Geometry & Topology 11 (2007) 315–427 **315**

Discrete models for the p–local homotopy theory of compact Lie groups and p–compact groups

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We define and study a certain class of spaces which includes p–completed classifying spaces of compact Lie groups, classifying spaces of p–compact groups, and p–completed classifying spaces of certain locally finite discrete groups. These spaces are determined by fusion and linking systems over "discrete p–toral groups"—extensions of $(\mathbb{Z}/p^\infty)^r$ by finite p–groups—in the same way that classifying spaces of p–local finite groups as defined in our paper [7] are determined by fusion and linking systems over finite p–groups. We call these structures "p–local compact groups".

55R35; 55R40, 57T10

In our earlier paper [7], we defined and studied a certain class of spaces which in many ways behave like p–completed classifying spaces of finite groups. These spaces occur as "classifying spaces" of certain algebraic objects called p–local finite groups. The purpose of this paper is to generalize the concept of p–local finite groups to what we call p–local compact groups. The motivation for introducing this family comes from the observation that p–completed classifying spaces of finite and compact Lie groups, as well as classifying spaces of p–compact groups (see Dwyer and Wilkerson [11]), share many similar homotopy theoretic properties, but earlier studies of these properties usually required different techniques for each case. Moreover, while p–completed classifying spaces of finite and, more generally, compact Lie groups arise from the algebraic and geometric structure of the groups in question, p–compact groups are purely homotopy theoretic objects. Unfortunately, many of the techniques used in the study of p–compact groups fail for p–completed classifying spaces of general compact Lie groups. With the approach presented here, we propose a framework general enough to include p–completed classifying spaces of arbitrary compact Lie groups as well as p–compact groups.

The new idea here is to replace fusion systems over finite p–groups, as handled in [7], by fusion systems over *discrete p–toral groups*. A discrete p–toral group is a group which contains a discrete p–torus (a group of the form $(\mathbb{Z}/p^\infty)^r$ for finite $r \geq 0$) as a normal subgroup of p–power index. A p–local compact group consists of a triple

Published: 16 March 2007 DOI: 10.2140/gt.2007.11.315

$(S,\mathcal{F},\mathcal{L})$, where S is a discrete p–toral group, $\mathcal{F}$ is a *saturated fusion system over* S (a collection of fusion data between subgroups of S arranged in the form of a category and satisfying certain axioms), and $\mathcal{L}$ is a *centric linking system associated to* $\mathcal{F}$ (a category whose objects are a certain distinguished subcollection of the object of $\mathcal{F}$, and of which the corresponding full subcategory of $\mathcal{F}$ is a quotient category). The linking system $\mathcal{L}$ allows us to define the *classifying space* of this p–local compact group to be the p–completed nerve $|\mathcal{L}|^\wedge_p$. If S is a finite p–group, then the theory reduces to the case of p–local finite groups as studied in [7].

We hope that working with this setup will make it possible to prove results of interest in a uniform fashion for the entire family. In this paper (Theorem 7.1), we give a combinatorial description of the space of self equivalences of $|\mathcal{L}|^\wedge_p$ in terms of automorphisms of the category $\mathcal{L}$, and a description of the group $\mathrm{Out}(|\mathcal{L}|^\wedge_p)$ of homotopy classes of self equivalences in terms of "fusion preserving automorphisms" of S. We also show that a p–local compact group $(S,\mathcal{F},\mathcal{L})$ is determined up to isomorphism by the homotopy type of its classifying space $|\mathcal{L}|^\wedge_p$. One future goal is to show that the mod p cohomology of the classifying space $|\mathcal{L}|^\wedge_p$ of a p–local compact group $(S,\mathcal{F},\mathcal{L})$ can always be described in terms of the fusion system $\mathcal{F}$, as a ring of "stable elements" in the cohomology of S. Other goals are to define *connected* p–local compact groups, and understand their properties and their relation to connected p–compact groups and to characterize algebraically (connected) p–compact groups among all (connected) p–local compact groups. Finally, a more general question which is still open is whether the p–completion of the classifying space of every finite loop space is the classifying space of a p–local compact group.

As one might expect, passing from a finite to an infinite setup introduces an array of problems one must deal with in order to produce a coherent theory. Some of the basic properties of fusion systems over discrete p–toral groups are analogous or even identical to the finite case, whereas other aspects are more delicate. Once the definition of a saturated fusion system over a discrete p–toral group is given and their basic properties are studied, one defines associated centric linking systems and p–local compact groups in a fashion more or less identical to the finite case. However, while in the finite case, any finite group G gives rise automatically to a saturated fusion system and an associated centric linking system, the corresponding construction for compact Lie groups is less obvious. Similar complications present themselves when dealing with the fusion system and the centric linking system associated to a p–compact group. It is for that reason that the only aims of this paper are to establish the setup, study some basic properties, and prove that the classifying spaces which are the obvious candidates to give rise to p–local compact groups indeed do so.

We proceed by describing the contents of the paper in some detail. In Section 1, we define and list some properties of discrete p–toral groups. We show why this class of groups is a natural one to consider for our purposes, and study some of its useful properties. Then in Section 2, we define saturated fusion systems over discrete p–toral groups. The definitions in this section are very similar to those given in [7] for the finite case, but some modifications are needed due to having given up finiteness.

Much of the work on p–local finite groups makes implicit use of the fact that the categories one works with are finite. If S is an infinite discrete p–toral group, then any fusion system over it will have infinitely many objects. In Section 3 we show that any saturated fusion system $\mathcal{F}$ over a discrete p–toral group S contains a full subcategory with finitely many objects, which in the appropriate sense determines $\mathcal{F}$ completely. More precisely, we show that $\mathcal{F}$ contains only finitely many objects which are both centric and radical, and then prove the appropriate analog of Alperin's fusion theorem. The latter, roughly speaking, says that in a saturated fusion system, every morphism can be factored into a sequence of morphisms each of which is the restriction of an automorphism of a centric radical subgroup.

Linking systems associated to fusion systems over discrete p–toral groups are defined in Section 4. In fact, the definition is identical to that used when working over a finite p–group, and the proof that the nerve $|\mathcal{L}|$ of a linking system is p–good is essentially identical to that in the finite case. The connection between linking systems associated to a given fusion system $\mathcal{F}$ and rigidifications of the homotopy functor $P \mapsto BP$ on the orbit category $\mathcal{O}^c(\mathcal{F})$ is then studied.

Higher limits over the orbit category of a fusion system are investigated in Section 5. We first describe how to reduce the general problem to one of higher limits over a finite subcategory, and then show how those can be computed with the help of the graded groups $\Lambda^*(\Gamma; M)$ introduced in Jackowski, McClure and Oliver [19; 20]. These general results are then applied to prove the acyclicity of certain explicit functors whose higher limits appear later as obstruction groups.

Spaces of maps $\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)$ are studied in Section 6, when Q is a discrete p–toral group and $|\mathcal{L}|^\wedge_p$ is the classifying space of a p–local compact group, and the space of self equivalences of $|\mathcal{L}|^\wedge_p$ is handled in Section 7. In both cases, the descriptions we obtain in this new situation (in Theorem 6.3 and Theorem 7.1) are the obvious generalizations of those obtained in [7] for linking systems over finite p–groups. We also prove (Theorem 7.4) that a p–local compact group is determined by the homotopy type of its classifying space: if $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are p–local compact groups such that $|\mathcal{L}|^\wedge_p \simeq |\mathcal{L}'|^\wedge_p$, then they are isomorphic as triples of groups and categories.

We finish with three sections of examples: certain infinite locally finite groups in Section 8, including linear torsion groups; compact Lie groups in Section 9; and p–compact groups in Section 10. In all cases, we show that the groups in question fit into our theory: they have saturated fusion systems and associated linking systems, defined in a unique way (unique up to isomorphism at least), and the classifying spaces of the resulting p–local compact groups are homotopy equivalent to the p–completed classifying spaces of the groups in the usual sense.

Acknowledgments The first and third authors would like to thank the University of Aberdeen for its hospitality during several visits; in particular, this work began when we got together there in January 2002. All three authors would like to thank the Mittag-Leffler Institute, where this work was finished over a period of several months.

Acknowledgments of financial support C Broto is partially supported by FEDER/MEC grant MTM2004–06686. R Levi is partially supported by EPSRC grant MA022 RGA0756. B Oliver is partially supported by UMR 7539 of the CNRS. All three of the authors were partially supported by the Mittag-Leffler Institute in Sweden, and also by the EU grant nr HPRN-CT-1999-00119.

1 Discrete p–toral groups

When attempting to generalize the theory of p–local finite groups to certain infinite groups, the first problem is to decide which groups should replace the finite p–groups over which we studied fusion systems in [7]. The following is the class of groups we have chosen for this purpose. Let $\mathbb{Z}/p^\infty \cong \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ denote the union of the cyclic p–groups $\mathbb{Z}/p^n$ under the obvious inclusions.

Definition 1.1 A *discrete p–toral group* is a group P, with normal subgroup $P_0 \lhd P$, such that P_0 is isomorphic to a finite product of copies of $\mathbb{Z}/p^\infty$, and P/P_0 is a finite p–group. The subgroup P_0 will be called the *identity component* of P, and P will be called *connected* if $P = P_0$. Set $\pi_0(P) \overset{\text{def}}{=} P/P_0$: the group of *components* of P.

The identity component P_0 of a discrete p–toral group P can be characterized as the subset of all infinitely p–divisible elements in P, and also as the minimal subgroup of finite index in P. Define $\mathrm{rk}(P) = k$ if $P_0 \cong (\mathbb{Z}/p^\infty)^k$, and set

$$|P| \overset{\text{def}}{=} (\mathrm{rk}(P), |\pi_0(P)|) = (\mathrm{rk}(P), |P/P_0|)\,.$$

We regard the order of a discrete p–toral group as an element of $\mathbb{N}^2$ with the lexicographical ordering. Thus $|P| \le |P'|$ if and only if $\mathrm{rk}(P) < \mathrm{rk}(P')$, or $\mathrm{rk}(P) = \mathrm{rk}(P')$

and $|\pi_0(P)| \le |\pi_0(P')|$. In particular, $P' \le P$ implies $|P'| \le |P|$, with equality only if $P' = P$.

The obvious motivation for choosing this class is the role they play as "Sylow p–subgroups" in compact Lie groups and p–compact groups. But in fact, it seems difficult to construct fusion systems with interesting properties over any larger class of subgroups. The reason for this is that discrete p–toral groups are characterized by certain finiteness properties, which are needed in order for fusion systems over them to be manageable, and for related homotopy theoretic phenomena to be controlled by p–local information.

A group G is *locally finite* if every finitely generated subgroup of G is finite, and is a *locally finite p–group* if every finitely generated subgroup of G is a finite p–group. The class of locally finite (p–)groups is closed under subgroups and quotient groups. It is also closed under group extensions, since finite index subgroups of finitely generated groups are again finitely generated.

A group G is *artinian* (satisfies the minimum condition in the terminology of Wehrfritz [29]) if every nonempty set of subgroups of G, partially ordered by inclusion, has a minimal element. Equivalently, G is artinian if its subgroups satisfy the descending chain condition. The class of artinian groups is closed under taking subgroups, quotients, and extensions. Every artinian group is a torsion group (since an infinite cyclic group is not artinian). If G is artinian and $\varphi \in \mathrm{Inj}(G, G)$ is an injective endomorphism of G, then φ is an automorphism, since otherwise $\{\varphi^n(G)\}$ would be an infinite descending chain. This is just one example of why it will be important that the groups we work with are artinian; the descending chain condition will be used in other ways later.

It is an open question whether every artinian group is locally finite (see Kegel and Wehrfritz [23, pp 31–32] for a discussion of this). If one restricts attention to groups all of whose elements have p–power order for some fixed prime p, then artinian groups are known to be locally finite if $p = 2$ [23, Theorem 1.F.6], but this seems to be unknown for odd primes. However, any counterexample to these questions would probably be far too wild for our purposes. Hence it is natural to restrict attention to locally finite groups, and since we are working with local structure at a prime p, to locally finite p–groups. The next proposition tells us that in fact, this restricts us to the class of discrete p–toral groups. It is included only as a way to help motivate this choice of groups to work with.

Proposition 1.2 *A group is a discrete p–toral group if and only if it is artinian and a locally finite p–group.*

Proof The group $\mathbb{Z}/p^\infty$ is clearly a locally finite p–group and artinian. Since both of these properties are preserved under extensions of groups, they are satisfied by every discrete p–toral group.

Conversely, assume G is artinian and a locally finite p–group. By [23, Theorem 5.8], every locally finite artinian group is a Černikov group; in particular, it contains a normal abelian subgroup with finite index. By [14, Theorems 25.1 & 3.1], every abelian artinian group is a finite product of groups of the form $\mathbb{Z}/q^m$ where q is a prime and $m \le \infty$. Thus G is an extension of the form

$$1 \longrightarrow A \longrightarrow G \longrightarrow \pi \longrightarrow 1,$$

where π is a finite p–group, and A is a finite product of groups $\mathbb{Z}/p^m$ for $m \le \infty$. The subgroup of A generated by the factors $\mathbb{Z}/p^\infty$ is the subgroup of infinitely p–divisible elements, thus a characteristic subgroup of A, and a normal subgroup of G of p–power index. It follows that G is a discrete p–toral group. □

We next note some of the other properties which make discrete p–toral groups convenient to work with.

Lemma 1.3 *Any subgroup or quotient group of a discrete p–toral group is a discrete p–toral group. Any extension of one discrete p–toral group by another is a discrete p–toral group.*

Proof These statements are easily checked directly. They also follow at once from Proposition 1.2, since the classes of locally finite p–groups and artinian groups are both closed under these operations. □

Clearly, the main difficulty when working with infinite discrete p–toral groups, instead of finite p–groups, is that they have infinitely many subgroups and infinite automorphism groups. We next investigate what finiteness properties these groups do have.

Lemma 1.4 *The following hold for each discrete p–toral group P.*

(a) *For each $n \ge 0$, P contains finitely many conjugacy classes of subgroups of order p^n.*

(b) *P contains finitely many conjugacy classes of elementary abelian p–subgroups.*

Proof Clearly, for each n, P_0 contains finitely many subgroups of order p^n, since they are all contained inside the p^n–torsion subgroup of P_0 which is finite. So to prove (a), it suffices, for each finite subgroup $A \le P_0$ and each subgroup $B = \widetilde{B}/P_0 \le P/P_0$, to show that there are finitely many P–conjugacy classes of subgroups $Q \le P$ such that $Q \cap P_0 = A$ and $QP_0 = \widetilde{B}$. Let $\mathcal{Q}$ be the set of all such subgroups, and assume $\mathcal{Q} \neq \varnothing$. Then $Q \in \mathcal{Q}$ if and only if $Q/A \cap P_0/A = 1$ and $QP_0/P_0 = B$; and this implies that $A \lhd QP_0 = \widetilde{B}$ and that Q/A is the image of a splitting of the extension

$$1 \longrightarrow P_0/A \longrightarrow \widetilde{B}/A \longrightarrow B \longrightarrow 1.$$

In other words, $\mathcal{Q}$ is in one-to-one correspondence with the set of splittings of this extension. The set of P_0–conjugacy classes of such splittings (if there are any) is in one-to-one correspondence with the elements of $H^1(B; P_0/A)$ (see Brown [8, Proposition IV.2.3]). Since this cohomology group is finite, so is the set of conjugacy classes of such extensions.

This proves point (a). Point (b) follows from (a), together with the observation that for any elementary abelian subgroup $E \le P$, $\mathrm{rk}(E) \le \mathrm{rk}(P) + \mathrm{rk}_p(P/P_0)$. □

We next check what can be said about finiteness in automorphism groups.

Proposition 1.5 *Let P be a discrete p–toral group.*

(a) *Any torsion subgroup of* $\mathrm{Aut}(P)$ *is an extension of an abelian group by a finite group.*

(b) *Any torsion subgroup of* $\mathrm{Out}(P)$ *is finite.*

(c) *For each* $Q \le P$, $\mathrm{Out}_P(Q)$ *is a finite p–group.*

Proof Assume first that $P \cong (\mathbb{Z}/p^\infty)^r$: a discrete p–torus of rank $r \ge 0$. Then $\mathrm{Aut}(P) \cong GL_r(\widehat{\mathbb{Z}}_p)$, and it is well known that the subgroup $(1 + p^2 M_r(\widehat{\mathbb{Z}}_p))^\times$ of matrices which are congruent modulo p^2 to the identity is torsion free. This follows, for example, from the inverse bijections

$$(1 + p^2 M_r(\widehat{\mathbb{Z}}_p))^\times \underset{\exp}{\overset{\log}{\rightleftarrows}} p^2 M_r(\widehat{\mathbb{Z}}_p)$$

defined by the usual power series: while log is not a homomorphism, it does satisfy the relation $\log(X^r) = r\log(X)$. So if H is a torsion subgroup of $\mathrm{Aut}(P)$ (equivalently, of $GL_r(\widehat{\mathbb{Z}}_p)$), then the composite

$$H \longrightarrow GL_r(\widehat{\mathbb{Z}}_p) \xrightarrow{/p^2} GL_r(\mathbb{Z}/p^2)$$

is injective, and thus H is finite.

Now let P be an arbitrary discrete p–toral group with connected component P_0 and group of components $\pi = P/P_0$. There is an exact sequence

$$0 \longrightarrow H^1(\pi; P_0) \longrightarrow \mathrm{Aut}(P)/\mathrm{Aut}_{P_0}(P) \longrightarrow \mathrm{Aut}(P_0) \times \mathrm{Aut}(\pi)$$

(cf Suzuki [28, 2.8.7]), where $\mathrm{Aut}_{P_0}(P) = \{c_x \in \mathrm{Aut}(P) \mid x \in P_0\}$. We have just seen that every torsion subgroup of $\mathrm{Aut}(P_0)$ is finite, and $H^1(\pi; P_0)$ and $\mathrm{Aut}(\pi)$ are clearly finite. Hence every torsion subgroup of $\mathrm{Aut}(P)/\mathrm{Inn}_{P_0}(P)$ is finite. This proves (b); and also proves (a) (every torsion subgroup of $\mathrm{Aut}(P)$ is an extension of an abelian group by a finite group) since $\mathrm{Aut}_{P_0}(P)$ is abelian. Point (c) follows immediately from (b), since P is a torsion group all of whose elements have p–power order. □

In the next section (in Definition 2.2), we will need some more precise bounds on the size of normalizers and centralizers.

Lemma 1.6 *Let S be any discrete p–toral group, and set $N = |\pi_0(S)|^{\mathrm{rk}(S)+1}$. Then for all $P \le S$,*

$$|\pi_0(C_S(P))| \le N, \quad |\pi_0(N_S(P)/P)| \le N, \quad \text{and} \quad |\pi_0(N_S(P))| \le N \cdot |\pi_0(P)|.$$

Proof Set $T = S_0$ for short, and set $Q = PT/T$. Let $\mathcal{N}_Q\colon T \longrightarrow T$ be the norm map for the Q action: $\mathcal{N}_Q(x) = \prod_{gT\in Q} gxg^{-1}$. The image of $\mathcal{N}_Q$ is connected and centralizes P, and thus $\mathrm{Im}(\mathcal{N}_Q) \le C_S(P)_0 = C_T(P)_0$. If $x \in C_T(P)$, then

$$x^{|Q|} = \mathcal{N}_Q(x) \in C_T(P)_0.$$

Thus every element in $C_T(P)/C_T(P)_0$ has order dividing $|Q|$, and it follows that

$$|\pi_0(C_T(P))| = |C_T(P)/C_T(P)_0| \le |Q|^{\mathrm{rk}(S)} \le |\pi_0(S)|^{\mathrm{rk}(S)}.$$

Thus $|\pi_0(C_S(P))| \le |\pi_0(C_T(P))| \cdot |S/T| \le N$.

If $x \in N_T(P)$, then

$$x^{|Q|} = \mathcal{N}_Q(x) \cdot \prod_{gT\in Q} [x,g] \in C_T(P)_0 \cdot P \le N_T(P)_0 \cdot P.$$

Thus $$\bigl|N_T(P)/\bigl(N_T(P)_0 \cdot (T \cap P)\bigr)\bigr| \le |Q|^{\mathrm{rk}(S)} \le |\pi_0(S)|^{\mathrm{rk}(S)},$$

and hence $|\pi_0(N_S(P)/P)| \le N$, by the same arguments as those used for $\pi_0(C_S(P))$. The last inequality is now immediate. □

Note that discrete p–toral groups are all solvable, but (in contrast to finite p–groups) need not be nilpotent. For instance, the infinite dihedral group, a split extension of $\mathbb{Z}/2^\infty$ by $\mathbb{Z}/2$, is a discrete 2–toral group which is not nilpotent (since the nilpotency class of D_{2^n} is $n-1$).

The following lemma contains some generalizations of a standard theorem about automorphisms of finite p–groups: if $\alpha \in \mathrm{Aut}(P)$ is the identity on $Q \lhd P$ and on P/Q, then it has p–power order.

Lemma 1.7 *The following hold for any discrete p–toral group P and any automorphism $\alpha \in \mathrm{Aut}(P)$.*

(a) *Assume, for some $Q \lhd P$, that $\alpha|_Q = \mathrm{Id}_Q$ and $\alpha \equiv \mathrm{Id}$ (mod Q). Then every α–orbit in P is finite of p–power order. If, in addition, $[P:Q] < \infty$, then α has finite order.*

(b) *α has finite order if and only if $\alpha|_{P_0}$ has finite order.*

(c) *Set $P_{(1)} = \{g \in P_0 \mid g^p = 1\}$. If $\alpha|_{P_{(1)}} = \mathrm{Id}$ and $\alpha \equiv \mathrm{Id}$ (mod P_0), then each orbit of α acting on P has p–power order.*

Proof **(a)** The proof is identical to the proof for finite p–groups (see Gorenstein [16, Theorem 5.3.2]), and in fact applies whenever all elements of Q have p–power order. For any $g \in P$, $\alpha(g) = gx$ for some $x \in Q$ (since $\alpha \equiv \mathrm{Id}$ (mod Q)), and $\alpha(x) = x$ since $\alpha|_Q = \mathrm{Id}$. Thus $\alpha^n(g) = gx^n$ for all n, and $\alpha^{p^k}(g) = g$ if $p^k = |x|$. Since the order of $\{\alpha^i(g)\}$ depends only on the coset gQ, this also shows that $|\alpha|$ is finite (and a power of p) if P/Q is finite.

(b) If $\alpha|_{P_0}$ has finite order, then there is $n \geq 1$ such that $\alpha^n|_{P_0} = \mathrm{Id}$ and $\alpha^n \equiv \mathrm{Id}$ (mod P_0). Then α^n has finite order by (a), so α also has finite order.

(c) For each $m \geq 1$, let $P_{(m)} \leq P_0$ be the p^m–torsion in P_0. Fix $g \in P$, and set $x = g^{-1}\alpha(g)$, $p^k = |x|$, and $Q = \langle g, P_{(k)} \rangle$. The $P_{(m)}$ are all α–invariant, and so Q is also α–invariant since $g^{-1}\alpha(g) \in P_{(k)}$. Also, α acts via the identity on $P_{(1)}$ by assumption, hence on $P_{(i)}/P_{(i-1)}$ for all $1 \leq i \leq k$, and also on $Q/P_{(k)}$. So by (a) (and since Q is a finite group), $\alpha|_Q$ has p–power order. In particular, the α–orbit of g has p–power order. □

The next lemma is another easy generalization of a standard result about finite p–groups.

Lemma 1.8 *If $P \lneqq Q$ are distinct discrete p–toral groups, then $P \lneqq N_Q(P)$.*

Proof When $[Q{:}P]<\infty$, this follows by the same proof as for finite p–groups. More precisely, when Q/P is finite, the action of P on Q/P (defined by $x(gP)=xgP$ for $x\in P$ and $g\in Q$) factors through a finite quotient group P/N of P. Also, P/N is a p–group since P is a p–torsion group. Thus

$$|N_Q(P)/P| = |(Q/P)^{P/N}| \equiv |Q/P| \equiv 0 \pmod p,$$

and so $N_Q(P)/P\neq 1$.

Now assume that $[Q{:}P]$ is infinite; ie that $P_0\precneqq Q_0$. Set $A_n=\{x\in Q_0\,|\,x^{p^n}=1\}$, for each n. Then $A_n\lhd Q$, and in particular is normalized by P. For n large enough, $A_n\not\leq P$, so $P\precneqq PA_n\leq Q$, $P\precneqq N_{PA_n}(P)$ since $[PA_n{:}P]<\infty$. Thus $P\precneqq N_Q(P)$. □

We will also need the following well known result about finite subgroups of discrete p–toral groups.

Lemma 1.9 *For any discrete p-toral group P, there is a finite subgroup $Q\leq P$ such that $P=QP_0$. There is also an increasing sequence $Q_1\leq Q_2\leq Q_3\leq\cdots$ of finite subgroups of P such that $P=\bigcup_{n=1}^{\infty}Q_n$. More generally, for any finite subgroup $K\leq \mathrm{Aut}(P)$, the Q_i can be chosen to be K–invariant.*

Proof Fix any (finite) set X of coset representatives for P_0 in P, and set $Q=\langle\alpha(g)\,|\,\alpha\in K,\ g\in X\rangle$. Then Q is K–invariant, Q is finite since P is locally finite, and $P=QP_0$ by construction. For each $n\geq 1$, let $P_n\leq P_0$ be the p^n–torsion subgroup, and set $Q_n=QP_n$. Then the Q_n are also finite and K–invariant, and $P=\bigcup_{n=1}^{\infty}Q_n$. □

To finish the section, we consider maps between the p–completed classifying spaces of discrete p–toral groups. This following lemma is implicit in Dwyer and Wilkerson [11; 12] (the spaces in question are classifying spaces of p–compact groups). But it does not seem to be stated explicitly anywhere there.

Lemma 1.10 *For any pair P,Q of discrete p–toral groups,*

$$B\colon \mathrm{Rep}(P,Q)\longrightarrow [BP^\wedge_p, BQ^\wedge_p]$$

is a bijection. In particular, any homotopy equivalence $BP^\wedge_p\xrightarrow{\simeq}BQ^\wedge_p$ is induced by an isomorphism $P\cong Q$. Also, for any homomorphism $\rho\colon P\longrightarrow Q$, the homomorphism

$$C_Q(\rho(P))\times P\xrightarrow{(\mathrm{incl},\rho)} Q$$

induces a homotopy equivalence

$$BC_Q(\rho(P))^\wedge_p\xrightarrow{\ \simeq\ }\mathrm{Map}(BP^\wedge_p, BQ^\wedge_p)_{B\rho}.$$

Proof For any pair G, H of discrete groups,

$$[BG, BH] \cong \mathrm{Rep}(G, H) \qquad \text{and} \qquad \mathrm{Map}(BG, BH)_{B\rho} \simeq BC_H(\rho(G))$$

for each $\rho \in \mathrm{Hom}(G, H)$. See, for example, Broto and Kitchloo [5, Proposition 7.1] for a proof.

By [12, Proposition 3.1], the homotopy fiber of the map $BQ \longrightarrow BQ^\wedge_p$ is a $K(V, 1)$ for some $\widehat{\mathbb{Q}}_p$–vector space V. Using this, together with standard obstruction theory and the fact that $\widetilde{H}^*(BQ; \mathbb{Q}) = 0$, one checks that

$$[BP^\wedge_p, BQ^\wedge_p] \cong [BP, BQ] \cong \mathrm{Rep}(P, Q).$$

Now fix some $\rho \in \mathrm{Hom}(P, Q)$. The space $\mathrm{Map}(BP^\wedge_p, BQ^\wedge_p)_{B\rho}$ is the classifying space of some p–compact group X by [11, Propositions 5.1 & 6.22], and in particular is p–complete. Since $\mathrm{Map}(BP, BQ)_{B\rho} \simeq C_Q(\rho(P))$ (P and Q are both discrete), we will be done upon showing that the completion map

$$\mathrm{Map}(BP, BQ)_{B\rho} \longrightarrow \mathrm{Map}(BP, BQ^\wedge_p)_{B\rho} \tag{1}$$

is a mod p homology equivalence.

Fix a sequence of finite subgroups $P_1 \le P_2 \le \cdots$ whose union is P. Since Q is artinian, $C_Q(\rho(P_n)) = C_Q(\rho(P))$ for n sufficiently large. Also, the space $\mathrm{Map}(BP, BQ^\wedge_p)_{B\rho}$ is the homotopy inverse limit of the mapping spaces $\mathrm{Map}(BP_n, BQ^\wedge_p)_{B\rho}$. So if (1) is a mod p equivalence upon replacing P by P_n for each n, it is also a mod p equivalence for P. In other words, it suffices to prove this when P is a finite p–group.

Let X be the homotopy fiber of the completion map $BQ \longrightarrow BQ^\wedge_p$. As noted above, X is a $K(V, 1)$ where V is a rational vector space. Since the map from $\mathrm{Map}(BP, BQ)$ to $\mathrm{Map}(BP, BQ^\wedge_p)$ is a bijection on components, the homotopy fiber of the map in (1) is X^{hP} for a proxy action of P on X (in the sense of Dwyer and Wilkerson [11]) induced by ρ.

Consider the fibration sequence

$$X^{hP} \longrightarrow \mathrm{Map}(BP, X_{hP})_{\{1\}} \xrightarrow{\mathrm{pr}\circ -} \mathrm{Map}(BP, BP)_{\mathrm{Id}},$$

where pr denotes the projection of X_{hP} to BP, and the total space is the set of all maps $f\colon BP \longrightarrow X_{hP}$ such that $\mathrm{pr}\circ f \simeq \mathrm{Id}$. Since X_{hP} is the total space of a fibration over BP with fiber X, it is a $K(\pi, 1)$ where $V \lhd \pi$ and $\pi/V \cong P$. Since P is a finite p–group and V is a rational vector space, this extension splits, and the splitting is unique up to conjugacy by elements of V.

It follows that $$\underset{\cong \mathrm{Rep}(P,\pi)}{[BP, X_{hP}]} \longrightarrow \underset{\cong \mathrm{Rep}(P,P)}{[BP, BP]}$$

is a bijection. Also, the induced map

$$\underbrace{\pi_1(\mathrm{Map}(BP, X_{hP})_{\{1\}})}_{\cong C_\pi(P)} \longrightarrow \underbrace{\pi_1(\mathrm{Map}(BP, BP)_{\mathrm{Id}})}_{\cong Z(P)}$$

is surjective, and its kernel V^P (where the action of P on V is induced by the action on X) is a rational vector space.

Thus $X^{hP} \simeq K(V^P, 1)$. It follows that X^{hP} is mod p acyclic, and hence that (1) is a mod p equivalence. This finishes the proof. □

2 Fusion systems over discrete p–toral groups

We now define saturated fusion systems over discrete p–toral groups and study their basic properties. The definitions are almost identical to those in the finite case [7, Section 1].

Definition 2.1 A *fusion system* $\mathcal{F}$ over a discrete p–toral group S is a category whose objects are the subgroups of S, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ satisfy the following conditions:

(a) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ for all $P, Q \le S$.

(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

Two subgroups $P, P' \le S$ are called *$\mathcal{F}$–conjugate* if $\mathrm{Iso}_{\mathcal{F}}(P,P') \ne \varnothing$.

Definition 2.2 Let $\mathcal{F}$ be a fusion system over a discrete p–toral group S.

- A subgroup $P \le S$ is *fully centralized in $\mathcal{F}$* if $|C_S(P)| \ge |C_S(P')|$ for all $P' \le S$ which is $\mathcal{F}$–conjugate to P.
- A subgroup $P \le S$ is *fully normalized in $\mathcal{F}$* if $|N_S(P)| \ge |N_S(P')|$ for all $P' \le S$ which is $\mathcal{F}$–conjugate to P.
- $\mathcal{F}$ is a *saturated fusion system* if the following three conditions hold:

  (I) For each $P \le S$ which is fully normalized in $\mathcal{F}$, P is fully centralized in $\mathcal{F}$, $\mathrm{Out}_{\mathcal{F}}(P)$ is finite, and $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$.

  (II) If $P \le S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ are such that $\varphi(P)$ is fully centralized, and if we set

  $$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\},$$

  then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_P = \varphi$.

(III) If $P_1 \le P_2 \le P_3 \le \cdots$ is an increasing sequence of subgroups of S, with $P_\infty = \bigcup_{n=1}^{\infty} P_n$, and if $\varphi \in \mathrm{Hom}(P_\infty, S)$ is any homomorphism such that $\varphi|_{P_n} \in \mathrm{Hom}_{\mathcal{F}}(P_n, S)$ for all n, then $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_\infty, S)$.

By Lemma 1.6, there is a global upper bound for $|\pi_0(C_S(P))|$ and $|\pi_0(N_S(P))|$, taken over all subgroups P of any given S. In particular, for any given subgroup $P \le S$, $|C_S(P')|$ and $|N_S(P')|$ take on maximal values among all P' which are $\mathcal{F}$–conjugate to P. This proves that the conjugacy class of P always contains fully centralized subgroups and fully normalized subgroups.

It is very convenient, in the above definition, to be working with a class of groups where the concept of "order" of subgroups is defined. However, there are other ways to define fully normalized and fully centralized subgroups in a fusion system, and hence to define saturation; and this property was not a factor in our decision to restrict attention to fusion systems over discrete p–toral groups. The crucial properties of these groups, which seem to be needed frequently when developing the theory, are that they are artinian and locally finite.

When $\mathcal{F}$ is a saturated fusion system over the discrete p–toral subgroup S, then by (I), $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ is finite for fully normalized $P \le S$, and hence for all $P \le S$. Since $\mathrm{Inn}(P)$ is discrete p–toral (being a quotient group of P), $\mathrm{Aut}_{\mathcal{F}}(P)$ inherits many of the properties of discrete p–toral groups. In particular, it is artinian, locally finite, and contains a unique conjugacy class of maximal discrete p–toral subgroups. This condition that $\mathrm{Out}_{\mathcal{F}}(P)$ be finite does simplify slightly the definition of a saturated fusion system, but it is in fact unnecessary, as is shown by the following proposition.

Proposition 2.3 *Let $\mathcal{F}$ be a fusion system over the discrete p–toral group S. Assume that axiom (II) in Definition 2.2 holds, and that (I) holds for all finite fully normalized subgroups of S. Then $\mathrm{Out}_{\mathcal{F}}(P)$ is finite for all $P \le S$.*

Proof Fix $P \le S$. For all $m \ge 1$, set $P_{(m)} = \{g \in P_0 \mid g^{p^m} = 1\}$. By Proposition 1.5(b), to show that $\mathrm{Out}_{\mathcal{F}}(P)$ is finite, it suffices to show that $\mathrm{Aut}_{\mathcal{F}}(P)$ is a torsion group.

Fix $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$. We want to show that α has finite order; by Lemma 1.7(b), it suffices to do this when $P = P_0$ is connected. After replacing α by α^n for some appropriate $n \ge 1$, we can assume that $\alpha|_{P_{(1)}} = \mathrm{Id}$. Then by Lemma 1.7(c), $\alpha_m \overset{\mathrm{def}}{=} \alpha|_{P_{(m)}}$ has p–power order for all m. For each m, there is $\varphi_m \in \mathrm{Hom}_{\mathcal{F}}(P_{(m)}, S)$ such that $\varphi_m(P_{(m)})$ is fully normalized, and by (I), $\varphi_m(P_{(m)})$ is fully centralized, and φ_m can be chosen such that $\varphi_m \alpha_m \varphi_m^{-1} \in \mathrm{Aut}_S(\varphi_m(P_{(m)}))$. Also, by (II), φ_m can be extended to $\overline{\varphi}_m \in \mathrm{Hom}_{\mathcal{F}}(S_0, S)$, so $\varphi_m(P_{(m)}) \le S_0$, and hence $|\mathrm{Aut}_S(\varphi_m(P_{(m)}))| \le |S/S_0|$. Thus $(\alpha_m)^{|S/S_0|} = \mathrm{Id}_{P_{(m)}}$ for each m, so $\alpha^{|S/S_0|} = \mathrm{Id}_P$, and α has finite order. □

In fact, one can show that in the definition of a saturated fusion system, it suffices to require that (I) holds for all finite fully normalized subgroups $P \le S$; it then follows that (I) holds for all fully normalized subgroups.

When $\mathcal{F}$ is a (saturated) fusion system over a discrete p–toral group S, we think of the identity component S_0 as the "maximal torus" of the fusion system, and think of $\mathrm{Aut}_{\mathcal{F}}(S_0)$ as its "Weyl group". The following lemma describes how morphisms between subgroups of the maximal torus are controlled by the Weyl group.

Lemma 2.4 *Let $\mathcal{F}$ be a saturated fusion system over a discrete p–toral group S with connected component $T = S_0$. Then the following hold for all $P \le T$.*

(a) *For every $P' \le S$ which is $\mathcal{F}$–conjugate to P and fully centralized in $\mathcal{F}$, $P' \le T$, and there exists some $w \in \mathrm{Aut}_{\mathcal{F}}(T)$ such that $w|_P \in \mathrm{Iso}_{\mathcal{F}}(P, P')$.*

(b) *Every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, T)$ is the restriction of some $w \in \mathrm{Aut}_{\mathcal{F}}(T)$.*

Proof We first prove the following statement.

(c) For each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ such that $P' \stackrel{\text{def}}{=} \varphi(P)$ is fully centralized in $\mathcal{F}$, there exists $w \in \mathrm{Aut}_{\mathcal{F}}(T)$ such that $w|_P = \varphi$.

By assumption, $P \le T \le C_S(P)$. By condition (II) in Definition 2.2, there is $\overline{\varphi}$ in $\mathrm{Hom}_{\mathcal{F}}(C_S(P), S)$ such that $\overline{\varphi}|_P = \varphi$. Then $\overline{\varphi}(T) \le T$ since T is connected (infinitely p–divisible), and so $\overline{\varphi}(T) = T$ since T is artinian. Thus $w \stackrel{\text{def}}{=} \overline{\varphi}|_T \in \mathrm{Aut}_{\mathcal{F}}(T)$ is such that $w|_P = \varphi$. This proves (c), and also proves (a) since $P' = w(P) \le T$.

Now fix any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, T)$. Let Q be a fully centralized subgroup of S in the $\mathcal{F}$–conjugacy class of P and $\varphi(P)$, and choose $\psi \in \mathrm{Iso}_{\mathcal{F}}(\varphi(P), Q)$. By (c), there are elements $u, v \in \mathrm{Aut}_{\mathcal{F}}(T)$ such that $u|_P = \psi \circ \varphi$ and $v|_{\varphi(P)} = \psi$. So if we set $w = v^{-1}u$, then $w|_P = \varphi$. $\square$

By Proposition 2.3, $\mathrm{Out}_{\mathcal{F}}(P)$ is finite for every subgroup $P \le S$. The following lemma extends this statement.

Lemma 2.5 *Let $\mathcal{F}$ be a saturated fusion system over a discrete p–toral group S. Then for all $P, Q \le S$, the set $\mathrm{Rep}_{\mathcal{F}}(P, Q) \stackrel{\text{def}}{=} \mathrm{Inn}(Q) \backslash \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is finite.*

Proof As just noted, $\mathrm{Out}_{\mathcal{F}}(P)$ is finite for all $P \le S$. Also, if $\varphi, \varphi' \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ and $\mathrm{Im}(\varphi) = \mathrm{Im}(\varphi')$, then $\varphi' = \varphi \circ \alpha$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ by condition (b) in Definition 2.1. So there is a bijection

(2) $$\mathrm{Rep}_{\mathcal{F}}(P, Q)/\,\mathrm{Out}_{\mathcal{F}}(P) \xrightarrow{\ \cong\ } \bigl\{P' \le Q \bigm| P' \ \mathcal{F}\text{–conjugate to } P\bigr\}\big/(Q\text{–conjugacy}),$$

which sends the class of a homomorphism to the conjugacy class of its image.

By Lemma 2.4, the $\mathcal{F}$–conjugacy class (P_0) of P_0 is just its orbit under the action of $\mathrm{Aut}_{\mathcal{F}}(S_0)$, and hence a finite set. By Lemma 1.4(a), for any given $Q \in (P_0)$, there are only finitely many $N_S(Q)/Q$–conjugacy classes of subgroups of order $|P/P_0|$ in $N_S(Q)/Q$. Hence there are only finitely many S–conjugacy classes of subgroups $P' \le S$ which are $\mathcal{F}$–conjugate to P and such that $P'_0 = Q$. This shows that the target set in (2) is finite, and hence that $\mathrm{Rep}_{\mathcal{F}}(P, Q)$ is also finite. □

The definitions of centric and radical subgroups in a fusion system over a discrete p–toral group are essentially the same as those in the finite case.

Definition 2.6 Let $\mathcal{F}$ be a fusion system over a discrete p–toral group S. A subgroup $P \le S$ is called *$\mathcal{F}$–centric* if P and all its $\mathcal{F}$–conjugates contain their S–centralizers. A subgroup $P \le S$ is called *$\mathcal{F}$–radical* if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$; ie if $\mathrm{Out}_{\mathcal{F}}(P)$ contains no nontrivial normal p–subgroup.

Notice that any $\mathcal{F}$–centric subgroup is fully centralized. Conversely, if $P \le S$ is fully centralized and centric in S; that is, $Z(P) = C_S(P)$, then it is $\mathcal{F}$–centric. The next proposition says that the set of $\mathcal{F}$–centric subgroups is closed under overgroups.

Proposition 2.7 *Let $\mathcal{F}$ be a saturated fusion system over the discrete p–toral group S, and let $P \le Q \le S$ be such that P is $\mathcal{F}$–centric. Then Q is also $\mathcal{F}$–centric.*

Proof Fix any Q' which is $\mathcal{F}$–conjugate to Q, choose $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, Q')$, and set $P' = \varphi(P)$. Then

$$C_S(Q') \le C_S(P') \le P' \le Q',$$

where the second inequality holds since P is $\mathcal{F}$–centric. So Q is also $\mathcal{F}$–centric. □

The next proposition gives another important property of $\mathcal{F}$–centric subgroups; one which is much less obvious.

Proposition 2.8 *Let $\mathcal{F}$ be a saturated fusion system over the discrete p–toral group S. Then for each $P \le Q \le S$ such that P is $\mathcal{F}$–centric, and each $\varphi, \varphi' \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$ such that $\varphi|_P = \varphi'|_P$, there is some $g \in Z(P)$ such that $\varphi = \varphi' \circ c_g$.*

Proof The hypothesis implies that $\varphi \circ \varphi'^{-1}|_{\varphi'(P)} = \mathrm{Id}_{\varphi'(P)}$, and we must show that $\varphi \circ \varphi'^{-1} = \mathrm{Id}_{\varphi'(Q)}$. It thus suffices to prove, for $P \le Q \le S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$ where P is $\mathcal{F}$–centric, that $\varphi|_P = \mathrm{Id}_P$ implies $\varphi = c_g$ for some $g \in Z(P)$.

Assume first that $P \lhd Q$. Then for each $x \in Q$, $c_{\varphi(x)}|_P = c_x|_P$. Thus $\varphi(x) \equiv x \pmod{C_S(P)}$, and $C_S(P) \le P$ since P is $\mathcal{F}$–centric. In particular, this shows that $\varphi(Q) = Q$, and thus that $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$. It also shows that φ induces the identity on Q/P. Since Q/P has finite order, φ has p–power order by Lemma 1.7(a).

Without loss of generality, we can replace Q by any other subgroup in its $\mathcal{F}$–conjugacy class. In particular, we can assume that Q is fully normalized, and hence that $\mathrm{Out}_S(Q) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(Q))$. So every p–subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$ is conjugate to a subgroup of $\mathrm{Aut}_S(Q)$. Thus there is $\chi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\chi \circ \varphi \circ \chi^{-1} = c_y$ for some $y \in N_S(Q)$. Since $\varphi|_P = \mathrm{Id}_P$, c_y acts as the identity on $\varphi(P)$, which is also $\mathcal{F}$–centric, hence $y \in C_S(\varphi(P)) = \varphi(Z(P))$. Set $x = \chi^{-1}(y)$; then $\varphi = c_x$.

Now assume P is not normal in Q. Let $\mathcal{Q}$ be the set of subgroups $Q' \le Q$ containing P such that $\varphi|_{Q'} = c_g|_{Q'}$ for some $g \in Z(P)$. If $P \le Q' \lneqq Q$ and $Q' \in \mathcal{Q}$, then $N_Q(Q') \gneqq Q'$ by Lemma 1.8, and $N_Q(Q') \in \mathcal{Q}$ since the proposition holds for the normal pair $Q' \lhd N_Q(Q')$. Hence if $\mathcal{Q}$ contains a maximal element, it must be Q itself.

Let $Q_1 \le Q_2 \le \cdots$ be any increasing chain in $\mathcal{Q}$, and set $Q_\infty = \bigcup_{n=1}^{\infty} Q_n$. Let $g_n \in Z(P)$ be such that $\varphi|_{Q_n} = c_{g_n}|_{Q_n}$. Since P is $\mathcal{F}$–centric, so are the Q_n, and thus $Z(Q_1) \ge Z(Q_2) \ge \cdots$ is a decreasing sequence of subgroups. Since S is artinian, there is some k such that $Z(Q_n) = Z(Q_k)$ for all $n \ge k$. This shows that $g_n \equiv g_k \pmod{Z(Q_k)}$ for all $n \ge k$, hence that $\varphi|_{Q_\infty} = c_{g_k}|_{Q_\infty}$, and hence that $Q_\infty \in \mathcal{Q}$. Thus by Zorn's lemma, $\mathcal{Q}$ contains a maximal element, so $Q \in \mathcal{Q}$, and this finishes the proof. □

3 A finite retract of a saturated fusion system

A fusion system $\mathcal{F}$ over a discrete p–toral group S generally has infinitely many isomorphism classes of objects. In this section, we construct a subcategory $\mathcal{F}^\bullet$ of $\mathcal{F}$ with only finitely many isomorphism classes of objects, together with a retraction functor from $\mathcal{F}$ to $\mathcal{F}^\bullet$ which is a left adjoint to the inclusion. This means that in many cases, it will suffice to work over the "finite" subcategory $\mathcal{F}^\bullet$ rather than the full fusion system $\mathcal{F}$. As a first application, we show that $\mathrm{Ob}(\mathcal{F}^\bullet)$ contains all $\mathcal{F}$–centric $\mathcal{F}$–radical subgroups, and hence that there are only finitely many conjugacy classes of such subgroups. A second application is Alperin's fusion theorem in this setting: restriction to $\mathcal{F}^\bullet$ allows us to repeat the same inductive argument as that used for fusion systems over a finite p–group.

Following the group theorists' usual notation, whenever Γ is a group of automorphisms of a group G and $H \le G$, we write

$$C_\Gamma(H) = \{\gamma \in \Gamma \mid \gamma|_H = \mathrm{Id}_H\}.$$

The following definitions were motivated by some constructions of Benson [2], which he in fact used to prove a version of Alperin's fusion theorem for compact Lie groups.

Definition 3.1 Let $\mathcal{F}$ be a saturated fusion system over a discrete p–toral group S, let $T = S_0$ be the identity component of S, and set $W = \mathrm{Aut}_{\mathcal{F}}(T) = \mathrm{Out}_{\mathcal{F}}(T)$ (the "Weyl group"). Set

$$p^m = \exp(S/T) \overset{\text{def}}{=} \min\{p^k \mid x^{p^k} \in T \text{ for all } x \in S\}.$$

(a) For each $P \le T$, set

$$I(P) = T^{C_W(P)} = \bigl\{t \in T \bigm| w(t) = t \text{ for all } w \in W \text{ such that } w|_P = \mathrm{Id}_P\bigr\},$$

and let $I(P)_0$ be the identity component of $I(P)$.

(b) For each $P \le S$, let $P^{[m]} = \langle g^{p^m} \mid g \in P\rangle \le T$, and set

$$P^\bullet = P \cdot I(P^{[m]})_0 \overset{\text{def}}{=} \{gt \mid g \in P,\ t \in I(P^{[m]})_0\}.$$

(c) Set $\quad \mathcal{H}(\mathcal{F}) = \{I(P)_0 \mid P \le T\} \qquad$ and $\qquad \mathcal{H}^\bullet(\mathcal{F}) = \{P^\bullet \mid P \le S\}$,

and let $\mathcal{F}^\bullet \subseteq \mathcal{F}$ be the full subcategory with object set $\mathcal{H}^\bullet(\mathcal{F})$.

Thus for $P \le T$, $I(P)$ is the maximal subgroup of T such that for all $w \in W$, $w|_P = \mathrm{Id}$ if and only $w|_{I(P)} = \mathrm{Id}$. In particular, for all v and w in W, $v|_P = w|_P$ if and only if $v|_{I(P)} = w|_{I(P)}$. Together with Lemma 2.4(b), this implies that every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,T)$ extends to a unique $I(\varphi) \in \mathrm{Hom}_{\mathcal{F}}(I(P),T)$, which is obtained by first extending φ to T and then restricting to $I(P)$. In other words, every $\mathcal{F}$–isomorphism $\varphi\colon P \longrightarrow Q$ between subgroups of T extends to a unique $\mathcal{F}$–isomorphism $I(\varphi)\colon I(P) \to I(Q)$.

For an arbitrary subgroup $P \le S$, $P^{[m]}$ is a subgroup of T, and the above arguments apply. Since $P^{[m]} \lhd P$, any $x \in P$ normalizes $P^{[m]}$, and hence also normalizes $I(P^{[m]})$. Thus P normalizes $I(P^{[m]})_0$, and this shows that the subset $P^\bullet \overset{\text{def}}{=} P \cdot I(P^{[m]})_0$ is a group.

More generally, for any $k \ge m$, we could define subgroups $P^{\bullet k} \ge P$ for each $P \le S$ by setting $P^{\bullet k} = P \cdot I(P^{[k]})$. This can be different from $P^\bullet$, but $P \mapsto P^{\bullet k}$ has all of the same properties which we prove here for $P^\bullet$. However, the only way in which this generalization might be needed would be if we wanted to compare these "bullet functors" for two different fusion systems over two different discrete p–toral groups, and that will not be needed in this paper.

Lemma 3.2 *The following hold for every saturated fusion system $\mathcal{F}$ over a discrete p–toral group S.*

(a) *The set $\mathcal{H}(\mathcal{F})$ is finite, and the set $\mathcal{H}^\bullet(\mathcal{F})$ contains finitely many S–conjugacy classes of subgroups of S.*

(b) *For all $P \le S$, $(P^\bullet)^\bullet = P^\bullet$.*

(c) *If $P \le Q \le S$, then $P^\bullet \le Q^\bullet$.*

(d) *If $P \le S$ is $\mathcal{F}$–centric, then $Z(P^\bullet) = Z(P)$.*

Proof Let $T = S_0 \lhd S$ be the identity component, and set $W = \mathrm{Aut}_{\mathcal{F}}(T)$ and $p^m = \exp(S/T)$. Note that for any $P \le Q \le T$, $C_W(P) \ge C_W(Q)$, and hence $I(P) \le I(Q)$. Also, $C_W(I(P)) = C_W(P)$ by definition, and hence $I(I(P)) = I(P)$.

