Consequently, if the sequence \( s_n(t_m) \) were bounded, then it would converge to \( t' \).

Assume now that the sequence \( s_n(t_m) \) is unbounded and take a divergent subsequence \( s_n(t_{m_{k_l}}) \). Consider the corresponding sequence,

\[
F_x(s_n(t_{m_k})) = g(s_n(t_{m_k}) - [s_n(t_{m_k})]) \circ f^{[s_n(t_{m_k})]}.
\]

Since \( s_n(t_{m_k}) - [s_n(t_{m_k})] \in [0, 1) \) for each \( m \), there exists a subsequence \( t_{m_{k_l}} \) of \( t_{m_k} \) such that \( s_n(t_{m_{k_l}}) - [s_n(t_{m_{k_l}})] \) converges to some \( t_0 \in [0, 1] \). Now since \( g \) is continuous and the sequence \( F_x(s_n(t_m)) \) converges to a homeomorphism \( F_x(s_n(t')) \) we have that \( f^{[s_n(t_{m_{k_l}})]} \) converges to a homeomorphism as \( l \to \infty \). However, as \( s_n(t_{m_k}) \) is divergent \( [s_n(t_{m_k})] \) will be divergent too.

Let \( S_f \) denote the set of fixed points of \( f \). Note that \( x \notin S_f \). Since we assume that \( f \) is not the identity we have that \( \mathbb{S}^1 \setminus S_f \) is non-empty. Let \( J \) be a component of \( \mathbb{S}^1 \setminus S_f \) and let \( a, b \in \mathbb{S}^1 \) be its endpoints. Since \( f \) fixes \( J \), and has no fixed points inside \( J \) we deduce that on compact subsets of \( J \) the sequence \( f^{[s_n(t_{m_{k_l}})]} \) converges.
to one of the endpoints and consequently, can not converge to a homeomorphism. This is a contradiction, so $s_n(t_m)$ can not be unbounded and $s_n$ is continuous.

Notice that $s_n(0) = 0$ and if $t \in \mathbb{Z}$ then $F_x(t)$ will commute with the $f^n$ so we have $s_n(m) = m$ for all $m \in \mathbb{Z}$. This yields that $s_n([0, 1]) = [0, 1]$ for every $n \in \mathbb{Z}$.

Let $U_f \subset S^1$ be the set defined as follows. We say that $y \in U_f$ if there exists an open interval $I$, $y \in I$, such that $|f^n(I)| \to 0$, $n \to \infty$. Here $|f^n(I)|$ denotes the length of the corresponding interval. The set $U_f$ is open. We show that $U_f$ is non-empty and not equal to $S^1$. As before, let $J$ be a component of $S^1 \setminus S_f$ and let $a, b \in S^1$ be its endpoints. Since $f$ fixes $J$, and has no fixed points inside $J$ we deduce that on compact subsets of $J$ the sequence $f^n$ converges to one of the endpoints, say $a$. This shows that $J \subset U_f$. Also, this shows that the point $b$ does not belong to $U_f$.

Let $y \in U_f$, and let $I$ be the corresponding open interval so that $y \in I$ and $|f^n(I)| \to 0$, $n \to \infty$. Set $f^n(I) = I_n$. Consider the interval $F_x(s_n(t))(I_n)$, $t \in [0, 1]$. Since $s_n([0, 1]) = [0, 1]$ we have that $F_x(s_n([0, 1]))$ is a compact family of homeomorphisms. This allows us to conclude that $|F_x(s_n(t))(I_n)| \to 0$, $n \to \infty$, uniformly in $n$ and $t \in [0, 1]$. Set $J_t = F_x(t)(I)$. From (***) we have that $|f^n(J_t)| \to 0$, $n \to \infty$, for a fixed $t \in [0, 1]$. This implies that the point $F_x(t)(y)$ belongs to the set $U_f$ for every $t \in [0, 1]$.

Let $J$ be a component of $U_f$, and let $a, b$ be its endpoints. Note that the points $a, b$ do not belong to $U_f$. Since $F_x(t)$ is a continuous path and $F_x(0) = \text{id}$, for small enough $t$ we have that $F_x(t)(J) \cap J \neq \emptyset$. Since $F_x(t)(J) \subset U_f$, and since $a, b$ are not in $U_f$ we have that $F_x(t)(J) = J$. By continuity this extends to hold for every $t \in [0, 1]$. But this means that $F_x(t)(a) = a$ for every $t \in [0, 1]$. However, for appropriately chosen inverse $t_0 = \pi_x^{-1}(a)$, we have that $F_x(t_0)(x) = a$, which contradicts the fact that $F_x(t_0)$ is a homeomorphism. This shows that $f = \text{id}$, and therefore we have proved that $G$ is a compact group.

To finish the argument, it remains to show that $G = \widehat{G}$. Let $\Phi \in \text{Homeo}(S^1)$ be a map which conjugates $\widehat{G}$ to $\text{SO}(2, \mathbb{R})$ and take $g \in G \setminus \widehat{G}$. Since $\widehat{G}$ is a normal subgroup of $G$, $\Phi \circ g \circ \Phi^{-1}$ conjugates rotations to rotations. Lifting to the universal cover we get that every lift of $\Phi \circ g \circ \Phi^{-1}$ conjugates translations to translations. If we choose one then by Lemma 3.9 it will be affine. On the other hand, it must be periodic, and hence is a translation. So that $\Phi \circ g \circ \Phi^{-1}$ is itself a rotation and we are done. \hfill \Box

**Theorem 3.10** If $J_x \neq \emptyset$ then one of the following is true:

1. $J_x = S^1 \setminus \{x\}$ in which case $G$ is continuously 2–transitive.
(2) There exists $R \in \text{Homeo}(\mathbb{S}^1)$ which is conjugate to a finite order rotation and satisfies $R \circ g = g \circ R$ for every $g \in G$. Moreover, $G$ is a cyclic cover of a group $G_{\Gamma}$ which is continuously 2–transitive, where the covering transformations are the cyclic group generated by $R$.

