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On the 2–loop polynomial of knots

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The 2–loop polynomial of a knot is a polynomial characterizing the 2–loop part of the Kontsevich invariant of the knot. An aim of this paper is to give a methodology to calculate the 2–loop polynomial. We introduce Gaussian diagrams to calculate the rational version of the Aarhus integral explicitly, which constructs the 2–loop polynomial, and we develop methodology of calculating Gaussian diagrams showing many basic formulas of them. As a consequence, we obtain an explicit presentation of the 2–loop polynomial for knots of genus 1 in terms of derivatives of the Jones polynomial of the knots.

Corresponding to quantum and related invariants of 3–manifolds, we can formulate equivariant invariants of the infinite cyclic covers of knots complements. Among such equivariant invariants, we can regard the 2–loop polynomial of a knot as an "equivariant Casson invariant" of the infinite cyclic cover of the knot complement. As an aspect of an equivariant Casson invariant, we show that the 2–loop polynomial of a knot is presented by using finite type invariants of degree ≤ 3 of a spine of a Seifert surface of the knot. By calculating this presentation concretely, we show that the degree of the 2–loop polynomial of a knot is bounded by twice the genus of the knot. This estimate of genus is effective, in particular, for knots with trivial Alexander polynomial, such as the Kinoshita–Terasaka knot and the Conway knot.

57M27; 57M25

Dedicated to Professor Yukio Matsumoto on the occasion of his 60th birthday

1 Introduction

The Kontsevich invariant is a very strong invariant of knots, which dominates all quantum invariants and all Vassiliev invariants, and it is expected that the Kontsevich invariant classifies knots. A problem when we study the Kontsevich invariant is that it is difficult to calculate the Kontsevich invariant for any knot concretely. That is, the value of the Kontsevich invariant is presented by an infinite linear sum of Jacobi diagrams (a certain kind of uni-trivalent graphs), and it is not known so far how to calculate all terms of such a linear sum at the same time for an arbitrarily given knot.

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Each term is a Vassiliev invariant, and there are algorithms to calculate it, but it is difficult to determine all terms at the same time.

The infinite sum of the terms of the Kontsevich invariant with a fixed loop number (the first Betti number of uni-trivalent graphs) is presented by a polynomial [11; 21; 39][1]; this presentation is called the "loop expansion". In particular, it is known[2] that the 1–loop part is presented by the Alexander polynomial. The polynomial presenting the 2–loop part is called the 2–loop polynomial. The 2–loop polynomial itself is a 2–variable polynomial invariant of knots.

A table of the 2–loop polynomial for knots with up to 7 crossings is given by Rozansky [40]. The 2–loop polynomial of knots with the trivial Alexander polynomial can often been calculated by surgery formulas (Garoufalidis and Kricker [11], Kricker [19], Marché [26]). The 2–loop polynomial for torus knots is explicitly presented by using a cabling formula (Marché [25] Ohtsuki [35]). However, it is still difficult to obtain an explicit presentation of the 2–loop polynomial for an arbitrarily given knot, because the "language" to calculate the 2–loop polynomial has not been enough.

An aim of this paper is to give a methodology to calculate the 2–loop polynomial for an arbitrarily given knot. We construct the 2–loop polynomial of a knot by calculating the rational version (Kricker [21]) of the Aarhus integral (Bar-Natan, Garoufalidis Rozansky and Thurston [2; 3; 4]) of a surgery presentation of the knot. The Lie algebra version of the Aarhus integral implies the perturbative expansion of a Gaussian integral, which is obtained by coupling the second-order part and higher-order part of the integral. In order to calculate the diagram version of the Aarhus integral explicitly, we introduce Gaussian diagrams, which present the second-order and higher-order parts of diagrams explicitly. We develop methodology to calculate Gaussian diagrams, showing many basic formulas of them.

Corresponding to quantum and related invariants of 3–manifolds, we can formulate equivariant invariants of the infinite cyclic covers of knots complements (Section 1.3). Among such equivariant invariants, the 2–loop polynomial of a knot can be regarded as an "equivariant Casson invariant" of the infinite cyclic cover of the knot complement (Marché [27] Ohtsuki [34]), while it is well known (see, eg, Lickorish [24]) that the Alexander polynomial can be regarded as an equivariant homology. One aspect of an equivariant Casson invariant is that a surgery formula for the 2–loop polynomial is given

[1] This was originally conjectured by Rozansky [39]. The existence of such rational presentations has been proved by Kricker [21] (though such a rational presentation itself is not necessarily a knot invariant in a general loop degree). Further, Garoufalidis and Kricker [11] defined a knot invariant in any loop degree, from which such a rational presentation can be deduced.

[2] This follows from the property called the Melvin–Morton–Rozansky conjecture (Bar-Natan and Garoufalidis [1]). See also [11; 21] and references therein.

by the equivariant linking number (Kojima and Yamasaki [17]) and equivariant finite type invariants of degree ≤ 3, while a surgery formula for the Alexander polynomial is given by the equivariant linking number. Another aspect of an equivariant Casson invariant is that the 2–loop polynomial of a knot is presented by using finite type invariants of degree ≤ 3 of a spine of a Seifert surface of the knot (Theorem 4.4), while the Alexander polynomial of a knot is presented by using finite type invariants of degree 1, ie, the Seifert form, of a spine of a Seifert surface of the knot.

By constructing the 2–loop polynomial using Gaussian diagrams along the latter aspect, in Theorem 4.7, we show the following estimate, which was conjectured by Rozansky [40], that

$$\big(\text{the degree of the 2–loop polynomial of a knot}\big) \;\leq\; 2\,\big(\text{the genus of the knot}\big),$$

where the *genus* of a knot is the minimal genus of a Seifert surface of the knot. This implies that the non-zero coefficients of the 2–loop polynomial of a knot lie in the hexagon whose edges are of length $2g$ for the genus g of the knot as shown in Table 1 and Table 2. This estimate is a refinement of the estimate of the genus by the degree of the Alexander polynomial (see, eg, [24]), and, in particular, this estimate is effective for knots with trivial Alexander polynomial. For example, we see, in Example 4.13, that our bound is sharp for the Kinoshita–Terasaka knot and the Conway knot whose Alexander polynomial is trivial, while genera of them and many knots has been determined by Gabai [8] geometrically and by Ozsváth and Szabó [38; 37] using the knot Floer homology.

n	-2	-1	0	1	2
$m=2$	$\cdot$	$\cdot$	γ	δ	γ
$m=1$	$\cdot$	δ	β	β	δ
$m=0$	γ	β	α	β	γ
$m=-1$	δ	β	β	δ	$\cdot$
$m=-2$	γ	δ	γ	$\cdot$	$\cdot$

Table 1: The non-zero coefficients of $t_1^n t_2^m$ in the 2–loop polynomial $\Theta_K(t_1,t_2)$ of a knot K of genus 1, where $\alpha,\beta,\gamma,\delta$ are some integers given by Theorem 3.1 and Theorem 3.7.

Further, by calculating Gaussian diagrams, we show explicit presentations of the 2–loop polynomial for some knots. We show, in Theorem 3.7, that the 2–loop polynomial

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$m=6$	·	·	·	·	·	·	3	−3	3	−3	3	−3	3
$m=5$	·	·	·	·	·	−3	·	·	·	·	·	·	−3
$m=4$	·	·	·	·	3	·	2	−2	2	−2	2	·	3
$m=3$	·	·	·	−3	·	−2	·	·	·	·	−2	·	−3
$m=2$	·	·	3	·	2	·	1	−1	1	·	2	·	3
$m=1$	·	−3	·	−2	·	−1	·	·	−1	·	−2	·	−3
$m=0$	3	·	2	·	1	·	·	·	1	·	2	·	3
$m=-1$	−3	·	−2	·	−1	·	·	−1	·	−2	·	−3	·
$m=-2$	3	·	2	·	1	−1	1	·	2	·	3	·	·
$m=-3$	−3	·	−2	·	·	·	·	−2	·	−3	·	·	·
$m=-4$	3	·	2	−2	2	−2	2	·	3	·	·	·	·
$m=-5$	−3	·	·	·	·	·	·	−3	·	·	·	·	·
$m=-6$	3	−3	3	−3	3	−3	3	·	·	·	·	·	·

Table 2: The non-zero coefficients of $t_1^n t_2^m$ in the 2–loop polynomial $\Theta_{T(7,2)}(t_1,t_2)$ of the torus knot $T(7,2)$ of type $(7,2)$, whose genus is 3;

$\Theta_K(t_1,t_2)$ of a knot K of genus 1 is presented by

$$\begin{aligned}\Theta_K(t_1,t_2) &= \tfrac{1}{24}\big(V_K'''(1)+3V_K''(1)\big)\Big(\big(d^2-d+\tfrac{1}{3}\big)(T_{2,1}-T_{1,0})-\tfrac{1}{2}d(d-1)T_{2,0}\Big)\\ &\quad -\tfrac{1}{16}V_K'(-1)\Big((5d^2-5d+1)T_{1,0}+\tfrac{1}{2}d(5d-1)T_{2,0}-\big(5d^2-\tfrac{7}{3}d+\tfrac{1}{3}\big)T_{2,1}\Big),\end{aligned}$$

where $V_K(t)$ denotes the Jones polynomial [15] of K, and $d=-\frac{1}{6}V_K''(1)$. Here, we put $T_{n,m}$, for integers n,m with $0\le 2m\le n$, by

$$\begin{aligned}(1)\quad T_{n,m} = \tfrac{1}{\varepsilon}\big(&t_1^n t_2^m + t_1^m t_2^n + t_1^{n-m}t_2^{-m} + t_1^{-m}t_2^{n-m} + t_1^{m-n}t_2^{-n} + t_1^{-n}t_2^{m-n}\\ &+ t_1^{-n}t_2^{-m} + t_1^{-m}t_2^{-n} + t_1^{m-n}t_2^{m} + t_1^{m}t_2^{m-n} + t_1^{n-m}t_2^{n} + t_1^{n}t_2^{n-m} - 12\big),\end{aligned}$$

where we put[3] $\varepsilon=1$ if $0<2m<n$, and $\varepsilon=2$ if $2m=0$ or n; for example,

$$\begin{aligned}T_{2,1} &= t_1^2t_2 + t_1t_2^2 + t_1t_2^{-1} + t_1^{-1}t_2 + t_1^{-1}t_2^{-2} + t_1^{-2}t_2^{-1} - 6,\\ T_{3,1} &= t_1^3t_2^2 + t_1^3t_2 + t_1^2t_2^3 + t_1t_2^3 + t_1^2t_2^{-1} + t_1t_2^{-2}\\ &\quad + t_1^{-1}t_2^2 + t_1^{-1}t_2^{-3} + t_1^{-2}t_2 + t_1^{-2}t_2^{-3} + t_1^{-3}t_2^{-1} + t_1^{-3}t_2^{-2} - 12.\end{aligned}$$

Further, we show, in Proposition 2.4 and Proposition 2.5, that the 2–loop polynomial for the Kinoshita–Terasaka knot K_m^{KT} and the Conway knot K_m^{C} are presented by

$$\Theta_{K_m^{\mathrm{KT}}}(t_1,t_2) = m\,(2T_{1,0}-2T_{2,0}-2T_{2,1}+T_{3,1}),$$

[3] That is, we set $\varepsilon=1,2$ so that the coefficient of $t_1^n t_2^m$ in $T_{n,m}$ is equal to 1.

$$\Theta_{K_m^{\mathrm{C}}}(t_1,t_2) = m\,(2T_{1,0} - 2T_{2,0} - 2T_{2,1} + T_{3,1}) \\ \qquad +2m^3\big(T_{1,0} + T_{2,1} - T_{3,1} + T_{4,0} + T_{4,2} + T_{5,0} - T_{5,1} - \tfrac{1}{2}T_{6,2} + T_{6,3}\big).$$

This implies that the 2–loop polynomial is sensitive to mutation, unlike the Alexander and Jones polynomials.

The paper is organized as follows. In Section 1, we review the definition of the 2–loop polynomial and its construction by the rational version of the Aarhus integral, introducing Gaussian diagrams. Further, we give a survey on equivariant invariants corresponding to quantum and related invariants. In Section 2, we give concrete presentation of the 2–loop polynomial for knots in terms of the Kontsevich invariant of their surgery presentation when the surgery is along knots, and calculate the 2–loop polynomial for the $(4nm+1, 2n)$ two-bridge knot, the Kinoshita–Terasaka knot, and the Conway knot. In Section 3, we give explicit presentations of the 2–loop polynomial for knots of genus 1 in terms of derivatives of the Jones polynomial of them. In Section 4, we show the genus bound of the degree of the 2–loop polynomial, calculating the 2–loop polynomial for knots of any genus. Further, we show clasper surgery formulas for the 2–loop polynomial. In Section 5, we show many formulas of Gaussian diagrams used in the other sections, developing methodology of calculating them. See Figure 1 for relations among these sections.

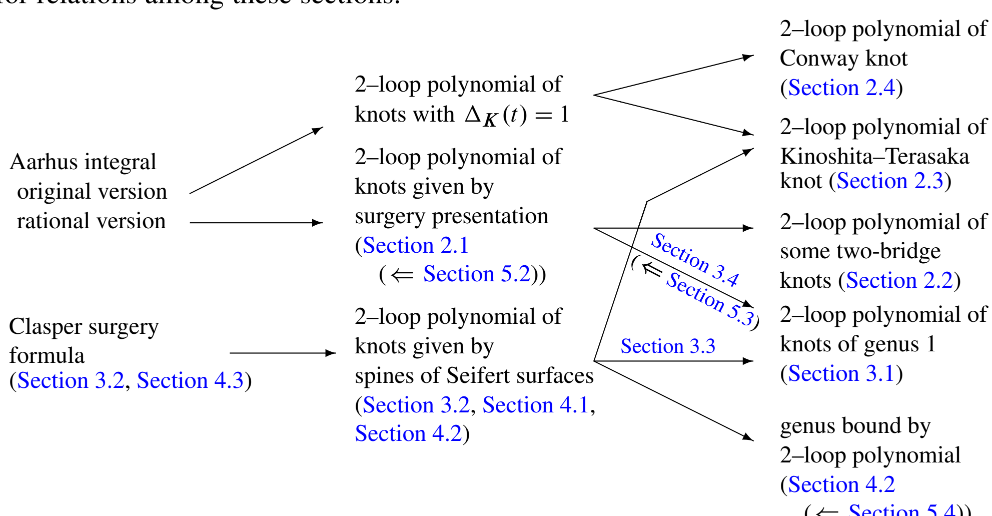

Figure 1: Relations among sections

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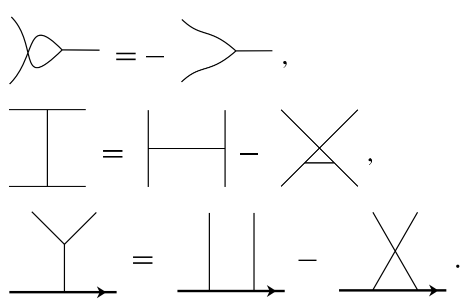
1.1 Definition and properties of the 2–loop polynomial

In this section, we review the definition of the 2–loop polynomial. Further, we introduce the reduced 2–loop polynomial, and show a property of it.

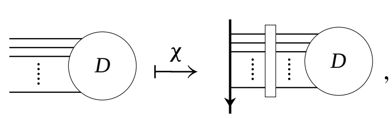
The Kontsevich invariant is defined in the space of Jacobi diagrams on S^1, which we define as follows. For a 1–manifold X, a *Jacobi diagram* on X is the manifold X together with a uni-trivalent graph such that univalent vertices of the graph are distinct points on X and each trivalent vertex is vertex-oriented, where a *vertex-oriented* trivalent vertex is a trivalent vertex such that a cyclic order of the three edges around the trivalent vertex is fixed. In figures we draw X by thick lines and the uni-trivalent graphs by thin lines, in such a way that each trivalent vertex is vertex-oriented in the counterclockwise order. We define the *degree* of a Jacobi diagram to be half the number of univalent and trivalent vertices of the uni-trivalent graph of the Jacobi diagram. We denote by $\mathcal{A}(X)$ the quotient vector space spanned by Jacobi diagrams on X subject to the following relations, called *the AS, IHX, and STU relations* respectively,



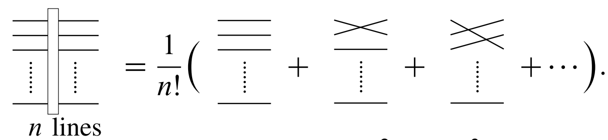
The *Kontsevich invariant* $Z(K)$ [18] of a knot K is defined to be in $\mathcal{A}(S^1)$; for details of its constructions, see, eg, [33].

The loop expansion of the Kontsevich invariant is defined in the space of open Jacobi diagrams. An *open Jacobi diagram* is a vertex-oriented uni-trivalent graph. We denote by $\mathcal{A}(*)$ the quotient vector space spanned by open Jacobi diagrams subject to the AS and IHX relations. The *Poincare–Birkhoff–Witt isomorphism* (*PBW isomorphism*) χ: $\mathcal{A}(*) \to \mathcal{A}(\downarrow)$ is defined by

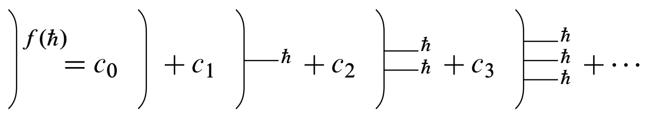
(2)

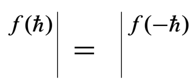


for any diagram D, where the box denotes the symmetrizer,

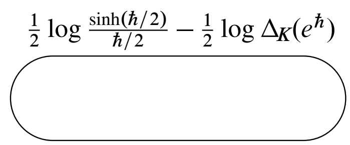
(3) 

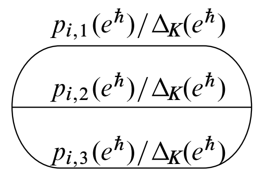
A label of a power series $f(\hbar) = c_0 + c_1\hbar + c_2\hbar^2 + c_3\hbar^3 + \cdots$ implies that

(4) 

Note that 

by the AS relation, in the notation of this paper. Any open Jacobi diagram can be presented by a trivalent graph with labels on its edges. It is known [11; 21; 39] that the Kontsevich invariant of a knot K has a presentation,

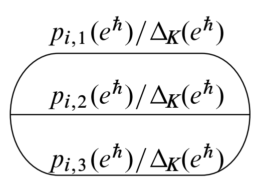
$\log_{\sqcup}\big(\chi^{-1}Z(K)\big) =$ 

$+ \sum_i^{\text{finite}}$ 

$+$ (terms of 3–loop, 4–loop, $\cdots$ presented in the same way),

where $\log_{\sqcup}$ denotes the logarithm with respect to the disjoint-union product of open Jacobi diagrams, $\Delta_K(t)$ denotes the Alexander polynomial, and $p_{i,j}(e^{\hbar})$ is a polynomial in $e^{\pm\hbar}$. This presentation is called the *loop expansion*.

We denote its 2–loop part by

$Z^{(2\text{–loop})}(K) = \sum_i^{\text{finite}}$  .

The 2–loop part is characterized by the polynomial,

$$\sum_i p_{i,1}(t_1)\,p_{i,2}(t_2)\,p_{i,3}(t_3) \quad \in \ \mathbb{Q}[t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}]/(t_1t_2t_3 = 1), \sim$$

where the equivalence "$\sim$" is generated by

$$f(t_1, t_2, t_3) \ \sim \ f(t^{\varepsilon}_{\sigma(1)}, t^{\varepsilon}_{\sigma(2)}, t^{\varepsilon}_{\sigma(3)}) \tag{5}$$

for any $\varepsilon = \pm 1$ and any permutation σ on $\{1,2,3\}$, which is derived from the symmetry of the θ graph, while the relation $t_1 t_2 t_3 = 1$ is derived from the IHX relation. The symmetrization of the above polynomial with respect to the symmetry of the θ graph is given by

$$\Theta_K(t_1,t_2,t_3) = \sum_{\substack{\varepsilon=\pm 1\\ \sigma}} p_{i,1}(t^{\varepsilon}_{\sigma(1)}) p_{i,2}(t^{\varepsilon}_{\sigma(2)}) p_{i,3}(t^{\varepsilon}_{\sigma(3)}) \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]/(t_1 t_2 t_3 = 1),$$

where σ runs all permutations on $\{1,2,3\}$. Putting $t_3 = t_1^{-1} t_2^{-1}$, we denote it by $\Theta_K(t_1,t_2)$, and call it the *2–loop polynomial* of K, (Note that this normalization of $\Theta_K(t_1,t_2)$ is 12 times the normalization in [40].) We also denote the 2–loop polynomial by $\Theta(K)$.

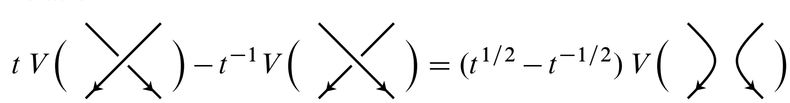
The 2–loop polynomial of the mirror image $\bar{K}$ of a knot K satisfies that $\Theta_{\bar{K}}(t_1,t_2) = -\Theta_K(t_1,t_2)$, since $Z(\bar{K})$ is obtained from $Z(K)$ by changing the sign of the part of odd degree.

A particular value $\Theta_K(t,1)$ is a symmetric polynomial in $t^{\pm 1}$ divisible by $t-1$ (since $\Theta_K(1,1) = 0$ by definition) and hence, divisible by $(t-1)^2$. As in [35], we define the *reduced 2–loop polynomial* by

$$\widehat{\Theta}_K(t) = \frac{\Theta_K(t,1)}{(t^{1/2} - t^{-1/2})^2} \quad \in \mathbb{Q}[t^{\pm 1}],$$

which is a symmetric polynomial in $t^{\pm 1}$. As shown in [35], this presents the sl_2 reduction of the 2–loop polynomial.

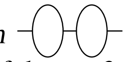
We obtain the formulas of the following proposition for $\widehat{\Theta}_K(1)$ and $\widehat{\Theta}_K(-1)$ by rewriting formulas in [35] and in [10; 28] respectively. Recall that the *Jones polynomial* $V_L(t) \in \mathbb{Z}[t^{\pm 1/2}]$ (which we also denote by $V(L)$) of an oriented link L is defined by the skein relation



and the normalization $V(\text{the trivial knot}) = 1$. Here, the three pictures in the formula imply three oriented links, which are identical except for a ball, where they differ as shown in the pictures.

Proposition 1.1 (See [35] and [10; 28]) *The reduced 2–loop polynomial of a knot K at $t = \pm 1$ satisfies that*

$$\widehat{\Theta}_K(1) = 2\, v_3(K) = \tfrac{1}{18} V_K'''(1) + \tfrac{1}{6} V_K''(1),$$
$$\widehat{\Theta}_K(-1) = -\tfrac{1}{12} V_K'(-1) V_K(-1),$$

where $v_3(K)$ *denotes* 4 *times the coefficient of the diagram*  *in* $\log_{\sqcup}\chi^{-1}Z(K)$*, which is an integer-valued primitive Vassiliev invariant of degree 3.*

Proof We obtain the first formula of the proposition, as follows. It is shown in [35] that $\widehat{\Theta}_K(1)$ is equal to $2v_3(K)$. In particular, it is a primitive Vassiliev invariant of degree 3, which is unique up to a scalar multiple. Hence, it is equal to a scalar multiple of the coefficient[4] of h^3 in the expansion of $V_K(e^h)$. We can determine the scalar of the multiple by calculating an example, say, see Example 3.6. It follows that

$$\widehat{\Theta}_K(1) = \tfrac{1}{3}\big(\text{the coefficient of } h^3 \text{ in } V_K(e^h)\big).$$

Further, since the third derivative of $V_K(e^h)$ is given by $\frac{d^3}{dh^3}V_K(e^h) = V_K'''(e^h)e^{3h} + 3V_K''(e^h)e^{2h} + V_K'(e^h)e^h$, the coefficient of $h^3/6$ in $V_K(e^h)$ is equal to $V_K'''(1) + 3V_K''(1)$. Therefore, we obtain the first formula of the proposition.

