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Cohomological estimates for cat(X, ξ)

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This paper studies the homotopy invariant $\mathrm{cat}(X, \xi)$ introduced by the first author in [6]. Given a finite cell-complex X, we study the function $\xi \mapsto \mathrm{cat}(X, \xi)$ where ξ varies in the cohomology space $H^1(X; \mathbb{R})$. Note that $\mathrm{cat}(X, \xi)$ turns into the classical Lusternik–Schnirelmann category $\mathrm{cat}(X)$ in the case $\xi = 0$. Interest in $\mathrm{cat}(X, \xi)$ is based on its applications in dynamics where it enters estimates of complexity of the chain recurrent set of a flow admitting Lyapunov closed 1–forms, see [6; 7].

In this paper we significantly improve earlier cohomological lower bounds for $\mathrm{cat}(X, \xi)$ suggested in [6; 7]. The advantages of the current results (see Theorems 5, 6 and 7 below) are twofold: firstly, we allow cohomology classes ξ of arbitrary rank (while in [6] the case of rank one classes was studied), and secondly, the theorems of the present paper are based on a different principle and give slightly better estimates even in the case of rank one classes. We introduce in this paper a new controlled version of $\mathrm{cat}(X, \xi)$ and find upper bounds for it (Theorems 11 and 16). We apply these upper and lower bounds in a number of specific examples where we explicitly compute $\mathrm{cat}(X, \xi)$ as a function of the cohomology class $\xi \in H^1(X; \mathbb{R})$.

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1 Introduction

This paper studies the homotopy invariant $\mathrm{cat}(X, \xi)$ originally introduced by the first author in [6] and investigated further in [7]. It is a generalization of the classical Lusternik–Schnirelman category, which appeared in connection with the problem of estimating the number of zeros of closed one-forms lying in a prescribed nonzero cohomology class. Assuming that the zeros are all Morse type the answer is provided by Novikov theory [16]. If the zeros are arbitrary, there always exists a closed 1–form with at most one zero [6; 9] and therefore one cannot expect to get meaningful lower bounds on the number of zeros in homotopic terms. However, such lower bounds exist if the gradient flow of the closed one form has no homoclinic cycles [6; 7]. More generally, there is always an interesting relation, governed by $\mathrm{cat}(X, \xi)$, between the topology of the chain recurrent set of the flow admitting a Lyapunov 1–form and the topology of the underlying manifold, see Farber and Kappeler [8], and Latour [14].

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Our goal in this paper is to find a cohomological lower bound for $\mathrm{cat}(X,\xi)$ generalizing the classical cup-length estimate for the usual category. Earlier cohomological lower bounds for $\mathrm{cat}(X,\xi)$ (see [6, Theorems 6.1 and 6.4]) were obtained only in the case of rank one cohomology classes. The advantages of the current results (see Theorems 5, 6 and 7 below) are twofold: firstly, we allow cohomology classes ξ of arbitrary rank and secondly, the theorems of the present paper are based on a different principle and give slightly better estimates even in the case of rank one classes.

The main result of the paper is Theorem 7 which we will now describe. Let (X,ξ) be a pair consisting of a finite cell-complex and a cohomology class $\xi\in H^1(X;\mathbb{R})$. We denote by $\mathrm{Ker}(\xi)\subset H_1(X;\mathbb{Z})$ the subgroup consisting of homology classes z such that $\langle\xi,z\rangle=0$. The factor group $H=H_1(X;\mathbb{Z})/\mathrm{Ker}(\xi)$ is a free abelian group of rank $r=\mathrm{rank}(\xi)$. We consider the set $\mathcal{V}_\xi$ of all complex flat line bundles L over X such that the monodromy of L is trivial along any homology class in $\mathrm{Ker}(\xi)$. Clearly, $\mathcal{V}_\xi=\mathrm{Hom}(H;\mathbb{C}^*)=(\mathbb{C}^*)^r$ has the structure of an algebraic variety. A bundle $L\in\mathcal{V}_\xi$ is called transcendental if the monodromy homomorphism $\mathrm{Mon}_L\colon\mathbb{Z}[H]\to\mathbb{C}$ is injective. In the case when $\mathrm{rank}H=1$, line bundles correspond to complex numbers (since $\mathcal{V}_\xi=\mathbb{C}^*$) and transcendental bundles correspond to numbers which are transcendental in the usual sense[1]. One of our main results is given by the following theorem:

Theorem 1 *Suppose that $L\in\mathcal{V}_\xi$ is transcendental and there exist cohomology classes $v_0\in H^*(X;L)$ and $v_i\in H^{d_i}(X;\mathbb{C})$ where $i=1,\dots,k$. If $d_i>0$ for $i=1,\dots,k$ and the cup-product $v_0\cup v_1\cup\cdots\cup v_k\in H^*(X;L)$ is nonzero then $\mathrm{cat}(X,\xi)>k$.*

Given a finite cell-complex X, we examine the function $\xi\mapsto\mathrm{cat}(X,\xi)$ where ξ varies in the cohomology space $H^1(X;\mathbb{R})$. We compute it explicitly in some examples and investigate its behavioral patterns.

Another new result worth mentioning is the introduction of a new controlled version of $\mathrm{cat}(X,\xi)$ which we denote $\mathrm{ccat}(X,\xi)$. It is a modification of the invariant $\mathrm{cat}(X,\xi)$ which coincides with $\mathrm{cat}(X,\xi)$ in all examples known to us. We find new upper bounds for $\mathrm{ccat}(X,\xi)$ (Theorems 11 and 16) and a product inequality generalizing the classical product inequality for the usual category.

There are interesting connections between the invariant $\mathrm{cat}(X,\xi)$ and its various modifications and the Bieri–Neumann–Strebel invariant of discrete groups [2], see also [1]. We discuss some of these relations in Section 13. We show that the Bieri–Neumann–Strebel invariant allows us to improve the upper bound for $\mathrm{cat}(X,\xi)$ in the case of manifolds. In particular we prove the following theorem (which is combination of Theorems 11 and 14 in the text):

[1] Numbers which are not roots of polynomial equations with integral coefficients

Theorem 2 *Let M be a closed connected smooth manifold. Then for any nonzero cohomology class $\xi \in H^1(M;\mathbb{R})$ one has $\mathrm{cat}(M,\xi) \leq n-1$ where $n = \dim M$. Moreover, if $n \geq 5$ and for some nonzero $\xi \in H^1(M;\mathbb{R})$ we have $\mathrm{cat}(M,\xi) = n-1$ then the fundamental group $\pi_1(M)$ contains a non-abelian free subgroup.*

In the last section of the paper we discuss several open problems.

2 Abel–Jacobi maps and neighborhoods of infinity

For convenience of the reader we give in Sections 2 and 3 a brief summary of our result [10] (see Theorem 3 below) which will play a crucial role in this paper.

Let X be a connected finite cell complex and $p\colon \widetilde{X} \to X$ a regular covering having a free abelian group of covering transformations $H \simeq \mathbb{Z}^r$. Denote $H_{\mathbb{R}} = H \otimes \mathbb{R}$; it is a vector space of dimension r containing H as a lattice.

Proposition 2.1 *There exists a canonical Abel–Jacobi map*

$$A\colon \widetilde{X} \to H_{\mathbb{R}} \tag{1}$$

having the following properties:

(a) *A is H–equivariant; here H acts on $\widetilde{X}$ by covering transformations and it acts on $H_{\mathbb{R}}$ by translations.*

(b) *A is proper (ie, the preimage of a compact subset of $H_{\mathbb{R}}$ is compact).*

(c) *A is determined uniquely up to replacing it by a map $A'\colon \widetilde{X} \to H_{\mathbb{R}}$ of the form $A' = A + F \circ p$ where $F\colon X \to H_{\mathbb{R}}$ is a continuous map.*

This fact is well-known; see for example [2; 10].

Let $\xi \in H^1(X;\mathbb{R})$ be a cohomology class with the property

$$p^*(\xi) = 0 \in H^1(\widetilde{X};\mathbb{R}).$$

Such a class ξ can be viewed either as a homomorphism $\xi\colon H \to \mathbb{R}$ or as a linear functional $\xi_{\mathbb{R}}\colon H_{\mathbb{R}} \to \mathbb{R}$.

Definition 2.2 A subset $N \subset \widetilde{X}$ is called a neighborhood of infinity in $\widetilde{X}$ with respect to the cohomology class ξ if N contains the set

$$\{x \in \widetilde{X};\ \xi_{\mathbb{R}}(A(x)) > c\} \subset N, \tag{2}$$

for some real $c \in \mathbb{R}$. Here $A\colon \widetilde{X} \to H_{\mathbb{R}}$ is an Abel–Jacobi map for the covering $p\colon \widetilde{X} \to X$.

This term was introduced in [10]. However, the object itself is certainly not new. It appeared earlier (with no name) in [2] and in other papers.

Neighborhoods of infinity $N \subset \widetilde{X}$ with respect to ξ can be characterized as follows. Let $\gamma\colon S^1 \to X$ be a continuous map such that evaluation of class ξ on the homology class $[\gamma] \in H_1(X;\mathbb{Z})$ is positive, $\langle \xi, [\gamma]\rangle > 0$. Consider a lift $\widetilde{\gamma}\colon \mathbb{R} \to \widetilde{X}$ of the composition $\mathbb{R} \stackrel{\exp}{\to} S^1 \stackrel{\gamma}{\to} \widetilde{X}$; see the commutative diagram:

(3)
$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\widetilde{\gamma}} & \widetilde{X} \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle p} \\ S^1 & \xrightarrow{\gamma} & X \end{array}$$

Then for all sufficiently large $t \in \mathbb{R}$ the point $\widetilde{\gamma}(t)$ lies in the neighborhood $N \subset \widetilde{X}$.

3 Homology classes movable to infinity

Let G be an abelian group (the coefficient system). We mainly have in mind the cases $G = \mathbb{Z}$ or $G = \mathbf{k}$ is a field.

Definition 3.1 A homology class $z \in H_i(\widetilde{X};G)$ is said to be movable to infinity of $\widetilde{X}$ with respect to a nonzero cohomology class $\xi \in H^1(X;\mathbb{R})$, $p^*(\xi) = 0$, if in any neighborhood N of infinity with respect to ξ there exists a (singular) cycle with coefficients in G representing z.

Equivalently, a homology class $z \in H_i(\widetilde{X};G)$ is movable to infinity with respect to $\xi \in H^1(X;\mathbb{R})$ if z lies in the intersection

(4)
$$\bigcap_N \operatorname{Im}\Big[H_i(N;G) \to H_i(\widetilde{X};G)\Big]$$

where N runs over all neighborhoods of infinity in $\widetilde{X}$ with respect to ξ. This can also be expressed by saying that z lies in the kernel of the natural homomorphism

(5)
$$H_i(\widetilde{X};G) \to \varprojlim H_i(\widetilde{X}, N;G)$$

where in the inverse limit N runs over all neighborhoods of infinity in $\widetilde{X}$ with respect to ξ.

The following theorem, proven in [10], gives an explicit description of all movable homology classes in the case when $G = \mathbf{k}$ is a field. It generalizes the result of [6, Section 5] treating the simplest case of infinite cyclic covers $q\colon \widetilde{X} \to X$.

Theorem 3 *Let X be a finite cell complex and $q\colon \widetilde{X}\to X$ be a regular covering having a free abelian group of covering transformations $H\simeq\mathbb{Z}^r$. Let $\xi\in H^1(X;\mathbb{R})$ be a nonzero cohomology class of rank r satisfying $q^*(\xi)=0$. The following properties* (A), (B), (C) *of a nonzero homology class $z\in H_i(\widetilde{X};\mathbf{k})$ (where $\mathbf{k}$ is a field) are equivalent:*

(A) *z is movable to infinity with respect to ξ.*

(B) *Any singular cycle c in $\widetilde{X}$ realizing the class z bounds an infinite singular chain c' in $\widetilde{X}$ containing only finitely many simplices lying outside every neighborhood of infinity $N\subset\widetilde{X}$ with respect to ξ.*

(C) *There exists a nonzero element $x\in\mathbf{k}[H]$ such that $x\cdot z=0$.*

4 Obstructions to movability to infinity of homology classes

In this section we show that *generic* flat line bundles can be used to detect movability to infinity of homology classes. This observation is used later in this paper when we study cohomological lower bounds for $\mathrm{cat}(X,\xi)$, see Section 5.