Proof If $J_x = \mathbb{S}^1 \setminus \{x\}$ then we are in case 4 of Lemma 2.5 with $n = 2$. In this situation we know that $G$ will be continuously 2–transitive.

We already know that $S_x = \mathbb{S}^1 \setminus J_x$ must contain $x$ and by Lemma 3.4 must be finite. Moreover, as $f(J_x) = J_{f(x)}$ the sets $S_x$ contain the same number of points for each $x \in \mathbb{S}^1$. Define $R: \mathbb{S}^1 \to \mathbb{S}^1$ by taking $R(x)$ to be the first point of $S_x$ you come to as you travel clockwise around $\mathbb{S}^1$. Now take $g \in G$ and $x \in \mathbb{S}^1$, then since $J_g(x) = g(J_x)$ and $g$ is orientation preserving $R \circ g(x) = g \circ R(x)$ for all $x \in \mathbb{S}^1$.

We now show that $R$ is a homeomorphism. To see this take any continuous path $x_t \in \mathbb{S}^1$, we will show that $R(x_t) \to R(x_0)$ as $t \to 0$. Since $G$ is continuously 1–transitive, there exists a continuous path $g_t \in G$ satisfying $g_t(x_t) = x_0$, so that,

$$
\lim_{t \to 0} R(x_t) = \lim_{t \to 0} (g_t)^{-1}(R(g_t(x_t))) = \lim_{t \to 0} (g_t)^{-1}(R(x_0)) = R(x_0),
$$

where the first equality follows from the fact that $R \circ g(x) = g \circ R(x)$ for all $x \in \mathbb{S}^1$. This shows that $R$ is continuous. If we take $y \notin J_x$ then $J_x \subset J_y$, and hence $S_x \supset S_y$ but in this case since $S_x$ and $S_y$ contain the same number of points they will be equal. Consequently, $R$ has an inverse defined by taking $R^{-1}(x)$ to be the first point of $S_x$ you come to by traveling clockwise around $\mathbb{S}^1$ and this inverse is continuous by the same argument as for $R$. Consequently, $R \in \text{Homeo}(\mathbb{S}^1)$. Furthermore, $R$ is of finite order equal to the number of points in $S_x$ and hence conjugate to a rotation.

Let $\Gamma$ denote the cyclic subgroup of $\text{Homeo}(\mathbb{S}^1)$ generated by $R$. Define $\pi: \mathbb{S}^1 \to \mathbb{S}^1/\Gamma \cong \mathbb{S}^1$, in the usual way with $\pi(x)$ being the orbit of $x$ under $\Gamma$. Since $R \circ g(x) = g \circ R(x)$ for all $x \in \mathbb{S}^1$, each $g \in G$ defines a well defined homeomorphism of the quotient space $\mathbb{S}^1/\Gamma$ which we call $g_{\Gamma}$. This gives us a homomorphism $\pi_{\Gamma}: G \to \text{Homeo}(\mathbb{S}^1)$, defined by $\pi_{\Gamma}(g) = g_{\Gamma}$. Let $G_{\Gamma}$ denote the image of $G$ under $\pi_{\Gamma}$, then $G_{\Gamma}$ is a cyclic cover of $G_{\Gamma}$.

It remains to see that $G_{\Gamma}$ is continuously 2–transitive. This follows from the fact that if we take $x_0 \in \mathbb{S}^1$ then $J_{\pi(x_0)} = \pi(J_{x_0})$, where $J_{\pi(x_0)}$ is the set of points that can be moved by continuous deformations of the identity in $G_{\Gamma}$ which fix $\pi(x_0)$. Consequently, $J_{\pi(x_0)} = \mathbb{S}^1 \setminus \{x_0\}$ so that $G_{\Gamma}$ is continuously 2–transitive by the first part of this proposition. \qed
4 Implications of continuous 2–transitivity

We now know that if \( G \) is transitive and contains a continuous deformation of the identity then it is either conjugate to the group of rotations \( \text{SO}(2, \mathbb{R}) \), is continuously 2–transitive, or is a cyclic cover of a group which is continuously 2–transitive. For the rest of the paper we assume that \( G \) is continuously 2–transitive and examine which possibilities arise.

For \( n \geq 2 \) and \((x_1 \ldots x_n) \in P_n\) we define \( J_{x_1 \ldots x_n} \) to be the subset of \( \mathbb{S}^1 \) containing the points \( x \in \mathbb{S}^1 \) which satisfy the following condition. There exists a continuous deformation of the identity \( f_t \in G \), with \( f_t(x_i) = x_i \) for each \( i \) and \( t \) and such that there exists \( t_0 \in [0, 1] \) with \( f_{t_0}(x) \neq x \). This generalizes the earlier definition of \( J_x \) and we get the following analogous results.

**Lemma 4.1** \( J_{f(x_1)\ldots f(x_n)} = f(J_{x_1 \ldots x_n}) \) for every \( f \in G \).

**Lemma 4.2** \( J_{x_1 \ldots x_n} \) is open.

We also have the following.

**Lemma 4.3** If \( J_{x_1 \ldots x_n} \) is nonempty and \( G \) is continuously \( n \)-transitive, then it is equal to \( \mathbb{S}^1 \setminus \{x_1 \ldots x_n\} \).

**Proof** Assume that \( J_{x_1 \ldots x_n} \subset \mathbb{S}^1 \setminus \{x_1, \ldots, x_n\} \) is nonempty. By Lemma 4.2 it is also open and hence is a countable union of open intervals. Pick one of these, and call its endpoints \( b_1 \) and \( b_2 \). Assume for contradiction that at least one of \( b_1 \) and \( b_2 \) is not one of the \( x_i \). Interchanging \( b_1 \) and \( b_2 \) if necessary we can assume that this point is \( b_1 \). Since \( G \) is continuously \( n \)-transitive there exist elements of \( G \) which cyclically permute the \( x_i \). Using these elements and the fact that \( J_{f(x_1)\ldots f(x_n)} = f(J_{x_1 \ldots x_n}) \) for every \( f \in G \), we can assume without loss of generality that \( b_1 \) and hence the whole interval lies in the component of \( \mathbb{S}^1 \setminus \{x_1, \ldots, x_n\} \) whose endpoints are \( x_1 \) and \( x_2 \).