We obtain the second formula of the proposition, as follows. The Casson–Walker invariant of the double branched cover of S^3 branched along K is presented in [28] (see also [9]) by a linear sum of the signature of K and $V_K'(-1)/V_K(-1)$. On the other hand, it is also presented in [10] by a linear sum of the signature of K and $\big(\Theta_K(1,-1)+\Theta_K(-1,1)+\Theta_K(-1,-1)\big)/\Delta_K(-1)^2$, noting that $\Theta_K(1,1)=0$. This is further equal to $-12\widehat{\Theta}(-1)/\Delta_K(-1)^2$ from the definition of the reduced 2–loop polynomial. Hence, $\widehat{\Theta}(-1)$ is equal to a scalar multiple of $V_K'(-1)V_K(-1)$, noting that $V_K(-1)=\Delta_K(-1)$. We can determine the scalar of the multiple by calculating an example, say, see Example 3.6, and this gives the second formula of the proposition. □

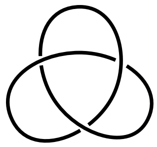
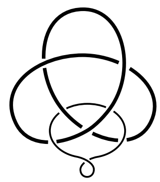
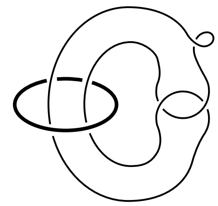
1.2 Construction of the 2–loop polynomial

In this section, we review a construction of the 2–loop polynomial by the rational version of the Aarhus integral. We introduce Gaussian diagrams to calculate the rational version of the Aarhus integral explicitly. Along this construction, we calculate the 2–loop polynomial in Sections 2–4.

Consider a framed link $K \cup L$, such that K is isotopic to the trivial knot with 0 framing, and the linking number of K and each component of L is equal to 0, and the 3–manifold obtained from S^3 by surgery along L is homeomorphic to S^3. We denote by K_L the knot obtained from K by surgery along L. The link $K \cup L$ is called a

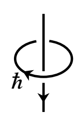
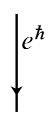
[4] Putting $V_K(e^h) = 1 + c_2(K)h^2 + c_3(K)h^3 + \cdots$, we can show that c_3 is primitive, ie, c_3 is additive with respect to the connected sum of knots, from the fact that $V_K(t)$ is multiplicative with respect to the connected sum of knots.

surgery presentation of the knot K_L. For example,

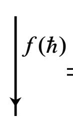
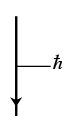
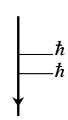
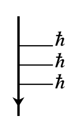
(6) $K_L =$  , $K \cup L =$  $=$ 

where we depict K and K_L by thick lines, and depict L by thin lines. We explain how to calculate the 2–loop polynomial of K_L from the Kontsevich invariant of $K \cup L$.

We review how to calculate the loop expansion of the Kontsevich invariant of K_L from the Kontsevich invariant of $K \cup L$, following an idea of Kricker [21], assuming, for simplicity, that L is a knot; for details see [11; 21]. The Kontsevich invariant $Z(K \cup L)$ of $K \cup L$ is defined to be in $\mathcal{A}(S^1 \sqcup S^1)$. We label the two components of $S^1 \sqcup S^1$ by $\hbar$ and x respectively. A *partially open Jacobi diagram* on $* \sqcup S^1$, where $*$ and S^1 are labeled by $\hbar$ and x respectively, is a vertex-oriented uni-trivalent graph such that some of the univalent vertices of the graph are labeled by $\hbar$ and the other univalent vertices are distinct points on S^1. The PBW isomorphism $\chi_\hbar\colon \mathcal{A}(* \sqcup S^1) \to \mathcal{A}(\downarrow \sqcup S^1)$ is defined by applying the map (2) to the univalent vertices labeled by $\hbar$. We identify $\mathcal{A}(\downarrow \sqcup S^1)$ and $\mathcal{A}(S^1 \sqcup S^1)$ by the isomorphism $\mathcal{A}(\downarrow \sqcup S^1) \to \mathcal{A}(S^1 \sqcup S^1)$, which is obtained by closing the two end points of $\downarrow$. It is shown that $\chi_\hbar^{-1} Z(K \cup L) \in \mathcal{A}(* \sqcup S^1)$ is equal to a linear sum of Jacobi diagrams whose $\hbar$–labeled vertices are given by labels of polynomials in $e^{\pm\hbar}$, by using the formula [6],

$\chi_\hbar^{-1} Z\Big($  $\Big) =$  ,

where a label of a power series $f(\hbar) = c_0 + c_1\hbar + c_2\hbar^2 + c_3\hbar^3 + \cdots$ implies that

 $= c_0$  $+ c_1$  $+ c_2$  $+ c_3$  $+ \cdots .$

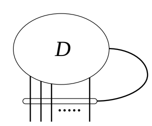
An *open Jacobi diagram on* $* \sqcup *$, where the two $*$'s of $* \sqcup *$ are labeled by $\hbar$ and x respectively, is a vertex-oriented uni-trivalent graph each of whose univalent vertices is labeled by either $\hbar$ or x. We denote by $\mathcal{A}(* \sqcup *)$ the quotient vector space spanned by Jacobi diagrams on $* \sqcup *$ subject to the AS and IHX relations. The PBW isomorphism $\chi\colon \mathcal{A}(* \sqcup *) \to \mathcal{A}(\downarrow \sqcup \downarrow)$ is defined by applying the map (2) to the $\hbar$–labeled vertices and the x–labeled vertices respectively. This isomorphism is equal to the composition of $\chi_\hbar$ and $\chi_x\colon \mathcal{A}(* \sqcup *) \to \mathcal{A}(* \sqcup \downarrow)$. We choose a pre-image of $\chi_\hbar^{-1} Z(K \cup L)$ in

$\mathcal{A}(*\sqcup *)$, which we denote by $\chi^{-1}Z(K\cup L)$, by choosing a 1–tangle $K\cup \check{L}$ whose closure is isotopic to $K\cup L$. The Kontsevich invariant of the surgery along L is described by the rational version [21] of the Aarhus integral [2; 3; 4], as follows,

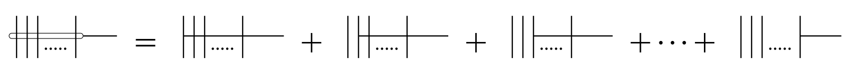
$$\chi^{-1}Z(K_L)=\frac{\displaystyle\int \chi^{-1}\check{Z}(K\cup L)dx}{\displaystyle\int \chi^{-1}\check{Z}(U_\pm)dx}, \tag{7}$$

where $U_\pm$ denotes the trivial knot with ± 1 framing whose sign we choose depending on the sign of the framing of L, and, by definition (see [23]), $\check{Z}(K\cup L)$ is obtained from $Z(K\cup L)$ by connect-summing ν to the L–component of $Z(K\cup L)$. An idea of Kricker [21] is to calculate the loop expansion of the Kontsevich invariant of K_L from (7).

When we calculate the Aarhus integral, it is often convenient to use the *link relation* "$\underset{(\circledast)}{\equiv}$" [2; 3; 4], which is defined by

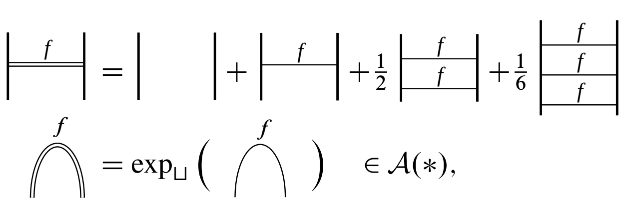

$\underset{(\circledast)}{\equiv} 0$

for any diagram D, where we put

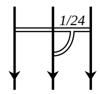


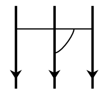
The link relations is a relation which relates pre-images in $\mathcal{A}(*\sqcup *)$ of an element in $\mathcal{A}(*\sqcup\downarrow)$ by the PBW map χ_x, and it is known [2; 3; 4] that the result of the Aarhus integral does not depend on the difference derived from the link relation.

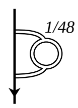
To calculate the rational version of the Aarhus integral explicitly, we use Gaussian diagrams, which we introduce as follows. We denote by a chord written in a double line an exponential chord; for example,

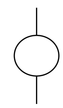
(8) 
$+\cdots\in\mathcal{A}$(two intervals),
$\in\mathcal{A}(*)$,

where $\exp_\sqcup$ denotes the exponential with respect to the disjoint-union product of open Jacobi diagrams. We call a Jacobi diagram with exponential chords a *Gaussian diagram*. Further, we denote by a uni-trivalent graph written in double lines its exponential; for example,

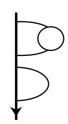
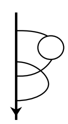
$\Phi \equiv$ 

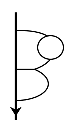
$\equiv \exp\Big(\frac{1}{24}$  $\Big),$

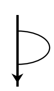
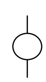
$\nu \equiv$ 

$\equiv$  $\times \Big(1 + \frac{1}{48}$  $\Big),$

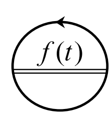
where ν is the element in $\mathcal{A}(\downarrow)$ whose closure in $\mathcal{A}(S^1)$ is equal to the Kontsevich invariant of the trivial knot, and Φ is the element called an *associator* which is the Kontsevich invariant of an elementary q–tangle; they are basic elements appearing in a combinatorial construction of the Kontsevich invariant; for details, see, eg, [33]. Here, in the above formulas (and throughout this paper), we write $\alpha \equiv \beta$ (resp. $\alpha \equiv_{(2)} \beta$) if $\alpha - \beta$ is a linear sum of Jacobi diagrams with at least 3 (resp. 2) trivalent vertices, where we do not count trivalent vertices generated by attached power series. If a uni-trivalent graph has 2 trivalent vertices, we can put it in any way in a Jacobi diagram modulo the equivalence; for example,

 $\equiv$ 

since their difference equals  $\equiv 0.$

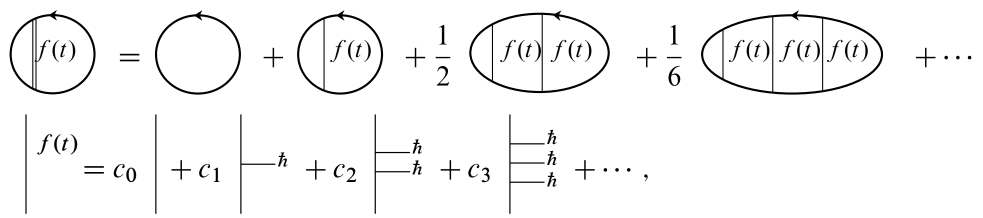
So, we write the previous diagrams as  $\times$  .

We explain how to calculate the rational version of the Aarhus integral in (7) concretely, using Gaussian diagrams. As mentioned above, $\chi_\hbar^{-1} Z(K \cup L)$ is presented by a linear sum of Jacobi diagrams whose $\hbar$–labeled vertices are given by labels of polynomials in $e^{\pm\hbar}$. For simplicity, as in Section 2.1 and Section 2.2, we assume that it is presented by

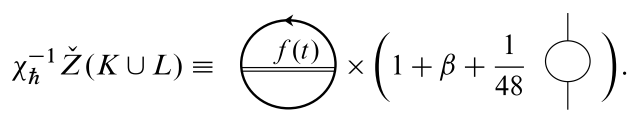
$\chi_\hbar^{-1} Z(K \cup L) \equiv$ 

$\times (1 + \beta)$

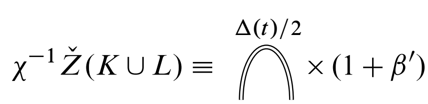
for some polynomial $f(t)$ in $t^{\pm 1}$ putting $t = e^\hbar$ and some β which is a linear sum of Jacobi diagrams each of which has at least two trivalent vertices; recall that a double line is defined in (8) and a label of a power series of $\hbar$ is defined in (4), that is,



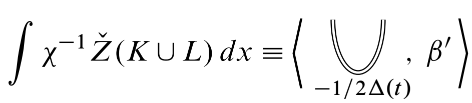
where $f(e^{\hbar}) = c_0 + c_1\hbar + c_2\hbar^2 + c_3\hbar^3 + \cdots$. By definition (see [23]), $\check{Z}(K \cup L)$ is obtained from $Z(K \cup L)$ by connect-summing ν to the L–component of $Z(K \cup L)$, that is,



By calculating χ^{-1} of a Gaussian diagram (for details see Section 2.1), we have that

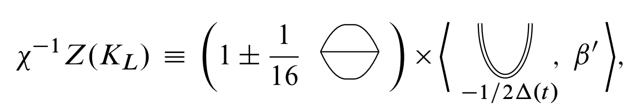


for some polynomial $\Delta(t)$ in $t^{\pm 1}$ with $\Delta(t) = \Delta(t^{-1})$, which gives the Alexander polynomial of K_L, and for some β' which is a linear sum of Jacobi diagrams each of which has at least one trivalent vertex. As in [11; 21], *the rational version of the Aarhus integral* is defined by

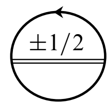
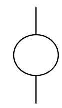


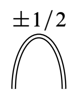
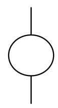
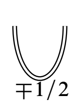
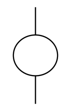
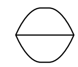
where the bracket $\langle D_1, D_2 \rangle$ is defined to be the sum of the diagrams obtained by connecting the univalent vertices of D_1 and the univalent vertices of D_2 if D_1 and D_2 have the same number of univalent vertices, and 0 otherwise. Hence, by (7),

(9)



where we choose the same sign in the formula as the sign of the framing of L, since the normalization factor in (7) is calculated as follows:

$$\int \chi^{-1}\check{Z}(U_\pm)dx \equiv \int \Big(\chi^{-1}$$  $$\times \Big(1+\frac{1}{24}$$  $$\Big)\Big)dx$$

$$\equiv \int \Big($$  $$\times \Big(1+\frac{1}{16}$$  $$\Big)\Big)dx = \Big\langle$$  $$,\ 1+\frac{1}{16}$$  $$\Big\rangle \equiv 1 \mp \frac{1}{16}$$  $$.$$

From (9), we obtain an explicit presentation of the 2–loop polynomial of K_L by calculating β' concretely.

1.3 Equivariant invariants of the infinite cyclic covers of knot complements

The 2–loop polynomial of a knot can be regarded as an "equivariant Casson invariant" of the infinite cyclic cover of the knot complement [27; 34]. The aim of this section is to see this from the viewpoint of quantum and related invariants of 3–manifolds and equivariant invariants corresponding to them. We give a survey on quantum and related invariants of 3–manifolds, and explain how Casson invariant behaves among them. Further, we see what are equivariant invariants corresponding to them, and consider relations of the 2–loop polynomial to these equivariant invariants, which would be meaningful for future directions of the study of these invariants.

We review quantum and related invariants of 3–manifolds; for details, see eg [33]. Let M be a closed 3–manifold, and let L be a framed link in S^3 such that M is obtained from S^3 by surgery along L. For simplicity, we consider the sl_2 case. Let V_n be the n–dimensional irreducible representation of the quantum group $U_q(sl_2)$, whose quantum dimension is $[n]=(q^{n/2}-q^{-n/2})/(q^{1/2}-q^{-1/2})$, and let r be an odd integer ≥ 3, and put $\zeta=\exp(2\pi\sqrt{-1}/r)$. Then, the *quantum $SO(3)$ invariant* of M is defined by

$$\tau_r^{SO(3)}(M) = (\text{normalization constant})\cdot \sum_{n_1,\cdots,n_l} [n_1]\cdots[n_l]\, Q(L;\, V_{n_1},\cdots,V_{n_l})\big|_{q=\zeta} \in \mathbb{C},$$

where the sum of each n_i runs over $n_i=1,3,\cdots,r-2$, and $Q(L;\, V_{n_1},\cdots,V_{n_l})$ denotes the quantum invariant of L whose components are associated with $V_{n_1},\cdots,V_{n_l}$, which is a polynomial in $q^{\pm 1}$. Further, for simplicity, let M be an integral homology 3–sphere. Then, the *perturbative $SO(3)$ invariant* $\tau^{SO(3)}(M)$ is defined to be an arithmetic limit of $\tau_r^{SO(3)}(M)$ as $r \to \infty$, ie,

$$\tau^{SO(3)}(M) = \sum_{n=0}^{\infty} \lambda_n (q-1)^n \ \in \mathbb{Z}[[q-1]]$$

where λ_n's are uniquely characterized by

$$\tau_r^{SO(3)}(M) \equiv \sum_{n=0}^{(r-3)/2} \lambda_n (\zeta-1)^n \quad \text{modulo } (\zeta-1)^{(r-1)/2} \text{ in } \mathbb{Z}[\zeta]$$

for any prime $r \geq 5$. Further, as shown in [13; 14], the perturbative invariant has an expansion of the form

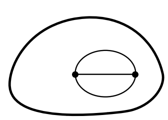
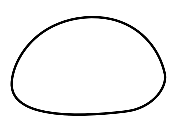
$$\tau^{SO(3)}(M) = \sum_{n=0}^{\infty} a_n (q-1)(q^2-1)\cdots(q^n-1)$$

for $a_n \in \mathbb{Z}[q]$, and each quantum invariant is derived from the perturbative invariant by substituting $q = \zeta$ in this expansion, ie,

$$\tau_r^{SO(3)}(M) = \tau^{SO(3)}(M)\big|_{q=\zeta},$$

where the substitution is taken in the above expansion. Each perturbative invariant is derived from the LMO invariant through the weight system (see [33]), and, in this sense, the LMO invariant is universal among all perturbative invariants. Further, as shown in [22], the LMO invariant of integral homology 3–spheres is universal among all finite type invariants; in particular, the degree d part of the LMO invariant are of finite type of degree d. See Figure 2 for relations among these invariants.

Casson invariant $\lambda(M)$ of an integral homology 3–sphere M appears in the degree 1 part of these invariants as follows; for details, see eg [33]. The degree 1 part of the LMO invariant is given by Casson invariant, and so is the finite type invariant of degree 1. In particular, Casson invariant has a clasper surgery formula,

(10) $\lambda\Big($  $\Big) - \lambda\Big($  $\Big) = 1,$

where the first picture implies a surgery on an integral homology 3–sphere along a graph clasper of the form of a θ graph (see Section 4.3, for the definition of a graph clasper). Further,

(11) $$\tau^{SO(3)}(M) = 1 + 6\,\lambda(M)\,(q-1) + (\text{higher terms}),$$

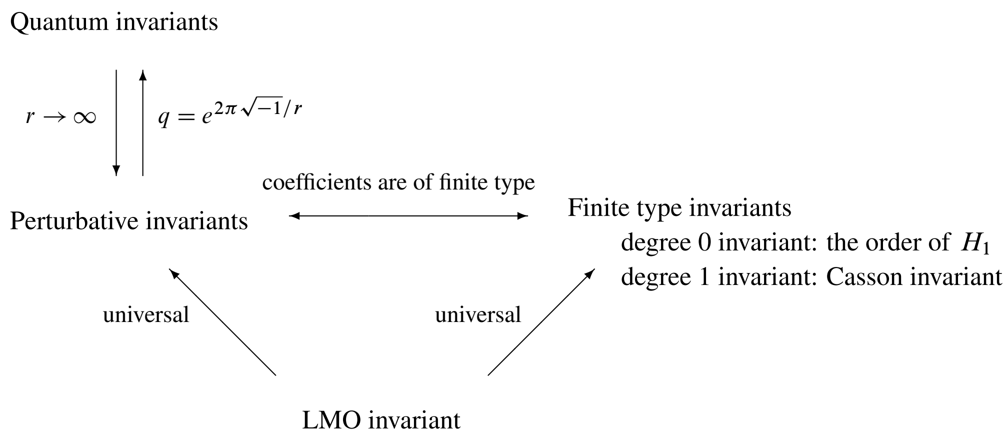


Figure 2: Quantum and related invariants of 3–manifolds

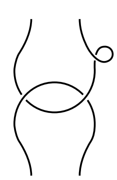
and hence,

$$\tau_r^{SO(3)}(M) \equiv 1 + 6\,\lambda(M)\,(\zeta - 1) \quad \text{modulo } (\zeta-1)^2 \text{ in } \mathbb{Z}[\zeta] \tag{12}$$

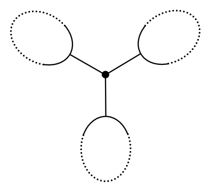
for any prime $r \geq 5$. In particular, $\big(\lambda(M) \text{ modulo } r\big) \in \mathbb{Z}/r\mathbb{Z}$ is determined by $\tau_r^{SO(3)}(M)$ for any prime $r \geq 5$, as shown in [29; 30].

Corresponding to the invariants shown in Figure 2, we consider equivariant invariants of the infinite cyclic covers of knot complements, as shown in Figure 3.

Corresponding to a quantum invariant of 3–manifolds defined from a modular category $\{V_i\}_{i\in I}$, an invariant $T^{V_m}(K)$ of a knot K is defined as an equivariant version of the quantum invariant for the infinite cyclic cover of the complement of K, as in [36]. Roughly speaking, $T^{V_m}(K)$ is defined to be the characteristic polynomial of the quantum invariant of a 3–cobordism obtained from $S^3 - K$ by cutting it along a Seifert surface whose boundary is associated with V_m. To be precise, when a link $K \cup L$ is a surgery presentation of a knot K_L as in (6), $T^{V_m}(K_L)$ is defined to be the characteristic polynomial of a matrix whose entries are quantum invariants of a tangle $\hat{L}$, where $\hat{L}$ is a tangle obtained from L by cutting along a disk bounded by K; for details, see [36]. For example, for $K \cup L$ shown in (6):

$\hat{L} =$ 

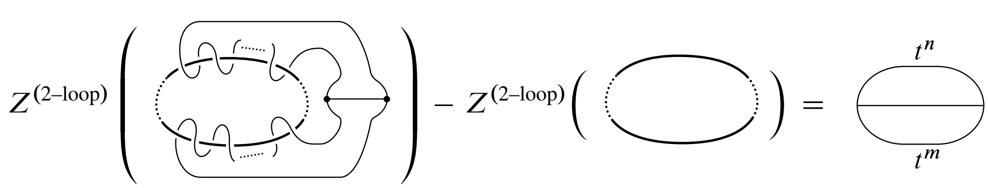
Corresponding to finite type invariants of 3–manifolds, loop finite type invariants of pairs of integral homology 3–spheres and knots are defined as follows. Consider pairs (M, K) such that M is an integral homology 3–sphere and K is an oriented knot in M, and consider a move between two pairs (M, K) and (M', K') such that (M', K') is obtained from (M, K) by surgery on a Y graph embedded in $M - K$ whose leaves have linking number zero with K, where a Y graph is a graph clasper of the form:



Loop finite type invariants of such pairs are defined similarly as a definition of Vassiliev invariants using this move instead of a crossing change; for details, see [11][5] (see also [32]). The weight systems of loop finite type invariants are based on the grading of Jacobi diagrams given by the loop-degree, where the *loop-degree* of a Jacobi diagram on S^1 is defined to be half of the number given by the number of trivalent vertices minus the number of univalent vertices of the uni-trivalent graph of the Jacobi diagram, ie, an n–loop diagram is a diagram of loop-degree $n-1$. The loop expansion of the Kontsevich invariant (the rational Z invariant) is universal among loop finite type invariants [11]. Further, the sl_2 reduction of the loop expansion of the Kontsevich invariant gives the loop expansion of the colored Jones polynomials.

See Figure 3 for relations among these invariants, corresponding to relations shown in Figure 2.

From the viewpoint that the 2–loop polynomial is an equivariant Casson invariant, we can expect some relations between the 2–loop polynomial and invariants shown in Figure 3, corresponding to the relations between Casson invariant and invariants in Figure 2 mentioned before. Corresponding to the clasper surgery formula (10), the 2–loop part of the Kontsevich invariant has a clasper surgery formula



as shown in [11], where the left picture implies a surgery on a knot along a graph clasper of the form of a θ graph whose upper and lower loops have linking numbers n

[5] The definition of loop finite type invariants also appears in the September 1999 version of [20].