(a) By definition, each subgroup in $\mathcal{H}(\mathcal{F})$ has the form $I(P)_0 = (T^K)_0$ for some $P \le T$, where $K = C_W(P) \le W$. Since the finite group $W = \mathrm{Out}_{\mathcal{F}}(T)$ has a finite number of subgroups, this shows that $\mathcal{H}(\mathcal{F})$ is finite. Also, for any $P \le S$, $P_0 \le P^{[m]} \le I(P^{[m]})$, and so $(P^\bullet)_0 = I(P^{[m]})_0 \in \mathcal{H}(\mathcal{F})$. In particular, there are only finitely many possibilities for identity components of subgroups in $\mathcal{H}^\bullet(\mathcal{F})$.

Fix $P \le S$, and set $K = C_W(P^{[m]})$. Since $P^{[m]}$ is generated by all p^m–powers in P (and $p^m = \exp(S/T)$), $P^{[m]} \le T$ and

$$[P{:}P^{[m]}] = [P{:}(P\cap T)]\cdot[(P\cap T){:}P^{[m]}] \le |S/T|\cdot p^{m\cdot\mathrm{rk}(T)}.$$

Here, the last inequality holds since $(P\cap T)/P^{[m]}$ is abelian with exponent at most p^m and rank at most $\mathrm{rk}(T)$. Also, since $P^{[m]}\cdot I(P^{[m]})_0 = P^{[m]}\cdot(T^K)_0 \le T^K$,

$$\begin{aligned}|\pi_0(P^\bullet)| = |\pi_0(P\cdot(T^K)_0)| &\le |\pi_0(P^{[m]}\cdot(T^K)_0)|\cdot|P/P^{[m]}|\\ &\le |\pi_0(T^K)|\cdot|P/P^{[m]}| \le |\pi_0(T^K)|\cdot|S/T|\cdot p^{m\cdot\mathrm{rk}(T)}.\end{aligned}$$

We have already seen that $(T^K)_0$ is the identity component of $P^\bullet$, and we have just shown that the number of components of $P^\bullet$ is bounded by an integer which depends only on K (and on S). Since $N_S((T^K)_0)/(T^K)_0$ has only finitely many conjugacy classes of finite subgroups of any given order (Lemma 1.4(a)), this shows that there are only finitely many conjugacy classes of subgroups in $\mathcal{H}^\bullet(\mathcal{F})$ corresponding to any given $K \le W$; and thus (since W is finite) only finitely many conjugacy classes of subgroups in $\mathcal{H}^\bullet(\mathcal{F})$.

(b) Fix $P \le S$. Since P normalizes $I(P^{[m]})_0$, for any $g \in P$ and any $x \in I(P^{[m]})_0$, $(gx)^{p^m} \in g^{p^m}\cdot I(P^{[m]})_0 \le P^{[m]}\cdot I(P^{[m]})_0$. This proves the second inequality on the following line:

$$P^{[m]} \le (P^\bullet)^{[m]} \le P^{[m]}\cdot I(P^{[m]})_0 \le I(P^{[m]}).$$

The others are clear. Since $I(-)$ is idempotent and preserves order, this shows that $I((P^\bullet)^{[m]}) = I(P^{[m]})$. Hence $(P^\bullet)^\bullet = P^\bullet \cdot I(P^{[m]})_0 = P^\bullet$.

(c) If $P \le Q$, then $P^{[m]} \le Q^{[m]}$, so $I(P^{[m]}) \le I(Q^{[m]})$, and hence $P^\bullet \le Q^\bullet$.

(d) For any $P \le S$, we have $P \le P^\bullet$. Thus if P is $\mathcal{F}$–centric, then so is $P^\bullet$, and $Z(P^\bullet) \le Z(P)$. To see that this is an equality, it suffices to show that every element in $Z(P)$ commutes with $I(P^{[m]})$. For all $x \in Z(P)$, c_x (as an element of $W = \mathrm{Aut}_{\mathcal{F}}(T)$) lies in $C_W(P^{[m]})$, hence commutes with all elements of $I(P^{[m]}) = T^{C_W(P^{[m]})}$, and in particular with all elements of $I(P^{[m]})_0$. □

We are now ready to prove the main, crucial, property of these subgroups $P^\bullet$.

Proposition 3.3 *Let $\mathcal{F}$ be a saturated fusion system over a discrete p–toral group S. Fix $P, Q \le S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. Then φ extends to a unique homomorphism $\varphi^\bullet \in \mathrm{Hom}_{\mathcal{F}}(P^\bullet, Q^\bullet)$; and this makes $P \mapsto P^\bullet$ into a functor from $\mathcal{F}$ to itself.*

Proof The functoriality of $P \mapsto P^\bullet$ and $\varphi \mapsto \varphi^\bullet$ (ie the fact that $(\mathrm{Id}_P)^\bullet = \mathrm{Id}_{P^\bullet}$ and $(\psi \circ \varphi)^\bullet = \psi^\bullet \circ \varphi^\bullet$) follows immediately from the existence and uniqueness of these extensions. So this is what we need to prove.

As usual, we set $T = S_0$ and $W = \mathrm{Aut}_{\mathcal{F}}(T)$. For all $Q \le T$, $C_W(Q) = C_W(I(Q))$ by definition of $I(-)$. This will be used frequently throughout the proof.

We first check that there is at most one morphism $\varphi^\bullet$ which extends φ. Assume that $\psi, \psi' \in \mathrm{Hom}_{\mathcal{F}}(P^\bullet, Q^\bullet)$ are two such extensions. By Lemma 2.4(b), there are elements $w, w' \in W$ such that

$$\psi|_{P^{[m]} \cdot I(P^{[m]})_0} = w|_{P^{[m]} \cdot I(P^{[m]})_0} \quad \text{and} \quad \psi'|_{P^{[m]} \cdot I(P^{[m]})_0} = w'|_{P^{[m]} \cdot I(P^{[m]})_0}.$$

Since $w|_{P^{[m]}} = w'|_{P^{[m]}}$, we have $w^{-1}w' \in C_W(P^{[m]}) = C_W(I(P^{[m]}))$, so $w|_{I(P^{[m]})} = w'|_{I(P^{[m]})}$ as well. It follows that $\psi = \psi'$, since they take the same values on P and on $I(P^{[m]})_0$.

It remains to prove the existence of $\varphi^\bullet$. By Lemma 3.2(c), it suffices to prove this when $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$. Recall that $P^\bullet = P \cdot I(P^{[m]})_0$. Fix $u \in W = \mathrm{Aut}_{\mathcal{F}}(T)$ such that $u|_{P^{[m]}} = \varphi|_{P^{[m]}}$. Define $\varphi^\bullet$ by setting, for all $g \in P$ and all $x \in I(P^{[m]})_0$,

$$\varphi^\bullet(gx) = \varphi(g)u(x).$$

After two preliminary steps, we show in Step 3 that $\varphi^\bullet$ is well defined and a homomorphism, and in Step 4 that it is a morphism in $\mathcal{F}$.

Step 1 Fix $A, A' \le T$, and $w \in W$ such that $w(A) = A'$. We show here that

$$A \le B \le I(A),\ \psi \in \mathrm{Hom}_{\mathcal{F}}(B, T),\ \psi|_A = w|_A \implies \psi = w|_B, \tag{3}$$

and also that

(4) $A \le B \le A\cdot I(A)_0,\ \psi \in \mathrm{Hom}_{\mathcal{F}}(B,S),\ \psi|_A = w|_A \implies \psi(B) \le T$ and $\psi = w|_B$.

If $\psi(B) \le T$, then $\psi = w'|_B$ for some $w' \in W$ by Lemma 2.4(b), $w^{-1}w' \in C_W(A) = C_W(I(A))$, and thus $\psi = w'|_B = w|_B$. This proves (3).

Now assume $B \le A\cdot I(A)_0$. By Lemma 2.4(a), there is $w' \in W$ such that $w'(B)$ is fully centralized in $\mathcal{F}$. It thus suffices to prove (4) when B is fully centralized. Set $B' = \psi(B)$ for short.

Now, $B' \ge A'$ and B' is abelian. So for all $x \in B'$, if we regard c_x as an element of $W = \mathrm{Aut}_{\mathcal{F}}(T)$, then $c_x \in C_W(A') = C_W(I(A'))$. Thus $I(A') = w(I(A)) \le C_S(B')$. By axiom (II) (and since $B = \psi^{-1}(B')$ is fully centralized), ψ^{-1} extends to an $\mathcal{F}$–morphism defined on $B'\cdot C_S(B')$, and in particular to $\beta \in \mathrm{Hom}_{\mathcal{F}}(B'\cdot I(A'), S)$. Since $\beta|_{A'} = w^{-1}|_{A'}$ and $\beta(I(A')_0) \le T$, $\beta|_{A'\cdot I(A')_0} = w^{-1}|_{A'\cdot I(A')_0}$ by (3).

Thus for all $x \in B'$, $\beta(x) = \psi^{-1}(x) \in B \le A\cdot I(A)_0 = \beta(A'\cdot I(A')_0)$. Since β is injective, this shows $x \in A'\cdot I(A')_0 \le T$. So $B' \le T$, and (4) now follows from (3).

Step 2 We next show that for all $x \in I(P^{[m]})$ and all $g \in P$, the following identity holds:

(5) $$u(gxg^{-1}) = \varphi(g)u(x)\varphi(g)^{-1},$$

or equivalently that $c_{\varphi(g)}^{-1} \circ u \circ c_g(x) = u(x)$. Set $w = c_{\varphi(g)}^{-1} \circ u \circ c_g \in W$ for short. Then (5) holds for $x \in P^{[m]}$ since $\varphi|_{P^{[m]}} = u|_{P^{[m]}}$, and thus $w|_{P^{[m]}} = u|_{P^{[m]}}$. So $w|_{I(P^{[m]})} = u|_{I(P^{[m]})}$ by (3), and this proves (5) for all $x \in I(P^{[m]})$.

Step 3 Recall that we defined $\varphi^\bullet(gx) = \varphi(g)u(x)$ for all $g \in P$ and $x \in I(P^{[m]})_0$. By assumption, $\varphi|_{P^{[m]}} = u|_{P^{[m]}}$. Hence the restrictions of φ and u to $P^{[m]}\cdot(P \cap I(P_0^{[m]}))$ are equal by (4), and this shows that $\varphi^\bullet$ is well defined.

For all $g, g' \in P$ and all $x, x' \in I(P^{[m]})_0$,

$$\begin{aligned}\varphi^\bullet((gx)(g'x')) &= \varphi(gg')\cdot u(g'^{-1}xg'x') = \varphi(gg')\cdot\big(\varphi(g')^{-1}u(x)\varphi(g')\big)\cdot u(x') \\ &= \varphi(g)u(x)\varphi(g')u(x') = \varphi^\bullet(gx)\cdot\varphi^\bullet(g'x'),\end{aligned}$$

where the second equality follows from Step 2. Thus $\varphi^\bullet$ is a homomorphism.

Step 4 It remains to show that $\varphi^\bullet \in \mathrm{Iso}_{\mathcal{F}}(P^\bullet, Q^\bullet)$; ie that $\varphi^\bullet$ is a morphism in the category $\mathcal{F}$. By condition (III) in Definition 2.2, together with Zorn's lemma, there is a maximal subgroup $P' \le P^\bullet$ containing P such that $\varphi^\bullet|_{P'} \in \mathrm{Hom}_{\mathcal{F}}(P', Q^\bullet)$. Assume $P' \lneqq P^\bullet$; and set $\varphi' = \varphi^\bullet|_{P'}$ and $P'' = N_{P^\bullet}(P') \gneqq P'$. By condition (II) in Definition 2.2, φ' extends to some morphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(P'', S)$ (the existence of the homomorphism $\varphi^\bullet$ shows that $N_{\varphi'} \geq P''$). By (4) again, the restrictions of ψ, u, and $\varphi^\bullet$ to $P'' \cap (P^{[m]} \cdot I(P^{[m]})_0)$ are equal. Since $P'' = P \cdot (P'' \cap I(P^{[m]})_0)$, this shows that $\psi = \varphi^\bullet|_{P''}$. This contradicts the maximality assumption about P'; so $P' = P^\bullet$, and we are done. □

Note in particular that by Lemma 3.2(c), the functor $\mathcal{F} \xrightarrow{(-)^\bullet} \mathcal{F}^\bullet$ of Proposition 3.3 sends inclusions of subgroups to inclusions.

Corollary 3.4 *The functor $(-)^\bullet$ is a left adjoint to the inclusion of $\mathcal{F}^\bullet$ as a full subcategory of $\mathcal{F}$.*

Proof Fix any P in $\mathcal{F}$ and any Q in $\mathcal{F}^\bullet$. Since $Q = Q^\bullet$ by Lemma 3.2(b), every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ extends to a unique $\varphi^\bullet \in \mathrm{Hom}_{\mathcal{F}}(P^\bullet, Q)$ by Proposition 3.3. The restriction map

$$\mathrm{Hom}_{\mathcal{F}}(P^\bullet, Q) \xrightarrow{\mathrm{Res}} \mathrm{Hom}_{\mathcal{F}}(P, Q)$$

is thus a bijection, and this proves adjointness. □

Corollary 3.4 will later be extended to orbit and linking categories associated to $\mathcal{F}$ and $\mathcal{F}^\bullet$.

Corollary 3.5 *Let $\mathcal{F}$ be a saturated fusion system over a discrete p–toral group S. Then all $\mathcal{F}$–centric $\mathcal{F}$–radical subgroups of S are in $\mathcal{H}^\bullet(\mathcal{F})$, and in particular there are only finitely many conjugacy classes of such subgroups.*

Proof Assume P is $\mathcal{F}$–centric and $\mathcal{F}$–radical. We claim that $I(P^{[m]})_0 \leq P$, and thus that $P = P^\bullet \in \mathcal{H}^\bullet(\mathcal{F})$.

Assume otherwise. Then $P^\bullet \gneqq P$, and hence $N_{P^\bullet}(P) \gneqq P$ by Lemma 1.8. Thus $N_{P^\bullet}(P)/P \neq 1$, and since P is $\mathcal{F}$–centric, this group can be identified with a p–subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$. By Proposition 3.3, any $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ extends to an automorphism of $P^\bullet$, and in particular to an automorphism of $N_{P^\bullet}(P)$. This shows that $N_{P^\bullet}(P)/P \lhd \mathrm{Out}_{\mathcal{F}}(P)$, which contradicts the assumption that P is $\mathcal{F}$–radical.

The last statement now follows since $\mathcal{H}^\bullet(\mathcal{F})$ contains only finitely many conjugacy classes by Lemma 3.2(a). □

As a third consequence of Proposition 3.3, we now prove Alperin's fusion theorem in our context. This theorem was originally formulated for finite groups in [1], and then for saturated fusion systems over finite p–groups by Puig [27] (see also our paper [7, Theorem A.10]). Our approach here (and our definition of $P^\bullet$) is modelled on Benson's proof of the theorem for fusion in compact Lie groups [2].

Theorem 3.6 (Alperin's fusion theorem) *Let $\mathcal{F}$ be a saturated fusion system over a discrete p–toral group S. Then for each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, P')$, there exist sequences of subgroups of S*

$$P = P_0, P_1, \dots, P_k = P' \qquad \text{and} \qquad Q_1, Q_2, \dots, Q_k,$$

and elements $\varphi_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$, such that

(a) *Q_i is fully normalized in $\mathcal{F}$, $\mathcal{F}$–radical, and $\mathcal{F}$–centric for each i;*

(b) *$P_{i-1}, P_i \le Q_i$ and $\varphi_i(P_{i-1}) = P_i$ for each i; and*

(c) *$\varphi = \varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1$.*

Proof For each $P \le S$, let $\nu(P)$ be the number of $\mathcal{F}$–conjugacy classes of subgroups in $\mathcal{H}^\bullet(\mathcal{F})$ which contain P. We prove the theorem by induction on $\nu(P)$. Using Proposition 3.3, we can assume that $P, P' \in \mathcal{H}^\bullet(\mathcal{F})$. The claim is clear when $\nu(P) = 1$ (ie $P = S$).

Assume $P \lneqq S$. Let $P'' \le S$ be any subgroup which is $\mathcal{F}$–conjugate to P and fully normalized in $\mathcal{F}$, and fix $\psi \in \mathrm{Iso}_{\mathcal{F}}(P, P'')$. The theorem holds for $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, P')$ if it holds for ψ and for $\psi \circ \varphi^{-1} \in \mathrm{Iso}_{\mathcal{F}}(P', P'')$. So we are reduced to proving the theorem when the target group P' is fully normalized in $\mathcal{F}$.

Since P' is fully normalized, the p–subgroup $\varphi \circ \mathrm{Aut}_S(P) \circ \varphi^{-1}$ of $\mathrm{Aut}_{\mathcal{F}}(P')$ is conjugate to a subgroup of $\mathrm{Aut}_S(P')$. Let $\chi \in \mathrm{Aut}_{\mathcal{F}}(P')$ be such that the subgroup $(\chi \circ \varphi) \circ \mathrm{Aut}_S(P) \circ (\chi \circ \varphi)^{-1} \le \mathrm{Aut}_S(P')$. By condition (II) in Definition 2.2, there exists $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), S)$ such that $\overline{\varphi}|_P = \chi \circ \varphi$. Since $N_S(P) \gneqq P$ (since $P \lneqq S$) and $P \in \mathcal{H}^\bullet(\mathcal{F})$, $\nu(N_S(P)) < \nu(P)$, and the theorem holds for $\overline{\varphi}$ (as an isomorphism to its image) by the induction hypothesis. So it holds for φ if and only if it holds for χ. Hence it now remains only to prove it when $P = P'$ is fully normalized in $\mathcal{F}$, $P \in \mathcal{H}^\bullet(\mathcal{F})$, and $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$.

In particular, this implies that P is fully centralized in $\mathcal{F}$. So if P is not $\mathcal{F}$–centric, then φ extends to an automorphism $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(C_S(P) \cdot P)$ by condition (II) in Definition 2.2. Since $\nu(C_S(P) \cdot P) < \nu(P)$, the theorem holds for φ by the induction hypothesis.

Now assume that P is not $\mathcal{F}$–radical. Let $K \le \mathrm{Aut}_{\mathcal{F}}(P)$ be the subgroup such that $K/\mathrm{Inn}(P) = O_p(\mathrm{Out}_{\mathcal{F}}(P)) \ne 1$. Since P is fully normalized in $\mathcal{F}$, $\mathrm{Out}_S(P)$ is contained in $\mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$, and so $K \le \mathrm{Aut}_S(P)$. In particular,

$$N_S^K(P) \overset{\text{def}}{=} \{g \in N_S(P) \,|\, c_g|_P \in K\} \gneqq P$$

since $K \gneqq \mathrm{Inn}(P)$. Also, for each $g \in N_S^K(P)$, we have $\varphi c_g \varphi^{-1} \in K$ (since K is normal in $\mathrm{Aut}_{\mathcal{F}}(P)$), and hence $\varphi c_g \varphi^{-1} = c_h$ for some $h \in N_S^K(P)$. So by condition (II) in Definition 2.2, φ extends to an automorphism of $N_S^K(P) \gneqq P$, and the theorem again holds for φ by the induction hypothesis.

Finally, if $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$ and $P \in \mathcal{H}^\bullet(\mathcal{F})$ is a fully normalized $\mathcal{F}$–centric $\mathcal{F}$–radical subgroup of S, then the theorem holds for trivial reasons. □

4 Linking systems over discrete p–toral groups

We are now ready to define linking systems associated to a fusion system over a discrete p–toral group, and to study the relationship between linking systems and certain finite full subcategories.

Definition 4.1 Let $\mathcal{F}$ be a fusion system over the discrete p–toral group S. A *centric linking system associated to* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$–centric subgroups of S, together with a functor

$$\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c$$

and "distinguished" monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$–centric subgroup $P \le S$, which satisfy the following conditions.

(A) π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \le \mathrm{Aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P,Q)/Z(P) \xrightarrow{\ \cong\ } \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

(B) For each $\mathcal{F}$–centric subgroup $P \le S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\delta_P(g)"] & Q \arrow[d, "\delta_Q(\pi(f)(g))"] \\ P \arrow[r, "f"] & Q \end{tikzcd}$$

More generally, if $\mathcal{F}_0 \subseteq \mathcal{F}^c$ is any subcategory, then a linking system associated to $\mathcal{F}_0$ is a category $\mathcal{L}_0$, together with a functor $\mathcal{L}_0 \xrightarrow{\pi_0} \mathcal{F}_0$ and distinguished monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}_0}(P)$ for $P \in \mathrm{Ob}(\mathcal{F}_0) = \mathrm{Ob}(\mathcal{L}_0)$, which satisfy conditions (A), (B), and (C) above.

It is now clear, by analogy with the finite case, how to define p–local compact groups.

Definition 4.2 A *p–local compact group* is a triple $(S,\mathcal{F},\mathcal{L})$, where S is a discrete p–toral group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a linking system associated to $\mathcal{F}$. The *classifying space* of such a triple $(S,\mathcal{F},\mathcal{L})$ is the p–completed nerve $|\mathcal{L}|^\wedge_p$.

The following very basic lemma about linking systems extends [7, Lemma 1.10] to this situation.

Lemma 4.3 *Fix a p–local compact group $(S,\mathcal{F},\mathcal{L})$, and let $\pi\colon \mathcal{L}\longrightarrow\mathcal{F}^c$ be the projection. Fix $\mathcal{F}$–centric subgroups P,Q,R in S. Then the following hold.*

(a) *Fix any sequence $P\xrightarrow{\varphi}Q\xrightarrow{\psi}R$ of morphisms in $\mathcal{F}^c$, and let $\widetilde{\psi}\in\pi^{-1}_{Q,R}(\psi)$ and $\widetilde{\psi\varphi}\in\pi^{-1}_{P,R}(\psi\varphi)$ be arbitrary liftings. Then there is a unique morphism $\widetilde{\varphi}\in\mathrm{Mor}_{\mathcal{L}}(P,Q)$ such that*
$$\widetilde{\psi}\circ\widetilde{\varphi}=\widetilde{\psi\varphi},$$
and furthermore $\pi_{P,Q}(\widetilde{\varphi})=\varphi$.

(b) *If $\widetilde{\varphi},\widetilde{\varphi}'\in\mathrm{Mor}_{\mathcal{L}}(P,Q)$ are such that the homomorphisms $\varphi\overset{\text{def}}{=}\pi_{P,Q}(\widetilde{\varphi})$ and $\varphi'\overset{\text{def}}{=}\pi_{P,Q}(\widetilde{\varphi}')$ are conjugate (differ by an element of $\mathrm{Inn}(Q)$), then there is a unique element $g\in Q$ such that $\widetilde{\varphi}'=\delta_Q(g)\circ\widetilde{\varphi}$ in $\mathrm{Mor}_{\mathcal{L}}(P,Q)$.*

Proof Part (a) is an easy application of axiom (A) for a linking system. Part (b) is first reduced to the case where $\varphi=\varphi'$ using axiom (B), and this case then follows from (A) and (C). For more detail, see the proof of [7, Lemma 1.10]. □

We next show that the nerve of a linking system is p–good, and hence that the classifying space of a p–local compact group is p–complete.

Proposition 4.4 *Let $(S,\mathcal{F},\mathcal{L})$ be any p–local compact group at the prime p. Then $|\mathcal{L}|$ is p–good. Also, the composite*
$$S\xrightarrow{\pi_1(\theta)}\pi_1(|\mathcal{L}|)\longrightarrow\pi_1(|\mathcal{L}|^\wedge_p)$$
induced by the inclusion $BS\xrightarrow{\theta}|\mathcal{L}|$, factors through a surjection
$$\pi_0(S)\twoheadrightarrow\pi_1(|\mathcal{L}|^\wedge_p).$$

Proof For each $\mathcal{F}$–centric subgroup $P \le S$, fix a morphism $\iota_P \in \mathrm{Mor}_{\mathcal{L}}(P,S)$ which lifts the inclusion (and set $\iota_S = \mathrm{Id}_S$). By Lemma 4.3(a), for each $P \le Q \le S$, there is a unique morphism $\iota_P^Q \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ such that $\iota_Q \circ \iota_P^Q = \iota_P$.

Regard the vertex S as the basepoint of $|\mathcal{L}|$. Define

$$\omega\colon \mathrm{Mor}(\mathcal{L}) \longrightarrow \pi_1(|\mathcal{L}|)$$

by sending each $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ to the loop formed by the edges ι_P, φ, and ι_Q (in that order). Clearly, $\omega(\psi\circ\varphi) = \omega(\psi)\cdot\omega(\varphi)$ whenever ψ and φ are composable, and $\omega(\iota_P^Q) = \omega(\iota_P) = 1$ for all $P \le Q \le S$. Also, $\pi_1(|\mathcal{L}|)$ is generated by $\mathrm{Im}(\omega)$ since any loop in $|\mathcal{L}|$ can be split up as a composite of loops of the above form.

By Theorem 3.6 (Alperin's fusion theorem), each morphism in $\mathcal{F}$, and hence each morphism in $\mathcal{L}$, is (up to inclusions) a composite of automorphisms of fully normalized $\mathcal{F}$–centric subgroups. Thus $\pi_1(|\mathcal{L}|)$ is generated by the subgroups $\omega(\mathrm{Aut}_{\mathcal{L}}(P))$ for all fully normalized $\mathcal{F}$–centric $P \le S$.

Let $K \lhd \pi_1(|\mathcal{L}|)$ be the subgroup generated by all infinitely p–divisible elements. For each fully normalized $\mathcal{F}$–centric $P \le S$, $\mathrm{Aut}_{\mathcal{L}}(P)$ is generated by its Sylow subgroup $N_S(P)$ together with elements of order prime to p. Hence $\pi_1(|\mathcal{L}|)$ is generated by K together with the subgroups $\omega(N_S(P))$; and $\omega(N_S(P)) \le \omega(S)$ for each P. This shows that ω sends S surjectively onto $\pi_1(|\mathcal{L}|)/K$, and hence (since the identity component of S is infinitely divisible) factors through a surjection of $\pi_0(S)$ onto $\pi_1(|\mathcal{L}|)/K$. In particular, this quotient group is a finite p–group.

Set $\pi = \pi_1(|\mathcal{L}|)/K$ for short. Since K is generated by infinitely p–divisible elements, the same is true of its abelianization, and hence $H_1(K;\mathbb{F}_p) = 0$. Thus, K is p–perfect. Let X be the cover of $|\mathcal{L}|$ with fundamental group K. Then X is p–good and $X^\wedge_p$ is simply connected since $\pi_1(X)$ is p–perfect [3, VII.3.2]. Also, since π is a finite p–group, it acts nilpotently on $H_i(X;\mathbb{F}_p)$ for all i. Hence $X^\wedge_p \longrightarrow |\mathcal{L}|^\wedge_p \longrightarrow B\pi$ is a fibration sequence and $|\mathcal{L}|^\wedge_p$ is p–complete [3, II.5.1]. So $|\mathcal{L}|$ is p–good, and $\pi_1(|\mathcal{L}|^\wedge_p) \cong \pi$ is a quotient group of $\pi_0(S)$. □

Recall, from Section 3, that for any saturated fusion system $\mathcal{F}$, we defined a finite subcategory $\mathcal{F}^\bullet$ such that the inclusion $\mathcal{F}^\bullet \subseteq \mathcal{F}$ has a left adjoint $(-)^\bullet$. We next show that we can do the same on the level of linking systems.

Proposition 4.5 *Let $\mathcal{F}$ be a saturated fusion system over a discrete p–toral group S, and let $\mathcal{F}^{c\bullet} \subseteq \mathcal{F}^c$ be the full subcategory whose objects are the $\mathcal{F}$–centric subgroups contained in $\mathcal{H}^\bullet(\mathcal{F})$.*

(a) *Let $\mathcal{L}$ be a centric linking system associated to $\mathcal{F}$, and let $\mathcal{L}^\bullet \subseteq \mathcal{L}$ be the full subcategory with $\mathrm{Ob}(\mathcal{L}^\bullet) = \mathrm{Ob}(\mathcal{F}^{c\bullet})$. Then the inclusion $\mathcal{L}^\bullet \hookrightarrow \mathcal{L}$ has a left adjoint, which sends P to $P^\bullet$ for each $\mathcal{F}$–centric $P \le S$. In particular, the inclusion $|\mathcal{L}^\bullet| \subseteq |\mathcal{L}|$ is a homotopy equivalence.*

(b) *Let $\mathcal{L}^\bullet$ be a linking system associated to $\mathcal{F}^{c\bullet}$. Let $\mathcal{L}$ be the category whose objects are the $\mathcal{F}$–centric subgroups of S, and where*

$$\mathrm{Mor}_{\mathcal{L}}(P,Q) = \bigl\{\varphi \in \mathrm{Mor}_{\mathcal{L}^\bullet}(P^\bullet, Q^\bullet) \bigm| \pi^\bullet(\varphi)(P) \le Q\bigr\}.$$

Let $\delta_P \colon P \longrightarrow \mathrm{Aut}_{\mathcal{L}}(P)$ be the restriction of $\Delta_{P^\bullet}$. In other words, $\mathcal{L}$ is the pullback category in the following square:

$$\begin{tikzcd}
\mathcal{L} \arrow[r] \arrow[d, "\pi"] & \mathcal{L}^\bullet \arrow[d, "\pi^\bullet"] \\
\mathcal{F}^c \arrow[r, "(-)^\bullet"] & \mathcal{F}^{c\bullet}
\end{tikzcd}$$

Then $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$.

Proof **(a)** For each $\mathcal{F}$–centric subgroup $P \le S$, fix a morphism $\iota_P \in \mathrm{Mor}_{\mathcal{L}}(P,S)$ such that $\pi(\iota_P)$ is the inclusion (and such that $\iota_S = \mathrm{Id}_S$). For any pair of $\mathcal{F}$–centric subgroups $P \le Q \le S$, the same group $Z(P)$ acts freely and transitively on the sets of morphisms in $\mathcal{L}$ covering the inclusions $P \subseteq Q$ and $P \subseteq S$, and hence there is a unique morphism $\iota_P^Q \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ such that $\iota_Q \circ \iota_P^Q = \iota_P$.

Now let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ be any morphism in $\mathcal{F}^c$. By Proposition 3.3, φ has a unique extension to $\varphi^\bullet \in \mathrm{Hom}_{\mathcal{F}}(P^\bullet, Q^\bullet)$. Also, by Lemma 3.2(d), $Z(P^\bullet) = Z(P)$. Hence by condition (A) in the definition of a linking system, restriction sends the morphisms in $\pi^{-1}(\varphi^\bullet)$ bijectively to the morphisms in $\pi^{-1}(\varphi)$. Thus for any $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ such that $\pi(\psi) = \varphi$, there is a unique "extension" $\psi^\bullet \in \mathrm{Mor}_{\mathcal{L}}(P^\bullet, Q^\bullet)$ of ψ; ie a unique morphism such that $\psi^\bullet \circ \iota_P^{P^\bullet} = \iota_Q^{Q^\bullet} \circ \psi$.

Thus, if we define $\theta \colon \mathcal{L} \longrightarrow \mathcal{L}^\bullet$ by setting $\theta(P) = P^\bullet$ and $\theta(\psi) = \psi^\bullet$, then θ is well defined. This also shows that $\mathrm{Mor}_{\mathcal{L}}(P,Q) = \mathrm{Mor}_{\mathcal{L}}(P^\bullet, Q)$ when $Q = Q^\bullet$, and thus that θ is a left adjoint functor to the inclusion. Since the inclusion has a left adjoint, it follows that it induces a homotopy equivalence $|\mathcal{L}^\bullet| \simeq |\mathcal{L}|$.

(b) Since $Z(P) = Z(P^\bullet)$ for all $\mathcal{F}$–centric $P \le S$ (Lemma 3.2(d) again), axiom (A) for $\mathcal{L}$ follows from the same axiom applied to $\mathcal{L}^\bullet$. Axioms (B) and (C) for $\mathcal{L}$ follow immediately from axioms (B) and (C) for $\mathcal{L}^\bullet$ by restriction. □

We finish the section with a description of the relation between linking systems associated to a full subcategory $\mathcal{F}_0 \subseteq \mathcal{F}^c$ of a given fusion system $\mathcal{F}$, and rigidifications of the homotopy functor $\mathbf{B}\colon \mathcal{O}(\mathcal{F}_0) \longrightarrow \mathrm{hoTop}$ defined by setting $\mathbf{B}(P) = BP$. Here, $\mathcal{O}(\mathcal{F}_0)$ stands for the orbit category of $\mathcal{F}_0$; that is, the quotient category of $\mathcal{F}_0$ with same objects and morphisms divided out by inner automorphisms of target groups (see Section 5). Each linking system $\mathcal{L}_0$ induces a rigidification of $\mathbf{B}$, which in turn defines a decomposition of $|\mathcal{L}_0|$ as a homotopy colimit. More precisely, by a "rigidification of the homotopy functor $\mathbf{B}$" in the following proposition is meant a functor $\widetilde{B}\colon \mathcal{O}(\mathcal{F}_0) \longrightarrow \mathrm{Top}$ *together with* a natural homotopy equivalence of functors (in hoTop) from $\mathbf{B}$ to $\mathrm{ho}\circ\widetilde{B}$; ie a natural transformation of functors to hoTop which defines a homotopy equivalence $BP \longrightarrow \widetilde{B}(P)$ for each P. A *natural homotopy equivalence of rigidifications* from $\widetilde{B}$ to $\widetilde{B}'$ is a natural transformation

$$\widetilde{B} \xrightarrow{\ \kappa\ } \widetilde{B}'$$

of functors to Top such that $\mathrm{ho}(\kappa)$ commutes with the functors from $\mathbf{B}$. Two rigidifications $\widetilde{B}_1$ and $\widetilde{B}_2$ are *equivalent* if there is a third rigidification $\widetilde{B}_0$ and natural homotopy equivalences $\widetilde{B}_1 \longrightarrow \widetilde{B}_0 \longleftarrow \widetilde{B}_2$; this is seen to be an equivalence relation by taking pushouts.

By a linking system $\mathcal{L}_0$ in the following proposition is always meant the category $\mathcal{L}_0$ together with the projection to the associated fusion system and the distinguished monomorphisms. Hence an isomorphism of linking systems means an isomorphism of the categories which is natural with respect to these other structures.

Proposition 4.6 *Fix a saturated fusion system $\mathcal{F}$ over a discrete p–toral group S, and let $\mathcal{F}_0 \subseteq \mathcal{F}^c$ be any full subcategory. Then there are mutually inverse bijections:*

$$\left\{\begin{array}{c}\text{linking systems}\\ \text{associated to } \mathcal{F}_0\\ \text{up to isomorphism}\end{array}\right\} \underset{\text{LS}}{\overset{\text{KE}}{\rightleftarrows}} \left\{\begin{array}{c}\text{rigidifications } \mathcal{O}(\mathcal{F}_0) \longrightarrow \mathrm{Top}\\ \text{of the homotopy functor } \mathbf{B}\\ \text{up to natural homotopy equivalence}\end{array}\right\}$$

More precisely, the following hold for any linking system $\mathcal{L}_0$ associated to $\mathcal{F}_0$ and any rigidification $\widetilde{B}$ of the homotopy functor $\mathbf{B}$ on $\mathcal{O}(\mathcal{F}_0)$.

(a) *The left homotopy Kan extension $\mathbf{ke}(\mathcal{L}_0)$ of the constant functor $\mathcal{L}_0 \xrightarrow{*} \mathrm{Top}$ along the projection $\tilde{\pi}_0\colon \mathcal{L}_0 \longrightarrow \mathcal{O}(\mathcal{F}_0)$ is a rigidification of $\mathbf{B}$, and there is a homotopy equivalence*

$$|\mathcal{L}_0| \simeq \underset{\mathcal{O}(\mathcal{F}_0)}{\underrightarrow{\mathrm{hocolim}}}(\mathbf{ke}(\mathcal{L}_0))\,. \tag{6}$$

(b) *There is a linking system $\mathbf{ls}(\widetilde{B})$ associated to $\mathcal{F}_0$, and a natural homotopy equivalence of functors*

$$\mathbf{ke}(\mathbf{ls}(\widetilde{B})) \xrightarrow{\ \simeq\ } \widetilde{B}.$$

Furthermore, if $\widetilde{B}'$ is another rigidification of $\mathbf{B}$, any natural homotopy equivalence of rigidifications $\kappa\colon \widetilde{B} \to \widetilde{B}'$ induces an isomorphism $\kappa_\sharp\colon \mathbf{ls}(\widetilde{B}) \to \mathbf{ls}(\widetilde{B}')$ of linking systems.

(c) *There is an isomorphism $\mathcal{L}_0 \cong \mathbf{ls}(\mathbf{ke}(\mathcal{L}_0))$ of linking systems associated to $\mathcal{F}_0$.*

We define $\mathrm{KE}([\mathcal{L}_0]) = [\mathbf{ke}(\mathcal{L}_0)]$ *for each* $\mathcal{L}_0$*, and* $\mathrm{LS}([\widetilde{B}]) = [\mathbf{ls}(\widetilde{B})]$ *for each* $\widetilde{B}$*.*

Proof The left homotopy Kan extension is natural with respect to isomorphisms $\mathcal{L}_0 \longrightarrow \mathcal{L}_0'$ of linking systems. Thus **ke** sends isomorphic systems to natural homotopy equivalent functors $\mathcal{O}(\mathcal{F}_0) \longrightarrow \mathrm{Top}$, these are rigidifications of $\mathbf{B}$ by (a), and hence KE is well defined. Point (b) implies that LS is well defined, and it also implies that $\mathrm{LS} \circ \mathrm{KE}$ is the identity. Finally, (c) implies that $\mathrm{KE} \circ \mathrm{LS}$ is the identity. Hence the Proposition follows once we prove (a), (b), and (c).

(a) Fix $\mathcal{L}_0$, and set $\widetilde{B} = \mathbf{ke}(\mathcal{L}_0)$ for short. Recall that we write $\mathrm{Rep}_{\mathcal{F}}(P,Q) = \mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P,Q)$. By definition, for each P in $\mathcal{F}_0$, $\widetilde{B}(P)$ is the nerve (homotopy colimit of the point functor) of the overcategory $\widetilde{\pi}_0{\downarrow}P$, whose objects are pairs (Q,α) for Q in $\mathcal{L}_0$ and $\alpha \in \mathrm{Rep}_{\mathcal{F}}(Q,P)$, and where

$$\mathrm{Mor}_{\widetilde{\pi}_0\downarrow P}\big((Q,\alpha),(R,\beta)\big) = \big\{\varphi \in \mathrm{Mor}_{\mathcal{L}}(Q,R) \,\big|\, \alpha = \beta \circ \widetilde{\pi}_0(\varphi)\big\}. \tag{7}$$

Since $|\mathcal{L}_0| \cong \operatorname{hocolim}_{\overrightarrow{\mathcal{L}_0}}(*)$, (6) holds by [17, Theorem 5.5].

It remains to show that $\widetilde{B}$ is a rigidification of the homotopy functor $\mathbf{B}$. Fix a section $\widetilde{\sigma}\colon \mathrm{Mor}(\mathcal{O}(\mathcal{F}_0)) \longrightarrow \mathrm{Mor}(\mathcal{L}_0)$ of $\widetilde{\pi}_0$ which sends identity morphisms to identity morphisms. For each P, let $\mathcal{B}(P)$ be the category with one object o_P and morphism group P (so $|\mathcal{B}(P)| \cong BP$), and define functors

$$\mathcal{B}(P) \xrightarrow{\ \theta_P\ } \widetilde{\pi}_0{\downarrow}P \xrightarrow{\ \Psi_P\ } \mathcal{B}(P)$$

as follows. Let $\theta_P(o_P) = (P,\mathrm{Id})$, and $\theta_P(g) = \delta_P(g)$ (as a morphism in $\widetilde{\pi}_0{\downarrow}P$ using (7)) for all $g \in P$. Set $\Psi_P(Q,\alpha) = o_P$; and let Ψ_P send each morphism $\varphi \in \mathrm{Mor}_{\widetilde{\pi}_0\downarrow P}((Q,\alpha),(R,\beta))$ to the unique element $g \in P$ (unique by Lemma 4.3(b)) such that the following square commutes:

$$\begin{tikzcd} Q \arrow[r, "\varphi"] \arrow[d, "\widetilde{\sigma}(\alpha)"'] & R \arrow[d, "\widetilde{\sigma}(\beta)"] \\ P \arrow[r, "\delta_P(g)"] & P \end{tikzcd}$$

Clearly, $\Psi_P \circ \theta_P = \mathrm{Id}_{\mathcal{B}(P)}$. As for the other composite, define $f\colon \mathrm{Id} \longrightarrow \theta_P \circ \Psi_P$ by sending each object (Q,α) to the morphism $\widetilde{\sigma}(\alpha) \in \mathrm{Mor}_{\mathcal{L}}(Q,P)$. This is clearly a natural transformation of functors, and thus

$$\widetilde{B}(P) = |\widetilde{\pi}_0{\downarrow}P| \simeq |\mathcal{B}(P)| \simeq BP.$$

To finish the proof that $\widetilde{B}$ is a rigidification of the homotopy functor $\mathbf{B}$, we must show, for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, that the following square commutes up to natural transformation:

$$\begin{tikzcd} \mathcal{B}(P) \arrow[r, "\theta_P"] \arrow[d, "B\varphi"] & \widetilde{\pi}_0{\downarrow}P \arrow[d, "\bar{\varphi}\circ-"] \\ \mathcal{B}(Q) \arrow[r, "\theta_Q"] & \widetilde{\pi}_0{\downarrow}Q \end{tikzcd}$$

Here, $[\varphi] \in \mathrm{Rep}_{\mathcal{F}}(P,Q)$ denotes the class of φ. This means constructing a natural transformation $F_1 \xrightarrow{\Phi} F_2$ of functors $\mathcal{B}(P) \longrightarrow \widetilde{\pi}_0{\downarrow}Q$, where $F_1 = ([\varphi]\circ-)\circ\theta_P$ and $F_2 = \theta_Q \circ B\varphi$ are given by the formulas:

$$F_1(o_P) = (P,[\varphi]) \qquad F_2(o_P) = (Q,\mathrm{Id})$$
$$F_1(g) = \delta_P(g) \qquad F_2(g) = \delta_Q(\varphi(g))$$

Let $\widetilde{\varphi} \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ be any lifting of φ. Then by condition (C), Φ can be defined by sending the object o_P to the morphism $\widetilde{\varphi} \in \mathrm{Mor}_{\widetilde{\pi}_0\downarrow P}\big((P,[\varphi]),(Q,\mathrm{Id})\big)$.

(b) We first fix some notation. For any space X and any $x,x' \in X$, $\pi_1(X;x,x')$ denotes the set of homotopy classes of paths in X (relative endpoints) from x to x'. For any $u \in \pi_1(X;x,x')$, u_* denotes the induced isomorphism from $\pi_1(X,x)$ to $\pi_1(X,x')$. Also, for any map of spaces $f\colon X \to Y$, f_* denotes the induced map from $\pi_1(X;x,x')$ to $\pi_1(Y;f(x),f(x'))$.

Now fix a rigidification $\widetilde{B}\colon \mathcal{O}(\mathcal{F}_0) \longrightarrow \mathrm{Top}$; we want to define a linking system $\mathcal{L}_0 = \mathbf{ls}(\widetilde{B})$ associated to $\mathcal{F}$. Since $\widetilde{B}$ is a rigidification of the homotopy functor $\mathbf{B}$, we are given homotopy equivalences $BP \xrightarrow{\epsilon_P} \widetilde{B}(P)$ such that the following square commutes up to homotopy for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$:

$$\begin{tikzcd} BP \arrow[r, "\epsilon_P"] \arrow[d, "B\varphi"'] & \widetilde{B}(P) \arrow[d, "\widetilde{B}([\varphi])"] \\ BQ \arrow[r, "\epsilon_Q"] & \widetilde{B}(Q) \end{tikzcd}$$

Here, $[\varphi] \in \mathrm{Rep}_{\mathcal{F}}(P,Q)$ denotes the class of φ (mod $\mathrm{Inn}(Q)$). For each P in $\mathcal{F}_0$, let $*_P \in \widetilde{B}(P)$ be the image under ϵ_P of the base point of BP, and let

$$\gamma_P\colon P \xrightarrow{\cong} \pi_1(\widetilde{B}(P),*_P)$$

be the isomorphism induced by ϵ_P on fundamental groups.

Let $\mathcal{L}_0 = \mathbf{ls}(\widetilde{B})$ be the category with $\mathrm{Ob}(\mathcal{L}_0) = \mathrm{Ob}(\mathcal{F}_0)$ and with

$$\mathrm{Mor}_{\mathcal{L}_0}(P,Q) = \bigl\{(\varphi,u) \bigm| \varphi \in \mathrm{Rep}_{\mathcal{F}}(P,Q),\ u \in \pi_1(\widetilde{B}(Q); \widetilde{B}\varphi(*_P), *_Q)\bigr\}.$$

Composition is defined by setting

$$(\psi,v)\circ(\varphi,u) = (\psi\varphi, v\cdot \widetilde{B}\psi_*(u)),$$

where paths are composed from right to left. Let $\pi_0\colon \mathcal{L}_0 \longrightarrow \mathcal{F}_0$ be the functor which is the identity on objects, and which sends $(\varphi,u)\in \mathrm{Mor}_{\mathcal{L}_0}(P,Q)$ to the composite

$$P \xrightarrow[\cong]{\gamma_P} \pi_1(\widetilde{B}(P), *_P) \xrightarrow{\widetilde{B}\varphi_*} \pi_1(\widetilde{B}(Q), \widetilde{B}\varphi(*_P)) \xrightarrow{u_*} \pi_1(\widetilde{B}(Q), *_Q) \xrightarrow[\cong]{\gamma_Q^{-1}} Q.$$

Also, for each P, define

$$\delta_P\colon P \longrightarrow \mathrm{Aut}_{\mathcal{L}_0}(P) \qquad \text{by setting} \qquad \delta_P(g) = (\mathrm{Id}_P, \gamma_P(g)).$$

Axioms (A), (B), and (C) for a centric linking system are easily seen to hold for $\mathcal{L}_0$. For example, (C) follows as an immediate consequence of the definition of π_0.

Now set $B_1 = \mathbf{ke}(\mathcal{L}_0) = \mathbf{ke}(\mathbf{ls}(\widetilde{B}))$, the left homotopy Kan extension along the projection $\widetilde{\pi}_0\colon \mathcal{L}_0 \longrightarrow \mathcal{O}(\mathcal{F}_0)$ of the constant point functor on $\mathcal{L}_0$. Thus for each P, we have $B_1(P) = |\mathcal{B}_1(P)|$, where $\mathcal{B}_1(P)$ is the category with objects the pairs (Q,α) for $\alpha \in \mathrm{Rep}_{\mathcal{F}}(Q,P)$, and with morphism sets

$$\begin{aligned}\mathrm{Mor}_{\mathcal{B}_1(P)}\bigl((Q,\alpha),(R,\beta)\bigr) &= \bigl\{\widehat{\varphi} \in \mathrm{Mor}_{\mathcal{L}_0}(Q,R) \bigm| \alpha = \beta\circ\widetilde{\pi}_0(\widehat{\varphi})\bigr\}\\ &= \bigl\{(\varphi,u) \bigm| \varphi\in \mathrm{Rep}_{\mathcal{F}}(Q,R),\ \alpha=\beta\circ\varphi,\ u\in\pi_1(\widetilde{B}(R); \widetilde{B}\varphi(*_Q), *_R)\bigr\}.\end{aligned}$$

We define a natural homotopy equivalence of functors $\Psi\colon B_1 \longrightarrow \widetilde{B}$ as follows. For all P, maps $\Psi_P\colon B_1(P) \longrightarrow \widetilde{B}(P)$ are defined inductively, one skeleton at a time, (and simultaneously for all P) as follows.

- Each vertex (Q,α) in $B_1(P) = |\mathcal{B}_1(P)|$ is sent to $\widetilde{B}(\alpha)(*_Q) \in \widetilde{B}(P)$.
- For each edge $\sigma = \bigl((Q,\varphi) \xrightarrow{(\varphi,u)} (P,\mathrm{Id})\bigr)$ in $B_1(P)$, where

  $$\varphi\in \mathrm{Rep}_{\mathcal{F}}(Q,P) \quad \text{and} \quad u \in \pi_1(\widetilde{B}(P); \widetilde{B}\varphi(*_Q), *_P),$$

  $\Phi_P|_\sigma = \widehat{u}$ for some path $\widehat{u}$ in the homotopy class of u.
- For each edge $\sigma = \bigl((Q,\varphi) \xrightarrow{(\varphi,u)} (R,\beta)\bigr)$ in $B_1(P)$, where $\beta \neq \mathrm{Id}_P$, write

  $$\sigma' = \bigl((Q,\varphi) \xrightarrow{(\varphi,u)} (R,\mathrm{Id})\bigr)$$

  (an edge in $B_1(R)$), and set $\Phi_P|_\sigma = \widetilde{B}(\beta)\circ(\Psi_R|_{\sigma'})$.

- Consider a simplex of dimension $m \geq 2$ in $B_1(P)$ of the form
$$\sigma = \Big((Q_0,\alpha_0) \longrightarrow (Q_1,\alpha_1) \longrightarrow \cdots \longrightarrow (Q_m,\alpha_m)\Big).$$
If $(Q_m,\alpha_m) = (P,\mathrm{Id})$, then let $\Psi_P|_\sigma$ be any singular simplex in $\widetilde{B}(P)$ whose boundary is as already defined. Otherwise, let σ' be the unique simplex in $B_1(Q_m)$ representing a chain ending in (Q_m,Id) such that $\sigma = B_1(\alpha_m)(\sigma')$, and set $\Psi_P|_\sigma = \widetilde{B}(\alpha_m)\circ(\Psi_{Q_m}|_{\sigma'})$.

Since $B_1(P) \simeq \widetilde{B}(P) \simeq BP$ (where P is given the discrete topology), the above construction is always possible, and defines a homotopy equivalence. It induces the identity on fundamental groups, under their given identifications with P. By construction, the Φ_P form a natural morphism of functors Ψ from B_1 to $\widetilde{B}$.

Let $\big(\widetilde{B}', \{\epsilon'_P\}\big)$ be another rigidification of $\mathbf{B}$, and let $\kappa\colon \widetilde{B} \longrightarrow \widetilde{B}'$ be a natural homotopy equivalence of rigidifications. We have already chosen our basepoint $*_P = \epsilon_P(*)$, where $* \in BP$ is a fixed basepoint, and we now set $*'_P = \epsilon'_P(*)$. Fix, for each P, a homotopy H_P between $\kappa_P\circ\epsilon_P$ and ϵ'_P. The restriction of H_P to the base point of BP provides a canonical path in $\widetilde{B}'(P)$ from $\kappa_P(*_P)$ to $*'_P$, whose homotopy class we denote $w_P \in \pi_1(\widetilde{P};\kappa_P(*_P),*'_P)$. We now define
$$\kappa_\sharp\colon \mathcal{L}_0 \longrightarrow \mathcal{L}'_0$$
to be the identity on objects, and for $(\varphi,u) \in \mathrm{Mor}_{\mathcal{L}_0}(P,Q)$,
$$\kappa_\sharp(\varphi,u) = (\varphi, w_P\cdot\kappa_{Q*}(u)\cdot\widetilde{B}'\varphi_*(w_P)^{-1}).$$
It is straightforward to show that κ is a well defined isomorphism of linking systems; ie an isomorphism of categories which is natural with respect to the projections to $\mathcal{F}_0$ and the distinguished monomorphisms.