Let X be a finite polyhedron and $\xi\in H^1(X;\mathbb{R})$. Denote by $\mathrm{Ker}(\xi)$ the kernel of the homomorphism $H_1(X;\mathbb{Z})\to\mathbb{R}$ given by evaluation on ξ. Then $H=H_1(X;\mathbb{Z})/\mathrm{Ker}(\xi)$ is a free abelian group of finite rank r where r denotes the rank of ξ. Consider the cover $p\colon\widetilde{X}\to X$ corresponding to $\mathrm{Ker}(\xi)$. It has H as the group of covering translations.

Let $\mathcal{V}_\xi=(\mathbb{C}^*)^r=\mathrm{Hom}(H,\mathbb{C}^*)$ denote the variety of all complex flat line bundles L over X such that the induced flat line bundle p^*L on $\widetilde{X}$ is trivial. If $t_1,\dots,t_r\in H$ is a basis, then the monodromy of $L\in\mathcal{V}_\xi$ along t_i is a nonzero complex number $x_i\in\mathbb{C}^*$ and the numbers $x_1,\dots,x_r\in\mathbb{C}^*$ form a coordinate system on $\mathcal{V}_\xi$. Given a flat line bundle $L\in\mathcal{V}_\xi$ the monodromy representation of L is the ring homomorphism

$$\mathrm{Mon}_L\colon\mathbb{C}[H]\to\mathbb{C} \tag{6}$$

sending each $t_i\in H$ to $x_i\in\mathbb{C}^*$. The dual bundle $L^*\in\mathcal{V}_\xi$ is such that $L\otimes L^*$ is trivial; if $x_1,\dots,x_r\in\mathbb{C}^*$ are coordinates of L then $x_1^{-1},\dots,x_r^{-1}\in\mathbb{C}^*$ are coordinates of L^*.

Any nontrivial element $p\in\mathbb{C}[H]$ lying in the kernel of Mon_L can be viewed as a (Laurent) polynomial equation between the variables $x_1,\dots,x_r$. Alternatively, we will consider algebraic subvarieties $V\subset\mathcal{V}_\xi$. Any such V is the set of all solutions of a system of equations of the form

$$q_i(x_1,\dots,x_r,x_1^{-1},\dots,x_r^{-1})=0,\quad i=1,\dots,m$$

where q_i is a Laurent polynomial with complex coefficients

$$p_i \in \mathbb{C}[x_1, \dots, x_r, x_1^{-1}, \dots, x_r^{-1}].$$

This is equivalent to fixing an ideal $J \subset \mathbb{Q}[H]$ and studying the set of all flat line bundles $L \in \mathcal{V}_\xi$ such that $\mathrm{Mon}_L(J) = 0$.

We are chiefly interested in the subset $A \subset \mathcal{V}_\xi$ consisting of all line flat bundles $L \in \mathcal{V}_\xi$ satisfying the following property: *for any homology class $z \in H_*(\widetilde{X}; \mathbb{C})$ which is movable to infinity with respect to ξ one has $p_*(z) = 0 \in H_*(X; L)$ where*

$$p_*\colon H_*(\widetilde{X}; \mathbb{C}) \to H_*(X; L) \tag{7}$$

is the map induced by the projection p. Homomorphism (7) exists because the induced flat line bundle $p^*(L)$ over $\widetilde{X}$ is trivial, $p^*(L) = \mathbb{C}$.

Lemma 4.1 *The complement $V = \mathcal{V}_\xi - A$ is contained in a proper algebraic subvariety of $\mathcal{V}_\xi$.*

Proof Let $T_q \subset H_q(\widetilde{X}; \mathbb{C})$ denote the subset of $\mathbb{C}[H]$–torsion homology classes. According to Theorem 3, T_q is exactly the set of homology classes in $H_q(\widetilde{X}; \mathbb{C})$ which are movable to infinity with respect to ξ.

One has $V = \cup V_q$ where $L \in V_q$ if and only if the composition

$$T_q \to H_q(\widetilde{X}; \mathbb{C}) \to H_q(X; L)$$

is nontrivial.

Let L_0 denote the fiber of L over the base point $x_0 \in X$. It is a one-dimensional complex vector space and the monodromy representation of L determines a structure of a right Λ–module on L_0; here $\Lambda = \mathbb{C}[H]$. Homomorphism (7) equals the composition

$$H_q(\widetilde{X}; \mathbb{C}) \to H_q(\widetilde{X}; \mathbb{C}) \otimes_\Lambda L_0 \to H_q(X; L).$$

Hence we find that $L \in V_q$ implies $T_q \otimes_\Lambda L_0 \neq 0$.

Let $\cdots \to F_1 \xrightarrow{d} F_0 \to T_q \to 0$ be a finitely generated free $\mathbb{C}[H]$–resolution of T_q. We obtain: $L \in V_q$ implies that the linear map

$$F_1 \otimes_\Lambda L_0 \xrightarrow{d} F_0 \otimes_\Lambda L_0 \tag{8}$$

is not onto.

Let M be the square matrix with entries in Λ representing $d\colon F_1 \to F_0$. Then V_q lies in the subvariety V_q' of $\mathcal{V}_\xi$ given by equating to zero all minors of M of size

$n_0 = \mathrm{rank} F_0$. Each of these minors is a polynomial in variables $x_1,\dots,x_r$ with complex coefficients.

Now we show that $V' = \cup V'_q \subset \mathcal{V}_\xi$ is proper. Since T_q is finitely generated and torsion there exists a nonzero Laurent polynomial $\Delta_q \in \mathbb{C}[H]$ such that the multiplication by Δ_q annihilates T_q. Looking at the commutative diagram

$$\begin{array}{ccccccccc}
\longrightarrow & F_1 & \xrightarrow{d} & F_0 & \longrightarrow & T_q & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle \Delta_q} & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle \Delta_q} & & \downarrow{\scriptstyle 0} & & \\
\longrightarrow & F_1 & \xrightarrow[d]{} & F_0 & \longrightarrow & T_q & \longrightarrow & 0
\end{array}$$

we find that there exists a Λ–morphism $g\colon F_0 \to F_1$ such that $d\circ g$ coincides with the multiplication by Δ_q. It is easy to see that $L \notin V'_q$ assuming that $\Delta_q(x_1,\dots,x_r) \neq 0 \in \mathbb{C}$. Hence V' is a proper subset of $\mathcal{V}_\xi$. □

Definition 4.2 The set $\mathcal{V}_\xi - A$ is called the support of (X,ξ). It is denoted by $\mathrm{Supp}(X,\xi)$.

As follows from the proof of the previous lemma, a flat line bundle $L \in \mathcal{V}_\xi$ lies in $\mathrm{Supp}(X,\xi)$ if and only if there exists a $\mathbb{C}[H]$–torsion homology class $z \in H_q(\widetilde{X},\mathbb{C})$ with $p_*(z) \neq 0 \in H_q(X;L)$. Here $p\colon \widetilde{X} \to X$ is the free abelian covering corresponding $\mathrm{Ker}(\xi)$. By Lemma 4.1, $\mathrm{Supp}(X,\xi) \subset \mathcal{V}_\xi$ is contained in a proper algebraic subvariety.

Example 4.3 In the case when ξ has rank one, $r = 1$, the variety $\mathcal{V}_\xi$ coincides with $\mathbb{C}^* = \mathbb{C} - \{0\}$ and the support $\mathrm{Supp}(X,\xi) \subset \mathbb{C}^*$ is a finite set. In this special case Definition 4.2 coincides with the definition given on [7, page 182].

Remark 4.4 Assume that the class $\xi \neq 0$ in nonzero and the polyhedron X is connected. Then the trivial line bundle $\mathbb{C} \in \mathcal{V}_\xi$ belongs to the support $\mathrm{Supp}(X,\xi)$. Indeed, the class $1 \in H_0(\widetilde{X};\mathbb{C})$ is torsion and $p_*(1) = 1 \in H_0(X;\mathbb{C})$.

As a corollary of Lemma 4.1 we obtain the following statement:

Theorem 4 *Let L be a flat line bundle $L \in \mathcal{V}_\xi$ which does not lie in the support $\mathrm{Supp}(X,\xi)$. Suppose that $\langle v, p_*(z)\rangle \neq 0$ where $v \in H^q(X;L)$, $z \in H_q(\widetilde{X};\mathbb{C})$ and $p_*(z) \in H_q(X;L^*)$. Here $L^* \in \mathcal{V}_\xi$ denotes the flat line bundle dual to L. Then z is not movable to infinity of $\widetilde{X}$ with respect to ξ.*

This theorem will be used in the next section in the proof of Theorem 5.

5 A cohomological lower bound for cat(X, ξ)

Let X be a finite simplicial polyhedron and let $\xi \in H^1(X;\mathbb{R})$ be a cohomology class. Our goal is to estimate from below the number $\mathrm{cat}(X,\xi)$ introduced in [6]. For convenience of the reader we recall the definition of $\mathrm{cat}(X,\xi)$.

Let ω be a closed 1–form on X representing ξ; we use the formalism of closed 1–forms on topological spaces suggested in [6]; see also [7].

Definition 5.1 A subset $A \subset X$ is N–movable with respect to ω (where $N \in \mathbb{Z}$ is an integer) if there exists a continuous homotopy $h_t\colon A \to X$, $t \in [0,1]$, such that $h_0\colon A \to X$ is the inclusion and for any point $x \in A$ one has

$$\int_{h_1(x)}^{x} \omega > N \tag{9}$$

where the integral is calculated along the path $t \mapsto h_{1-t}(x) \in X$, $t \in [0,1]$.

Intuitively, an N–movable subset can be continuously deformed inside X such that each of its points is winding around yielding a large quantity (9) measured in terms of the form ω.

It is easy to see that (assuming that $\xi \neq 0$ and the space X is connected) any subset $A \subset X$ such that the inclusion $A \to X$ is null-homotopic is N–movable for any N with respect to any closed 1–form ω representing the class ξ.

Example 5.2 Assume that $X = Y \times S^1$ and $\xi \in H^1(X;\mathbb{R})$ is such that $\xi|_{S^1} \neq 0$. Then for any integer N the total space $A = X$ is N–movable with respect to any closed 1–form ω representing class ξ. A homotopy $h_t\colon X \to X$ as above can be described as follows. Identify $S^1 \subset \mathbb{C}$ with the set of complex numbers having norm one. Then for $y \in Y$ and $z \in S^1$ one sets

$$h_t(y,z) = (y, e^{iMt} \cdot z), \quad t \in [0,1]$$

with suitable real M.

Definition 5.3 Fix a closed 1–form ω representing ξ. The number $\mathrm{cat}(X,\xi)$ is the minimal integer k with the property that for any $N > 0$ there exists an open cover $F, F_1, \dots, F_k \subset X$ such that each inclusion $F_j \to X$ is null-homotopic and such that F is N–movable with respect to ω.

Example 5.4 Assume that $X = Y \times S^1$ and $\xi \in H^1(X;\mathbb{R})$ is such that $\xi|_{S^1} \neq 0$. Then $\mathrm{cat}(X,\xi) = 0$; compare Example 5.2.

It is known that $\mathrm{cat}(X,\xi)$ is homotopy invariant, see [7, Lemma 10.12]. In particular, it is independent of the choice of ω, see [7, page 166].

The next result gives a lower bound for $\mathrm{cat}(X,\xi)$ in terms of cohomological information.