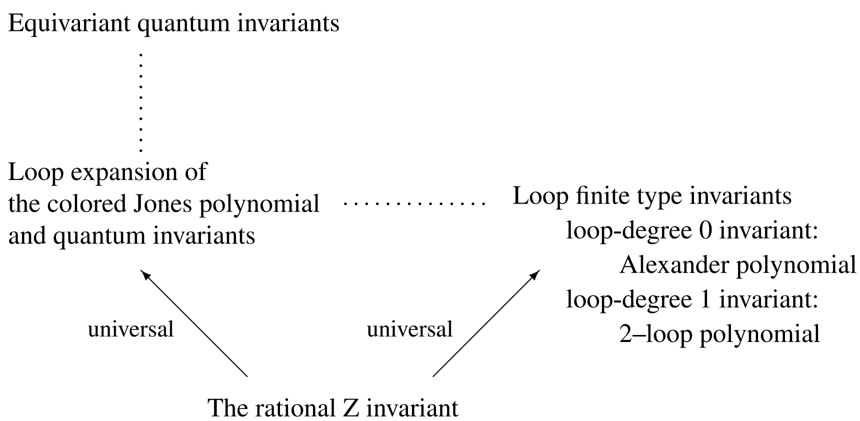


Figure 3: Equivariant invariants of the infinite cyclic covers of knot complements corresponding to the invariants shown in Figure 2

and m with the knot. Further, the 2–loop polynomial has other clasper surgery formulas shown in Section 4.3. Corresponding to the relation (11) between Casson invariant and the perturbative invariant, we have a relation between the reduced 2–loop polynomial and the 2–loop part of the loop expansion of the colored Jones polynomials as in [35]. Corresponding to the relation (12) between Casson invariant and the quantum invariant, we can expect that there would be some relations between the 2–loop polynomial and equivariant quantum invariants, though such relations are not formulated yet so far. Thus, the 2–loop polynomial would have a central role in the study of these invariants.

2 The 2–loop polynomial calculated from surgery presentations

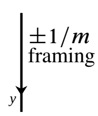
In this section, we calculate the 2–loop polynomial of knots from their surgery presentations. We give concrete presentations of the 2–loop polynomial for knots in terms of the Kontsevich invariant of their surgery presentations of surgery along knots in Section 2.1. Further, we calculate the 2–loop polynomial for the $(4nm+1, 2n)$ two-bridge knot, the Kinoshita–Terasaka knot, and the Conway knot in Section 2.2, Section 2.3, and Section 2.4, respectively.

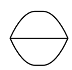
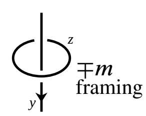
2.1 Surgery presentations of surgery along knots

In this section, we give a concrete presentation of the 2–loop polynomial of a knot in terms of the Kontsevich invariant of a surgery presentation of the knot whose surgery is along a knot.

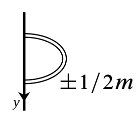
Consider a link $K \cup L$, such that K is isotopic to the trivial knot with 0 framing, L is a knot with $\pm 1/m$ framing, and the linking number of K and L is equal to 0. We denote by K_L the knot obtained from K by surgery along L. The link $K \cup L$ is a surgery presentation of the knot K_L. The aim of this section is to give a concrete presentation of the 2–loop polynomial of K_L in terms of the Kontsevich invariant of $K \cup L$.

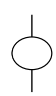
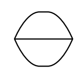
An idea of [5] to compute a rational surgery is to define the Kontsevich invariant of a string with a rational framing to be the Kontsevich invariant of a Hopf chain, for example, as follows,

$Z\Big($ 

$\Big) = \Big(1 \pm \frac{1}{16}$  $\Big)^{-1} \int \chi_z^{-1} \check{Z}\Big($ 

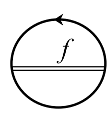
$\Big) dz$

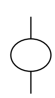
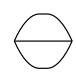
$\equiv$ 

$\times \Big(1 + \frac{1/m^2 - 1}{48}$  $\pm \frac{(m-1)(m-2)}{48m}$  $\Big).$

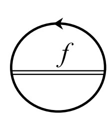
For detailed and general formulas, see [5]. Hence, by putting the Kontsevich invariant of a string of a rational framing in this way, we can also apply the rational version of the Aarhus integral.

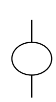
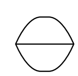
Let us calculate the 2–loop polynomial of K_L in a simple case that $\chi_\hbar^{-1} Z(K \cup L)$ is given by

$\chi_\hbar^{-1} Z(K \cup L) \equiv$ 

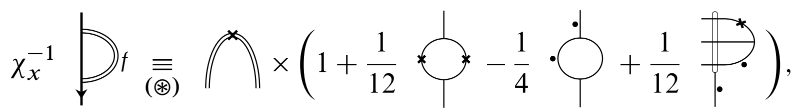
$\times \Big(1 + \frac{1}{48m^2}$  $\pm \frac{(m-1)(m-2)}{48m}$  $\Big),$

for some polynomial $f(e^\hbar)$ in $e^\hbar, e^{-\hbar}$, satisfying that $f(1) = \pm 1/2m$, since L has the framing $\pm 1/m$. The reason why we add the last factor in the above formula is that this term is natural in the sense that, if f was a scalar in $\frac{1}{2}\mathbb{Z}$, the above formula presents, modulo "$\equiv$", the Kontsevich invariant of the trivial knot with $2f$ framing. From the definition of $\check{Z}$,

$\chi_\hbar^{-1} \check{Z}(K \cup L) \equiv$ 

$\times \Big(1 + \frac{1/m^2 + 1}{48}$  $\pm \frac{(m-1)(m-2)}{48m}$  $\Big).$

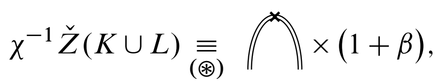
Further, by Lemma 5.13,



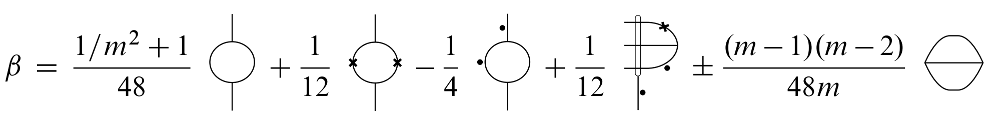
where we introduce two markings by

(13)

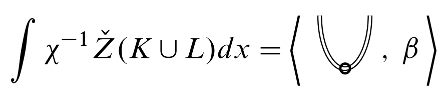
Hence,



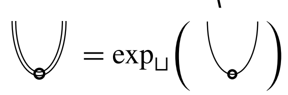
where



The rational version of the Aarhus integral is calculated as follows,

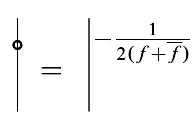


where we put

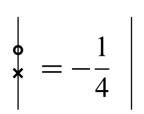


Here, we define the marking of a circle by

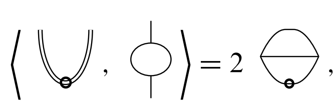
(14)

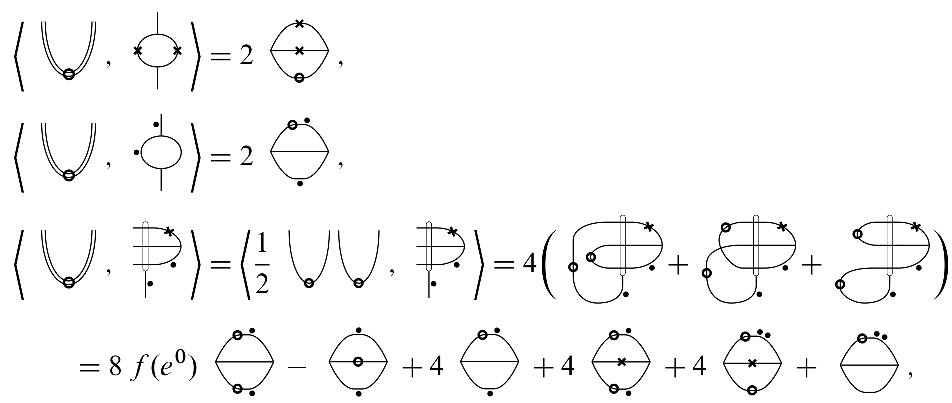


for f and its conjugate $\overline{f}$ defined by $\overline{f}(e^{\hbar}) = f(e^{-\hbar})$. In particular, we have that

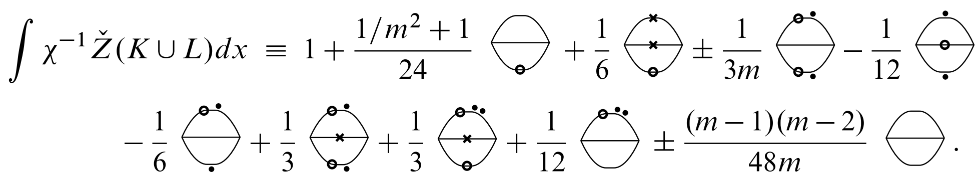


For the diagram of β, we have that



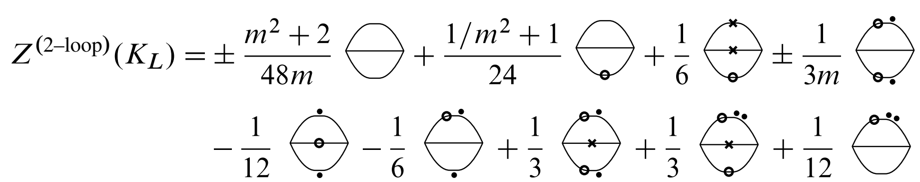


where $f(e^0) = \pm 1/2m$. Hence,



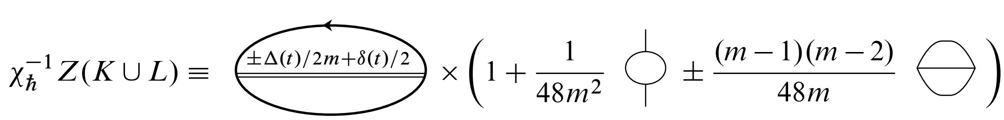
Further, as we explained in Section 1.2, we have, from (9), that

(15)



Therefore, from the definition of the 2–loop polynomial, we obtain the following proposition, by putting $\Delta(t) = \pm m\left(f(t) + f(t^{-1})\right)$ and $\delta(t) = f(t) - f(t^{-1})$.

Proposition 2.1 *Let $K \cup L$ and K_L be as above, satisfying that*



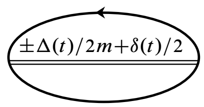
for polynomials $\Delta(t)$ and $\delta(t)$ in t, t^{-1} satisfying that $\Delta(1) = 1$, $\Delta(t^{-1}) = \Delta(t)$ and $\delta(t^{-1}) = -\delta(t)$. Then, the 2–loop polynomial of K_L is presented, modulo the *equivalence (5), by*

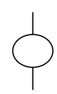
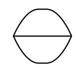
$$\Theta_{K_L}(t_1,t_2) \sim \pm\frac{1}{4m}\Delta(t_1)\Delta(t_2)\big((m^2+2)\Delta(t_3)-\Delta(t_1)\Delta(t_2)-m^2-1\big) \mp\frac{m\,\delta(t_1)\delta(t_2)}{4}\big(\Delta(t_3)+\tfrac{1}{2}\Delta(t_1)\Delta(t_2)+\Delta(t_1)\Delta(t_3)+\tfrac{1}{2}\Delta(t_3)^2\big) \pm\frac{m\,\delta(t_1)^2}{8}\Delta(t_2)\,\big(\Delta(t_2)-\Delta(t_3)\big), \tag{16}$$

and its reduced 2–loop polynomial is presented by

$$\widehat{\Theta}_{K_L}(t) = \mp\frac{1}{12m}\Delta(t)\,\big(\Delta(t)^3-(2m^2+3)\Delta(t)+2m^2+2\big)\pm\frac{m\,\delta(t)^2}{12}\big(\Delta(t)^2+2\big).$$

Remark 2.2 More generally, if $\chi_\hbar^{-1}Z(K\cup L)$ is given by the following form,

$$\chi_\hbar^{-1}Z(K\cup L)\equiv$$


$$\times\Big(1+\frac{1}{48m^2}$$

$$\pm\frac{(m-1)(m-2)}{48m}$$

$$+\beta\Big),$$

where β is a linear sum of Jacobi diagrams each of which has two trivalent vertices, then we can show in a similar way as above that $Z^{(2\text{–loop})}(K_L)$ is presented by

$$Z^{(2\text{–loop})}(K_L) = \big(\text{the right-hand side of (15)}\big)+\Big\langle$$

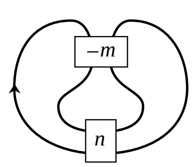
$$,\ \beta\Big\rangle,$$

and the 2–loop polynomial $\Theta_{K_L}(t_1,t_2)$ is given by the sum of the right-hand side of (16) and the polynomial corresponding to the last term of the above formula.

2.2 The 2–loop polynomial of the $(4nm+1, 2n)$ two-bridge knot

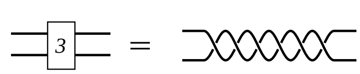
In this section, we calculate the 2–loop polynomial of the $(4nm+1, 2n)$ two-bridge knot as an application of Proposition 2.1 (and Remark 2.2).

The $(4nm+1, 2n)$ *two-bridge knot* is the knot given by



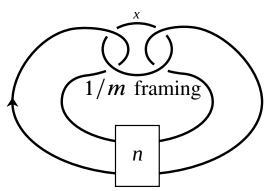
,

where we mean k full twists by a boxed "k"; for example,

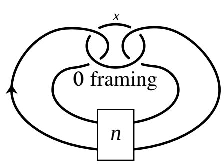


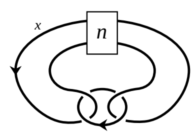
.

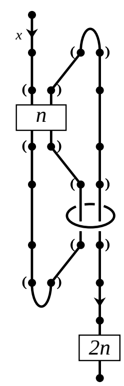
The above knot is obtained from the following link $K \cup L$ by $1/m$ surgery along the component L labeled by x:

$K \cup L =$ 

Let $K \cup L_0$ be $K \cup L$ putting the framing of L to be 0. Then,

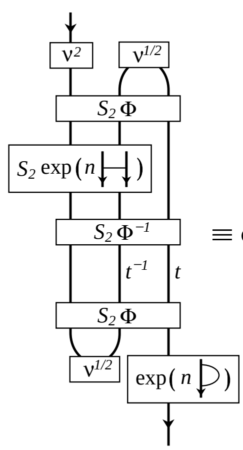
$K \cup L_0 =$ 

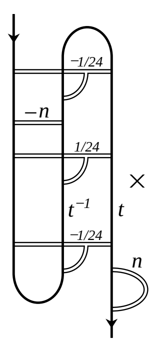
$=$ 

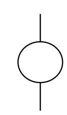
$=$ closure of 

.

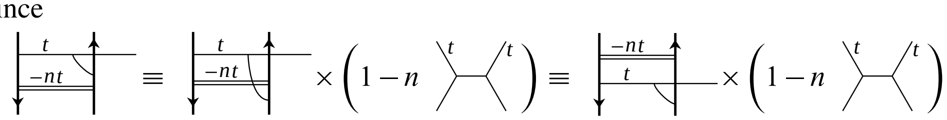
Hence,

$\chi_{\hbar}^{-1}\check{Z}(K \cup L_0) =$ cl. of 

$\equiv$ cl. of 

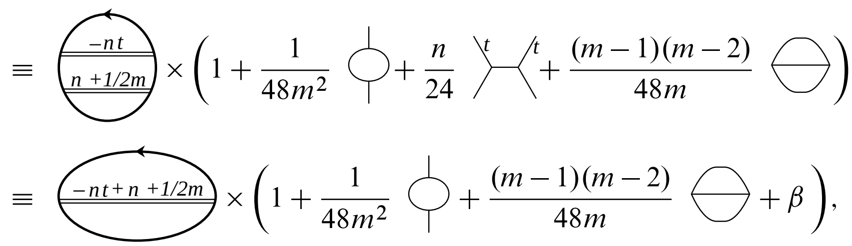
$\times \Big(1 + \dfrac{1}{16}$  $\Big)$.

Since

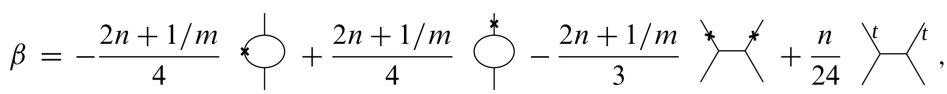


we have, by Lemma 5.8, that

$$\chi_{\hbar}^{-1} Z(K \cup L)$$



where

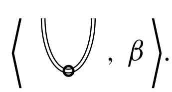


under the notation (13), putting $f = -nt + n + \frac{1}{2m}$.

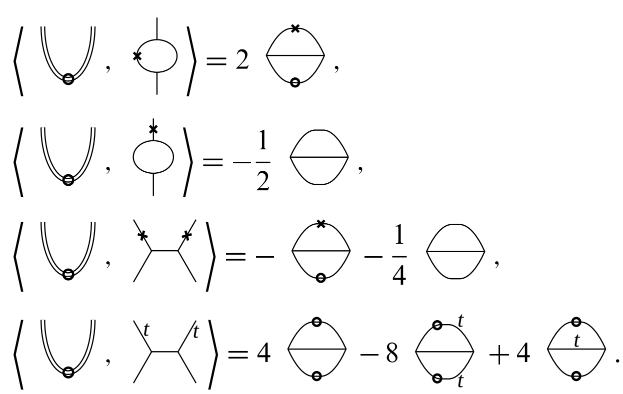
We apply Proposition 2.1 (and Remark 2.2) to the above formula of $\chi_{\hbar}^{-1} Z(K \cup L)$, putting $\Delta(t) = 1 - nm(t + t^{-1} - 2)$ and $\delta(t) = -n(t - t^{-1})$, to obtain

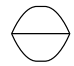
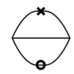
$$
\begin{aligned}
(17)\quad \Theta_{K_L}(t_1, t_2) \sim{} & \frac{1}{4m} \Delta(t_1)\Delta(t_2)\big((m^2+2)\Delta(t_3) - \Delta(t_1)\Delta(t_2) - m^2 - 1\big) \\
& - \frac{m\,\delta(t_1)\delta(t_2)}{4}\big(\Delta(t_3) + \tfrac{1}{2}\Delta(t_1)\Delta(t_2) + \Delta(t_1)\Delta(t_3) + \tfrac{1}{2}\Delta(t_3)^2\big) \\
& + \frac{m\,\delta(t_1)^2}{8}\Delta(t_2)\big(\Delta(t_2) - \Delta(t_3)\big)
\end{aligned}
$$

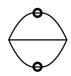
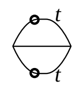
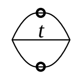
where the additional part is the polynomial corresponding to



Here, the marking of a circle is defined in (14). We calculate this for each diagram of β as



Hence $\left\langle\right.$  $, \ \beta \left.\right\rangle = -\dfrac{2n+1/m}{24}$  $-\dfrac{2n+1/m}{6}$ 

$+\dfrac{n}{6}\Bigg($  -2  $+$  $\Bigg).$

Since the markings of a crossing and a circle stand for the labels of $\Delta(t)/2m$ and $-m/2\Delta(t)$ respectively, from the definition of the 2–loop polynomial, the additional part of (17) is presented, modulo the equivalence (5), by

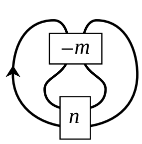
$$\frac{2n+1/m}{2}\Delta(t_1)\Delta(t_2)\big(\Delta(t_1)-\Delta(t_3)\big)+\frac{nm^2}{2}\Delta(t_1)\big(1-2t_2t_3^{-1}+t_1\big).$$

Therefore,

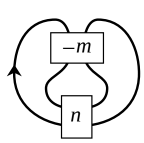
$$\begin{aligned}\Theta_{K_L}(t_1,t_2) \ \sim \ & \frac{1}{4m}\Delta(t_1)\Delta(t_2)\big((m^2+2)\Delta(t_3)-\Delta(t_1)\Delta(t_2)-m^2-1\big)\\ &-\frac{m\,\delta(t_1)\delta(t_2)}{4}\big(\Delta(t_3)+\tfrac{1}{2}\Delta(t_1)\Delta(t_2)+\Delta(t_1)\Delta(t_3)+\tfrac{1}{2}\Delta(t_3)^2\big)\\ &+\frac{m\,\delta(t_1)^2}{8}\Delta(t_2)\big(\Delta(t_2)-\Delta(t_3)\big)\\ &+\frac{2n+1/m}{2}\Delta(t_1)\Delta(t_2)\big(\Delta(t_1)-\Delta(t_3)\big)+\frac{nm^2}{2}\Delta(t_1)\big(1-2t_2t_3^{-1}+t_1\big).\end{aligned}$$

By substituting $\Delta(t)=1-nm(t+t^{-1}-2)$ and $\delta(t)=-n(t-t^{-1})$ and by symmetrizing the formula, we obtain the following proposition.

Proposition 2.3 *The 2–loop polynomial of the* $(4nm+1,2n)$ *two-bridge knot is presented by*

$\Theta\Bigg($  $\Bigg) = \dfrac{nm(n-m)}{2}\Big(nm(nm+1)T_{1,0}+\dfrac{nm(nm-1)}{2}\cdot T_{2,0} - \dfrac{3n^2m^2-nm-1}{3}\cdot T_{2,1}\Big),$

where $T_{n,m}$*'s are defined in* (1)*, and its reduced 2–loop polynomial is presented by*

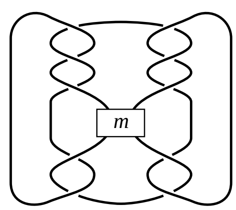
$\widehat{\Theta}\Bigg($  $\Bigg) = \dfrac{nm(n-m)}{6}\big(6-(2nm-1)(t+t^{-1}-2)\big).$

2.3 The 2–loop polynomial of the Kinoshita–Terasaka knot

In this section, we calculate the 2–loop polynomial for the Kinoshita–Terasaka knot, from a surgery presentation of the knot. Unlike the case of the previous sections, we need, not the rational version, but the original version of the Aarhus integral to calculate the surgery, since the equivariant linking number of the surgery presentation is a scalar in this case.

The *Kinoshita–Terasaka knot* K_m^{KT} [16] is the knot given by

$K_m^{\mathrm{KT}} =$

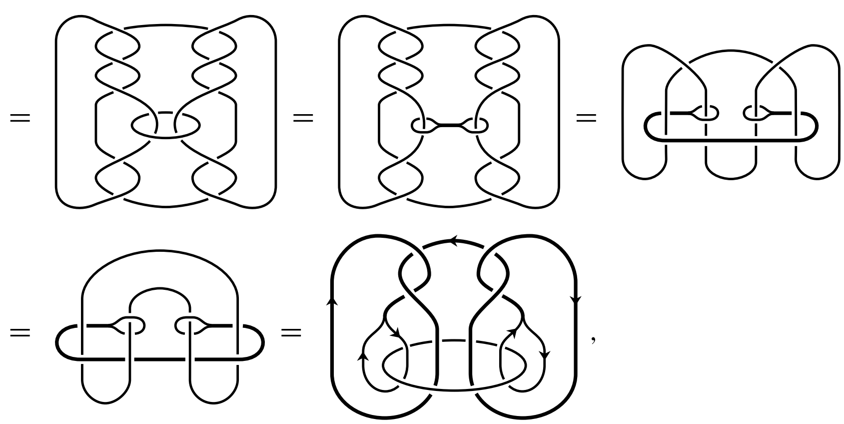


.

The 2–loop polynomial $\Theta(K_m^{\mathrm{KT}})$ of K_m^{KT} is a polynomial in m, as we see later. Note, in advance, that this polynomial consists only of terms of odd power of m, since $\Theta(K_{-m}^{\mathrm{KT}}) = \Theta(\overline{K_m^{\mathrm{KT}}}) = -\Theta(K_m^{\mathrm{KT}})$, where $\overline{K_m^{\mathrm{KT}}}$ denotes the mirror image of K_m^{KT}.

The Kinoshita–Terasaka knot K_m^{KT} is obtained from the following link $K \cup L_0$ by $-1/m$ surgery along the middle component L_0,

$K \cup L_0 =$

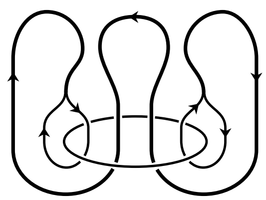


,

where in these pictures we mean by a thick line 2 parallel copies of the line.