(c) Now assume $\mathcal{L}_0$ is given; it remains to construct an isomorphism $\mathcal{L}_0 \cong \mathbf{ls}(\mathbf{ke}(\mathcal{L}_0))$ of linking systems associated to $\mathcal{F}_0$. Set $\widetilde{B} = \mathbf{ke}(\mathcal{L}_0)$ and $\mathcal{L}_1 = \mathbf{ls}(\widetilde{B})$ for short. By definition, $\mathcal{L}_0$ and $\mathcal{L}_1$ have the same objects, and a morphism in $\mathcal{L}_1$ from P to Q is a pair (φ,u), where $\varphi \in \mathrm{Rep}_{\mathcal{F}}(P,Q)$ and $u \in \pi_1(\widetilde{B}(Q);\widetilde{B}\varphi(*_P),*_Q)$. Also, $\widetilde{B}(P) = |\widetilde{\pi}_0{\downarrow}P|$ where $\widetilde{\pi}_0$ is the projection of $\mathcal{L}_0$ onto $\mathcal{O}(\mathcal{F}_0)$; in particular, we choose $*_P$ to be the vertex of (P,Id). Define $\Psi\colon \mathcal{L}_0 \longrightarrow \mathcal{L}_1$ by sending each object to itself, and by sending $\alpha \in \mathrm{Mor}_{\mathcal{L}_0}(P,Q)$ to $(\widetilde{\pi}_0(\alpha),[\alpha])$, where $[\alpha]$ is the homotopy class of α, regarded as an edge in $|\widetilde{\pi}_0{\downarrow}Q|$ from $(P,\widetilde{\pi}_0(\alpha)) = \widetilde{\pi}_0\alpha(*_P)$ to $(Q,\mathrm{Id}) = *_Q$. This is easily checked to be an isomorphism of categories, and to commute with the distinguished monomorphisms and the projections to $\mathcal{F}_0$. □

5 Higher limits over orbit categories

If $\mathcal{F}$ is any fusion system over a discrete p–toral group S, then $\mathcal{O}(\mathcal{F})$ will denote its orbit category: the category whose objects are the subgroups of S, and where

$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P,Q)=\mathrm{Rep}_{\mathcal{F}}(P,Q)\stackrel{\text{def}}{=}\mathrm{Inn}(Q)\backslash\mathrm{Hom}_{\mathcal{F}}(P,Q).$$

Also, we write $\mathcal{O}^c(\mathcal{F})=\mathcal{O}(\mathcal{F}^c)$ to denote the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are the $\mathcal{F}$–centric subgroups of S; and more generally write $\mathcal{O}(\mathcal{F}_0)$ to denote the full subcategory of $\mathcal{O}(\mathcal{F})$ corresponding to any full subcategory $\mathcal{F}_0$ of $\mathcal{F}$.

By Lemma 2.5, the morphism sets in the orbit category are all finite. There is a canonical projection functor $\mathcal{F}\to\mathcal{O}(\mathcal{F})$ which is the identity on objects and the natural projection $\mathrm{Hom}_{\mathcal{F}}(P,Q)\to\mathrm{Rep}_{\mathcal{F}}(P,Q)$ on morphisms.

Throughout this section, when $\mathcal{C}$ is a category, we frequently write $\mathcal{C}$–mod to denote the category of functors $\mathcal{C}^{\mathrm{op}}\longrightarrow\mathrm{Ab}$. This notation will not be used in the statements of results here, but it is used in several of the proofs.

Lemma 5.1 *Let $\mathcal{F}$ be a saturated fusion system over a discrete p–toral group S, and let $\mathcal{F}_0\subseteq\mathcal{F}$ be any full subcategory such that $P\in\mathrm{Ob}(\mathcal{F}_0)$ implies $P^\bullet\in\mathrm{Ob}(\mathcal{F}_0)$. Set $\mathcal{F}_0^\bullet=\mathcal{F}_0\cap\mathcal{F}^\bullet$. Then there are well defined functors*

$$\mathcal{O}^c(\mathcal{F})\ \underset{\text{incl}}{\overset{(-)^\bullet}{\rightleftarrows}}\ \mathcal{O}(\mathcal{F}^{c\bullet}),$$

where $(-)^\bullet$ sends P to $P^\bullet$ and $[\varphi]$ to $[\varphi^\bullet]$. Also, $(-)^\bullet$ is a left adjoint to the inclusion.

Proof This follows from Corollary 3.4. The only thing to check is that $(-)^\bullet$ is well defined on morphisms in the orbit category. If $\varphi_1,\varphi_2\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$ represent the same morphism in the orbit category, then $\varphi_1=c_g\circ\varphi_2$ for some $g\in Q$, so $\varphi_1^\bullet=c_g\circ\varphi_2^\bullet$ by functoriality, and hence $[\varphi_1^\bullet]=[\varphi_2^\bullet]$ in $\mathrm{Rep}_{\mathcal{F}}(P^\bullet,Q^\bullet)$. $\square$

The following proposition shows that the problem of describing higher limits over the orbit categories we are considering can always be reduced to one over a finite subcategory.

Proposition 5.2 *Let $\mathcal{F}$ be a saturated fusion system over a discrete p–toral group S. Let $\mathcal{F}_0\subseteq\mathcal{F}$ be any full subcategory such that $P\in\mathrm{Ob}(\mathcal{F}_0)$ implies $P^\bullet\in\mathrm{Ob}(\mathcal{F}_0)$, and set $\mathcal{F}_0^\bullet=\mathcal{F}_0\cap\mathcal{F}^\bullet$. Then for any $F\colon\mathcal{O}(\mathcal{F}_0)^{\mathrm{op}}\longrightarrow\mathbb{Z}_{(p)}$–$\mathrm{mod}$, restriction to $\mathcal{F}_0^\bullet$ induces an isomorphism*

$$\varprojlim_{\mathcal{O}(\mathcal{F}_0)}{}^{*}(F)\cong\varprojlim_{\mathcal{O}(\mathcal{F}_0^\bullet)}{}^{*}(F|_{\mathcal{O}(\mathcal{F}_0^\bullet)}).$$

Proof Consider the functors

$$\mathcal{O}(\mathcal{F}_0)\text{–}\mathrm{mod} \underset{T}{\overset{R}{\rightleftarrows}} \mathcal{O}(\mathcal{F}_0^\bullet)\text{–}\mathrm{mod}\,,$$

where R is given by restriction and T by composition with the functor $(-)^\bullet$. Then T is a left adjoint to R, since $(-)^\bullet$ is a left adjoint to the inclusion by Lemma 5.1. Also, T and R are both exact functors, and R sends injectives to injectives since it is right adjoint to an exact functor.

Let $\underline{\mathbb{Z}}$ be the constant functor on $\mathcal{O}(\mathcal{F}_0^\bullet)$ which sends all objects to $\mathbb{Z}$. Then $T(\underline{\mathbb{Z}})$ is the constant functor on $\mathcal{O}(\mathcal{F}_0)$, and hence for any functor F on $\mathcal{O}(\mathcal{F}_0)$,

$$\begin{aligned}\varprojlim_{\mathcal{O}(\mathcal{F}_0)}(F) &= \mathrm{Hom}_{\mathcal{O}(\mathcal{F}_0)\text{–}\mathrm{mod}}(T(\underline{\mathbb{Z}}), F)\\ &\cong \mathrm{Hom}_{\mathcal{O}(\mathcal{F}_0^\bullet)\text{–}\mathrm{mod}}(\underline{\mathbb{Z}}, R(F)) = \varprojlim_{\mathcal{O}(\mathcal{F}_0^\bullet)}(R(F))\,.\end{aligned}$$

Since R is exact and sends injectives to injectives, it sends injective resolutions to injective resolutions, and thus induces an isomorphism between higher limits over $\mathcal{O}(\mathcal{F}_0)$ and over $\mathcal{O}(\mathcal{F}_0^\bullet)$. □

We next want to show that the techniques which we have already developed for handling higher limits over orbit categories in the finite case [7, Section 3] also apply in this new situation. The proof of this is similar to the proof in [7] of the analogous result for fusion systems over finite p–groups, and is in fact a special case of a very general result which we prove here.

For any group Γ (not necessarily finite), and any set $\mathcal{H}$ of subgroups of Γ, we define $\mathcal{O}_{\mathcal{H}}(\Gamma)$ to be the corresponding orbit category of Γ: the category with $\mathrm{Ob}(\mathcal{O}_{\mathcal{H}}(\Gamma)) = \mathcal{H}$, and with morphism sets

$$\mathrm{Mor}_{\mathcal{O}_{\mathcal{H}}(\Gamma)}(H, H') = H'\backslash N_\Gamma(H, H') \cong \mathrm{Map}_\Gamma(\Gamma/H, \Gamma/H').$$

Here, $N_\Gamma(H, H')$ is the transporter set

$$N_\Gamma(H, H') = \{g \in \Gamma \mid gHg^{-1} \le H'\}.$$

If $1 \in \mathcal{H}$, then for any $\mathbb{Z}[\Gamma]$–module M, we define

$$\Lambda^*_{\mathcal{H}}(\Gamma; M) = \varprojlim_{\mathcal{O}_{\mathcal{H}}(\Gamma)}{}^*(F_M)\,,$$

where $F_M\colon \mathcal{O}_{\mathcal{H}}(\Gamma)^{\mathrm{op}} \longrightarrow \mathrm{Ab}$ is the functor defined by setting $F_M(H) = 0$ if $H \ne 1$, and $F_M(1) = M$.

It is important to distinguish between the orbit category of a group and the orbit category of a fusion system. When G is a finite group and $S \in \mathrm{Syl}_p(G)$, the orbit category of the fusion system $\mathcal{F}_S(G)$ is not the same as the orbit category $\mathcal{O}_S(G)$ (the orbit category of G with objects the subgroups of S).

Proposition 5.3 *Fix a category $\mathcal{C}$, a group Γ, a set $\mathcal{H}$ of subgroups of Γ such that $1 \in \mathcal{H}$, and a functor*

$$\alpha\colon \mathcal{O}_{\mathcal{H}}(\Gamma) \longrightarrow \mathcal{C}.$$

Set $c_0 = \alpha(1)$. For each object d in $\mathcal{C}$, we regard the set $\mathrm{Mor}_{\mathcal{C}}(c_0, d)$ as a Γ–set via α and composition. Assume that the following conditions hold:

- (a) *α sends $\Gamma = \mathrm{Aut}_{\mathcal{O}_{\mathcal{H}}(\Gamma)}(1)$ bijectively to $\mathrm{End}_{\mathcal{C}}(c_0)$.*
- (b) *For each $d \in \mathrm{Ob}(\mathcal{C})$ such that $d \not\cong c_0$, all isotropy subgroups of the Γ–action on $\mathrm{Mor}_{\mathcal{C}}(c_0, d)$ are nontrivial and conjugate to subgroups in $\mathcal{H}$.*
- (c) *For each $\xi \in \mathrm{Mor}(\mathcal{O}_{\mathcal{H}}(\Gamma))$, $\alpha(\xi)$ is an epimorphism in the categorical sense: $\varphi \circ \alpha(\xi) = \psi \circ \alpha(\xi)$ implies $\varphi = \psi$.*
- (d) *For any $H \in \mathcal{H}$, any $d \in \mathrm{Ob}(\mathcal{C})$, and any $\varphi \in \mathrm{Mor}_{\mathcal{C}}(c_0, d)$ which is H–invariant, there is some $\overline{\varphi} \in \mathrm{Mor}_{\mathcal{C}}(\alpha(H), d)$ such that $\varphi = \overline{\varphi} \circ \alpha(\mathrm{incl}_1^H)$.*

Let

$$\Phi\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Ab}$$

be any functor which vanishes except on the isomorphism class of c_0. Then the natural map

$$\varprojlim_{\mathcal{C}}{}^*(\Phi) \xrightarrow[\cong]{\alpha^*} \varprojlim_{\mathcal{O}_{\mathcal{H}}(\Gamma)}{}^*(\Phi \circ \alpha) = \Lambda^*_{\mathcal{H}}(\Gamma; \Phi(c_0))$$

is an isomorphism.

Proof Consider the functors

$$\mathcal{O}_{\mathcal{H}}(\Gamma)\text{–}\mathsf{mod} \underset{R_\alpha}{\overset{\alpha^*}{\rightleftarrows}} \mathcal{C}\text{–}\mathsf{mod},$$

where α^* is composition with α^{op}, and R_α is the right Kan extension of α^{op}. Specifically, for $d \in \mathrm{Ob}(\mathcal{C})$, let $\alpha{\downarrow}d$ be the overcategory whose objects are pairs (H, φ) for $\varphi \in \mathrm{Mor}_{\mathcal{C}}(\alpha(H), d)$, and where a morphism from (H, φ) to (K, ψ) is a morphism χ in $\mathrm{Mor}_{\mathcal{O}_{\mathcal{H}}(\Gamma)}(H, K)$ such that $\psi \circ \alpha(\chi) = \varphi$. Let $\kappa_d\colon \alpha{\downarrow}d \longrightarrow \mathcal{O}_{\mathcal{H}}(\Gamma)$ be the forgetful functor. Then $(\alpha{\downarrow}d)^{\mathrm{op}} = d{\downarrow}\alpha^{\mathrm{op}}$ (the undercategory), and for $F\colon \mathcal{O}_{\mathcal{H}}(\Gamma)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$, $R_\alpha(F)$ is defined by setting

$$R_\alpha(F)(d) = \varprojlim_{(\alpha\downarrow d)^{\mathrm{op}}} (F \circ {\kappa_d}^{\mathrm{op}}).$$

On morphisms, $R_\alpha(F)$ sends $f\in\mathrm{Mor}_{\mathcal{C}}(d,d')$ to the morphism induced by the functor $\bigl(\alpha{\downarrow}d \xrightarrow{f\circ-} \alpha{\downarrow}d'\bigr)$. By Mac Lane [24, Section X.3, Theorem 1], R_α is right adjoint to α^*. In particular, since α^* preserves exact sequences, R_α sends injectives to injectives.

Fix $H\in\mathcal{H}$ and $d\in\mathrm{Ob}(\mathcal{C})$. Consider the map

$$\mu\colon \mathrm{Mor}_{\mathcal{C}}(\alpha(H),d) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(c_0,d)$$

defined by composition with the "inclusion" morphism $\alpha(\mathrm{incl}_1^H)$. This map is injective by (c), and $\mathrm{Im}(\mu)\supseteq \mathrm{Mor}_{\mathcal{C}}(c_0,d)^H$ by (d). Also, $\mathrm{Im}(\mu)$ is contained in $\mathrm{Mor}_{\mathcal{C}}(c_0,d)^H$ since $\mathrm{incl}_1^H\circ x=\mathrm{incl}_1^H$ for all $x\in H$. Thus μ induces a bijection

$$\mathrm{Mor}_{\mathcal{C}}(\alpha(H),d) \xrightarrow[\cong]{\mu_0} \mathrm{Mor}_{\mathcal{C}}(c_0,d)^H . \tag{8}$$

Fix representatives $\{\varphi_i^d\}_{i\in I_d}$ for the Γ–orbits in $\mathrm{Mor}_{\mathcal{C}}(c_0,d)$, and let $\Gamma_i^d\le\Gamma$ be the stabilizer subgroup of φ_i^d. By (b), we can choose the φ_i^d such that $\Gamma_i^d\in\mathcal{H}$ for all i. By (8), each φ_i^d has a unique "extension" to $\psi_i^d\in\mathrm{Mor}_{\mathcal{C}}(\alpha(\Gamma_i^d),d)$; ie there is a unique ψ_i^d such that $\varphi_i^d=\psi_i^d\circ\alpha(\mathrm{incl}_1^{\Gamma_i^d})$. Also, for any (H,χ) in $\alpha{\downarrow}d$, there is a unique $i\in I_d$ and a unique morphism $\chi_0\in\mathrm{Mor}_{\mathcal{O}_{\mathcal{H}}(\Gamma)}(H,\Gamma_i^d)$ such that $\chi=\psi_i^d\circ\chi_0$. So each object (Γ_i^d,ψ_i^d) is a final object in its connected component of the overcategory $\alpha{\downarrow}d$. Thus for any F in $\mathcal{O}_{\mathcal{H}}(\Gamma)$–$\mathrm{mod}$,

$$R_\alpha(F)(d)\cong\prod_{i\in I_d}F(\Gamma_i^d). \tag{9}$$

In particular, R_α is an exact functor.

Let $\underline{\mathbb{Z}}$ denote the constant functor on $\mathcal{C}^{\mathrm{op}}$ which sends each object to $\mathbb{Z}$ and each morphism to the identity. Then $\alpha^*\underline{\mathbb{Z}}$ is the constant functor on $\mathcal{O}_{\mathcal{H}}(\Gamma)^{\mathrm{op}}$. If $F\colon\mathcal{C}^{\mathrm{op}}\to\mathrm{Ab}$ is any functor, then

$$\varprojlim_{\mathcal{C}}(F)\cong\mathrm{Hom}_{\mathcal{C}\text{–}\mathrm{mod}}(\underline{\mathbb{Z}},F)$$

and similarly for functors in $\mathcal{O}_{\mathcal{H}}(\Gamma)$–$\mathrm{mod}$.

Assume $H\in\mathcal{H}$ is such that $\alpha(H)\cong\alpha(1)=c_0$. Since all endomorphisms of c_0 are automorphisms (by (a)), $\mathrm{Mor}_{\mathcal{C}}(c_0,\alpha(H))$ contains only isomorphisms, and in particular $\alpha(\mathrm{incl}_1^H)$ is an isomorphism. Also, $\mathrm{incl}_1^H\circ x=\mathrm{incl}_1^H$ for all $x\in H$, so $\alpha(x)=\mathrm{Id}_{c_0}$ for all $x\in H$. By (a) again, this implies that $H=1$.

The functor $\alpha^*\Phi=\Phi\circ\alpha^{\mathrm{op}}\colon\mathcal{O}_{\mathcal{H}}(\Gamma)^{\mathrm{op}}\longrightarrow\mathbb{Z}_{(p)}$–$\mathrm{mod}$ thus sends the object 1 to $\Phi(c_0)$ (with the given action of Γ), and sends all other objects to 0. Then R_α sends an injective resolution I_* of $\alpha^*\Phi$ to an injective resolution $R_\alpha(I_*)$ of $R_\alpha(\alpha^*\Phi)$. It follows that

$$\begin{aligned}\Lambda^*_{\mathcal{H}}(\Gamma;\Phi(c_0)) &\overset{\text{def}}{=} \varprojlim_{\mathcal{O}_{\mathcal{H}}(\Gamma)}{}^{*}(\alpha^*\Phi) \cong H^*\big(\mathrm{Mor}_{\mathcal{O}_{\mathcal{H}}(\Gamma)\text{–}\mathsf{mod}}(\alpha^*\underline{\mathbb{Z}}, I_*)\big)\\ &\cong H^*\big(\mathrm{Mor}_{\mathcal{C}\text{–}\mathsf{mod}}(\underline{\mathbb{Z}}, R_\alpha(I_*))\big) \cong \varprojlim_{\mathcal{C}}{}^{*}(R_\alpha(\alpha^*\Phi)).\end{aligned}$$

It remains only to show that $R_\alpha(\alpha^*\Phi)\cong\Phi$. For each $d\in\mathrm{Ob}(\mathcal{C})$, if $d\not\cong c_0$, then $\mathrm{Mor}_{\mathcal{C}}(c_0,d)$ is a disjoint union of orbits Γ/Γ_i^d, where $1\neq\Gamma_i^d\in\mathcal{H}$ by (b). So by (9),

$$R_\alpha(\alpha^*\Phi)(d) = R_\alpha(\Phi\circ\alpha)(d) \cong \prod_i \Phi(\alpha(H_i)) = 0\,,$$

where the last equality holds since we already showed that $H\neq 1$ implies $\alpha(H)\not\cong c_0$. If $d\cong c_0$, then $\mathrm{Mor}_{\mathcal{C}}(c_0,d)$ consists of one free orbit of Γ (by (a)), and hence $R_\alpha(\alpha^*\Phi)(d)\cong\Phi(\alpha(1))\cong\Phi(c_0)$. This finishes the proof that $R_\alpha(\alpha^*\Phi)\cong\Phi$. □

Our first application of Proposition 5.3 is to the case where $\mathcal{C}$ is the orbit category of a saturated fusion system over a discrete p–toral group. As in [19; 20] and [7], when Γ is finite and $\mathcal{H}$ is the set of p–subgroups of Γ (or the set of subgroups of a given Sylow p–subgroup), we write $\Lambda^*(\Gamma;M)=\Lambda^*_{\mathcal{H}}(\Gamma;M)$ (and the prime p is understood).

Proposition 5.4 *Let $\mathcal{F}$ be a saturated fusion system over S. Let*

$$\Phi\colon \mathcal{O}^c(\mathcal{F})^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{–}\mathsf{mod}$$

be any functor which vanishes except on the isomorphism class of some fixed $\mathcal{F}$–centric subgroup $Q\le S$. Then

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^{*}(\Phi) \cong \Lambda^*(\mathrm{Out}_{\mathcal{F}}(Q);\Phi(Q)).$$

Proof It suffices to do this when Q is fully normalized. Set $\Gamma=\mathrm{Out}_{\mathcal{F}}(Q)$ and $\Sigma=\mathrm{Out}_S(Q)\in\mathrm{Syl}_p(\Gamma)$, and let $\mathcal{H}$ be the set of subgroups of Σ. Since $\Sigma\cong N_S(Q)/Q$, each subgroup of Σ has the form $\mathrm{Out}_P(Q)$ for some unique $P\le N_S(Q)$ containing Q. Define

$$\alpha\colon \mathcal{O}_\Sigma(\Gamma) \longrightarrow \mathcal{O}^c(\mathcal{F})$$

on objects by setting $\alpha(\mathrm{Out}_P(Q))=P$ for $Q\le P\le N_S(Q)$. If $\varphi\in\mathrm{Aut}_{\mathcal{F}}(Q)$ is such that $[\varphi]\in N_\Gamma(\mathrm{Out}_P(Q),\mathrm{Out}_{P'}(Q))$ (the set of elements which conjugate $\mathrm{Out}_P(Q)$ into $\mathrm{Out}_{P'}(Q)$), then φ can be extended to some $\overline{\varphi}\in\mathrm{Hom}_{\mathcal{F}}(P,P')$ by axiom (II), the class of $\overline{\varphi}$ in the orbit category is uniquely determined by φ by Proposition 2.8, and α sends the class of $[\varphi]$ to the class of $\overline{\varphi}$.

We apply Proposition 5.3 to this functor α. Condition (a) is clear, (c) holds for $\mathcal{O}^c(\mathcal{F})$ by Proposition 2.8, and (d) holds by axiom (II) of a saturated fusion system. As for (b), since every morphism in $\mathcal{F}$ is the composite of an isomorphism followed by an inclusion, it suffices to prove that the stabilizer in Γ of an inclusion $\mathrm{incl}_Q^P \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$, where $Q \subsetneqq P$, is a nontrivial p–subgroup. But the stabilizer is $\mathrm{Out}_P(Q) \cong N_P(Q)/Q$, which is nontrivial by Lemma 1.8. All of the hypotheses of Proposition 5.3 thus hold, and the result follows. □

Using the terminology of [7], we say that a category $\mathcal{C}$ has *bounded limits at* p if there is $k > 0$ such that for any functor $\Phi\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{–mod}$, $\varprojlim^i(\Phi) = 0$ for all $i > k$. The following is a first corollary of Proposition 5.4.

Corollary 5.5 *Let $\mathcal{F}$ be a saturated fusion system over a discrete p–toral group S, and let $\mathcal{F}_0 \subseteq \mathcal{F}^c$ be a full subcategory such that $P \in \mathrm{Ob}(\mathcal{F}_0)$ implies $P^\bullet \in \mathrm{Ob}(\mathcal{F}_0)$. Then the orbit category $\mathcal{O}(\mathcal{F}_0)$ has bounded limits at p.*

Proof By Proposition 5.2, it suffices to prove this when $\mathcal{F}_0 \subseteq \mathcal{F}^\bullet$; in particular, when $\mathcal{F}_0$ has only finitely many isomorphism classes. By [21, Proposition 4.11], for each finite group Γ, there is some k_Γ such that $\Lambda^i(\Gamma; M) = 0$ for all $\mathbb{Z}_{(p)}[\Gamma]$–modules M and all $i > k_\Gamma$. Let k be the maximum of the $k_{\mathrm{Out}_{\mathcal{F}}(P)}$ for all $P \in \mathrm{Ob}(\mathcal{F}_0)$. Then by Proposition 5.4, for each functor $\Phi\colon \mathcal{O}(\mathcal{F}_0)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{–mod}$ which vanishes except on one orbit type, $\varprojlim^i(\Phi) = 0$ for $i > k$. The same result for an arbitrary p–local functor Φ on $\mathcal{O}(\mathcal{F}_0)$ now follows from the exact sequences of higher limits associated to short exact sequences of functors. □

In practice, when computing higher limits over orbit categories $\mathcal{O}^c(\mathcal{F})$, it is useful to combine Proposition 5.2 and Proposition 5.4, as illustrated by the following corollary.

Corollary 5.6 *Let $\mathcal{F}$ be a saturated fusion system over a discrete p–toral group S. Let $F\colon \mathcal{O}^c(\mathcal{F})^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{–mod}$ be a functor with the property that for each $\mathcal{F}$–centric subgroup $P \in \mathcal{H}^\bullet(\mathcal{F})$, $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); F(P)) = 0$. Then $\varprojlim^*(F) = 0$.*

Proof Let $F_0\colon \mathcal{O}^c(\mathcal{F}^\bullet)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{–mod}$ be the restriction of F. By Proposition 5.2,

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^*(F) \cong \varprojlim_{\mathcal{O}^c(\mathcal{F}^\bullet)}{}^*(F_0).$$

Assume first that F_0 vanishes except on the conjugacy class of one subgroup P in $\mathcal{H}^\bullet(\mathcal{F})$. Let F' be the functor on $\mathcal{O}^c(\mathcal{F})$ which takes the same value on the conjugacy class of P and vanishes on all other subgroups. Then

$$\varprojlim_{\mathcal{O}^c(\mathcal{F}^\bullet)}{}^*(F_0) \cong \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^*(F') \cong \Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); F(P))$$

by Proposition 5.2 and Proposition 5.4, and this is zero by assumption.

By Lemma 3.2(a), the category $\mathcal{O}^c(\mathcal{F}^\bullet)$ contains only finitely many isomorphism classes. Hence there is a sequence

$$0 = \Phi_0 \subseteq \Phi_1 \subseteq \cdots \subseteq \Phi_k = F_0$$

of subfunctors defined on $\mathcal{O}^c(\mathcal{F}^\bullet)$, with the property that for each i, Φ_i/Φ_{i-1} vanishes except on the conjugacy class of one subgroup P, and $(\Phi_i/\Phi_{i-1})(P) \cong F(P)$. We have just seen that $\varprojlim^*(\Phi_i/\Phi_{i-1}) = 0$ for all i; and hence $\varprojlim^*(F_0) = 0$ by the relative long exact sequences of higher limits. □

The following lemma will be useful in showing that certain functors on the orbit category are acyclic. As usual, when $\mathcal{F}$ is a fusion system over S, a subgroup $P \le S$ will be called *weakly closed* in $\mathcal{F}$ if it is the only subgroup in its $\mathcal{F}$–conjugacy class.

Lemma 5.7 *Let $\mathcal{F}$ be any saturated fusion system over a discrete p–toral group S, and let $Q \lhd S$ be any $\mathcal{F}$–centric subgroup which is weakly closed in $\mathcal{F}$. Set $\Gamma = \mathrm{Out}_{\mathcal{F}}(Q)$, and let $\mathcal{F}_{\ge Q} \subseteq \mathcal{F}^c$ be the full subcategory whose objects are the subgroups which contain Q. Define the functor*

$$\Theta\colon \mathcal{O}(\mathcal{F}_{\ge Q})^{\mathrm{op}} \longrightarrow \mathcal{O}_p(\Gamma)$$

by sending an object P to $\mathrm{Out}_P(Q) \le \Gamma$, and by sending a morphism $\varphi \in \mathrm{Rep}_{\mathcal{F}}(P, P')$ to the class of $\varphi|_Q \in N_\Gamma(\Theta(P), \Theta(P'))$. Then for any pair of functors

$$F\colon \mathcal{O}^c(\mathcal{F})^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{–mod} \qquad \text{and} \qquad \Phi\colon \mathcal{O}_p(\Gamma)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{–mod}$$

such that $\Phi \circ \Theta \cong F|_{\mathcal{O}(\mathcal{F}_{\ge Q})}$, and such that $\mathrm{Out}_Q(P) \cong N_{PQ}(P)/P$ acts trivially on $F(P)$ for all $P \le S$,

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^*(F) \cong \varprojlim_{\mathcal{O}_p(\Gamma)}{}^*(\Phi).$$

Proof Define a functor

$$F'\colon \mathcal{O}^c(\mathcal{F})^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{–mod}$$

by setting $F'(P) = F(P)$ if $P \ge Q$ and $F'(P) = 0$ otherwise. Regard F' as a quotient functor of F, and set $F'' = \mathrm{Ker}[F \twoheadrightarrow F']$.

If $P \le S$ is $\mathcal{F}$–centric and $P \not\ge Q$, then $\mathrm{Out}_Q(P) \cong N_{PQ}(P)/P \ne 1$, and by assumption this group acts trivially on $F(P) \cong F''(P)$. Hence the kernel of the action of $\mathrm{Out}_{\mathcal{F}}(P)$ on $F''(P)$ has order a multiple of p, and so $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); F''(P)) = 0$ by [20, Proposition 5.5]. Thus $\varprojlim^*(F'') = 0$ by Corollary 5.6, and hence

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^*(F) \cong \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^*(F').$$

Recall that $\Gamma = \mathrm{Out}_{\mathcal{F}}(Q)$. Since Q is fully normalized in $\mathcal{F}$ (it is the unique subgroup in its $\mathcal{F}$–conjugacy class), $\Theta(S) = \mathrm{Out}_S(Q) \in \mathrm{Syl}_p(\Gamma)$. Also, Θ defines a bijection between subgroups of $\Theta(S) \cong S/Q$ and subgroups of S which contain Q. For all $Q \le P, P' \le S$,

$$\mathrm{Rep}_{\mathcal{F}}(P, P') \xrightarrow{(-)|_Q} \mathrm{Mor}_{\mathcal{O}_p(\Gamma)}(\Theta(P), \Theta(P'))$$

is injective by Proposition 2.8. If $g \in N_\Gamma(\Theta(P), \Theta(P'))$ is any element in the transporter, and $g = [\varphi]$ for $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$, then for all $x \in P$ there is $y \in P'$ such that $\varphi c_x \varphi^{-1} = c_y$ as automorphisms of Q. Hence by condition (II) in Definition 2.2, φ extends to a homomorphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(P, P')$, and Θ sends $[\overline{\varphi}] \in \mathrm{Rep}_{\mathcal{F}}(P, P')$ to the class of g.

This proves that Θ induces bijections on all morphism sets, and thus is an equivalence of categories. Hence if Φ is such that $\Phi \circ \Theta \cong F|_{\mathcal{O}(\mathcal{F}_{\ge Q})}$, then

$$\varprojlim_{\mathcal{O}_p(\Gamma)}{}^*(\Phi) \cong \varprojlim_{\mathcal{O}(\mathcal{F}_{\ge Q})}{}^*(F|_{\mathcal{O}(\mathcal{F}_{\ge Q})}) \cong \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^*(F') \cong \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^*(F). \qquad \square$$

This can now be applied to prove the acyclicity of certain explicit functors.

Proposition 5.8 *Let $\mathcal{F}$ be any saturated fusion system over a discrete p–toral group S. Define*

$$F_1, F_2\colon \mathcal{O}^c(\mathcal{F})^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{–mod}$$

on objects by setting $F_1(P) = Z(P)_0$ and $F_2(P) = \pi_2(B(Z(P))^\wedge_p)$. On morphisms, each F_i sends the class of $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, P')$ to the homomorphism induced by the inclusion of $Z(P')$ into $Z(\varphi(P))$ followed by $\varphi^{-1}|_{Z(\varphi(P))}$. Then F_1 and F_2 are both acyclic.

Proof Set $T = S_0$ (the "maximal torus" in $\mathcal{F}$), $Q = C_S(T) \lhd S$, and $\Gamma = \mathrm{Out}_{\mathcal{F}}(Q)$. Then Q is $\mathcal{F}$–centric, and is weakly closed in $\mathcal{F}$ since T is. Let

$$\Theta\colon \mathcal{O}(\mathcal{F}_{\ge Q}) \longrightarrow \mathcal{O}_p(\Gamma)$$

be the functor of Lemma 5.7. For each p–subgroup $\Pi \le \Gamma$, regarded as a group of automorphisms of Q, let $\mathcal{N}_\Pi$ be the norm map for the action of Π on T; ie

$\mathcal{N}_\Pi(t) = \prod_{\gamma\in\Pi} \gamma(t)$ for $t \in T$. Define

$$\Phi_1(\Pi) = \mathcal{N}_\Pi(T) \qquad \text{and} \qquad \Phi_2(\Pi) = \operatorname{Hom}(\mathbb{Z}/p^\infty, T)^\Pi.$$

These define functors $\Phi_i\colon \mathcal{O}_p(\Gamma)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{–}\mathfrak{mod}$.

For each $P \le S$ which contains Q, $\mathcal{N}_{P/Q}(T)$ is connected (ie infinitely p–divisible), and has finite index in $Z(P)$ since $Z(P) \cap T = T^P$ and $T^P/\mathcal{N}_{P/Q}(T)$ has exponent at most $|P/Q|$. Hence $\mathcal{N}_{P/Q}(T)$ is equal to the identity component $Z(P)_0$, and we have

$$F_1(P) = Z(P)_0 = \mathcal{N}_{P/Q}(T) = \Phi_1(\Theta(P)).$$

In general, for any discrete p–toral group P,

$$\pi_2(BP^\wedge_p) = [S^2, BP^\wedge_p] \cong [BS^1, BP^\wedge_p] \cong \operatorname{Hom}(\mathbb{Z}/p^\infty, P).$$

Here, the last equivalence follows from Lemma 1.10, while the middle one follows by obstruction theory (since $\pi_i(BP^\wedge_p) = 0$ for $i > 2$). Hence for any $P \le S$ which contains Q,

$$F_2(P) = \pi_2(BZ(P)^\wedge_p) \cong \operatorname{Hom}(\mathbb{Z}/p^\infty, Z(P)) \cong \operatorname{Hom}(\mathbb{Z}/p^\infty, T)^{P/Q} = \Phi_2(\Theta(P)).$$

Thus $\Phi_i \circ \Theta \cong F_i|_{\mathcal{O}(\mathcal{F}_{\ge Q})}$ (for $i = 1, 2$). Also, for each $P \le S$, $\operatorname{Out}_Q(P)$ acts trivially on $F_i(P)$ for $i = 1, 2$ since Q centralizes $Z(P)_0 \le T$. So by Lemma 5.7,

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^{*}(F_i) \cong \varprojlim_{\mathcal{O}_p(\Gamma)}{}^{*}(\Phi_i).$$

The functors Φ_1 and Φ_2 are both Mackey functors on $\mathcal{O}_p(\Gamma)$ [18, Proposition 5.14; 20, Proposition 5.2], and hence are acyclic. □

As in Section 4, when $\mathcal{F}$ is a saturated fusion system over S, we let $\mathbf{B}$ denote the homotopy functor $\mathbf{B}(P) = BP$, and by extension let $\mathbf{B}^\wedge_p$ denote the functor $\mathbf{B}^\wedge_p(P) = BP^\wedge_p$. The following proposition is a first application of Proposition 5.8. It shows that there is a bijective correspondence between rigidifications of these two functors.

Proposition 5.9 *Let $\mathcal{F}$ be a saturated fusion system over a discrete p–toral group S, and let $\mathcal{F}_0 \subseteq \mathcal{F}^c$ be any full subcategory which contains $\mathcal{F}^{c\bullet}$. Let $\widehat{B}\colon \mathcal{O}(\mathcal{F}_0) \to \mathrm{Top}$ be any rigidification of the homotopy functor $\mathbf{B}^\wedge_p$. Then there is a functor $\widetilde{B}\colon \mathcal{O}(\mathcal{F}_0) \to \mathrm{Top}$ such that $\widetilde{B}(P) \simeq BP$ for all P, together with a natural transformation of functors $\widetilde{B} \to \widehat{B}$ which is a homotopy equivalence after p–completion. Moreover, there is a bijection between equivalence classes of rigidifications of $\mathbf{B}$ and equivalence classes of rigidifications of $\mathbf{B}^\wedge_p$.*

Proof Let $\chi\colon \mathbf{B} \longrightarrow \mathbf{B}^\wedge_p$ be the natural transformation of homotopy functors which sends BP to $BP^\wedge_p$ by the canonical map. We want to apply Theorem A.3, which is a relative version of the Dwyer–Kan theorem [10] for rigidifying centric homotopy diagrams. We first check that χ is relatively centric in the sense of Theorem A.3. This means showing, for each $\varphi \in \mathrm{Mor}_{\mathcal{O}(\mathcal{F}_0)}(P,Q)$, that the square

$$\begin{tikzcd} \mathrm{Map}(BP,BP)_{\mathrm{Id}} \arrow[r, "B\varphi\circ-"] \arrow[d, "\chi(P)\circ-"'] & \mathrm{Map}(BP,BQ)_{B\varphi} \arrow[d, "\chi(Q)\circ-"'] \\ \mathrm{Map}(BP,BP^\wedge_p)_{\chi(P)} \arrow[r, "B\varphi\circ-"] & \mathrm{Map}(BP,BQ^\wedge_p)_{\chi(Q)\circ B\varphi} \end{tikzcd}$$

is a homotopy pullback. By a classical result, the top row is a homotopy equivalence, and both mapping spaces have the homotopy type of $BZ(P)$ (cf [5, Proposition 7.1]). By Lemma 1.10, the second row is also a homotopy equivalence, and both mapping spaces have the homotopy type of $BZ(P)^\wedge_p$. So the square is a homotopy pullback.

For each $i \geq 1$, let $\beta_i\colon \mathcal{O}(\mathcal{F}_0)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ be the functor defined in Theorem A.3, where for each P,

$$\beta_i(P) = \pi_i\Bigl(\mathrm{hofiber}\bigl(\underbrace{\mathrm{Map}(BP,BP)_{\mathrm{Id}}}_{\simeq BZ(P)} \xrightarrow{\chi(P)\circ-} \underbrace{\mathrm{Map}(BP,BP^\wedge_p)_{\chi(P)}}_{\simeq BZ(P)^\wedge_p}\bigr)\Bigr).$$

By [12, Proposition 3.1], this homotopy fiber is a $K(V,1)$ for some $\widehat{\mathbb{Q}}_p$–vector space V. In particular, the fiber is connected, $\beta_1(P)$ is abelian for all P, and $\beta_i = 0$ for all $i \geq 2$. Also, by the homotopy exact sequence for the fibration, there is a short exact sequence of functors

$$0 \longrightarrow F_2 \longrightarrow \beta_1 \longrightarrow F_1 \longrightarrow 0,$$

where F_1 and F_2 are the functors of Proposition 5.8. By Proposition 5.2, for all $i \geq 1$ and $j = 1,2$,

$$\varprojlim_{\mathcal{O}(\mathcal{F}_0)}{}^i(F_j) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^\bullet)}{}^i(F_j) \cong \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^i(F_j),$$

where the last group vanishes by Proposition 5.8. Thus $\varprojlim^i(\beta_1) = 0$ for all $i \geq 1$.

The proposition now follows directly from Theorem A.3. □

In Section 8, we will also need to work with higher limits over orbit categories of certain infinite groups. For any (discrete) group G, let $\mathcal{O}_{\mathrm{dpt}}(G)$ denote the orbit category of G whose objects are the discrete p–toral subgroups of G; and define (for any

$\mathbb{Z}[G]$–module M),

$$\Lambda^*_{\mathrm{dpt}}(G;M) = \varprojlim_{\mathcal{O}_{\mathrm{dpt}}(G)} (F_M) \qquad \text{where} \quad F_M(P) = \begin{cases} M & \text{if } P=1 \\ 0 & \text{if } P \neq 1. \end{cases}$$

We are now ready to give a second application of Proposition 5.3.

Lemma 5.10 *Fix a group G, a discrete p–toral subgroup $Q \le G$, and a functor $\Phi\colon \mathcal{O}_{\mathrm{dpt}}(G)^{\mathrm{op}} \longrightarrow \mathrm{Ab}$ with the property that $\Phi(P)=0$ except when P is G–conjugate to Q. Let $\Phi'\colon \mathcal{O}_{\mathrm{dpt}}(N_G(Q)/Q)^{\mathrm{op}} \longrightarrow \mathrm{Ab}$ be the functor $\Phi'(P/Q) = \Phi(P)$. Then*

$$\varprojlim_{\mathcal{O}_{\mathrm{dpt}}(G)}{}^{*} (\Phi) \cong \varprojlim_{\mathcal{O}_{\mathrm{dpt}}(N_G(Q)/Q)}{}^{*} (\Phi') \cong \Lambda^*_{\mathrm{dpt}}(N_G(Q)/Q;\Phi(Q)).$$

Proof We apply Proposition 5.3, where $\mathcal{C} = \mathcal{O}_{\mathrm{dpt}}(G)$, $\Gamma = N_G(Q)/Q$, and $\mathcal{H}$ is the set of discrete p–toral subgroups of Γ. A functor

$$\alpha\colon \mathcal{O}_{\mathrm{dpt}}(\Gamma) \longrightarrow \mathcal{O}_{\mathrm{dpt}}(G)$$

is defined by setting $\alpha(P/Q) = P$, and by sending each morphism set

$$(P'/Q)\backslash N_\Gamma(P/Q, P'/Q)$$

to $P'\backslash N_G(P,P')$ in the obvious way.

The hypotheses of Proposition 5.3 follow easily from the definition of the orbit categories, and so the isomorphisms between higher limits follow from the proposition. □

The following very general lemma will help in certain cases to reduce computations of higher limits to those taken over finite subcategories.

Lemma 5.11 *Let $\mathcal{C}$ be a (small) category, and let $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \cdots$ be an increasing sequence of subcategories of $\mathcal{C}$ whose union is $\mathcal{C}$. Let $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$ be a functor such that for each k,*

$$\varprojlim_{i}{}^{1}\Big(\varprojlim_{\mathcal{C}_i}{}^{k}(F|_{\mathcal{C}_i})\Big) = 0.$$

Then the homomorphism

$$\varprojlim_{\mathcal{C}}{}^{k}(F) \xrightarrow{\ \cong\ } \varprojlim_{i}\Big(\varprojlim_{\mathcal{C}_i}{}^{k}(F|_{\mathcal{C}_i})\Big)$$

induced by the restrictions is an isomorphism for all k.

Proof For any category $\mathcal{D}$ and any functor $\Phi\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{A}b$, $\varprojlim^*(\Phi)$ is the homology of the chain complex $(C^*(\mathcal{D};\Phi),d)$, defined by setting

$$C^n(\mathcal{D};\Phi) = \prod_{c_0\to\cdots\to c_n} \Phi(c_0),$$

where the product is taken over composable n–tuples of morphisms in $\mathcal{D}$, and where

$$d(\xi)(c_0 \xrightarrow{\alpha} c_1 \to \cdots \to c_{n+1}) = \alpha^*\xi(c_1 \to \cdots \to c_{n+1}) + \sum_{i=1}^{n+1} \xi(c_0 \to \cdots \widehat{c_i} \cdots \to c_{n+1}).$$

See, for example, Gabriel and Zisman [15, Appendix II, Proposition 3.3] or Oliver [26, Lemma 2]. If $\mathcal{D}_0 \subseteq \mathcal{D}$ is a subcategory, then the restriction homomorphism from $\varprojlim_{\mathcal{D}}^*(\Phi)$ to $\varprojlim_{\mathcal{D}_0}^*(\Phi|_{\mathcal{D}_0})$ is induced by the obvious surjections $C^*(\mathcal{D};\Phi) \twoheadrightarrow C^*(\mathcal{D}_0;\Phi)$.

In the above situation, the chain complex $(C^*(\mathcal{C};F),d)$ is the limit of an inverse system of chain complexes $(C^*(\mathcal{C}_i;F|_{\mathcal{C}_i}),d)$ with surjections, where the inverse system of homology groups of these chain complexes has vanishing $\varprojlim^1(-)$. Since $\varprojlim^1(-)$ vanishes for a (countable directed) inverse system with surjections, we conclude that the cohomology of $(C^*(\mathcal{C};F),d)$ is isomorphic to the inverse limit of the cohomology of the complexes $(C^*(\mathcal{C}_i;F|_{\mathcal{C}_i}),d)$. □

The next lemma describes how, in some cases, the computation of $\Lambda^*_{\mathrm{dpt}}(G;M)$ can be reduced to the case where G is finite. When G is a finite group and M is a $\mathbb{Z}[G]$–module, we let $\Lambda^*(G;M)$ denote the Λ–functor taken with respect to p–subgroups of G.

Lemma 5.12 *Let G be a locally finite group. Assume there is a discrete p–toral subgroup $S \le G$ such that every discrete p–toral subgroup of G is conjugate to a subgroup of S. Fix a $\mathbb{Z}[G]$–module M, and assume that for some finite subgroup $H_0 \le G$, $\Lambda^*(H;M)=0$ for all finite subgroups $H \le G$ which contain H_0. Then $\Lambda^*_{\mathrm{dpt}}(G;M)=0$. In particular, $\Lambda^*_{\mathrm{dpt}}(G;M)=0$ if M is a $\mathbb{Z}_{(p)}[G]$–module and the kernel of the action of G on M contains an element of order p.*

Proof By [20, Proposition 5.5], for any finite group H and any $\mathbb{Z}_{(p)}[H]$–module M such that the kernel of the H–action on M has order a multiple of p, $\Lambda^*(H;M)=0$. Hence the last statement follows as a special case of the first.

Fix a Sylow p–subgroup $S \in \mathrm{Syl}_p(G)$, and let $\mathcal{O}_S(G) \subseteq \mathcal{O}_{\mathrm{dpt}}(G)$ be the full subcategory whose objects are the subgroups of S. Since each discrete p–toral subgroups of

G is G–conjugate to a subgroup of S, these categories are equivalent, and so we can work over $\mathcal{O}_S(G)$ instead. Define

$$F_M\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Ab} \qquad \text{by setting} \quad F_M(P) = \begin{cases} M & \text{if } P = 1 \\ 0 & \text{if } P \neq 1. \end{cases}$$

By definition, $\Lambda^*_{\mathrm{dpt}}(G; M) = \varprojlim{}^*(F_M)$, and we must show that this vanishes in all degrees.

Step 1 To simplify the notation, we write $\mathcal{C} = \mathcal{O}_S(G)$, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory whose objects are the finite subgroups of S. For each subgroup $Q \le S$ and each abelian group A, let $\mathfrak{I}^A_Q$ in $\mathcal{C}$–mod be the functor

$$\mathfrak{I}^A_Q(P) = \mathrm{Map}(\mathrm{Mor}_{\mathcal{C}}(Q, P), A) \cong \prod_{\mathrm{Mor}_{\mathcal{C}}(Q,P)} A.$$

For any F in $\mathcal{C}$–mod, $\mathrm{Hom}_{\mathcal{C}-\mathrm{mod}}(F, \mathfrak{I}^A_Q) \cong \mathrm{Hom}_{\mathbb{Z}}(F(Q), A)$. Hence $\mathfrak{I}^A_Q$ is injective if A is injective as an abelian group, and each functor on $\mathcal{C}$ injects into a product of such injectives. Also, when Q is finite,

$$\varprojlim_{\mathcal{C}_0}(\mathfrak{I}^A_Q|_{\mathcal{C}_0}) \cong \varprojlim_{\mathcal{C}}(\mathfrak{I}^A_Q) \cong A,$$

where the second isomorphism holds for arbitrary $Q \le S$.

Choose a sequence of functors

$$0 \longrightarrow F_M \xrightarrow{d_0} \mathfrak{I}_0 \xrightarrow{d_1} \mathfrak{I}_1 \xrightarrow{d_2} \cdots \tag{10}$$

where each $\mathfrak{I}_k$ is a product of injective functors $\mathfrak{I}^A_Q$ for *finite* subgroups $Q \le S$ and injective abelian groups A, and where (10) is exact after restriction to $\mathcal{C}_0$. We claim that this is an injective resolution of F_M. In other words, the sequence

$$0 \longrightarrow F_M(P) \longrightarrow \mathfrak{I}_0(P) \longrightarrow \mathfrak{I}_1(P) \longrightarrow \cdots \tag{11}$$

is exact for all finite $P \le S$, and we want to show it is exact for all $P \le S$. Fix an infinite subgroup $P \le S$, and choose finite subgroups $P_1 \le P_2 \le \cdots$ such that $P = \bigcup_{j=1}^{\infty} P_j$ (Lemma 1.9). Then $F_M(P) = 0 = \varprojlim_j F_M(P_j)$. For all finite $Q \le S$ and all A,

$$\mathfrak{I}^A_Q(P) = \mathrm{Map}(\mathrm{Mor}_{\mathcal{C}}(Q, P), A) = \varprojlim_j \bigl(\mathrm{Map}(\mathrm{Mor}_{\mathcal{C}}(Q, P_j), A)\bigr)$$

since $\mathrm{Mor}_{\mathcal{C}}(Q, P)$ is the union of the $\mathrm{Mor}_{\mathcal{C}}(Q, P_j)$; and furthermore this is an inverse system of surjections. Hence (11) is the inverse limit of the corresponding exact sequences for the P_j, all restriction maps $\mathfrak{I}_k(P_{j+1}) \longrightarrow \mathfrak{I}_k(P_j)$ are surjective, and so (11) is also exact. Thus

$$\begin{aligned}\Lambda^*_{\mathrm{dpt}}(G;M) = \varprojlim_{\mathcal{C}}{}^*(F_M) &\cong H^*(\varprojlim_{\mathcal{C}}(\mathfrak{I}_k), d_k)\\ &\cong H^*(\varprojlim_{\mathcal{C}_0}(\mathfrak{I}_k|_{\mathcal{C}_0}), d_k) \cong \varprojlim_{\mathcal{C}_0}{}^*(F_M|_{\mathcal{C}_0}).\end{aligned}$$

Step 2 Fix a sequence $S_1 \le S_2 \le S_3 \le \cdots$ of finite subgroups of S such that $S = \bigcup_{j=1}^{\infty} S_j$ (Lemma 1.9). We first construct inductively a sequence of finite subgroups $H_1 \le H_2 \le \cdots$ of G containing H_0 such that for each $j \ge 1$, $H_j \ge S_j$, and $\mathcal{O}_p(H_j)$ contains the full subcategory with object set the p–subgroups of H_{j-1}. Fix $j \ge 1$, and assume that H_{j-1} has been constructed. Let $\mathcal{C}_j$ be the full subcategory of $\mathcal{O}_p^{\mathrm{fin}}(G)$ whose objects are the p–subgroups of $\langle H_{j-1}, S_j\rangle$ (a finite group since G is locally finite). Choose a finite set of morphisms in $\mathcal{C}_j$ which generate it, let $X_j \subseteq G$ be a finite set of elements which induce those morphisms, and set $H_j = \langle X_j\rangle$. Since G is locally finite, H_j is a finite subgroup. By construction, $\mathcal{O}_p(H_j) \supseteq \mathcal{C}_j$; and hence contains both $\mathcal{O}(S_j)$ and the full subcategory with the same objects as $\mathcal{O}_p(H_{j-1})$.