Denote by $p\colon \widetilde{X} \to X$ the covering corresponding to $\mathrm{Ker}\xi$.

Theorem 5 *Suppose that $L_0, L_1, \dots, L_k \in \mathcal{V}_\xi$ are complex flat vector bundles with the following properties:*

(1) *L_0 does not belong to the support $\mathrm{Supp}(X,\xi)$.*

(2) *For a homology class $z \in H_d(\widetilde{X};\mathbb{C})$ and some cohomology classes $v_i \in H^{d_i}(X;L_i)$, $i = 0,1,\dots,k$, where*

$$d_i > 0, \quad \text{for } i = 1,\dots,k \tag{10}$$

the cup-product $v_0 \cup v_1 \cup \cdots \cup v_k \in H^d(X;L)$ evaluates nontrivially on the homology class $p_*(z) \in H_d(X;L^*)$ ie,

$$\langle v_0 \cup \cdots \cup v_k,\, p_*(z)\rangle \neq 0. \tag{11}$$

Here $z \in H_d(\widetilde{X};\mathbb{C})$, $d = \sum_{i=0}^{k} d_i$, $L = L_0 \otimes L_1 \otimes \cdots \otimes L_k$ *and L^* is the dual of L.*

Then

$$\mathrm{cat}(X,\xi) > k. \tag{12}$$

Remark 5.5 Theorem 5 is meaningful for $k = 0$ as well; in this case it gives $\mathrm{cat}(X,\xi) > 0$.

Remark 5.6 In the case $\xi = 0$ (ie, when one deals with functions) the statement above turns into the usual cup-length estimate for the Lusternik–Schnirelmann category $\mathrm{cat}(X)$, see Section 8 below.

Proof of Theorem 5 Assume that (12) is false, ie, $\mathrm{cat}(X,\xi) \leq k$. Let ω be a continuous closed 1–form on X representing ξ.

Then for any $N > 0$ there exists an open cover $F, F_1, \dots, F_k \subset X$ such that each inclusion $F_j \to X$ is null-homotopic and such that F is N–movable with respect to ω.

Fix a singular cycle c in $\widetilde{X}$ representing the class $z \in H_d(\widetilde{X}; \mathbb{C})$. One may find a compact polyhedron $K \subset \widetilde{X}$ containing c.

Let us show that there exists a neighborhood of infinity $U \subset \widetilde{X}$ with the following property: any homology class in $H_*(K; \mathbb{C})$, which is homologous in $\widetilde{X}$ to a cycle lying in U, is movable to infinity of $\widetilde{X}$ with respect to ξ. Indeed, start with an arbitrary neighborhood of infinity N satisfying [10, Lemma 3]. For a covering translation $g\colon \widetilde{X} \to \widetilde{X}$ the intersection

$$V_g = \mathrm{Im}[H_*(gN) \to H_*(\widetilde{X})] \cap \mathrm{Im}[H_*(K) \to H_*(\widetilde{X})]$$

(we use homology with $\mathbb{C}$ coefficients) is a finite dimensional complex vector space. Therefore one may find a covering translation $g_0\colon \widetilde{X} \to \widetilde{X}$ such that for any covering translation g one has $V_{g_0} \subset V_g$. Then clearly the neighborhood of infinity $U = g_0 N$ satisfies the above requirement.

One has $p^*\omega = df$ where $f\colon \widetilde{X} \to \mathbb{R}$ is a continuous function. Then $f(K) \subset [a, b]$ and $U \supset f^{-1}(-\infty, c)$ where $c < a < b$.

Pick a number $N > b - c$ and apply Definition 5.3. We obtain an open cover $F \cup F_1 \cup \dots \cup F_k = X$ where F is N–movable with respect to ω and each inclusion $F_j \to X$ is null-homotopic, $j = 1, 2, \dots, k$.

Find subsets $A' \subset A \subset F$ and $B' \subset B \subset F_1 \cup F_2 \cup \dots \cup F_k$ with the following properties: (i) A' and B' are open and cover X, ie, $A' \cup B' = X$; (ii) A and B are compact sub-polyhedra of X.

The restriction of the cup-product $v = v_1 \cup \dots \cup v_k$ on the set $F_1 \cup \dots \cup F_k$ vanishes[2] and so v can be realized by a singular cochain g vanishing on all singular simplices lying entirely in B'. The cochain g takes values in the local system $L_1 \otimes \dots \otimes L_k$.

We may assume that the topology of X is given by a metric d. Let $\epsilon > 0$ be the Lebesgue number of the open cover $A' \cup B'$.

Subdivide the singular chain c representing z such that it is a linear combination of finitely many singular simplices, each of diameter $< \epsilon$.

From (11) we have

$$\langle v_0, p_*(p^*(v) \cap z) \rangle \neq 0. \tag{13}$$

The class

$$z_0 = p^*(v) \cap z \in H_*(\widetilde{X}; \mathbb{C})$$

[2] Here we use (10) and the assumption that $F_j \to X$ is null-homotopic, $j = 1, \dots k$.

is represented by the singular cycle $p^*(g)\cap c$ (see Spanier [18, Chapter 5, Section 6]) having support in K. From the construction above, we see that the projected cycle $p_*(p^*(g)\cap c)$ lies entirely in A'. We obtain (because of our choice of N and U) that the homology class z_0 is movable to infinity of $\widetilde{X}$ with respect to ξ. But this contradicts Theorem 4 since $\langle v_0, p_*(z_0)\rangle \neq 0$ and the bundle L_0 is assumed to be generic, ie, $L_0 \notin \mathrm{Supp}(X,\xi)$. □

6 Transcendental line bundles

In this section we improve Theorem 5 by showing that under certain conditions one may avoid mentioning explicitly the homology class $z\in H_*(\widetilde{X})$ in the statement.

First we recall our notations. Let $\xi\in H^1(X;\mathbb{R})$ be a real cohomology where X is a finite cell complex. ξ determines a homomorphism $H_1(X;\mathbb{Z})\to\mathbb{R}$; we denote by $\mathrm{Ker}(\xi)$ its kernel. We set

$$H = H_1(X;\mathbb{Z})/\mathrm{Ker}(\xi).$$

It is a free abelian group of finite rank $r=\mathrm{rank}\xi$. Any flat line bundle $L\in\mathcal{V}_\xi$ determines a monodromy homomorphism (6).

Definition 6.1 We say that a bundle $L\in\mathcal{V}_\xi$ is algebraic if the monodromy homomorphism $\mathrm{Mon}_L\colon \mathbb{Z}[H]\to\mathbb{C}$ has nontrivial kernel. We say that L is transcendental[3] if $\mathrm{Mon}_L\colon \mathbb{Z}[H]\to\mathbb{C}$ is injective.

If $t_1,\dots,t_r\in H$ is a basis and if $a_i\in\mathbb{C}$ denotes the monodromy of L along t_i, ie, $a_i=\mathrm{Mon}_L(t_i)$, then L is algebraic iff there exist a nontrivial Laurent polynomial equation with integral coefficients $q(t_1,\dots,t_r)$ such that $q(a_1,\dots,a_r)=0$.

There exist countably many nonzero Laurent polynomials q with integral coefficients and for each such q the set of solutions $q(a_1,\dots,a_r)=0$ is nowhere dense in $\mathcal{V}_\xi$. Since $\mathcal{V}_\xi=(\mathbb{C}^*)^r$ is homeomorphic to a complete metric space we obtain:

Lemma 6.2 *The set of all transcendental bundles $L\in\mathcal{V}_\xi$ is of Baire category 2. In particular, the set of transcendental $L\in\mathcal{V}_\xi$ is dense in the variety $\mathcal{V}_\xi$.*

Lemma 6.3 *The dimension of the vector space $H^q(X;L)$ is constant on the set of transcendental flat line bundles $L\in\mathcal{V}_\xi$. In other words, for any two transcendental $L,L'\in\mathcal{V}_\xi$ one has*

$$\dim H^q(X;L)=\dim H^q(X;L').$$

[3] Perhaps it would be better to call such bundles ξ*–algebraic* and ξ*–transcendental* as both properties depend on the class ξ (they actually depend only on $\mathrm{Ker}\xi$).

Proof The monodromy homomorphism $\mathrm{Mon}_L\colon \mathbb{Z}[H] \to \mathbb{C}$ defines a left $\mathbb{Z}[H]$–module structure $\mathbb{C}_L$ on $\mathbb{C}$ and by definition

$$H^q(X;L) = H^q(\mathrm{Hom}_{\mathbb{Z}[H]}(C_*(\widetilde{X}), \mathbb{C}_L)) \tag{14}$$

where $C_*(\widetilde{X})$ is the cellular chain complex of the covering $\widetilde{X} \to X$ corresponding to $\mathrm{Ker}(\xi)$. If L is transcendental then Mon_L gives a field extension $Q(H) \to \mathbb{C}$ where $Q(H)$ is the field of fractions of $\mathbb{Z}[H]$. We obtain therefore (using finiteness of $C_*(\widetilde{X})$ over $\mathbb{Z}[H]$ and (14)):

$$\begin{aligned} H^q(X;L) &\simeq H^q(\mathrm{Hom}_{\mathbb{Z}[H]}(C_*(\widetilde{X}); Q(H))) \otimes_{Q(H)} \mathbb{C}_L \\ &\simeq H^q(X; Q(H)) \otimes_{Q(H)} \mathbb{C}_L. \end{aligned}$$

This implies that for any transcendental L one has

$$\dim_{\mathbb{C}} H^q(X;L) \;=\; \dim_{Q(H)} H^q(X; Q(H)) \tag{15}$$

and the right hand side[4] is independent of L. □

Proposition 6.4 *Let X be a finite complex and $\xi \in H^1(X;\mathbb{R})$. If the flat line bundle $L \in \mathcal{V}_\xi$ is transcendental then $L \notin \mathrm{Supp}(X,\xi)$.*

Proof If L is transcendental then the monodromy homomorphism $\mathrm{Mon}_{\mathrm{L}}$ can be decomposed into $\mathbb{Z}[H] \to Q(H) \to \mathbb{C}$ (where we use the notations introduced in the proof of the previous lemma) and hence the homomorphism $p_*\colon H_q(\widetilde{X};\mathbb{Z}) \to H_q(X;L)$ can be decomposed into $H_q(\widetilde{X};\mathbb{Z}) \to Q(H) \otimes_{\mathbb{Z}[H]} H_q(\widetilde{X};\mathbb{Z}) \to H_q(X;L)$. This shows that all $\mathbb{Z}[H]$–torsion classes $z \in H_q(\widetilde{X};\mathbb{Z})$ satisfy $p_*(z) = 0$.

Let $T_q \subset H_q(\widetilde{X};\mathbb{Z})$ be the subgroup of all $\mathbb{Z}[H]$–torsion classes. We claim that $\mathbb{C}[H]$–torsion of $H_q(\widetilde{X};\mathbb{C})$ coincides with $\mathbb{C}[H] \otimes_{\mathbb{Z}[H]} T_q$. Together with the remark of the previous paragraph this would imply that for any $\mathbb{C}[H]$–torsion class $z \in H_q(\widetilde{X};\mathbb{C})$ one has $p_*(z) = 0$, ie, $L \notin \mathrm{Supp}(X,\xi)$.