We now claim that \( J_{b_1, b_2, x_3, \ldots, x_n} \supset J_{x_1 \ldots x_n} \). To see this, take \( x \in J_{x_1 \ldots x_n} \), then there exists a continuous deformation of the identity \( f_t \) which fixes \( x_1, \ldots, x_n \) and for which there exists \( t_0 \) such that \( f_{t_0}(x) \neq x \). Now since \( b_1, b_2 \notin J_{x_1 \ldots x_n} \), \( f_t \) must also fix \( b_1 \) and \( b_2 \) for all \( t \), consequently we can use \( f_t \) to show that \( x \in J_{b_1, b_2, x_3, \ldots, x_n} \). In particular, this means that \( J_{b_1, b_2, x_3, \ldots, x_n} \) contains the whole interval between \( b_1 \) and \( b_2 \).

Take \( g \in G \) which maps \{\( b_1, b_2 \)\} to \{\( x_1, x_2 \)\} and fixes the other \( x_i \), such an element exists as \( G \) is continuously \( n \)-transitive. Then,

\[
J_{x_1 x_2 x_3 \ldots x_n} = J_{g(b_1), g(b_2), g(x_3), \ldots, g(x_n)} = g(J_{b_1, b_2, x_3 \ldots x_n})
\]
so that $J_{x_1\ldots x_n}$ must contain the whole interval between $x_1$ and $x_2$. This is a contradiction, since $b_1$ lies between $x_1$ and $x_2$ but is not in $J_{x_1\ldots x_n}$.

\textbf{Proposition 4.4} Let $G$ be continuously $n$–transitive for some $n \geq 2$ and suppose there exist $n$ distinct points $a_1, \ldots, a_n \in S^1$ and a continuous deformation of the identity $g_t \in G$, which fixes each $a_i$ for all $t$. Then $G$ is continuously $n+1$–transitive.

\textbf{Proof} $J_{a_1\ldots a_n} \neq \emptyset$ so by Lemma 4.3 $J_{a_1\ldots a_n} = S^1 \setminus \{a_1, \ldots, a_n\}$. We can now apply Lemma 2.5 to see that $G$ is continuously $n+1$–transitive.

\textbf{Corollary 4.5} If $G$ is continuously 2–transitive and there exists $g \in G \setminus \{id\}$ with an open interval $I \subset S^1$ such that the restriction of $g$ to $I$ is the identity, then $G$ is continuously $n$–transitive for every $n \geq 2$.

\textbf{Proof} Let $I \subset S^1$ be a maximal interval on which $g$ acts as the identity, so that if $I' \supset I$ is another interval containing $I$ then $g$ doesn’t act as the identity on $I'$. Let $a$ and $b$ be the endpoints of $I$ and let $a_t$ and $b_t$ be continuous injective paths with $a_0 = a, b_0 = b$ and $a_t, b_t \notin I$ for each $t \neq 0$. This is possible because $g \neq id$ so that $S^1 \setminus I$ will be a closed interval containing more than one point. Let $g_t$ be a continuous path in $G$ so that $g_0 = id, g_t(a) = a_t$ and $g_t(b) = b_t$, such a path exists as $G$ is continuously 2–transitive.

Consider the path $h_t = g^{-1} \circ g_t \circ g \circ g_t^{-1}$ since $g_0 = id$ we get $h_0 = id$. Now $g_t \circ g \circ g_t^{-1}$ acts as the identity on the interval between $a_t$ and $b_t$ and by maximality of $I$, $g^{-1}$ will not act as the identity for $t \neq 0$. Consequently, $h_t$ is a continuous deformation of the identity which acts as the identity on $I$. So if $G$ is continuously $k$–transitive for $k \geq 2$, by taking $k$–points in $I$ and using Proposition 4.4 we get that $G$ is $k+1$–transitive. As a result, since $G$ is continuously 2–transitive it will be $n$–transitive for every $n \geq 2$.

SO($2, \mathbb{R}$) is an example of a subgroup of Homeo($S^1$) which is continuously 1–transitive but not continuously 2–transitive. However, as the next result shows, there are no subgroups of Homeo($S^1$) which are continuously 2–transitive but not continuously 3–transitive.

\textbf{Proposition 4.6} If $G$ is continuously 2–transitive, then it is continuously 3–transitive.

\textbf{Proof} Let $a, b \in S^1$ be distinct points. Construct two injective paths $a(t), b(t)$ in $S^1$ with disjoint images, such that $a(0) = a, b(0) = b$ and such that $a(t)$ and $b(t)$ lie in
the same component of $\mathbb{S}^1 \setminus \{a, b\}$ for $t \in (0, 1]$. We label this component $I$ and the other $I'$.  

Since $G$ is continuously 2–transitive, there exists a path $g(t) \in G$ such that $g(0) = \text{id}$, $g(t)(a) = a(t)$ and $g(t)(b) = b(t)$ for every $t$. Now for every $t$ the restriction of $g(t)$ to the closure of $I$, is a continuous map of a closed interval into itself, and hence must have a fixed point, $c(t)$. This point will normally not be unique, but since $g(t)$ is continuous, for a small enough time interval we can choose it to depend continuously on $t$. Likewise for the restriction of $g(t)^{-1}$ to the closure of $I'$, for a small enough time interval we can choose a path of fixed points $d(t)$, which must therefore also be fixed points for $g(t)$.