Set $\mathcal{C}' \overset{\mathrm{def}}{=} \bigcup_{j=1}^{\infty} \mathcal{O}_p(H_j)$. This is a full subcategory of $\mathcal{O}_p^{\mathrm{fin}}(G)$ which contains all finite subgroups of S as objects. In particular, $\mathcal{C}'$ is equivalent to $\mathcal{C}_0$, and hence

$$\varprojlim_{\mathcal{C}_0}{}^*(F_M) \cong \varprojlim_{\mathcal{C}'}{}^*(F_M|_{\mathcal{C}'}).$$

Since $\varprojlim_{\mathcal{O}_p(H_j)}{}^*(F_M|_{\mathcal{O}_p(H_j)}) = 0$ for all j, $\varprojlim_{\mathcal{C}'}{}^*(F_M|_{\mathcal{C}'}) = 0$ by Lemma 5.11. □

6 Mapping spaces

We now look at the spaces of maps from BQ to $|\mathcal{L}|_p^\wedge$, when Q is a discrete p–toral group and $\mathcal{L}$ is a linking system. In general, for any p–local compact group $(S,\mathcal{F},\mathcal{L})$ and any discrete p–toral group Q, we define

$$\mathrm{Rep}(Q,\mathcal{L}) = \mathrm{Hom}(Q,S)/\sim,$$

where $\sim$ is the equivalence relation setting $\rho \sim \rho'$ if there is $\chi \in \mathrm{Hom}_{\mathcal{F}}(\rho(Q), \rho'(Q))$ such that $\rho' = \chi \circ \rho$. We want to show that $[BQ, |\mathcal{L}|_p^\wedge] \cong \mathrm{Rep}(Q,\mathcal{L})$.

The following lemma will be needed to reduce this to the case where Q is finite. The functor $(-)^\bullet$ of Section 3 plays an important role when doing this.

Lemma 6.1 *Fix a discrete p–toral group Q, and let $Q_1 \le Q_2 \le \cdots \le Q$ be a sequence of finite subgroups such that $Q = \bigcup_{n=1}^{\infty} Q_n$. Let $(S,\mathcal{F},\mathcal{L})$ be a p–local compact group. Then the following hold.*

(a) *The natural map*

$$R\colon \mathrm{Rep}(Q,\mathcal{L}) \xrightarrow{\;\cong\;} \varprojlim_n \mathrm{Rep}(Q_n,\mathcal{L})\,,$$

induced by restriction, is a bijection.

(b) *Assume $Q\le S$. Then for n large enough, $Q_n^\bullet=Q^\bullet\ge Q$, and hence restriction induces a bijection $\mathrm{Hom}_{\mathcal{F}}(Q,P)\cong\mathrm{Hom}_{\mathcal{F}}(Q_n,P)$ for all $P\in\mathrm{Ob}(\mathcal{F}^\bullet)$.*

Proof In general, for any homomorphism $\varphi\in\mathrm{Hom}(H,K)$, we let $[\varphi]$ denote its class in $\mathrm{Rep}(H,K)$.

(a) Assume first that $\varphi,\psi\in\mathrm{Hom}(Q,S)$ are such that $R([\varphi])=R([\psi])$. Thus $\varphi|_{Q_n}$ and $\psi|_{Q_n}$ are $\mathcal{F}$–conjugate for each n; ie $\psi|_{Q_n}=\alpha_n\circ\varphi|_{Q_n}$ for some unique $\alpha_n\in\mathrm{Iso}_{\mathcal{F}}(\varphi(Q_n),\psi(Q_n))$. In particular, $\mathrm{Ker}(\varphi)\cap Q_n=\mathrm{Ker}(\psi)\cap Q_n$ for each n, so $\mathrm{Ker}(\varphi)=\mathrm{Ker}(\psi)$, and $\psi=\alpha\circ\varphi$ for some unique $\alpha\in\mathrm{Iso}(\varphi(Q),\psi(Q))$. Then $\alpha|_{Q_n}=\alpha_n$ is in $\mathcal{F}$ for each n, so $\alpha\in\mathrm{Iso}_{\mathcal{F}}(\varphi(Q),\psi(Q))$ by axiom (III), and $[\psi]=[\varphi]\in\mathrm{Rep}(Q,\mathcal{F})$.

This proves the injectivity of R, and it remains to prove surjectivity. Fix some

$$\{[\varphi_n]\}_{n\ge1}\in\varprojlim_n \mathrm{Rep}(Q_n,\mathcal{L}).$$

Thus for each n, $\varphi_n\in\mathrm{Hom}(Q_n,S)$, and $\varphi_{n+1}|_{Q_n}$ is $\mathcal{F}$–conjugate to φ_n. By Lemma 3.2(a), the set $\{\varphi_n(Q_n)^\bullet\,|\,n\ge1\}$ contains finitely many conjugacy classes. Since for all n, $\varphi_n(Q_n)$ is $\mathcal{F}$–conjugate to a subgroup of $\varphi_{n+1}(Q_{n+1})$, $\varphi_n(Q_n)^\bullet$ is $\mathcal{F}$–conjugate to a subgroup of $\varphi_{n+1}(Q_{n+1})^\bullet$ by Lemma 3.2(b) and Proposition 3.3. Hence for some m, $\varphi_n(Q_n)^\bullet$ is $\mathcal{F}$–conjugate to $\varphi_m(Q_m)^\bullet$ for all $n\ge m$.

We now construct inductively homomorphisms $\varphi'_n\in\mathrm{Hom}(Q_n,S)$ for all $n>m$ such that $[\varphi'_n]=[\varphi_n]$ in $\mathrm{Rep}(Q_n,\mathcal{L})$, and $\varphi'_n|_{Q_{n-1}}=\varphi'_{n-1}$. Assume φ'_{n-1} has been constructed, and set $\alpha_n=\varphi_n\circ(\varphi'_{n-1})^{-1}\in\mathrm{Hom}_{\mathcal{F}}(\varphi'_{n-1}(Q_{n-1}),\varphi_n(Q_n))$. Again by Proposition 3.3, this extends to a unique morphism

$$\alpha_n^\bullet\in\mathrm{Hom}_{\mathcal{F}}(\varphi'_{n-1}(Q_{n-1})^\bullet,\varphi_n(Q_n)^\bullet),$$

which must be an isomorphism since it is injective and the two groups are abstractly isomorphic and artinian. Set $\varphi'_n=(\alpha_n^\bullet)^{-1}\circ\varphi_n$; then $\varphi'_n|_{Q_{n-1}}=\varphi'_{n-1}$. Let $\varphi\in\mathrm{Hom}(Q,S)$ be the union of the φ'_n; then $[\varphi]\in R^{-1}(\{[\varphi_n]\})$, and this proves the surjectivity of R.

(b) Now assume $Q\le S$. By Lemma 3.2(a,b), for all n, $Q_n^\bullet\le Q_{n+1}^\bullet\le Q^\bullet$, and the set $\{Q_n^\bullet\,|\,n\ge1\}$ is finite. Hence $Q_n^\bullet\ge Q$ for n sufficiently large, and this implies $Q_n^\bullet=Q^\bullet$. If $P=P^\bullet\le S$, then every $\varphi\in\mathrm{Hom}_{\mathcal{F}}(Q_n,P)$ extends to a unique

$\varphi^\bullet \in \mathrm{Hom}_{\mathcal{F}}(Q_n^\bullet, P)$ by Proposition 3.3, and thus $\mathrm{Hom}_{\mathcal{F}}(Q, P) \cong \mathrm{Hom}_{\mathcal{F}}(Q_n, P)$ whenever $Q_n^\bullet = Q^\bullet$. □

For any linking system $\mathcal{L}$ and any discrete p–toral group Q, we let $\mathcal{L}^Q$ be the category whose objects are the pairs (P, α) for $P \in \mathrm{Ob}(\mathcal{L})$ and $\alpha \in \mathrm{Hom}(Q, P)$, and where

$$\mathrm{Mor}_{\mathcal{L}^Q}\big((P,\alpha),(P',\alpha')\big) = \big\{\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,P') \,\big|\, \alpha' = \pi(\varphi)\circ\alpha \in \mathrm{Hom}(Q,P')\big\}.$$

We next show that $\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p) \simeq |\mathcal{L}^Q|^\wedge_p$ in this situation.

Proposition 6.2 *Fix a p–local compact group $(S,\mathcal{F},\mathcal{L})$ and a discrete p–toral group Q. Let $\mathcal{F}_0 \subseteq \mathcal{F}^c$ be any full subcategory which contains all $\mathcal{F}$–centric $\mathcal{F}$–radical subgroups of S, and such that $P \in \mathrm{Ob}(\mathcal{F}_0)$ implies $P^\bullet \in \mathrm{Ob}(\mathcal{F}_0)$. Let $\mathcal{L}_0 \subseteq \mathcal{L}$ and $\mathcal{L}_0^Q \subseteq \mathcal{L}^Q$ be the full subcategories where $\mathrm{Ob}(\mathcal{L}_0) = \mathrm{Ob}(\mathcal{F}_0)$, and $\mathrm{Ob}(\mathcal{L}_0^Q)$ is the set of pairs $(P,\alpha) \in \mathrm{Ob}(\mathcal{L}^Q)$ such that $P \in \mathrm{Ob}(\mathcal{L}_0)$. Then there is a bijection*

$$\pi_0(|\mathcal{L}_0^Q|) \xrightarrow{\ \cong\ } \mathrm{Rep}(Q,\mathcal{L}) \tag{12}$$

which sends a vertex (P,α) to the class of α as a homomorphism to S. If, furthermore, we define $\Phi\colon \mathcal{L}_0^Q \times \mathcal{B}(Q) \longrightarrow \mathcal{L}_0$ by setting

$$\Phi\big((P,\alpha), o_Q\big) = P \qquad \text{and} \qquad \Phi\Big((P,\alpha) \xrightarrow{\varphi} (P',\alpha')\,,\ x\Big) = \varphi\circ\delta_P(\alpha(x))\,,$$

then the map

$$|\Phi|'\colon |\mathcal{L}_0^Q|^\wedge_p \longrightarrow \mathrm{Map}(BQ, |\mathcal{L}_0|^\wedge_p) \tag{13}$$

adjoint to $|\Phi|$ is a homotopy equivalence.

Proof Every vertex (P,α) in $|\mathcal{L}_0^Q|$ is connected by an edge to the vertex $(S, \mathrm{incl}_P^S\circ\alpha)$. Furthermore, by the assumption that $\mathcal{F}_0$ contains all $\mathcal{F}$–centric $\mathcal{F}$–radical subgroups, together with Alperin's fusion theorem (Theorem 3.6), two vertices (S,α) and (S,α') in $|\mathcal{L}_0^Q|$ are in the same connected component if and only if α and α' represent the same element of $\mathrm{Rep}(Q,\mathcal{L})$. This proves (12).

Since $\Phi(\varphi,x) = \varphi\circ\delta_P(\alpha(x)) = \delta_{P'}(\alpha'(x))\circ\varphi$ by condition (C), Φ is a well defined functor. It remains to prove the homotopy equivalence (13). Step 1, where we handle the case Q is finite, is essentially the same as the corresponding proof in [7]. In Step 2, we extend this to the general case.

By assumption, for each $P \in \mathrm{Ob}(\mathcal{L}_0)$, $P^\bullet \in \mathrm{Ob}(\mathcal{L}_0)$. So the functor $(-)^\bullet$ of Proposition 4.5 restricts to a functor from $\mathcal{L}_0$ to $\mathcal{L}_0^\bullet$, and also induces a functor from $\mathcal{L}_0^Q$ to $\mathcal{L}_0^{\bullet Q}$. All of these are left adjoint to the inclusion functors, and hence induce homotopy equivalences between their geometric realizations. Thus, without loss of generality, we can assume that $\mathcal{L}_0 = \mathcal{L}_0^\bullet$; ie that $P = P^\bullet$ for all P in $\mathcal{L}_0$. This assumption will be needed at the end of each of Steps 1 and 2 below.

Step 1 Assume that Q is a finite p–group. Let $\mathcal{O}(\mathcal{F}_0) \subseteq \mathcal{O}^c(\mathcal{F})$ be the full subcategory with $\mathrm{Ob}(\mathcal{O}(\mathcal{F}_0)) = \mathrm{Ob}(\mathcal{F}_0) = \mathrm{Ob}(\mathcal{L}_0)$, and let $\widetilde{\pi}\colon \mathcal{L}_0 \longrightarrow \mathcal{O}(\mathcal{F}_0)$ be the projection functor. Let $\widetilde{\pi}_Q\colon \mathcal{L}_0^Q \longrightarrow \mathcal{O}(\mathcal{F}_0)$ be the functor $\widetilde{\pi}_Q(P,\alpha) = P$ and $\widetilde{\pi}_Q(\varphi) = \widetilde{\pi}(\varphi)$. Let

$$\widetilde{B}_Q, \widetilde{B}\colon \mathcal{O}(\mathcal{F}_0) \longrightarrow \mathrm{Top}$$

be the left homotopy Kan extensions over $\widetilde{\pi}_Q$ and $\widetilde{\pi}$, respectively, of the constant functors $*$. Then

$$|\mathcal{L}_0| \simeq \underset{\mathcal{O}(\mathcal{F}_0)}{\underrightarrow{\mathrm{hocolim}}}(\widetilde{B}) \qquad \text{and} \qquad |\mathcal{L}_0^Q| \simeq \underset{\mathcal{O}(\mathcal{F}_0)}{\underrightarrow{\mathrm{hocolim}}}(\widetilde{B}_Q) \tag{14}$$

(cf Hollender and Vogt [17, Theorem 5.5]).

For each P in $\mathcal{O}(\mathcal{F}_0)$, $\widetilde{B}(P)$ is the nerve of the overcategory $\widetilde{\pi}{\downarrow}P$, whose objects are the pairs (R,χ) for $R \in \mathrm{Ob}(\mathcal{L}_0) = \mathrm{Ob}(\mathcal{O}(\mathcal{F}_0))$ and $\chi \in \mathrm{Rep}_{\mathcal{F}}(R,P)$, and where

$$\mathrm{Mor}_{\widetilde{\pi}\downarrow P}\big((R,\chi),(R',\chi')\big) = \big\{\varphi \in \mathrm{Mor}_{\mathcal{L}_0}(R,R') \mid \chi = \chi' \circ \widetilde{\pi}(\varphi)\big\}.$$

Let $\mathcal{B}'(P)$ be the full subcategory of $\widetilde{\pi}{\downarrow}P$ with the unique object (P,Id), and with morphisms the group of all $\delta_P(g)$ for $g \in P$.

Similarly, $\widetilde{B}_Q(P)$ is the nerve of the category $\widetilde{\pi}_Q{\downarrow}P$, whose objects are the triples (R,α,χ) for $R \in \mathrm{Ob}(\mathcal{L}_0) = \mathrm{Ob}(\mathcal{O}(\mathcal{F}_0))$, $\alpha \in \mathrm{Hom}(Q,R)$, and $\chi \in \mathrm{Rep}_{\mathcal{F}}(R,P)$; and where

$$\mathrm{Mor}_{\widetilde{\pi}_Q\downarrow P}\big((R,\alpha,\chi),(R',\alpha',\chi')\big) = \big\{\varphi \in \mathrm{Mor}_{\mathcal{L}_0}(R,R') \mid \alpha' = \pi(\varphi)\circ\alpha,\ \chi = \chi'\circ\widetilde{\pi}(\varphi)\big\}.$$

Let $\mathcal{B}'_Q(P)$ be the full subcategory of $\widetilde{\pi}_Q{\downarrow}P$ with objects the triples (P,α,Id) for $\alpha \in \mathrm{Hom}(Q,P)$.

Fix a section $\widetilde{\sigma}\colon \mathrm{Mor}(\mathcal{O}(\mathcal{F}_0)) \longrightarrow \mathrm{Mor}(\mathcal{L}_0)$ which sends identity morphisms to identity morphisms. Retractions

$$\widetilde{\pi}{\downarrow}P \xrightarrow{\ \Psi\ } \mathcal{B}'(P) \qquad \text{and} \qquad \widetilde{\pi}_Q{\downarrow}P \xrightarrow{\ \Psi_Q\ } \mathcal{B}'_Q(P)$$

are defined by setting

$$\Psi(R,\chi) = (P,\mathrm{Id}) \qquad \text{and} \qquad \Psi_Q(R,\alpha,\chi) = (P, \pi\widetilde{\sigma}(\chi)\circ\alpha, \mathrm{Id});$$

and by sending φ in $\mathrm{Mor}_{\widetilde{\pi}\downarrow P}((R,\chi),(R',\chi'))$ or $\mathrm{Mor}_{\widetilde{\pi}_Q\downarrow P}((R,\alpha,\chi),(R',\alpha',\chi'))$ to the automorphism $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}_0}(P)$, where $g \in P$ is the unique element such that $\widetilde{\sigma}(\chi')\circ\varphi=\delta_P(g)\circ\widetilde{\sigma}(\chi)$ in $\mathrm{Mor}_{\mathcal{L}_0}(R,P)$ (Lemma 4.3(b)). There are natural transformations

$$\mathrm{Id}_{\widetilde{\pi}\downarrow P}\xrightarrow{\qquad}\mathrm{incl}\circ\Psi\qquad\text{and}\qquad\mathrm{Id}_{\widetilde{\pi}_Q\downarrow P}\xrightarrow{\qquad}\mathrm{incl}\circ\Psi_Q$$

of functors which send an object (R,χ) to $\chi\in\mathrm{Mor}_{\widetilde{\pi}\downarrow P}((R,\chi),(P,\mathrm{Id}))$ and similarly for an object (R,α,χ). This shows that $|\mathcal{B}'(P)|\subseteq|\widetilde{\pi}\downarrow P|$ and $|\mathcal{B}'_Q(P)|\subseteq|\widetilde{\pi}_Q\downarrow P|$ are deformation retracts.

We have now shown that for all $P\in\mathrm{Ob}(\mathcal{L}_0)$,

$$\widetilde{B}(P)\simeq|\mathcal{B}'(P)|\simeq BP\qquad\text{and}\qquad\widetilde{B}_Q(P)\simeq|\mathcal{B}'_Q(P)|\,.\tag{15}$$

All morphisms in $\mathcal{B}'_Q(P)$ are isomorphisms, two objects (P,α,Id) and (P,α',Id) are isomorphic if and only if α and α' are conjugate in P, and the automorphism group of (P,α,Id) is isomorphic to $C_P(\alpha Q)$. Thus

$$\widetilde{B}_Q(P)\simeq\coprod_{\alpha\in\mathrm{Rep}(Q,P)}BC_P(\alpha Q).\tag{16}$$

Denote by $\widetilde{B}^\wedge_p$ and $\widetilde{B}_Q{}^\wedge_p$ be the p–completions of $\widetilde{B}$ and $\widetilde{B}_Q$; ie $(\widetilde{B}^\wedge_p)(P)=(\widetilde{B}(P))^\wedge_p$ and $(\widetilde{B}_Q{}^\wedge_p)(P)=(\widetilde{B}_Q(P))^\wedge_p$. By (14), and since the spaces $\widetilde{B}(P)$ and $\widetilde{B}_Q(P)$ are all p–good by (15) and (16),

$$|\mathcal{L}_0|^\wedge_p\simeq\Big(\underset{\mathcal{O}(\mathcal{F}_0)}{\underrightarrow{\mathrm{hocolim}}}(\widetilde{B}^\wedge_p)\Big)^\wedge_p\qquad\text{and}\qquad|\mathcal{L}_0^Q|^\wedge_p\simeq\Big(\underset{\mathcal{O}(\mathcal{F}_0)}{\underrightarrow{\mathrm{hocolim}}}(\widetilde{B}_Q{}^\wedge_p)\Big)^\wedge_p\,.$$

Consider the commutative triangle:

$$\begin{tikzcd}
\mathcal{L}_0^Q\times\mathcal{B}(Q) \arrow[rr, "\Phi"] \arrow[dr, "\widetilde{\pi}_Q\circ\mathrm{pr}_1"'] & & \mathcal{L}_0 \arrow[dl, "\widetilde{\pi}"] \\
& \mathcal{O}(\mathcal{F}_0) &
\end{tikzcd}$$

The left homotopy Kan extension over $\widetilde{\pi}_Q\circ\mathrm{pr}_1$ of the constant functor $*$ is the functor $\widetilde{B}_Q\times BQ$, and so the triangle induces a natural transformation of functors

$$\Phi'\colon\widetilde{B}_Q\times BQ\xrightarrow{\qquad}\widetilde{B}.$$

The map $\widetilde{\Phi}\colon \widetilde{B}_Q \longrightarrow \mathrm{Map}(BQ, \widetilde{B})$ adjoint to Φ' is also a natural transformation of functors from $\mathcal{O}(\mathcal{F}_0)$ to Top, and induces a commutative diagram:

$$\begin{tikzcd}
\Big(\underset{\mathcal{O}(\mathcal{F}_0)}{\mathrm{hocolim}}(\widetilde{B}_Q)^\wedge_p\Big)^\wedge_p \arrow[r, "\mathrm{hocolim}(\widetilde{\Phi})"] \arrow[d, "\simeq"'] & \Big(\underset{\mathcal{O}(\mathcal{F}_0)}{\mathrm{hocolim}}\,\mathrm{Map}(BQ, \widetilde{B}^\wedge_p)\Big)^\wedge_p \arrow[r, "\omega"] & \mathrm{Map}\big(BQ, \underset{\mathcal{O}(\mathcal{F}_0)}{\mathrm{hocolim}}(\widetilde{B})^\wedge_p\big) \arrow[d, "\simeq"'] \\
|\mathcal{L}_0^Q|^\wedge_p \arrow[rr, "|\Phi|'"] & & \mathrm{Map}(BQ, |\mathcal{L}_0|^\wedge_p)
\end{tikzcd}$$

For each $P \le S$ and $Q_0 \le Q$, Lemma 1.10 (together with (15)) implies that each component of $\mathrm{Map}(BQ_0, \mathcal{B}(P)^\wedge_p)$ has the form $BC_P(\rho(Q_0))^\wedge_p$ for some $\rho \in \mathrm{Hom}(Q_0, P)$. So all such mapping spaces are p–complete and have finite mod p cohomology in each degree. Also, $\mathcal{O}(\mathcal{F}_0)$ is a finite category (it has finitely many isomorphism classes of objects by Lemma 3.2(a) and has finite morphism sets by Lemma 2.5), and it has bounded limits at p by Corollary 5.5. Hence ω is a homotopy equivalence by [7, Proposition 4.2].

It remains only to show that $\widetilde{\Phi}(P)$ is a homotopy equivalence for each $P \in \mathrm{Ob}(\mathcal{L}_0)$. By (15), this means showing that $\widetilde{\Phi}(P)$ restricts to a homotopy equivalence

$$\widetilde{\Phi}'(P)\colon |\mathcal{B}'_Q(P)| \longrightarrow \mathrm{Map}(BQ, |\mathcal{B}'(P)|)\,.$$

Since $|\mathcal{B}'(P)| \cong BP$, and since $\widetilde{\Phi}'(P)$ is induced by the homomorphisms $(\mathrm{incl}\cdot\alpha)$ from $C_P(\alpha(Q)) \times Q$ to P, this follows from (16).

Step 2 Now let Q be an arbitrary p–toral group. Let $Q_1 \le Q_2 \le \cdots Q$ be an increasing sequence of finite subgroups whose union is Q (Lemma 1.9). Then

$$\pi_0(|\mathcal{L}_0^Q|) \cong \varprojlim_n \pi_0(|\mathcal{L}_0^{Q_n}|) \cong \varprojlim_n [BQ_n, |\mathcal{L}_0|^\wedge_p]\,.$$

The first bijection holds by Lemma 6.1 and (12), and the second by Step 1.

Fix $\varphi \in \mathrm{Hom}(Q, S)$, and set $\varphi_n = \varphi|_{Q_n}$. Let $\mathrm{Map}(BQ, |\mathcal{L}_0|^\wedge_p)_{\widehat{\varphi}}$ be the space of maps $f\colon BQ \longrightarrow |\mathcal{L}_0|^\wedge_p$ such that $f|_{BQ_n} \simeq B\varphi_n$ for each n. (This contains the connected component of $B\varphi$, but could, *a priori*, contain other components.) Let $(\mathcal{L}_0^Q)_\varphi \subseteq \mathcal{L}_0^Q$ and $(\mathcal{L}_0^{Q_n})_\varphi \subseteq \mathcal{L}_0^{Q_n}$ be the full subcategories with objects those (P, α) such that α is $\mathcal{F}$–conjugate to φ or to φ_n, respectively. Thus $|(\mathcal{L}_0^Q)_\varphi|$ is the connected component of $|\mathcal{L}_0^Q|$ which contains (S, φ), and $|(\mathcal{L}_0^{Q_n})_\varphi|$ is the connected component which contains (S, φ_n).

Consider the following commutative diagram, for all $n \geq 1$:

$$\tag{17} \begin{tikzcd} |(\mathcal{L}_0^Q)_\varphi|_p^\wedge \arrow[r] \arrow[d] & \mathrm{Map}(BQ, |\mathcal{L}_0|_p^\wedge)_{\widehat{\varphi}} \arrow[d] \\ |(\mathcal{L}_0^{Q_n})_\varphi|_p^\wedge \arrow[r, "\simeq"] & \mathrm{Map}(BQ_n, |\mathcal{L}_0|_p^\wedge)_{\varphi_n} \end{tikzcd}$$

We want to show that the top row is a homotopy equivalence; the proposition then follows by taking the union of such maps as φ runs through representatives of all elements of $\mathrm{Rep}(Q,\mathcal{L})$. The bottom row is a homotopy equivalence by Step 1. So we will be done if we can show that the vertical maps are homotopy equivalences for n large enough.

By Lemma 6.1(b), there is some m such that for all $n \geq m$, $\varphi(Q_n)^\bullet = \varphi(Q)^\bullet$, and restriction induces a bijection $\mathrm{Rep}_{\mathcal{F}}(\varphi(Q),P) \cong \mathrm{Rep}_{\mathcal{F}}(\varphi(Q_n),P)$ for all $P \in \mathrm{Ob}(\mathcal{L}_0)$. (Recall that we are assuming $\mathcal{L}_0 = \mathcal{L}_0^\bullet$.) This implies that $|(\mathcal{L}_0^Q)_\varphi| \cong |(\mathcal{L}_0^{Q_n})_\varphi|$ for all $n \geq m$. Hence the components $\mathrm{Map}(BQ_n, |\mathcal{L}_0|_p^\wedge)_{B\varphi_n}$ are all homotopy equivalent for $n \geq m$ by Step 1, so $\mathrm{Map}(BQ, |\mathcal{L}|_p^\wedge)_{\widehat{\varphi}} \simeq \mathrm{Map}(BQ_n, |\mathcal{L}|_p^\wedge)_{B\varphi_n}$ for $n \geq m$, and this proves that the vertical maps in (17) are equivalences. □

The following theorem gives a more explicit description of the set $[BQ, |\mathcal{L}|_p^\wedge]$ of homotopy classes of maps, as well as of the individual components in certain cases.

Theorem 6.3 *Fix a p–local compact group $(S,\mathcal{F},\mathcal{L})$, and let $\theta\colon BS \to |\mathcal{L}|_p^\wedge$ be the natural inclusion followed by completion. Then the following hold, for any discrete p–toral group Q.*

(a) *The natural map*

$$\mathrm{Rep}(Q,\mathcal{L}) \xrightarrow{\ \cong\ } [BQ, |\mathcal{L}|_p^\wedge]$$

is a bijection. Thus each map $BQ \longrightarrow |\mathcal{L}|_p^\wedge$ is homotopic to $\theta \circ B\rho$ for some $\rho \in \mathrm{Hom}(Q,S)$. If $\rho, \rho' \in \mathrm{Hom}(Q,S)$ are such that $\theta \circ B\rho \simeq \theta \circ B\rho'$ as maps from BQ to $|\mathcal{L}|_p^\wedge$, then there is $\chi \in \mathrm{Hom}_{\mathcal{F}}(\rho(Q),\rho'(Q))$ such that $\rho' = \chi \circ \rho$.

(b) *For each $\rho \in \mathrm{Hom}(Q,S)$ such that $\rho(Q)$ is $\mathcal{F}$–centric, the composite*

$$BZ(\rho(Q)) \times BQ \xrightarrow{\mathrm{incl}\cdot B\rho} BS \xrightarrow{\ \theta\ } |\mathcal{L}|_p^\wedge$$

induces a homotopy equivalence

$$BZ(\rho(Q))_p^\wedge \xrightarrow{\ \simeq\ } \mathrm{Map}(BQ, |\mathcal{L}|_p^\wedge)_{\theta \circ B\rho}.$$

(c) *The evaluation map induces a homotopy equivalence*

$$\mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)_{\mathrm{triv}} \simeq |\mathcal{L}|^\wedge_p.$$

Proof We refer to the category $\mathcal{L}^Q$ and to the homotopy equivalence

$$|\Phi|' \colon |\mathcal{L}^Q|^\wedge_p \xrightarrow{\ \simeq\ } \mathrm{Map}(BQ, |\mathcal{L}|^\wedge_p)$$

of Proposition 6.2. Point (a) is an immediate consequence of point (12) in the proposition, and (c) holds since the component of $\mathcal{L}_Q$ which contains the objects $(P,1)$ is equivalent to $\mathcal{L}$.

If $\rho \in \mathrm{Hom}(Q,S)$ is such that $\rho(Q)$ is $\mathcal{F}$–centric, then the connected component of $|\mathcal{L}_Q|$ which contains the vertex $(\rho(Q),\rho)$ contains as deformation retract the nerve of the full subcategory with that as its only object. Since $\mathrm{Aut}_{\mathcal{L}_Q}(\rho(Q),\rho) \cong Z(\rho(Q))$, this component has the homotopy type of $BZ(\rho(Q))$, which proves point (b). □

7 Equivalences of classifying spaces

We next describe the monoid $\mathrm{Aut}(|\mathcal{L}|^\wedge_p)$ of self homotopy equivalences of $|\mathcal{L}|^\wedge_p$ in Theorem 7.1; and also show that p–local compact groups which have homotopy equivalent classifying spaces are themselves isomorphic (Theorem 7.4). There is some overlap between the proofs in this section and those of the corresponding results for p–local finite groups in [7, Sections 8 & 7]; but they differ in some key respects, mostly due to the fact that we do not have a way to recover the category $\mathcal{L}$ from the space $|\mathcal{L}|^\wedge_p$ via a functor from spaces to categories.

We first recall some notation from [6] and [7]. For any space X, $\mathrm{Aut}(X)$ denotes the monoid of self homotopy equivalences of X, and $\mathrm{Out}(X) = \pi_0(\mathrm{Aut}(X))$ is the group of homotopy classes of self equivalences. For any discrete category $\mathcal{C}$, $\mathcal{A}ut(\mathcal{C})$ is the category whose objects are the self equivalences of $\mathcal{C}$ and whose morphisms are the natural isomorphisms between self equivalences, and $\mathrm{Out}(\mathcal{C}) = \pi_0(|\mathcal{A}ut(\mathcal{C})|)$ is the group of isomorphism classes of self equivalences. We consider $\mathcal{A}ut(\mathcal{C})$ as a discrete strict monoidal category, in the sense that composition defines a strictly associative functor

$$\mathcal{A}ut(-) \times \mathcal{A}ut(-) \longrightarrow \mathcal{A}ut(-)$$

with strict identity. The nerve of $\mathcal{A}ut(\mathcal{C})$ is thus a simplicial monoid, and its realization $|\mathcal{A}ut(\mathcal{C})|$ is a topological monoid.

Consider the evaluation functor

$$\mathrm{ev} \colon \mathcal{A}ut(\mathcal{C}) \times \mathcal{C} \longrightarrow \mathcal{C}$$

which sends a pair of objects (Ψ, c) to $\Psi(c) \in \mathrm{Ob}(\mathcal{C})$, and which is defined on morphisms by setting

$$\mathrm{ev}\Big(\Psi \xrightarrow{\ \chi\ } \Psi', c \xrightarrow{\ \varphi\ } d\Big) = \Big(\Psi(c) \xrightarrow[=\chi(d)\circ\Psi(\varphi)]{\Psi'(\varphi)\circ\chi(c)} \Psi'(d)\Big).$$

Upon taking geometric realizations, this defines a map of spaces from $|\mathcal{A}ut(\mathcal{C})| \times |\mathcal{C}|$ to $|\mathcal{C}|$, which is adjoint to a homomorphism of topological groups

$$\Omega_{\mathcal{C}}\colon |\mathcal{A}ut(\mathcal{C})| \longrightarrow \mathrm{Aut}(|\mathcal{C}|).$$

Recall that part of the structure of a centric linking system $\mathcal{L}$ associated to a fusion system is a homomorphism $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each P in $\mathcal{L}$. We write $P_{\boldsymbol{\delta}} = \mathrm{Im}(\delta_{\boldsymbol{P}})$, which we think of as a "distinguished subgroup" of $\mathrm{Aut}_{\mathcal{L}}(P)$ which can be identified with P. For the purposes of this paper, an equivalence of categories $\mathcal{L} \xrightarrow{\Psi} \mathcal{L}$ will be called *isotypical* if for each P, $\Psi_{\boldsymbol{P},\boldsymbol{P}}$ sends the subgroup $P_{\boldsymbol{\delta}} \le \mathrm{Aut}_{\mathcal{L}}(P)$ to the subgroup $\Psi(P)_{\boldsymbol{\delta}} \le \mathrm{Aut}_{\mathcal{L}}(\Psi(P))$. Let $\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})$ be the full subcategory of $\mathcal{A}ut(\mathcal{L})$ whose objects are the isotypical equivalences, and set $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) = \pi_0(|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})|)$.

By [7, Lemma 8.2], when $\mathcal{L}$ is a linking system over a *finite* p–group, an equivalence $\Psi\colon \mathcal{L} \longrightarrow \mathcal{L}$ is isotypical if and only if the triangle involving Ψ and the forgetful functor from $\mathcal{L}$ to groups commutes up to natural isomorphism. The same proof applies for linking systems over discrete p–toral groups, although we won't be using that here.

Clearly, any equivalence which is naturally isomorphic to an isotypical equivalence is itself isotypical, and any inverse to an isotypical equivalence (inverse up to natural isomorphism of functors) is also isotypical. The subcategory $\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})$ is thus a union of connected components of $\mathcal{A}ut(\mathcal{L})$, and $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ is a subgroup of $\mathrm{Out}(\mathcal{L})$.

The main result of this section is the following theorem:

Theorem 7.1 *Fix a p–local compact group $(S, \mathcal{F}, \mathcal{L})$, and set $\Omega = \Omega_{\mathcal{L}}$. Then the composite*

$$\Omega^\wedge_p\colon |\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})| \xrightarrow{\ \Omega\ } \mathrm{Aut}(|\mathcal{L}|) \xrightarrow{(-)^\wedge_p} \mathrm{Aut}(|\mathcal{L}|^\wedge_p)$$

induces a homotopy equivalence of topological monoids from

$$|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})|^\wedge_p \quad \text{to} \quad \mathrm{Aut}(|\mathcal{L}|^\wedge_p).$$

In particular, if we let $\pi_i(B\mathcal{Z}^\wedge_p)$ denote the functor $\mathcal{O}^c(\mathcal{F})^{\mathrm{op}} \to \mathcal{A}b$ which sends P to $\pi_i(BZ(P)^\wedge_p)$ (each $i \ge 1$), then

$$\mathrm{Out}(|\mathcal{L}|^\wedge_p) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}),$$

$$\pi_i(\mathrm{Aut}(|\mathcal{L}|^\wedge_p)) \cong \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^0(\pi_i(B\mathcal{Z}^\wedge_p)) \ \text{ for } i = 1, 2, \quad \text{and} \quad \pi_i(\mathrm{Aut}(|\mathcal{L}|^\wedge_p)) = 0 \ \text{ for } i \ge 3.$$

Proof We prove the isomorphism between groups of components in Step 2, and the homotopy equivalence between the individual components in Step 3. In Step 1, we outline the general procedure for describing the mapping space $\mathrm{Aut}(|\mathcal{L}|_p^\wedge)$.

Assume we have fixed inclusion morphisms $\iota_P \in \mathrm{Mor}_{\mathcal{L}}(P,S)$ for each P. If Ψ is an isotypical self equivalence of $\mathcal{L}$, then clearly $\Psi(S)=S$, and hence $\Psi_{S,S}$ is an automorphism of $\mathrm{Aut}_{\mathcal{L}}(S)$ which sends S_δ $(=\mathrm{Im}(\delta_S))$ to itself. Set

$$\psi = \delta_S^{-1}\circ \Phi_{S,S}|_{S_\delta}\circ\delta_S \in \mathrm{Aut}(S).$$

For each $P\in\mathrm{Ob}(\mathcal{L})$, axiom (C) and the functoriality of Ψ imply that the following diagram commutes for all $g\in P$:

$$\begin{tikzcd}
\Psi(P) \arrow[rr, "\Psi(\iota_P)"] \arrow[d, "\Psi(\delta_P(g))"', "\in\Psi(P)_\delta"] & & S \arrow[d, "\Psi(\delta_S(g))=\delta_S(\psi(g))"] \\
\Psi(P) \arrow[rr, "\Psi(\iota_P)"] & & S
\end{tikzcd}$$

Hence $\pi(\Psi(\iota_P))(\Psi(P))=\psi(P)$ (by axiom (C) again). So $\Psi(\iota_P)=\iota_{\psi(P)}\circ\alpha_P$ for a unique $\alpha_P\in\mathrm{Iso}_{\mathcal{L}}(\Psi(P),\psi(P))$ by Lemma 4.3(a). Thus Ψ is naturally isomorphic to an automorphism Ψ' of $\mathcal{L}$ such that $\Psi'_{S,S}=\Psi_{S,S}$, and $\Psi'(P)=\psi(P)$ and $\Psi'_{P,S}(\iota_P)=\iota_{\psi(P)}$ for each P. This shows that every object in $\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})$ is isomorphic to an isotypical *automorphism* of $\mathcal{L}$ which sends inclusions to inclusions, and from now on we restrict attention to such automorphisms.

Step 1 Consider the decomposition

$$\mathrm{pr}\colon \underset{\mathcal{O}^c(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}(\widetilde{B}) \xrightarrow{\ \simeq\ } |\mathcal{L}|$$

of Proposition 4.6(a), where $\widetilde{B}\colon \mathcal{O}^c(\mathcal{F})\longrightarrow \mathrm{Top}$ is a rigidification of the homotopy functor $P\mapsto BP$. In the following constructions, we regard $\underrightarrow{\mathrm{hocolim}}(\widetilde{B})$ as the union of skeleta:

$$\underset{\mathcal{O}^c(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}{}^{(n)}(\widetilde{B}) = \Big(\coprod_{i=0}^{n}\ \coprod_{P_0\to\cdots\to P_n}\widetilde{B}(P_0)\times D^i\Big)\Big/\sim$$

where we divide out by the usual face and degeneracy relations.

Define functors $\mathcal{Z},\mathcal{Z}_0\colon \mathcal{O}^c(\mathcal{F})^{\mathrm{op}}\longrightarrow \mathrm{Ab}$ and $B\mathcal{Z}_p^\wedge\colon \mathcal{O}^c(\mathcal{F})^{\mathrm{op}}\longrightarrow\mathrm{Top}$ by setting

$$\mathcal{Z}(P)=Z(P),\qquad \mathcal{Z}_0(P)=Z(P)_0,\qquad\text{and}\qquad B\mathcal{Z}_p^\wedge(P)=BZ(P)_p^\wedge,$$

and by sending $[\varphi] \in \mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ to $\varphi^{-1}|_{Z(Q)}$ or $B\big(\varphi^{-1}|_{Z(Q)}\big)^\wedge_p$. For any element

$$\mathbf{f} = \big(f_P\big)_{P\in\mathcal{O}^c(\mathcal{F})} \in \varprojlim_{\mathcal{O}^c(\mathcal{F})} [B-, |\mathcal{L}|^\wedge_p],$$

let $\mathrm{Map}(|\mathcal{L}|^\wedge_p, |\mathcal{L}|^\wedge_p)_{\mathbf{f}}$ be the union of the components of the mapping space which restrict to $\mathbf{f}$. By Wojtkowiak [30], the obstructions to this space being nonempty lie in the groups

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^{i+1}\big(\pi_i(\mathrm{Map}(B-, |\mathcal{L}|^\wedge_p)_{f_-})\big) \cong \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^{i+1}(\pi_i(B\mathcal{Z}^\wedge_p))$$

for $i \geq 1$; the functor vanishes for $i > 2$, and the higher limits vanish for $i = 2$ by Proposition 5.8. Also, if $\mathrm{Map}(|\mathcal{L}|, |\mathcal{L}|^\wedge_p)_{\mathbf{f}} \neq \varnothing$, then the filtration of the mapping space

$$\mathrm{Map}(|\mathcal{L}|^\wedge_p, |\mathcal{L}|^\wedge_p) \simeq \mathrm{Map}\Big(\operatorname*{hocolim}_{\overrightarrow{\mathcal{O}^c(\mathcal{F})}}(\widetilde{B}),\ |\mathcal{L}|^\wedge_p\Big)$$

by the skeleta of the homotopy colimit defines a spectral sequence with E^2–term

$$E^2_{-i,j} = \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^{i}\big(\pi_j(\mathrm{Map}(B-, |\mathcal{L}|^\wedge_p)_{f_-})\big),$$

which converges to $\pi_{j-i}\big(\mathrm{Map}(|\mathcal{L}|, |\mathcal{L}|^\wedge_p)_{\mathbf{f}}\big)$.

By Theorem 6.3(b),

$$\pi_j(\mathrm{Map}(B-, |\mathcal{L}|^\wedge_p)_{f_-}) \cong \pi_j(BZ(-)^\wedge_p) \cong \begin{cases} \mathcal{Z}/\mathcal{Z}_0 & \text{if } j=1 \\ \pi_2(B\mathcal{Z}^\wedge_p) & \text{if } j=2 \\ 0 & \text{if } j \geq 3. \end{cases}$$

Since $\pi_2(B\mathcal{Z}^\wedge_p)$ is acyclic by Proposition 5.8, the only obstruction to the space $\mathrm{Map}(|\mathcal{L}|^\wedge_p, |\mathcal{L}|^\wedge_p)_{\mathbf{f}}$ being nonempty lies in $\varprojlim^2(\mathcal{Z}/\mathcal{Z}_0)$; while the spectral sequence takes the form

$$E^2_{-i,j} \cong \begin{cases} \varprojlim^i(\mathcal{Z}/\mathcal{Z}_0) & \text{if } j=1 \\ \varprojlim^0(\pi_2(B\mathcal{Z}^\wedge_p)) & \text{if } (i,j)=(0,2) \\ 0 & \text{otherwise.} \end{cases}$$

Step 2 Let $\mathrm{Aut_{fus}}(S)$ be the group of fusion preserving automorphisms of S; ie the group of those $\alpha \in \mathrm{Aut}(S)$ which induce an automorphism of the fusion system $\mathcal{F}$ by sending P to $\alpha(P)$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ to $(\alpha|_Q)\circ\varphi\circ(\alpha|_P)^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\alpha(P),\alpha(Q))$.

The proof that $\operatorname{Out}(|\mathcal{L}|_p^\wedge) \cong \operatorname{Out}_{\mathrm{typ}}(\mathcal{L})$ is based on the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & \varprojlim^1(\mathcal{Z}) \arrow[r, "\lambda'"] \arrow[d, "\omega_1"', "\cong"] & \operatorname{Out}_{\mathrm{typ}}(\mathcal{L}) \arrow[r, "\mu'"] \arrow[d, "\pi_0(\Omega_p^\wedge)"'] & \operatorname{Out}_{\mathrm{fus}}(S) \arrow[r, "\omega'"] \arrow[d, "\omega_2"', "\cong"] & \varprojlim^2(\mathcal{Z}) \arrow[d, "\omega_3"', "\cong"] \\
1 \arrow[r] & \varprojlim^1(\mathcal{Z}/\mathcal{Z}_0) \arrow[r, "\lambda"] & \operatorname{Out}(|\mathcal{L}|_p^\wedge) \arrow[r, "\mu"] & \varprojlim \operatorname{IRep}(-,\mathcal{F}) \arrow[r, "\omega"] & \varprojlim^2(\mathcal{Z}/\mathcal{Z}_0)
\end{tikzcd} \tag{18}$$

Here, $\operatorname{IRep}(P,\mathcal{F}) \subseteq \operatorname{Rep}(P,\mathcal{F})$ denotes the set of classes of injective homomorphisms. All limits are taken over $\mathcal{O}^c(\mathcal{F})$, and ω_1 and ω_3 are induced by the natural surjection of functors from $\mathcal{Z}$ onto $\mathcal{Z}/\mathcal{Z}_0$. They are isomorphisms since $\varprojlim^i(\mathcal{Z}_0) = 0$ for all $i \geq 1$ (Proposition 5.8). Also, ω_2 is induced by the inclusion of the outer automorphisms $\operatorname{Out}_{\mathrm{fus}}(S) = \operatorname{Aut}_{\mathrm{fus}}(S)/\operatorname{Aut}_{\mathcal{F}}(S)$ into $\operatorname{IRep}(S,\mathcal{F}) = \operatorname{Aut}(S)/\operatorname{Aut}_{\mathcal{F}}(S)$, and by definition of fusion preserving $\operatorname{Im}(\omega_2) = \varprojlim \operatorname{IRep}(-,\mathcal{F})$ (thus ω_2 is a bijection). It remains to define the two rows, and prove that they are exact and the diagram commutes. It will then follow immediately that $\pi_0(\Omega_p^\wedge)$ is an isomorphism. Note that this does not require us to know that $\varprojlim \operatorname{IRep}(-,\mathcal{F})$ is a group or that ω' is a homomorphism; only that $\operatorname{Im}(\mu) = \omega^{-1}(0)$, $\operatorname{Im}(\mu') = \omega'^{-1}(0)$, and the inverse image under μ of each element in the target is a coset of $\operatorname{Im}(\lambda)$.

We first consider the top row, where μ' is defined by restricting an isotypical equivalence of $\mathcal{L}$ to the image of δ_S. Any fusion preserving automorphism $\alpha \in \operatorname{Aut}_{\mathrm{fus}}(S)$ defines an isotypical automorphism $\overline{\alpha}$ of $\mathcal{F}$, and $\omega'(\alpha)$ is the obstruction of [7, Proposition 3.1] to lifting $\overline{\alpha}$ to an automorphism of $\mathcal{L}$. (The proof in [7] applies without change to the case of a linking system over a discrete p–toral group.) Finally, the description of $\operatorname{Ker}(\mu')$ is identical to that shown in [6, Theorem 6.2]. More specifically, a reduced 1–cocycle $\varepsilon \in Z^1(\mathcal{O}^c(\mathcal{F}); \mathcal{Z})$ sends each morphism $[\varphi] \in \operatorname{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q)$ to $\varepsilon(\varphi) \in Z(P)$ (where $\varepsilon(\mathrm{Id}_P) = 1$), and $\lambda'([\varepsilon])$ is represented by the automorphism $A_\varepsilon \in \operatorname{Aut}(\mathcal{L})$ defined by setting $A_\varepsilon(P) = P$ for all P, and $A_\varepsilon(\psi) = \psi \circ \delta_P(\varepsilon([\pi(\psi)]))^{-1}$ for all $\psi \in \operatorname{Mor}_{\mathcal{L}}(P,Q)$. This proves the exactness of the top row.

As for the bottom row in (18), let μ be the homomorphism defined by restriction:

$$\mu\colon \operatorname{Out}(|\mathcal{L}|_p^\wedge) \xrightarrow{\mathrm{Res}} [BS, |\mathcal{L}|_p^\wedge] \cong \operatorname{IRep}(S,\mathcal{F}).$$

We want to compare

$$\operatorname{Map}(|\mathcal{L}|_p^\wedge, |\mathcal{L}|_p^\wedge) \simeq \operatorname{Map}\Big(\underset{\mathcal{O}^c(\mathcal{F})}{\operatorname{hocolim}}(\widetilde{B}),\ |\mathcal{L}|_p^\wedge\Big)$$

with
$$\varprojlim_{\mathcal{O}^c(\mathcal{F})} [B(-), |\mathcal{L}|_p^\wedge] \cong \varprojlim_{\mathcal{O}^c(\mathcal{F})} \operatorname{IRep}(-,\mathcal{F}).$$

By Step 1, the only obstruction to extending any given α in this last set to an automorphism of $|\mathcal{L}|^\wedge_p$ lies in $\varprojlim^2(\mathcal{Z}/\mathcal{Z}_0)$, while if there are liftings, then the set of homotopy classes is in bijective correspondence with $\varprojlim^1(\mathcal{Z}/\mathcal{Z}_0)$. This proves the exactness of the bottom row in the sense explained above.

The second square in (18) clearly commutes. To prove that the first square commutes, fix some $\varepsilon \in Z^1(\mathcal{O}^c(\mathcal{F}); \mathcal{Z})$. Then $\lambda'([\varepsilon]) = [A_\varepsilon]$ where $A_\varepsilon \in \mathrm{Aut}(\mathcal{L})$ is the automorphism defined above; and $|A_\varepsilon| \in \mathrm{Aut}(|\mathcal{L}|)$ sends each $BP \subseteq B(\mathrm{Aut}_{\mathcal{L}}(P)) \subseteq |\mathcal{L}|$ to $|\mathcal{L}|$ by the inclusion. For each $\varphi \in \mathrm{Hom}_{\mathcal{F}^c}(P, Q)$, let $\mathcal{C}_\varphi \subseteq \mathcal{L}$ be the subcategory with two objects P and Q, whose morphisms are those morphisms in $\mathcal{L}$ which get sent to $[\mathrm{Id}_P]$, $[\mathrm{Id}_Q]$, or $[\varphi]$ in $\mathcal{O}^c(\mathcal{F})$. Then $|\mathcal{C}_\varphi| \subseteq |\mathcal{L}|$ is homeomorphic to the mapping cylinder of $B\varphi\colon BP \longrightarrow BQ$; and $|A_\varepsilon|$ sends $|\mathcal{C}_\varphi|$ to itself by a map which differs from the identity via a loop in $\mathrm{Map}(BP, BQ)_{B\varphi} \simeq BZ(P)$ which represents the element $\varepsilon([\varphi]) \in Z(P)$. After taking the p–completion, this shows that $[|A_\varepsilon|^\wedge_p] = \lambda([\overline{\varepsilon}])$, where $\overline{\varepsilon} \in Z^1(\mathcal{O}^c(\mathcal{F}); \mathcal{Z}/\mathcal{Z}_0)$ is the class of ε modulo $\mathcal{Z}_0$. This proves that the first square in (18) commutes.