We have the exact sequence

$$0 \to \mathbb{C}[H] \otimes T_q \to \mathbb{C}[H] \otimes H_q(\widetilde{X};\mathbb{Z}) \to \mathbb{C}[H] \otimes M \to 0 \tag{16}$$

where M is $H_q(\widetilde{X};\mathbb{Z})/T_q$ and the tensor product is over $\mathbb{Z}[H]$. The sequence (16) is exact since $\mathbb{C}[H]$ is flat as a $\mathbb{Z}[H]$–module. Using [3, Exercise 20 from Chapter 1, Section 2 on page 46] we find that $\mathbb{C}[H] \otimes_{\mathbb{Z}[H]} M$ has no $\mathbb{C}[H]$–torsion. The sequence (16) implies now that $\mathbb{C}[H] \otimes_{\mathbb{Z}[H]} T_q$ coincides with the $\mathbb{C}[H]$–torsion of $H_q(\widetilde{X};\mathbb{C}) = \mathbb{C}[H] \otimes_{\mathbb{Z}[H]} H_q(\widetilde{X};\mathbb{Z})$. □

[4] The common value $\dim_{\mathbb{C}} H^q(X;L)$ for L transcendental which appears in (15) equals the Novikov–Betti number $b_q(\xi)$, see [16] and [7, Proposition 1.30].

The authors wish to thank Holger Brenner for valuable comments leading to Proposition 6.4.

Proposition 6.5 *Assume that $L \in \mathcal{V}_\xi$ is transcendental. Let $v \in H^q(X;L)$ be a nonzero cohomology class. Then there exists a homology class $z \in H_q(\widetilde{X};\mathbb{Z})$ such that $\langle v, p_*(z)\rangle \neq 0$. Here*

$$p_*\colon H_q(\widetilde{X};\mathbb{Z}) \to H_q(X;L^*) \tag{17}$$

denotes homomorphism similar to (7). Moreover, the kernel of (17) coincides with the set of $\mathbb{Z}[H]$–torsion classes in $H_q(\widetilde{X};\mathbb{Z})$.

Proof Obviously, if the cohomology class v is nonzero then there exists a homology class $z' \in H_q(X;L^*)$ with $\langle v, z'\rangle \neq 0$. Here $L^* \in \mathcal{V}_\xi$ denotes the dual bundle to L.

If L is transcendental then the dual bundle L^* is transcendental as well, ie, $\mathrm{Mon}_{L^*}\colon \mathbb{Z}[H] \to \mathbb{C}$ is injective. Let $\mathbb{C}_{L^*}$ denote the field of complex numbers $\mathbb{C}$ viewed as a right $\mathbb{Z}[H]$–module via the ring homomorphism Mon_{L^*}. We want to show that

$$H_q(X;L^*) \simeq \mathbb{C}_{L^*} \otimes_{\mathbb{Z}[H]} H_q(\widetilde{X};\mathbb{Z}). \tag{18}$$

Assuming (18) we would be able to argue that the class z' can be represented as a finite sum $z' = \sum c_i\, p_*(z_i)$ where $c_i \in \mathbb{C}$ and $z_i \in H_q(\widetilde{X};\mathbb{Z})$. Since $\langle v, z'\rangle \neq 0$ the number $\langle v, p_*(z_i)\rangle$ is nonzero for some i which implies our statement.

Observe that $\mathbb{Z}[H]$ is a Laurent polynomial ring. Denote by $Q(H)$ its field of fractions. The ring homomorphism $\mathrm{Mon}_{L^*}\colon \mathbb{Z}[H] \to \mathbb{C}_{L^*}$ extends to a field embedding $Q(H) \to \mathbb{C}_{L^*}$. Let $C_*(\widetilde{X})$ be the cellular chain complex of $\widetilde{X}$ with integral coefficients. The homomorphism (17) is induced by the chain map

$$C_*(\widetilde{X}) \to \mathbb{C}_{L^*} \otimes_{\mathbb{Z}[H]} C_*(\widetilde{X}).$$

The latter can be decomposed as

$$\begin{aligned} C_*(\widetilde{X}) \to Q(H) \otimes_{\mathbb{Z}[H]} C_*(\widetilde{X}) &\to \mathbb{C}_{L^*} \otimes_{\mathbb{Z}[H]} C_*(\widetilde{X}) \\ &= \mathbb{C}_{L^*} \otimes_{Q(H)} (Q(H) \otimes_{\mathbb{Z}[H]} C_*(\widetilde{X})). \end{aligned} \tag{19}$$

The left map in (19) is a localization and hence it induces localization on homology

$$H_q(\widetilde{X};\mathbb{Z}) \to Q(H) \otimes_{\mathbb{Z}[H]} H_q(\widetilde{X};\mathbb{Z}).$$

The right map in (19) is induced by a field extension $Q(H) \to \mathbb{C}_{L^*}$. Note that $\mathbb{C}_{L^*}$ viewed as a $\mathbb{Z}[H]$–module is a direct sum of infinite number of copies of $Q(H)$. Hence the right homomorphism in (19) induces

$$Q(H) \otimes_{\mathbb{Z}[H]} H_q(\widetilde{X};\mathbb{Z}) \to \mathbb{C}_{L^*} \otimes_{\mathbb{Z}[H]} H_q(\widetilde{X};\mathbb{Z}).$$

Therefore (17) coincides with $H_q(\widetilde{X};\mathbb{Z}) \to \mathbb{C}_{L^*} \otimes_{\mathbb{Z}[H]} H_q(\widetilde{X};\mathbb{Z})$ proving (18). □

Next we state an improved version of Theorem 5 taking into account Proposition 6.5.

Theorem 6 *Suppose that $L_0, L_1, \dots, L_k \in \mathcal{V}_\xi$ are complex flat vector bundles with the following properties:*

(1) *L_0 does not belong to the support* $\mathrm{Supp}(X,\xi)$.

(2) *For some cohomology classes $v_i \in H^{d_i}(X; L_i)$, $i = 0, 1, \dots, k$, where $d_i > 0$ for $i = 1, \dots, k$ the cup-product*

$$v_0 \cup v_1 \cup \cdots \cup v_k \neq 0 \in H^d(X; L) \tag{20}$$

is nontrivial; here $d = \sum_{i=0}^k d_i$ and $L = L_0 \otimes L_1 \otimes \cdots \otimes L_k$.

(3) *L is transcendental, ie, the monodromy homomorphism* $\mathrm{Mon}_L\colon \mathbb{Z}[H] \to \mathbb{C}$ *is injective.*

Then

$$\mathrm{cat}(X,\xi) > k. \tag{21}$$

Proof This follows by combining Theorem 5 with Proposition 6.5. □

As a useful special case of Theorem 6 and Proposition 6.4 we mention the following statement:

Theorem 7 *Let X be a finite cell complex and $\xi \in H^1(X;\mathbb{R})$. Let $L \in \mathcal{V}_\xi$ be transcendental. Assume that there exist cohomology classes $v_0 \in H^{d_0}(X; L)$ and $v_i \in H^{d_i}(X;\mathbb{C})$ where $i = 1, \dots, k$ such that $d_i > 0$ for $i \in \{1, \dots, k\}$ and the cup-product*

$$v_0 \cup v_1 \cup \cdots \cup v_k \neq 0 \in H^*(X; L) \tag{22}$$

is nontrivial. Then $\mathrm{cat}(X,\xi) > k$.

Theorem 7, combining simplicity with remarkable efficiency, has a very satisfying statement. We view this theorem as being the main result of the paper. In the following sections we test this theorem in many specific examples. Besides, we compare theorems of this section with cohomological lower bounds for $\mathrm{cat}(X,\xi)$ obtained earlier in [7, Theorem 6.1] and in [6, Theorem 10.23].

7 The notion of cup-length $\mathrm{cl}(X,\xi)$

In view of Theorem 7 we introduce the following notation.

Let X be a finite cell complex and $\xi \in H^1(X;\mathbb{R})$. We denote by $\mathrm{cl}(X,\xi)$ the maximal integer $k \geq 0$ such that Theorem 7 could be applied to (X,ξ); if Theorem 7 is not applicable (ie, if $H^*(X;L)=0$ for any transcendental $L \in \mathcal{V}_\xi$, compare Lemma 6.3) we set $\mathrm{cl}(X,\xi) = -1$. Hence,

$$\mathrm{cl}(X,\xi) \in \{-1,0,1,\dots\}. \tag{23}$$

In other words, $\mathrm{cl}(X,\xi) \geq k$ where $k \geq 0$ iff there exists a transcendental flat line bundle $L \in \mathcal{V}_\xi$ and there exist cohomology classes $v_0 \in H^{d_0}(X;L)$ and $v_i \in H^{d_i}(X;\mathbb{C})$ where $i = 1,\dots,k$ and $d_i > 0$ for $i \in \{1,\dots,k\}$ such that the cup-product

$$v_0 \cup v_1 \cup \cdots \cup v_k \neq 0 \in H^*(X;L) \tag{24}$$

is nontrivial.

Note that for $\xi = 0$ the number $\mathrm{cl}(X,\xi)$ coincides with the usual cup-length $\mathrm{cl}(X)$; recall that the later is defined as the largest integer r such that there exist cohomology classes $u_i \in H^{d_i}(X;\mathbb{C})$ where $i = 1,\dots,r$ of positive degree such that their cup-product $u_1 \cup \cdots \cup u_k \neq 0 \in H^*(X;L)$ is nontrivial. Indeed, in the case $\xi = 0$ the trivial bundle $L = \mathbb{C}$ is not in $\mathrm{Supp}(X,\xi)$ and therefore one may take $L = \mathbb{C}$ and $v_0 = 1 \in H^0(X;\mathbb{C})$.

One can restate Theorem 7 as follows:

Theorem 8 *One has* $\mathrm{cat}(X,\xi) \geq \mathrm{cl}(X,\xi) + 1$.

The next useful Lemma suggests several different ways to characterize the number $\mathrm{cl}(X,\xi)$. This Lemma plays an important role in the sequel.

Lemma 7.1 *Let X be a finite cell complex and $\xi \in H^1(X;\mathbb{R})$. The following statements regarding an integer $k \geq 0$ are equivalent:*

(A) $\mathrm{cl}(X,\xi) \geq k$

(B) *There exists cohomology class $v_0 \in H^{d_0}(X;L)$ where $L \in \mathcal{V}_\xi$ is a transcendental flat line bundle and there exist k integral cohomology classes $v_i \in H^{d_i}(X;\mathbb{Z})$ where $i = 1,\dots,k$ and $d_i > 0$ for $i \in \{1,\dots,k\}$, such that the cup-product (24) is nontrivial.*

(C) *Let H denote $H_1(X;\mathbb{Z})/\mathrm{Ker}\xi$; it is a free abelian group of finite rank. Let $Q(H)$ denote the field of fractions of the group ring $\mathbb{Z}[H]$. Then there exist cohomology classes $w_0 \in H^{d_0}(X; Q(H))$ (the latter denotes homology with twisted coefficients) and $v_i \in H^{d_i}(X;\mathbb{Z})$ where $d_i > 0$ for $i = 1,\dots,k$ such that the cup-product*

$$w_0 \cup v_1 \cup \cdots \cup v_k \neq 0 \in H^*(X; Q(H))$$

is nontrivial.

(D) *For any transcendental flat line bundle $L \in \mathcal{V}_\xi$ there exists cohomology class $v_0 \in H^{d_0}(X; L)$ such that the cup-product (24) is nontrivial for some integral cohomology classes $v_i \in H^{d_i}(X;\mathbb{Z})$ with $d_i > 0$ for $i \in \{1,\dots,k\}$.*

(E) *For any transcendental flat line bundle $L \in \mathcal{V}_\xi$ there exist cohomology classes $v_0 \in H^{d_0}(X; L)$ and $v_i \in H^{d_i}(X;\mathbb{C})$ where $i = 1,\dots,k$ and $d_i > 0$ for $i \in \{1,\dots,k\}$ such that the cup-product (24) is nontrivial.*

Proof Let us show that (A) $\Longrightarrow$ (B). Fix a transcendental bundle $L \in \mathcal{V}_\xi$, and $v_0 \in H^*(X; L)$ such that (24) is nontrivial for some $v_i \in H^{d_i}(X;\mathbb{C})$ with $d_i > 0$. Consider now the cup-products

$$v_0 \cup v_1' \cup \cdots \cup v_k' \tag{25}$$

with arbitrary integral cohomology classes $v_i' \in H^{d_i}(X;\mathbb{Z})$; here the degrees d_i are assumed to be fixed. (25) is a multi-linear function of the classes v_i'. Since the integral classes generate $H^{d_i}(X;\mathbb{C})$ over $\mathbb{C}$ we obtain that (25) must be nonzero for some choice of classes v_i', ie, (B) holds.