We calculate the Kontsevich invariant of $K \cup L_0$ comparing it to the Kontsevich invariant of the following link,

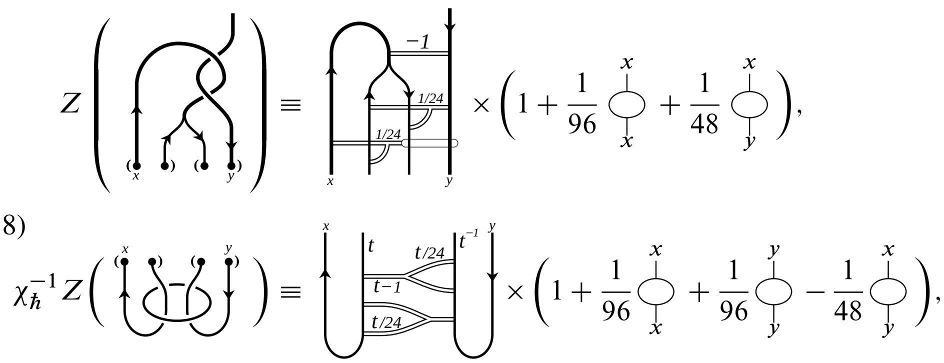
$K \cup L_0' =$



,

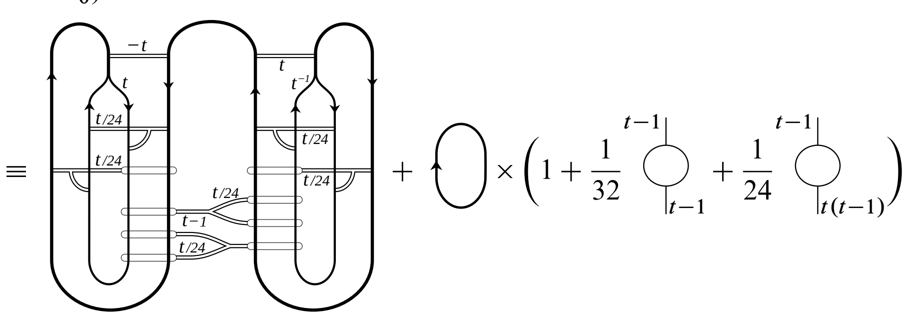
which is isotopic to the trivial 2–component link. Since

(18)



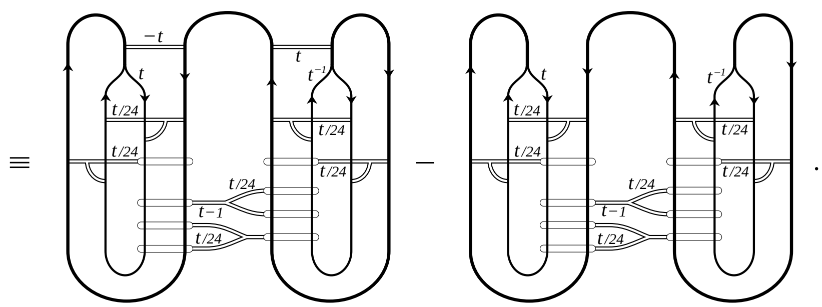
the Kontsevich invariant of $K \cup L_0$ is presented by

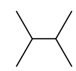
$\chi_\hbar^{-1} Z(K \cup L_0)$

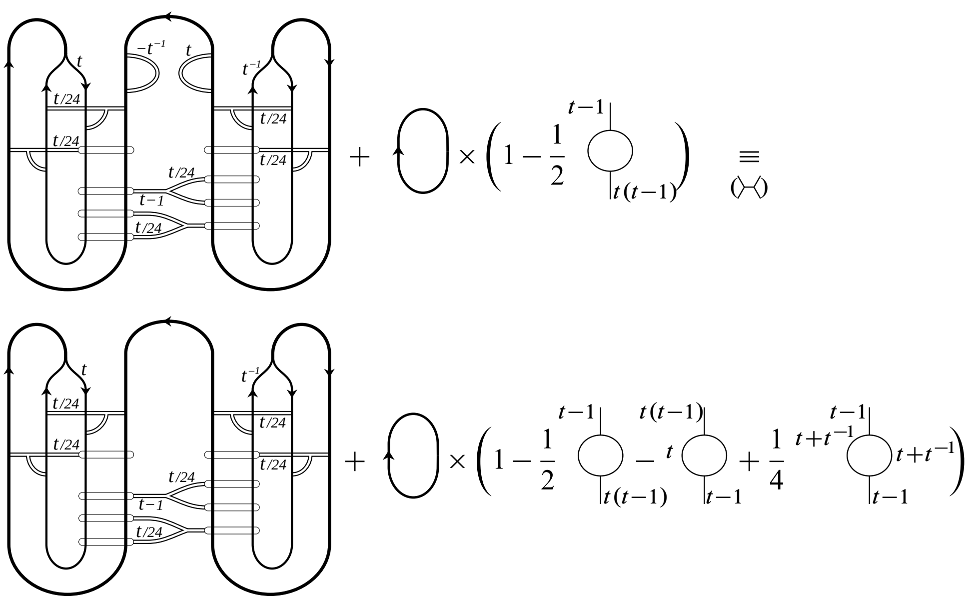


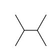
Hence, the difference between the Kontsevich invariant of $K \cup L_0$ and $K \cup L_0'$ is given by

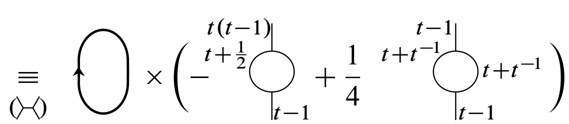
$$\chi_\hbar^{-1} Z(K \cup L_0) - \chi_\hbar^{-1} Z(K \cup L_0')$$



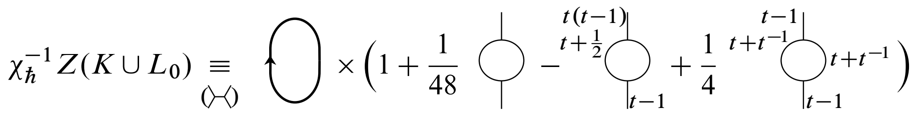
By Lemma 5.3, the first diagram is equivalent, modulo diagrams of the form  , to



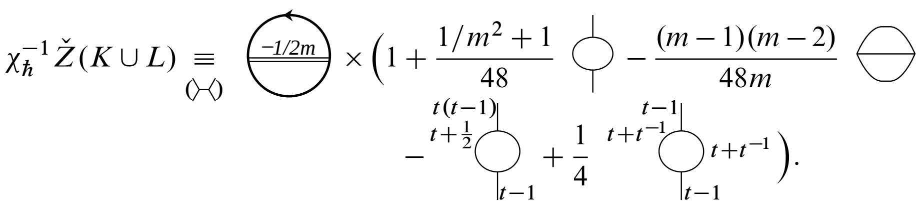
where we obtain the equivalence by Lemma 5.9, noting that diagrams of the form  contribute to $\Theta(K_m^{\mathrm{KT}})$ by terms of m^2, though such terms vanish in $\Theta(K_m^{\mathrm{KT}})$ as we mentioned before. It follows from the above formulas that

$$\chi_\hbar^{-1} Z(K \cup L_0) - \chi_\hbar^{-1} Z(K \cup L_0')$$ 

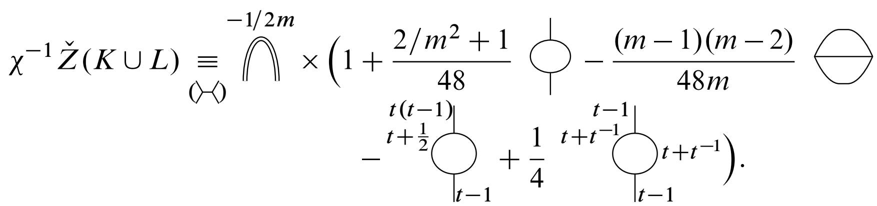
Since $K \cup L_0'$ is isotopic to the trivial link, we have that



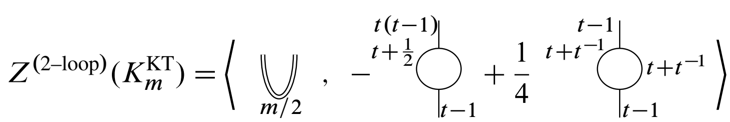
Let L be L_0 with $-1/m$ framing. Then, similarly as in Section 2.1,

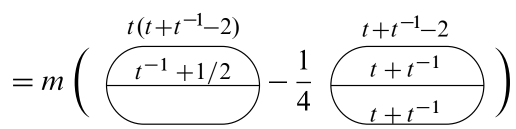


Further, from the definition of χ,



The 2–loop part of the Kontsevich invariant of K_m^{KT} is given by the Aarhus integral, as follows,





Hence, by definition, the 2–loop polynomial is presented, modulo the equivalence (5), by

$$\Theta_{K_m^{\mathrm{KT}}}(t_1,t_2) \sim 12m\Big(t_1(t_1+t_1^{-1}-2)\big(t_2^{-1}+\tfrac{1}{2}\big) - \tfrac{1}{4}(t_1+t_1^{-1}-2)(t_2+t_2^{-1})(t_1t_2+t_1^{-1}t_2^{-1})\Big).$$

By symmetrizing this polynomial, we obtain the following proposition.

Proposition 2.4 *The 2–loop polynomial of the Kinoshita–Terasaka knot K_m^{KT} is presented by*

$$\Theta_{K_m^{\mathrm{KT}}}(t_1,t_2) = m\,(2T_{1,0} - 2T_{2,0} - 2T_{2,1} + T_{3,1}),$$

where $T_{n,m}$'s are defined in (1), and its reduced 2–loop polynomial is presented by

$$\widehat{\Theta}_{K_m^{\mathrm{KT}}}(t) = 2m\,(t^2 + t^{-2}).$$

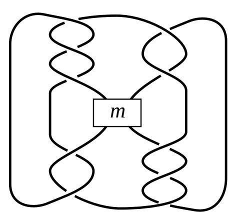
We verify the proposition in Example 4.21 by using a surgery formula for the 2–loop polynomial. We can also verify that the special values $\widehat{\Theta}_{K_m^{\mathrm{KT}}}(1) = \widehat{\Theta}_{K_m^{\mathrm{KT}}}(-1) = 4m$ satisfy Proposition 1.1, noting that

$$V_{K_m^{\mathrm{KT}}}(t) = 1 + (t^{2m} - 1)(t + t^{-1} + 1)(t + t^{-1})(t + t^{-1} - 1)(t + t^{-1} - 2).$$

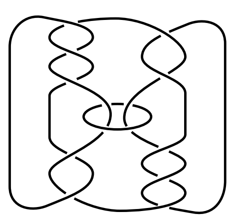
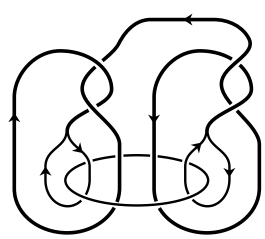
2.4 The 2–loop polynomial of the Conway knot

In this section, we calculate the 2–loop polynomial for the Conway knot, from a surgery presentation of the knot. Similarly as the case of the Kinoshita–Terasaka knot in the previous section, we need, not the rational version, but the original version of the Aarhus integral to calculate the surgery in this case.

The *Conway knot* K_m^{C} (see [24]) is the knot given by

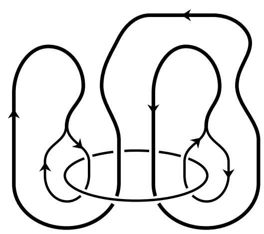
$K_m^{\mathrm{C}} =$  ,

which is obtained from the Kinoshita–Terasaka knot by mutation. Similarly to the case of the Kinoshita–Terasaka knot, the Conway knot is obtained from the following link $K \cup L_0$ by $-1/m$ surgery along the middle component L_0,

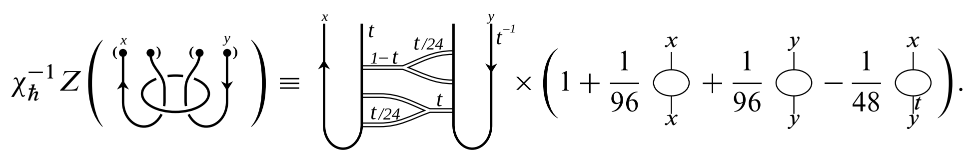
$K \cup L_0 =$  $=$  ,

where we mean by a thick line 2 parallel copies of the line.

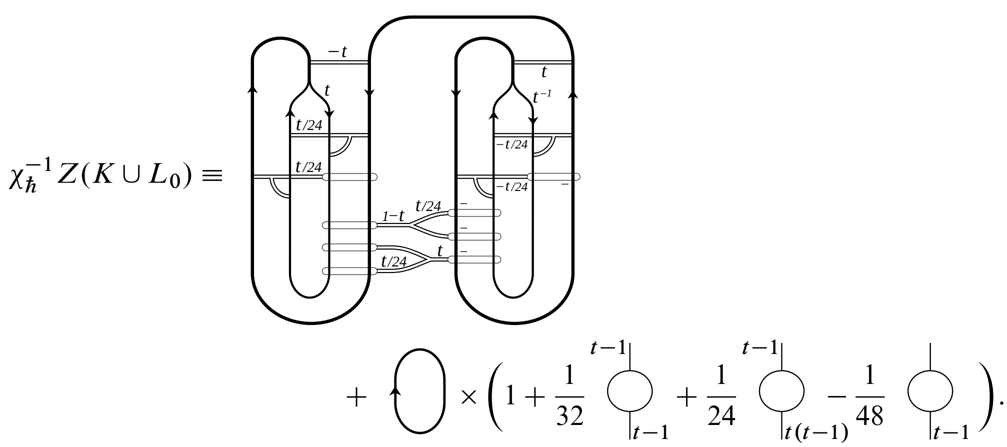
We calculate the Kontsevich invariant of $K \cup L_0$ comparing it to the Kontsevich invariant of the link

$K \cup L_0' =$ 

which is isotopic to the trivial 2–component link. In a similar way to the case of the Kinoshita–Terasaka knot, we calculate $\chi_\hbar^{-1} Z(K \cup L_0)$, where, instead of (18), we use an equivalent form of it,



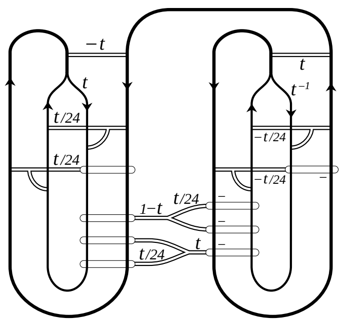
It follows that



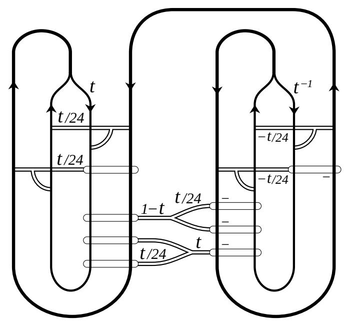
Hence

$$\chi_\hbar^{-1} Z(K\cup L_0) - \chi_\hbar^{-1} Z(K\cup L_0')$$

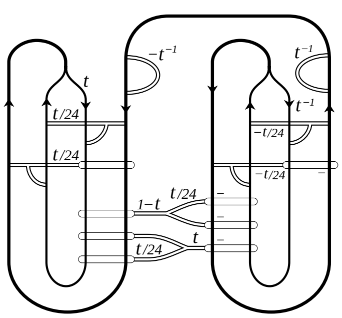
$\equiv$



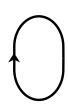
$-$



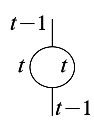
By Lemma 5.3, the first diagram is equivalent, modulo diagrams of the form $\rangle\!\langle$, to



$+$

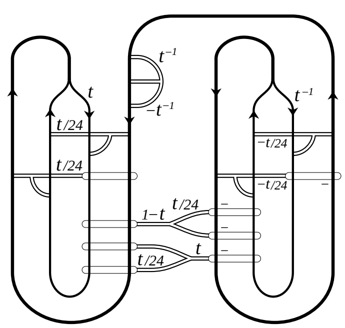


$\times \left(1-\frac{1}{2}\right.$



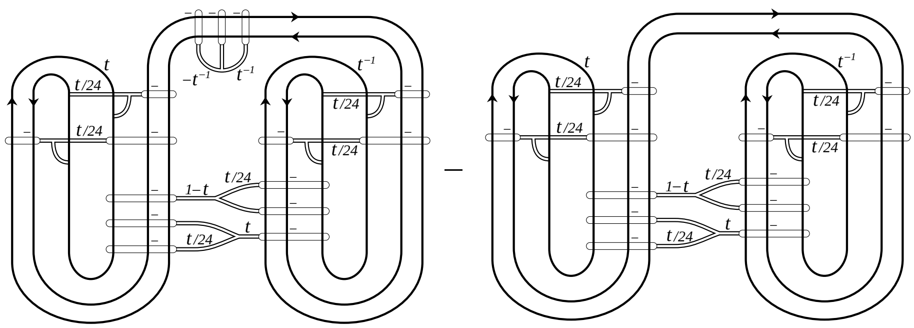
$\left.\right)$

$\underset{(\rangle\!\langle)}{\equiv}$

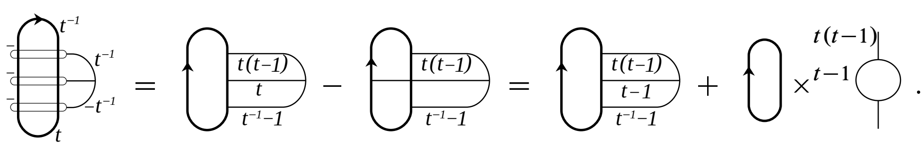


where the equivalence is obtained by Lemma 5.10. Hence

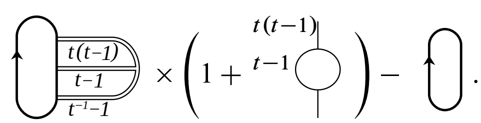
$$\chi_\hbar^{-1} Z(K \cup L_0) - \chi_\hbar^{-1} Z(K \cup L_0') \underset{(\rangle\!\!-\!\!\langle)}{\equiv}$$



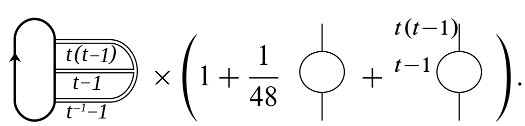
Since the second diagram is related to the trivial link, the diagrams of the form  in the second diagram cancel with each other. Hence, the degree 2 part of the first diagram is calculated, as follows,



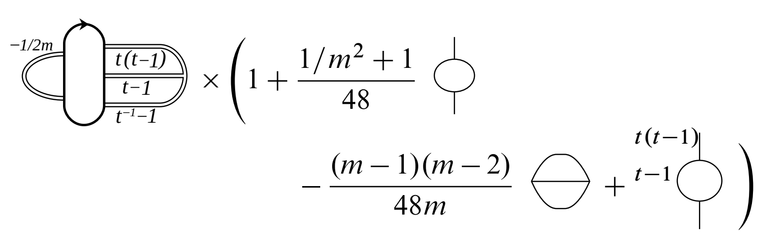
Therefore,

$$\chi_\hbar^{-1} Z(K \cup L_0) - \chi_\hbar^{-1} Z(K \cup L_0') \underset{(\rangle\!\!-\!\!\langle)}{\equiv}$$ 

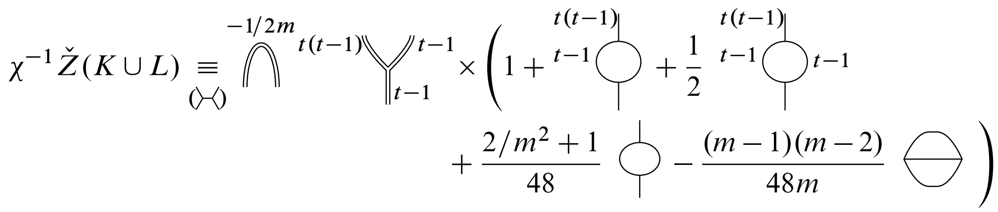
Since $K \cup L_0'$ is isotopic to the trivial link, we have that

$$\chi_\hbar^{-1} Z(K \cup L_0) \underset{(\rangle\!\!-\!\!\langle)}{\equiv}$$ 

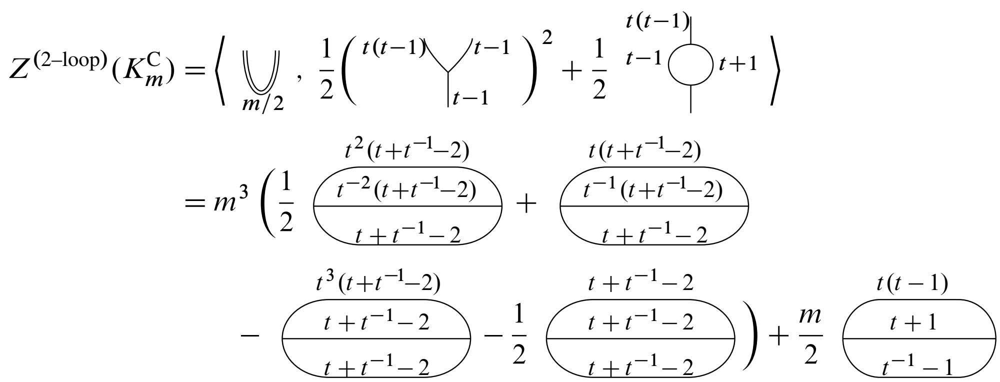
Let L be L_0 with $-1/m$ framing. Then, similarly as in Section 2.1,

$$\chi_\hbar^{-1} \check{Z}(K \cup L) \underset{(\rangle\!\!-\!\!\langle)}{\equiv}$$ 

Further, from the definition of χ,



The 2–loop part of the Kontsevich invariant of K_m^{C} is given by the Aarhus integral, as follows,



Hence, by definition, the 2–loop polynomial is presented, modulo the equivalence (5), by

$$\begin{aligned}
&\Theta_{K_m^{\mathrm{C}}}(t_1,t_2)\\
&\quad\sim\ 12m^3(t_1+t_1^{-1}-2)(t_2+t_2^{-1}-2)(t_1t_2+t_1^{-1}t_2^{-1}-2)\big(\tfrac{1}{2}t_1^2t_2^{-2}+t_1t_2^{-1}-t_1^3-\tfrac{1}{2}\big)\\
&\qquad\qquad + 6m\,t_1(t_1-1)(t_2+1)(t_1t_2-1).
\end{aligned}$$

By symmetrizing this polynomial, we obtain the following proposition.

Proposition 2.5 *The 2–loop polynomial of the Conway knot K_m^{C} is presented by*

$$\begin{aligned}
\Theta_{K_m^{\mathrm{C}}}(t_1,t_2) &= m\,(2T_{1,0}-2T_{2,0}-2T_{2,1}+T_{3,1})\\
&\quad + m^3(t_1t_2^{-1}+t_1^{-1}t_2-2)(t_1^2t_2+t_1^{-2}t_2^{-1}-2)(t_1t_2^2+t_1^{-1}t_2^{-2}-2)\\
&\qquad \times(t_1+t_1^{-1}-2)(t_2+t_2^{-1}-2)(t_1t_2+t_1^{-1}t_2^{-1}-2)\\
&= m\,(2T_{1,0}-2T_{2,0}-2T_{2,1}+T_{3,1})\\
&\quad +2m^3\big(T_{1,0}+T_{2,1}-T_{3,1}+T_{4,0}+T_{4,2}+T_{5,0}-T_{5,1}-\tfrac{1}{2}T_{6,2}+T_{6,3}\big),
\end{aligned}$$

where $T_{n,m}$'s are defined in (1), and its reduced 2–loop polynomial is presented by

$$\widehat{\Theta}_{K_m^{\mathrm{C}}}(t) = 2m\,(t^2 + t^{-2}).$$

We can verify that the special values $\widehat{\Theta}_{K_m^{\mathrm{C}}}(1) = \widehat{\Theta}_{K_m^{\mathrm{C}}}(-1) = 4m$ satisfy Proposition 1.1, since the Conway knot and the Kinoshita–Terasaka knot are mutant and have the same Jones polynomial. Proposition 2.4 and Proposition 2.5 show that the 2–loop polynomial is sensitive to mutation, unlike the Alexander, Jones, HOMFLY and Kauffman polynomials.