Now pick points $c \in I$ and $d \in I'$. Using continuous 2–transitivity of $G$ construct a path $h(t) \in G$ such that $h(t)(c) = c(t)$ and $h(t)(d) = d(t)$. Then $h_t^{-1} \circ g(t) \circ h_t$ is only the identity when $t = 0$ because the same is true of $g(t)$ and we have constructed a continuous deformation of the identity which fixes $c$ and $d$ for all $t$. Consequently we can use Proposition 4.4 to show that $G$ is continuously 3–transitive.

\section{Convergence Groups}

**Definition 5.1** A subgroup $G$ of Homeo($\mathbb{S}^1$) is a convergence group if for every sequence of distinct elements $g_n \in G$, there exists a subsequence $g_{n_k}$ satisfying one of the following two properties:

1. There exists $g \in G$ such that,
   \[ \lim_{k \to \infty} g_{n_k} = g \quad \text{and} \quad \lim_{k \to \infty} g_{n_k}^{-1} = g^{-1} \]
   uniformly in $\mathbb{S}^1$.

2. There exist points $x_0, y_0 \in \mathbb{S}^1$ such that,
   \[ \lim_{k \to \infty} g_{n_k} = x_0 \quad \text{and} \quad \lim_{k \to \infty} g_{n_k}^{-1} = y_0 \]
   uniformly on compact subsets of $\mathbb{S}^1 \setminus \{y_0\}$ and $\mathbb{S}^1 \setminus \{x_0\}$ respectively.

The notion of convergence groups was introduced by Gehring and Martin [4] and they have proceeded to play a central role in geometric group theory. The following theorem has been one of the most important and we shall make frequent use of it.

**Theorem 5.2** $G$ is a convergence group if and only if it is conjugate in Homeo($\mathbb{S}^1$) to a subgroup of PSL(2, $\mathbb{R}$).

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This Theorem was proved by Gabai in [3]. Prior to that, Tukia [7] proved this result in many cases and Hinkkanen [6] proved it for non discrete groups. Casson and Jungreis proved it independently using different methods [2]. See [2], [3], [7] for references to other papers in this subject.

For the rest of this section we shall assume that \( G \) is continuously \( n \)-transitive, but not continuously \( n+1 \)-transitive for some \( n \geq 3 \).

Take \((x_1, \ldots, x_{n-1}) \in P_{n-1}\) and define

\[
G_0 = \{ g \in G : g(x_i) = x_i \quad i = 1, \ldots, n-1 \}.
\]

Choose a component \( I \) of \( S^1 \setminus \{x_1, \ldots, x_{n-1}\} \) and denote its closure by \( \overline{I} \). We construct a homomorphism \( \Phi : G_0 \to \text{Homeo}(S^1) \) as follows. Take \( g \in G_0 \), then since \( g \) fixes the endpoints of \( I \) and is orientation preserving, we can restrict it to a homeomorphism \( g' \) of \( \overline{I} \). By identifying the endpoints of \( \overline{I} \) we get a copy of \( S^1 \) and we define \( \Phi(g) \) to be the homeomorphism of \( S^1 \) that \( g' \) descends to under this identification. We label the identification point \( \overline{I} \) and set \( G_0 = \Phi(G_0) \) to be the image of \( G_0 \) under \( \Phi \).

In this situation Lemma 2.5 implies the following. For every \( x \in I \), there exists a continuous map \( F_x : S^1 \setminus \overline{I} \to G_0 \) satisfying the properties,

1. \( (F_x(y))(x) = y \quad \forall \ y \in S^1 \setminus \overline{I} \)
2. \( F_x(x) = \text{id} \).

**Proposition 5.3** \( \Phi : G_0 \to G_0 \) is an isomorphism.

**Proof** Surjectivity is trivial. If we assume that \( \Phi \) is not injective then there will exist \( g \in G_0 \) which is non-trivial and acts as the identity on \( I \). Then by Corollary 4.5 \( G \) will be \( n+1 \) transitive, a contradiction.

Let \( \widehat{G}_0 \) denote the path component of the identity in \( G_0 \), we now analyze the group \( \widehat{G}_0 = \Phi(\widehat{G}_0) \).

**Proposition 5.4** \( \widehat{G}_0 \) is a convergence group.

**Proof** Choose \( x \in I \) then we know there exists a continuous map \( F_x : S^1 \setminus \overline{I} \to G_0 \) satisfying the properties,

1. \( (F_x(y))(x) = y \quad \forall \ y \in S^1 \setminus \overline{I} \)
2. \( F_x(x) = \text{id} \).
Now since $F_x(x) = \text{id}$ and $F_x$ is continuous, the image of $F_x$ will lie entirely in $\hat{G}_0$. In fact, $F_x$ gives a bijection between $S^1 \setminus \bar{x}$ and $\hat{G}_0$. To see this we first observe that injectivity follows directly from condition 1. To see that it is also surjective, take $g \in \hat{G}_0$. Then there exists a path $g_t \in \hat{G}_0$ for $t \in [0, 1]$ with $g_0 = \text{id}$ and $g_1 = g$. So that $g_t(x)$ is a path in $S^1 \setminus \bar{x}$ from $x$ to $g(x)$. Consider the path $(F_x(g_t(x)))^{-1} \circ g_t$ in $\hat{G}_0$, it fixes $x$ for every $t$, and so must be the identity for each $t$. Otherwise, by Proposition 5.4, $G$ would be continuously $n+1$-transitive, which would contradict our assumptions. As a result $g = F_x(g(x))$ so $F_x$ is a bijection, with inverse given by evaluation at $x$.

Fix $x_0 \in S^1 \setminus \bar{x}$, let $g_n$ be a sequence of elements of $\hat{G}_0$ and consider the sequence of points $g_n(x_0)$, since $S^1$ is compact $g_n(x_0)$ has a convergent subsequence $g_{n_k}(x_0)$ converging to some point $x'$. If $x' \neq \bar{x}$ then by continuity of $F_{x_0}$, $g_{n_k}$ will converge to $F_{x_0}(x')$. Now if there does not exist a subsequence of $g_n(x_0)$ converging to some $x' \neq \bar{x}$, then take a subsequence $g_{n_k}$ such that $g_{n_k}(x_0)$ converges to $\bar{x}$. If we can show that $g_{n_k}(x)$ converges to $\bar{x}$ for every $x \in S^1 \setminus \bar{x}$ then we shall be done.