Fix $\alpha \in \mathrm{Out}_{\mathrm{fus}}(S)$, and let $\widehat{\alpha}$ be the automorphism of the fusion system $\mathcal{F}$ induced by α. Choose maps

$$\mathrm{Mor}_{\mathcal{L}}(P, Q) \xrightarrow{\alpha^*_{P,Q}} \mathrm{Mor}_{\mathcal{L}}(\alpha(P), \alpha(Q))$$

which lift those defined by $\widehat{\alpha}$; then $\omega'(\alpha)$ is the class of the 2–cocycle $\beta \in Z^2(\mathcal{O}^c(\mathcal{F}); \mathcal{Z})$ which measures the deviation of the $\alpha^*_{P,Q}$ from defining a functor. These same liftings $\alpha^*_{P,Q}$ allow us to define a map of spaces

$$\alpha^*\colon \underset{\mathcal{O}^c(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}{}^{(1)}(\widetilde{B}) \longrightarrow |\mathcal{L}|,$$

and the obstruction to extending this to $\underrightarrow{\mathrm{hocolim}}^{(2)}(\widetilde{B})$ is precisely the class of the same 2–cocycle β: but regarded as a 2–cocycle with coefficients in

$$\mathcal{Z}/\mathcal{Z}_0 \cong \pi_1(\mathrm{Map}(B-, |\mathcal{L}|^\wedge_p)_\alpha).$$

This proves that the third square commutes, and finishes the proof that $\pi_0(\Omega^\wedge_p)$ is an isomorphism.

Step 3 Set $Z(\mathcal{F}) = \varprojlim(\mathcal{Z})$, regarded as a subgroup of S. Let

$$\lambda\colon \mathcal{B}(Z(\mathcal{F})) \times \mathcal{L} \longrightarrow \mathcal{L}$$

be the functor which sends $(x, P \xrightarrow{\varphi} Q)$ to $\varphi \circ \delta_P(x)$. This is adjoint to a functor from $\mathcal{B}(Z(\mathcal{F}))$ to $\mathcal{A}ut(\mathcal{L})$, which in turn induces a map

$$\eta'\colon BZ(\mathcal{F}) \longrightarrow |\mathcal{A}ut(\mathcal{L})|_{\mathrm{Id}}$$

upon taking geometric realizations. On the other hand, if we first take geometric realizations, then p–complete, and then take the adjoint, we get a map η from $BZ(\mathcal{F})^\wedge_p$ to $\mathrm{Aut}(|\mathcal{L}|^\wedge_p)_{\mathrm{Id}}$. These maps now fit together in the following commutative square:

$$
\begin{tikzcd}
BZ(\mathcal{F}) \arrow[r, "\eta'", "\simeq"'] \arrow[d, "(-)^\wedge_p"'] & {|\mathcal{A}ut(\mathcal{L})|_{\mathrm{Id}}} \arrow[d, "\Omega"'] \\
BZ(\mathcal{F})^\wedge_p \arrow[r, "\eta", "\simeq"'] & \mathrm{Aut}(|\mathcal{L}|^\wedge_p)_{\mathrm{Id}}
\end{tikzcd}
\tag{19}
$$

Since we are restricting attention to automorphisms of $\mathcal{L}$ (as opposed to working with all equivalences), $\mathcal{A}ut(\mathcal{L})$ is a groupoid, and so $\pi_1(|\mathcal{A}ut(\mathcal{L})|)$ is the group of natural isomorphisms of functors from $\mathrm{Id}_{\mathcal{L}}$ to itself. A natural equivalence α sends each object P to an element $\alpha(P)\in\mathrm{Aut}_{\mathcal{L}}(P)$, such that for each $\varphi\in\mathrm{Mor}_{\mathcal{L}}(P,Q)$, $\varphi\circ\alpha(P)=\alpha(Q)\circ\varphi$. In particular, upon restricting to the case $P=Q$ and $\varphi\in\delta_P(P)$, we see that $\pi(\alpha(P))=\mathrm{Id}_P$ for each P, and thus $\alpha(P)\in\delta_P(Z(P))\cong\mathcal{Z}(P)$. The other relations are equivalent to requiring that $\alpha\in\varprojlim^0(\mathcal{Z})=Z(\mathcal{F})$. This proves that $\pi_1(|\mathcal{A}ut(\mathcal{L})|)\cong Z(\mathcal{F})$; and since $|\mathcal{A}ut(\mathcal{L})|_{\mathrm{Id}}$ is aspherical, shows that η' is a homotopy equivalence.

The E^2–term of the spectral sequence for maps defined on a homotopy colimit was described in Step 1: it vanishes except for the row coming from $\varprojlim^*(\mathcal{Z}/\mathcal{Z}_0)$, and the position $E^2_{0,2}\cong\varprojlim^0(\pi_2(B\mathcal{Z}^\wedge_p))$. Hence from the spectral sequence, one sees immediately that for $i\geq 1$,

$$\pi_i(\mathrm{Aut}(|\mathcal{L}|^\wedge_p))\cong\varprojlim_{\mathcal{O}^c(\mathcal{F})}(\pi_i(B\mathcal{Z}^\wedge_p))\cong\pi_i(BZ(\mathcal{F})^\wedge_p).$$

By naturality, these isomorphisms are induced by η, and so η is a homotopy equivalence.

It now follows from (19) and from Step 1 that $\Omega^\wedge_p$ induces a homotopy equivalence $\mathrm{Aut}(|\mathcal{L}|^\wedge_p)\simeq|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})|^\wedge_p$. □

We also note here the following result, which was shown while proving Theorem 7.1.

Proposition 7.2 *For any p–local compact group $(S,\mathcal{F},\mathcal{L})$, there is an exact sequence*

$$0\longrightarrow\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^1(\mathcal{Z}/\mathcal{Z}_0)\longrightarrow\mathrm{Out}(|\mathcal{L}|^\wedge_p)\longrightarrow\mathrm{Out}_{\mathrm{fus}}(S)\longrightarrow\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^2(\mathcal{Z}/\mathcal{Z}_0),$$

where $\mathcal{Z}_0\subseteq\mathcal{Z}\colon\mathcal{O}^c(\mathcal{F})^{\mathrm{op}}\longrightarrow\mathrm{Ab}$ are the functors $\mathcal{Z}(P)=Z(P)$ and $\mathcal{Z}_0(P)=Z(P)_0$.

In Section 9, we will show that for any compact Lie group G, there is a p–local compact group $(S,\mathcal{F},\mathcal{L})=(S,\mathcal{F}_S(G),\mathcal{L}^c_S(G))$ such that $|\mathcal{L}|^\wedge_p\simeq BG^\wedge_p$. Hence when

G is connected, the exact sequence of Proposition 7.2 gives a new way to describe $\mathrm{Out}(BG^\wedge_p)$, which is different from but closely related to the descriptions in [19; 20] and [22].

We now turn our attention to maps between p–completed nerves of different linking systems. We first look at the case where the linking systems in question are associated to the same fusion system. As usual, when we talk about an isomorphism of linking systems, we mean an isomorphism of categories which is natural with respect to the projections to the fusion system and with respect to the distinguished monomorphisms.

Lemma 7.3 *Let $\mathcal{F}$ be a saturated fusion system over a discrete p–toral group S, and let $\mathcal{F}_0 \subseteq \mathcal{F}^c$ be any full subcategory which contains $\mathcal{F}^{c\bullet}$. Let $\mathcal{L}_0$ and $\mathcal{L}_0'$ be two linking systems associated to $\mathcal{F}_0$. Assume that there is a map $f\colon |\mathcal{L}_0|^\wedge_p \longrightarrow |\mathcal{L}_0'|^\wedge_p$ such that the triangle*

$$\begin{tikzcd} & BS \arrow[dl, "\theta"'] \arrow[dr, "\theta'"] & \\ |\mathcal{L}_0|^\wedge_p \arrow[rr, "f"] & & |\mathcal{L}_0'|^\wedge_p \end{tikzcd}$$

is homotopy commutative. Here, θ and θ' are the maps induced by the inclusion of $\mathcal{B}(S)$ into $\mathcal{L}_0$ or $\mathcal{L}_0'$. Then $\mathcal{L}_0$ and $\mathcal{L}_0'$ are isomorphic linking systems associated to $\mathcal{F}_0$. Furthermore, we can choose an isomorphism $\mathcal{L}_0 \xrightarrow{\cong} \mathcal{L}_0'$ of linking systems that induces f on p–completed nerves.

Proof Let $\mathbf{ke}(\mathcal{L}_0)$ and $\mathbf{ke}(\mathcal{L}_0')$ be the left homotopy Kan extensions of the constant point functors along the projections $\widetilde{\pi}_0\colon \mathcal{L}_0 \to \mathcal{O}(\mathcal{F}_0)$ and $\widetilde{\pi}_0'\colon \mathcal{L}_0' \to \mathcal{O}(\mathcal{F}_0)$ respectively. Let $\kappa_P\colon \mathbf{ke}(\mathcal{L}_0)(P) \longrightarrow |\mathcal{L}_0|^\wedge_p$ be induced by the forgetful functor from $\widetilde{\pi}_0{\downarrow}P$ to $\mathcal{L}_0$, and similarly for $\kappa_P'\colon \mathbf{ke}(\mathcal{L}_0)(P) \longrightarrow |\mathcal{L}_0'|^\wedge_p$. Then θ and θ' factor through κ_S and κ_S', and we have a homotopy commutative diagram:

$$\begin{tikzcd} & \mathbf{ke}(\mathcal{L}_0)(P) \arrow[r] & \mathbf{ke}(\mathcal{L}_0)(S) \arrow[r, "\kappa_S"] & |\mathcal{L}_0|^\wedge_p \arrow[dd, "f"] \\ BP \arrow[ur, "\simeq"] \arrow[r] \arrow[dr, "\simeq"'] & BS \arrow[ur, "\simeq"] \arrow[urr, "\theta"'] \arrow[drr, "\theta'"] \arrow[dr, "\simeq"'] & & \\ & \mathbf{ke}(\mathcal{L}_0')(P) \arrow[r] & \mathbf{ke}(\mathcal{L}_0')(S) \arrow[r, "\kappa_S'"] & |\mathcal{L}_0'|^\wedge_p \end{tikzcd}$$

Hence the maps $f_P\colon \mathbf{ke}(\mathcal{L}_0)(P) \longrightarrow |\mathcal{L}_0'|^\wedge_p$ and $f_P'\colon \mathbf{ke}(\mathcal{L}_0')(P) \longrightarrow |\mathcal{L}_0'|^\wedge_p$, defined as the obvious composites shown in the above diagram satisfy the following:

(a) The composites
$$BP \xrightarrow{\simeq} \mathbf{ke}(\mathcal{L}_0)(P) \xrightarrow{f_P} |\mathcal{L}_0'|{}_p^\wedge \quad \text{and} \quad BP \xrightarrow{\simeq} \mathbf{ke}(\mathcal{L}_0')(P) \xrightarrow{f_P'} |\mathcal{L}_0'|{}_p^\wedge$$
are homotopic, and are centric after p–completion by Theorem 6.3(b).

(b) $f_Q \circ \mathbf{ke}(\mathcal{L}_0)(\varphi) \simeq f_P$ and $f_Q' \circ \mathbf{ke}(\mathcal{L}_0')(\varphi) \simeq f_P'$ for each morphism $\varphi\colon P \longrightarrow Q$ of $\mathcal{O}(\mathcal{F}_0)$.

Thus $\mathbf{ke}(\mathcal{L}_0)_p^\wedge$ and $\mathbf{ke}(\mathcal{L}_0')_p^\wedge$ are equivalent rigidifications of $\mathbf{B}_p^\wedge$ by Corollary A.5; and so $\mathbf{ke}(\mathcal{L}_0)$ and $\mathbf{ke}(\mathcal{L}_0')$ are equivalent rigidifications of $\mathbf{B}$ by Proposition 5.9. Hence by Proposition 4.6, $\mathcal{L}_0$ and $\mathcal{L}_0'$ are isomorphic linking systems associated to $\mathcal{F}_0$.

More precisely, there is a third rigidification $\widetilde{B}$ of $\mathbf{B}$, and a commutative diagram of natural transformations between functors $\mathcal{O}(\mathcal{F}_0) \longrightarrow \mathrm{Top}$ of the following form:

$$\begin{tikzcd}
\mathbf{ke}(\mathcal{L}_0) \arrow[rr, "\psi"] \arrow[d] & & \widetilde{B} \arrow[d] & \mathbf{ke}(\mathcal{L}_0') \arrow[l, "\psi'"'] \arrow[d] \\
|\mathcal{L}_0|{}_p^\wedge \arrow[r, "f"] & |\mathcal{L}_0'|{}_p^\wedge \arrow[r, "f_1", "\simeq"'] & X & |\mathcal{L}_0'|{}_p^\wedge \arrow[l, "f_2"', "\simeq"]
\end{tikzcd}$$

Here, $\psi(P)$ and $\psi'(P)$ are homotopy equivalences for each P; X is some space homotopy equivalent to $|\mathcal{L}_0'|{}_p^\wedge$; all functors in the bottom row of the diagram are constant functors on $\mathcal{O}(\mathcal{F}_0)$ (sending all objects to the given space and all morphisms to the identity); and f_1 and f_2 are homotopy equivalences. Upon taking homotopy colimits of the functors in the top row, we get the homotopy commutative diagram:

$$\begin{tikzcd}
\underset{\mathcal{O}(\mathcal{F}_0)}{\underrightarrow{\mathrm{hocolim}}}\big(\mathbf{ke}(\mathcal{L}_0)\big) \arrow[rr, "\simeq"] \arrow[d, "\simeq"'] & & \underset{\mathcal{O}(\mathcal{F}_0)}{\underrightarrow{\mathrm{hocolim}}}(\widetilde{B}) \arrow[d] & \underset{\mathcal{O}(\mathcal{F}_0)}{\underrightarrow{\mathrm{hocolim}}}\big(\mathbf{ke}(\mathcal{L}_0')\big) \arrow[l, "\simeq"'] \arrow[d, "\simeq"'] \\
|\mathcal{L}_0|{}_p^\wedge \arrow[r, "f"] & |\mathcal{L}_0'|{}_p^\wedge \arrow[r, "f_1", "\simeq"'] & X & |\mathcal{L}_0'|{}_p^\wedge \arrow[l, "f_2"', "\simeq"]
\end{tikzcd}$$

Here, the left and right vertical maps are homotopy equivalences by Proposition 4.6(a). This proves that f is a homotopy equivalence. The last statement (an isomorphism $\mathcal{L}_0 \cong \mathcal{L}_0'$ can be chosen to induce f) now follows since by Theorem 7.1, every homotopy equivalence from $|\mathcal{L}_0'|{}_p^\wedge$ to itself is induced by some self equivalence of $\mathcal{L}_0'$. □

An *isomorphism* $(S,\mathcal{F},\mathcal{L}) \longrightarrow (S',\mathcal{F}',\mathcal{L}')$ of p–local compact groups consists of a triple $(\alpha,\alpha_{\mathcal{F}},\alpha_{\mathcal{L}})$, where
$$S \xrightarrow{\alpha} S', \qquad \mathcal{F} \xrightarrow{\alpha_{\mathcal{F}}} \mathcal{F}', \qquad \text{and} \qquad \mathcal{L} \xrightarrow{\alpha_{\mathcal{L}}} \mathcal{L}'$$

are isomorphisms of groups and categories which satisfy the following compatibility conditions:

(a) $\alpha_{\mathcal{F}}(P)=\alpha_{\mathcal{L}}(P)=\alpha(P)$ for all $P\le S$.

(b) $\alpha_{\mathcal{F}}$ and $\alpha_{\mathcal{L}}$ commute with the projections $\pi\colon \mathcal{L}\to\mathcal{F}$ and $\pi'\colon \mathcal{L}'\to\mathcal{F}'$.

(c) $\alpha_{\mathcal{L}}$ commutes with the distinguished monomorphisms $\delta_P\colon P\to \mathrm{Aut}_{\mathcal{L}}(P)$ and $\delta'_P\colon P\to \mathrm{Aut}_{\mathcal{L}'}(P)$.

We are now ready to show that the isomorphism class of a p–local compact group is determined by the homotopy type of its classifying space. This was shown for p–local finite groups in [7, Theorem 7.4].

Theorem 7.4 *If $(S,\mathcal{F},\mathcal{L})$ and $(S',\mathcal{F}',\mathcal{L}')$ are two p–local compact groups such that $|\mathcal{L}|^\wedge_p\simeq|\mathcal{L}'|^\wedge_p$, then $(S,\mathcal{F},\mathcal{L})$ and $(S',\mathcal{F}',\mathcal{L}')$ are isomorphic as p–local compact groups.*

Proof If $|\mathcal{L}|^\wedge_p\xrightarrow{f}|\mathcal{L}'|^\wedge_p$ is a homotopy equivalence, then by Theorem 6.3(a), there are homomorphisms $\alpha\in\mathrm{Hom}(S,S')$ and $\alpha'\in\mathrm{Hom}(S',S)$ such that the squares

$$\begin{tikzcd}
BS \arrow[r, "B\alpha"] \arrow[d, "\theta"'] & BS' \arrow[r, "B\alpha'"] \arrow[d, "\theta'"'] & BS \arrow[d, "\theta"'] \\
|\mathcal{L}|^\wedge_p \arrow[r, "f"] & |\mathcal{L}'|^\wedge_p \arrow[r, "f'"] & |\mathcal{L}|^\wedge_p
\end{tikzcd}$$

commute up to homotopy, where f' is any homotopy inverse to f. The composites $\alpha'\circ\alpha$ and $\alpha\circ\alpha'$ are $\mathcal{F}$–conjugate to Id_S and $\mathrm{Id}_{S'}$ by Theorem 6.3(a) again, and thus α is an isomorphism.

By yet another application of Theorem 6.3(a), for any $P,Q\le S$,

$$\mathrm{Hom}_{\mathcal{F}}(P,Q)=\bigl\{\varphi\in\mathrm{Inj}(P,Q)\bigm|\theta|_{BQ}\circ B\varphi\simeq\theta|_{BP}\bigr\}.$$

From this, and the corresponding result for $\mathrm{Hom}_{\mathcal{F}'}(\alpha(P),\alpha(Q))$, we see that α induces an isomorphism of categories from $\mathcal{F}$ to $\mathcal{F}'$.

Upon replacing S' and $\mathcal{F}'$ by S and $\mathcal{F}$, we can now assume that $\mathcal{L}$ and $\mathcal{L}'$ are two linking systems associated to $\mathcal{F}$, for which there is a homotopy equivalence $|\mathcal{L}|^\wedge_p\xrightarrow{f}|\mathcal{L}'|^\wedge_p$ such that $f\circ\theta\simeq\theta'$. Then $\mathcal{L}\cong\mathcal{L}'$ (as linking systems associated to $\mathcal{F}$) by Lemma 7.3. □

8 Fusion and linking systems of infinite groups

We now want to find some general conditions on an infinite group G which guarantee that we can associate to G a p–local compact group $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ such that $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$. This will be done in as much generality as possible. For example, we prove the saturation of the fusion system $\mathcal{F}_S(G)$ in sufficient generality so that the result also applies to the case where G is a compact Lie group.

At the end of the section, to show that the theory we have built up does contain some interesting examples, we show that it applies in particular to all linear torsion groups in characteristic different from p.

We say that a group G "has Sylow p–subgroups" if there is a discrete p–toral subgroup $S \le G$ which contains all discrete p–toral subgroups of G up to conjugacy. For any such G, we let $\mathrm{Syl}_p(G)$ be the set of such maximal discrete p–toral subgroups.

Lemma 8.1 *Fix a group G, a normal discrete p–toral subgroup $Q \lhd G$, and a subgroup $K \le G$ such that $G = QK$. Assume that K has Sylow p–subgroups. Then G has Sylow p–subgroups, and*

$$\mathrm{Syl}_p(G) = \{QS \mid S \in \mathrm{Syl}_p(K)\}.$$

Proof Let $\mathrm{Syl}'_p(G) = \{QS \mid S \in \mathrm{Syl}_p(K)\}$. All subgroups in $\mathrm{Syl}'_p(G)$ are G–conjugate since all subgroups in $\mathrm{Syl}_p(K)$ are K–conjugate. If $P \le G$ is an arbitrary discrete p–toral subgroup, then QP is also discrete p–toral (since Q and QP/Q are discrete p–toral), and

$$QP = QK \cap QP = Q \cdot (K \cap QP).$$

Thus $P \le QP \le QS \in \mathrm{Syl}'_p(G)$ for any $S \in \mathrm{Syl}_p(K)$ which contains $K \cap QP$. This shows that G has Sylow p–subgroups, and that they are precisely the subgroups in $\mathrm{Syl}'_p(G)$. □

We first establish some general conditions on an infinite group G with Sylow p–subgroups, which imply that $\mathcal{F}_S(G)$ is a saturated fusion system for $S \in \mathrm{Syl}_p(G)$. The following technical lemma will be needed when doing this.

Lemma 8.2 *Fix a group G, and normal subgroups $N, Q \lhd G$, with the following properties:*

(a) *Q is a discrete p–toral group.*

(b) *G/QN is a finite group.*

(c) *For each $H \le G$ such that $H \ge N$ and H/N is finite, H has Sylow p–subgroups.*

(d) *Each coset $gN \in G/N$ contains at least one element of finite order.*

Then G has Sylow p–subgroups. If $P \le G$ is any discrete p–toral subgroup, then $P \in \mathrm{Syl}_p(G)$ if and only if $P \ge Q$, $P \cap N \in \mathrm{Syl}_p(N)$ and $PN/QN \in \mathrm{Syl}_p(G/QN)$.

Proof Fix any $G' \le G$ such that $G' \ge QN$ and $G'/QN \in \mathrm{Syl}_p(G/QN)$. For every discrete p–toral subgroup $P \le G$, PQN/QN is conjugate to a subgroup of G'/QN, hence P is G–conjugate to a subgroup of G'. Hence G has Sylow p–subgroups if G' does, and in that case, $\mathrm{Syl}_p(G')$ is the set of subgroups of G' which are in $\mathrm{Syl}_p(G)$. It thus suffices to prove the lemma when $G = G'$; ie when G/QN is a finite p–group; and we assume this from now on.

Step 1 Assume first that $Q = 1$, and thus that $|G/N|$ is a finite p–group. Then G has Sylow p–subgroups by (c). Throughout this step, we fix some $S \in \mathrm{Syl}_p(G)$. We first prove that $NS = G$ (hence $NS/N \in \mathrm{Syl}_p(G/N)$) and $S \cap N \in \mathrm{Syl}_p(N)$. Afterwards, we prove the converse: $P \cap N \in \mathrm{Syl}_p(N)$ and $NP = G$ imply P is G–conjugate to S, and hence $P \in \mathrm{Syl}_p(G)$.

If $NS \lneqq G$, then $NS/N \lneqq G/N$, where the latter is a finite p–group. Since every proper subgroup of a p–group is contained in a proper normal subgroup, there is a proper normal subgroup $\widehat{N} \lhd G$ which contains NS. By (d), there is an element $g \in G \smallsetminus \widehat{N}$ of finite order. Write $|g| = mp^k$ where $p \nmid m$, and set $g' = g^m$. Then $g' \in G \smallsetminus \widehat{N}$ since $\widehat{N}$ has p–power index, and $|g'| = p^k$. This means that $\langle g' \rangle$ is a finite p–subgroup of G which is not conjugate to a subgroup of S, which contradicts the assumption that $S \in \mathrm{Syl}_p(G)$. Thus $NS = G$.

For all $S' \in \mathrm{Syl}_p(N)$, there are elements $x \in G$ and $y \in N$ such that $xS'x^{-1} \le S \cap N$ and $y(S \cap N)y^{-1} \le S'$. Thus $(yx)S'(yx)^{-1} \le S'$, and this must be an equality since S' is artinian. It follows that $S \cap N = xS'x^{-1} \in \mathrm{Syl}_p(N)$.

Now let $P \le G$ be any subgroup such that $P \cap N \in \mathrm{Syl}_p(N)$ and $PN = G$. Fix $x \in G$ such that $xPx^{-1} \le S$. Then $(xPx^{-1})N = xPNx^{-1} = G$, $xPx^{-1} \cap N = x(P \cap N)x^{-1} \le S \cap N$, and this last must be an equality since $P \cap N \in \mathrm{Syl}_p(N)$. It follows that

$$\begin{aligned} |G/N| = |xPx^{-1} \cdot N/N| &= |xPx^{-1}/(xPx^{-1} \cap N)| \\ &\le |S/(S \cap N)| = |SN/N| \le |G/N|, \end{aligned}$$

So these are all equalities, and $P = x^{-1}Sx \in \mathrm{Syl}_p(G)$.

Step 2 Now consider the general case. By assumption, G/N is an extension of the discrete p–toral group QN/N by the finite p–group G/QN, and hence is discrete p–toral. So by Lemma 1.9, there is a finite p–subgroup $G_0/N \le G/N$ such that $(G_0/N)\cdot(QN/N) = G/N$, and thus $QG_0 = G$ (since $G_0 \ge N$). Then G_0 has Sylow p–subgroups by (c). Hence G has Sylow p–subgroups by Lemma 8.1. Also, by Step 1 applied to the pair $N \lhd G_0$ (recall G_0/N is a p–group),

$$P \in \mathrm{Syl}_p(G_0) \iff P \cap N \in \mathrm{Syl}_p(N) \text{ and } PN = G_0.$$

Let $P \le G$ be any discrete p–toral subgroup which contains Q, and set $P_0 = P \cap G_0$. In general, for any $A, B \le G$ and $C \lhd G$ with $C \le A$, $C\cdot(A \cap B) = A \cap CB$. Thus

$$\begin{aligned} QP_0 &= Q\cdot(P \cap G_0) = P \cap QG_0 = P \cap G = P \\ P_0N &= (P \cap G_0)\cdot N = PN \cap G_0\,. \end{aligned} \tag{20}$$

Also, by Lemma 8.1 again, $\mathrm{Syl}_p(G) = \{QS \mid S \in \mathrm{Syl}_p(G_0)\}$. Hence

$$\begin{aligned} P \in \mathrm{Syl}_p(G) &\iff P_0 = P \cap G_0 \in \mathrm{Syl}_p(G_0) \\ &\iff P_0 \cap N \in \mathrm{Syl}_p(N) \text{ and } P_0N = G_0 \\ &\iff P \cap N \in \mathrm{Syl}_p(N) \text{ and } PN = G\,, \end{aligned} \tag{Step 1}$$

where the last equivalence holds by (20) and since $P_0 \cap N = P \cap (G_0 \cap N) = P \cap N$. □

Let G be any group which has Sylow p–subgroups. For any $S \in \mathrm{Syl}_p(G)$, we let $\mathcal{F}_S(G)$ be the fusion system over S with objects the subgroups of S and morphisms

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)\,.$$

Proposition 8.3 *Let G be a group for which the following conditions hold:*

(a) *For each discrete p–toral subgroup $P \le G$, each element of $\mathrm{Aut}_G(P)$ is conjugation by some $x \in N_G(P)$ of finite order.*

(b) *For each discrete p–toral subgroup $P \le G$, and each finite subgroup $H/C_G(P)$ in $N_G(P)/C_G(P)$, H has Sylow p–subgroups.*

(c) *For each increasing sequence $P_1 \le P_2 \le P_3 \le \cdots$ of discrete p–toral subgroups of G, there is some k such that $C_G(P_n) = C_G(P_k)$ for all $n \ge k$.*

Then for each $S \in \mathrm{Syl}_p(G)$, $\mathcal{F}_S(G)$ is a saturated fusion system. Furthermore, the following hold for each subgroup $P \le S$.

(21) *$C_G(P)$ has Sylow p–subgroups, and P is fully centralized in $\mathcal{F}_S(G)$ if and only if $C_S(P) \in \mathrm{Syl}_p(C_G(P))$.*

(22) *$N_G(P)$ has Sylow p–subgroups, and P is fully normalized in $\mathcal{F}_S(G)$ if and only if $N_S(P) \in \mathrm{Syl}_p(N_G(P))$.*

Proof Note first that G has Sylow p–subgroups by (b), applied with $P=1$.

Fix $S\in \mathrm{Syl}_p(G)$, and let $P\le S$ be any subgroup. By (a), $\mathrm{Aut}_G(P)$ is a torsion group, so $\mathrm{Out}_G(P)$ is a torsion group, and hence is finite by Proposition 1.5(b). We first claim that

$$P\cdot C_G(P)\le \Gamma\le N_G(P) \implies \Gamma \text{ has Sylow } p\text{–subgroups,} \tag{23}$$

and that if we set $S_0=S\cap\Gamma$, then

$$S_0\in \mathrm{Syl}_p(\Gamma) \iff C_S(P)\in \mathrm{Syl}_p(C_G(P)) \text{ and } \frac{S_0\cdot C_G(P)}{P\cdot C_G(P)}\in \mathrm{Syl}_p(\Gamma/P\cdot C_G(P)). \tag{24}$$

Points (23) and (24) follow from Lemma 8.2, applied with $N=C_G(P)$ and $Q=P$. Conditions (c) and (d) of Lemma 8.2 follow from conditions (b) and (a) above. Note that Γ/QN is finite since $\mathrm{Out}_G(P)\cong N_G(P)/QN$ is finite.

We next prove (21) and (22). For all $P\le S$, (23) (applied with $\Gamma=N_G(P)$) implies that there is $Q\in \mathrm{Syl}_p(N_G(P))$ such that $N_S(P)\le Q\le N_G(P)$. Choose $g\in G$ such that $gQg^{-1}\le S$; then gQg^{-1} is a Sylow p–subgroup of $gN_G(P)g^{-1}=N_G(gPg^{-1})$. Since $gQg^{-1}\le S$, $gQg^{-1}=S\cap N_G(gPg^{-1})=N_S(gPg^{-1})$. Hence $N_S(gPg^{-1})$ is a Sylow p–subgroup of $N_G(gPg^{-1})$. If P is fully normalized, then $|N_S(P)|\ge |N_S(gPg^{-1})|=|Q|$. Since $N_S(P)\le Q$, this implies that $N_S(P)=Q$ is in $\mathrm{Syl}_p(N_G(P))$.

Conversely, suppose that $N_S(P)\in\mathrm{Syl}_p(N_G(P))$. Choose $g\in G$ such that $gPg^{-1}\le S$, and gPg^{-1} is fully normalized in $\mathcal{F}_S(G)$. Then $N_S(gPg^{-1})\in \mathrm{Syl}_p(N_G(gPg^{-1}))$, so $N_S(P)\cong N_S(gPg^{-1})$ since $N_G(P)\cong N_G(gPg^{-1})$, and P is also fully normalized.

This proves (22). The proof of (21) (the condition for P to be fully centralized) is similar, except that $C_G(P)$ has Sylow p–subgroups by (b).

We now prove that $\mathcal{F}_S(G)$ is saturated.

(I) Assume that $P\le S$ is fully normalized in $\mathcal{F}_S(G)$. We have already seen that $\mathrm{Out}_G(P)$ is finite (since it is a torsion group by (a)). Also, $N_S(P)\in\mathrm{Syl}_p(N_G(P))$ by (22). So by (24), applied with $\Gamma=N_G(P)$, $C_S(P)\in\mathrm{Syl}_p(C_G(P))$ (hence P is fully centralized by (21)), and $\mathrm{Out}_S(P)\in\mathrm{Syl}_p(\mathrm{Out}_G(P))$.

(II) Let $P\le S$ be an arbitrary subgroup, and let $g\in G$ be such that $P'\overset{\text{def}}{=}gPg^{-1}\le S$ is fully centralized. Set $\Gamma=N_S(P')\cdot C_G(P')$, and define

$$N=\left\{x\in N_S(P)\,\middle|\, c_g\circ c_x\circ c_g^{-1}\in \mathrm{Aut}_S(P')\right\}=\left\{x\in N_S(P)\,\middle|\, gxg^{-1}\in\Gamma\right\}.$$

Then $C_S(P')\in\mathrm{Syl}_p(C_G(P'))$ by (21), and so by (24), $N_S(P')$ is a Sylow p–subgroup of $\Gamma=N_S(P')\cdot C_G(P')$ ($S_0\cdot C_G(P)=\Gamma$ in the notation of (24)). Since gNg^{-1} is a discrete p–toral subgroup of Γ, it is Γ–conjugate to a subgroup of $N_S(P')$. Thus there are elements $x\in N_S(P')$ and $y\in C_G(P')$ such that $(xyg)N(xyg)^{-1}\le N_S(P')$. Therefore $(yg)N(yg)^{-1}\le N_S(P')\le S$, and $c_{yg}\in\mathrm{Hom}_G(N,S)$ is an extension of $c_g\in\mathrm{Hom}_G(P,S)$.

(III) Let $P_1\le P_2\le P_3\le\cdots$ be a sequence of subgroups of S, and let P_∞ denote $\bigcup_{n=1}^\infty P_n$. Assume $\varphi\in\mathrm{Hom}(P_\infty,S)$ is a monomorphism such that for each n $\varphi|_{P_n}\in\mathrm{Hom}_G(P_n,S)$. Fix elements $g_n\in G$ for each n, such that $\varphi(x)=g_nxg_n^{-1}$, for $x\in P_n$. Then for all $1\le k<n$, $g_n^{-1}g_k\in C_G(P_k)$.

By (c), there is a k such that $C_G(P_k)=C_G(P_n)=C_G(P_\infty)$ for all $n\ge k$. Hence for all $n\ge k$ and all $x\in P_n$, $\varphi(x)=g_nxg_n^{-1}=g_kxg_k^{-1}$. Thus $\varphi=c_{g_k}\in\mathrm{Hom}_G(P_\infty,S)$. □

In general, for any group G, we define a *p–centric subgroup* of G to be a discrete p–toral subgroup $P\le G$ such that $Z(P)$ is the unique Sylow p–subgroup of $C_G(P)$ (ie every discrete p–toral subgroup of $C_G(P)$ is contained in $Z(P)$). Equivalently, P is p–centric if and only if $C_G(P)/Z(P)$ has no elements of order p.

Proposition 8.4 *Let G be any group which has Sylow p–subgroups, and fix a group $S\in\mathrm{Syl}_p(G)$. Then a subgroup $P\le S$ is $\mathcal{F}_S(G)$–centric if and only if P is p–centric in G.*

Proof Assume P is p–centric in G; ie that $Z(P)\in\mathrm{Syl}_p(C_G(P))$. For every $g\in G$ such that $gPg^{-1}\le S$, $C_S(gPg^{-1})$ is a discrete p–toral subgroup of $C_G(gPg^{-1})=gC_G(P)g^{-1}$, and $Z(gPg^{-1})=gZ(P)g^{-1}$ is a Sylow p–subgroup (hence the unique one) of $gC_G(P)g^{-1}$. It follows that $Z(gPg^{-1})=C_S(gPg^{-1})$ for all such g, and so P is $\mathcal{F}_S(G)$–centric.

Conversely, suppose that $P\le S$ is $\mathcal{F}_S(G)$–centric. Let Q be any discrete p–toral subgroup of $C_G(P)$. Then QP is a discrete p–toral subgroup, and hence there exists an element $g\in G$ such that $g(QP)g^{-1}\le S$. Therefore $gPg^{-1}\le S$, and $gQg^{-1}\le S\cap C_G(gPg^{-1})=C_S(gPg^{-1})$. Since P is $\mathcal{F}_S(G)$–centric, this shows that $gQg^{-1}\le Z(gPg^{-1})$, and thus that $Q\le Z(P)$. In other words, every discrete p–toral subgroup of $C_G(P)$ is contained in $Z(P)$, and so P is p–centric in G. □

We now restrict attention to locally finite groups. For any such group G, for the purposes of this section, we define $O^p(G)\lhd G$ be the subgroup generated by all elements of order prime to p. This clearly generalizes the usual definition of $O^p(G)$ for finite G (although it is not the only generalization).

Proposition 8.5 *If G is locally finite, then a discrete p–toral subgroup $P \le G$ is p–centric if and only if $C_G(P) = Z(P) \times O^p(C_G(P))$ and all elements of $O^p(C_G(P))$ have order prime to p.*

Proof By the above definition, a discrete p–toral subgroup $P \le G$ is p–centric if and only if $C_G(P)/Z(P)$ has no elements of order p. So if P is not p–centric, then either $O^p(C_G(P))$ has p–torsion, or $C_G(P)$ is not generated by $Z(P)$ and $O^p(C_G(P))$.

Conversely, assume that P is p–centric, and thus that $C_G(P)/Z(P)$ has no p–torsion. Consider the universal coefficient exact sequence:

$$\begin{aligned} 0 \longrightarrow \operatorname{Ext}\bigl(H_1(C_G(P)/Z(P)), Z(P)\bigr) &\longrightarrow H^2(C_G(P)/Z(P); Z(P)) \\ &\longrightarrow \operatorname{Hom}\bigl(H_2(C_G(P)/Z(P)), Z(P)\bigr) \longrightarrow 0 \end{aligned}$$

By assumption, all elements of $C_G(P)/Z(P)$ have order prime to p, all elements of $Z(P)$ have p–power order, and both groups are locally finite. Hence for $i = 1, 2$, $H_i(C_G(P)/Z(P))$ is a direct limit of finite abelian groups of order prime to p, and thus a torsion group all of whose elements have order prime to p. This shows that all terms in the above sequence vanish. Hence the following central extension splits:

$$1 \longrightarrow Z(P) \longrightarrow C_G(P) \longrightarrow C_G(P)/Z(P) \longrightarrow 1.$$

So $C_G(P) \cong Z(P) \times (C_G(P)/Z(P))$, and all elements of the group $O^p(C_G(P)) \cong C_G(P)/Z(P)$ have order prime to p. □

When working with fusion systems over discrete p–toral groups and their orbit categories, we are able to reduce certain problems to ones involving finite categories using the functor $(-)^\bullet$ constructed in Section 3. This is not a functor on the orbit category of a group, and so we need a different way to make such reductions. For any group G with Sylow p–subgroups, we let $\mathfrak{X} = \mathfrak{X}(G)$ denote the set of all subgroups of G which are intersections of (nonempty) subsets of $\mathrm{Syl}_p(G)$. Since discrete p–toral groups are artinian, it makes no difference whether we require finite intersections or allow infinite intersections.

Lemma 8.6 *Let G be a group such that for each discrete p–toral subgroup $P \le G$, $N_G(P)$ has Sylow p–subgroups. Assume that for every increasing sequence*

$$P_{(1)} \le P_{(2)} \le P_{(3)} \le \cdots$$

of discrete p–toral subgroups of G the union of the $P_{(i)}$ is again a discrete p–toral group, and that there is some k such that $C_G(P_{(n)}) = C_G(P_{(k)})$ for all $n \ge k$. Then the set $\mathfrak{X}(G)$ contains finitely many G–conjugacy classes.

Proof For each discrete p–toral subgroup $P \le G$, we let $P^\circ \ge P$ denote the intersection of all Sylow p–subgroups of G which contain P. We first prove

(25) For each discrete p–toral subgroup $P \le G$, there is a finite subgroup $P' \le P$ such that $P'^\circ = P^\circ$.

To see this, set $p^n = \exp(\pi_0(S))$ for $S \in \mathrm{Syl}_p(G)$. The discrete p–torus S_0 is the union of an increasing sequence of finite p–subgroups, and since centralizers stabilize by assumption, there is a finite subgroup $Q \le P_0$ such that $C_G(Q) = C_G(P_0)$. Set $Q' = \{x \in P_0 \mid x^{p^n} \in Q\}$: also a finite p–subgroup. By Lemma 1.9, there is a finite subgroup $P' \le P$ such that $P' \ge Q'$ and $P' P_0 = P$.

Fix $S \in \mathrm{Syl}_p(G)$ which contains P'. Then $S_0 \ge Q$ (since $S \ge Q'$), and hence $S_0 \le C_G(Q) = C_G(P_0)$. Since S_0 is a maximal discrete p–torus in G and $S_0 \cdot P_0$ is also a discrete p–torus, this implies that $S_0 \ge P_0$. Hence $S \ge P' P_0 = P$. Since this holds for all $S \in \mathrm{Syl}_p(G)$ which contains P', we have shown that $P'^\circ = P^\circ$; and this finishes the proof of (25).

Let $\overline{\mathfrak{X}}(G)$ be the set of G–conjugacy classes of subgroups in $\mathfrak{X}(G)$. We let (P) denote the conjugacy class of the subgroup P, and make $\overline{\mathfrak{X}}(G)$ into a poset by setting $(P) \le (Q)$ if $P \le xQx^{-1}$ for some $x \in G$. Let $\mathcal{P} \subseteq \overline{\mathfrak{X}}(G)$ be the set of all classes (P) which are contained in infinitely many other classes. We claim that $\mathcal{P} = \varnothing$. Since $\overline{\mathfrak{X}}(G)$ contains a smallest element which is contained in all the others (the class of the intersection of all Sylow p–subgroups of G), $\mathcal{P} = \varnothing$ implies that $\overline{\mathfrak{X}}(G)$ is finite, which is what we want to prove.

Assume otherwise: assume $\mathcal{P} \ne \varnothing$. We claim that $\mathcal{P}$ has a maximal element. For any totally ordered subset $\mathcal{P}_0$ of $\mathcal{P}$, upon restricting to those subgroups of maximal rank, we obtain a sequence of subgroups $P_{(1)} \le P_{(2)} \le P_{(3)} \le \cdots$ whose conjugacy classes are cofinal in $\mathcal{P}_0$. If this sequence is finite, then $\mathcal{P}_0$ clearly has a maximal element. Otherwise, set $P_{(\infty)} = \bigcup_{i=1}^{\infty} P_{(i)}$, and let $P' \le P_{(\infty)}$ be a finite subgroup such that $P'^\circ = P_{(\infty)}$ (apply (1)). Then $P' \le P_{(k)}$ for some k, and so $(P_{(k)}) = (P_{(\infty)})$ is a maximal element in $\mathcal{P}_0$.

Thus by Zorn's lemma, $\mathcal{P}$ contains a maximal element (Q), and clearly $Q \notin \mathrm{Syl}_p(G)$. Since $N_G(Q)$ has Sylow p–subgroups, there is some $S \in \mathrm{Syl}_p(G)$ such that every p–toral subgroup of G containing Q with index p is G–conjugate to a subgroup of $N_S(Q)$; and hence by Lemma 1.4, there are finitely many G–conjugacy classes of such subgroups. Hence since (Q) is contained in infinitely many classes in $\overline{\mathfrak{X}}(G)$, the same holds for (Q') for some $Q' \le G$ such that $Q \lhd Q'$ with index p. Then $(Q'^\circ) \in \mathcal{P}$, which contradicts the maximality assumption about Q. So $\mathcal{P}$ contains no maximal element, hence must be empty, and so $\mathfrak{X}(G)$ has finitely many G–conjugacy classes. □

Now, for any discrete group G which has Sylow p–subgroups, let $\mathcal{L}_p^c(G)$ be the category whose objects are the p–centric subgroups of G, and where

$$\mathrm{Mor}_{\mathcal{L}_p^c(G)}(P,Q) = N_G(P,Q)/O^p(C_G(P)).$$

For any $S \in \mathrm{Syl}_p(G)$, let $\mathcal{L}_S^c(G) \subseteq \mathcal{L}_p^c(G)$ be the equivalent full subcategory whose objects are the subgroups of S which are p–centric in G.

It will be convenient, throughout the rest of this section, to use the term "p–group" to mean any group each of whose elements has p–power order. It is not hard to show that if G is locally finite, and has Sylow p–subgroups in the sense described above, then every p–subgroup of G is a discrete p–toral subgroup. Hence there is no loss of generality to assume this in the following theorem.

Theorem 8.7 *Let G be any group which satisfies the following conditions:*

(a) *G is locally finite.*

(b) *Each p–subgroup of G is a discrete p–toral group.*

(c) *For any increasing sequence $A_{(1)} \le A_{(2)} \le A_{(3)} \le \cdots$ of finite abelian p–subgroups of G, there is some k such that $C_G(A_{(n)}) = C_G(A_{(k)})$ for all $n \ge k$.*

Then G has a unique conjugacy class $\mathrm{Syl}_p(G)$ of maximal discrete p–toral subgroups. For any $S \in \mathrm{Syl}_p(G)$, $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ is a p–local compact group, with classifying space $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$.

Proof We first apply Proposition 8.3 to show that $\mathcal{F}_S(G)$ is a saturated fusion system over S. Once this has been checked, then it easily follows that $\mathcal{L}_S^c(G)$ is a centric linking system associated to $\mathcal{F}_S(G)$: condition (A) in Definition 4.1 holds by Proposition 8.5 and Proposition 8.4, and conditions (B) and (C) are immediate. It then will remain only to show that $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$.

By [23, Theorem 3.4], conditions (a) and (c) above imply that all maximal p–subgroups of G are conjugate, and hence (by (b)) that G has Sylow p–subgroups. Since these three conditions are carried over to subgroups of G, this also shows that each subgroup of G has Sylow p–subgroups. This proves condition (b) in Proposition 8.3, and condition (a) holds since G is locally finite.

It remains to prove condition (c) in Proposition 8.3, which we state here as:

(d) For any increasing sequence $P_{(1)} \le P_{(2)} \le P_{(3)} \le \cdots$ of discrete p–toral subgroups of G, there is some k such that $C_G(P_{(n)}) = C_G(P_{(k)})$ for all $n \ge k$.

To see this, fix any such sequence, and let $P_{(\infty)}$ be its union. Let $A=(P_{(\infty)})_0$ be the identity component, and set $A_{(i)}=A\cap P_{(i)}$ for all i. Let r be such that $P_{(i)}$ surjects onto $\pi_0(P_{(\infty)})$ for all $i\ge r$; equivalently, $P_{(i)}\cdot A=P_{(\infty)}$ for all $i\ge r$. For each i, let $A'_i\le A_{(i)}$ be the finite subgroup of elements of order at most p^i. Then $A=\bigcup_{i=1}^{\infty}A'_i$; and so by (c) there is r such that $C_G(A)=C_G(A'_r)$. Hence $C_G(A_{(i)})=C_G(A_{(r)})$ for all $i\ge r$ (since $A'_r\le A_{(r)}\le A_{(i)}\le A$). We can assume that r is chosen large enough so that $P_{(r)}$ surjects onto $P_{(\infty)}/A$; ie such that $P_{(r)}\cdot A=P_{(\infty)}$. Then for all $i\ge r$, $P_{(i)}=A_{(i)}\cdot P_{(r)}$,

$$C_G(P_{(i)})=C_G(A_{(i)})\cap C_G(P_{(r)})=C_G(A_{(r)})\cap C_G(P_{(r)})=C_G(P_{(r)}),$$

and this finishes the proof of (d).

We have now shown that the hypotheses of Proposition 8.3 hold, and thus that $\mathcal{F}_S(G)$ is a saturated fusion system over S. We have already seen that $\mathcal{L}_S^c(G)$ is a linking system associated to $\mathcal{F}_S(G)$, and it remains only to show that $|\mathcal{L}_S^c(G)|_p^\wedge\simeq BG_p^\wedge$.

As in Section 5, for any discrete group G, we let $\mathcal{O}_p(G)$ be the category whose objects are the discrete p–toral subgroups $P\le G$ and where

$$\mathrm{Mor}_{\mathcal{O}_p(G)}(P,Q)=Q\backslash N_G(P,Q)\cong \mathrm{Map}_G(G/P,G/Q).$$

Let $\mathcal{O}_{\mathfrak{X}}(G)\subseteq\mathcal{O}_p(G)$ be the full subcategory with object set $\mathfrak{X}=\mathfrak{X}(G)$: the set of all intersections of subgroups in $\mathrm{Syl}_p(G)$. For each discrete p–toral subgroup $P\le G$, we let $P^\circ\in\mathfrak{X}$ denote the intersection of all subgroups in $\mathrm{Syl}_p(G)$ which contain P. Clearly, for any P and Q, $N_G(P,Q)\subseteq N_G(P^\circ,Q^\circ)$, and so this defines a functor $(-)^\circ$ from $\mathcal{O}_p(G)$ to $\mathcal{O}_{\mathfrak{X}}(G)$. Since $N_G(P^\circ,Q)=N_G(P,Q)$ when $Q\in\mathfrak{X}(G)$, the two functors

$$\mathcal{O}_{\mathfrak{X}}(G)\ \underset{(-)^\circ}{\overset{\mathrm{incl}}{\rightleftarrows}}\ \mathcal{O}_p(G) \tag{26}$$

are adjoint.

Step 1 Let $\mathcal{I}$ and Φ be the following functors from $\mathcal{O}_p(G)$ to (G-)spaces:

$$\mathcal{I}(P)=G/P \qquad\text{and}\qquad \Phi(P)=EG\times_G\mathcal{I}(P)\cong EG/P.$$

Then for any full subcategory $\mathcal{C}\subseteq\mathcal{O}_p(G)$,

$$\underset{\mathcal{C}}{\underrightarrow{\mathrm{hocolim}}}(\mathcal{I})=\Bigl(\coprod_{n=0}^{\infty}\ \coprod_{G/P_0\to\cdots\to G/P_n} G/P_0\times\Delta^n\Bigr)\Big/\!\sim$$

is the nerve of the category whose objects are the cosets gP for all $P\in\mathrm{Ob}(\mathcal{C})$, and with a unique morphism $gP\to hQ$ exactly when $gPg^{-1}\le hQh^{-1}$. When $\mathcal{C}=\mathcal{O}_{\mathfrak{X}}(G)$, this category has as initial object the intersection of all Sylow p–subgroups of G, and hence $\operatorname{hocolim}_{\mathcal{O}_{\mathfrak{X}}(G)}(\mathcal{I})$ is contractible. Since the Borel construction commutes with homotopy colimits in this situation (being itself a special case of a homotopy colimit),

$$\underset{\mathcal{O}_{\mathfrak{X}}(G)}{\underrightarrow{\operatorname{hocolim}}}(\Phi) \cong EG \times_G \Big(\underset{\mathcal{O}_{\mathfrak{X}}(G)}{\underrightarrow{\operatorname{hocolim}}}(\mathcal{I})\Big) \simeq BG. \tag{27}$$

Step 2 Fix some $Q \in \mathfrak{X}$ which is not p–centric. For each $i \geq 0$, consider the functor $F_i^{[Q]}\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathrm{Ab}$, defined by setting

$$F_i^{[Q]}(P) = \begin{cases} H^i(BP;\mathbb{F}_p) & \text{if } P \text{ is } G\text{–conjugate to } Q \\ 0 & \text{otherwise.} \end{cases}$$

The subgroup $C_G(Q)\cdot Q/Q \cong C_G(Q)/Z(Q)$ of $\operatorname{Aut}_{\mathcal{O}_p(G)}(Q) = N(Q)/Q$ acts trivially on $F_i^{[Q]}(Q)$, and contains an element of order p since Q is not p–centric. Hence by Lemma 5.10 and Lemma 5.12,

$$\underset{\mathcal{O}_{\mathfrak{X}}(G)}{\underleftarrow{\lim}}{}^*(F_i^{[Q]}) \cong \underset{\mathcal{O}_p(G)}{\underleftarrow{\lim}}{}^*(F_i^{[Q]}) \cong \Lambda^*\big(N_G(Q)/Q; F_i^{[Q]}(P)\big) = 0 \quad \text{for all } i,$$

where the first isomorphism follows from the adjoint functors (26).