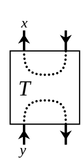
3 The 2–loop polynomial for knots of genus 1

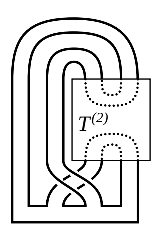
In this section, we give explicit presentations of the 2–loop polynomial for knots of genus 1 in Theorem 3.1 and Theorem 3.7. We give the presentations in Section 3.1 and prove Theorem 3.1 in Section 3.2–Section 3.4.

3.1 Presentation of the 2–loop polynomial for knots of genus 1

In this section, we give explicit presentations of the 2–loop polynomial for knots of genus 1, in terms of finite type invariants of a tangle which gives a part of a spine of a Seifert surface of the knot in Theorem 3.1, and in terms of derivatives of the Jones polynomial in Theorem 3.7.

Let T be a 2–component framed tangle, and let K_T be the knot obtained from T as follows,

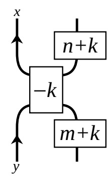
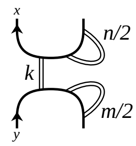
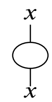
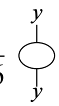
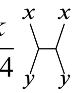
(19) $T =$ 

, $K_T =$ 

,

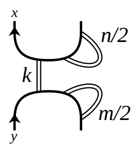
where dotted lines in the picture of T imply strands possibly knotted and linked in some fashion, and $T^{(2)}$ denotes the tangle obtained from T by replacing each component of T with 2 parallel copies of it. Any knot of genus 1 can be presented by K_T for some T.

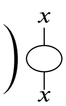
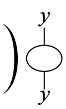
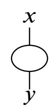
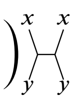
In particular, when T is of the following form (recalling that a boxed "n" implies the n full twists), its Kontsevich invariant is presented by

(20) $Z\Bigg($  $\Bigg) \equiv$  $\times \Big(1 + \frac{1}{96}$  $+ \frac{1}{96}$  $- \frac{k}{24}$  $\Big),$

where we obtain the formula in a similar way as in Section 2.1.

Theorem 3.1 *Let T and K_T be as in (19). We present $Z(T)$ by*

$Z(T) \equiv$ 

$\times \Bigg(1 + \Big(\frac{v_2^{xx}}{2} + \frac{1}{96} \Big)$  $+ \Big(\frac{v_2^{yy}}{2} + \frac{1}{96} \Big)$  $+ \frac{v_2^{xy}}{2}$  $+ \Big(-\frac{k}{24} + v_3 \Big)$  $\Bigg)$

with some integers $n, m, k, v_2^{xx}, v_2^{yy}, v_2^{xy}, v_3$. Then, the 2–loop polynomial of K_T is presented by

$$\begin{aligned}\Theta_{K_T}(t_1,t_2) = &\big((n+m)(d-\tfrac{nm}{2}) - k(k+\tfrac{1}{2})(k+1) + 12v_3\big) \\ &\times \big(-d(d-1)T_{1,0} - \tfrac{1}{2}d(d+1)T_{2,0} + \big(d^2 + \tfrac{1}{3}d - \tfrac{1}{3}\big)T_{2,1}\big) \\ &+ 12\big(-mv_2^{xx} - nv_2^{yy} + (k+\tfrac{1}{2})v_2^{xy} + 3v_3\big) \\ &\times \big(\big(d^2 - d + \tfrac{1}{6}\big)T_{1,0} + \tfrac{1}{2}d^2T_{2,0} - \big(d^2 - \tfrac{1}{3}d\big)T_{2,1}\big),\end{aligned} \tag{21}$$

where we put $d = nm - k^2 - k$, and $T_{n,m}$'s are defined in (1).

We give a proof of the theorem in Section 3.2.

Remark 3.2 Rozansky [40] conjectured that $\text{degree}_{t_1}\big(\Theta_K(t_1,t_2)\big) \le 2g(K)$ for the genus $g(K)$ of K, which we prove in Theorem 4.7. In particular, we can verify this formula for knots of genus 1 concretely by Theorem 3.1; see also Table 1.

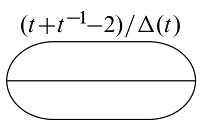
Remark 3.3 Rozansky [40] also conjectured that the 2–loop polynomial is a polynomial with integer coefficients. We show this for the knots of genus 1 by Theorem 3.1, as follows. Since the latter half of (21) has integer coefficients, it is sufficient to show that the former half has integer coefficients. Note that the first line of (21) is an integer. If d

is congruent to 1 modulo 3, then the second line of (21) has integer coefficients, and so does $\Theta(K_T)$. Suppose that d is not congruent to 1 modulo 3. Since $k(k+\frac{1}{2})(k+1)$ is divisible by 3, it is sufficient to show that $(n+m)(d-\frac{nm}{2})$ is divisible by 3. Suppose that $n+m$ is not divisible by 3. Then, nm is congruent to 0 or 1 modulo 3. If nm is congruent to 0, then d is congruent to 0. If nm is congruent to 1, then d is congruent to 2. In any case, $d-\frac{nm}{2}$ is divisible by 3. Therefore, $\Theta(K_T)$ has integer coefficients.

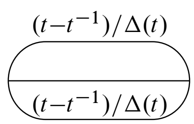
An alternative presentation of the formula of $\Theta(K_T)$ in Theorem 3.1 is that the 2–loop part of $\log_\sqcup \chi^{-1} Z(K_T)$ is given by

$$Z^{(2\text{–loop})}(K_T) = \Big(\frac{n+m}{12}\big(d-\frac{nm}{2}\big) - \frac{1}{12}k\big(k+\frac{1}{2}\big)(k+1) + v_3\Big)$$

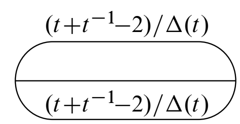
$$\times\Big(-\frac{3}{2}$$



$$+\frac{3}{4}$$

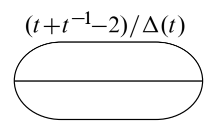


$$+\Big(d-\frac{1}{4}\Big)$$



$$\Big)$$

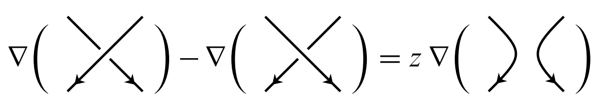
$$-\frac{1}{2}\Big(mv_2^{xx} + nv_2^{yy} - \Big(k+\frac{1}{2}\Big)v_2^{xy} - 3v_3\Big)$$



,

where we put $\Delta(t) = 1 + d(t+t^{-1}-2)$, which is equal to the Alexander polynomial of K_T.

Recall that the *Conway polynomial* $\nabla_L(z) \in \mathbb{Z}[z]$ (which we also denote by $\nabla(L)$) of an unframed oriented link L is defined by the skein relation



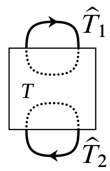
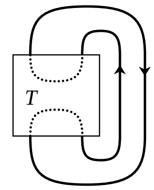
and the normalization $\nabla(\text{the trivial knot}) = 1$. Note that $\nabla_L(t^{1/2} - t^{-1/2}) = \Delta_K(t)$. The scalars in the formula of $Z(T)$ in Theorem 3.1 are elementarily calculated by the following proposition.

Proposition 3.4 *Under the assumption of Theorem 3.1,*

$$n = (\text{framing of } \widehat{T}_1), \qquad m = (\text{framing of } \widehat{T}_2), \qquad k = lk(\widehat{T}_1, \widehat{T}_2),$$

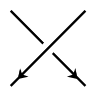
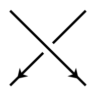
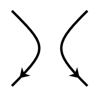
$$\nabla(\widehat{T}_1) = 1 - v_2^{xx}z^2 + O(z^4), \qquad \nabla(\widehat{T}_2) = 1 - v_2^{yy}z^2 + O(z^4),$$
$$\nabla(\widehat{T}_1 \cup \widehat{T}_2) = kz - 2v_3z^3 + O(z^5), \quad \nabla(\widehat{T}) = 1 - (v_2^{xx} + v_2^{yy} + v_2^{xy})z^2 + O(z^4),$$

where $\widehat{T}_1 \cup \widehat{T}_2$ and $\widehat{T}$ are framed links given by

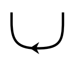
$$\widehat{T}_1 \cup \widehat{T}_2 =$$  , $$\widehat{T} =$$  .

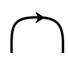
When we calculate the Conway polynomial in the proposition, we forget the framing. The proof of the proposition is an elementary exercise; for a reference, see eg [33].

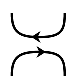
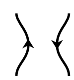
Alternatively, in the linear skein modulo the relation

 $-$  $= z$  ,

some formulas of Proposition 3.4 are rewritten

$$T_1 = \bigl(1 - v_2^{xx} z^2 + O(z^4)\bigr)$$  ,

$$T_2 = \bigl(1 - v_2^{yy} z^2 + O(z^4)\bigr)$$  ,

$$T = \bigl(1 - (v_2^{xx} + v_2^{yy} + v_2^{xy}) z^2 + O(z^4)\bigr)$$  $+ \bigl(kz - 2v_3 z^3 + O(z^5)\bigr)$  .

Corollary 3.5 *Under the assumption of Theorem 3.1, the reduced 2–loop polynomial is presented by*

$$\begin{aligned}
&\widehat{\Theta}_{K_T}(t) \\
&\quad= \Bigl((n+m)\bigl(d - \tfrac{nm}{2}\bigr) - k\bigl(k+\tfrac{1}{2}\bigr)(k+1) + 12v_3\Bigr)\Bigl(-2 - \tfrac{2d+1}{3}(t + t^{-1} - 2)\Bigr) \\
&\qquad - 4\Bigl(mv_2^{xx} + nv_2^{yy} - \bigl(k+\tfrac{1}{2}\bigr)v_2^{xy} - 3v_3\Bigr)\Delta_{K_T}(t),
\end{aligned}$$

where $\Delta_{K_T}(t) = 1 + d(t + t^{-1} - 2)$ *is the Alexander polynomial of* K_T *.*

Proof By definition, we obtain $\widehat{\Theta}_K(t)$ from $\Theta_K(t_1, t_2)$ by replacing

$$T_{1,0} \longmapsto 2, \qquad T_{2,0} \longmapsto 2(t + t^{-1} + 2), \qquad T_{2,1} \longmapsto t + t^{-1} + 4.$$

Hence, we obtain the corollary from Theorem 3.1. □

Example 3.6 For the pretzel knot K of type (p, q, r), the reduced 2–loop polynomial and the Jones polynomial are given by

$$\widehat{\Theta}_K(t) = \tfrac{1}{16}\bigl((p+q+r)(4d+1) + pqr\bigr)\Bigl(-2 - \frac{2d+1}{3}(t + t^{-1} - 2)\Bigr),$$

$$V_K(t) = 1 - \frac{t^2+t+1}{(t+1)^2}(t^{p+q+r+1} + t^{p+q+r-1} - t^{p+q} - t^{p+r} - t^{q+r} + 1),$$

where $d = (pq + qr + rp + 1)/4$; see Section 3.3 for a presentation of the 2–loop polynomial of the pretzel knot in terms of p, q, r. By the following formulas

$$\begin{aligned}
&\widehat{\Theta}_K(1) = -\tfrac{1}{8}\big((p+q+r)(4d+1) + pqr\big),\\
&\big(\text{the coefficient of } h^3 \text{ in } V_K(e^h)\big) = -\tfrac{3}{8}\big((p+q+r)(4d+1) + pqr\big),\\
&\widehat{\Theta}_K(-1) = -\tfrac{1}{24}\big((p+q+r)(4d+1) + pqr\big)(1-4d),\\
&V_K(-1) = \Delta_K(-1) = 1-4d,\\
&V_K'(-1) = \tfrac{1}{2}\big((p+q+r)(4d+1) + pqr\big),
\end{aligned}$$

we can verify the formula in Proposition 1.1.

Theorem 3.7 *The 2–loop polynomial $\Theta_K(t_1,t_2)$ of a knot K of genus 1 is presented by*

$$\begin{aligned}
\Theta_K(t_1,t_2) = \tfrac{1}{24}\big(V_K'''(1) + 3V_K''(1)\big)&\Big(\big(d^2 - d + \tfrac{1}{3}\big)(T_{2,1} - T_{1,0}) - \tfrac{1}{2}d(d-1)T_{2,0}\Big)\\
- \tfrac{1}{16}V_K'(-1)&\Big((5d^2 - 5d + 1)T_{1,0} + \tfrac{1}{2}d(5d-1)T_{2,0} - \big(5d^2 - \tfrac{7}{3}d + \tfrac{1}{3}\big)T_{2,1}\Big),
\end{aligned}$$

where $V_K(t)$ is the Jones polynomial of K, and $d = \frac{1}{2}\Delta_K''(1) = -\frac{1}{6}V_K''(1)$, and $T_{n,m}$'s are defined in (1).

Proof We choose a tangle T in (19) such that K_T is isotopic to K. By the formula of Corollary 3.5 at $t = \pm 1$, we have that

$$\begin{aligned}
(n+m)(d - \tfrac{nm}{2}) - k(k+\tfrac{1}{2})(k+1) + 12v_3 &= -\tfrac{3}{4}\Big(\widehat{\Theta}_K(1) + \frac{1}{4d-1}\widehat{\Theta}_K(-1)\Big)\\
&= -\tfrac{1}{24}\big(V_K'''(1) + 3V_K''(1) + \tfrac{3}{2}V_K'(-1)\big),\\
4\big(mv_2^{xx} + nv_2^{yy} - (k+\tfrac{1}{2})v_2^{xy} - 3v_3\big) &= -\tfrac{1}{2}\Big(\widehat{\Theta}_K(1) + \frac{3}{4d-1}\widehat{\Theta}_K(-1)\Big)\\
&= \tfrac{1}{36}\big(V_K'''(1) + 3V_K''(1) + \tfrac{9}{2}V_K'(-1)\big),
\end{aligned}$$

where we obtain the second and fourth equality by Proposition 1.1, noting that $V_K(-1) = \Delta_K(-1) = 1-4d$. By substituting those formulas into the formula of Theorem 3.1, we obtain the required formula. □

From the theorem, we obtain the following corollary.

Corollary 3.8 *The reduced 2–loop polynomial* $\widehat{\Theta}_K(t)$ *of a knot* K *of genus 1 is presented by*

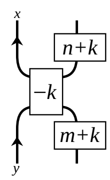
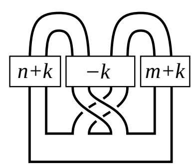
$$\widehat{\Theta}_K(t) = \tfrac{1}{72}(t+t^{-1}+2)\big(V_K'''(1)+3V_K''(1)\big) + \tfrac{1}{48}(t+t^{-1}-2)V_K'(-1)V_K(-1),$$

where $V_K(t)$ *is the Jones polynomial of* K.

3.2 Proof of Theorem 3.1

The aim of this section is to prove Theorem 3.1. We reduce the proof to Proposition 3.9 below.

We denote by T_0 the tangle in (20), that is, we put

(22) $T_0 =$ , $K_{T_0} =$ ,

for integers n, m, k.

Proposition 3.9 *The 2–loop polynomial of the knot* K_{T_0} *is presented by*

$$\begin{aligned}\Theta(K_{T_0}) = &\big((n+m)(d-\tfrac{nm}{2}) - k(k+\tfrac{1}{2})(k+1)\big)\\ &\times\big(-d(d-1)T_{1,0} - \tfrac{1}{2}d(d+1)T_{2,0} + \big(d^2+\tfrac{1}{3}d-\tfrac{1}{3}\big)T_{2,1}\big),\end{aligned}$$

where we put $d = nm - k^2 - k$ *as before, and* $T_{n,m}$*'s are defined in* (1).

We give two proofs of the proposition; one is a proof using the symmetry of K_{T_0}, which we give in Section 3.3, and the other is a proof using a surgery presentation of K_{T_0}, which we give in Section 3.4.

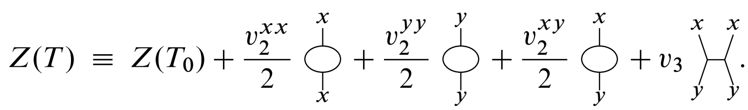
Example 3.10 When $k=0$, the knot K_{T_0} is isotopic to the $(-4nm+1, 2n)$ two-bridge knot. Hence, by Proposition 2.3,

$$\begin{aligned}\Theta(K_{T_0})\big|_{k=0} = &\frac{1}{2}nm(n+m)\\ &\times\big(-nm(nm-1)T_{1,0} - \tfrac{1}{2}nm(nm+1)T_{2,0} + \big(n^2m^2+\tfrac{1}{3}nm-\tfrac{1}{3}\big)T_{2,1}\big).\end{aligned}$$

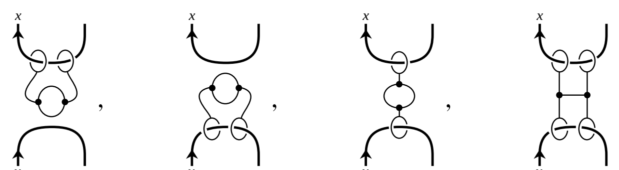
This is a particular value of the proposition.

Proof of Theorem 3.1 By definition, $\Theta(K_T)$ is determined by $Z(K_T)$. Further, from properties of the Kontsevich invariant, $Z(K_T)$ is determined by $Z(T)$. Hence, $\Theta(K_T)$ can be determined by $Z(T)$. In particular, $\Theta(K_T)$ does not depend on the degree > 2 part of $Z(T)$, because such a part can be presented by Jacobi diagrams with at least 3 trivalent vertices, and we can show, in the same way as the proof of Lemma 3.11 below, that such a Jacobi diagram changes $Z(K_T)$ by (> 2)–loop diagrams. Therefore, $\Theta(K_T)$ can be determined by the degree ≤ 2 part of $Z(T)$.

Let T_0 be as in (22). Then,



Therefore, T_0 is related to some T', satisfying that $Z(T) \equiv Z(T')$ $\big($hence, $\Theta(K_T) = \Theta(K_{T'})\big)$, by clasper surgery along graph claspers of the form,



See Section 4.3 for graph claspers. Hence, by Lemma 3.11 and Lemma 3.12,

$$\begin{aligned}\Theta(K_T) \;\sim\; & \Theta(K_{T_0}) - 6\,\big(m v_2^{xx} + n v_2^{yy} - (k+\tfrac{1}{2}) v_2^{xy}\big)(t_1 + t_1^{-1} - 2)\,\Delta(t_2)\,\Delta(t_3) \\ & + v_3 \cdot 12\,\big(\tfrac{3}{4}(t_1 - t_1^{-1})(t_2 - t_2^{-1}) + \big(d - \tfrac{1}{4}\big)(t_1 + t_1^{-1} - 2)(t_2 + t_2^{-1} - 2)\big)\Delta(t_3).\end{aligned}$$

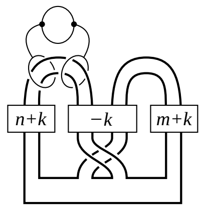
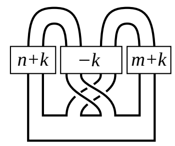
By symmetrizing it, we have that

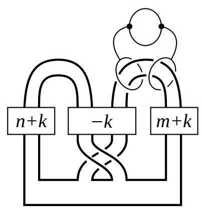
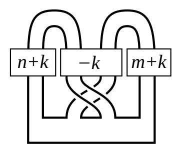
$$\begin{aligned}\Theta(K_T) = \Theta(K_{T_0}) & - \big(m v_2^{xx} + n v_2^{yy} - (k+\tfrac{1}{2}) v_2^{xy}\big) \\ & \times 12\big(\big(d^2 - d + \tfrac{1}{6}\big)T_{1,0} + \tfrac{1}{2} d^2 T_{2,0} - \big(d^2 - \tfrac{1}{3} d\big) T_{2,1}\big) \\ & + v_3 \cdot 12\big(\big(2d^2 - 2d + \tfrac{1}{2}\big)T_{1,0} + \big(d^2 - \tfrac{1}{2} d\big)T_{2,0} + \big(-2d^2 + \tfrac{4}{3} d - \tfrac{1}{3}\big)T_{2,1}\big).\end{aligned}$$

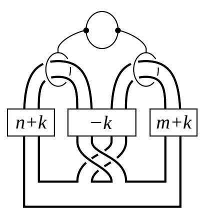
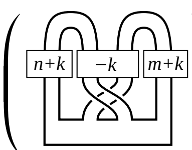
Therefore, by Proposition 3.9, we obtain the required formula. $\square$

A general form of the following lemma is given in Proposition 4.17.

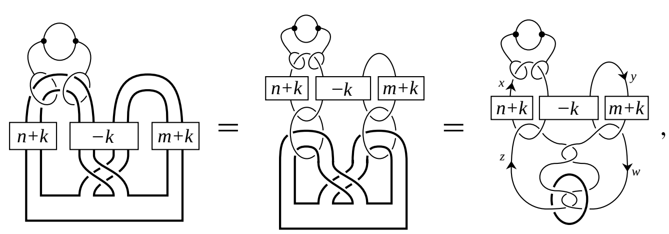
Lemma 3.11 *The changes of the 2–loop polynomial for surgery along the following graph claspers are presented, modulo the equivalence* (5), *by*

$\Theta\Bigg($  $\Bigg) - \Theta\Bigg($  $\Bigg) \sim -12\, m\, (t_1+t_1^{-1}-2)\, \Delta(t_2)\, \Delta(t_3),$

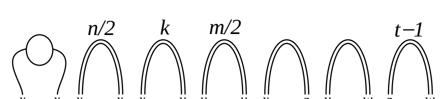
$\Theta\Bigg($  $\Bigg) - \Theta\Bigg($  $\Bigg) \sim -12\, n\, (t_1+t_1^{-1}-2)\, \Delta(t_2)\, \Delta(t_3),$

$\Theta\Bigg($  $\Bigg) - \Theta\Bigg($  $\Bigg) \sim 12\left(k+\tfrac{1}{2}\right) (t_1+t_1^{-1}-2)\, \Delta(t_2)\, \Delta(t_3).$

Proof We show the first formula of the lemma. The first knot of the formula is rewritten as



where components depicted in thin lines imply surgery along the components. Hence, by Lemma 4.16, the difference of the Kontsevich invariant of the two knots in the first formula is equivalent (modulo "≡") to

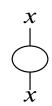
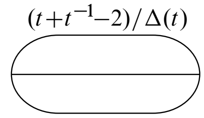


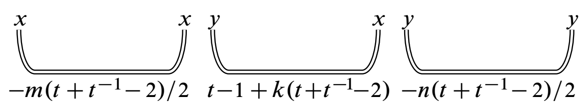
The equivariant linking matrix and its negative inverse are given by

$$M=\begin{pmatrix} n & k & 1 & 0\\ k & m & 0 & 1\\ 1 & 0 & 0 & t-1\\ 0 & 1 & t^{-1}-1 & 0\end{pmatrix},$$

$$-M^{-1}=\frac{1}{\Delta(t)}\begin{pmatrix} -m(t+t^{-1}-2) & t-1+k(t+t^{-1}-2) & \cdot & \cdot\\ t^{-1}-1+k(t+t^{-1}-2) & -n(t+t^{-1}-2) & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix},$$

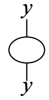
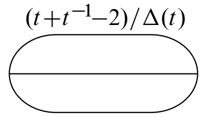
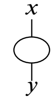
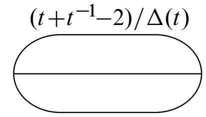
where $\Delta(t)=1+(nm-k^2-k)(t+t^{-1}-2)$, noting that we do not need the omitted entries in this proof. Hence, the 2–loop part of the Kontsevich invariant of the result of the surgery is given by the rational version of the Aarhus integral as follows,

$\Big\langle \alpha\ ,$  $\Big\rangle = -m$ ,

where $\alpha =$ .

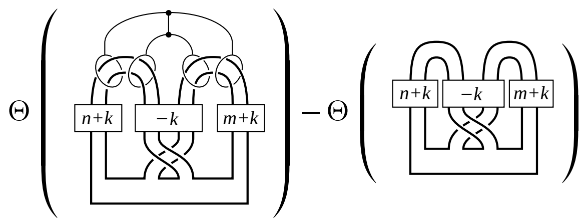
The corresponding 2–loop polynomial gives the right-hand side of the first formula of the lemma.