Now we show that (B) $\Longrightarrow$ (C). Fix $L \in \mathcal{V}_\xi$ and the classes $v_0 \in H^{d_0}(X; L)$ and $v_i \in H^{d_i}(X;\mathbb{Z})$ satisfying conditions described in (B). The monodromy homomorphism $\mathrm{Mon}_L\colon \mathbb{Z}[H] \to \mathbb{C}$ is an injective ring homomorphism, it extends to the field of fractions $Q(H) \to \mathbb{C}$. The image of the induced homomorphism on cohomology

$$\psi\colon H^{d_0}(X; Q(H)) \to H^{d_0}(X; L) \tag{26}$$

generates $H^{d_0}(X; L)$ over $\mathbb{C}$ and (26) is injective (for reasons mentioned in the proof of Lemma 6.3). Fix cohomology classes $v_i \in H^*(X;\mathbb{Z})$ where $i = 1,\dots,k$. For a cohomology class $w_0 \in H^{d_0}(X; Q(H))$ the function

$$\psi(w_0) \cup v_1 \cup \cdots \cup v_k = \psi(w_0 \cup v_1 \cup \cdots \cup v_k) \in H^*(X; L) \tag{27}$$

extends to a $\mathbb{C}$–linear function of $v_0 \in H^{d_0}(X; L)$

$$v_0 \mapsto v_0 \cup v_1 \cup \cdots \cup v_k \in H^*(X; L).$$

If this function is nonzero then it may not vanish on the image of ψ, ie, (C) holds.

The implication (C) $\Longrightarrow$ (D) follows from injectivity of homomorphism (26) and from (27).

Implications (D) $\Longrightarrow$ (E) and (E) $\Longrightarrow$ (A) are obvious. This completes the proof. □

Remark 7.2 Finding a specific transcendental line bundle may represent a problem in applications. In this respect we emphasize that condition (C) of Lemma 7.1 does not involve such a choice.

Lemma 7.3 *Assume that X and Y are path connected finite cell complexes and $\xi \in H^1(X\times Y;\mathbb{R})$. Then*

$$\mathrm{cl}(X\times Y,\xi) \geq \mathrm{cl}(X,\xi|_X) + \mathrm{cl}(Y,\xi|_Y). \tag{28}$$

Proof Denote $\mathrm{cl}(X,\xi|_X) = k$ and $\mathrm{cl}(Y,\xi|_Y) = r$. Any flat line bundle L over $X\times Y$ has the form $L_1 \boxtimes L_2$ (exterior tensor product) where L_1 and L_2 are flat line bundles over X and Y respectively. Note that if L lies in the variety $\mathcal{V}_\xi = \mathrm{Hom}(H_1(X\times Y;\mathbb{Z})/\mathrm{Ker}(\xi),\mathbb{C}^*)$ then L_1 and L_2 are obtained by restrictions and hence $L_1 \in \mathcal{V}_{\xi|_X}$ and $L_2 \in \mathcal{V}_{\xi|_Y}$. We will use equivalence between (A), (E) and (F) of Lemma 7.1. Fix a transcendental bundle $L = L_1\boxtimes L_2 \in \mathcal{V}_\xi$ over $X\times Y$. Then both L_1 and L_2 are transcendental. Find classes $v_0 \in H^*(X;L_1)$, $v_1,\dots,v_k \in H^*(X;\mathbb{C})$, $u_0 \in H^*(Y;L_2)$, $u_1,\dots,u_r \in H^*(Y;\mathbb{C})$ such that $v_0\cup v_1\cup\cdots\cup v_k \neq 0$ and $u_0\cup u_1\cup\cdots\cup u_r \neq 0$. Now we have cohomology classes $v_0\times u_0 \in H^*(X\times Y;L)$ and $v_i\times 1, 1\times u_j \in H^*(X\times Y;\mathbb{C})$ and the product

$$(v_0\times u_0)\cup\prod_{i=1}^{k}(v_i\times 1)\cup\prod_{j=1}^{r}(1\times u_j) \neq\ 0 \in H^*(X\times Y;L)$$

is nonzero. Here we use the Künneth formula which states

$$H^*(X\times Y;L_1\boxtimes L_2) \simeq H^*(X;L_1)\otimes H^*(Y;L_2). \tag{29}$$

This proves (28). □

8 First examples

In this section we test Theorem 7 in a number of simple examples.

8.1 First consider the case $\xi = 0$. We know that for $\xi = 0$ the number $\mathrm{cat}(X,\xi)$ coincides with the classical LS category $\mathrm{cat}(X)$; see [7, Example 10.8]. Let us examine

what Theorem 7 gives in this case. The variety $\mathcal{V}_\xi$ has only one point – the trivial flat line bundle $\mathbb{C}$ over X. The support $\mathrm{Supp}(X,\xi)=\varnothing$ is always empty for $\xi=0$. We may therefore take $v_0=1\in H^0(X;\mathbb{C})$ applying Theorem 7. Thus, we see that Theorem 7 claims in the special case $\xi=0$ that if there exist cohomology classes $v_1,\dots,v_k\in H^{>0}(X;\mathbb{C})$ with $v_1\cup\dots\cup v_k\neq 0$ then $\mathrm{cat}(X)>k$. This claim is the classical cup-length estimate for the LS category $\mathrm{cat}(X)$.

8.2 Note that if $\xi\neq 0$ and X is connected then $H^0(X;L)=0$ for any nontrivial $L\in\mathcal{V}_\xi$. Note also that the trivial flat line bundle $\mathbb{C}\in\mathcal{V}_r$ always lies in the support $\mathrm{Supp}(X,\xi)$ for $\xi\neq 0$, see Remark 4.4, and it is never transcendental. Therefore the degree of the class v_0 (which appears in Theorem 5) in the case $\xi\neq 0$ must be positive. Hence for $\xi\neq 0$ the number k in Theorem 6 satisfies $k\leq \dim X-1$. This explains why Theorem 7 cannot give $\mathrm{cat}(X,\xi)\geq \dim X+1$ for $\xi\neq 0$.

Inequality (10.8) in [7] yields

$$\mathrm{cat}(X,\xi)\leq \mathrm{cat}(X)-1\leq \dim X \tag{30}$$

assuming that X is connected and $\xi\neq 0$. This is consistent with the remark of the previous paragraph.

8.3 The following example shows that (30) can be satisfied as an equality ie, that $\mathrm{cat}(X,\xi)=\dim X$ is possible[5]. Consider the bouquet $X=Y\vee S^1$ where Y is a finite polyhedron, and assume that the class $\xi\in H^1(X;\mathbb{R})$ satisfies $\xi|_Y=0$ and $\xi|_{S^1}\neq 0$. We know that in this case

$$\mathrm{cat}(X,\xi)=\mathrm{cat}(Y)-1, \tag{31}$$

see [7, Example 10.11].

We are going to apply Theorem 7. The variety $\mathcal{V}_\xi$ in this case coincides with the set $\mathbb{C}^*=\mathbb{C}-\{0\}$. The support $\mathrm{Supp}(X,\xi)$ contains in this case only the trivial line bundle. $L\in\mathcal{V}_\xi$ is transcendental if the monodromy along the circle S^1 is a transcendental complex number. For any $L\in\mathcal{V}_\xi$ the restriction $L|_Y$ is trivial and the restriction homomorphism $H^i(X;L)\to H^i(Y;\mathbb{C})$ is onto.

Suppose that the cohomological cup-length of Y with $\mathbb{C}$–coefficients equals ℓ, ie, there exist cohomology classes of positive degree $u_0,u_1,\dots,u_{\ell-1}\in H^{>0}(Y;\mathbb{C})$ such that the product $u_0\cup\dots\cup u_{\ell-1}\neq 0$ is nonzero. By the above remark, for a nontrivial $L\in\mathcal{V}_\xi$ we obtain cohomology classes $v_0\in H^*(X;L)$ and $v_1,\dots,v_{\ell-1}\in H^{>0}(X;\mathbb{C})$ such that $v_i|_Y=u_i$. Hence $v_0\cup v_1\cup\dots\cup v_{\ell-1}\neq 0\in H^*(X;L)$. By Theorem 7 we

[5] Note that if X is a closed smooth manifold then $\mathrm{cat}(X,\xi)<\dim X$, see Section 12.

obtain $\mathrm{cat}(X,\xi) > \ell - 1$ which is equivalent (taking into account (31)) to $\mathrm{cat}(Y) > \ell$. The last inequality is the classical cup-length estimate for the usual category.

8.4 Comparing the previous example with [6, Example 6.6] we find that Theorem 5 and Theorem 7 are stronger than the cohomological lower bounds given in [6], even in the case of rank one classes ξ. The reason for this is that in Theorem 5 we have only one generic bundle (compared with two in [6, Theorem 6.1]) but non-vanishing of the product $v_0 \cup \cdots \cup v_k$ is understood in a stronger sense.

8.5 Let us now consider a very specific example: $X = T^2 \vee S^1$. In this case $H^1(X;\mathbb{R}) = \mathbb{R}^3$ and we describe $\mathrm{cat}(X,\xi)$ as function of $\xi \in H^1(X;\mathbb{R}) = \mathbb{R}^3$. We denote by $\ell \subset \mathbb{R}^3$ the set of all classes ξ such that $\xi|_{T^2} = 0$. Clearly ℓ is a line through the origin in $\mathbb{R}^3$. We claim that:

$$\mathrm{cat}(X,\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^3 - \ell, \\ 2, & \text{if } \xi \in \ell - \{0\}, \\ 3, & \text{if } \xi = 0. \end{cases} \tag{32}$$

Indeed, consider first the case $\xi \notin \ell$, ie, $\xi|_{T^2} \neq 0$. Let us show that $\mathrm{cat}(X,\xi) \leq 1$. Denote $p = T^2 \cap S^1$ and let $q \in S^1$ be a point distinct from p. Set $F = X - \{q\}$ and $F_1 = S^1 - \{p\}$. Then $F \cup F_1 = X$ is an open cover of X with $F_1 \to X$ null-homotopic and with F being N–movable in X for any N (assuming that $\xi|_{T^2} \neq 0$; this follows from homotopy invariance of $\mathrm{cat}(X,\xi)$ and from Examples 5.2 and 5.4.

Since $\mathrm{cat}(X,\xi) = 0$ would imply $\chi(X) = 0$, by Theorem 10 stated below we obtain that $\mathrm{cat}(X,\xi) > 0$ for any ξ (as $\chi(X) = -1 \neq 0$). This proves the first line of (32).

If $\xi \in \ell - \{0\}$ we apply the result of [6, Example 3.5] which gives $\mathrm{cat}(X,\xi) = \mathrm{cat}(T^2) - 1 = 2$.

For $\xi = 0$ we easily find $\mathrm{cat}(X,\xi) = \mathrm{cat}(X) = 3$.

9 A controlled version of $\mathrm{cat}(X,\xi)$

In this section we introduce a new controlled version of $\mathrm{cat}(X,\xi)$ which has some advantages, for example it behaves better under products.

Let ω be a continuous closed 1–form on a finite cell complex X. Let $\xi = [\omega] \in H^1(X;\mathbb{R})$ be the cohomology class represented by ω. We refer to [6] for the formalism of closed 1–forms on topological spaces.