Suppose for contradiction that there exists $x \in S^1 \setminus \bar{x}$ such that $g_{n_k}(x)$ does not converge to $\bar{x}$. Then there exists a subsequence of $g_{n_k}(x)$ which converges to $x' \neq \bar{x}$, but then by the previous argument the corresponding subsequence of $g_{n_k}$ will converge to the homeomorphism $F_x(x')$. This is a contradiction since $F_x(x')(x_0)$ would have to equal $\bar{x}$. 

**Corollary 5.5** Let $g$ be an element of $\hat{G}_0$. If $g$ fixes a point in $S^1 \setminus \bar{x}$ then it is the identity.

**Proof** Let $x \in S^1 \setminus \bar{x}$ be a fixed point of $g$. From the previous proof we know that $F_x: I \to \hat{G}_0$ is a bijection. So that $F_x(g(x)) = g$, but $g$ fixes $x$ so that $g = F_x(x) = \text{id}$. 

**Corollary 5.6** The restriction of the action of $\hat{G}_0$ to $S^1 \setminus \bar{x}$ is conjugate to the action of $\mathbb{R}$ on itself by translation.

**Proof** By Theorem 5.2 and Proposition 5.4 $\hat{G}_0$ is conjugate in $\text{Homeo}(S^1)$ to a subgroup of $\text{PSL}(2, \mathbb{R})$ which fixes the point $\bar{x}$. Moreover, from Corollary 5.5 this is the only point fixed by a non trivial element. By identifying $S^1$ with $\mathbb{R} \cup \{\infty\}$ so that $\bar{x}$ is identified with $\{\infty\}$ in the usual way, we see that $\hat{G}_0$ is conjugate to a subgroup of the Möbius group acting on $\mathbb{R} \cup \{\infty\}$. Since every element will fix $\{\infty\}$, their restriction to $\mathbb{R}$ will be an element of $\text{Aff}(\mathbb{R})$ acting without fixed points, so can only be a translation. On the other hand the group must act transitively on $\mathbb{R}$ and so must be the full group of translations. This gives the result.
We finish this section by comparing the directions that a non-trivial element of $G$ and hence every $x$ shall say that $x$ acts on $I$ with at most one fixed point.

**Proposition 5.7** The restriction of the action of $G_0$ to $I$ is conjugate to the action of a subgroup of the affine group $\text{Aff}(\mathbb{R})$ on $\mathbb{R}$. In particular, each non-trivial element of $G_0$ can act on $I$ with at most one fixed point.

**Proof** The restriction of $\hat{G}_0$ to $S^1 \setminus \bar{x}$ is isomorphic to the restriction of $\hat{G}_0$ to $I$. So that by Corollary 5.6 there exists a homeomorphism $\phi: I \to \mathbb{R}$ which conjugates the restriction of $\hat{G}_0$ to $I$, to the action of $\mathbb{R}$ on itself by translation. Take $h \in G_0 \setminus \hat{G}_0$ then $h' = \phi \circ h \circ \phi^{-1}$ is a self-homeomorphism of $\mathbb{R}$. Since $\hat{G}_0$ is a normal subgroup of $G_0$, $h'$ conjugates every translation to another one and so by Lemma 3.9 is itself an affine map and the proof is complete.

Let $g$ be a non-trivial element of $G_0$, then $g \in \hat{G}_0$ if and only if it acts on each component of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$ as a conjugate of a non-trivial translation. Furthermore, if $g \notin \hat{G}_0$ then it acts on each component of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$ as a conjugate of an affine map which is not a translation, each of which must have a fixed point. This situation cannot actually arise as the next proposition will show.

**Proposition 5.8** $G_0 = \hat{G}_0$

**Proof** Let $g \in G_0 \setminus \hat{G}_0$, then $g$ acts on each component of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$ as a conjugate of an affine map which is not a translation. Consequently, $g$ will have a fixed point in each component of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$. Label the fixed points of $g$ in the components of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$ whose boundaries both contain $x_1$ as $y_1$ and $y_2$. Since $G$ is $n$-transitive, there exists a map $g'$ which sends $y_1$ to $x_1$ and fixes all the other $x_i$. Then $g' \circ g \circ (g')^{-1}$ fixes all the $x_i$ and hence is an element of $G_0$. On the other hand, $g' \circ g \circ (g')^{-1}$ also fixes $g'(x_1)$ and $g'(y_2)$ which lie in the same component of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$, this is impossible since every non-trivial element of $G_0$ can only have one fixed point in each component of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$.

**Corollary 5.9** The restriction of the action of $G_0$ to $I$ is conjugate to the action of $\mathbb{R}$ on itself by translation. In particular the action is free.

We finish this section by comparing the directions that a non-trivial element of $G_0$ moves points in different components of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$. So endow $S^1$ with the anti-clockwise orientation, this gives us an ordering on any interval $I \subset S^1$, where for distinct points $x, y \in I$, $x < y$ if one travels in an anti-clockwise direction to get from $x$ to $y$ in $I$. Let $g \in G_0 \setminus \{\text{id}\}$ if $I$ is a component of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$ then we shall say that $g$ acts positively on $I$ if $x < g(x)$ and negatively if $x > g(x)$ for one and hence every $x \in I$. 

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Let $I$ and $I'$ be the two components of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$ whose boundaries contain $x_i$. Labeled so that in the order on the closure of $I$, $x < x_i$ for each $x \in I$, whereas in the order on the closure of $I'$, $x_i < x$ for each $x \in I'$. Then we have the following,

**Proposition 5.10** Let $g$ be a non trivial element of $G_0$, if $g$ acts positively on $I$ then it acts negatively on $I'$ and if $g$ acts negatively on $I$ then it acts positively on $I'$.