The other formulas of the lemma are obtained, in the same way, from

$\Big\langle \alpha\ ,$  $\Big\rangle = -\,n$ , $\quad\Big\langle \alpha\ ,$  $\Big\rangle = \big(k+\tfrac{1}{2}\big)$ . $\square$

A general form of the following lemma is given in Proposition 4.18.

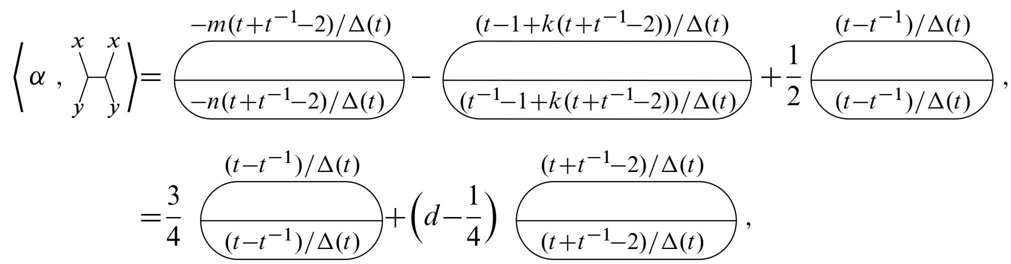
Lemma 3.12 *The change of the 2–loop polynomial for surgery along the following graph clasper is presented, modulo the equivalence (5), by*



$$\sim\ 12\left(\tfrac{3}{4}(t_1-t_1^{-1})(t_2-t_2^{-1})+\left(d-\tfrac{1}{4}\right)(t_1+t_1^{-1}-2)(t_2+t_2^{-1}-2)\right)\Delta(t_3),$$

where $d=nm-k^2-k$.

Proof Similarly to the proof of Lemma 3.11, the left-hand side of the formula of the lemma is given by

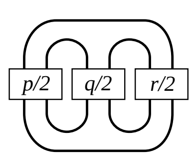


where α is the one given in the proof of Lemma 3.11. The corresponding 2–loop polynomial gives the right-hand side of the formula of the lemma. □

3.3 Proof of Proposition 3.9 by using symmetry

In this section, we give a proof of Proposition 3.9 using the symmetry of a pretzel knot, which is isotopic to K_{T_0}.

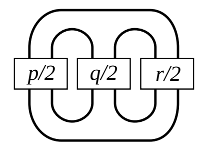
The *pretzel knot* of type (p,q,r) for odd integers p,q,r is given by



,

where we mean k half twists by a boxed "$k/2$". This is isotopic to K_{T_0} given in (22) putting $p = 2n+2k+1$, $q = -2k-1$, $r = 2m+2k+1$. We put

$$P(p,q,r) = \Theta\Bigg($$



$$\Bigg).$$

Then, the formula of Proposition 3.9 is rewritten as

$$(23)\quad P(p,q,r) = \tfrac{1}{16}\big((p+q+r)(4d+1)+pqr\big) \\ \times\big(-d(d-1)T_{1,0} - \tfrac{1}{2}d(d+1)T_{2,0} + \big(d^2+\tfrac{1}{3}d-\tfrac{1}{3}\big)T_{2,1}\big),$$

where $d = (pq+qr+rp+1)/4$. The aim of this section is to prove this formula.

Proof of Proposition 3.9 We show (23), which is equivalent to Proposition 3.9. Let $P_1(p,q,r)$ be the coefficient of $T_{1,0}$ in $P(p,q,r)$. Putting

$$\widehat{P}_1(p,q,r) = P_1(p,q,r) - \frac{1}{16}\big((p+q+r)(4d+1)+pqr\big)\big(-d(d-1)\big),$$

we show that $\widehat{P}_1(p,q,r) = 0$; the other part of (23) can be shown in the same way.

By the symmetry of the pretzel knot, $P(p,q,r)$ (hence, $\widehat{P}_1(p,q,r)$) is a symmetric polynomial in p,q,r. Hence, $\widehat{P}_1(p,q,r)$ can be presented by some polynomial $F(\sigma_1,\sigma_2,\sigma_3)$ in the elementary symmetric polynomials $\sigma_1 = p+q+r$, $\sigma_2 = pq+qr+rp$, $\sigma_3 = pqr$. By comparing the pretzel knot with its mirror image, $P(p,q,r) = -P(-p,-q,-r)$, hence, $\widehat{P}_1(p,q,r) = -\widehat{P}_1(-p,-q,-r)$. Therefore, $F(\sigma_1,\sigma_2,\sigma_3) = -F(-\sigma_1,\sigma_2,-\sigma_3)$. Further, by Lemma 3.15,

$$P_1(p,q,r) = P_1(p+2r,-r,q+2r) + \frac{r(r^2-1)}{4}\cdot\big(-d(d-1)\big),$$

and hence, $$\widehat{P}_1(p,q,r) = \widehat{P}_1(p+2r,-r,q+2r).$$

Let $\sigma_1',\sigma_2',\sigma_3'$ be the elementary symmetric polynomials in $p' = p+2r$, $q' = -r$, $r' = q+2r$. Then, putting $\alpha = p+q+2r$, $\beta = (p+r)(q+r)$, we have that

$$\begin{aligned} &\sigma_1 = \alpha - r, && \sigma_1' = \alpha + r, \\ &\sigma_2 = \sigma_2' = \beta - r^2, && \\ &\sigma_3 = (\beta - \alpha r + r^2)r, && \sigma_3' = -(\beta + \alpha r + r^2)r. \end{aligned}$$

It follows that

$$F\big(\alpha - r, \beta - r^2, (\beta - \alpha r + r^2) r\big) = F\big(\alpha + r, \beta - r^2, -(\beta + \alpha r + r^2) r\big).$$

Therefore, by Lemma 3.13 below, $F(\sigma_1, \sigma_2, \sigma_3)$ (hence, $\widehat{P}_1(p, q, r)$) is equal to 0, as required. □

Lemma 3.13 *Let $F(\sigma_1, \sigma_2, \sigma_3)$ be a polynomial in indeterminates $\sigma_1, \sigma_2, \sigma_3$ with rational coefficients, satisfying that*

$$F(\sigma_1, \sigma_2, \sigma_3) = -F(-\sigma_1, \sigma_2, -\sigma_3),$$
$$F\big(\alpha - r, \beta - r^2, (\beta - \alpha r + r^2) r\big) = F\big(\alpha + r, \beta - r^2, -(\beta + \alpha r + r^2) r\big),$$

where we also regard α, β, r as indeterminates in this lemma. Then, $F(\sigma_1, \sigma_2, \sigma_3) = 0$.

Proof Put $F(\sigma_1, \sigma_2, \sigma_3) = \sum_{n=0}^{N} f_n(\sigma_1, \sigma_3) \sigma_2^n$. Putting $\sigma_2 = 0$ and $\beta = r^2$, we have that $f_0(\sigma_1, \sigma_3) = -f_0(-\sigma_1, -\sigma_3)$ and $f_0\big(\alpha - r, (2r - \alpha) r^2\big) = f_0\big(\alpha + r, -(2r + \alpha) r^2\big)$. Hence, by Lemma 3.14 below, $f_0(\sigma_1, \sigma_3) = 0$. Since $F(\sigma_1, \sigma_2, \sigma_3)/\sigma_2$ satisfies the assumption of the lemma, the lemma is shown by induction on N. □

Lemma 3.14 *Let $G(\sigma_1, \sigma_3)$ be a polynomial in σ_1, σ_3 with rational coefficients, satisfying that*

$$G(\sigma_1, \sigma_3) = -G(-\sigma_1, -\sigma_3),$$
$$G\big(\alpha - r, (2r - \alpha) r^2\big) = G\big(\alpha + r, -(2r + \alpha) r^2\big),$$

where we regard α, r as indeterminates in this lemma. Then, $G(\sigma_1, \sigma_3) = 0$.

Proof From the assumption of the lemma, $G\big(\alpha - r, (2r - \alpha) r^2\big)$ is equal to a linear sum of $r^{\text{even}} \alpha^{\text{odd}}$ with rational coefficients. Putting $\alpha = cr$, it is a polynomial in $(c-1)r$, $(c-2)r^3$, which is equal to a linear sum of $r^{\text{odd}} c^{\text{odd}}$ with rational coefficients. In particular, the coefficient of r^k for each odd k is equal to a linear sum of

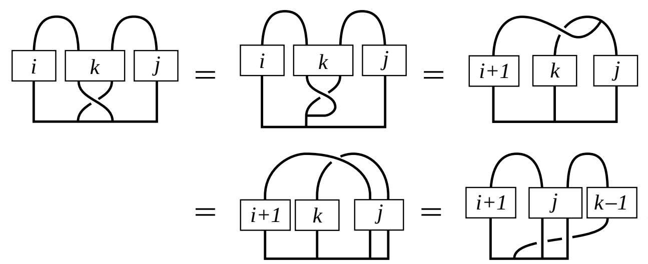
$$(c-1)^k, \quad (c-1)^{k-3}(c-2), \quad (c-1)^{k-6}(c-2)^2, \quad \cdots,$$

which is equal to a linear sum of c^{odd}. Hence, it is equal to 0. □

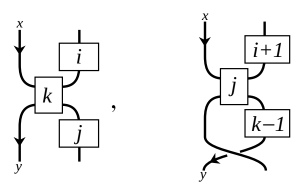
Lemma 3.15

$$P(p, q, r) = P(p + 2r, -r, q + 2r) + \frac{r(r^2 - 1)}{4} \times \big(-d(d-1) T_{1,0} + \big(-\tfrac{1}{2} d^2 - \tfrac{1}{2} d\big) T_{2,0} + \big(d^2 + \tfrac{1}{3} d - \tfrac{1}{3}\big) T_{2,1}\big).$$

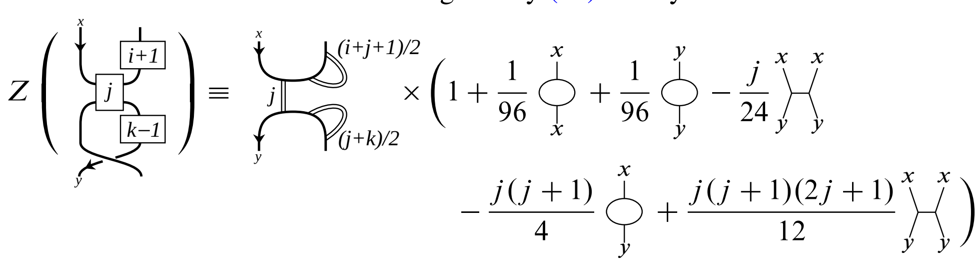
Proof Recall a presentation (22) of the pretzel knot of type (p,q,r). We deform it by isotopy as follows,



where these pictures present knotted framed graphs (with blackboard framing) such that by a graph we mean the knot of the boundary of a ribbon graph given by the framed graph. As in (19), the first and the last terms are given by the following tangles,



The Kontsevich invariant of them are given by (20) and by

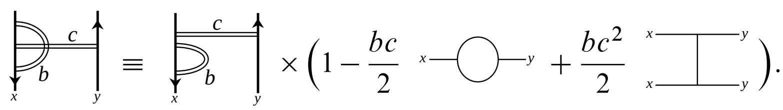


where we obtain the formula in a similar way as in Section 2.1; Lemma 3.16 and Lemma 5.3 are useful when we calculate the formula. Hence, by Lemma 3.11 and Lemma 3.12 and by putting $j=(r-1)/2$,

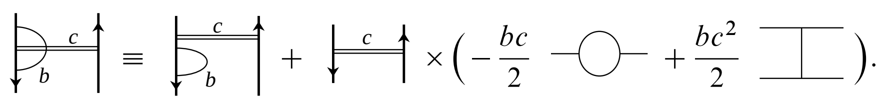
$$
\begin{aligned}
P(p,q,r) = P(p+2r,-r,q+2r)& \\
&- \frac{r^2-1}{4}\cdot 3(-r)\big(-\big(d^2-d+\tfrac{1}{6}\big)T_{1,0}-\tfrac{1}{2}d^2T_{2,0}+\big(d^2-\tfrac{1}{3}d\big)T_{2,1}\big) \\
&+\frac{r(r^2-1)}{4}\cdot\big(\big(2d^2-2d+\tfrac{1}{2}\big)T_{1,0}+\big(d^2-\tfrac{1}{2}d\big)T_{2,0}+\big(-2d^2+\tfrac{4}{3}d-\tfrac{1}{3}\big)T_{2,1}\big).
\end{aligned}
$$

This gives the required formula. $\square$

Lemma 3.16 *For scalars b and c,*



Proof By induction on the number of chords labeled by c, we can show that

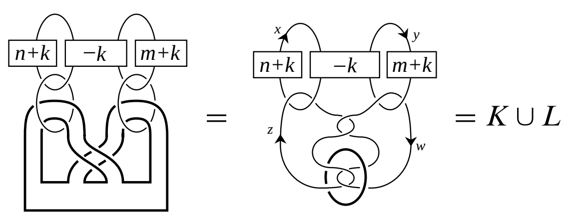


The required formula can be shown from the above formula by induction on the number of chords labeled by b. A detailed proof is left to the reader. □

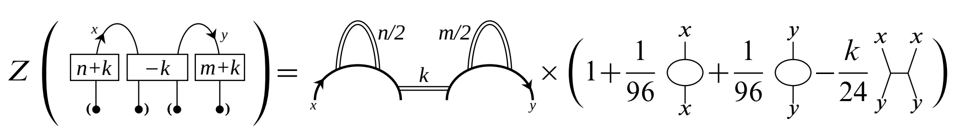
3.4 Proof of Proposition 3.9 from surgery presentations

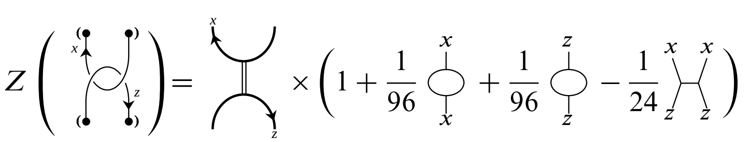
In this section, we give another proof of Proposition 3.9 by using a surgery presentation of K_{T_0}. Indeed this proof might be somehow tedious comparing to the previous proof, but the way of this proof is generalized to the case of higher genus later in Section 4.

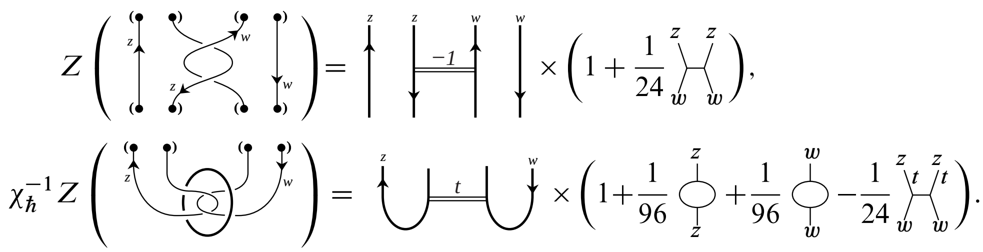
Proof of Proposition 3.9 The knot K_{T_0} given in (22) is obtained from the following link by surgery along the components of thin lines. This link is isotopic to the second link, which we denote by $K \cup L$, where K denotes the knot of thick line and L denotes the link of thin lines.



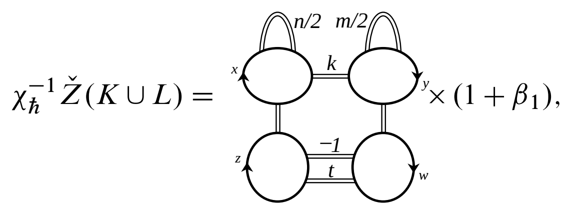
We calculate the Kontsevich invariant of $K \cup L$ by decomposing it into the following parts; each part can be calculated in a similar way as in Section 2.1,



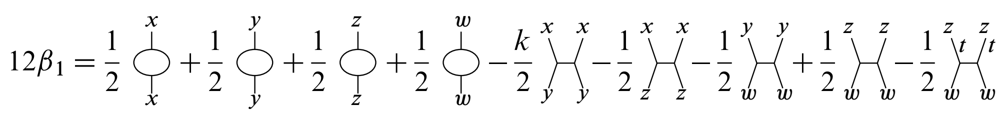




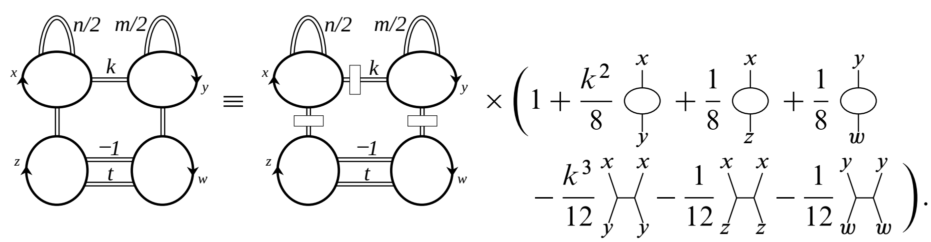
By composing them, we obtain $Z(K\cup L)$. Since $\check{Z}(K\cup L)$ is obtained from $Z(K\cup L)$ by connect-summing ν to each component of L by definition, we have that



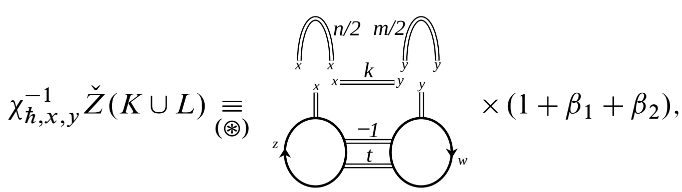
where



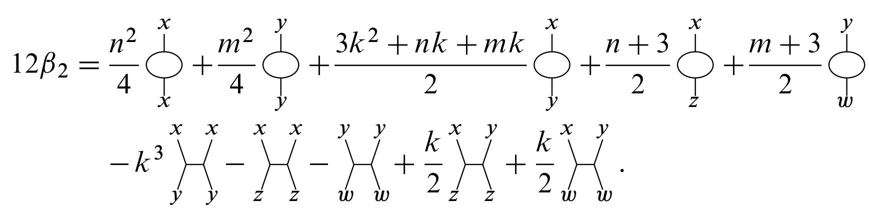
Further, by Lemma 5.2,



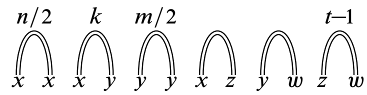
Hence, by Lemma 5.15,



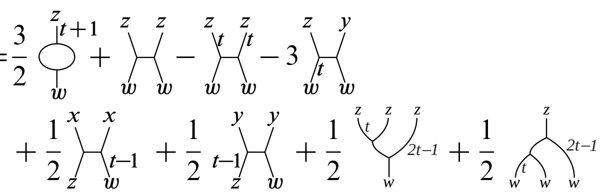
where



Further, by Lemma 5.17, Lemma 5.19, and Lemma 5.20,

$$\chi^{-1}\check{Z}(K\cup L) \underset{(\circledast)}{\equiv}$$


$$\times(1+\beta_1+\beta_2+\beta_3),$$

where $12\beta_3 =$


The equivariant linking matrix M is given by

$$M=\begin{pmatrix} n & k & 1 & 0\\ k & m & 0 & 1\\ 1 & 0 & 0 & t-1\\ 0 & 1 & t^{-1}-1 & 0\end{pmatrix},\qquad -M^{-1}=\frac{1}{\Delta(t)}\big(F_{XY}(t)\big),$$

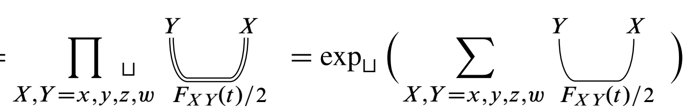
where $\Delta(t)=1+(nm-k^2-k)(t+t^{-1}-2)$ and $F_{XY}(t)$ is given by

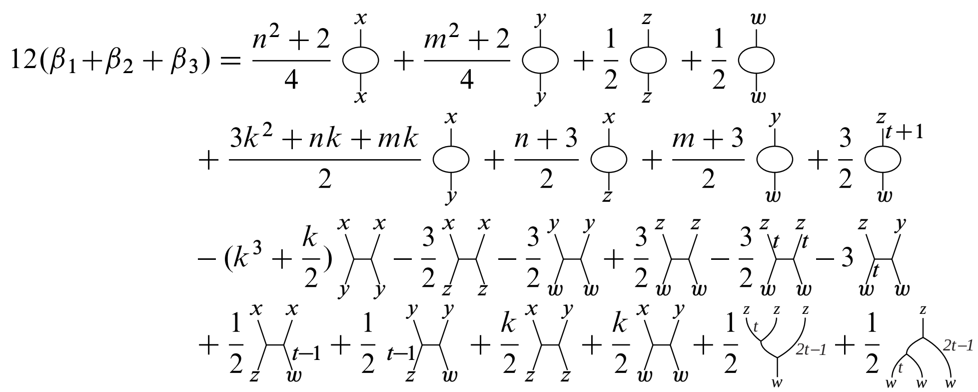
$$\begin{aligned}
F_{xx}(t)&=-m(t+t^{-1}-2), & F_{xy}(t)&=t-1+k(t+t^{-1}-2),\\
F_{xz}(t)&=-1+k(t^{-1}-1), & F_{xw}(t)&=-m(t-1),\\
F_{yy}(t)&=-n(t+t^{-1}-2), & F_{yz}(t)&=-n(t^{-1}-1),\\
F_{yw}(t)&=-1+k(t-1), & F_{zz}(t)&=n,\\
F_{zw}(t)&=k+(nm-k^2)(t-1), & F_{ww}(t)&=m,
\end{aligned}$$

and $F_{YX}(t)=F_{XY}(t^{-1})$. Therefore, we obtain $\chi^{-1}Z(K_L)$ from $\chi^{-1}\check{Z}(K\cup L)$ by surgery along L using the rational version of the Aarhus integral,

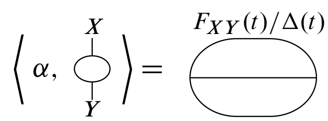
$$\chi^{-1}Z(K_L)=\int \chi^{-1}\check{Z}(K\cup L)\;dx\;dy\;dz\;dw \equiv \big\langle\, \alpha,\ (1+\beta_1+\beta_2+\beta_3)\big\rangle$$

where

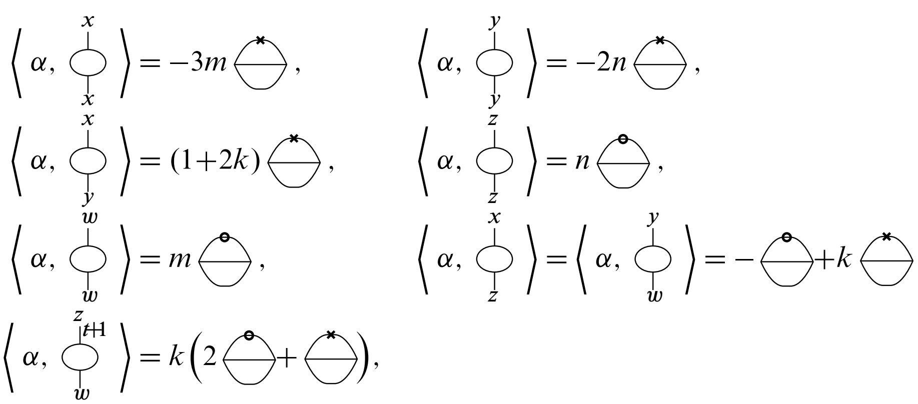
(24) $\alpha =$




We calculate the corresponding part of each diagram of $12(\beta_1+\beta_2+\beta_3)$ in the formula of $\chi^{-1}Z(K_L)$ as follows. Noting that, by definition,