Definition 9.1 Let N and C be two real numbers. A subset $A \subset X$ is N–movable with respect to ω with control C if there exists a continuous homotopy $h_t\colon A \to X$, $t \in [0,1]$, such that

(1) $h_0\colon A \to X$ is the inclusion;

(2) for any point $x \in A$ one has

$$\int_x^{h_1(x)} \omega < -N \tag{33}$$

where the integral is calculated along the path $t \mapsto h_t(x) \in X$, $t \in [0,1]$ and

(3) for any point $x \in A$ and for any $t \in [0,1]$ one has

$$\int_x^{h_t(x)} \omega \leq C. \tag{34}$$

Geometrically, an N–movable subset with control C can be continuously deformed inside X such that each of its points is winding around yielding a small quantity (33) measured in terms of the form ω and such that for all times t the integral (34) remains controlled, ie, it is smaller than a fixed quantity C.

Example 9.2 Let X be a closed smooth manifold and let ω be a smooth closed 1–form on X having no zeroes. Find a vector field v on X such that $\omega(v) < 0$. Then the whole space X is N–movable with control $C = 0$ for any $N > 0$. The homotopy $h_t\colon X \to X$ as in Definition 9.1 can be easily constructed using the flow generated by v.

Example 9.3 Assume that X is a path connected polyhedron and the cohomology class of ω is nonzero $\xi = [\omega] \neq 0 \in H^1(X;\mathbb{R})$. For any subset $F \subset X$ such that the inclusion $F \to X$ is homotopic to a constant map $F \to X$ there exists a constant $C > 0$ so that for any $N > 0$ the set F is N–movable with respect to ω with control C. Indeed, find a closed loop γ in X such that $\langle \xi, [\gamma] \rangle = \int_\gamma \omega < 0$. Define the homotopy $h_t\colon F \to X$ as concatenation of contraction of F to a point in X, then moving this point towards the loop γ and finally traversing γ many times. The integral (33) becomes smaller when one increases the number of turns around γ. However the estimate (34) will hold independently of the number of turns, ie, independently of N.

Definition 9.4 Fix a closed 1–form ω representing ξ. The number

$$\operatorname{ccat}(X, \xi) \tag{35}$$

is the minimal integer k with the property that there exists $C>0$ such that for any $N>0$ there exists an open cover $F, F_1, \dots, F_k \subset X$ with the property that each inclusion $F_j \to X$ is null-homotopic and such that F is N–movable with control C with respect to ω.

Example 9.5 Assume that $X = Y \times S^1$ and $\xi \in H^1(X;\mathbb{R})$ is such that $\xi|_{S^1} \neq 0$. Then $\mathrm{ccat}(X,\xi) = 0$, compare Example 5.2.

Lemma 9.6 *If $\xi \neq 0$ and X is connected then* $\mathrm{ccat}(X,\xi) \leq \mathrm{cat}(X) - 1$.

This follows from the remark of Example 9.3

Lemma 9.7 *If $\xi = 0$ then* $\mathrm{ccat}(X,\xi) = \mathrm{cat}(X,\xi) = \mathrm{cat}(X)$.

The proof of Lemma 9.7 is identical to the argument of [7, Example 10.8].

Next we state the homotopy invariance property of $\mathrm{ccat}(X,\xi)$.

Lemma 9.8 *Let $\phi\colon X_1 \to X_2$ be a homotopy equivalence, $\xi_2 \in H^1(X_2;\mathbb{R})$, and $\xi_1 = \phi^*(\xi_2) \in H^1(X_1;\mathbb{R})$. Then*

$$\mathrm{ccat}(X_1,\xi_1) = \mathrm{ccat}(X_2,\xi_2). \tag{36}$$

The proof repeats the arguments of [7, Lemma 10.12].

It is obvious that in general

$$\mathrm{ccat}(X,\xi) \geq \mathrm{cat}(X,\xi). \tag{37}$$

In all examples known to us we have that (37) is an equality.

Remark 9.9 Note that the potentially larger quantity $\mathrm{ccat}(X,\xi)$ can replace $\mathrm{cat}(X,\xi)$ in the applications to dynamics described in [6; 7; 14; 17]. This gives an additional incentive to be interested in the new invariant $\mathrm{ccat}(X,\xi)$. All homotopies $h_t\colon F \to X$ which appear in applications to dynamics are induced by gradient flows of closed 1–forms and hence they satisfy conditions of Definition 9.1 with the control constant $C = 0$.

Example 9.10 Examining Example 8.5 we easily find that for $X = T^2 \vee S^1$ one has:

$$\mathrm{cat}(X,\xi) \;=\; \mathrm{ccat}(X,\xi) \;=\; \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^3 - \ell, \\ 2, & \text{if } \xi \in \ell - \{0\}, \\ 3, & \text{if } \xi = 0 \end{cases} \tag{38}$$

(as all homotopies described in Example 8.5 are in fact with control). Here $\ell \subset \mathbb{R}^3 = H^1(X;\mathbb{R})$ denotes the set of all classes $\xi \in H^1(X;\mathbb{R})$ such that $\xi|_{T^2} = 0$.

10 Product inequality

The classical product inequality for the Lusternik–Schnirelmann category states that

$$\operatorname{cat}(X \times Y) \leq \operatorname{cat}(X) + \operatorname{cat}(Y) - 1,$$

see [4, Theorem 1.37]. In this section we show that a similar inequality holds for $\operatorname{ccat}(X, \xi)$.

Theorem 9 *Let X and Y be finite cell complexes and let $\xi_X \in H^1(X; \mathbb{R})$ and $\xi_Y \in H^1(Y; \mathbb{R})$ be real cohomology classes. Assume that*

$$\operatorname{ccat}(X, \xi_X) > 0 \quad \text{or} \quad \operatorname{ccat}(Y, \xi_Y) > 0. \tag{39}$$

Then

$$\operatorname{ccat}(X \times Y, \xi) \leq \operatorname{ccat}(X, \xi_X) + \operatorname{ccat}(Y, \xi_Y) - 1, \tag{40}$$

where

$$\xi = \xi_X \times 1 \,+\, 1 \times \xi_Y. \tag{41}$$

The C–control assumption which appears in Definitions 9.1 and 9.4 is used in an essential way in the proof. Theorem 9 is the main motivation for introducing $\operatorname{ccat}(X, \xi)$ as an alternative to $\operatorname{cat}(X, \xi)$.

We will start with some auxiliary statements.

Lemma 10.1 *Let ω be a continuous closed 1–form on a finite cell complex X. Let $F \subset X$ be an open subset which is N–movable with respect to ω with control C (see Definition 9.1). Given a closed subset $A \subset F$ there exists an open set F' such that $A \subset F' \subset F$ and a homotopy $H_t\colon X \to X$, $t \in [0, 1]$ satisfying the following properties:*

(1) *$H_0(x) = x$ for all $x \in X$;*

(2) *for any point $x \in F'$ one has*

$$\int_x^{H_1(x)} \omega \,<\, -N, \tag{42}$$

and

(3) *for any $x \in X$ and for any $t \in [0, 1]$ the following holds*

$$\int_x^{H_t(x)} \omega \,<\, C + 1. \tag{43}$$

The statement of the Lemma can be rephrased as follows: the homotopy which appears in Definition 5.1 can be extended to the whole space X such that the control condition (ie, inequality (34)) will hold for all $x \in X$ and the main inequality (33) will hold on a slightly smaller set.

Proof of Lemma 10.1 Let $h_t\colon F \to X$, $t \in [0,1]$ be a homotopy given by Definition 9.1. Since X is an ANR and $A \subset X$ is a closed subset, the homotopy $h_t|_A\colon A \to X$ can be extended to a homotopy $h'_t\colon X \to X$, $t \in [0,1]$, see [18, Chapter 1, Exercise D2]. We have $h'_0(x) = x$ for all $x \in X$ and $h'_t(x) = h_t(x)$ for all $x \in A$ and $t \in [0,1]$. Using continuity, we may find a neighborhood $F'' \subset F$ of A such that (42) holds for all $x \in F''$ and such that (43) holds for all $x \in F''$ and for any $t \in [0,1]$.

Now we want to change the homotopy h'_t such that (43) holds for all. Find an open set F' with $A \subset F' \subset \overline{F'} \subset F''$. Find a continuous function $\phi\colon X \to [0,1]$ such that $\phi(x) = 1$ for $x \in \overline{F'}$ and $\phi(x) = 0$ for $x \in (X - F'')$. Define $h''_t\colon X \to X$ by $h''_t(x) = h'_{\phi(x)t}(x)$, where $x \in X$ and $t \in [0,1]$. Now it is quite obvious that the obtained homotopy h''_t satisfies all required conditions. □

Proof of Theorem 9 Let ω_X (correspondingly, ω_Y) be a continuous closed 1–form on X (correspondingly, Y) representing ξ_X (correspondingly ξ_Y). Then $\omega = \omega_X + \omega_Y$ represents ξ. More precisely, if $U \subset X$ and $V \subset Y$ are simply connected open subsets and if $\xi_X|_U = df_U$, $\xi_Y|_V = dg_V$ then $\omega|_{U\times V} = df_U + f\,d_V$, see [7, Section 10.2].

Denote $\mathrm{ccat}(X,\xi_X) = r$ and $\mathrm{ccat}(Y,\xi_Y) = s$. There exists $C > 0$ such that for any $N > 0$ there exist an open cover $F_X \cup F_1^X \cup \cdots \cup F_r^X = X$ with the set $F_X \subset X$ being $(N + C + 1)$–movable with respect to ω_X with control C and with each inclusion $F_j^X \to X$ null-homotopic. Let $F_Y \cup F_1^Y \cup \cdots \cup F_s^Y = Y$ be a cover of Y having similar properties.

Denote $A_X = X - (F_1^X \cup \cdots \cup F_r^X)$. It is a closed subset of F_X. We denote by $H_t^X\colon X \to X$ the homotopy given by Lemma 10.1 applied to $A_X \subset F_X$.

Similarly, consider the set $A_Y = Y - (F_1^Y \cup \cdots \cup F_s^Y)$ and denote by $H_t^X\colon X \to X$ the homotopy given by Lemma 10.1 applied to $A_Y \subset F_Y$.

We obtain that the homotopy $H_t\colon X \times Y \to X \times Y$ given by

$$H_t(x,y) = (H_t^X(x), H_t^Y(y)), \quad x \in X, y \in Y$$

restricted to a neighborhood $F \subset X \times Y$ of the set $A_X \times Y \cup X \times A_Y$ satisfies conditions of Definition 9.1.

Now the set F together with $F_i^X \times F_j^Y$ where $i = 1, \dots r$ and $j = 1, \dots, s$ cover $X \times Y$. It is well known that the union of the sets $F_i^X \times F_j^Y$ can be covered by $r+s-1$ open sets[6] each null homotopic in $X \times Y$, see [4; 12]. Hence we obtain (40). □

11 Spaces of category zero

In this section we collect some simple observations about spaces with $\operatorname{cat}(X, \xi) = 0$ for some $\xi \in H^1(X; \mathbb{R})$. More information can be found in [17, Section 3].

Lemma 11.1 *Let X be a finite CW complex and $\xi \in H^1(X; \mathbb{R})$. The following properties are equivalent:*

(1) $\operatorname{cat}(X, \xi) = 0$

(2) $\operatorname{ccat}(X, \xi) = 0$

(3) *There exists a continuous closed 1–form ω on X representing ξ (in the sense of [7, Section 10.2]) and a homotopy $h_t\colon X \to X$, where $t \in [0,1]$, such that for any point $x \in X$ one has*

$$\int_x^{h_1(x)} \omega < 0. \tag{44}$$

In (44) the integral is calculated along the curve $t \mapsto h_t(x)$, $t \in [0,1]$.

(4) *For any continuous closed 1–form ω on X representing ξ there exists a homotopy $h_t\colon X \to X$, where $t \in [0,1]$, such that for any point $x \in X$ inequality (44) holds.*

Proof By Definition 5.3, $\operatorname{cat}(X, \xi) = 0$ means that the whole space X is N–movable for any $N > 0$, ie, given $N > 0$, there exists a homotopy $H_t\colon X \to X$, where $t \in [0,1]$, such that $H_0(x) = x$ and

$$\int_x^{H_1(x)} \omega < -N \tag{45}$$

for any $x \in X$. Hence, (1) implies (3).