**Proof** Let $x, x' \in I$ and $y, y' \in I'$ be points such that $x < x'$ and $y > y'$. There exists $g \in G$ fixing $x_1, \ldots, x_{i-1}$ and $x_{i+1}, \ldots, x_{n-1}$ and sending $x$ to $x'$ and $y$ to $y'$. This map will have a fixed point $\bar{x}$ between $x'$ and $y'$, since it maps the interval between them into itself.

Let $g' \in G$ fix $x_1, \ldots, x_{i-1}$ and $x_{i+1}, \ldots, x_{n-1}$ and send $\bar{x}$ to $x_i$. Then $g_0 = g' \circ g \circ (g')^{-1}$ will fix $x_1, \ldots, x_{n-1}$ and hence lie in $G_0$. Moreover, $g_0$ acts positively on $I$ and negatively on $I'$.

Now let $g_1 \in G_0$ be any non-trivial element which acts positively on $I$. Then there exists a path $g_t$ in $G_0$ from $g_0 = g' \circ g \circ (g')^{-1}$ to $g_1$, so that $g_t \neq \text{id}$ for any $t$. Since $g_t$ is never the identity and $g_0$ acts negatively on $I'$, $g_1$ must also act negatively on $I'$.

If $h \in G_0$ is a non-trivial element which acts negatively on $I$, then $h^{-1}$ will act positively on $I$. So that, by the above argument, $h^{-1}$ will act negatively on $I'$. This means that $h$ will act positively on $I'$ as required.

**Corollary 5.11** If $G$ is $n$–transitive but not $n + 1$–transitive for $n \geq 3$ then $n$ is odd.

**Proof** Let $g$ be a non-trivial element of $G_0$ which acts positively on some component $I$ of $S^1 \setminus \{x_1, \ldots, x_{n-1}\}$. Then by Proposition 5.10 as we travel around $S^1$ in an anti-clockwise direction the manner in which it acts on each component will alternate between negative and positive. Consequently, if $n$ was even, when we return to $I$ we would require that $g$ acted negatively on $I$, a contradiction, so $n$ is odd.

### 6 Continuous 3–transitivity and beyond

We begin this section by analyzing the case where $G$ is continuously 3–transitive but not continuously 4–transitive. We shall show that such a group is a convergence group and consequently conjugate to a subgroup of $\text{PSL}(2, \mathbb{R})$.

Fix distinct points $x_0, y_0 \in S^1$ and define

$$G_0 = \{g \in G : g(x_0) = x_0, g(y_0) = y_0\}$$
\[ \tilde{G} = \{ g \in G : g(x_0) = x_0 \} \]

then we have the following propositions.

**Proposition 6.1** \( G_0 \) is a convergence group.

**Proof** From Corollary 5.9, we know that the restriction of \( G_0 \) to each of the components of \( S^1 \setminus \{x_0, y_0\} \) is conjugate to the action of \( \mathbb{R} \) on itself by translation. Let \( g_n \) be a sequence of distinct elements of \( G_0 \) and take a point \( x \in S^1 \setminus \{x_0, y_0\} \). Then the sequence of points \( g_n(x) \) will have a convergent subsequence \( g_{n_k}(x) \). If this sequence converges to \( x_0 \) or \( y_0 \), then from Proposition 5.10 so will the sequences \( g_{n_k}(y) \) for all \( y \in S^1 \setminus \{y_0\} \) or \( S^1 \setminus \{x_0\} \) respectively.

Let \( I_x \) be the component of \( S^1 \setminus \{x_0, y_0\} \) containing \( x \). Assume that the sequence of points \( g_{n_k}(x) \) converges to a point \( x' \in I_x \). Now let \( y \) be a point in the other component, \( I_y \) of \( S^1 \setminus \{x_0, y_0\} \), and consider the sequence of points \( g_{n_k}(y) \) in \( I_y \). If it had a subsequence which converged to \( x_0 \) or \( y_0 \) then the sequence \( g_{n_k}(x) \) would have to as well. This is impossible so \( g_{n_k}(y) \) must stay within a compact subset of \( I_y \) and hence \( g_{n_k} \) has a subsequence, \( g_{n_{k_l}} \), for which \( g_{n_{k_l}}(y) \) converges to some point \( y' \in I_y \).

By Corollary 5.9 there exist self homeomorphisms of \( I_x \) and \( I_y \) to which the sequence \( g_{n_{k_l}} \) converges uniformly on \( I_x \) and \( I_y \) respectively. Gluing these together at \( x_0 \) and \( y_0 \) gives us an element of \( \text{Homeo}(S^1) \) which \( g_{n_k} \) converges to uniformly. Consequently, \( G_0 \) is a convergence group. \( \square \)

**Proposition 6.2** \( \tilde{G} \) is a convergence group.

**Proof** Let \( f_n \) be a sequence of elements of \( \tilde{G} \). If for every \( y \in S^1 \setminus \{x_0\} \) every convergent subsequence of \( f_n(y) \) converges to \( x_0 \) then we would be done. So assume that this is not the case, take \( y \in S^1 \setminus \{x_0\} \) such that the sequence of points \( f_n(y) \) has a convergent subsequence \( f_{n_k}(y) \) converging to some point \( \tilde{y} \neq x_0 \). Let \( I \) be a small open interval around \( \tilde{y} \), not containing \( x_0 \) then since \( G \) is continuously 3–transitive, there exists a map \( F_{\tilde{y}} : I \to \tilde{G} \) satisfying the following,

1. \( F_{\tilde{y}}(x)(\tilde{y}) = x \) for all \( x \in I \)
2. \( F_{\tilde{y}}(\tilde{y}) \) is the identity.