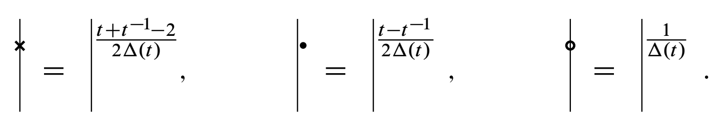


for $X, Y = x, y, z, w$, we have that

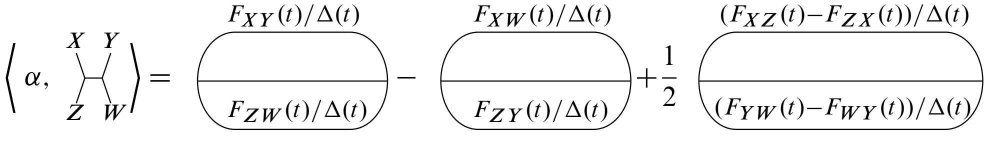


where, in this section, we define the markings by

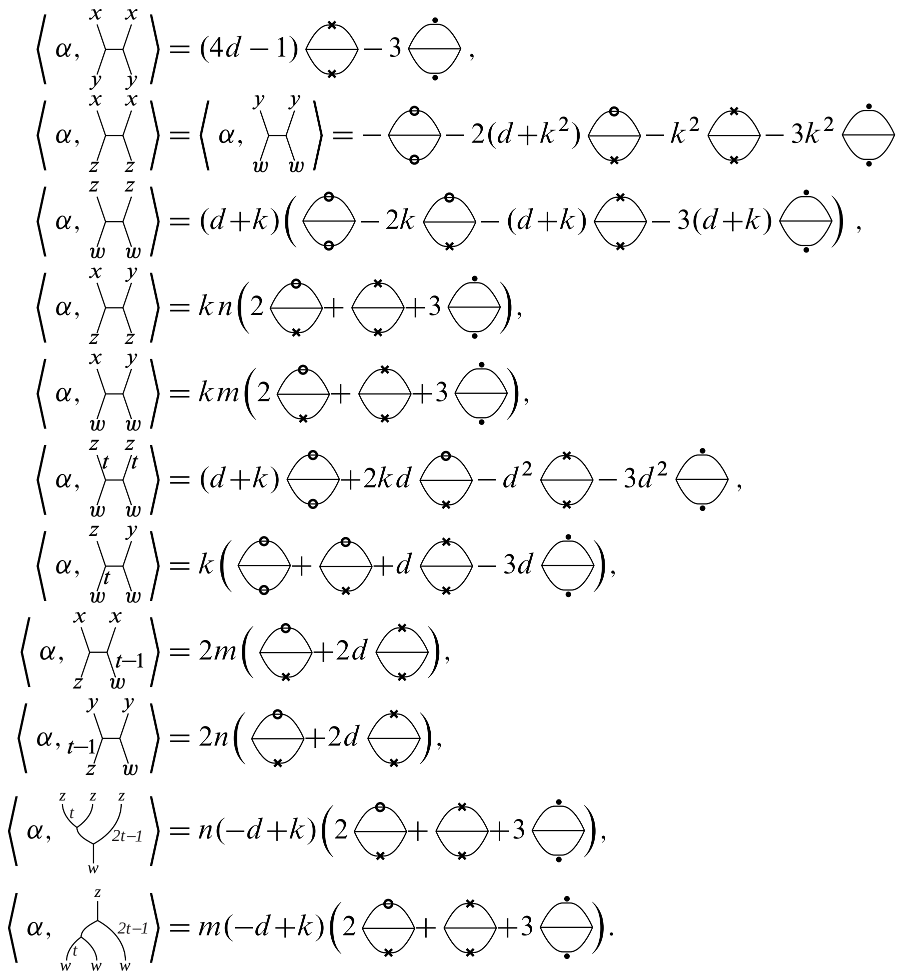
(25)



Further, noting that, by definition,

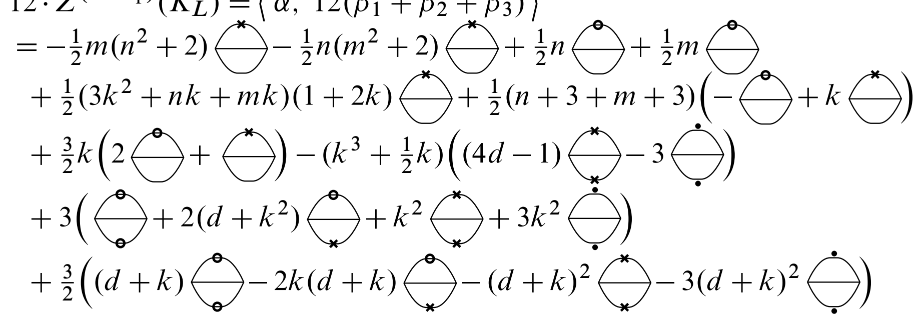


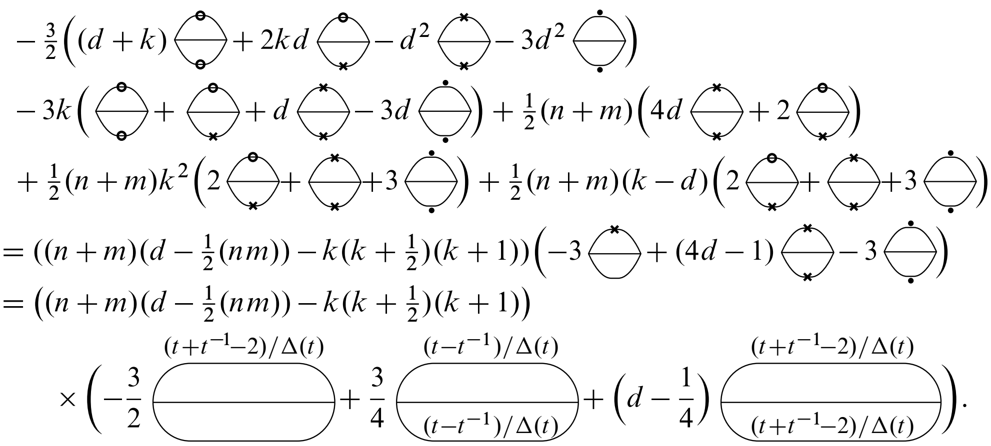
for $X, Y = x, y, z, w$, we have that



Hence, the 2–loop part of the Kontsevich invariant of K_L is presented by

$$12 \cdot Z^{(2\text{–loop})}(K_L) = \langle\, \alpha,\ 12(\beta_1 + \beta_2 + \beta_3)\,\rangle$$





This gives the formula of Proposition 3.9. ◻

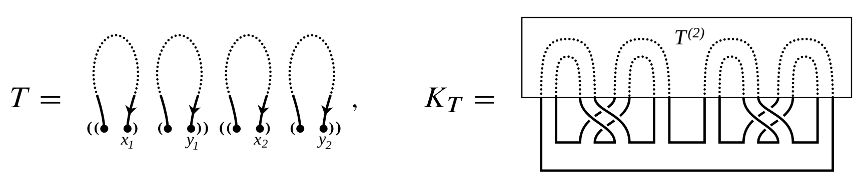
4 The 2–loop polynomial for knots of any genus

In this section, we give a presentation of the 2–loop part of the Kontsevich invariant for knots of any genus. By using the presentation, we show that the degree of the 2–loop polynomial of a knot is bounded by twice the genus of the knot in Section 4.2. Further, we show clasper surgery formulas for the 2–loop polynomial in Section 4.3.

4.1 The 2–loop polynomial for knots of genus 2

Before we calculate the case of any genus, we calculate the case of genus 2 in this section. The approach of the calculations for both cases are almost the same.

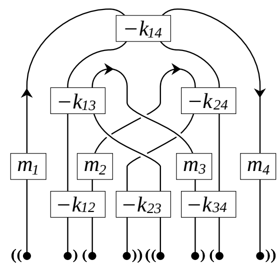
Similarly to the case of genus 1, we let T be a 4–component framed tangle, and let K_T be the knot obtained from T as follows,



where dotted lines in the picture of T imply strands possibly knotted and linked in some fashion, and $T^{(2)}$ denotes the tangle obtained from T by replacing each component of T with 2 parallel copies of it. Any knot of genus 2 can be presented by K_T for some T.

We consider a particular form T_0 of T for integers m_i and k_{ij}; recall that a boxed "k" implies k full twists:

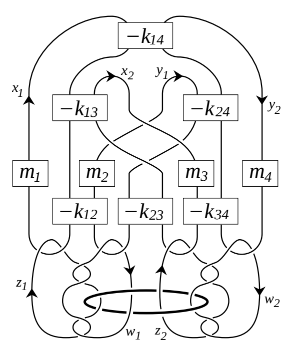
(26) $T_0 =$



Note that $\Theta(K_T)$ for any T can be obtained from $\Theta(K_{T_0})$ by Proposition 4.3; the proposition does not change the linking matrix of T, while T_0 represents a class of T having an arbitrarily fixed linking matrix.

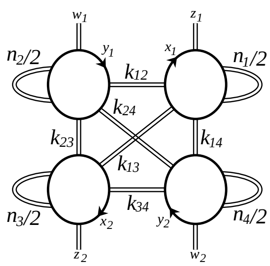
By a similar argument as in Section 3.4, we obtain K_{T_0} from the following link $K \cup L$ by surgery along L:

$K \cup L =$



where K denotes the knot of thick line and L denotes the link of thin lines.

We calculated the Kontsevich invariant of $K \cup L$, in a similar way as in Section 3.4, as follows. We rename the labels by $X_1 = x_1$, $X_2 = y_1$, $X_3 = x_2$, $X_4 = y_2$, and $Z_1 = z_1$, $Z_2 = w_1$, $Z_3 = z_2$, $Z_4 = w_2$, and use both names in formulas of this section, to simplify their presentations. By a similar argument as in Section 3.4, the upper part of $\chi_{\hbar}^{-1}\check{Z}(K\cup L)$ is given by



$$\times\Big(1+\sum_{1\le i\le 4}\frac{1}{24}\begin{smallmatrix}X_i\\ \ominus\\ X_i\end{smallmatrix}-\sum_{1\le i<j\le 4}\frac{k_{ij}}{24}\begin{smallmatrix}X_i & X_i\\ \mathrm{H}\\ X_j & X_j\end{smallmatrix}+\frac{k_{13}k_{24}}{2}\begin{smallmatrix}X_2 & X_1\\ \bowtie\\ X_3 & X_4\end{smallmatrix}\Big),$$

where $n_i=m_i-\sum_{j\neq i}k_{ij}$ putting $k_{ij}=k_{ji}$. Further, the middle part of $\chi_{\hbar}^{-1}\check{Z}(K\cup L)$ is given by

$$\prod_{1\le i\le 4}\sqcup\ \underset{X_i\ Z_i}{\frown}\times\Big(1+\sum_{1\le i\le 4}\Big(\frac{1}{8}\begin{smallmatrix}X_i\\ \ominus\\ Z_i\end{smallmatrix}-\frac{1}{8}\begin{smallmatrix}X_i & X_i\\ \mathrm{H}\\ Z_i & Z_i\end{smallmatrix}\Big)\Big).$$

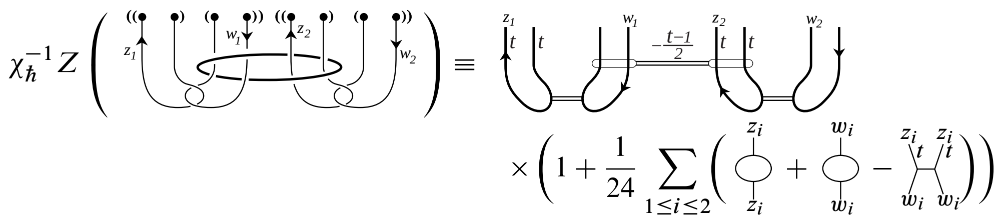
Hence, by Lemma 5.28, $\chi^{-1}\check{Z}(K\cup L)$ of the above parts is given by

$$\prod_{1\le i\le 4}\sqcup\ \overset{n_i/2}{\underset{X_i\ X_i}{\frown}}\times\prod_{1\le i<j\le 4}\sqcup\ \overset{k_{ij}}{\underset{X_i\ X_j}{\frown}}\times\prod_{1\le i\le 4}\sqcup\ \underset{X_i\ Z_i}{\frown}\times\prod_{1\le i<j<l\le 4}\sqcup\ \begin{smallmatrix}X_i & X_l\\ \mathrm{Y}_{\kappa_{ijl}}\\ X_j\end{smallmatrix}\times(1+\beta_1),$$

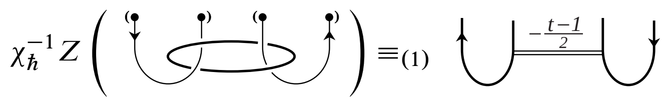
where we put $\kappa_{ijl}=(k_{ij}k_{il}+k_{ij}k_{jl}+k_{il}k_{jl})/2$ and

$$\begin{aligned}
12\beta_1=&\sum_{1\le i\le 4}\Big(\frac{n_i^2+2}{4}\begin{smallmatrix}X_i\\ \ominus\\ X_i\end{smallmatrix}+\frac{n_i+3}{2}\begin{smallmatrix}X_i\\ \ominus\\ Z_i\end{smallmatrix}-\frac{3}{2}\begin{smallmatrix}X_i & X_i\\ \mathrm{H}\\ Z_i & Z_i\end{smallmatrix}\Big)\\
&+\sum_{1\le i<j\le 4}\Big(\frac{3k_{ij}^2+n_ik_{ij}+n_jk_{ij}}{2}\begin{smallmatrix}X_i\\ \ominus\\ X_j\end{smallmatrix}-\Big(k_{ij}^3+\frac{k_{ij}}{2}\Big)\begin{smallmatrix}X_i & X_i\\ \mathrm{H}\\ X_j & X_j\end{smallmatrix}+\frac{k_{ij}}{2}\begin{smallmatrix}X_i & X_j\\ \mathrm{H}\\ Z_i & Z_i\end{smallmatrix}+\frac{k_{ij}}{2}\begin{smallmatrix}X_i & X_j\\ \mathrm{H}\\ Z_j & Z_j\end{smallmatrix}\Big)\\
&+\sum_{1\le i\le 4}\sum_{\substack{j<l\\ j,l\neq i}}\Big(3k_{ij}k_{il}k_{jl}+k_{jl}(k_{ij}^2+k_{il}^2)\Big)\begin{smallmatrix}X_i & X_i\\ \mathrm{H}\\ X_j & X_l\end{smallmatrix}\\
&+2(k_{12}k_{13}k_{14}+k_{21}k_{23}k_{24}+k_{31}k_{32}k_{34}+k_{41}k_{42}k_{43})\Big(\begin{smallmatrix}X_2 & X_1\\ \mathrm{H}\\ X_3 & X_4\end{smallmatrix}+\begin{smallmatrix}X_2 & X_1\\ \mathrm{I}\\ X_3 & X_4\end{smallmatrix}\Big)\\
&+3(k_{12}+k_{34}-k_{14}-k_{23}-2)k_{13}k_{24}\Big(\begin{smallmatrix}X_2 & X_1\\ \mathrm{H}\\ X_3 & X_4\end{smallmatrix}-\begin{smallmatrix}X_2 & X_1\\ \mathrm{I}\\ X_3 & X_4\end{smallmatrix}\Big)\\
&+3(k_{12}+k_{13}+k_{24}+k_{34})k_{14}k_{23}\begin{smallmatrix}X_2 & X_1\\ \mathrm{H}\\ X_3 & X_4\end{smallmatrix}+3(k_{13}+k_{14}+k_{23}+k_{24})k_{12}k_{34}\begin{smallmatrix}X_2 & X_1\\ \mathrm{I}\\ X_3 & X_4\end{smallmatrix}.
\end{aligned}$$

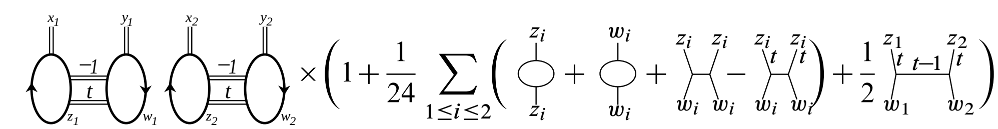
The lower part of $\chi_\hbar^{-1}\check{Z}(K\cup L)$ is calculated similarly as in Section 3.4, but a different point is to use



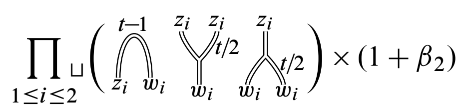
which is obtained from



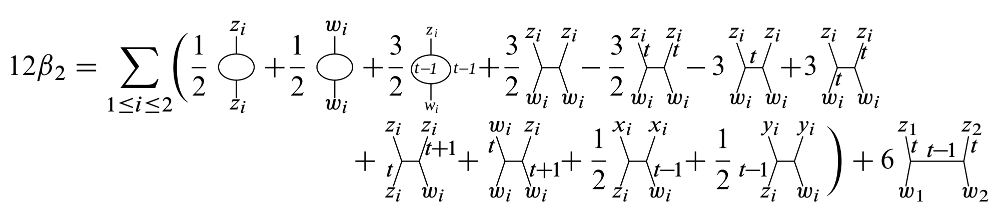
where we write $\gamma_1 \equiv_{(1)} \gamma_2$ if $\gamma_1-\gamma_2$ is equal to a linear sum of diagrams with at least 1 trivalent vertex. It follows that the lower part of $\chi_\hbar^{-1}\check{Z}(K\cup L)$ is given by



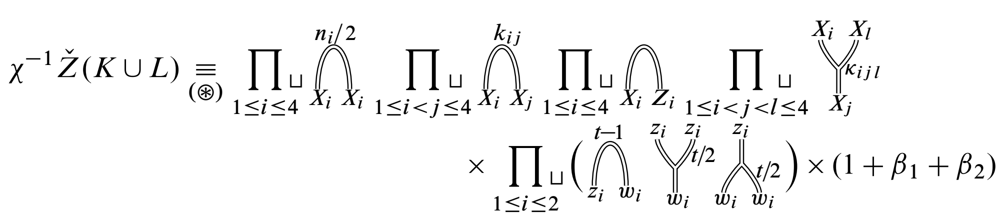
Hence, by Lemma 5.17, $\chi^{-1}\check{Z}(K\cup L)$ of the lower part is given by



where



By composing the above resulting formulas for the parts of $\chi^{-1}\check{Z}(K\cup L)$, we have



Its equivariant linking matrix is presented by

$$M = \left(\begin{array}{c|c} A & I \\ \hline I & J \end{array} \right),$$

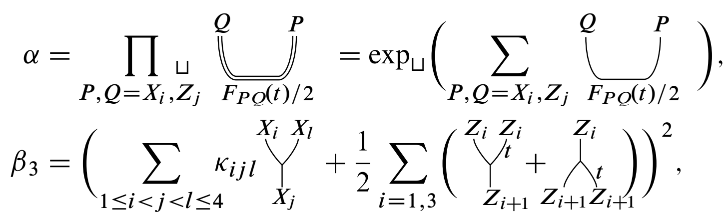
where I is the unit matrix of size 4, and A is the linking matrix of T_0, and

$$J = \begin{pmatrix} 0 & t-1 & 0 & 0 \\ t^{-1}-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & t-1 \\ 0 & 0 & t^{-1}-1 & 0 \end{pmatrix}.$$

As in Section 3.4, we obtain $\chi^{-1} Z(K_{T_0})$ from $\chi^{-1} \check{Z}(K \cup L)$ by the rational version of the Aarhus integral; in particular, the 2–loop part is presented by

$$Z^{(2\text{–loop})}(K_{T_0}) = \big\langle\, \alpha,\ (\beta_1 + \beta_2 + \beta_3)\,\big\rangle,$$

where

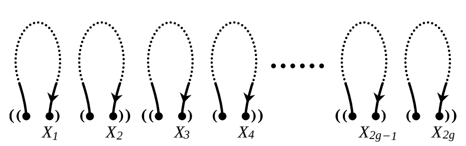


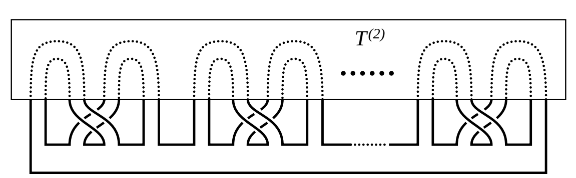
and $\big(F_{PQ}(t)\big) = -M^{-1}$; see also Lemma 4.2 for a presentation of its entries.

4.2 The 2–loop polynomial for knots of any genus

In this section, we give a presentation of the 2–loop part of the Kontsevich invariant for knots of any genus g. By using the presentation, we show that the degree of the 2–loop polynomial of a knot is bounded by twice the genus of the knot.

Similarly to the case of genus ≤ 2, we let T be a $2g$–component framed tangle, and let K_T be the knot obtained from T, as follows,

$T =$ 

(27) $K_T =$ 

where dotted lines in the picture of T imply strands possibly knotted and linked in some fashion, and $T^{(2)}$ denotes the tangle obtained from T by replacing each component of T with 2 parallel copies of it. Any knot of genus g can be presented by K_T for some T.

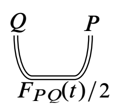
Similarly to the case of genus 2, we let T_0 be a particular tangle of T such that its linking matrix is (k_{ij}), where we put $k_{ii} = n_i$, and any 4–component sub-tangle of T_0 is of the form (26). Further, we set $K \cup L$ as in Section 4.1, and label the components of L by x_i, y_i, z_i, w_i similarly to the case of genus 2, and rename them by $X_{2i-1} = x_i$, $X_{2i} = y_i$, $Z_{2i-1} = z_i$, and $Z_{2i} = w_i$.

In the same way as in Section 4.1, we obtain the following proposition.