Conversely, given property (3), using compactness of X we find $\epsilon > 0$ such that (44) can be replaced by $\int_x^{h_1(x)} \omega < -\epsilon$. Now, one may iterate this deformation as follows. The k-th iteration is a homotopy $H_t^k\colon X \to X$, where $t \in [0,1]$, defined as follows.

[6] Here we use our assumption (39).

Denote by $h_1^{(i)}\colon X\to X$ the i–fold composition $h_1^{(i)}=h_1\circ h_1\circ\cdots\circ h_1$ (i times). Then for $t\in[i/k,(i+1)/k]$ one has

$$H_t^k(x)=h_{kt-i}(h_1^{(i)}(x)). \tag{46}$$

If it is known that $\int_x^{h_1(x)}\omega<-\epsilon$ for any $x\in X$ then for the k-th iteration one has $\int_x^{H_1^k(x)}\omega<-k\epsilon$ and (3) follows assuming that $k>N/\epsilon$. This shows equivalence between (1) and (3).

(4) $\Longrightarrow$ (3) is obvious. Now suppose that (3) holds for ω and let ω_1 be another continuous closed one-form lying in the same cohomology class, ie, $\omega_1=\omega+df$ where $f\colon X\to\mathbb{R}$ is continuous, see [7, Section 10.2]. Using compactness of X we may find C such that for any path $\gamma\colon[0,1]\to X$ one has $|\int_\gamma df|<C$. Fix $N>C$ and apply equivalence between (1) and (3) to find a homotopy $h_t\colon X\to X$ with $\int_x^{h_1(x)}\omega<-N$. Then one has $\int_x^{h_1(x)}\omega_1<0$, ie, (4) holds.

It is obvious that (2) $\Longrightarrow$ (1). Hence we are left to show that (1) implies (2). Given (2) fix a deformation as described in (3). Let $C>0$ be such that for any $x\in X$ and for any $t\in[0,1]$ one has $\int_1^{h_t(x)}\omega<C$. Then for any iteration $H_t^k\colon X\to X$ (see above) one has $\int_x^{H_t^k(x)}\omega<C$ and the result follows. □

In the case when X is a closed smooth manifold a deformation as appearing in (2) can be constructed as the flow generated by a vector field v on X satisfying $\omega(v)<0$.

The remark of the previous paragraph explains why the following statement can be viewed as an analogue of the classical Euler–Poincaré theorem:

Theorem 10 $\mathrm{cat}(X,\xi)=0$ *implies* $\chi(X)=0$.

Proof Suppose that $\mathrm{cat}(X,\xi)=0$. Then $\xi\neq0$, ie, the rank r of class ξ is positive. By Lemma 6.2 there exists transcendental bundle $L\in\mathcal{V}_\xi=(\mathbb{C}^*)^r$. If $H^q(X;L)\neq0$ for some q then one may apply Theorem 7 with $k=0$ obtaining $\mathrm{cat}(X,\xi)>0$ and contradicting our hypothesis. Hence $H^q(X;L)=0$ for all q which implies

$$\chi(X)=\sum_q(-1)^q\dim H^q(X;L)=0. \qquad \square$$

Pairs (X,ξ) with $\mathrm{cat}(X,\xi)=0$ form an *"ideal"* in the following sense:

Lemma 11.2 *Let* X_1 *and* X_2 *be finite cell complexes and* $\xi_1\in H^1(X_1;\mathbb{R})$, $\xi_2\in H^1(X_2;\mathbb{R})$. *If* $\mathrm{cat}(X_1,\xi_1)=0$ *then* $\mathrm{cat}(X_1\times X_2,\xi)=0$ *where* $\xi=\xi_1\times1+1\times\xi_2\in H^1(X_1\times X_2;\mathbb{R})$.

Proof The statement follows directly by applying the definitions. □

12 An upper bound for ccat(M,ξ)

If X is a connected cell complex and $\xi \in H^1(X;\mathbb{R})$ is nonzero then $\operatorname{cat}(X,\xi) \leq \dim X$ and we have seen examples when this inequality is sharp, ie, is an equality. However, it can be improved assuming that X is a manifold.

Theorem 11 *Let M be a closed connected smooth n–dimensional manifold and let $\xi \in H^1(M;\mathbb{R})$ be nonzero. Then*[7]

$$\operatorname{ccat}(M,\xi) \leq n-1. \tag{47}$$

Proof Choose a closed 1–form ω representing ξ having Morse type zeros and having no zeros of Morse index 0 or n; here we use our assumption that $\xi \neq 0$ and M is connected; see [15].

Theorem 11 will be proven once we show that for any $N>0$ there exist closed subsets $A_1, A_2, \dots, A_{n-1} \subset M$ such that each inclusion $A_j \to M$ is null-homotopic and such that the complement $F = M - \cup_{j=1}^{n-1} A_j$ is N–movable with respect to ω. It is well-known (see [4, Lemma 1.11]) that each such A_j can be enlarged to an open subset $A_j \subset F_j \subset M$ such that $F_j \to M$ is null-homotopic; thus one obtains an open cover $F \cup F_1 \cup \cdots \cup F_{n-1}$ satisfying conditions of Definition 5.3.

Fix a gradient-like vector field v for ω. As usual, for a zero $p \in M$ of ω we denote by $W^s(p)$ and $W^u(p)$ its stable and unstable manifolds with respect to the flow $x \mapsto x\cdot t$ generated by $-v$. The closed 1–form ω "locally decreases" along the flow. More precisely, this means that

$$\int_x^{x\cdot t} \omega < 0$$

assuming that $t>0$ and x not a fixed point of the flow; the integral is calculated along the trajectory of the flow.

We will assume that the stable and unstable manifolds of zeros of ω intersect transversely; in particular we will require that for any pair of distinct zeros $p \neq q$ of the same index $\operatorname{ind}(p) = \operatorname{ind}(q)$ one has $W^s(p) \cap W^u(q) = \varnothing$. Such v exists by the Kupka–Smale theorem.

Given $N>0$ and a zero $p \in M$ we denote by $A_N(p)$ the set of all points $x \in W^s(p)$ such that

$$\int_x^p \omega \geq -N. \tag{48}$$

[7] It is curious to mention that inequality (47) becomes false if one replaces $\operatorname{ccat}(X,\xi)$ by another modification of $\operatorname{cat}(X,\xi)$ which was introduced in [8], denoted $\operatorname{cat}^1(X,\xi)$. We discuss these issues in detail in [11]. Another related invariant, $\operatorname{Cat}(X,\xi)$, was introduced in [5].

As above, the integral is calculated along the integral trajectory $x\cdot t$ where $t\in[0,\infty)$ (or, equivalently, along an arbitrary curve lying in the stable manifold). Note that $A_N(p)$ is closed in the stable manifold $W^s(p)$ but it may be not closed in M; the closure of $A_N(p)$ in M may contain zeros of ω of index greater than $\operatorname{ind} p$. Indeed, suppose that there is an integral trajectory of the flow connecting a zero q with p where $\operatorname{ind} q > \operatorname{ind} p$. Then all points of this trajectory except q lie in the stable manifold $W^s(p)$ and the set $A_N(p)$ is not closed for $\int_q^p \omega \geq -N$.

For each zero $p\in M$ choose a small compact disc $B_N(p)$ containing p in its interior such that (i) $B_N(p)\cap B_N(p')=\varnothing$ for $p\neq p'$; and (ii) the set

$$C_N(p)=A_N(p)\cup B_N(p)\subset M$$

is contractible; see Figure 1.

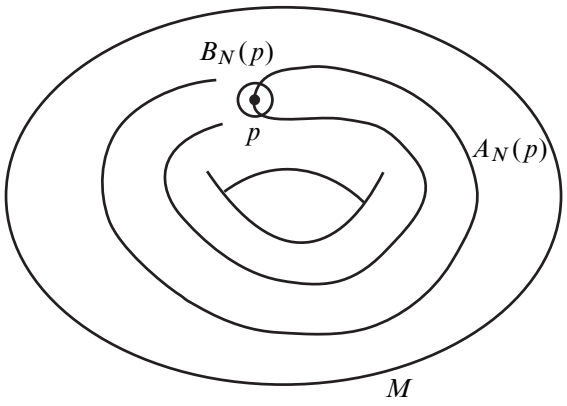

Figure 1

Now we claim:

(a) *Small discs* $B_N(p)$ *satisfying (i) and (ii) exist.*

(b) *The set*

$$C'_N(p)=C_N(p)-\bigcup_{\operatorname{ind}(q)>j}\operatorname{Int}B_N(q)$$

is compact. Here $j=\operatorname{ind}p$.

For $j=1,2,\dots,n-1$ let A_j be the union of the sets $C'_N(p)$ where p runs over all zeros of ω of index j. We see that each A_j is a closed subset which is null-homotopic in M.

Next we want to show that the complement

$$F = M - \bigcup_{\omega_p = 0} A_N(p)$$

is N–movable with respect to ω. For any point $x \in F$ there exists $T_x > 0$ such that

$$\int_x^{x \cdot T_x} \omega < -N.$$

Hence there is a neighborhood $V_x \subset F$ of x such that

$$\int_y^{y \cdot T_x} \omega < -N$$

for all $y \in V_x$. Without loss of generality we may assume that for some sequence of points $x_n \in F$ the sets V_{x_n} form a locally finite cover of F. Then we may find continuous functions $\rho_n \colon F \to [0,1]$ such that the support of ρ_n is contained in V_{x_n} and $\max_n \rho_n(y) = 1$ for any $y \in F$. Then $T = \sum_n T_{x_n} \rho_n$ is a continuous function on F and for any point $y \in F$ one has

$$\int_y^{yT(y)} \omega < -N.$$

Now a homotopy $h_\tau \colon F \to M$ as in Definition 9.1 (with constant $C = 0$) can be defined by the formula $h_\tau(y) = y \cdot (\tau T(y))$, where $\tau \in [0,1]$. □

13 Relations with the Bieri–Neumann–Strebel invariant

Bieri, Neumann and Strebel introduced in [2] a geometric invariant of discrete groups G which captures information about the finite generation of kernels of abelian quotients of G. In this section we describe a relation between this invariant and properties of $\operatorname{cat}(X, \xi)$.

Let us recall the definition. We always assume that G is finitely presented as this is sufficient for our purposes. Let $S(G)$ denote $(\operatorname{Hom}(G, \mathbb{R}) - \{0\})/\mathbb{R}_+$ where $\mathbb{R}_+$ acts on $\operatorname{Hom}(G, \mathbb{R})$ by multiplication. Clearly $S(G)$ is a sphere of dimension $n-1$ where n is the rank of the abelianization of G. Denote by $[\chi]$ the equivalence class of a nonzero homomorphism $\chi \colon G \to \mathbb{R}$. The Bieri–Neumann–Strebel invariant associates to G a subset $\Sigma(G) \subset S(G)$ defined as follows[8]. Let X be a finite cell complex with $\pi_1(X) = G$ and let $p \colon \widetilde{X} \to X$ be the universal abelian cover of X. A homomorphism

[8] We rely on [2, Theorem 5.1] which states that the definition of $\Sigma(G)$ given above coincides in the case of finitely presented G with the main definition of [2].

$\chi \in \mathrm{Hom}(G,\mathbb{R})$ can be viewed as a cohomology class lying in $H^1(X;\mathbb{R})$. One has $\chi \in \Sigma(G)$ if and only if the inclusion $N \to \widetilde{X}$ induces an epimorphism $\pi_1(N,x_0) \to \pi_1(\widetilde{X},x_0)$ where $N \subset \widetilde{X}$ is a connected neighborhood of infinity with respect to χ, see Section 2 and [2, Lemma 5.2].