Step 3 Now let $\mathcal{O}^c_{\mathfrak{X}}(G) \subseteq \mathcal{O}_{\mathfrak{X}}(G)$ be the full subcategory with objects the p–centric subgroups which lie in $\mathfrak{X}$. Let $P_1, P_2, \dots, P_m \leq S$ be representatives for those G–conjugacy classes in $\mathfrak{X}(G)$ which are not p–centric (a finite set by Lemma 8.6). We assume these are ordered such that $|P_i| \leq |P_{i+1}|$ for each i.

For each $i \geq 0$, consider the functor $F_i\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathrm{Ab}$, defined by setting $F_i(P) = H^i(BP;\mathbb{F}_p)$ for all P. For all $k = 0,\dots,m$, define functors

$$F_{i,k}\colon \mathcal{O}_{\mathfrak{X}}(G)^{\mathrm{op}} \longrightarrow \mathrm{Ab} \quad \text{where} \quad F_{i,k}(P) = \begin{cases} 0 & \text{if } P \sim_G P_j, \text{ some } j \geq k \\ F_i(P) & \text{otherwise.} \end{cases}$$

Here, "$\sim_G$" means "G–conjugate", and these are all defined to be quotient functors of $F_i|_{\mathcal{O}_{\mathfrak{X}}(G)}$. In particular, $F_{i,0} = F_i|_{\mathcal{O}_{\mathfrak{X}}(G)}$ and $F_{i,m} = F_i|_{\mathcal{O}^c_{\mathfrak{X}}(G)}$. Also, for all k,

$$\operatorname{Ker}\big[F_{i,k} \twoheadrightarrow F_{i,k+1}\big] \cong F_i^{[P_k]}|_{\mathcal{O}_{\mathfrak{X}}(G)},$$

and the higher limits of this last functor vanish by Step 2. So there are isomorphisms

$$\underset{\mathcal{O}_{\mathfrak{X}}(G)}{\underleftarrow{\lim}}{}^*(F_i) = \underset{\mathcal{O}_{\mathfrak{X}}(G)}{\underleftarrow{\lim}}{}^*(F_{i,0}) \cong \underset{\mathcal{O}_{\mathfrak{X}}(G)}{\underleftarrow{\lim}}{}^*(F_{i,1}) \cong \cdots \cong \underset{\mathcal{O}_{\mathfrak{X}}(G)}{\underleftarrow{\lim}}{}^*(F_{i,m}) \cong \underset{\mathcal{O}^c_{\mathfrak{X}}(G)}{\underleftarrow{\lim}}{}^*(F_i)$$

whose composite is induced by restriction from $\mathcal{O}_{\mathfrak{X}}(G)$ to $\mathcal{O}^c_{\mathfrak{X}}(G)$.

The spectral sequence for the cohomology of a homotopy colimit now implies that the inclusion of $\mathcal{O}^c_{\mathfrak{X}}(G)$ into $\mathcal{O}_{\mathfrak{X}}(G)$ induces a mod p homology isomorphism of homotopy colimits of Φ, and hence a homotopy equivalence

$$\Big(\underset{\mathcal{O}^c_{\mathfrak{X}}(G)}{\underrightarrow{\mathrm{hocolim}}}(\Phi)\Big)^\wedge_p \xrightarrow{\ \simeq\ } \Big(\underset{\mathcal{O}_{\mathfrak{X}}(G)}{\underrightarrow{\mathrm{hocolim}}}(\Phi)\Big)^\wedge_p . \tag{28}$$

Also, the adjoint functors in (26) restrict to adjoint functors between $\mathcal{O}^c_{\mathfrak{X}}(G)$ and $\mathcal{O}^c_p(G)$, and hence induce a homotopy equivalence

$$\Big(\underset{\mathcal{O}^c_{\mathfrak{X}}(G)}{\underrightarrow{\mathrm{hocolim}}}(\Phi)\Big)^\wedge_p \simeq \Big(\underset{\mathcal{O}^c_p(G)}{\underrightarrow{\mathrm{hocolim}}}(\Phi)\Big)^\wedge_p . \tag{29}$$

Step 4 Let $\mathcal{T}^c_p(G)$ be the centric transporter category for G: the category whose objects are the p–centric subgroups of G, and where the set of morphisms from P to Q is the transporter $N_G(P,Q)$. By exactly the same argument as in [6, Lemma 1.2],

$$\underset{\mathcal{O}^c_p(G)}{\underrightarrow{\mathrm{hocolim}}}(\Phi) \simeq |\mathcal{T}^c_p(G)|. \tag{30}$$

The canonical projection functor $\mathcal{T}^c_p(G) \longrightarrow \mathcal{L}^c_p(G)$ satisfies all of the hypotheses of the functor in [6, Lemma 1.3], *except* that we only know that

$$K(P) \overset{\mathrm{def}}{=} \mathrm{Ker}\big[\mathrm{Aut}_{\mathcal{T}^c_p(G)}(P) \twoheadrightarrow \mathrm{Aut}_{\mathcal{L}^c_p(G)}(P)\big] = O^p(C_G(P))$$

is a locally finite group all of whose elements have order prime to p (not necessarily a finite group). But this suffices to ensure that coinvariants preserve exact sequences of $\mathbb{Z}_{(p)}[K_P]$–modules, which is the only way this property of K_P is used in the proof of [6, Lemma 1.3]. Hence the induced map

$$|\mathcal{T}^c_p(G)| \longrightarrow |\mathcal{L}^c_p(G)|.$$

is a mod p homology equivalence. Together with (27), (28), (29), and (30), this shows that

$$|\mathcal{L}^c_S(G)|^\wedge_p \simeq |\mathcal{L}^c_p(G)|^\wedge_p \simeq |\mathcal{T}^c_p(G)|^\wedge_p \simeq BG^\wedge_p. \qquad \square$$

We now finish the section by exhibiting a more concrete class of groups which satisfy the hypotheses of Theorem 8.7. A *linear torsion group* is a torsion subgroup of $GL_n(k)$, for any positive integer n and any (commutative) field k. These are also referred to as "periodic linear groups", since their elements are all periodic transformations (automorphisms of finite order) of a finite dimensional vector space.

The following facts about linear torsion groups are the starting point of our work here.

Proposition 8.8 *The following hold for every field k, and every linear torsion group $G \le GL_n(k)$.*

(a) *G is locally finite.*

(b) *For $p \ne \mathrm{char}(k)$, every p–subgroup of G is a discrete p–toral group.*

Proof Point (a) is a theorem of Schur, and is shown in Wehrfritz [29, Corollary 4.9]. By [29, 2.6], every locally finite p–subgroup of $GL_n(k)$ is artinian (when $p \ne \mathrm{char}(k)$), and hence is discrete p–toral by Proposition 1.2. □

In order to apply Theorem 8.7, it remains only to check that centralizers of discrete p–toral subgroups of linear torsion groups stabilize in the sense of Theorem 8.7.

Proposition 8.9 *Let $A_1 \le A_2 \le A_3 \le \dots$ be an increasing sequence of finite abelian p–subgroups of a linear torsion group $G \le GL_n(k)$, where $\mathrm{char}(k) \ne p$. Then there is r such that $C_G(A_i) = C_G(A_r)$ for all $i \ge r$.*

Proof Upon replacing k by its algebraic closure if necessary, we can assume that k is algebraically closed. Hence any representation over k of a finite abelian p–group A splits as a sum of 1–dimensional irreducible representations. Moreover, if $A \le GL_n(k)$, and $k^n = U_1 \oplus \cdots \oplus U_m$ is the unique decomposition with the property that each U_i is a sum of irreducible modules with the same character and different U_i correspond to different characters of A, then

$$(d_i = \dim(U_i)) \qquad C_{GL_n(k)}(A) \cong \prod_{i=1}^{m} \mathrm{Aut}_k(U_i) \cong \prod_{i=1}^{m} GL_{d_i}(k).$$

From this observation, it is clear that for any increasing sequence of such subgroups A_i, the centralizers $C_{GL_n(k)}(A_i)$ stabilize for i sufficiently large, and hence the stabilizers $C_G(A_i)$ also stabilize. □

Proposition 8.8 and Proposition 8.9 show that linear torsion groups satisfy all of the hypotheses of Theorem 8.7. So as an immediate consequence, we get:

Theorem 8.10 *Fix a linear torsion group G, a prime p not equal to the defining characteristic of G, and a Sylow subgroup $S \in \mathrm{Syl}_p(G)$. Then the triple $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ is a p–local compact group, with classifying space $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$.* □

9 Compact Lie groups

Throughout this section, we fix a compact Lie group G and a prime p. Our main result is to show that G defines a p–local compact group whose classifying space has the homotopy type of $BG^\wedge_p$.

A compact Lie group P is called p–toral if its identity component is a torus and if its group of components is a p–group. The closure $\overline{P}$ of a discrete p–toral subgroup $P \le G$ is a p–toral group, since $\overline{P_0}$ is abelian and connected, hence a torus, and has p–power index in $\overline{P}$. We will generally denote p–toral groups (including tori) by $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{T}$, etc., to distinguish them from discrete p–toral groups P, Q, T, etc. Our first task is to identify the maximal (discrete) p–toral subgroups of G.

Definition 9.1 (a) For any p–toral group $\boldsymbol{P}$, $\mathrm{Syl}_p(\boldsymbol{P})$ denotes the set of discrete p–toral subgroups $P \le \boldsymbol{P}$ such that $P\cdot\boldsymbol{P}_0 = \boldsymbol{P}$ and P contains all p–power torsion in $\boldsymbol{P}_0$.

(b) A discrete p–toral subgroup $P \le G$ is *snugly embedded* if $P \in \mathrm{Syl}_p(\overline{P})$.

(c) $\overline{\mathrm{Syl}}_p(G)$ denotes the set of all p–toral subgroups $\boldsymbol{S} \le G$ such that the identity component $\boldsymbol{S}_0$ is a maximal torus of G and $\boldsymbol{S}/\boldsymbol{S}_0 \in \mathrm{Syl}_p(N(\boldsymbol{S}_0)/\boldsymbol{S}_0)$.

(d) $\mathrm{Syl}_p(G)$ denotes the set of all discrete p–toral subgroups $P \le G$ such that $\overline{P} \in \overline{\mathrm{Syl}}_p(G)$ and $P \in \mathrm{Syl}_p(\overline{P})$.

For any discrete p–toral subgroup $P \le G$, $\overline{P_0}$ is a torus, as noted above, and has finite index in $\overline{P}$. Hence $\overline{P_0} = (\overline{P})_0$, and $\pi_0(\overline{P}) \cong P/P_0$. So P is snugly embedded in G if and only if P_0 is snugly embedded, and this holds exactly when $\mathrm{rk}(P) = \mathrm{rk}(\overline{P})$. As an example of a subgroup which is not snugly embedded, one can construct a rank one subgroup $P \cong \mathbb{Z}/p^\infty$ which is densely embedded in a torus $(S^1)^r$ for $r > 1$.

Clearly, when $\mathrm{rk}(P) < \mathrm{rk}(\overline{P})$, we cannot expect $BP^\wedge_p$ and $B\overline{P}^\wedge_p$ to have the same homotopy type. But we do get a homotopy equivalence when P is snugly embedded.

Proposition 9.2 *If $P \le G$ is snugly embedded, then the inclusion of P in $\overline{P}$ induces a homotopy equivalence $BP^\wedge_p \simeq B\overline{P}^\wedge_p$.*

Proof This means showing that the inclusion of BP into $B\overline{P}$ induces an isomorphism on mod p cohomology. See, for example, Feshbach [13, Proposition 2.3]. □

The following proposition is well known. It says that $\overline{\mathrm{Syl}}_p(G)$ is the set of maximal p–toral subgroups of G, that $\mathrm{Syl}_p(G)$ is the set of maximal discrete p–toral subgroups of G, and that each of these sets contains exactly one G–conjugacy class. Note in particular the case where $G = \boldsymbol{P}$ is p–toral: there is a unique conjugacy class of discrete p–toral subgroups snugly embedded in $\boldsymbol{P}$, and every discrete p–toral subgroup of $\boldsymbol{P}$ is contained in a snugly embedded subgroup.

Proposition 9.3 *The following hold for any compact Lie group G and any p–toral group $\boldsymbol{P}$.*

(a) *Any two subgroups in $\overline{\mathrm{Syl}}_p(G)$ are G–conjugate, and each p–toral subgroup $\boldsymbol{P} \le G$ is contained in some subgroup $\boldsymbol{S} \in \overline{\mathrm{Syl}}_p(G)$.*

(b) *Any two subgroups in $\mathrm{Syl}_p(G)$ are G–conjugate, and each discrete p–toral subgroup $P \le G$ is contained in some subgroup $S \in \mathrm{Syl}_p(G)$.*

Proof **(a)** The subgroups in $\overline{\mathrm{Syl}}_p(G)$ are clearly all conjugate to each other, since all maximal tori in G are conjugate. For any $\boldsymbol{S} \in \overline{\mathrm{Syl}}_p(G)$ with identity component the maximal torus $\boldsymbol{T} = \boldsymbol{S}_0$, $\chi(G/N(\boldsymbol{T})) = 1$ (see Bredon [4, Proposition 0.6.3]), and hence $\chi(G/\boldsymbol{S})$ is prime to p. If $\boldsymbol{Q}$ is an arbitrary p–toral subgroup of G, then $\chi((G/\boldsymbol{S})^{\boldsymbol{Q}})$ is congruent mod p to $\chi(G/\boldsymbol{S})$, so $(G/\boldsymbol{S})^{\boldsymbol{Q}} \neq \varnothing$, and hence $\boldsymbol{Q} \le g\boldsymbol{S}g^{-1} \in \overline{\mathrm{Syl}}_p(G)$ for some $g \in G$.

(b) Assume first that $G = \boldsymbol{P}$ is p–toral. Set $\boldsymbol{T} = \boldsymbol{P}_0$, and let $T \le \boldsymbol{T}$ be the subgroup of elements of p–power torsion. By definition, $\mathrm{Syl}_p(\boldsymbol{P})$ is the set of all subgroups $P \ge T$ such that P/T is the image of a splitting of the extension

$$1 \longrightarrow \boldsymbol{T}/T \longrightarrow \boldsymbol{P}/T \longrightarrow \boldsymbol{P}/\boldsymbol{T} \longrightarrow 1. \tag{31}$$

The cohomology groups $H^i(\boldsymbol{P}/\boldsymbol{T}; \boldsymbol{T}/T)$ vanish for all $i > 0$, since $\boldsymbol{P}/\boldsymbol{T}$ is a p–group and $\boldsymbol{T}/T$ is uniquely p–divisible. Hence the extension (31) is split, and any two splittings are conjugate by an element of $\boldsymbol{T}/T$. Thus $\mathrm{Syl}_p(\boldsymbol{P}) \neq \varnothing$, and its elements are conjugate to each other by elements of $\boldsymbol{T}$.

Now let $Q \le \boldsymbol{P}$ be an arbitrary discrete p–toral subgroup. Then QT is also a discrete p–toral subgroup (since $T \lhd \boldsymbol{P}$), and QT/T is the image of a splitting of the extension of $\boldsymbol{T}/T$ by $Q\boldsymbol{T}/\boldsymbol{T}$. We have seen that any two such splittings are conjugate by elements of $\boldsymbol{T}/T$, and hence they all extend to splittings of the extension by $\boldsymbol{P}/\boldsymbol{T}$. In other words, there is a subgroup $P \in \mathrm{Syl}_p(\boldsymbol{P})$ which contains QT, and hence contains Q.

Now let G be an arbitrary compact Lie group. For any $S, S' \in \mathrm{Syl}_p(G)$, $\overline{S}$ is G–conjugate to $\overline{S'}$ by (a), so $\overline{S} = \overline{gS'g^{-1}}$ for some $g \in G$, and $S, gS'g^{-1} \in \mathrm{Syl}_p(\overline{S})$. We have just shown that all subgroups in $\mathrm{Syl}_p(\overline{S})$ are conjugate, and hence S and S' are conjugate. If $P \le G$ is an arbitrary discrete p–toral subgroup, then its closure $\overline{P}$ is a p–toral subgroup, and hence contained in some maximal subgroup $\boldsymbol{S} \in \overline{\mathrm{Syl}}_p(G)$ by (a) again. So there is some $S \in \mathrm{Syl}_p(\boldsymbol{S}) \subseteq \mathrm{Syl}_p(G)$ which contains P. □

We next need some information about the outer automorphisms of (discrete) p–toral subgroups of G.

Lemma 9.4 *The following hold for all discrete p–toral subgroups $P, Q \le G$.*

(a) *If $P \le Q$, then $\mathrm{Out}_{\bar{Q}}(P)$ is a finite p–group, and $\mathrm{Out}_{\bar{Q}}(P) = \mathrm{Out}_Q(P)$ if Q is snugly embedded in G. In particular, $\mathrm{Out}_{\bar{Q}}(Q) = 1$ if Q is snugly embedded in G.*

(b) *$\mathrm{Out}_G(P)$ and $\mathrm{Out}_G(\bar{P})$ are both finite.*

(c) *If Q is snugly embedded, then the natural map*

$$\mathrm{Rep}_G(P, Q) \xrightarrow{\cong} \mathrm{Rep}_G(\bar{P}, \bar{Q})$$

is a bijection.

Proof **(a)** Choose $Q' \le \bar{Q}$ such that $\overline{Q'} = \bar{Q}$ and Q' is snugly embedded. Then $\mathrm{Out}_{Q'}(P)$ is a finite p–group by Proposition 1.5(c). The first statement thus follows from the second.

Now assume Q is snugly embedded. We must show that $\mathrm{Out}_{\bar{Q}}(P) = \mathrm{Out}_Q(P)$; or equivalently that $\mathrm{Aut}_{\bar{Q}}(P) = \mathrm{Aut}_Q(P)$. Fix $x \in N_{\bar{Q}}(P)$, and set $|P| = p^k$.

Let $\bar{Q}/Q$ be the set of *left* cosets gQ for $g \in \bar{Q}$, and let $(\bar{Q}/Q)^P$ be the fixed point set of the left P–action. Then for $g \in \bar{Q}$, $gQ \in (\bar{Q}/Q)^P$ if and only if $g^{-1}Pg \le Q$. In particular, $xQ \in (\bar{Q}/Q)^P$ since x normalizes P and $P \le Q$. Since $\bar{Q}/Q = \bar{Q}_0/Q_0$ and the latter group is uniquely p–divisible (since Q is snugly embedded), there is $y \in \bar{Q}_0$ such that $y^{p^k} \in xQ$ and $yQ \in (\bar{Q}/Q)^P$.

Set $$\widehat{y} = \prod_{a \in P} (aya^{-1}) = y^{p^k} \cdot \prod_{a \in P} ((y^{-1}ay) \cdot a^{-1}) \in y^{p^k} Q = xQ,$$

where the inclusion holds since $P \le Q$ and $y^{-1}Py \le Q$. Then $\widehat{y} \in C_{\bar{Q}}(P)$. Since x was arbitrary, this proves that $N_{\bar{Q}}(P) = C_{\bar{Q}}(P) \cdot N_Q(P)$, and finishes the proof that $\mathrm{Aut}_{\bar{Q}}(P) = \mathrm{Aut}_Q(P)$. In the case $P = Q$, this shows that $\mathrm{Out}_{\bar{Q}}(Q) = \mathrm{Out}_Q(Q) = 1$.

(b) The kernel of the homomorphism

$$\mathrm{Out}_G(P) \longrightarrow \mathrm{Out}_G(\bar{P}) \tag{32}$$

is $\mathrm{Out}_{\bar{P}}(P)$. By (a), this is always finite, and is trivial if P is snugly embedded. If P is snugly embedded, ie if $P \in \mathrm{Syl}_p(\bar{P})$, then $N_G(\bar{P}) = \bar{P} \cdot N_G(P)$ (any subgroup of $\bar{P}$ which is G–conjugate to P is also $\bar{P}$–conjugate to P), and hence the map in (32) is also surjective. Thus in this case, $\mathrm{Out}_G(P) \cong \mathrm{Out}_G(\bar{P}) \cong N_G(\bar{P})/\bar{P} \cdot C_G(\bar{P})$ is a compact Lie group, all torsion subgroups of which are finite by Proposition 1.5. Thus $\mathrm{Out}_G(\overline{P})$ is finite (since otherwise it would contain S^1). If P is an arbitrary discrete p–toral subgroup of G, then the kernel and the image of the map in (32) are finite, and hence $\mathrm{Out}_G(P)$ is also finite in this case.

(c) Assume $P, Q \le S$, where Q is snugly embedded. We must show that the map from $\mathrm{Rep}_G(P,Q)$ to $\mathrm{Rep}_G(\overline{P},\overline{Q})$ which sends a homomorphism to its unique continuous extension is a bijection. For any $\varphi \in \mathrm{Hom}_G(\overline{P},\overline{Q})$, $\varphi(P)$ is $\overline{Q}$–conjugate to a subgroup of $Q \in \mathrm{Syl}_p(\overline{Q})$, and hence φ is $\overline{Q}$–conjugate to a homomorphism which sends P into Q. This proves surjectivity. To prove injectivity, fix $\varphi_1, \varphi_2 \in \mathrm{Hom}_G(P,Q)$ which induce the same class in $\mathrm{Rep}_G(\overline{P},\overline{Q})$, and set $P_i = \mathrm{Im}(\varphi_i)$. Then $\varphi_2 = \chi\circ\varphi_1$ for some $\chi \in \mathrm{Iso}_{\overline{Q}}(P_1,P_2)$. We must show that $\chi \in \mathrm{Iso}_Q(P_1,P_2)$, and it suffices to do this when $\chi \in \mathrm{Iso}_{\overline{Q}_0}(P_1,P_2)$. In this case, $P_1Q_0 = P_2Q_0$, so χ extends to $\overline{\chi} \in \mathrm{Aut}_{\overline{Q}_0}(P_1Q_0)$. Also, P_1Q_0 is snugly embedded since Q is, so $\mathrm{Out}_{\overline{Q}_0}(P_1Q_0) = 1$ by (b), and hence χ is conjugation by an element of $\overline{Q}_0 \cap P_1Q_0 = Q_0$. □

The fusion system of a compact Lie group is defined exactly as in Section 8. For any $S \in \mathrm{Syl}_p(G)$, $\mathcal{F}_S(G)$ is the fusion system over S where for $P, Q \le S$,

$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q) \cong N_G(P,Q)/C_G(P)$$

is the set of homomorphisms from P to Q induced by conjugation by elements of G. Here, as usual,

$$N_G(P,Q) = \{x \in G \mid xPx^{-1} \le Q\}$$

denotes the transporter set.

Lemma 9.5 *For each maximal discrete p–toral subgroup $S \in \mathrm{Syl}_p(G)$, $\mathcal{F}_S(G)$ is a saturated fusion system over S. Also, a subgroup $P \le S$ is fully centralized in $\mathcal{F}_S(G)$ if and only if $C_S(P) \in \mathrm{Syl}_p(C_G(P))$.*

Proof We must show that conditions (a), (b), and (c) of Proposition 8.3 all hold. For each discrete p–toral subgroup $P \le G$, $\mathrm{Out}_G(P)$ is finite by Lemma 9.4(b), so $\mathrm{Aut}_G(P) \cong N_G(P)/C_G(P)$ is a torsion group. Hence for each $g \in N_G(P)$, $\langle g\rangle\cdot C_G(P)$ is a finite extension of $C_G(P)$, thus a closed subgroup, and so the coset $gC_G(P)$ contains elements of finite order. Also, for each finite subgroup $H/C_G(P)$ in $N_G(P)/C_G(P)$, H is a closed subgroup of G, and hence has Sylow p–subgroups in the sense of Section 8. If $P_1 \le P_2 \le P_3 \le \cdots$ is an increasing sequence of discrete p–toral subgroups of G, then the centralizers $C_G(P_i)$ form a decreasing sequence of closed subgroups of G, and hence is constant for i sufficiently large.

Thus Proposition 8.3 applies: for any $S \in \mathrm{Syl}_p(G)$, $\mathcal{F}_S(G)$ is a saturated fusion system over S, and a subgroup $P \le S$ is fully centralized in $\mathcal{F}_S(G)$ if and only if $C_S(P) \in \mathrm{Syl}_p(C_G(P))$. □

Recall from Section 8 that a discrete p–toral subgroup $P \le G$ is called *p–centric in G* if $Z(P) \in \mathrm{Syl}_p(C_G(P))$. By analogy with this definition, a p–toral subgroup $\boldsymbol{P} \le G$ is called *p–centric* if $Z(\boldsymbol{P}) \in \overline{\mathrm{Syl}}_p(C_G(\boldsymbol{P}))$. We next note some conditions which characterize p–toral and discrete p–toral subgroups of G which are p–centric.

Lemma 9.6 *The following hold for any discrete p–toral subgroup $P \le G$.*

(a) *If $\overline{P}$ is p–centric in G, then $N_G(\overline{P})/\overline{P}$ is finite, and $C_G(\overline{P})/Z(\overline{P})$ is finite of order prime to p.*

(b) *If P is p–centric in G, then $\overline{P}$ is p–centric in G.*

(c) *If $\overline{P}$ is p–centric in G and P is snugly embedded, then P is p–centric in G.*

Proof **(a)** Assume $\boldsymbol{P}$ is p–centric in G, and consider the groups

$$C_G(\boldsymbol{P})/Z(\boldsymbol{P}) \cong \boldsymbol{P}\cdot C_G(\boldsymbol{P})/\boldsymbol{P} \qquad \text{and} \qquad \mathrm{Out}_G(\boldsymbol{P}) \cong N_G(\boldsymbol{P})/(\boldsymbol{P}\cdot C_G(\boldsymbol{P})).$$

The first group is finite of order prime to p since $Z(\boldsymbol{P})$ is a maximal p–toral subgroup of $C_G(\boldsymbol{P})$ which is also central. The second group is finite by Lemma 9.4(b). Hence $N_G(\boldsymbol{P})/\boldsymbol{P}$ is also finite.

(b) If P is p–centric in G, then $\overline{Z(P)} \le Z(\overline{P})$ is a maximal p–toral subgroup in $C_G(P) = C_G(\overline{P})$, and hence $\overline{P}$ is also p–centric in G.

(c) Assume $P \in \mathrm{Syl}_p(\boldsymbol{P})$. If $x \in Z(\boldsymbol{P})$ has p–power order, then since $[x, \boldsymbol{P}_0] = 1$, the only elements of p–power order in $x\boldsymbol{P}_0$ are those in xP_0. Since some element of $x\boldsymbol{P}_0$ lies in P and has p–power order, this shows that $x \in P$, and hence that $x \in Z(P)$. In other words, $Z(P) \in \mathrm{Syl}_p(Z(\boldsymbol{P}))$. So if $\boldsymbol{P}$ is p–centric in G, then $Z(P)$ is a maximal discrete p–toral subgroup of $C_G(P) = C_G(\boldsymbol{P})$, and hence P is also p–centric in G. □

We want to apply Proposition 4.6, to construct a centric linking system $\mathcal{L}_S^c(G)$ associated to $\mathcal{F}_S(G)$, and to show that $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$. This means constructing a rigidification of the homotopy functor $\mathbf{B}$: $P \mapsto BP$, which by Proposition 5.9 is equivalent to constructing a rigidification of the homotopy functor $\mathbf{B}_p^\wedge$: $P \mapsto BP_p^\wedge$. This last is closely related to the homotopy decomposition of BG constructed in [19; 20].

For any $\boldsymbol{S} \in \overline{\mathrm{Syl}}_p(G)$, we let $\mathcal{O}_S(G)$ denote the category whose objects are the p–toral subgroups of $\boldsymbol{S}$, and where

$$\mathrm{Mor}_{\mathcal{O}_S(G)}(\boldsymbol{P}, \boldsymbol{Q}) = \boldsymbol{Q}\backslash N_G(\boldsymbol{P}, \boldsymbol{Q}).$$

Define $\boldsymbol{B}\colon \mathcal{O}_{\boldsymbol{S}}(G) \longrightarrow \mathrm{Top}$ by setting

$$\boldsymbol{B}(\boldsymbol{P}) = EG/\boldsymbol{P} \qquad \text{and} \qquad \boldsymbol{B}(\boldsymbol{P} \xrightarrow{Qx} \boldsymbol{Q}) = (EG/\boldsymbol{P} \xrightarrow{\cdot x^{-1}} EG/\boldsymbol{Q}).$$

Let
$$\boldsymbol{\Phi}\colon \underset{\mathcal{O}_{\boldsymbol{S}}(G)}{\underrightarrow{\mathrm{hocolim}}}(\boldsymbol{B}) \longrightarrow EG/G = BG$$

be the map induced by the obvious surjections from $\boldsymbol{B}(\boldsymbol{P}) = EG/\boldsymbol{P}$ onto $BG = EG/G$.

Lemma 9.7 *Fix a maximal p–toral subgroup $\boldsymbol{S} \in \mathrm{Syl}_p(G)$. Let $\mathcal{O}^c_{\boldsymbol{S}}(G) \subseteq \mathcal{O}_{\boldsymbol{S}}(G)$ be the full subcategory whose objects are those p–toral subgroups of $\boldsymbol{S}$ which are p–centric in G, and let*

$$\boldsymbol{B}_c\colon \mathcal{O}^c_{\boldsymbol{S}}(G) \longrightarrow \mathrm{Top} \qquad \text{and} \qquad \boldsymbol{\Phi}_c\colon \underset{\mathcal{O}^c_{\boldsymbol{S}}(G)}{\underrightarrow{\mathrm{hocolim}}}(\boldsymbol{B}_c) \longrightarrow BG$$

be the restrictions of $\boldsymbol{B}$ and $\boldsymbol{\Phi}$, respectively. Then $\boldsymbol{\Phi}_c$ is a mod p homology equivalence.

Proof Define

$$X = \big\{\boldsymbol{P} \le \boldsymbol{S}\ p\text{–toral} \bigm| |N_G(\boldsymbol{P})/\boldsymbol{P}| < \infty,\ O_p(N_G(\boldsymbol{P})/\boldsymbol{P}) = 1 \text{ or } \boldsymbol{P} \text{ is } p\text{–centric}\big\}.$$

Let $\mathcal{O}^X_{\boldsymbol{S}}(G) \subseteq \mathcal{O}_{\boldsymbol{S}}(G)$ be the full subcategory with object set X, and let

$$\boldsymbol{B}_X\colon \mathcal{O}^X_{\boldsymbol{S}}(G) \longrightarrow \mathrm{Top} \qquad \text{and} \qquad \boldsymbol{\Phi}_X\colon \underset{\mathcal{O}^X_{\boldsymbol{S}}(G)}{\underrightarrow{\mathrm{hocolim}}}(\boldsymbol{B}_X) \longrightarrow BG$$

be the restrictions of $\boldsymbol{B}$ and $\boldsymbol{\Phi}$.

By [19, Theorem 1.4], $\boldsymbol{\Phi}_X$ is a mod p homology equivalence. So to prove the proposition, we must show that the inclusion of $\underrightarrow{\mathrm{hocolim}}(\boldsymbol{B}_c)$ in $\underrightarrow{\mathrm{hocolim}}(\boldsymbol{B}_X)$ is a mod p homology equivalence. Set $F = H^*(\boldsymbol{B}_X(-);\mathbb{F}_p)$, regarded as a functor on $\mathcal{O}^X_{\boldsymbol{S}}(G)^{\mathrm{op}}$. Let $F_0 \subseteq F$ be the subfunctor defined by setting $F_0(\boldsymbol{P}) = 0$ if $\boldsymbol{P}$ is p–centric in G and $F_0(\boldsymbol{P}) = F(\boldsymbol{P})$ otherwise. We claim that $\underleftarrow{\lim}^*(F_0) = 0$. Assuming this, we see that

$$\underset{\mathcal{O}^X_{\boldsymbol{S}}(G)}{\underleftarrow{\lim}^*}(H^*(\boldsymbol{B}_X(-);\mathbb{F}_p)) \cong \underset{\mathcal{O}^X_{\boldsymbol{S}}(G)}{\underleftarrow{\lim}^*}(F/F_0) \cong \underset{\mathcal{O}^c_{\boldsymbol{S}}(G)}{\underleftarrow{\lim}^*}(H^*(\boldsymbol{B}_c(-);\mathbb{F}_p)),$$

where the last step follows since there are no morphisms from any object of the subcategory to any object not in the subcategory. This shows that the spectral sequences for the cohomology of $\underrightarrow{\mathrm{hocolim}}(\boldsymbol{B}_X)$ and $\underrightarrow{\mathrm{hocolim}}(\boldsymbol{B}_c)$ have isomorphic E_2–terms, and hence that the inclusion is a mod p homology equivalence.

It remains to prove that $\varprojlim^*(F_0) = 0$. By [19, Proposition 1.6], X_0 contains finitely many G–conjugacy classes. Hence by [20, Proposition 5.4] and an appropriate finite filtration of F_0, it suffices to show that $\Lambda^*(N_{\boldsymbol{G}}(\boldsymbol{P})/\boldsymbol{P}; H^*(B\boldsymbol{P}; \mathbb{F}_p))$ vanishes for each p–toral subgroup $\boldsymbol{P}$ in $\mathcal{O}^X_{\boldsymbol{S}}(G)$ which is not p–centric. For each such $\boldsymbol{P}$, $C_{\boldsymbol{G}}(\boldsymbol{P}) \cdot \boldsymbol{P}/\boldsymbol{P} \cong C_{\boldsymbol{G}}(\boldsymbol{P})/Z(\boldsymbol{P})$ is a finite group of order a multiple of p which acts trivially on $H^*(EG/\boldsymbol{P}; \mathbb{F}_p) \cong H^*(B\boldsymbol{P}; \mathbb{F}_p)$, and hence $\Lambda^*(N_{\boldsymbol{G}}(\boldsymbol{P})/\boldsymbol{P}; H^*(B\boldsymbol{P}; \mathbb{F}_p)) = 0$ by [20, Proposition 5.5]. □

We are now ready to construct a rigidification of the homotopy functor $\mathbf{B}^\wedge_p$.

Proposition 9.8 *Fix a maximal discrete p–toral subgroup $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$ for short. Let $\mathcal{F}^{cs} \subseteq \mathcal{F}^c$ be the full subcategory of subgroups $P \le S$ which are p–centric in G and snugly embedded, and let $\mathcal{O}^{cs}(\mathcal{F}) \subseteq \mathcal{O}^c(\mathcal{F})$ be its orbit category. Then there is a functor*

$$\widehat{B}\colon \mathcal{O}^{cs}(\mathcal{F}) \longrightarrow \mathsf{Top}$$

which is a rigidification of the homotopy functor $\mathbf{B}^\wedge_p$, and a homotopy equivalence

$$\widehat{\Phi}\colon \Big(\underset{\mathcal{O}^{cs}(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}(\widehat{B})\Big)^\wedge_p \longrightarrow BG^\wedge_p.$$

Proof Set $\boldsymbol{S} = \overline{S} \in \overline{\mathrm{Syl}}_p(G)$. We will construct orbit categories and functors as indicated in the diagram

$$\begin{tikzcd}
\mathcal{O}^{cs}(\mathcal{F}) \arrow[r, "\mathrm{cl}"] \arrow[dr, "B_{cs}"'] & \overline{\mathcal{O}}^c_{\boldsymbol{S}}(G) \arrow[d, "\overline{\boldsymbol{B}}_c"] & \mathcal{O}^c_{\boldsymbol{S}}(G) \arrow[l, two heads, "\mathrm{pr}"'] \arrow[dl, "\boldsymbol{B}_c"] \\
& \mathsf{Top} &
\end{tikzcd}$$

together with mod p homology equivalences:

$$\begin{tikzcd}
\underrightarrow{\mathrm{hocolim}}(B_{cs}) \arrow[r, "\mathrm{cl}_*"] \arrow[dr, "\Phi_{cs}"'] & \underrightarrow{\mathrm{hocolim}}(\overline{\boldsymbol{B}}_c) \arrow[r, "\nu", "\simeq"'] \arrow[d, "\overline{\boldsymbol{\Phi}}_c"] & \underrightarrow{\mathrm{hocolim}}(\boldsymbol{B}_c) \arrow[dl, "\boldsymbol{\Phi}_c"] \\
& BG &
\end{tikzcd}$$

We then set $\widehat{B} = (B_{cs})^\wedge_p$.

The category $\mathcal{O}_S^c(G)$, together with $\boldsymbol{B}_c$ and $\boldsymbol{\Phi}_c$, were already constructed in Lemma 9.7. We construct $\mathcal{O}_S^c(G)$, $\boldsymbol{B}_c$, and $\overline{\boldsymbol{\Phi}}_c$ in Step 1 (and prove the properties we need); and then do the same for $\mathcal{O}^{cs}(\mathcal{F})$, B_{cs}, and Φ_{cs} in Step 2.

Step 1 Define $\overline{\mathcal{O}}_S^c(G)$ by setting

$$\mathrm{Ob}(\overline{\mathcal{O}}_S^c(G)) = \mathrm{Ob}(\mathcal{O}_S^c(G)) = \{\boldsymbol{P} \le \boldsymbol{S} \mid \boldsymbol{P} \ p\text{–toral and } p\text{–centric in } G\}$$

and
$$\mathrm{Mor}_{\overline{\mathcal{O}}_S^c(G)}(\boldsymbol{P},\boldsymbol{Q}) = \boldsymbol{Q}\backslash N_G(\boldsymbol{P},\boldsymbol{Q})/C_G(\boldsymbol{P}) \cong \mathrm{Rep}_G(\boldsymbol{P},\boldsymbol{Q}).$$

Let $\overline{\boldsymbol{B}}_c$ be the left homotopy Kan extension of $\boldsymbol{B}_c$ along the projection functor. Let $\overline{\boldsymbol{\Phi}}_c$ be the composite of $\boldsymbol{\Phi}_c$ with the standard homotopy equivalence

$$\nu\colon \underset{\overline{\mathcal{O}}_S^c(G)}{\underrightarrow{\mathrm{hocolim}}}(\overline{\boldsymbol{B}}_c) \xrightarrow{\ \simeq\ } \underset{\mathcal{O}_S^c(G)}{\underrightarrow{\mathrm{hocolim}}}(\boldsymbol{B}_c)$$

of [17, Proposition 5.5]. Thus $\overline{\boldsymbol{\Phi}}_c$ is a mod p homology equivalence by Lemma 9.7, and it remains only to show that $\overline{\boldsymbol{B}}_c$ is a rigidification of $\mathbf{B}_p^\wedge$ (after p–completion).

This means showing that the natural morphism of functors

$$\boldsymbol{B}_c \longrightarrow \overline{\boldsymbol{B}}_c \circ \mathrm{pr}$$

(natural up to homotopy) is a mod p homology equivalence on all objects. By definition, for each $\boldsymbol{P}$,

$$\overline{\boldsymbol{B}}_c(\boldsymbol{P}) = \underset{\mathrm{pr}\downarrow \boldsymbol{P}}{\underrightarrow{\mathrm{hocolim}}}(\boldsymbol{B}_c \circ \xi).$$

Here, $\mathrm{pr}\downarrow\boldsymbol{P}$ is the overcategory whose objects are the morphisms $\boldsymbol{Q} \xrightarrow{\alpha} \boldsymbol{P}$ in $\overline{\mathcal{O}}_S^c(G)$, and where a morphism from $(\boldsymbol{Q},\alpha)$ to $(\boldsymbol{R},\beta)$ is a morphism $\varphi \in \mathrm{Mor}_{\mathcal{O}_S^c(G)}(\boldsymbol{Q},\boldsymbol{R})$ such that $\alpha = \beta\circ\mathrm{pr}(\varphi)$. Also, ξ is the forgetful functor from $\mathrm{pr}\downarrow\boldsymbol{P}$ to $\mathcal{O}_S^c(G)$.

Consider the spectral sequence

$$E_2^{i,j} \cong \underset{\mathrm{pr}\downarrow\boldsymbol{P}}{\underleftarrow{\lim}}{}^i\big(H^j(\boldsymbol{B}_c\circ\xi(-);\mathbb{F}_p)\big) \implies H^{i+j}(\overline{\boldsymbol{B}}_c(\boldsymbol{P});\mathbb{F}_p)\,. \tag{33}$$

For each $\boldsymbol{Q}$ in $\mathcal{O}_S^c(G)$, set

$$K(\boldsymbol{Q}) = \mathrm{Ker}\big[\mathrm{Aut}_{\mathcal{O}_S^c(G)}(\boldsymbol{Q}) \xrightarrow{\mathrm{pr}_{\boldsymbol{P}}} \mathrm{Aut}_{\overline{\mathcal{O}}_S^c(G)}(\boldsymbol{Q})\big] \cong C_G(\boldsymbol{Q})/Z(\boldsymbol{Q}).$$

This is a finite group of order prime to p, and it acts trivially on $H^*(EG/\boldsymbol{Q})$. Since $K(\boldsymbol{Q})$ acts trivially on the homology

$$H^*(\boldsymbol{B}_c\circ\xi(\boldsymbol{Q},\alpha);\mathbb{F}_p) = H^*(EG/\boldsymbol{Q};\mathbb{F}_p)$$

for each object $(\boldsymbol{Q},\alpha)$, this functor factors through the overcategory $\mathrm{Id}\downarrow\boldsymbol{P}$. The projection of $\mathrm{pr}\downarrow\boldsymbol{P}$ onto $\mathrm{Id}\downarrow\boldsymbol{P}$ satisfies the hypotheses of [6, Lemma 1.3] (in particular, the target category is obtained from the source by dividing out by these automorphism groups $K(\boldsymbol{Q})$ of order prime to p), and hence

$$\varprojlim_{\mathrm{pr}\downarrow\boldsymbol{P}}{}^{i}\big(H^*(\boldsymbol{B}_c\circ\xi(-);\mathbb{F}_p)\big)\cong \varprojlim_{\mathrm{Id}\downarrow\boldsymbol{P}}{}^{i}\big(H^*(\boldsymbol{B}_c\circ\xi(-);\mathbb{F}_p)\big)\cong \begin{cases} H^*(\boldsymbol{B}_c(\boldsymbol{P});\mathbb{F}_p) & \text{if } i=0\\ 0 & \text{if } i>0.\end{cases}$$

Here, the last isomorphism holds since $\mathrm{Id}\downarrow P$ has final object $(\boldsymbol{P},\mathrm{Id})$. The spectral sequence (33) thus collapses, and so $H^*(\overline{\boldsymbol{B}}_c(\boldsymbol{P});\mathbb{F}_p)\cong H^*(\boldsymbol{B}_c(\boldsymbol{P});\mathbb{F}_p)$ (and the isomorphism is induced by the natural inclusion of $\boldsymbol{B}_c(\boldsymbol{P})$ into $\overline{\boldsymbol{B}}_c(\boldsymbol{P})$).

Step 2 The "closure functor"

$$\mathcal{O}^{cs}(\mathcal{F})\xrightarrow{\ \mathrm{cl}\ }\bar{\mathcal{O}}^{c}_{S}(G)$$

is defined to send P to $\overline{P}$. It induces a bijection between isomorphism classes of objects by definition of $\mathcal{F}^{cs}$ and Lemma 9.6(b,c), and induces bijections on morphism sets by Lemma 9.4(c). So this is an equivalence of categories.

Set $B_{cs}=\overline{\boldsymbol{B}}_c\circ\mathrm{cl}$. Since $\overline{\boldsymbol{B}}_c$ is (after p–completion) a rigidification of the homotopy functor $\boldsymbol{P}\mapsto B\boldsymbol{P}^\wedge_p$ by Step 1, and since $BP^\wedge_p\simeq B\overline{P}^\wedge_p$ when P is snugly embedded (Proposition 9.2), B_{cs} is a rigidification of the homotopy functor $\mathbf{B}^\wedge_p\colon P\mapsto BP^\wedge_p$ (again, up to p–completion).

Now let Φ_{cs} be the composite of $\overline{\boldsymbol{\Phi}}_c$ with the map

$$\operatorname*{hocolim}_{\mathcal{O}^{cs}(\mathcal{F})}(B_{cs})\xrightarrow{\ \mathrm{cl}_*\ }\operatorname*{hocolim}_{\bar{\mathcal{O}}^{c}_{S}(G)}(\overline{\boldsymbol{B}}_c)$$

induced by cl. Then $\operatorname{hocolim}(\overline{\boldsymbol{B}}_c)\simeq\operatorname{hocolim}(B_{cs})$ since cl is an equivalence of categories, and thus Φ_{cs} is a mod p homology equivalence since $\overline{\boldsymbol{\Phi}}_c$ is by Step 1.

Now set $\widehat{B}=(B_{cs})^\wedge_p$ and let

$$\widehat{\Phi}\colon \Big(\operatorname*{hocolim}_{\mathcal{O}^{cs}(\mathcal{F})}(\widehat{B})\Big)^\wedge_p\simeq\Big(\operatorname*{hocolim}_{\mathcal{O}^{cs}(\mathcal{F})}(B_{cs})\Big)^\wedge_p\xrightarrow{\ (\Phi_{cs})^\wedge_p\ }BG^\wedge_p$$

be the completion of Φ_{cs}. Then $\widehat{B}$ is a rigidification of the homotopy functor $\mathbf{B}^\wedge_p$ (see Proposition 9.2), and $\widehat{\Phi}$ is a homotopy equivalence. □

We also need the following result about snugly embedded subgroups:

Lemma 9.9 *For each discrete p–toral subgroup $P\le G$, $P^\bullet$ is snugly embedded.*

Proof Fix $S \in \mathrm{Syl}_p(G)$, and set $T = S_0$ and $p^m = \exp(S/T)$. By Definition 3.1, $P^\bullet = P \cdot I(Q)$, where $Q = P^{[m]} = \langle g^{p^m} \mid g \in P \rangle \geq P_0$, and $I(Q) = T^{C_W(Q)}$. Then $I(Q)$ contains all elements of p–power order in $\overline{I(Q)} = \overline{T}^{C_W(Q)}$, and hence is snugly embedded. Since $[P^\bullet : I(Q)]$ is finite, $P^\bullet$ is also snugly embedded. □

We are now ready to prove the main result.

Theorem 9.10 *Fix a compact Lie group G and a maximal discrete p–toral subgroup $S \in \mathrm{Syl}_p(G)$. Then there exists a centric linking system $\mathcal{L}_S^c(G)$ associated to $\mathcal{F}_S(G)$ such that $(S, \mathcal{F}_S(G), \mathcal{L}_S(G))$ is a p–local compact group with classifying space $|\mathcal{L}_S(G)|_p^\wedge \simeq BG_p^\wedge$.*

Proof Set $\mathcal{F} = \mathcal{F}_S(G)$ for short; a saturated fusion system by Lemma 9.5. Let $\mathcal{F}^{cs} \subseteq \mathcal{F}^c$ be the full subcategory with objects the set of all $P \leq S$ which are p–centric and snugly embedded in G, and let $\mathcal{O}^{cs}(\mathcal{F})$ be its orbit category. By Lemma 9.9, $\mathcal{F}^{cs} \supseteq \mathcal{F}^{c\bullet}$.

By Proposition 9.8, there is a functor

$$\widehat{B}\colon \mathcal{O}^{cs}(\mathcal{F}) \longrightarrow \mathrm{Top}$$

which is a rigidification of the homotopy functor $\mathbf{B}_p^\wedge$ and a homotopy equivalence

$$\widehat{\Phi}\colon \Big(\underset{\mathcal{O}^{cs}(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}(\widehat{B})\Big)_p^\wedge \longrightarrow BG_p^\wedge.$$

By Proposition 5.9, there is a functor $\widetilde{B}\colon \mathcal{O}^{cs}(\mathcal{F}) \longrightarrow \mathrm{Top}$ which is a rigidification of the homotopy functor $\mathbf{B}$, and a natural transformation of functors $\chi\colon \widetilde{B} \longrightarrow \widehat{B}$ such that $\chi(P)$ is homotopic to the completion map for each P. By Proposition 4.6, there is a centric linking system $\mathcal{L}_S^{cs}(G)$ associated to $\mathcal{F}^{cs}$ whose nerve has the homotopy type of $\underrightarrow{\mathrm{hocolim}}(\widetilde{B})$, and thus

$$|\mathcal{L}_S^{cs}(G)|_p^\wedge \simeq \Big(\underset{\mathcal{O}^{cs}(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}(\widetilde{B})\Big)_p^\wedge \simeq \Big(\underset{\mathcal{O}^{cs}(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}(\widehat{B})\Big)_p^\wedge \simeq BG_p^\wedge.$$

Define $\mathcal{F}^c \xrightarrow{\Psi} \mathcal{F}^{cs}$ by setting $\Psi(P) = P \cdot (\overline{P}_0)_{(p)}$, where $(\overline{P}_0)_{(p)}$ denotes the subgroup of elements of p–power order in the torus $\overline{P}_0$. By Lemma 9.4(c), for each $P \in \mathrm{Ob}(\mathcal{F}^c)$ and $Q \in \mathrm{Ob}(\mathcal{F}^{cs})$,

$$\mathrm{Rep}_G(P, Q) \cong \mathrm{Rep}_G(\overline{P}, \overline{Q}) \cong \mathrm{Rep}_G(\Psi(P), Q),$$

and thus Ψ is left adjoint to the inclusion. Also, for each P, $C_G(\Psi(P)) = C_G(\overline{P}) = C_G(P)$, and hence $Z(\Psi(P)) = Z(P)$. So if we define $\mathcal{L}_S^c(G)$ to be the pullback of

$\mathcal{L}_S^{cs}(G)$ and $\mathcal{F}^c$ over $\mathcal{F}^{cs}$, then it is a centric linking system, and $|\mathcal{L}_S^c(G)| \simeq |\mathcal{L}_S^{cs}(G)|$. (Compare this argument with the proof of Proposition 4.5(b).) □

The above construction of the linking system of G has the disadvantage that it seems rather arbitrary. We know, by Theorem 7.4, that there is (up to isomorphism) at most one linking system $\mathcal{L}_S^c(G)$ such that $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$, but we would really like to have a more obvious algebraic connection between $\mathcal{L}_S^c(G)$ and the group G itself. We end this section by showing that $\mathcal{L}_S^c(G)$ can, in fact, be obtained as a subquotient of the transporter category of G—although not in a completely canonical way.