Let \( g_1, g_2 \in \tilde{G} \) satisfy \( g_1(\tilde{y}) = y_0 \) and \( g_2(y_0) = y \) consider the sequence,

\[ h_k = g_1 \circ F_{\tilde{y}}(f_{n_k}(y))^{-1} \circ f_{n_k} \circ g_2 \]

of elements of \( \tilde{G} \). They all fix \( y_0 \), and since \( g_1 \circ F_{\tilde{y}}(f_{n_k}(y))^{-1} \) converges to \( g_1 \) as \( k \to \infty \) we have the following.
(1) If $h_k$ contains a subsequence $h_{k_l}$ such that there exists a homeomorphism $h$ with,
\[ \lim_{l \to \infty} h_{k_l} = h \quad \text{and} \quad \lim_{l \to \infty} (h_{k_l})^{-1} = h^{-1} \]
then so does $f_{n_k}$.

(2) Furthermore, if there exist points $x', y' \in \mathbb{S}^1$ and a subsequence $h_{k_l}$ of $h_k$ such that,
\[ \lim_{l \to \infty} h_{k_l} = x' \quad \text{and} \quad \lim_{l \to \infty} (h_{k_l})^{-1} = y' \]
uniformly on compact subsets of $\mathbb{S}^1 \setminus \{y'\}$ and $\mathbb{S}^1 \setminus \{x'\}$ respectively, then so does $f_{n_k} (x' \text{ and } y'$ will be replaced by $g_1^{-1}(x')$ and $g_1^{-1}(y')$).

Now, since $G_0$ is a convergence group, one of the above situations must occur. Consequently, $\tilde{G} = \{ g \in G : g(x_0) = x_0 \}$ is a convergence group.

**Proposition 6.3** If $G$ is a subgroup of $\text{Homeo}(\mathbb{S}^1)$ which is continuously 3–transitive but not continuously 4–transitive then $G$ is a convergence group.

**Proof** This proof is almost identical to the previous one but we write it out in full for clarity.

Choose $x_0 \in \mathbb{S}^1$ and let $f_n$ be a sequence of elements of $G$. Then since $\mathbb{S}^1$ is compact, the sequence of points $f_n(x_0)$ will have a convergent subsequence, $f_{n_k}(x_0)$, converging to some point $\tilde{x}$. Let $I$ be a small open interval around $\tilde{x}$, then since $G$ is continuously 3–transitive, there exists a map $F_{\tilde{x}} : I \to G$ satisfying the following,

1. $F_{\tilde{x}}(x) = x$ for all $x \in I$
2. $F_{\tilde{x}}(\tilde{x})$ is the identity.

Let $g \in G$ send $\tilde{x}$ to $x_0$ and consider the sequence,

\[ h_k = g \circ F_{\tilde{x}}(f_{n_k}(x_0))^{-1} \circ f_{n_k} \]

of elements of $G$. They all fix $x_0$, and since $g \circ F_{\tilde{x}}(f_{n_k}(x_0))^{-1}$ converges to $g$ as $k \to \infty$ we have the following.

(1) If $h_k$ contains a subsequence $h_{k_l}$ such that there exists a homeomorphism $h$ with,
\[ \lim_{l \to \infty} h_{k_l} = h \quad \text{and} \quad \lim_{l \to \infty} (h_{k_l})^{-1} = h^{-1} \]
then so does $f_{n_k}$.
Furthermore, if there exist points \( x', y' \in \mathbb{S}^1 \) and a subsequence \( h_{k_l} \) of \( h_k \) such that,
\[
\lim_{l \to \infty} h_{k_l} = x' \quad \text{and} \quad \lim_{l \to \infty} (h_{k_l})^{-1} = y'
\]
uniformly on compact subsets of \( \mathbb{S}^1 \setminus \{y'\} \) and \( \mathbb{S}^1 \setminus \{x'\} \) respectively, then so does \( f_{nk} \) (\( x' \) and \( y' \) will be replaced by \( g^{-1}(x') \) and \( g^{-1}(y') \)).

Now, since \( \bar{G} = \{ g \in G : g(x_0) = x_0 \} \) is a convergence group \( G \) is too.

We now look at the case where \( G \) is continuously 4–transitive. In this case, we show that \( G \) must be \( n–\)transitive for every \( n \in \mathbb{N} \).

**Theorem 6.4** If \( G \) is continuously \( n–\)transitive for \( n \geq 4 \), then it is continuously \( n + 1–\)transitive.

**Proof** Fix \( n \geq 4 \) and assume for contradiction that \( G \) is continuously \( n–\)transitive but not continuously \( n + 1–\)transitive. Take \( (a_1, \ldots, a_{n-2}) \in P_{n-2} \) and define,
\[
\bar{G} = \{ g \in G : g(a_i) = a_i \ \forall i \}
\]
Let \( I \) be a component of \( \mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-2}\} \). Construct a homomorphism \( \Psi: \bar{G} \to \text{Homeo}(\mathbb{S}^1) \) in the same way as \( \Phi: G_0 \to \text{Homeo}(\mathbb{S}^1) \) was constructed in Section 5. Explicitly, take \( g \in \bar{G} \), restrict it to a self homeomorphism of \( \bar{I} \) and identify the endpoints to get an element of \( \text{Homeo}(\mathbb{S}^1) \).

Let \( \bar{G} \) denote the image of \( \bar{G} \) under \( \Psi \). Then as in Proposition 5.3 \( \bar{G} \) is isomorphic to \( \bar{G} \). Using the arguments from the earlier Propositions in this section we can show that \( \bar{G} \) is a convergence group and hence conjugate to a subgroup of \( \text{PSL}(2, \mathbb{R}) \). On the other hand, \( \bar{G} \) is 2–transitive on \( I \) and every element fixes the identification point. This means that the action of \( \bar{G} \) on \( I \) must be conjugate to the action of \( \text{Aff}(\mathbb{R}) \) on \( \mathbb{R} \).