The following result improves the estimate given by Theorem 11.

Theorem 12 *Let M be a closed connected smooth manifold of dimension $n \geq 5$ and $G = \pi_1(M)$. If for a nonzero cohomology class $\xi \in H^1(X;\mathbb{R})$ either $[\xi] \in \Sigma(G)$ or $[-\xi] \in \Sigma(G)$ then*

$$\mathrm{ccat}(M,\xi) \leq n-2. \tag{49}$$

Proof By Latour [13, Propositions 5.8 and 4.2] the assumption $[-\xi] \in \Sigma(G)$ implies that ξ can be realized by a smooth Morse closed 1–form ω having no zeros of Morse index $0,1,n$. Having such ω one repeats the argument of the proof of Theorem 11 (without modifications) which leads to (49).

In the case $[\xi] \in \Sigma(G)$ one applies the above argument to $-\xi$ and obtains a smooth Morse closed 1–form ω having no zeros of Morse index $0, n-1, n$. Then one applies the argument of Theorem 11. □

Theorem 13 *Let M be a closed connected smooth manifold of dimension $n \geq 5$ such that $\pi_1(M)$ has no non-abelian free subgroups. Then for any nonzero cohomology class $\xi \in H^1(X;\mathbb{R})$ one has*

$$\mathrm{ccat}(M,\xi) \leq n-2. \tag{50}$$

Proof This follows from the previous statement and from Bieri, Neumann, Strebel [2, Theorem C]. For convenience of the reader we recall that [2, Theorem C] claims that $\Sigma(G) \cup -\Sigma(G) = S(G)$ assuming that G has no non-abelian free subgroups. □

Here is another statement which is obviously equivalent to Theorem 13:

Theorem 14 *Let M be a closed connected smooth manifold of dimension $n \geq 5$ such that for some nonzero $\xi \in H^1(X;\mathbb{R})$ one has $\mathrm{ccat}(M,\xi) = n-1$. Then $\pi_1(M)$ contains a non-abelian free subgroup.*

In this theorem one can replace $\mathrm{ccat}(M,\xi)$ by $\mathrm{cat}(M,\xi)$.

Examples of manifolds satisfying $\mathrm{cat}(M,\xi) = \dim M - 1$ for some nonzero $\xi \in H^1(M;\mathbb{R})$ are known, see [6, Example 6.5]. Here M is obtained from n–dimensional torus by adding a handle of index one, i.e $M = T^n \# (S^1 \times S^{n-1})$ and ξ is nonzero restricted to the handle. In this case it is obvious that $\pi_1(M)$ contains a free subgroup on two generators.

Theorem 15 *Let M be a closed connected smooth manifold of dimension $n \geq 5$ and $G = \pi_1(M)$. If for a nonzero class $\xi \in H^1(X;\mathbb{R})$ both $[\xi] \in \Sigma(G)$ and $[-\xi] \in \Sigma(G)$ then*

$$\operatorname{ccat}(X, \xi) \leq n - 3. \tag{51}$$

Proof The proof is similar to the proof of Theorem 12. Namely, using Latour [13, Propositions 5.8 and 4.2] we see that the assumptions

$$[\xi], [-\xi] \in \Sigma(G)$$

imply that ξ can be realized by a smooth closed 1–form ω having no zeros of Morse index $0, 1, n-1, n$. Then one repeats the arguments of the proof of Theorem 11 obtaining (51). □

Theorem 15 is similar to [17, Corollary 6.9]. It is worth mentioning that [17, Theorem 6.8] states that if M is a closed smooth manifold of dimension $n \geq 5$ and $\xi \in H^1(M;\mathbb{R})$ is such that $[\xi], [-\xi] \in \Sigma(\pi_1(M))$ then M admits a closed 1–form ω lying in the class ξ which has at most $n-3$ zeros and such that the gradient flow of ω has no homoclinic cycles.

14 Improved upper bound for products

Combining the product inequality and the upper bound of Theorem 11 we obtain:

Theorem 16 *Let $M_1, M_2, \ldots, M_k$ be closed connected smooth manifolds,* $\dim M_i \geq 2$. *Then*

$$\operatorname{ccat}(M_1 \times M_2 \times \cdots \times M_k, \xi) \leq 1 - 2k + \sum_{i=1}^{k} \dim M_i \tag{52}$$

assuming that the cohomology class

$$\xi \in H^1(M_1 \times \cdots \times M_k;\mathbb{R})$$

is such that the restriction

$$\xi_i = \xi|_{M_i} \in H^1(M_i;\mathbb{R})$$

is nonzero for every $i = 1, \ldots, k$.

Proof First we consider the case when $\mathrm{ccat}(M_i,\xi_i)=0$ for some $i=1,\dots,k$. Then the left hand side of (52) vanishes (see Lemma 11.2) and inequality (52) is true due to our assumption $\dim M_i \geq 2$.

In the case when $\mathrm{ccat}(M_i,\xi_i)>0$ for all $i=1,\dots,k$, Theorems 9 and 11 applied repeatedly imply the result by induction. □

15 Calculation of $\mathrm{cat}(X,\xi)$ for products of surfaces

Theorem 17 *Let M^{2k} denote the product $\Sigma_1\times\Sigma_2\times\cdots\times\Sigma_k$ where each Σ_i is a closed orientable surface of genus $g_i>1$. Given a cohomology class $\xi\in H^1(M^{2k};\mathbb{R})$, one has*

$$\mathrm{cat}(M^{2k},\xi)=\mathrm{ccat}(M^{2k},\xi)=1+2r \tag{53}$$

where r is the number of indices $i\in\{1,2,\dots,k\}$ such that the cohomology class $\xi|_{\Sigma_i}\in H^1(\Sigma_i,\mathbb{R})$ vanishes. In particular

$$\mathrm{cat}(M^{2k},\xi)=\mathrm{ccat}(M^{2k},\xi)=1 \tag{54}$$

assuming that $\xi|_{\Sigma_i}\neq 0\in H^1(\Sigma_i;\mathbb{R})$ for any $i=1,\dots,k$.

Proof If Σ_i is a surface of genus $g_i>1$ and $\xi_i\in H^1(\Sigma_i;\mathbb{R})$ is nonzero then $\mathrm{ccat}(\Sigma_i,\xi)\leq 1$ by Theorem 11 and $\mathrm{ccat}(\Sigma_i,\xi_i)\geq 1$ by Theorem 10 (since $\chi(\Sigma_i)=2-2g_i\neq 0$). Hence we obtain $\mathrm{ccat}(\Sigma_i,\xi_i)=1$. Using Theorem 16 we find that $\mathrm{ccat}(M^{2k},\xi)=1$ assuming that $\xi|_{\Sigma_i}\neq 0$ for all $i=1,\dots,k$.

Now assume that $\xi_i|_{\Sigma_i}\neq 0$ for $i=1,\dots,k-r$ and $\xi_i|_{\Sigma_i}=0$ for $i=k-r+1,\dots,k$. Denote

$$M'=\prod_{i=1}^{k-r}\Sigma_i,\quad M''=\prod_{i=k-r+1}^{k}\Sigma_i,\quad \xi'=\xi|_{M'},\quad \xi''=\xi|_{M''}=0.$$

As in the previous paragraph we find $\mathrm{ccat}(M',\xi')=1$. Clearly,

$$\mathrm{ccat}(M'',\xi'')=\mathrm{cat}(M'')=2r+1,$$

see Lemma 9.7. Using the product inequality (Theorem 9) we obtain

$$\mathrm{ccat}(M,\xi)\leq\mathrm{ccat}(M',\xi')+\mathrm{ccat}(M'',\xi'')-1=2r+1.$$

To complete the proof we apply Lemma 7.3 to get a lower bound. We have (since the genus of Σ_i is greater than 1)

$$\mathrm{cl}(\Sigma_i, \xi_i) = \begin{cases} 0, & \text{if } \xi_i \neq 0 \in H^1(\Sigma_i; \mathbb{R}), \\ 2, & \text{if } \xi_i = 0 \in H^1(\Sigma_i; \mathbb{R}). \end{cases} \tag{55}$$

By Lemma 7.3 we obtain $\mathrm{cl}(M^{2k}, \xi) \geq 2r$ where r. Hence $\mathrm{cat}(M^{2k}, \xi) \geq 2r + 1$ by Theorem 8.

This completes the proof. □

16 Another example

Let X denote $T^2 \vee S^1$ as in Example 8.5. Formula (38) expresses $\mathrm{ccat}(X, \xi) = \mathrm{cat}(X, \xi)$ as function of $\xi \in H^1(X; \mathbb{R}) = \mathbb{R}^3$.

Consider the product $X^k = X \times \cdots \times X$ of k–copies of X. For any index $i = 1, \ldots, k$ denote by $p_i \colon X \to X^k$ the inclusion $x \mapsto (x_0, \ldots, x_0, x, x_0, \ldots, x_0)$ where x stands on the place number i and on other places is the base point $x_0 \in X$. Let $q_i \colon T^2 \to X^k$ be the composition of the inclusion $T^2 \to X$ and of $p_i \colon X \to X^k$.

Theorem 18 *For any $\xi \in H^1(X^k; \mathbb{R})$ one has*

$$\mathrm{cat}(X^k, \xi) = \mathrm{ccat}(X^k, \xi) = 1 + a(\xi) + 2b(\xi) \tag{56}$$

where $a(\xi)$ denotes the number of indexes $i \in \{1, \ldots, k\}$ such that $p_i^*(\xi) \neq 0$ and $q_i^*(\xi) = 0$, and $b(\xi)$ denotes the the number of indexes $i \in \{1, \ldots, k\}$ such that $p_i^*(\xi) = 0$.

Proof Denote by X_i the i-th factor X in the product $X^k = X \times \cdots \times X$. Let $\xi_i \in H^1(X_i; \mathbb{R})$ denote $p_i^*(\xi)$. Finally we denote by $\ell_i \subset H^1(X_i; \mathbb{R})$ the set of cohomology classes such that their restriction onto the torus $T^2 \subset X_i$ vanishes. We have

$$\mathrm{cat}(X_i, \xi_i) = \mathrm{ccat}(X_i, \xi_i) = \begin{cases} 1, & \text{if } \xi_i \notin \ell_i, \\ 2, & \text{if } \xi_i \in \ell_i - \{0\}, \\ 3, & \text{if } \xi_i = 0; \end{cases} \tag{57}$$

see (38). It is easy to check directly using the definitions that

$$\mathrm{cl}(X_i, \xi_i) = \begin{cases} 0, & \text{if } \xi_i \notin \ell_i, \\ 1, & \text{if } \xi_i \in \ell_i - \{0\}, \\ 2, & \text{if } \xi_i = 0. \end{cases} \tag{58}$$

Applying Theorem 9 we find

$$\operatorname{cat}(X^{2k},\xi) \le \operatorname{ccat}(X^{2k},\xi) \le 1 + a(\xi) + 2b(\xi).$$

Applying Theorem 8 and Lemma 7.3 inductively we obtain the inverse inequality

$$\operatorname{cat}(X^{2k},\xi) \ge 1 + a(\xi) + 2b(\xi).$$

This completes the proof. □

17 Questions

Finally we raise two challenging questions which are inspired by problems discussed in this article.

Question 1 Is it always true that $\operatorname{ccat}(X,\xi) = \operatorname{cat}(X,\xi)$?

In all examples discussed in this paper the function $\xi \mapsto \operatorname{cat}(X,\xi)$ was upper semi-continuous, ie, the sets of the form $\{\xi \in H^1(X;\mathbb{R});\ \operatorname{cat}(X,\xi) \ge r\}$ were closed.

Question 2 Is it true in general that the functions $H^1(X;\mathbb{R}) \to \mathbb{R}$ given by

$$\xi \mapsto \operatorname{cat}(X,\xi), \quad \xi \mapsto \operatorname{ccat}(X,\xi)$$

are upper semi-continuous?

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