Fix a compact Lie group G, and choose $S \in \mathrm{Syl}_p(G)$. The transporter category $\mathcal{T}_S^c(G)$ of G over S is the category whose objects are the subgroups of S that are p–centric in G, and where $\mathrm{Mor}_{\mathcal{T}_S^c(G)}(P,Q) = N_G(P,Q)$, for each pair of objects P and Q of $\mathcal{T}_S^c(G)$. Let $C_G\colon \mathcal{O}^c(\mathcal{F})^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ be the functor which sends P to its centralizer. For any subfunctor $\Phi \subseteq C_G$, $\mathcal{T}_S^c(G)/\Phi$ denotes the quotient category with the same objects as $\mathcal{T}_S^c(G)$, and where

$$\mathrm{Mor}_{\mathcal{T}_S^c(G)/\Phi}(P,Q) = \mathrm{Mor}_{\mathcal{T}_S^c(G)}(P,Q)/\Phi(P) = N_G(P,Q)/\Phi(P).$$

For example, in this notation, $\mathcal{F}_S^c(G) = \mathcal{T}_S^c(G)/C_G$.

For each $P \in \mathrm{Ob}(\mathcal{F}^c)$, there is a central extension

$$1 \longrightarrow \overline{Z(P)} \longrightarrow C_G(P) \longrightarrow C_G(P)/\overline{Z(P)} \longrightarrow 1,$$

where $\overline{Z(P)}$ is abelian and p–toral and $C_G(P)/\overline{Z(P)}$ is finite of order prime to p (by definition of p–centric). Hence the set of elements of $C_G(P)$ of finite order prime to p forms a subgroup, which we denote here $\nu_{p'}(P)$. Also, $Z(P)$ and $\nu_{p'}(P)$ are both normal subgroups of $C_G(P)$, and the quotient group $C_G(P)/(Z(P)\times\nu_{p'}(P))$ is a $\mathbb{Q}$–vector space. As earlier, we write $\mathcal{Z}(P) = Z(P)$, and regard $\mathcal{Z}$, $\nu_{p'}$, and $\mathcal{Z}\times\nu_{p'}$ as subfunctors of C_G.

Lemma 9.11 *The extension*

$$\mathcal{T}_S^c(G)/(\mathcal{Z}\times\nu_{p'}) \xrightarrow{\ \mathrm{pr}\ } \mathcal{F}^c$$

is split, by a splitting which sends $\mathrm{Out}_P(P)$ to $P/Z(P)$ for each object P; and such a splitting is unique up to natural isomorphism of functors.

Proof For all $P,Q \in \mathrm{Ob}(\mathcal{F}^c)$, choose maps

$$\sigma_{P,Q}\colon \mathrm{Hom}_{\mathcal{F}}(P,Q) \longrightarrow N_G(P,Q)\big/\big(Z(P)\times\nu_{p'}(P)\big) = \mathrm{Mor}_{\mathcal{T}_S^c(G)/(\mathcal{Z}\times\nu_{p'})}(P,Q)$$

which split the natural projection. This can be done in such a way that for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ and each $g \in Q$, $\sigma_{P,Q}(c_g\circ\varphi) = [g]\circ\sigma_{P,Q}(\varphi)$ (define it first on orbit representatives for the action of $\mathrm{Inn}(Q)$ and then extend it appropriately). Also, when $Q = P$, we let $\sigma_{P,P}(\mathrm{Id}_P)$ be the class of $1 \in N_G(P)$.

The "deviation" of $\{\sigma_{P,Q}\}$ from being a functor is a 2–cocycle with values in the functor $C_G/(\mathcal{Z}\times\nu_{p'})$, and the assumption that they commute with the $\mathrm{Inn}(Q)$–actions implies that we get a cocycle over the orbit category $\mathcal{O}^c(\mathcal{F})$. If, furthermore, this cocycle is a coboundary, then the $\sigma_{P,Q}$ can be replaced by maps $\sigma'_{P,Q}$ which define a splitting functor. The obstruction to the existence of such a splitting thus lies in

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^2(C_G/(\mathcal{Z}\times\nu_{p'})).$$

In a similar (but simpler) way, the obstruction to uniqueness is seen to lie in

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^1(C_G/(\mathcal{Z}\times\nu_{p'})).$$

We will show that both of these groups vanish, using Lemma 5.7 (and an argument similar to that used to prove Proposition 5.8). Let F be the functor $C_G/(\mathcal{Z}\times\nu_{p'})$. As in the proof of Proposition 5.8, set $T = S_0$, $Q = C_S(T)$, and $\Gamma = \mathrm{Out}_{\mathcal{F}}(Q)$. Set $M = \overline{T}/(\text{torsion})$, regarded as a $\mathbb{Q}[\Gamma]$–module. Let

$$\Phi\colon \mathcal{O}_p(\Gamma)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{–mod}$$

be the functor $\Phi(\Pi) = M^{\Pi}$ for all p–subgroups $\Pi \le \Gamma$. Then $F(P) \cong \Phi(\mathrm{Out}_P(Q))$ (functorially) for all $P \le S$ containing Q, and $\mathrm{Out}_Q(P)$ acts trivially on $F(P) \cong \overline{Z(P)}/(\text{torsion})$ for each P. The hypotheses of Lemma 5.7 thus hold, and so

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^i(F) \cong \varprojlim_{\mathcal{O}_p(\Gamma)}{}^i(\Phi)$$

for all i. Since Φ is a Mackey functor, by [18, Proposition 5.14] these groups vanish for all $i \ge 1$. □

We are now ready to construct a more explicit linking system $\mathcal{L}^c_S(G)$, and prove it is isomorphic to the one already constructed in Theorem 9.10.

Proposition 9.12 *Let G be a compact Lie group, and choose $S \in \mathrm{Syl}_p(G)$. Fix a splitting s of $\mathcal{T}^c_S(G)/(\mathcal{Z}\times\nu_{p'}) \xrightarrow{\mathrm{pr}} \mathcal{F}^c$, and define $\mathcal{L}^c_S(G)$ to be the pullback category in the following pullback diagram:*

$$\begin{tikzcd}
\mathcal{L}^c_S(G) \arrow[r, "\overline{s}"] \arrow[d, "\pi"'] & \mathcal{T}^c_S(G)/\nu_{p'} \arrow[d, "\mathrm{pr}"] \\
\mathcal{F}^c_S(G) \arrow[r, "s"] & \mathcal{T}^c_S(G)/(\mathcal{Z}\times\nu_{p'})
\end{tikzcd}$$

Then $\mathcal{L}_S^c(G)$ is a centric linking system associated to $\mathcal{F}_S^c(G)$, and is isomorphic to the centric linking system of Theorem 9.10. In particular, the map $\mathcal{L}_S^c(G) \xrightarrow{\overline{s}} \mathcal{T}_S^c(G)/\nu_{p'}$ describes the linking system $\mathcal{L}_S^c(G)$ as a subquotient of the transporter category.

Proof We will first show that the pullback category $\mathcal{L}_S^c(G)$ is a centric linking system associated to $\mathcal{F}_S(G)$. Since s and pr are the identity on objects, we can as well assume that the pullback category $\mathcal{L}_S^c(G)$ has the same objects, and that $\overline{s}$ and π are the identity on objects. Then for any pair of objects $P, Q \le S$ p–centric in G, we have

$$\begin{aligned}&\mathrm{Mor}_{\mathcal{L}_S^c(G)}(P,Q) = \\ &\qquad \bigl\{ (\varphi,\psi) \bigm| \varphi \in \mathrm{Mor}_{\mathcal{F}_S^c(G)}(P,Q),\ \psi \in \mathrm{Mor}_{\mathcal{T}_S^c(G)/\nu_{p'}}(P,Q), \text{ and } \pi(\varphi) = \mathrm{pr}(\psi) \bigr\}\end{aligned}$$

Now, for each $P \le S$ which is p–centric in G, we have $P \le \mathbb{N}_G(P)/\nu_{p'}(P)$ and then we can define distinguished homomorphisms

$$\delta_P \colon P \longrightarrow \mathrm{Aut}_{\mathcal{L}_S^c(G)}(P)$$

by setting $\delta_P(g) = (c_g, g)$. Conditions (A), (B), and (C) in the definition of a centric linking system are easily checked.

Next we will find a map $|\mathcal{L}_S^c(G)| \longrightarrow BG_p^\wedge$ that commutes with the respective natural maps from BS. To do this, we first lift $\mathcal{L}_S^c(G)$ to a subcategory $\widetilde{\mathcal{L}}_S^c(G)$ of the transporter category $\mathcal{T}_S^c(G)$, defined via the pullback square:

$$\begin{tikzcd} \widetilde{\mathcal{L}}_S^c(G) \arrow[r, "\mathrm{incl}"] \arrow[d] & \mathcal{T}_S^c(G) \arrow[d] \\ \mathcal{L}_S^c(G) \arrow[r, "\overline{s}"] & \mathcal{T}_S^c(G)/\nu_{p'} \end{tikzcd}$$

We will then construct the maps in the following commutative diagram:

$$\begin{tikzcd} & BS \arrow[dl] \arrow[d] \arrow[dr] & \\ |\mathcal{L}_S^c(G)|_p^\wedge & |\widetilde{\mathcal{L}}_S^c(G)|_p^\wedge \arrow[l, "\simeq"'] \arrow[r] & BG_p^\wedge \end{tikzcd}$$

We proceed in two steps.

(a) A map $|\widetilde{\mathcal{L}}_S^c(G)| \longrightarrow |\mathcal{L}_S^c(G)|$ commuting with the respective natural maps from BS is induced by the functor $\widetilde{\mathcal{L}}_S^c(G) \longrightarrow \mathcal{L}_S^c(G)$. We will show that it is a mod p homology equivalence.

By definition of $\widetilde{\mathcal{L}}^c_S(G)$, for all $P, Q \le S$ centric, we have that $\nu'_p(P)$ acts freely on $\mathrm{Mor}_{\widetilde{\mathcal{L}}^c_S(G)}(P,Q)$ and the orbit set if $\mathrm{Mor}_{\mathcal{L}^c_S(G)}(P,Q)$. In particular,

$$\nu'_p(P) = \mathrm{Ker}\Big[\,\mathrm{Aut}_{\widetilde{\mathcal{L}}^c_S(G)}(P) \longrightarrow \mathrm{Aut}_{\mathcal{L}^c_S(G)}(P)\Big].$$

Recall that $\nu_{p'}(P)$ is the subgroup of elements of $C_G(P)$ of finite order prime to p. It sits in an extension $\nu_{p'}(P)^0 \longrightarrow \nu_{p'}(P) \longrightarrow C'_G(P)$, where $\nu_{p'}(P)^0$ is the set of elements of the maximal torus of $\overline{Z(P)}$ of finite order prime to p and $C'_G(P) = C_G(P)/\overline{Z(P)}$. Therefore $\nu_{p'}(P)$ is locally finite and can be written as a union $\bigcup_{m\ge 0}\nu_{p'}(P)^m$ of finite groups of order prime to p. A generalized version of [6, Lemma 1.3] now applies to the constant functor defined on $\mathcal{L}^c_S(G)$ and the result follows.

In fact, [6, Lemma 1.3] generalizes to allow that the kernels $K(-)$ be countable increasing unions of finite groups of order prime to p. Proving this requires showing, for any such K, that $H_0(K;-)$ is an exact functor on the category of $\mathbb{Z}_{(p)}[K]$–modules. But if K is the union of a sequence of subgroups $K_1 \le K_2 \le \cdots$ each of which is finite, then $H_0(K_i;-)$ is exact for each i, $H_0(K;M) \cong \operatorname*{colim}_{\overrightarrow{i}} H_0(K_i;M)$ for each M, and hence $H_0(K;-)$ is exact since direct limits of this type are exact.

(b) A map $|\mathcal{L}^c_S(G)| \longrightarrow BG$ that commutes up to homotopy with the respective maps from BS is defined by composing the inclusion $\widetilde{\mathcal{L}}^c_S(G) \xrightarrow{\text{incl}} \mathcal{T}^c_S(G)$ with the functor $\mathcal{T}^c_S(G) \longrightarrow \mathcal{B}(G)$. Here, $\mathcal{B}(G)$ is the topological category with one object and the Lie group G as morphisms (and all other categories are discrete), and the functor sends the morphism $g \in N_G(P,Q)$ to $g \in G$ for all objects P, Q of $\mathcal{T}^c_S(G)$. The nerve of $\mathcal{B}(G)$ is the topological bar construction $\mathbf{B}G \simeq BG$, and the composite functor induces a map $|\widetilde{\mathcal{L}}^c_S(G)| \longrightarrow |\mathcal{B}(G)| \simeq BG$.

Finally, Theorem 9.10 defines the centric linking system of G over S and shows that the p–completed nerve is homotopy equivalent to $BG^\wedge_p$. This combines with the map constructed above, so that Lemma 7.3 implies that the pullback category $\mathcal{L}^c_S(G)$ is isomorphic to the centric linking system of Theorem 9.10, and then, also, that the map $|\widetilde{\mathcal{L}}^c_S(G)| \longrightarrow BG$ constructed in step (b) is actually a homotopy equivalence after p–completion. □

10 *p*–compact groups

A p–compact group is a p–complete version of a finite loop space. As defined by Dwyer and Wilkerson in [11], a p–compact group is a triple (X, BX, e), where X is a space such that $H^*(X;\mathbb{F}_p)$ is finite, BX is a pointed p–complete space, and

$e\colon X \longrightarrow \Omega(BX)$ is a homotopy equivalence. If G is a compact Lie group such that the group of components $\pi_0(G)$ is a finite p–group, then upon setting $B\widehat{G} = BG^\wedge_p$ and $\widehat{G} = \Omega(B\widehat{G})$, the triple $(\widehat{G}, B\widehat{G}, \mathrm{Id})$ is a p–compact group. For general references on p–compact groups, we refer to the original papers by Dwyer and Wilkerson [11] and [12], and also to the survey article by Jesper Møller [25].

When $T \cong (S^1)^r$ is a torus of rank r, then the p–completion $\widehat{T} = \Omega(BT^\wedge_p)$ of T is called a *p–compact torus* of rank r. Both $B\widehat{T} \simeq K((\widehat{\mathbb{Z}}_p)^r, 2)$ and $\widehat{T} \simeq K((\widehat{\mathbb{Z}}_p)^r, 1)$ are Eilenberg–Mac Lane spaces. A *p–compact toral group* is a p–compact group $(\widehat{P}, B\widehat{P}, e)$ such that $\pi_1(B\widehat{P})$ is a p–group, and the identity component of $\widehat{P}$ is a p–compact torus with classifying space the universal cover of $B\widehat{P}$.

If X is either a discrete p–toral group or a p–compact group, and Y is a p–compact group, a homomorphism $f\colon X \to Y$ is by definition a pointed map $Bf\colon BX \to BY$. Two homomorphisms $f, f'\colon X \to Y$ are conjugate if Bf and Bf' are freely homotopic, ie via a homotopy which need not preserve basepoints. If $f\colon X \to Y$ is a homomorphism, the homotopy fibre of Bf is denoted $Y/f(X)$, or just Y/X if f is understood from the context. With this notation, f is called a *monomorphism* if $H^*(Y/f(X); \mathbb{F}_p)$ is finite. By [11, Proposition 9.11], a homomorphism f is a monomorphism if and only if $H^*(BX; \mathbb{F}_p)$ is a finitely generated $H^*(BY; \mathbb{F}_p)$–module via $H^*(f; \mathbb{F}_p)$.

If $\widehat{P}$ is an arbitrary p–compact toral group, a *discrete approximation* to $\widehat{P}$ is a pair (P, f), where P is a discrete p–toral group and $Bf\colon BP \to B\widehat{P}$ induces an isomorphism in mod p cohomology. By [11, Proposition 6.9], every compact p–toral group has a discrete approximation. Each discrete p–toral group P is a discrete approximation of $(\widehat{P}, B\widehat{P}, \mathrm{Id})$, where $B\widehat{P} = BP^\wedge_p$ and $\widehat{P} = \Omega(B\widehat{P})$. Hence every monomorphism $f\colon P \to X$ from a discrete p–toral group to a p–compact group factors as $P \longrightarrow \widehat{P} \xrightarrow{\hat{f}} X$: a discrete approximation followed by a monomorphism of p–compact groups. Lemma 1.10 says, among other things, that any two discrete approximations of a p–compact toral group are isomorphic.

If $f\colon X \to Y$ is a homomorphism of p–compact groups, the centralizer of f in Y is defined to be the triple $(C_Y(X, f), BC_Y(X, f), \mathrm{Id})$, where

$$BC_Y(X, f) = \mathrm{Map}(BX, BY)_{Bf} \quad \text{and} \quad C_Y(X, f) = \Omega(BC_Y(X, f)).$$

Whenever f is understood, we simply write $C_Y(X)$ for $C_Y(X, f)$.

A *discrete p–toral subgroup* of a p–compact group X is a pair (P, f), where P is a discrete p–toral group and $\widehat{P} \xrightarrow{f} X$ is a monomorphism. We write $BC_X(P, f) = BC_X(\widehat{P}, f) = \mathrm{Map}(BP, BX)_{Bf}$ and $C_X(P, f) = C_X(\widehat{P}, f)$ for short. The group $C_X(P)$ is p–compact by [11, Section 5–6], and the homomorphism $C_X(P) \longrightarrow X$

(induced by evaluation at the basepoint of BP) is a monomorphism. The subgroup (P,f) is called *central* if this monomorphism $C_X(P) \longrightarrow X$ is an equivalence.

Proposition 10.1 *Let X be any p–compact group.*

(a) *X has a maximal discrete p–toral subgroup $S \xrightarrow{f} X$. If $P \xrightarrow{u} X$ is any other discrete p–toral subgroup of X, then $Bu \simeq Bf \circ B\psi$ for some $\psi \in \mathrm{Hom}(P,S)$; and (P,u) is maximal if and only if $p \nmid \chi(X/u(\widehat{P}))$. Here, Euler characteristics are taken with respect to homology with coefficients in $\mathbb{F}_p$.*

(b) *The centralizer $C_X(P,f)$ of any discrete p–toral subgroup $P \xrightarrow{f} X$ is again a p–compact group, and a subgroup of X. Also, if $P = \bigcup_{n=1}^{\infty} P_n$, then $BC_X(P) \simeq BC_X(P_n)$ for n large enough.*

(c) *A discrete p–toral subgroup $P \xrightarrow{f} X$ is central if and only if there is a map $BP \times BX \longrightarrow BX$ whose restriction to $BP \times *$ is Bf and whose restriction to $* \times BX$ is the identity. When this is the case, then P is abelian, and there is a fibration sequence $BP^{\wedge}_p \xrightarrow{f} BX \longrightarrow B(X/P)$ where $B(X/P)$ is the classifying space of a p–compact group X/P.*

Proof Point (a) follows mostly from [12, Propositions 2.10 & 2.14] together with Lemma 1.10. If (P,u) is not maximal, then since u factors through S, $\chi(X/u(\widehat{P})) = \chi(X/f(\widehat{S})) \cdot \chi(\widehat{S}/\widehat{P})$, and the last factor is a multiple of p.

Point (b) is shown in [11, Proposition 5.1 & Theorem 6.1]. In point (c), a central subgroup is abelian by [12, Theorem 1.2], while the other two claims are shown in [11, Lemma 8.6 & Proposition 8.3]. □

As in other contexts, the maximal discrete p–toral subgroups of a p–compact group X will be referred to as Sylow p–subgroups of X.

The fusion system of a p–compact group is easily defined: it is just the fusion system of the space BX, as defined in [7, Definition 7.1].

Definition 10.2 For any p–compact group X with Sylow p–subgroup $S \xrightarrow{f} X$, let $\mathcal{F}_{S,f}(X)$ be the category whose objects are the subgroups of S, and where for $P, Q \le S$,

$$\mathrm{Mor}_{\mathcal{F}_{S,f}(X)}(P,Q) = \mathrm{Hom}_X(P,Q) \overset{\mathrm{def}}{=} \bigl\{\varphi \in \mathrm{Hom}(P,Q) \bigm| Bf|_{BQ} \circ B\varphi \simeq Bf|_{BP}\bigr\}.$$

We next want to show that $\mathcal{F}_{S,f}(X)$ is saturated. Before doing this, we need to define and study normalizers of discrete p–toral subgroups of p–compact groups. We also need to establish an "adjointness" relation which corresponds to the equivalence (for groups) between homomorphisms $Q \longrightarrow N_G(P)$ and homomorphisms $P \rtimes Q \longrightarrow G$.

Fix a p–compact group X and a Sylow p–subgroup $f\colon S \longrightarrow X$. For any subgroup $P \le S$ and any discrete p–toral subgroup $K \le \mathrm{Aut}_X(P)$, set

$$BN_X^K(P) = \big(EK \times_K BC_X(P)\big)^\wedge_p,$$

where K acts on $BC_X(P) = \mathrm{Map}(BP, BX)_{B\alpha}$ via the action on P. Set $N_X^K(P) = \Omega(BN_X^K(P))$. Since the action of K on BP fixes the basepoint, evaluation at the basepoint of BP defines a map

$$\mathrm{ev}\colon BN_X^K(P) = \big(EK \times_K \mathrm{Map}(BP, BX)_{Bf}\big)^\wedge_p \longrightarrow BX.$$

If Q is any discrete p–toral group, and $\overline{\rho} \in \mathrm{Hom}(Q, K)$, then any homomorphism

$$EQ \times_Q BP \cong B(P \rtimes_{\overline{\rho}} Q) \longrightarrow BX$$

is adjoint to a Q–equivariant map

$$EQ \longrightarrow \mathrm{Map}(BP, BX)_{f|BP} = BC_X(P)\,,$$

where Q acts on $BC_X(P)$ via the action on P defined by $\overline{\rho}$ (and via the trivial action on BX). After taking the Borel construction, this defines a map

$$BQ \longrightarrow BN_X^K(P) = \big(EK \times_K BC_X(P)\big)^\wedge_p\,.$$

In particular, when Q is the group

$$N_S^K(P) = \{g \in N_S(P) \mid c_g \in K\}$$

and
$$B(P \rtimes N_S^K(P)) \longrightarrow BS \xrightarrow{\ f\ } BX$$

is induced by the inclusions and f, then this construction is denoted

$$B\gamma_P^K\colon BN_S^K(P) \longrightarrow BN_X^K(P)\,.$$

Lemma 10.3 *Fix a p–compact group X, a Sylow p–subgroup $S \xrightarrow{f} X$, and subgroups $P \le S$ and $K \le \mathrm{Aut}_X(P)$ where K is discrete p–toral. Then the induced sequence*

$$BC_X(P) \longrightarrow BN_X^K(P) \xrightarrow{\ \tau\ } BK^\wedge_p \tag{34}$$

is a fibration sequence. If Q is another discrete p–toral group, then for any homomorphism $\rho\colon Q \longrightarrow N_X^K(P)$, there is a fibration sequence

$$\mathrm{Map}(B(P \rtimes_{\overline{\rho}} Q), BX)_{f,\rho} \longrightarrow \mathrm{Map}(BQ, BN_X^K(P))_{B\rho} \longrightarrow \mathrm{Map}(BQ, BK_p^\wedge)_{B\overline{\rho}},$$

where $\overline{\rho} \in \mathrm{Hom}(Q,K)$ is any homomorphism such that $B\overline{\rho}_p^\wedge \simeq \tau \circ B\rho$, $P \rtimes_{\overline{\rho}} Q$ is the semidirect product for the action $Q \xrightarrow{\overline{\rho}} K \le \mathrm{Aut}(P)$, and the fiber is the space of all maps $B(P \rtimes_{\overline{\rho}} Q) \longrightarrow BX$ which restrict (up to homotopy) to $BP \xrightarrow{Bf|_P} BX$ and are adjoint to $B\rho$ in the sense described above.

Proof The action of K on each cohomology group $H^i(BC_X(P);\mathbb{F}_p)$ factors through a finite quotient group of K, thus through the p–group $\pi_0(K)$, and hence is nilpotent. So by Bousfield and Kan [3, II.5.1], the usual fibration sequence

$$BC_X(P) \longrightarrow EK \times_K BC_X(P) \longrightarrow BK$$

for the Borel construction over BK is still a fibration sequence after p–completion. Thus (34) is a fibration sequence.

Since $[BQ, BK_p^\wedge] \cong [BQ, BK] \cong \mathrm{Rep}(Q,K)$ (Lemma 1.10), $\overline{\rho} \in \mathrm{Hom}(Q,K)$ is uniquely determined up to conjugacy by ρ.

For any fixed homomorphism $\overline{\rho}\colon Q \longrightarrow K$, (34) induces a fibration sequence

$$BC_X(P)^{hQ} \longrightarrow \mathrm{Map}(BQ, BN_X^K(P))_{\widetilde{\rho}} \longrightarrow \mathrm{Map}(BQ, BK_p^\wedge)_{B\overline{\rho}_p^\wedge},$$

where $\widetilde{\rho}$ denotes the set of connected components of $\mathrm{Map}(BQ, BN_X^K(P))$ that map into $\mathrm{Map}(BQ, BK_p^\wedge)_{B\overline{\rho}_p^\wedge}$; and (if $\widetilde{\rho} \neq \varnothing$) $BC_X(P)^{hQ}$ is the homotopy fixed point set of the action of Q induced by the pullback of (34) over

$$BQ \xrightarrow{B\overline{\rho}_p^\wedge \simeq \tau \circ B\rho} BK_p^\wedge.$$

We need to identify this action of Q on $BC_X(P)$ with that induced by the action of Q on P via $\overline{\rho}$. This follows by comparing the fibrations

$$\begin{tikzcd}
BC_X(P) \arrow[r] \arrow[d, equal] & EK \times_K BC_X(P) \arrow[r] \arrow[d] & BK \arrow[d] \\
BC_X(P) \arrow[r] & BN_X^K(P) \arrow[r] & BK_p^\wedge
\end{tikzcd}$$

since the action of Q on $BC_X(P)$ induced by $BQ \xrightarrow{B\overline{\rho}_p^\wedge} BK_p^\wedge$ in the bottom fibration coincides with that induced by $\overline{\rho}$ in the fibration sequence of the top row. By construction, the action of K on $BC_X(P)$ induced by the top row is just the action of K on $BC_X(P) = \mathrm{Map}(BP, BX)_{Bf|_P}$ induced by the original action of K on P.

Now set $f_P = f|_P \colon P \longrightarrow X$ for short. We can identify

$$\begin{aligned} BC_X(P)^{hQ} &= \big(\mathrm{Map}(BP, BX)_{Bf_P}\big)^{hQ} = \mathrm{Map}_Q(EQ, \mathrm{Map}(BP, BX)_{Bf_P}) \\ &\simeq \mathrm{Map}_Q(BP \times EQ, BX)_{\widetilde{f}} \simeq \mathrm{Map}(BP \times_Q EQ, BX)_{\widetilde{f}} \\ &\simeq \mathrm{Map}(B(P \rtimes_{\overline{\rho}} Q), BX)_{\widetilde{f}}, \end{aligned}$$

where $\widetilde{f}$ is the set of connected components of maps whose restriction to BP is homotopic to Bf_P. Here, $BP \times_Q EQ \simeq B(P \rtimes_{\overline{\rho}} Q)$ because the action of Q in BP is induced from the action described above of $\mathrm{Aut}(P)$ on BP, and this has a fixed point, providing a section of the fibration

$$BP \longrightarrow BP \times_{\mathrm{Aut}(P)} E\,\mathrm{Aut}(P) \longrightarrow B\,\mathrm{Aut}(P)\,.$$

Finally, upon restricting to one component of $\mathrm{Map}(BQ, BN_X^K(P))_{\widetilde{\rho}}$, we obtain the fibration in the statement of the proposition. □

Notice that in the particular case where $K = 1$, $\mathrm{Map}(BQ, BK)$ is contractible, and the fibration of Lemma 10.3 reduces to the equivalence

$$\mathrm{Map}(BP \times BQ, BX) \simeq \mathrm{Map}(BQ, \mathrm{Map}(BP, BX)).$$

Proposition 10.4 *Let X be a p–compact group, let $S \xrightarrow{f} X$ be a Sylow p–subgroup, and set $\mathcal{F} = \mathcal{F}_{S,f}(X)$ for short. Fix a subgroup $P \le S$, and a discrete p–toral group of automorphisms $K \le \mathrm{Aut}_{\mathcal{F}}(P)$. Then the following hold.*

(a) *$BN_X^K(P)$ is the classifying space of a p–compact group which we denote $N_X^K(P)$, and*

$$N_S^K(P) \xrightarrow{\gamma_P^K} N_X^K(P)$$

is a discrete p–toral subgroup. Furthermore, the square

$$\begin{array}{ccc} BN_S^K(P) & \xrightarrow{B\gamma_P^K} & BN_X^K(P) \\ {\scriptstyle B\,\mathrm{incl}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}} \\ BS & \xrightarrow{\quad f \quad} & BX \end{array} \tag{35}$$

commutes up to pointed homotopy.

(b) *There is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ such that $\varphi(P)$ is fully $\varphi K \varphi^{-1}$–normalized in $\mathcal{F}$.*

(c) *P is fully K–normalized in $\mathcal{F}$ if and only if $N_S^K(P)$ is a Sylow p–subgroup of $N_X^K(P)$.*

Proof **(a)** By Lemma 10.3, $BC_X(P) \longrightarrow BN_X^K(P) \longrightarrow BK_p^\wedge$ is a fibration sequence. The loop spaces of the fiber and base of this sequence have finite mod p cohomology, so the same is true of $N_X^K(P) \overset{\text{def}}{=} \Omega(BN_X^K(P))$. Thus $N_X^K(P)$ is a p–compact group.

The map $$BN_S^K(P) \xrightarrow{B\gamma_P^K} BN_X^K(P) = \big(EK \times_K \mathrm{Map}(BP, BX)_{Bf|_P}\big)_p^\wedge$$

is defined to be adjoint to the composite

$$B\big(P \rtimes N_S^K(P)\big) \xrightarrow{B(\mathrm{incl} \rtimes \mathrm{incl})} BS \xrightarrow{f} BX. \tag{36}$$

Hence the composite of $B\gamma_P^K$ with the evaluation map from $\mathrm{Map}(BP, BX)_{Bf_P}$ to BX (evaluation at the basepoint of BP) equals the restriction of (36) to $BN_S^K(P)$. This proves that (35) is commutative.

If γ_P^K factored through a quotient group $N_S^K(P)/R$ for some $R \neq 1$, then the restriction of $Bf\colon BS \longrightarrow BX$ to BR would be homotopically trivial, but this cannot happen. So if $S_0 \xrightarrow{f_0} N_S^K(P)$ is a maximal discrete p–toral subgroup, then γ_P^K factors through a monomorphism from $N_S^K(P)$ to S_0 (Proposition 10.1(a)), and thus γ_P^K is itself a monomorphism.

(b,c) By Lemma 10.3, any discrete p–toral subgroup $BQ \longrightarrow BN_X^K(P)$ lifts to a map $B(P \rtimes Q) \longrightarrow BX$ which factors through a homomorphism $P \rtimes Q \xrightarrow{\beta} S$. Set $P' = \beta(P \rtimes 1) \le S$, $\varphi = \beta|_{P\rtimes 1} \in \mathrm{Iso}_{\mathcal{F}}(P, P')$, and $K' = \varphi K \varphi^{-1} \le \mathrm{Aut}_X(P')$. Then $\beta(Q) \le N_S^{K'}(P')$, and $\beta|_{1\rtimes Q}$ is injective since otherwise $BQ \longrightarrow BN_X^K(P)$ would factor through a quotient group of Q and hence wouldn't be a subgroup. Thus, the largest possible K–normalizer $N_S^{K'}(P')$ occurs when it is a Sylow p–subgroup of $N_X^K(P)$, so P' is fully K'–normalized in $\mathcal{F}$, and P is fully K–normalized if and only if $N_S^K(P)$ is a Sylow p–subgroup of $N_X^K(P)$. □

We are now ready to show that $\mathcal{F}_{S,f}(X)$ is saturated.

Proposition 10.5 *Let X be a p–compact group, and let $S \xrightarrow{f} X$ be a Sylow p–subgroup. Then $\mathcal{F}_{S,f}(X)$ is a saturated fusion system over S.*

Proof Write $\mathcal{F} = \mathcal{F}_{S,f}(X)$ for short.

Proof of (I) Fix a subgroup $P \le S$ which is fully normalized in $\mathcal{F}$. Let $K \le \mathrm{Aut}_{\mathcal{F}}(P)$ be such that $K \ge \mathrm{Aut}_S(P)$ and $K/\mathrm{Inn}(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$. Then P is fully K–normalized, as it is fully normalized and $N_S^K(P) = N_S(P)$. So by Proposition 10.4(c), $N_S^K(P)$ is a Sylow p–subgroup of $N_X^K(P)$.

Set $K' = \mathrm{Aut}_S(P)$ for short, and consider the following commutative diagram of connected spaces:

$$\begin{tikzcd} BC_S(P)^\wedge_p \arrow[r] \arrow[d, "f_1"'] & BN_S^K(P)^\wedge_p \arrow[r] \arrow[d, "f_2"'] & BK'^\wedge_p \arrow[d, "f_3"'] \\ BC_X(P) \arrow[r] & BN_X^K(P)^\wedge_p \arrow[r] & BK^\wedge_p \end{tikzcd}$$

Let F_i be the homotopy fiber of f_i (for $i = 1, 2, 3$). Each row is a fibration sequence before p–completion; and the actions of K' on $H^*(BC_S(P);\mathbb{F}_p)$ and of K on $H^*(BC_X(P);\mathbb{F}_p)$ factor through finite p–group quotients and hence are nilpotent. So the rows are still fibration sequences after p–completion by [3, II.5.1].

Each of the maps f_i is a monomorphism of p–compact groups, and hence $H^*(F_i;\mathbb{F}_p)$ is finite for each i. Since $BC_X(P)$ is connected, $\pi_1(BN_X^K(P)^\wedge_p)$ surjects onto $\pi_1(BK^\wedge_p) \cong \pi_0(K)$, and hence $\pi_0(F_2)$ surjects onto $\pi_0(F_3)$. Thus F_1 is the homotopy fiber of the map $F_2 \longrightarrow F_3$, and so $\chi(F_2) = \chi(F_1)\cdot\chi(F_3)$.

Since $N_S^K(P) \in \mathrm{Syl}_p(N_X^K(P))$, $\chi(F_2)$ is prime to p by Proposition 10.1(a). Thus $\chi(F_1)$ and $\chi(F_3)$ are both prime to p, and hence $C_S(P) \in \mathrm{Syl}_p(C_X(P))$ and (since K is discrete p–toral) $K' = K$. Hence $\mathrm{Out}_S(P) = K/\mathrm{Inn}(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$. Also, since $C_S(P) \in \mathrm{Syl}_p(C_X(P))$, we can again apply Proposition 10.4(c) (this time with $K = 1$), to show that P is fully centralized in $\mathcal{F}$. This finishes the proof of (I).

Proof of (II) Fix $P \le S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, and set $P' = \varphi(P)$. Assume that P' is fully centralized in $\mathcal{F}$. Set

$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(P')\},$$

and set $K = \mathrm{Aut}_{N_\varphi}(P)$, $K' = \varphi K \varphi^{-1} \le \mathrm{Aut}_S(P')$, and $N'_\varphi = N_S^{K'}(P')$. Then P' is fully K'–normalized in $\mathcal{F}$, since it is fully centralized and $K' \le \mathrm{Aut}_S(P')$. Consider the following diagram:

$$\begin{tikzcd} BN_\varphi \arrow[r, "B\gamma_P^K"] \arrow[d, dotted, "B\overline{\varphi}"'] & BN_X^K(P) \arrow[r, "\mathrm{proj}"] \arrow[d, "\simeq"', "Ec_\varphi\times(\varphi^*)^{-1}"] & BK^\wedge_p \arrow[d, "\cong"', "Bc_\varphi"] \\ BN'_\varphi \arrow[r, "B\gamma_{P'}^{K'}"] & BN_X^{K'}(P') \arrow[r, "\mathrm{proj}"] & BK'^\wedge_p \end{tikzcd} \tag{37}$$

The composites in the two rows are induced by the epimorphisms $N_\varphi \xrightarrow{\omega} K$ and $N'_\varphi \xrightarrow{\omega'} K'$ (exactly, not just up to homotopy). By Proposition 10.4(c), N'_φ is a Sylow p–subgroup of $N_X^{K'}(P') \simeq N_X^K(P)$, and hence there exists a homomorphism $\overline{\varphi} \in \mathrm{Hom}(N_\varphi, N'_\varphi)$ which makes the left hand square commute up to homotopy.

Since $[BN_\varphi, BK'^\wedge_p] \cong \mathrm{Rep}(N_\varphi, K')$ (Lemma 1.10), the homotopy commutativity of (37) implies that there is $\overline{g} \in K'$ such that $c_\varphi \circ \omega = c_{\overline{g}} \circ \omega' \circ \overline{\varphi}$. Since ω' is onto, there is $g \in N'_\varphi$ such that $\omega'(g) = \overline{g}$; and upon replacing $\overline{\varphi}$ by $c_g \circ \overline{\varphi}$ we can assume that $c_\varphi \circ \omega = \omega' \circ \overline{\varphi}$.

Fix a homotopy H which makes the left hand square in (37) commute. Then the composite $\mathrm{proj} \circ H$ is a loop in $\mathrm{Map}(BN_\varphi, BK'^\wedge_p)$ based at $B(\omega' \circ \overline{\varphi})$, and this component has the homotopy type of $BC_{K'}(\omega' \circ \overline{\varphi}(N_\varphi))^\wedge_p$ by Lemma 1.10 again. So after replacing $\overline{\varphi}$ by $c_{g'} \circ \overline{\varphi}$ for some appropriate $g' \in N'_\varphi$, and after modifying H using the homotopy from $B\overline{\varphi}$ to $B(c_{g'} \circ \overline{\varphi})$ determined by g', we can arrange that $\mathrm{proj} \circ H$ is nullhomotopic in $\mathrm{Map}(BN_\varphi, BK'^\wedge_p)$. We can now apply Lemma 10.3, to show that the following diagram commutes up to homotopy:

$$\begin{tikzcd}
B(P \rtimes N_\varphi) \arrow[r, "\mathrm{incl} \rtimes \mathrm{incl}"] \arrow[d, "B(\varphi \rtimes \overline{\varphi})"] & BS \arrow[r, "Bf"] & BX \arrow[d, equal] \\
B(P' \rtimes N'_\varphi) \arrow[r, "\mathrm{incl} \rtimes \mathrm{incl}"] & BS \arrow[r, "Bf"] & BX
\end{tikzcd}$$

In particular, $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, N'_\varphi)$. Also, the two homomorphisms from $P \rtimes N_\varphi$ to S induced by inclusions and by $\varphi \rtimes \overline{\varphi}$ have the same kernel $\{(g, g^{-1}) \mid g \in P\}$, and this implies that $\varphi = \overline{\varphi}|_P$.

Proof of (III) Fix $P = \bigcup_{n=1}^\infty P_n$, where $P_1 \le P_2 \le \cdots$ is an increasing sequence of subgroups. Let $\varphi \in \mathrm{Inj}(P, S)$ be such that $\varphi|_{P_n} \in \mathrm{Hom}_{\mathcal{F}}(P_n, S)$ for all n. Thus for each n, $(Bf \circ B\varphi)|_{BP_n} \simeq Bf|_{BP_n}$. Also, $\mathrm{Map}(BP_n, BX)_{Bf|BP_n} \simeq \mathrm{Map}(BP, BX)_{Bf|BP}$ for n sufficiently large, by Proposition 10.1(b). We can thus choose homotopies H_n from $(Bf \circ B\varphi)|_{BP_n}$ to $Bf|_{BP_n}$ such that $H_n = H_{n+1}|_{BP_n \times I}$, and set $H = \bigcup H_n$. This shows that $Bf \circ B\varphi \simeq Bf|_{BP}$, and hence that $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. □

In Castellana, Levi and Notbohm [9], a p–compact toral subgroup P of a p–compact group X is called centric if the inclusion map $BP \xrightarrow{Bf} BX$ is a centric map; ie if $\mathrm{Map}(BP, BP)_{\mathrm{Id}} \xrightarrow{Bf \circ -} \mathrm{Map}(BP, BX)_{Bf}$ is an equivalence. We must check that this is equivalent to the concept of $\mathcal{F}$–centricity (applied to discrete p–toral subgroups) used here.

Lemma 10.6 *Let X be a p–compact group, and let $S \xrightarrow{f} X$ be a Sylow p–subgroup. Set $\mathcal{F} = \mathcal{F}_{S,f}(X)$. Then for any subgroup $P \le S$,*

$$BP^\wedge_p \xrightarrow{Bf|_{BP}} BX$$

is a centric map if and only if P is $\mathcal{F}$–centric.

Proof Assume P is $\mathcal{F}$–centric. In particular, P is fully centralized in $\mathcal{F}$. By Proposition 10.4(a,c) (applied with $K=1$), $C_X(P)$ is a p–compact group with Sylow p–subgroup $C_S(P)=Z(P)$. Also, composition defines a map

$$\underbrace{\mathrm{Map}(BP,BP)_{\mathrm{Id}}}_{\simeq BZ(P)} \times \mathrm{Map}(BP,BX)_{Bf|BP} \longrightarrow \mathrm{Map}(BP,BX)_{Bf|BP}.$$

So by Proposition 10.1(c), $Z(P)$ is central in $C_X(P)$, and there is a p–compact group $C_X(P)/Z(P)$ whose Euler characteristic is prime to p and a fibration sequence

$$BZ(P)^\wedge_p \longrightarrow BC_X(P) \longrightarrow B(C_X(P)/Z(P)) .$$

Then $C_X(P)/Z(P)$ must be trivial, so $B(C_X(P)/Z(P)) \simeq *$,

$$BZ(P)^\wedge_p \simeq \mathrm{Map}(BP,BX)_{Bf|BP},$$

and hence $Bf|_{BP}$ is a centric map.

Conversely, if $Bf|_{BP}$ is a centric map, then $BC_X(P)\simeq BZ(P)$ by Lemma 1.10, so $C_S(P')=Z(P')$ for all $P'\le S$ which is $\mathcal{F}$–conjugate to P, and P is $\mathcal{F}$–centric. □

It remains to construct a linking system associated to $\mathcal{F}_{S,f}(X)$ whose p–completed nerve has the homotopy type of BX. This will be done using Proposition 4.6, together with a construction by Castellana, Levi and Notbohm [9].

Theorem 10.7 *Let X be a p–compact group, and let $S \xrightarrow{f} X$ be a Sylow p–subgroup. Set $\mathcal{F}=\mathcal{F}_{S,f}(X) \overset{\mathrm{def}}{=} \mathcal{F}_{S,Bf}(BX)$ for short. Then there is a centric linking system $\mathcal{L}=\mathcal{L}^c_{S,f}(X)$ associated to $\mathcal{F}$ such that*

$$|\mathcal{L}^c_{S,f}(X)|^\wedge_p \simeq BX .$$

In other words, $(S,\mathcal{F},\mathcal{L})$ is a p–local compact group whose classifying space is homotopy equivalent to BX.

Proof By Proposition 10.5, the fusion system $\mathcal{F}$ is saturated.

In [9], the authors define a category $\mathcal{O}^c(\mathcal{F})_+$ by adding a final object to $\mathcal{O}^c(\mathcal{F})$; ie the category $\mathcal{O}^c(\mathcal{F})_+$ consists of $\mathcal{O}^c(\mathcal{F})$ together with an additional object $*$, and a unique morphism from each object in $\mathcal{O}^c(\mathcal{F})$ to $*$. (The actual category they work with contains the same objects as $\mathcal{O}^c(\mathcal{F})$ by Lemma 10.6.) They then define a homotopy functor

$$\mathbf{B}_+ \colon \mathcal{O}^c(\mathcal{F})_+ \longrightarrow \mathrm{hoTop}$$

by setting $\mathbf{B}_+(P)=BP^\wedge_p$ for all $\mathcal{F}$–centric $P\le S$, and $\mathbf{B}_+(*)=BX$ (with the obvious maps between them). By Lemma 1.10 and Lemma 10.6, this is a centric diagram in the sense of [10]. Since $\mathcal{O}^c(\mathcal{F})$ has a final object, the Dwyer–Kan obstructions to rigidifying $\mathbf{B}_+$ to a functor to Top all vanish [10] (see also Corollary A.4), and so this functor can be lifted. In particular, this restricts to a functor $\widehat{B}$ from $\mathcal{O}^c(\mathcal{F})$ to Top, together with a map from $\underrightarrow{\mathrm{hocolim}}(\widehat{B})$ to BX. (See also Corollary A.5.)

By [9, Theorem A], this map from $\underrightarrow{\mathrm{hocolim}}(\widehat{B})$ to BX induces a homotopy equivalence

$$\big(\underrightarrow{\mathrm{hocolim}}(\widehat{B})\big)^\wedge_p \simeq BX$$

(the collection of $\mathcal{F}$–centric subgroups of X is "subgroup ample"). By Proposition 5.9, there is then a functor $\widetilde{B}\colon \mathcal{O}^c(\mathcal{F}) \longrightarrow \mathrm{Top}$ which is a rigidification of the homotopy functor $\mathbf{B}$, and a natural transformation of functors $\chi\colon \widetilde{B} \longrightarrow \widehat{B}$ which is the completion map on each object. Proposition 4.6 now applies to show that there is a centric linking system $\mathcal{L} = \mathcal{L}^c_{S,f}(X)$ associated to $\mathcal{F}$ such that

$$|\mathcal{L}|^\wedge_p \simeq \big(\underrightarrow{\mathrm{hocolim}}_{\mathcal{O}^c(\mathcal{F})}(\widetilde{B})\big)^\wedge_p \simeq \big(\underrightarrow{\mathrm{hocolim}}_{\mathcal{O}^c(\mathcal{F})}(\widehat{B})\big)^\wedge_p \simeq BX^\wedge_p. \qquad \square$$

In fact, by Theorem 7.4, there is at most one centric linking system $\mathcal{L}$ associated to $\mathcal{F}_{S,f}(X)$ with the property that $|\mathcal{L}|^\wedge_p \simeq BX^\wedge_p$. Thus the system constructed above is unique.

Appendix A Lifting diagrams in the homotopy category

As elsewhere in the paper, we let Top denote the category of spaces, and hoTop the homotopy category. Let $\mathrm{ho}\colon \mathrm{Top} \longrightarrow \mathrm{hoTop}$ be the forgetful functor. When $\mathcal{C}$ is a small category, a functor F from $\mathcal{C}$ to Top or hoTop is called *centric* if for each morphism $\varphi \in \mathrm{Mor}_{\mathcal{C}}(c,d)$, the natural map

$$\mathrm{Map}(F(c),F(c))_{\mathrm{Id}} \xrightarrow[\simeq]{F(\varphi)\circ-} \mathrm{Map}(F(c),F(d))_{F(\varphi)}$$

is a homotopy equivalence. In [10], Dwyer and Kan identify the obstructions to rigidifying a centric functor $F\colon \mathcal{C} \longrightarrow \mathrm{hoTop}$ to a functor $\widetilde{F}\colon \mathcal{C} \longrightarrow \mathrm{Top}$; and also describe the space of such rigidifications. We prove here a relative version of their result which is needed in Section 5. This result can, in fact, be derived from the main theorem in [10], but that argument is so indirect that we find it helpful to give a more direct, and also more elementary, proof.

More precisely, a *rigidification* of F is a functor $\widetilde{F}\colon \mathcal{C} \longrightarrow \mathrm{Top}$, together with a natural transformation of functors $F \longrightarrow \mathrm{ho}\circ\widetilde{F}$ which is a homotopy equivalence on each object. Two rigidifications $\widetilde{F}$ and $\widetilde{F}'$ are *equivalent* if there is a third rigidification

$\widetilde{F}''$, together with natural transformations of functors $\widetilde{F} \longrightarrow \widetilde{F}'' \longleftarrow \widetilde{F}'$ which commute with the natural transformations from F, and hence which define homotopy equivalences $\widetilde{F}(c) \simeq \widetilde{F}''(c) \simeq \widetilde{F}'(c)$ for each $c \in \mathrm{Ob}(\mathcal{C})$. This is easily seen to be an equivalence relation by taking pushouts.

The main idea here is to construct a rigidification of $F\colon \mathcal{C} \longrightarrow \mathrm{hoTop}$ by first constructing a space which looks like a "homotopy colimit" of F, and then show that this homotopy colimit automatically induces a rigidification $\widetilde{F}$. Recall that the nerve of a small category $\mathcal{C}$ is defined by setting

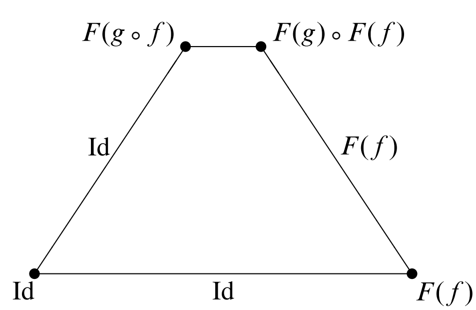
$$B\mathcal{C} = \Big(\coprod_{n\geq 0} \coprod_{x_0\to\cdots\to x_n} \Delta^n\Big)\Big/\sim$$

and that the homotopy colimit of any functor $F\colon \mathcal{C} \longrightarrow \mathrm{Top}$ is the space:

$$\underset{\mathcal{C}}{\underrightarrow{\mathrm{hocolim}}}(F) = \Big(\coprod_{n\geq 0} \coprod_{x_0\to\cdots\to x_n} F(x_0)\times\Delta^n\Big)\Big/\sim$$

Here, in both cases, we divide out by the usual face and degeneracy identifications. Let $p_F\colon \underrightarrow{\mathrm{hocolim}}(F) \longrightarrow B\mathcal{C}$ be the projection. It will be convenient to refer to the "skeleta" of the homotopy colimit: let $\underrightarrow{\mathrm{hocolim}}^{(n)}(F)$ denote the union of the $F(x_0)\times\Delta^i$ for all $i \leq n$ (and all $x_0 \to \cdots \to x_i$ in $\mathcal{C}$).

Now assume that $F\colon \mathcal{C} \longrightarrow \mathrm{hoTop}$ is a functor to the homotopy category instead. We assume that for each $f\colon x \longrightarrow y$ in $\mathcal{C}$, a concrete map $F(f)\colon F(x) \longrightarrow F(y)$ has been chosen. The 1–skeleton $\underrightarrow{\mathrm{hocolim}}^{(1)}(F)$ is defined in the same way as before: it is the union of the mapping cylinders of the $F(f)$ taken over all $f \in \mathrm{Mor}(\mathcal{C})$. It is also straightforward to define the 2–skeleton; but it is convenient at this stage to replace Δ^2 by a truncated triangle Δ^2_t. More precisely, for each sequence $x_0 \xrightarrow{f} x_1 \xrightarrow{g} x_2$, $F(x_0)\times\Delta^2_t$ is attached to $\underrightarrow{\mathrm{hocolim}}^{(1)}(F)$ via the following picture:



where the small segment at the top is mapped using a homotopy between $F(g\circ f)$ and $F(g)\circ F(f)$.