Let \( I \) and \( I' \) be two components of \( \mathbb{S}^1 \setminus \{a_1, \ldots, a_{n-2}\} \) and let \( \phi: I \to \mathbb{R} \) be a homeomorphism which conjugates the action of \( \bar{G} \) on \( I \) to the action of \( \text{Aff}(\mathbb{R}) \) on \( \mathbb{R} \). Let \( a_{n-1}, a'_{n-1} \) be two distinct points in \( I' \). Consider the groups
\[
G_0 = \{ g \in \bar{G} : g(a_{n-1}) = a_{n-1} \}
\]
and
\[
G'_0 = \{ g \in \bar{G} : g(a'_{n-1}) = a'_{n-1} \}
\]
They each act transitively on \( I \) and by Corollary 5.5 and Proposition 5.8 without fixed points. Consequently, \( \phi \) conjugates both of these actions to the action of \( \mathbb{R} \) on itself by translation. Let \( g \in G_0 \) and \( g' \in G'_0 \) be elements which are conjugated to \( x \mapsto x + 1 \)
by $\phi$. Then $g^{-1} \circ g'$ acts on $I$ as the identity. However, if it is equal to the identity, then $g' = g$ fixes $a_{n-1}$ and $a_{n-1}'$, this is impossible as non-trivial elements of $\widetilde{G}$ can have at most one fixed point in $I'$. So $g^{-1} \circ g$ is a non-trivial element of $G$ which acts as the identity on $I$ and so by Corollary 4.5 we have that $G$ is continuously $n + 1$–transitive.

\section{Summary of Results}

\textbf{Theorem 7.1} Let $G$ be a transitive subgroup of $\text{Homeo}(\mathbb{S}^1)$ which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds:

1. $G$ is conjugate to $\text{SO}(2, \mathbb{R})$ in $\text{Homeo}(\mathbb{S}^1)$.
2. $G$ is conjugate to $\text{PSL}(2, \mathbb{R})$ in $\text{Homeo}(\mathbb{S}^1)$.
3. For every $f \in \text{Homeo}(\mathbb{S}^1)$ and each finite set of points $x_1, \ldots, x_n \in \mathbb{S}^1$ there exists $g \in G$ such that $g(x_i) = f(x_i)$ for each $i$.
4. $G$ is a cyclic cover of a conjugate of $\text{PSL}(2, \mathbb{R})$ in $\text{Homeo}(\mathbb{S}^1)$ and hence conjugate to $\text{PSL}_k(2, \mathbb{R})$ for some $k > 1$.
5. $G$ is a cyclic cover of a group satisfying condition 3 above.

\textbf{Proof} Let $f : [0, 1] \to G$ be a non constant continuous path. Then

$$f(0)^{-1} \circ f : [0, 1] \to G$$

is a continuous deformation of the identity in $G$. Consequently, Proposition 2.6 tells us that $G$ is continuously 1–transitive.

If $J_x = \emptyset$ for every $x \in \mathbb{S}^1$ then by Theorem 3.8 $G$ is conjugate to $\text{SO}(2, \mathbb{R})$ in $\text{Homeo}(\mathbb{S}^1)$. If $J_x \neq \emptyset$ for some and hence all $x \in \mathbb{S}^1$ then by Theorem 3.10 $G$ is either continuously 2–transitive or is a cyclic cover of a group $G'$ which is continuously 2–transitive.

So assume that $G$ is continuously 2–transitive, then by Proposition 4.6 it is continuously 3–transitive. If moreover $G$ is not continuously 4–transitive, then by Proposition 6.3 it is a convergence group and hence conjugate to a subgroup of $\text{PSL}(2, \mathbb{R})$. On the other hand, since $G$ is continuously 3–transitive, it is 3–transitive, and hence must be conjugate to the whole of $\text{PSL}(2, \mathbb{R})$.

If we now assume that $G$ is continuously 4–transitive then by Theorem 6.4 it is continuously $n$–transitive and hence $n$–transitive for every $n \in \mathbb{N}$. So if we take $f \in \text{Homeo}(\mathbb{S}^1)$ and a finite set of points $x_1, \ldots, x_n \in \mathbb{S}^1$ there exists $g \in G$ such that $g(x_i) = f(x_i)$ and we are done. \qed

Theorem 7.2 Let $G$ be a closed transitive subgroup of $\text{Homeo}(\mathbb{S}^1)$ which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds:

1. $G$ is conjugate to $\text{SO}(2, \mathbb{R})$ in $\text{Homeo}(\mathbb{S}^1)$.
2. $G$ is conjugate to $\text{PSL}_k(2, \mathbb{R})$ in $\text{Homeo}(\mathbb{S}^1)$ for some $k \geq 1$.
3. $G$ is conjugate to $\text{Homeo}_k(\mathbb{S}^1)$ in $\text{Homeo}(\mathbb{S}^1)$ for some $k \geq 1$.

Proof Since $G$ is a transitive subgroup of $\text{Homeo}(\mathbb{S}^1)$ which contains a non constant continuous path, Theorem 7.1 applies. It remains to show that if $G$ satisfies condition 3 in Theorem 7.1 then its closure is $\text{Homeo}(\mathbb{S}^1)$.

To see this, let $f$ be an arbitrary element of $\text{Homeo}(\mathbb{S}^1)$. If we can find a sequence of elements of $G$ which converges uniformly to $f$ then we shall be done. So let \{a_n : n \in \mathbb{N}\} be a countable and dense set of points in $\mathbb{S}^1$. Choose a sequence of maps $g_n \in G$ so that $g_n(a_k) = f(a_k)$ for $1 \leq k \leq n$. Then $g_n$ will converge uniformly to $f$ so that the closure of $G$ will equal $\text{Homeo}(\mathbb{S}^1)$.

Theorem 7.3 $\text{PSL}(2, \mathbb{R})$ is a maximal closed subgroup of $\text{Homeo}(\mathbb{S}^1)$.

Proof Let $G$ be a closed subgroup of $\text{Homeo}(\mathbb{S}^1)$ containing $\text{PSL}(2, \mathbb{R})$. Then $G$ is 3–transitive and by applying Theorem 7.2 we can see that $\text{Homeo}(\mathbb{S}^1)$ and $\text{PSL}(2, \mathbb{R})$ are the only possibilities for $G$.

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