The first obstructions arise when constructing the 3–skeleton. For each $x_0 \to \cdots \to x_3$, we want to attach $F(x_0)\times\Delta^3_t$ to $\underrightarrow{\mathrm{hocolim}}^{(2)}(F)$, where Δ^3_t (the "truncated 3–simplex") is the cone over Δ^2_t with its vertex cut off. The attachment map is easily defined, except on the "top" surface resulting from truncating the cone vertex. Hence, the obstruction to defining the attachment map lies in the group

$$\pi_1\Big(\mathrm{Map}(F(x_0), F(x_3))\,,\ F(x_0 \to x_3)\Big).$$

At this point, it becomes necessary to switch from the intuitive picture to formal definitions, by replacing the truncated simplices Δ^n_t by cubes I^n, and regarding simplices as cubes modulo certain identifications. This correspondence will be made explicit later.

Let $\mathbf{\Delta}$ be the simplicial category, with objects the sets $[n]=\{0,\dots,n\}$ for $n\ge 0$, and morphisms the order preserving maps between sets. We let $\partial_i \in \mathrm{Mor}_{\mathbf{\Delta}}([n-1],[n])$ denote the i–th face map (with image $[n]\smallsetminus\{i\}$). Define a functor

$$I^\bullet\colon\ \mathbf{\Delta} \longrightarrow \mathsf{Top}$$

by setting $I^\bullet([n]) = I^{n+1}$ (where I is the closed interval $I=[0,1]$), and

$$I^\bullet(\sigma)(t_0,\dots,t_n) = \Big(\prod_{i\in\sigma^{-1}(0)} t_i\,,\ \prod_{i\in\sigma^{-1}(1)} t_i\,,\ \dots\,,\ \prod_{i\in\sigma^{-1}(m)} t_i\Big)$$

for $\sigma \in \mathrm{Mor}_{\mathbf{\Delta}_j}([n],[m])$. Here, the product over the empty set is always 1.

Let $\mathbf{\Delta}_1 \subseteq \mathbf{\Delta}_0 \subseteq \mathbf{\Delta}$ be the subcategories with the same objects, where

$$\mathrm{Mor}_{\mathbf{\Delta}_0}([m],[n]) = \{\sigma \in \mathrm{Mor}_{\mathbf{\Delta}}([m],[n]) \mid \sigma(0)=0\}$$
$$\mathrm{Mor}_{\mathbf{\Delta}_1}([m],[n]) = \{\sigma \in \mathrm{Mor}_{\mathbf{\Delta}}([m],[n]) \mid \sigma(0)=0,\ \sigma(m)=n\}.$$

For each $n\ge 0$, let $I^\bullet_1([n]) \subseteq I^\bullet_0([n]) \subseteq I^\bullet([n])$ be the subspaces

$$I^\bullet_0([n]) = \{(0,x_1,\dots,x_n)\in I^\bullet([n])\} \cong I^n$$
$$I^\bullet_1([n]) = \{(0,x_1,\dots,x_{n-1},0)\in I^\bullet([n])\} \cong I^{n-1}\,.$$

Then for each $j=0,1$, $I^\bullet|_{\mathbf{\Delta}_j}$ restricts to a subfunctor $I^\bullet_j\colon\ \mathbf{\Delta}_j \longrightarrow \mathsf{Top}$.

Throughout the rest of this section, $\mathcal{C}$ denotes a fixed small category. For each $n\ge 0$, define $\mathrm{Mor}^n = \mathrm{Mor}^n(\mathcal{C})$ to be the set of all sequences $c_0 \to c_1 \to \cdots \to c_n$ of composable morphisms in $\mathcal{C}$. In particular, $\mathrm{Mor}^0(\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}^1(\mathcal{C}) = \mathrm{Mor}(\mathcal{C})$. For $\sigma \in \mathrm{Mor}_{\mathbf{\Delta}}([n],[m])$, $\sigma^*\colon \mathrm{Mor}^m(\mathcal{C}) \longrightarrow \mathrm{Mor}^n(\mathcal{C})$ is defined as usual by taking compositions, inserting identity morphisms, and (if $\sigma\notin\mathrm{Mor}(\mathbf{\Delta}_1)$) dropping morphisms at one or both ends of the chain. For example, ∂_i^* (from $\mathrm{Mor}^n(\mathcal{C})$ to $\mathrm{Mor}^{n-1}(\mathcal{C})$) is defined by composing two morphisms in the sequence, or by dropping one of them if $i=0$ or n. Also, for each $\xi=(c_0\to\cdots\to c_n)$ in $\mathrm{Mor}^n(\mathcal{C})$ and each $0\le i\le j\le n$, we write

$$\xi_{ij}=(c_i\to\cdots\to c_j)\in\mathrm{Mor}^{j-i}(\mathcal{C}),$$

let $\mathring{\xi}_{ij}\in\mathrm{Mor}_{\mathcal{C}}(c_i,c_j)$ denote the composite of this sequence of maps, and set $\mathring{\xi}=\mathring{\xi}_{0n}$.

In order to simplify the notation in what follows, whenever $F\colon\mathcal{C}\longrightarrow\mathrm{hoTop}$ is a functor and $\varphi\in\mathrm{Mor}(\mathcal{C})$, we let $F(\varphi)$ denote some chosen representative of the homotopy class of maps defined by F, not the homotopy class itself.

Definition A.1 Fix a functor $F\colon\mathcal{C}\longrightarrow\mathrm{hoTop}$. An Rg_∞*–structure* $\overline{F}$ on F is a space $\overline{F}(c)$ and a homotopy equivalence $F(c)\xrightarrow{\nu(c)}\overline{F}(c)$, defined for each $c\in\mathrm{Ob}(\mathcal{C})$; together with maps

$$\overline{F}(\xi)\colon I^{n-1}=I_1^\bullet([n])\longrightarrow\mathrm{Map}\big(\overline{F}(c_0),\overline{F}(c_n)\big),$$

defined for each $n\ge1$ and each $\xi=(c_0\to c_1\to\cdots\to c_n)\in\mathrm{Mor}^n(\mathcal{C})$, which satisfy the following relations.

(a) For all $\varphi\in\mathrm{Mor}_{\mathcal{C}}(c_0,c_1)$, $\overline{F}(c_0\xrightarrow{\varphi}c_1)\circ\nu(c_0)\simeq\nu(c_1)\circ F(\varphi)$.

(b) For all $m,n\ge1$, $\sigma\in\mathrm{Mor}_{\mathbf{\Delta}_1}([m],[n])$, $\xi\in\mathrm{Mor}^n(\mathcal{C})$, and $\boldsymbol{t}\in I^{m-1}$,

$$\overline{F}(\sigma^*\xi)(\boldsymbol{t})=\overline{F}(\xi)(I_1^\bullet(\sigma)(\boldsymbol{t})).$$

(c) For all $n\ge2$, $\xi\in\mathrm{Mor}^n(\mathcal{C})$, $1\le i\le n-1$, $\boldsymbol{t}_1\in I^{i-1}$, and $\boldsymbol{t}_2\in I^{n-i-1}$,

$$\overline{F}(\xi)(\boldsymbol{t}_1,0,\boldsymbol{t}_2)=\overline{F}(\xi_{in})(\boldsymbol{t}_2)\circ\overline{F}(\xi_{0i})(\boldsymbol{t}_1).$$

Schematically, relation (b) can be described via the commutative diagram

$$\begin{tikzcd} I^{m-1} \arrow[rr, "I_1^\bullet(\sigma)"] \arrow[dr, "\overline{F}(\sigma^*(\xi))"'] & & I^{n-1} \arrow[dl, "\overline{F}(\xi)"] \\ & \mathrm{Map}(\overline{F}(c_0),\overline{F}(c_n)) & \end{tikzcd}$$

while relation (c) can be described via the following diagram:

$$\begin{tikzcd} I^{i-1}\times I^{n-i-1} \arrow[r, "t_1,t_2\mapsto(t_1,0,t_2)"] \arrow[d, "\overline{F}(\xi_{0i})\times\overline{F}(\xi_{in})"'] & I^{n-1} \arrow[d, "\overline{F}(\xi)"] \\ \mathrm{Map}(\overline{F}(c_0),\overline{F}(c_i))\times\mathrm{Map}(\overline{F}(c_i),\overline{F}(c_n)) \arrow[r, "\text{composition}"] & \mathrm{Map}(\overline{F}(c_0),\overline{F}(c_n)) \end{tikzcd}$$

These relations are more easily understood when one thinks of $I_1^\bullet([n]) \cong I^{n-1}$ as the space of all $(t_0,\dots,t_n) \in I^\bullet([n]) \cong I^{n+1}$ such that $t_0 = 0 = t_n$. Each coordinate in $I^\bullet([n])$ corresponds to one of the objects in the chain $\xi = (c_0 \to \cdots \to c_n)$. When $t_i = 1$ for some $0 < i < n$, t_i and c_i can be removed, giving the face relation

$$\overline{F}(\xi)(t_1,\dots,t_{i-1},1,t_{i+1},\dots,t_{n-1}) = \overline{F}(\partial_i\xi)(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{n-1}).$$

When $t_i = 0$ for $0 < i < n$, then $\overline{F}(\xi)(\boldsymbol{t})$ can be split as a composite at the object c_i (relation (c)). If one of the morphisms in ξ is an identity, then one can remove it and multiply the coordinates corresponding to its two objects.

For instance, when $m = 2$ and $n = 1$ (and σ is one of the surjections), condition (b) says that the following maps are both the constant maps to $\overline{F}(\varphi)$:

$$\overline{F}(c_0 \xrightarrow{\mathrm{Id}} c_0 \xrightarrow{\varphi} c_1) \qquad \text{and} \qquad \overline{F}(c_0 \xrightarrow{\varphi} c_1 \xrightarrow{\mathrm{Id}} c_1)$$

In particular, $\overline{F}(\varphi) \circ \overline{F}(\mathrm{Id}_{c_0}) = \overline{F}(\varphi) = \overline{F}(\mathrm{Id}_{c_1}) \circ \overline{F}(\varphi)$.

When $n=2$, condition (c) says that $\overline{F}\big(c_0 \xrightarrow{\varphi} c_1 \xrightarrow{\psi} c_2\big)$ is a homotopy from $\overline{F}(\psi)\circ\overline{F}(\varphi)$ to $\overline{F}(\psi\circ\varphi)$. More generally, when $\xi \in \mathrm{Mor}^n(\mathcal{C})$ for $n \ge 2$,

$$\overline{F}(\xi)(0,\dots,0) = \overline{F}(\xi_{n-1,n}) \circ \cdots \circ \overline{F}(\xi_{12}) \circ \overline{F}(\xi_{01}) \quad \text{and} \quad \overline{F}(\xi)(1,\dots,1) = \overline{F}(\mathring{\xi}).$$

At the other vertices of I^{n-1}, we get all of the other possible composites of the $\overline{F}(\mathring{\xi}_{ij})$. An Rg_∞–structure on F is thus a collection of higher homotopies connecting given homotopies $F(\psi) \circ F(\varphi) \simeq F(\psi\circ\varphi)$.

From this point of view, one sees that when defining an Rg_∞–structure on F, it suffices to define it on all nondegenerate sequences $\xi \in \mathrm{Mor}^n(\mathcal{C})$ (ie those containing no identity morphisms), inductively for increasing n, where at each step $\overline{F}(\xi)$ has already been defined on ∂I^{n-1} and must be extended in some way to I^{n-1}. The starting point can be any choice of maps $\overline{F}(\varphi)$, for all $\varphi \in \mathrm{Mor}(\mathcal{C})$, in the given homotopy class determined by $F(\varphi)$, such that

$$\overline{F}(\varphi) \circ \overline{F}(\mathrm{Id}_c) = \overline{F}(\varphi) = \overline{F}(\mathrm{Id}_d) \circ \overline{F}(\varphi)$$

for each morphism $\varphi \in \mathrm{Mor}_{\mathcal{C}}(c,d)$ in $\mathcal{C}$.

If $\overline{F}$ and $\overline{F}'$ are both Rg_∞–structures on F, then a *morphism* $\Theta\colon \overline{F} \longrightarrow \overline{F}'$ consists of homotopy equivalences $\theta(c)\colon \overline{F}(c) \xrightarrow{\simeq} \overline{F}'(c)$ (for each $c \in \mathrm{Ob}(\mathcal{C})$) such that $\theta(c)\circ\nu(c) \simeq \nu'(c)$, and such that for each $\xi = (c_0 \to \cdots \to c_n)$ and each $\boldsymbol{t} \in I^{n-1}$,

$$\theta(c_n) \circ \overline{F}(\xi)(\boldsymbol{t}) = \overline{F}'(\xi)(\boldsymbol{t}) \circ \theta(c_0) \in \mathrm{Map}(\overline{F}(c_0, \overline{F}'(c_n)).$$

Two Rg_∞–structures on F are *equivalent* if there is a third to which they both have morphisms. One easily sees that (homotopy) pushouts exist for morphisms of Rg_∞–structures on F, and hence that this defines an equivalence relation among Rg_∞–structures.

For any given Rg_∞–structure $\overline{F}$ on $F\colon \mathcal{C} \longrightarrow \mathfrak{hoTop}$, we define its "homotopy colimit" $\mathfrak{Sp}(\overline{F})$ to be the space

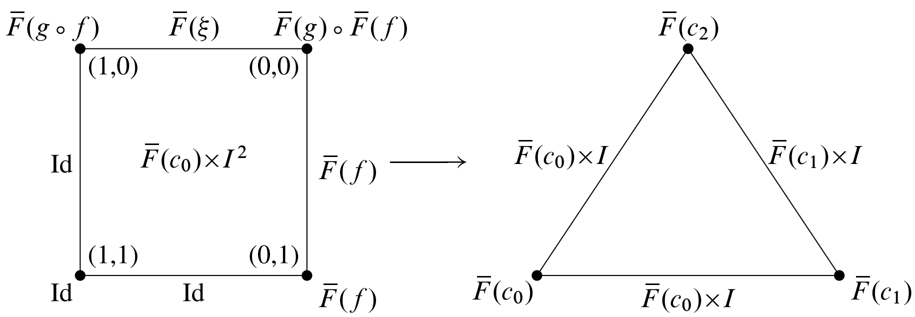
$$(I^n = I_0^\bullet([n])) \qquad \mathfrak{Sp}(\overline{F}) = \Big(\coprod_{n\geq 0} \coprod_{c_0\to\cdots\to c_n} \overline{F}(c_0)\times I^n\Big)\Big/\!\sim$$

where the identifications below are made for each $n \geq 1$, each $\xi = (c_0 \to \cdots \to c_n)$ in $\mathrm{Mor}^n(\mathcal{C})$, and each $x \in \overline{F}(c_0)$:

$$(\sigma \in \mathrm{Mor}_{\mathbf{\Delta}_0}([m],[n]), \boldsymbol{t} \in I^m) \qquad \big(x; I_0^\bullet(\sigma)(\boldsymbol{t})\big)_{[\xi]} \sim \big(x;\boldsymbol{t}\big)_{[\sigma^*\xi]}$$

$$(1 \leq i \leq n, \boldsymbol{t}_1 \in I^{i-1}, \boldsymbol{t}_2 \in I^{n-i}) \qquad \big(x;(\boldsymbol{t}_1,0,\boldsymbol{t}_2)\big)_{[\xi]} \sim \big(\overline{F}(\xi_{0i})(\boldsymbol{t}_1)(x);\boldsymbol{t}_2\big)_{[\xi_{in}]}$$

For example, in the case of a sequence $\xi = (c_0 \xrightarrow{f} c_1 \xrightarrow{g} c_2)$ in $\mathrm{Mor}^2(\mathcal{C})$, the corresponding square $I_0^\bullet([2])$ is attached to the 1–skeleton in the following way:



The labels in the first picture describe the maps by which a vertex $\overline{F}(c_0)$ or an edge $\overline{F}(c_0)\times I$ is attached to the space represented by the second picture. Thus the trapezoid in the earlier picture has now been replaced by a square.

One way to understand these relations and their connection with those in Definition A.1 is to think of $I_0^\bullet([n]) \cong I^n$ as the subspace of all $(n+1)$–tuples $(0, t_1, \ldots, t_n)$ in $I^\bullet([n]) \cong I^{n+1}$. For $\xi \in \mathrm{Mor}^n(\mathcal{C})$, each coordinate in $I^\bullet([n])$ corresponds to one of the objects in the chain $\xi = (c_0 \to \cdots \to c_n)$. When $t_i = 1$ for $0 < i \leq n$, t_i and c_i can be removed, giving the face relation

$$\big(x;(t_1,\ldots,t_{i-1},1,t_{i+1},\ldots,t_n)\big)_{[\xi]} \sim \big(x;(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n)\big)_{[\partial_i\xi]}.$$

When $t_i = 0$ for $0 < i \leq n$, then ξ splits as a composite at the object c_i, we apply $\overline{F}(\xi_{0i})(t_1,\ldots,t_{i-1})$ to x, and get the second of the above relations. If one of the morphisms in ξ is an identity, then we get a degeneracy relation by removing it and multiplying the two corresponding coordinates.

Consider the maps $\sigma_n\colon I^n \longrightarrow \Delta^n$ defined by

$$\sigma_n(t_1,\dots,t_n) = \Big(t_1t_2\cdots t_n,\ (1-t_1)t_2\cdots t_n,\ (1-t_2)t_3\cdots t_n,\ \dots,\ (1-t_{n-1})t_n, 1-t_n\Big).$$

When F is a functor to Top and $\overline{F}$ is the corresponding locally constant Rg_∞–structure (ie for each ξ, $\overline{F}(\xi)$ is the constant map with value $F(\xi)$), then the σ_n define a homeomorphism from $\mathfrak{Sp}(\overline{F})$ to the usual homotopy colimit $\underrightarrow{\mathrm{hocolim}}(F)$. More generally, when $\overline{F}$ is an arbitrary Rg_∞–structure, then there is a map

$$\mathrm{pr}_{\overline{F}}\colon \mathfrak{Sp}(\overline{F}) \xrightarrow{\qquad} |\mathcal{C}|$$

defined on each subspace $\overline{F}(c_0)\times I^n$ by first projecting to the I^n and then to Δ^n via σ_n.

We now define a functor $\mathfrak{Rg}(\overline{F})\colon \mathcal{C} \longrightarrow \mathrm{Top}$ by letting $\mathfrak{Rg}(\overline{F})(c)$ be the pullback space

$$\begin{array}{ccc} \mathfrak{Rg}(\overline{F})(c) & \xrightarrow{\qquad} & \mathfrak{Sp}(\overline{F}) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{pr}_{\overline{F}}} \\ |\mathcal{C}{\downarrow}c| & \xrightarrow{\qquad\quad} & |\mathcal{C}| \end{array}$$

(the setwise pullback, not the homotopy pullback). A morphism $\varphi \in \mathrm{Mor}_{\mathcal{C}}(c,d)$ induces a map from $|\mathcal{C}{\downarrow}c|$ to $|\mathcal{C}{\downarrow}d|$ via composition with φ in the usual way, and hence induces a map from $\mathfrak{Rg}(\overline{F})(c)$ to $\mathfrak{Rg}(\overline{F})(d)$. Equivalently,

$$(I^n = I_0^\bullet([n])) \qquad \mathfrak{Rg}(\overline{F})(c) = \Big(\coprod_{n\ge 0}\ \coprod_{c_0\to\cdots\to c_n\to c} \overline{F}(c_0)\times I^n\Big)\Big/\!\sim$$

where the identifications are analogous to those used to define $\mathfrak{Sp}(\overline{F})$. This clearly makes $\mathfrak{Rg}(\overline{F})$ into a functor from $\mathcal{C}$ to Top.

For each c, $\overline{F}(c)$ can be identified as a subspace of $\mathfrak{Rg}(\overline{F})(c)$: the inverse image under the projection to $|\mathcal{C}{\downarrow}c|$ of the vertex $(c \xrightarrow{\mathrm{Id}} c)$. The composite

$$F(c) \xrightarrow[\simeq]{\nu(c)} \overline{F}(c) \subseteq \mathfrak{Rg}(\overline{F})(c)$$

defines a natural transformation $F \longrightarrow \mathfrak{ho}\circ\mathfrak{Rg}(\overline{F})$ of functors $\mathcal{C}\longrightarrow \mathfrak{ho}\mathrm{Top}$. The following proposition now shows that this is a natural equivalence, and hence that $\mathfrak{Rg}(\overline{F})$ is a rigidification of F.

Proposition A.2 *For any Rg_∞–structure $\overline{F}$ on $F\colon \mathcal{C} \longrightarrow \mathfrak{hoTop}$, for each $c \in \mathrm{Ob}(\mathcal{C})$, $\overline{F}(c)$ is a deformation retract of $\mathfrak{Rg}(\overline{F})(c)$. Thus $\mathfrak{Rg}(\overline{F})$ is a rigidification of F.*

Proof Define $\Phi\colon \mathfrak{Rg}(\overline{F})(c) \times I \longrightarrow \mathfrak{Rg}(\overline{F})(c)$ by setting

$$\Phi\big((x;\boldsymbol{t})_{[\xi \to c]}, s\big) = (x;(\boldsymbol{t},s))_{[\xi \to c \xrightarrow{\mathrm{Id}} c]}$$

for all $\xi \in \mathrm{Mor}^n(\mathcal{C})$, $\boldsymbol{t} \in I^n$, and $s \in I$. Then $\Phi(u,1) = u$ and $\Phi(u,0) \in \overline{F}(c)$ for all $u \in \mathfrak{Rg}(\overline{F})(c)$ by definition of $\mathfrak{Rg}(\overline{F})(c)$. Furthermore, the homotopy is the identity on $\overline{F}(c)$, and thus $\overline{F}(c)$ is a deformation retract. □

For any given $F\colon \mathcal{C} \longrightarrow \mathfrak{hoTop}$, let $\mathrm{Rigid}(F)$ be the set of equivalence classes of rigidifications of F, and let $\mathrm{Rg}_\infty(F)$ be the set of equivalence classes of Rg_∞–structures on F. A rigidification of F can be regarded as a "locally constant" Rg_∞–structure on F; ie an Rg_∞–structure $\overline{F}$ where each of the maps $\overline{F}(\xi)$ (for $\xi \in \mathrm{Mor}^n(\mathcal{C})$) is constant on I^{n-1}. We thus have maps

$$\mathrm{Rigid}(F) \underset{\mathfrak{Rg}}{\overset{\mathrm{const}}{\rightleftarrows}} \mathrm{Rg}_\infty(F).$$

One easily checks that for any rigidification $\widetilde{F}$, there is a natural transformation of functors from $\mathfrak{Rg}(\mathrm{const}(\widetilde{F}))$ to $\widetilde{F}$, and hence these are equal in $\mathrm{Rigid}(F)$. We do not know whether the other composite is the identity on $\mathrm{Rg}_\infty(F)$, but that will not be needed here.

A natural transformation $\chi\colon F \longrightarrow F'$ of functors $F, F'\colon \mathcal{C} \longrightarrow \mathfrak{hoTop}$ will be called *relatively centric* if for each morphism $\varphi \in \mathrm{Mor}_{\mathcal{C}}(c,d)$ in $\mathcal{C}$, the homotopy commutative square

$$\begin{tikzcd}
\mathrm{Map}(F(c),F(c))_{\mathrm{Id}} \arrow[r, "F(\varphi)\circ-"] \arrow[d, "\chi(c)\circ-"'] & \mathrm{Map}(F(c),F(d))_{F(\varphi)} \arrow[d, "\chi(d)\circ-"'] \\
\mathrm{Map}(F(c),F'(c))_{\chi(c)} \arrow[r, "F'(\varphi)\circ-"] & \mathrm{Map}(F(c),F'(d))_{F'(\varphi)\circ\chi(c)}
\end{tikzcd}$$

is a homotopy pullback. For example, when F' is the functor which sends every object to a point, then χ is relatively centric if and only if the functor F defines a centric diagram. Assume we are given a relatively centric natural transformation $\chi\colon F \longrightarrow F'$ where F' is a functor to $\mathfrak{Top}$, and assume furthermore that for each $c \in \mathrm{Ob}(\mathcal{C})$, the homotopy fiber

$$\Gamma(c) \overset{\mathrm{def}}{=} \mathrm{hofiber}\Big(\mathrm{Map}(F(c),F(c))_{\mathrm{Id}} \xrightarrow{\chi(c)\circ-} \mathrm{Map}(F(c),F'(c))_{\chi(c)}\Big)$$

is connected. We claim that this determines functors

$$\text{(all } i \ge 1) \qquad \beta_i \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$$

such that $\beta_i(c) \cong \pi_i(\Gamma(c))$ for all c. To show this, we can assume without loss of generality that $\chi(c)$ is a fibration for all c, and let $\Gamma(c)$ be the space of all maps $f \in \mathrm{Map}(F(c), F(c))$ such that $\chi(c) \circ f = \chi(c)$. Then $\Gamma(c)$ is a monoid under composition, and in particular, $\pi_1(\Gamma(c))$ is abelian. For each morphism $\varphi \in \mathrm{Mor}_{\mathcal{C}}(c, d)$ in $\mathcal{C}$, we can choose a representative $F(\varphi)$ such that the following square commutes:

$$\begin{tikzcd} F(c) \arrow[r, "F(\varphi)"] \arrow[d, "\chi(c)"'] & F(d) \arrow[d, "\chi(d)"'] \\ F'(c) \arrow[r, "F'(\varphi)"] & F'(d) \end{tikzcd}$$

Since χ is relatively centric, the fibers of the map

$$(\chi(d) \circ -)\colon \mathrm{Map}(F(c), F(d))_{F(\varphi)} \longrightarrow \mathrm{Map}(F(c), F'(d))_{F'(\varphi)\circ\chi(c)}$$

have the homotopy type of $\Gamma(c)$ and hence are connected. Hence any two choices for $F(\varphi)$ differ by a path in the fiber over the point $F'(\varphi) \circ \chi(c)$; ie by a homotopy $\{F_t(\varphi)\}_{t\in I}$ such that $\chi(d) \circ F_t(\varphi) = F'(\varphi) \circ \chi(c)$ for each t.

For each $\varphi \in \mathrm{Mor}_{\mathcal{C}}(c, d)$, consider the following diagram:

$$\begin{tikzcd} \mathrm{Map}(F(d), F(d))_{\mathrm{Id}} \arrow[r, "-\circ F(\varphi)", "=w_1"'] \arrow[d, "\chi(d)\circ-"', "=u_1"] & \mathrm{Map}(F(c), F(d))_{F(\varphi)} \arrow[d, "\chi(d)\circ-"', "=u_2"] & \mathrm{Map}(F(c), F(c))_{\mathrm{Id}} \arrow[l, "F(\varphi)\circ-"', "=w_2"] \arrow[d, "\chi(c)\circ-"', "=u_3"] \\ \mathrm{Map}(F(d), F'(d))_{\chi(d)} \arrow[r, "-\circ F(\varphi)"] & \mathrm{Map}(F(c), F'(d))_{\chi(d)F(\varphi)} & \mathrm{Map}(F(c), F'(c))_{\chi(c)} \arrow[l, "F'(\varphi)\circ-"'] \end{tikzcd}$$

where the right hand square commutes by the assumption on $F(\varphi)$ (and the other since composition is associative). Set $\Gamma(c) = u_3^{-1}(\chi(c))$ and $\beta_i(c) = \pi_i(\Gamma(c), \mathrm{Id}_{F(c)})$ (and similarly for d). By assumption, w_2 sends $\Gamma(c)$ to $u_2^{-1}(F'(\varphi) \circ \chi(c))$ by a homotopy equivalence, and we let $\beta_i(\varphi)$ be the composite

$$\underbrace{\pi_i(\Gamma(d), \mathrm{Id}_{F(d)})}_{=\beta_i(d)} \xrightarrow{\;w_1\circ-\;} \pi_i(u_2^{-1}(F'(\varphi) \circ \chi(c)), F(\varphi)) \xrightarrow[\cong]{(w_2\circ-)^{-1}} \underbrace{\pi_i(\Gamma(c), \mathrm{Id}_{F(c)})}_{=\beta_i(c)}.$$

By the above remarks, this is independent of the choice of map $F(\varphi)$. Hence this defines a functor on $\mathcal{C}^{\mathrm{op}}$: the relations $\beta_i(\psi \circ \varphi) = \beta_i(\psi) \circ \beta_i(\varphi)$ follow using any choice of homotopy from $F(\psi \circ \varphi)$ to $F(\psi) \circ F(\varphi)$ which covers $F'(\psi \circ \varphi)$. (Recall that we are assuming F' is a functor to Top, so $F'(\psi \circ \varphi) = F'(\psi) \circ F'(\varphi)$.)

The following theorem is our main result giving a relative version of the Dwyer–Kan obstruction theory. The special case where $F'(c)$ is a point for all $c \in \mathrm{Ob}(\mathcal{C})$ is the case shown by Dwyer and Kan in [10].

Theorem A.3 *Fix functors* $F\colon \mathcal{C} \longrightarrow \mathtt{hoTop}$ *and* $F'\colon \mathcal{C} \longrightarrow \mathtt{Top}$, *and let*

$$\chi\colon F \longrightarrow \mathtt{ho} \circ F'$$

be a relatively centric natural transformation of functors. For each $c \in \mathrm{Ob}(\mathcal{C})$, *assume that the homotopy fiber*

$$\Gamma(c) = \mathrm{hofiber}\big(\mathrm{Map}(F(c), F(c))_{\mathrm{Id}} \xrightarrow{\chi(c)\circ -} \mathrm{Map}(F(c), F'(c))_{\chi(c)}\big)$$

is connected. Let $\beta_i\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathtt{Ab}$ *(all* $i \geq 1$*) be the functors defined above. Then the obstructions to the existence of a rigidification* $\widetilde{F} \xrightarrow{\widetilde{\chi}} F'$ *of* $F \xrightarrow{\chi} F'$ *lie in the groups* $\varprojlim_{\mathcal{C}}^{n+2}(\beta_n)$ *for* $n \geq 1$*; while the obstructions to the uniqueness of* $(\widetilde{F}, \widetilde{\chi})$ *up to equivalence of rigidifications lie in* $\varprojlim_{\mathcal{C}}^{n+1}(\beta_n)$ *for* $n \geq 1$.

Proof We use here the general description of the higher limits of a functor $\alpha\colon \mathcal{C}^{\mathrm{op}} \to \mathtt{Ab}$ as the homology groups of the normalized cochain complex

$$\overline{C}^n(\mathcal{C};\alpha) = \prod_{c_0 \to \cdots \to c_n} \alpha(c_0),$$

where the product is taken over all composable sequences of nonidentity morphisms. For $\xi \in \overline{C}^n(\mathcal{C};\alpha)$, define

$$d(\xi)(c_0 \xrightarrow{\varphi} c_1 \to \cdots \to c_{n+1}) = F(\varphi)(\xi(c_1 \to \cdots \to c_{n+1})) + \sum_{i=1}^{n+1} (-1)^i \xi(c_0 \to \cdots \widehat{c_i} \cdots \to c_{n+1}) \ .$$

Then we have $\varprojlim_{\mathcal{C}}^*(\alpha) \cong H^*(\overline{C}^*(\mathcal{C};\alpha), d)$ (cf [15, Appendix II, Proposition 3.3] or [26, Lemma 2].)

Proof of existence As above, we replace each $\chi(c)$ by a fibration, and replace each $F(\varphi)$ (for $\varphi \in \mathrm{Mor}_{\mathcal{C}}(c,d)$ by a map such that $\chi(d) \circ F(\varphi) = F'(\varphi) \circ \chi(c)$. We also assume that $\overline{F}(\mathrm{Id}_c) = \mathrm{Id}_{\overline{F}(c)}$ for each c. Then

$$\Gamma(c) \overset{\mathrm{def}}{=} \big\{ f \in \mathrm{Map}(F(c), F(c)) \mid \chi(c) \circ f = \chi(c) \big\}$$

is a topological monoid under composition, and is connected by assumption. So we can ignore basepoints when working in the homotopy groups $\beta_i(c) = \pi_i(\Gamma(c))$.

We want to construct an Rg_∞–structure $\bar{F}$ such that $\bar{F}(c) = F(c)$ for all $c \in \mathrm{Ob}(\mathcal{C})$, $\bar{F}(\varphi) = F(\varphi)$ for all $\varphi \in \mathrm{Mor}(\mathcal{C})$, and such that for each $n \ge 2$ and each chain $\xi = (c_0 \to \cdots \to c_n) \in \mathrm{Mor}^n(\mathcal{C})$, the following square commutes (exactly) for each $t \in I^{n-1}$:

$$\begin{tikzcd} F(c_0) \arrow[r, "\bar{F}(\xi)(t)"] \arrow[d, "\chi(c_0)"'] & F(c_n) \arrow[d, "\chi(c_n)"'] \\ F'(c_0) \arrow[r, "F'(\mathring{\xi})"] & F'(c_n) \end{tikzcd} \tag{38}$$

By Proposition A.2, any such structure induces a rigidification $\widetilde{F}$ of F, together with a natural transformation of functors $\widetilde{\chi}$ from $\widetilde{F}$ to F'.

Assume, for some $n \ge 2$, that $\bar{F}$ has been defined on $\mathrm{Mor}^i(\mathcal{C})$ for all $i < n$. Fix $\xi \in \mathrm{Mor}^n(\mathcal{C})$, a composite of (nonidentity) maps from c_0 to c_n. Consider the following commutative square, which is a homotopy pullback by assumption:

$$\begin{tikzcd} \mathrm{Map}(F(c_0), F(c_n))_{F(\mathring{\xi})} \arrow[d, "\chi(c_n)\circ-"', "=u"] & \mathrm{Map}(F(c_0), F(c_0))_{\mathrm{Id}} \arrow[l, "F(\mathring{\xi})\circ-"', "=w"] \arrow[d, "\chi(c_0)\circ-"'] \\ \mathrm{Map}(F(c_0), F'(c_n))_{F'(\mathring{\xi})\circ\chi(c_0)} & \mathrm{Map}(F(c_0), F'(c_0))_{\chi(c_0)} \arrow[l, "F'(\mathring{\xi})\circ-"'] \end{tikzcd} \tag{39}$$

Conditions (b) and (c) in Definition A.1 determine a map $\bar{F}(\xi)_0$ from ∂I^{n-1} to $\mathrm{Map}(F(c_0), F(c_n))_{F(\mathring{\xi})}$ whose image lies in $u^{-1}(F'(\mathring{\xi})\circ\chi(c_0))$. Hence the obstruction to defining $\bar{F}(\xi)$ on I^{n-1} is an element

$$\eta(\xi) \in \pi_{n-2}\big(u^{-1}(F'(\mathring{\xi}) \circ \chi(c_0)),\big) \xleftarrow[\cong]{w\circ-} \pi_{n-2}(\Gamma(c_0)) = \beta_{n-2}(c_0).$$

If one of the morphisms in the sequence ξ is an identity morphism, then we define $\bar{F}(\xi)$ using the appropriate formula in Definition A.1(b), and $\eta(\xi) = 0$. Thus η is in $\bar{C}^n(\mathcal{C}; \beta_{n-2})$.

We claim that $d\eta = 0$. Fix $\omega = (c_0 \to \cdots \to c_{n+1}) \in \mathrm{Mor}^{n+1}(\mathcal{C})$. Consider the face maps on the n–cube

$$\delta_i^t\colon I^{n-1} \longrightarrow I^n \qquad \text{where} \qquad \delta_i^t(t_1, \dots, t_{n-1}) = (t_1, \dots, t_{i-1}, t, t_i, \dots, t_{n-1})$$

(for all $i = 1, \dots, n$ and $t = 0, 1$). The conditions in Definition A.1(b,c) define a map

$$\begin{aligned} \bar{F}_\bullet(\omega)\colon (I^n)^{(n-2)} \longrightarrow \mathrm{Map}(F(c_0), F(c_{n+1}))_{F(\mathring{\omega})} &\simeq \mathrm{Aut}(F(c_0))_1 \\ &= \mathrm{Map}(F(c_0), F(c_0))_{\mathrm{Id}}, \end{aligned}$$

and hence

$$\sum_{i=1}^{n}(-1)^i\Big([\bar F_\bullet(\omega)\circ\delta_i^1|_{\partial I^{n-1}}]-[\bar F_\bullet(\omega)\circ\delta_i^0|_{\partial I^{n-1}}]\Big)=0\in\pi_1(\mathrm{Aut}(F(c_0))_1). \tag{40}$$

Furthermore, $\bar F_\bullet(\omega)$ extends to the faces $\delta_i^0(I^{n-1})$ for $2\le i\le n-1$ (again, by the conditions in Definition A.1(c)), and so those terms vanish in (40). So we are left with the equality

$$\begin{aligned}0&=[\bar F_\bullet(\omega)\circ\delta_1^0|_{\partial I^{n-1}}]+\sum_{i=1}^{n}(-1)^i[\bar F_\bullet(\omega)\circ\delta_i^1|_{\partial I^{n-1}}]+(-1)^{n+1}[\bar F_\bullet(\omega)\circ\delta_n^0|_{\partial I^{n-1}}]\\&=F(\omega_{01})^*(\eta(\partial_0\omega))+\sum_{i=1}^{n+1}(-1)^i\eta(\partial_i\omega)=d\eta(\omega).\end{aligned}$$

Thus $d\eta=0$, and so $[\eta]\in\varprojlim^n_{\mathcal C}(\beta_{n-2})$.

If $[\eta]=0$, then there is $\rho\in\bar C^{n-1}(\mathcal C;\beta_{n-2})$ such that $\eta=d\rho$. Similar (but simpler) arguments to those used above now show that $\bar F$ can be "changed by ρ" on elements of $\mathrm{Mor}^{n-1}(\mathcal C)$, in a way so that the obstruction η vanishes. We can thus arrange that $\bar F$ can be extended to $\mathrm{Mor}^n(\mathcal C)$. Upon continuing this procedure, we obtain the Rg_∞–structure $\bar F$.

Proof of uniqueness Now assume that

$$\widetilde F_1\xrightarrow{\widetilde\chi_1}F'\xleftarrow{\widetilde\chi_2}\widetilde F_2$$

are two rigidifications of $\chi\colon F\longrightarrow F'$. In other words, we have a homotopy commutative diagram

$$\begin{tikzcd} F \arrow[r, "\nu_1"] \arrow[d, "\nu_2"'] \arrow[dr, "\chi"] & \mathrm{ho}\circ\widetilde F_1 \arrow[d, "\mathrm{ho}(\widetilde\chi_1)"] \\ \mathrm{ho}\circ\widetilde F_2 \arrow[r, "\mathrm{ho}(\widetilde\chi_2)"] & \mathrm{ho}\circ F' \end{tikzcd} \tag{41}$$

of functors $\mathcal C\to\mathrm{hoTop}$ and natural transformations between them. We can assume that the maps $\widetilde\chi(c)$ and $\widetilde\chi'(c)$ are fibrations for each $c\in\mathrm{Ob}(\mathcal C)$; otherwise we replace them by fibrations using one of the canonical constructions.

For each $c\in\mathrm{Ob}(\mathcal C)$, let $\theta(c)\colon\widetilde F_1(c)\to\widetilde F_2(c)$ be any map such that $\theta(c)\circ\nu_1(c)\simeq\nu_2(c)$ as maps from $\widetilde F_1(c)$ to $\widetilde F_2(c)$. Using the homotopy commutativity of (41), and the homotopy lifting property for $\widetilde\chi_2(c)$, we can assume that $\widetilde\chi_2(c)\circ\theta(c)=\widetilde\chi_1(c)$

(exactly, not just up to homotopy). Let $\widehat{F}(c)$ be the mapping cylinder of $\theta(c)$, and let $\widehat{\chi}(c)\colon \widehat{F}(c) \longrightarrow F'(c)$ be the projection induced by $\widetilde{\chi}_1(c)$ and $\widetilde{\chi}_2(c)$.

Regard $\widetilde{F}_1(c)$ and $\widetilde{F}_2(c)$ as subspaces of $\widehat{F}(c)$. We want to extend the locally finite Rg_∞–structures $\widetilde{F}$ and $\widetilde{F}'$ to an Rg_∞–structure $\widehat{F}$ covering F'. For each morphism $\varphi \in \mathrm{Mor}_{\mathcal{C}}(c,d)$ in $\mathcal{C}$, $\theta(d)\circ\widetilde{F}_1(\varphi) \simeq \widetilde{F}_2(\varphi)\circ\theta(c)$, and hence $\widetilde{F}_1(\varphi)$ and $\widetilde{F}_2(\varphi)$ can be extended to a map $\widehat{F}(\varphi)$ from $\widehat{F}(c)$ to $\widehat{F}(d)$. Using the homotopy lifting property again, this can be chosen such that $\widehat{\chi}(d)\circ\widehat{F}(\varphi) = F'(\varphi)\circ\widehat{\chi}(c)$.

Assume, for some $n \geq 2$, that $\widehat{F}$ has been defined on $\mathrm{Mor}^i(\mathcal{C})$ for $i < n$ in a way so that (38) commutes (with F and $\overline{F}$ replaced by $\widehat{F}$) for each ξ. Then for each $\xi = (c_0 \to \cdots \to c_n)$ in $\mathrm{Mor}^n(\mathcal{C})$, $\widehat{F}(\xi)$ has been defined on

$$\Big((\widetilde{F}_1(c_0)\cup\widetilde{F}_2(c_0))\times I^{n-1}\Big)\cup\Big(\widehat{F}(c_0)\times\partial I^{n-1}\Big),$$

and must be extended to $\widehat{F}(c_0)\times I^{n-1}$ while covering

$$\widehat{\chi}(c_n)\circ\widehat{F}(\mathring{\xi}) \in \mathrm{Map}(\widehat{F}(c_0), F'(c_n)).$$

So with the help of diagram (39) again, the obstruction to defining $\widehat{F}(\xi)$ is seen to be an element $\tau(\xi) \in \pi_{n-1}(\Gamma(c_0)) = \beta_{n-1}(c_0)$. Together, these define a cochain $\tau \in \overline{C}^n(\mathcal{C};\beta_{n-1})$. Just as in the proof of existence, one then shows that $d\tau = 0$, and hence that τ represents a class $[\tau] \in \varprojlim^n(\beta_{n-1})$. If $[\tau] = 0$, then $\tau = d\rho$ for some $\rho \in \overline{C}^{n-1}(\mathcal{C};\beta_{n-1})$, and $\widehat{F}$ can be modified on $\mathrm{Mor}^{n-1}(\mathcal{C})$ using ρ in such a way that it can then be extended to $\mathrm{Mor}^n(\mathcal{C})$. Upon continuing this procedure, we construct an Rg_∞–structure $\widehat{F}$ on F, together with a natural transformation to F' and morphisms of Rg_∞–structures

$$\widetilde{F}_1 \longrightarrow \widehat{F} \longleftarrow \widetilde{F}_2.$$

So by Proposition A.2,

$$\widetilde{F}_1 \simeq \mathfrak{Rg}(\widetilde{F}_1) \simeq \mathfrak{Rg}(\widehat{F}) \simeq \mathfrak{Rg}(\widetilde{F}_2) \simeq \widetilde{F}_2. \qquad \square$$

We finish the section with two corollaries to Theorem A.3. The first is the main theorem of Dwyer and Kan in [10]. It is the "absolute case" of Theorem A.3: the case where F' is the constant functor which sends each object to a point.

A functor F from $\mathcal{C}$ to Top or hoTop will be called *centric* if for each morphism $\varphi \in \mathrm{Mor}_{\mathcal{C}}(c,d)$ in $\mathcal{C}$, the induced map

$$\mathrm{Map}(F(c), F(c)_{\mathrm{Id}} \xrightarrow{\varphi\circ-} \mathrm{Map}(F(c), F(d))_{\varphi}$$

is a homotopy equivalence. This is what Dwyer and Kan call a centric diagram.

Corollary A.4 *Fix a centric functor $F\colon \mathcal{C} \longrightarrow \mathtt{hoTop}$. Define $\alpha_i\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathtt{Ab}$ (all $i \ge 1$) by setting $\alpha_i(c) = \pi_i\big(\mathrm{Map}(F(c),F(c))_{\mathrm{Id}}\big)$ and by letting $\alpha_i\big(c \xrightarrow{\varphi} d\big)$ be the composite*

$$\pi_i\big(\mathrm{Map}(F(d),F(d))_{\mathrm{Id}}\big) \xrightarrow{(-\circ F(\varphi))_*} \pi_i\big(\mathrm{Map}(F(c),F(d))_{F(\varphi)}\big)$$
$$\xleftarrow[\cong]{(F(\varphi)\circ -)_*} \pi_i\big(\mathrm{Map}(F(c),F(c))_{\mathrm{Id}}\big).$$

Then the obstructions to the existence of a rigidification $\widetilde{F}$ of F lie in the groups $\varprojlim_{\mathcal{C}}^{n+2}(\alpha_n)$ for $n \ge 1$; while the obstructions to the uniqueness of $\widetilde{F}$ up to equivalence of rigidifications lie in $\varprojlim_{\mathcal{C}}^{n+1}(\alpha_n)$ for $n \ge 1$.

The second corollary is a generalization of [9, Proposition B], and follows upon combining Corollary A.4 with an idea taken from the proof of that proposition.

Corollary A.5 *Fix a space X, and a centric functor $F\colon \mathcal{C} \longrightarrow \mathtt{hoTop}$. We also let X denote the constant functor $X\colon \mathcal{C} \longrightarrow \mathtt{Top}$ which sends each object to X and each morphism to Id_X.*

(a) *Assume there is a natural transformation of functors $\chi\colon F \longrightarrow \mathtt{ho}\circ X$ such that the map $\chi(c)\colon F(c) \longrightarrow X$ is centric for each $c \in \mathrm{Ob}(\mathcal{C})$. Then there is a rigidification $\widetilde{F}$ of F, together with a rigidification $\widetilde{\chi}\colon \widetilde{F} \longrightarrow X$ of χ.*

(b) *Assume $\widetilde{F}_1$ and $\widetilde{F}_2$ are two rigidifications of F. Let $\psi_i\colon F \longrightarrow \mathtt{ho}\circ \widetilde{F}_i$ be natural equivalences, and let $\widetilde{\chi}_i\colon \widetilde{F}_i \longrightarrow X$ be natural transformations of functors such that for all $c \in \mathrm{Ob}(\mathcal{C})$, $\widetilde{\chi}_i(c) \in \mathrm{Map}(\widetilde{F}_i(c),X)$ is centric, and the square*

$$\begin{tikzcd} F(c) \arrow[r, "\psi_1(c)"] \arrow[d, "\psi_2(c)"'] & \widetilde{F}_1(c) \arrow[d, "\widetilde{\chi}_1(c)"] \\ \widetilde{F}_2(c) \arrow[r, "\widetilde{\chi}_2(c)"] & X \end{tikzcd} \tag{42}$$

commutes up to homotopy. Then $\widetilde{F}_1$ and $\widetilde{F}_2$ are equivalent rigidifications. More precisely, there is a third rigidification $\widetilde{F}_0$ of F, natural transformations of functors

$$\widetilde{F}_1 \xrightarrow{\widetilde{\psi}_1} \widetilde{F}_0 \xleftarrow{\widetilde{\psi}_2} \widetilde{F}_2$$

such that $\widetilde{\psi}_i(c)$ is a homotopy equivalence for each $c \in \mathrm{Ob}(\mathcal{C})$, a space X_0 together with a natural transformation $\widetilde{\chi}_0\colon \widetilde{F}_0 \longrightarrow X_0$ to the constant functor, *and homotopic homotopy equivalences $f_1 \simeq f_2\colon X \longrightarrow X_0$, such that the following diagram commutes for each $c \in \mathrm{Ob}(\mathcal{C})$:*

$$\begin{tikzcd}
\widetilde{F}_1(c) \arrow[r, "\widetilde{\psi}_1(c)", "\simeq"'] \arrow[d, "\widetilde{\chi}_1(c)"'] & \widetilde{F}_0(c) \arrow[d, "\widetilde{\chi}_0(c)"'] & \widetilde{F}_2(c) \arrow[l, "\widetilde{\psi}_2(c)"', "\simeq"] \arrow[d, "\widetilde{\chi}_2(c)"'] \\
X \arrow[r, "f_1"] & X_0 & X \arrow[l, "f_2"']
\end{tikzcd} \tag{43}$$

Proof Let $\mathcal{C}_+$ be the category $\mathcal{C}$ with an additional final object $*$ added. For any functor $\alpha\colon \mathcal{C}_+{}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$, $\varprojlim^i(\alpha) = 0$ for all $i \geq 1$ since $\mathcal{C}_+{}^{\mathrm{op}}$ has an initial object. A functor $F_+\colon \mathcal{C}_+ \longrightarrow \mathrm{hoTop}$ can be thought of as a triple $F_+ = (F, X, \chi)$, where $F = F_+|_{\mathcal{C}}$ is a functor from $\mathcal{C}$ to hoTop, $X = F_+(*)$ is a space, and χ is a natural transformation of functors from F to the constant functor X. Functors from $\mathcal{C}_+$ to Top are described in an analogous way.

In the situation of (a), (F, X, χ) is a functor from $\mathcal{C}_+$ to hoTop. The obstruction groups of Corollary A.4 vanish, and hence it has a rigidification $(\widetilde{F}, \widetilde{X}, \widetilde{\chi})$. Upon composing with an appropriate homotopy equivalence $\widetilde{X} \xrightarrow{\simeq} X$, we can arrange that $\widetilde{X} = X$.

In the situation of (b), $(\widetilde{F}_1, X, \widetilde{\chi}_1)$ and $(\widetilde{F}_2, X, \widetilde{\chi}_2)$ are two functors from $\mathcal{C}_+$ to Top which are rigidifications of the same functor $(F, X, \widetilde{\chi}_1 \circ \psi_1)$ by the homotopy commutativity of (42). Since the uniqueness obstructions of Corollary A.4 all vanish, there is a third homotopy lifting $(\widetilde{F}_0, X_0, \widetilde{\chi}_0)$, together with natural transformations of functors

$$(\widetilde{F}_1, X, \widetilde{\chi}_1) \xleftarrow{\ \widetilde{\psi}_1\ } (\widetilde{F}_0, X_0, \widetilde{\chi}_0) \xrightarrow{\ \widetilde{\psi}_2\ } (\widetilde{F}_2, X, \widetilde{\chi}_2)$$

which induce homotopy equivalences on all objects. Thus upon setting $f_i = \widetilde{\psi}_i(*)$, we obtain the commutative diagram (43), where all horizontal maps are homotopy equivalences. Finally, $\widetilde{\psi}_1(*) \simeq \widetilde{\psi}_2(*)$, since they come from equivalences between liftings of the same homotopy functor, and this finishes the proof of (b). □

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Received: 19 July 2006
Accepted: 20 November 2006