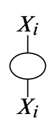
Proposition 4.1 *The 2–loop part of the Kontsevich invariant of K_{T_0} is given by*

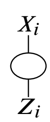
$$Z^{(2\text{–loop})}(K_{T_0}) = \big\langle\, \alpha,\ (\beta_1 + \beta_2 + \beta_3)\,\big\rangle,$$

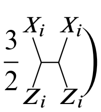
where

$\alpha = \prod_{P,Q = X_i, Z_j} \sqcup$ 

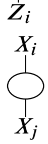
for $F_{PQ}(t)$ given in Lemma 4.2,

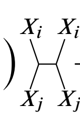
$12\beta_1 = \sum_{1 \le i \le 2g} \Big(\frac{n_i^2 + 2}{4}$ 

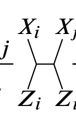
$+ \frac{n_i + 3}{2}$ 

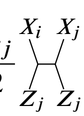
$- \frac{3}{2}$ 

$\Big)$

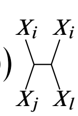
$+ \sum_{1 \le i < j \le 2g} \Big(\frac{3k_{ij}^2 + n_i k_{ij} + n_j k_{ij}}{2}$ 

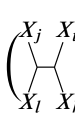
$- \big(k_{ij}^3 + \frac{k_{ij}}{2}\big)$ 

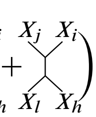
$+ \frac{k_{ij}}{2}$ 

$+ \frac{k_{ij}}{2}$ 

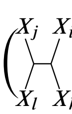
$\Big)$

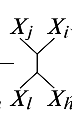
$+ \sum_{1 \le i \le 2g} \sum_{\substack{j<l \\ j,l \ne i}} \big(3k_{ij}k_{il}k_{jl} + k_{jl}(k_{ij}^2 + k_{il}^2)\big)$ 

$+ \sum_{i,j,l,h} \Big(2(k_{ij}k_{il}k_{ih} + k_{ji}k_{jl}k_{jh} + k_{li}k_{lj}k_{lh} + k_{hi}k_{hj}k_{hl}) \Big($ 

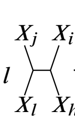
$+$ 

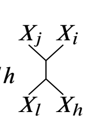
$\Big)$

$+ 3(k_{ij} + k_{lh} - k_{ih} - k_{jl} - 2)k_{il}k_{jh} \Big($ 

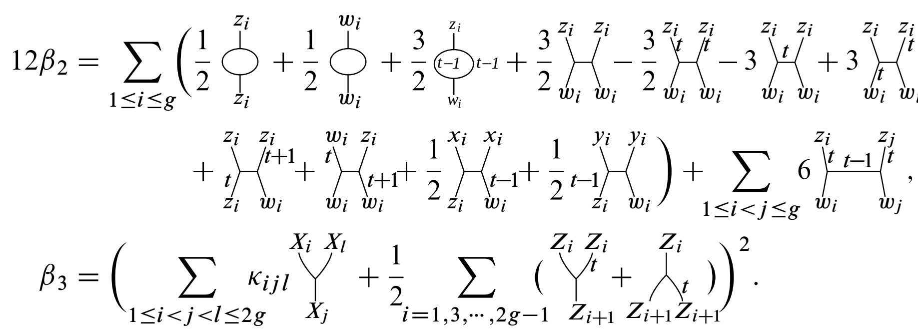
$-$ 

$\Big)$

$+ 3(k_{ij} + k_{il} + k_{jh} + k_{lh})k_{ih}k_{jl}$ 

$+ 3(k_{il} + k_{ih} + k_{jl} + k_{jh})k_{ij}k_{lh}$ 

$\Big),$



Lemma 4.2 *The* $F_{PQ}(t)$*'s in Proposition 4.1 are presented by*

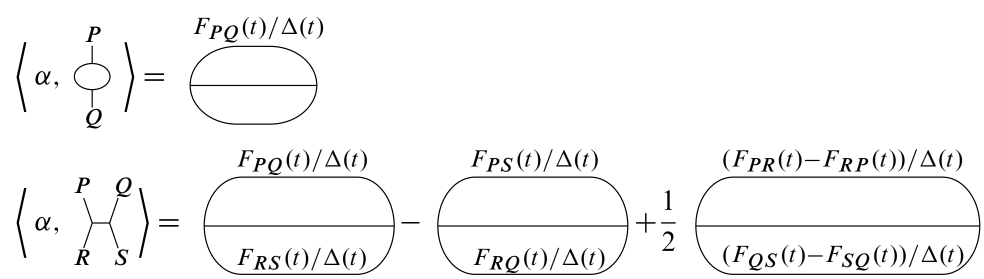
$$\frac{F_{X_iX_j}(t)}{\Delta(t)} = -(t^{1/2}-t^{-1/2})\, e_i^T\, (t^{1/2}V - t^{-1/2}V^T)^{-1}\, e_j,$$

$$\frac{F_{X_iZ_j}(t)}{\Delta(t)} = -e_i^T\, (t^{1/2}V - t^{-1/2}V^T)^{-1} \begin{pmatrix} 0 & -t^{-1/2} \\ t^{1/2} & 0 \end{pmatrix}^{\oplus g} e_j,$$

$$\frac{F_{Z_iX_j}(t)}{\Delta(t)} = -e_i^T \begin{pmatrix} 0 & -t^{-1/2} \\ t^{1/2} & 0 \end{pmatrix}^{\oplus g} (t^{1/2}V - t^{-1/2}V^T)^{-1}\, e_j,$$

$$\frac{F_{Z_iZ_j}(t)}{\Delta(t)} = e_i^T \begin{pmatrix} 0 & -t^{-1/2} \\ t^{1/2} & 0 \end{pmatrix}^{\oplus g} (t^{1/2}V - t^{-1/2}V^T)^{-1}\, A\, e_j,$$

where A *is the linking matrix of* T*, and* V *is the Seifert matrix of a natural Seifert surface of* K_T *in (27), and* $\Delta(t) = \det\big(t^{1/2}V - t^{-1/2}V^T\big)$*. Further, for* $P, Q, R, S \in \{X_i, Z_j\}$ *and* α *given in Proposition 4.1:*



Proof In a similar way to Section 4.1, $F_{PQ}(t)$'s are the entries of $\big(F_{PQ}(t)\big) = -M^{-1}$, where

$$M = \left(\begin{array}{c|c} A & I \\ \hline I & (t^{1/2}-t^{-1/2})\, J \end{array}\right), \qquad J = \begin{pmatrix} 0 & t^{1/2} \\ -t^{-1/2} & 0 \end{pmatrix}^{\oplus g},$$

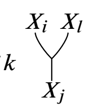
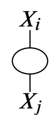
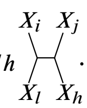
and I is the unit matrix of size $2g$. Hence,

$$-M^{-1}=\frac{1}{\Delta(t)}\left(\begin{array}{c|c} -(t^{1/2}-t^{-1/2})\,S^{-1} & -S^{-1}J \\ \hline -J\,S^{-1} & J\,S^{-1}A \end{array}\right),$$

where we put $S=t^{1/2}V-t^{-1/2}V^T$ and $\Delta(t)=\det S$, noting that $V=A+\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)^{\oplus g}$. Therefore, we obtain the former four formulas of the lemma.

We obtain the latter two formulas from the definition of the bracket. $\square$

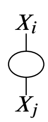
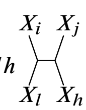
Proposition 4.3 *Let T, T_0, and K_T be as above, such that the linking matrices of T and T_0 are equal. We present $\chi^{-1}Z(T)$ by*

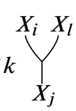
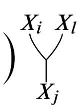
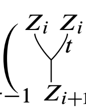
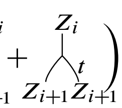
$$\log_{\sqcup}\chi^{-1}Z(T)\equiv\log_{\sqcup}\chi^{-1}Z(T_0)$$
$$+\sum_{1\le i<j<l\le 2g} a_{ijk}$$

$$+\sum_{1\le i<j\le 2g} b_{ij}$$

$$+\sum_{i,j,l,h} c_{ijlh}$$


Then, the 2–loop part of the Kontsevich invariant of K_T is given by

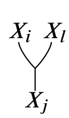
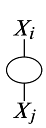
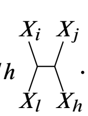
$$Z^{(2\text{–loop})}(K_T)=Z^{(2\text{–loop})}(K_{T_0})+\langle\,\alpha,\ \beta_1'+\beta_3'\,\rangle,$$

where α is as in Proposition 4.1, and

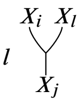
$$\beta_1'=\sum_{1\le i<j\le 2g} b_{ij}$$

$$+\sum_{i,j,l,h} c_{ijlh}$$


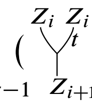
$$\beta_3'=\Bigg(\sum_{1\le i<j<l\le 2g} a_{ijk}$$

$$\Bigg)\times\Bigg(\sum_{1\le i<j<l\le 2g}\Big(\kappa_{ijl}+\frac{a_{ijk}}{2}\Big)$$

$$+\frac{1}{2}\sum_{i=1,3,\cdots,2g-1}\Bigg($$

$$+$$

$$\Bigg)\Bigg).$$

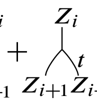
Proof As in Section 4.1, K_T (resp. K_{T_0}) is obtained from a link $K\cup L$ (resp. $K_0\cup L_0$) by surgery along L (resp. L_0). By Lemma 5.24, the Kontsevich invariant of $K\cup L$ and $K_0\cup L_0$ are related by

$$\log_{\sqcup}\chi^{-1}\check{Z}(K\cup L)-\log_{\sqcup}\chi^{-1}\check{Z}(K_0\cup L_0)$$
$$\equiv\sum_{1\le i<j<l\le 2g} a_{ijk}$$

$$+\sum_{1\le i<j\le 2g} b_{ij}$$

$$+\sum_{i,j,l,h} c_{ijlh}$$


Since $\chi^{-1}\check{Z}(K\cup L)\equiv_{(2)}\chi^{-1}\check{Z}(K_0\cup L_0)$

$$\equiv_{(2)} 1+\sum_{1\le i<j<l\le 2g}\kappa_{ijl}$$


$$+\frac{1}{2}\sum_{i=1,3,\cdots,2g-1}\Big($$


$$+$$


$$\Big),$$

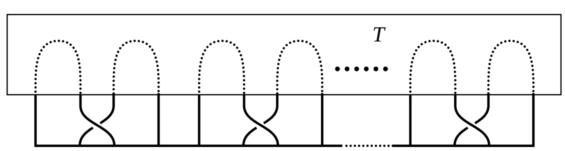
the part of $\chi^{-1}\check{Z}(K\cup L)-\chi^{-1}\check{Z}(K_0\cup L_0)$ which contributes to the 2–loop part is equal to $\beta_1'+\beta_3'$. Hence, we obtain the required formula. □

We mean by a *finite type invariant* of a knotted trivalent graph of degree d a coefficient of the Kontsevich invariant of the knotted trivalent graph of degree d, where the Kontsevich invariant of a knotted trivalent graph is defined in [31]. It is a problem proposed by Kricker [19] to express the 2–loop polynomial of a knot in terms of finite type invariants of degree ≤ 3 of the links obtained by pushing off the curves on a Seifert surface of the knot.

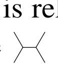
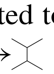
Theorem 4.4 (see a problem in [19]) *For a tangle T and the knot K_T given in (27), the 2–loop polynomial $\Theta(K_T)$ of K_T can be presented by finite type invariants of T of degree ≤ 3. In other words, the 2–loop polynomial of a knot can be presented by finite type invariants of degree ≤ 3 of a knotted trivalent graph which is a spine of a Seifert surface of the knot.*

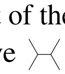
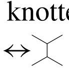
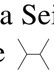
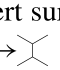
Proof By Proposition 4.3, the 2–loop part of the Kontsevich invariant of K_T is presented by finite type invariants of T of degree ≤ 3, hence, so is the 2–loop polynomial of K_T. This is the former statement of the theorem.

We put the knotted trivalent graph G_T by

$G_T =$ 

.

Then, G_T is related to a spine of a Seifert surface of the knot by a sequence of the local move  $\leftrightarrow$  . Further, when two knotted trivalent graphs are related by this move, finite type invariants of one of them can be presented by finite type invariants of the other. Hence, we obtain the latter statement of the theorem. □

Remark 4.5 The 2–loop polynomial of the knot given as the boundary of the ribbon graph of a knotted trivalent graph is a finite type invariant of the knotted trivalent graph of degree ≤ 3 which is unchanged under the local move  $\leftrightarrow$  . In general, the n–loop part of $\log_{\sqcup}\chi^{-1}Z(K)$ can be presented by finite type invariants of degree $\le (2n-1)$ of the knotted trivalent graph which is a spine of a Seifert surface of the knot such that the invariant is unchanged under the local move  $\leftrightarrow$  .

Remark 4.6 Rozansky [40] conjectured that the coefficients of the 2–loop polynomial are integers. Marché [26] proved a weaker statement that the coefficients are in $\frac{1}{12}\mathbb{Z}$. From Proposition 4.1, we can obtain an integrality result, but it is a further weaker statement that the coefficients are in $\frac{1}{24}\mathbb{Z}$.

Theorem 4.7 (a conjecture in [40]) *The degree of the 2–loop polynomial of a knot K is bounded by twice the genus g of K,*

$$\text{degree}_{t_1}\big(\Theta_K(t_1,t_2)\big) \le 2g.$$

Proof We choose a tangle T whose K_T in (27) is isotopic to K. By Proposition 4.1 and Proposition 4.3, it is sufficient to show that

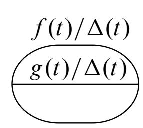
$$\Big\langle\, \alpha,\ (\beta_1+\beta_2+\beta_3+\beta_1'+\beta_3')\,\Big\rangle \underset{(\deg\le 2g)}{\equiv} 0, \tag{28}$$

where $\alpha, \beta_1, \beta_2, \beta_3, \beta_1', \beta_3'$ are given in the propositions, and we write

$$\theta_1 \underset{(\deg\le 2g)}{\equiv} \theta_2$$

if the t_1–degree of the corresponding 2–loop polynomial of $\theta_1-\theta_2$ is at most $2g$.

We calculate the $\beta_1+\beta_1'$ part of (28) as follows. For each diagram of $\beta+\beta_1'$, by Lemma 4.2, its corresponding part in (28) is equal to a linear sum of diagrams of the form



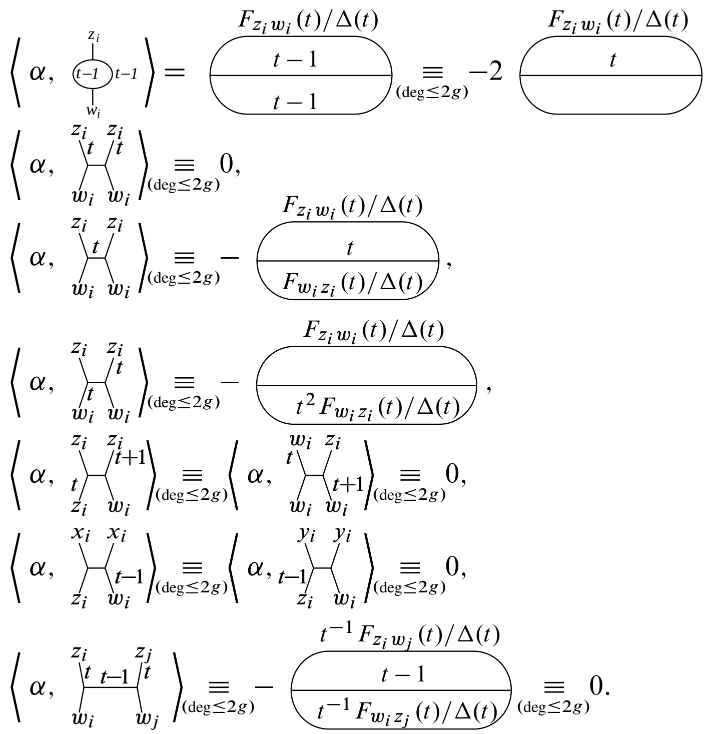
where $f(t)$ and $g(t)$ are either $F_{X_iX_j}(t)$, $F_{X_iZ_j}(t)$, $F_{Z_iX_j}(t)$, or $F_{Z_iZ_j}(t)$. The corresponding 2–loop polynomial of this form is

$$\sum_{\substack{\varepsilon=\pm1\\ \{i,j,k\}=\{1,2,3\}}} f(t_i^{\varepsilon})g(t_j^{\varepsilon})\Delta(t_k^{\varepsilon}) \ \in\ \mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1}]/(t_1t_2t_3=1).$$

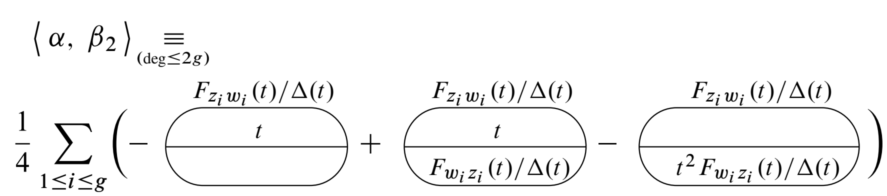
By Lemma 4.8, its t_1–degree is at most $2g$. Hence,

$$\Big\langle\, \alpha,\ \beta+\beta_1'\,\Big\rangle \underset{(\deg\le 2g)}{\equiv} 0.$$

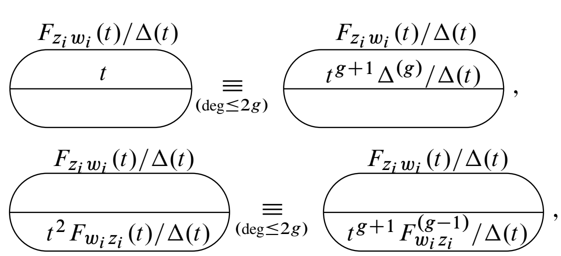
We calculate the β_2 part of (28) as follows. By the same argument as above, diagrams without labels of polynomials in $t^{\pm1}$ vanish in (28). For the other diagrams, the corresponding parts in (28) are calculated as follows,



Hence



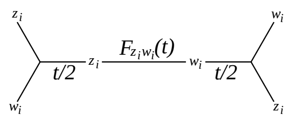
The first and third diagrams are further calculated as follows,



where we denote by $\Delta^{(g)}$ the coefficient of t^g in $\Delta(t)$ and denote by $F^{(g-1)}_{w_i z_i}$ the coefficient of t^{g-1} in $F_{w_i z_i}(t)$. Since they cancel with each other by Lemma 4.9, we have that

$$\left\langle \alpha,\ \beta_2 \right\rangle \underset{(\deg\le 2g)}{\equiv} \frac{1}{4} \sum_{1\le i\le g} \begin{array}{c} F_{z_i w_i}(t)/\Delta(t) \\ \hline t \\ \hline F_{w_i z_i}(t)/\Delta(t) \end{array}.$$

We calculate the $\beta_3+\beta_3'$ part of (28) as follows. By the same argument as in the case of β_1, most diagrams in $\beta_3+\beta_3'$ vanish in (28). The surviving case is graphically shown as follows.



Hence, we have that

$$\left\langle \alpha,\ \beta_3+\beta_3' \right\rangle \underset{(\deg\le 2g)}{\equiv} \frac{1}{4} \sum_{1\le i\le g} \begin{array}{c} F_{z_i w_i}(t)/\Delta(t) \\ \hline t^{-2} F_{z_i w_i}(t)/\Delta(t) \\ \hline F_{w_i z_i}(t)/\Delta(t) \end{array} \underset{(\deg\le 2g)}{\equiv} \frac{1}{4} \sum_{1\le i\le g} \begin{array}{c} F_{z_i w_i}(t)/\Delta(t) \\ \hline t^{2} F_{w_i z_i}(t)/\Delta(t) \\ \hline F_{w_i z_i}(t)/\Delta(t) \end{array}$$

Therefore, by summing the resulting formulas of the $\beta_1+\beta_1'$, β_2, $\beta_3+\beta_3'$ cases, we have that

$$\begin{aligned} &\left\langle \alpha,\ (\beta_1+\beta_1'+\beta_2+\beta_3+\beta_3') \right\rangle \\ &\underset{(\deg\le 2g)}{\equiv} \frac{1}{4} \sum_{1\le i\le g} \left(\begin{array}{c} F_{z_i w_i}(t)/\Delta(t) \\ \hline t \\ \hline F_{w_i z_i}(t)/\Delta(t) \end{array} + \begin{array}{c} F_{z_i w_i}(t)/\Delta(t) \\ \hline t^{2} F_{w_i z_i}(t)/\Delta(t) \\ \hline F_{w_i z_i}(t)/\Delta(t) \end{array} \right) \\ &\underset{(\deg\le 2g)}{\equiv} \frac{1}{4} \sum_{1\le i\le g} \left(\begin{array}{c} F_{z_i w_i}(t)/\Delta(t) \\ \hline t^{g+1}\Delta^{(g)}/\Delta(t) \\ \hline F_{w_i z_i}(t)/\Delta(t) \end{array} + \begin{array}{c} F_{z_i w_i}(t)/\Delta(t) \\ \hline t^{g+1} F^{(g-1)}_{w_i z_i}/\Delta(t) \\ \hline F_{w_i z_i}(t)/\Delta(t) \end{array} \right). \end{aligned}$$

Since this vanishes by Lemma 4.9, we obtain (28), as required. □

Lemma 4.8 *The minimal and maximal degrees of polynomials given in Lemma 4.2 are bounded, as follows,*

min-deg $\Delta(t) \ge -g$,	max-deg $\Delta(t) \le g$,
min-deg $F_{x_i x_j}(t) \ge -g$,	max-deg $F_{x_i x_j}(t) \le g$,
min-deg $F_{z_i z_j}(t) \ge -(g-1)$,	max-deg $F_{z_i z_j}(t) \le g-1$,
min-deg $F_{w_i w_j}(t) \ge -(g-1)$,	max-deg $F_{w_i w_j}(t) \le g-1$,
min-deg $F_{z_i w_j}(t) \ge -(g-1)$,	max-deg $F_{z_i w_j}(t) \le g$,
min-deg $F_{z_i x_j}(t) \ge -(g-1)$,	max-deg $F_{z_i x_j}(t) \le g$,
min-deg $F_{w_i x_j}(t) \ge -g$,	max-deg $F_{w_i x_j}(t) \le g-1$.

Further, if $i \neq j$ *, then min-deg* $F_{z_i w_j} \geq -(g-2)$ *.*

Proof The bound for $\Delta(t)$ is obtained from the fact that $\Delta(t) = \det(t^{1/2} V^T - t^{-1/2} V)$ and V is a $(2g) \times (2g)$ matrix.

The bound for $F_{X_i X_j}(t)$ is obtained as follows. From its definition in Lemma 4.2,

$$F_{X_i X_j}(t) = -(t^{1/2} - t^{-1/2}) \times e_i^T \cdot \Delta(t)(t^{1/2} V - t^{-1/2} V^T)^{-1} \cdot e_j .$$

The second factor in the right-hand side is equal to the (i, j) entry of $\Delta(t)(t^{1/2} V - t^{-1/2} V^T)^{-1}$, which is equal to a minor determinant of $(t^{1/2} V - t^{-1/2} V^T)$. Hence, we obtain the required bound for $F_{X_i X_j}(t)$.

In the same way, we obtain the bounds of the other polynomials except for the last one.

The last bound of the lemma is obtained as follows. By definition,

$$\begin{aligned} F_{z_i w_j}(t) = F_{Z_{2i-1} Z_{2j}}(t) &= \Delta(t)\, t^{1/2}\, e_{2i}^T\, (t^{1/2} V - t^{-1/2} V^T)^{-1}\, A\, e_{2j} \\ &= t^{1/2}\, e_{2i}^T\, (t^{1/2} V - t^{-1/2} V^T)'\, A\, e_{2j} , \end{aligned}$$

where we denote $(\det M) M^{-1}$ by M' for a regular matrix M, noting that entries of M' are presented by minor determinants of M. Hence, similarly as in the above case, we have that min-deg $F_{z_i w_j}(t) \geq -(g-1)$. Further, the coefficient $F_{z_i w_j}^{(-g+1)}$ of t^{-g+1} in $F_{z_i w_j}(t)$ is calculated as follows,

$$F_{z_i w_j}^{(-g+1)} = e_{2i}^T\, (-V^T)'\, A\, e_{2j} = e_{2i}^T\, (-V^T)'\, V^T\, e_{2j} = -(\det V)\, e_{2i}^T\, e_{2j} = 0.$$

This implies the required bound. $\square$

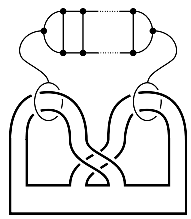
Lemma 4.9 *We have that* $F_{w_i z_i}^{(g-1)} = -\Delta^{(g)}$, *where* $\Delta^{(g)}$ *denotes the coefficient of* t^g *in* $\Delta(t)$, *and* $F_{w_i z_i}^{(g-1)}$ *denotes the coefficient of* t^{g-1} *in* $F_{w_i z_i}(t)$.

Proof In the same way as in the proof of Lemma 4.8,

$$F_{w_i z_i}^{(g-1)} = -(\det V)\, e_{2i}^T\, e_{2i} = -(\det V).$$

Further, $\Delta^{(g)} = \det V$, since $\Delta(t) = \det(t^{1/2} V - t^{-1/2} V^T)$. Hence, we obtain the lemma. $\square$

Remark 4.10 The bound of Theorem 4.7 might be sharp for most knots as we see in Table 1, Table 2, Example 4.11 and Example 4.13. However, we can construct examples which give inequality of the formula of Theorem 4.7. Such examples can be constructed by surgery along graph claspers with at least 3 trivalent vertices; for graph claspers, see Section 4.3. More generally, it is known [7] that we can construct knots which have the same $(<n)$–loop part of the Kontsevich invariant, but have different n–loop part, for any n. For example, the following knot and the trivial knot give such examples, when this graph clasper has $2n$ trivalent vertices.

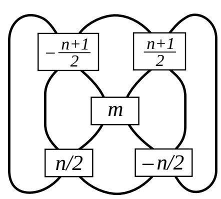


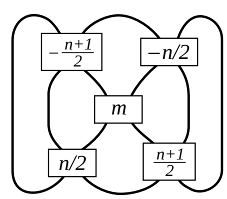
In particular, this knot gives the inequality of the formula of Theorem 4.7.

Example 4.11 It is shown in [35] that the degree of the 2–loop polynomial of the torus knot of type (p,q) equals $(p-1)(q-1)$. Since the genus of the torus knot of type (p,q) equals $(p-1)(q-1)/2$ (see eg [24]), torus knots give the equality of the formula of Theorem 4.7.

Example 4.12 In Theorem 3.7, we gave a presentation of the 2–loop polynomial for knot of genus 1. Its degree satisfies the formula of Theorem 4.7, and its equality holds when $V'_K(-1) \neq 0$ or $v_3(K) \neq 0$ (see also Proposition 1.1).

Example 4.13 The *generalized Kinoshita–Terasaka knot* $K^{\mathrm{KT}}_{m,n}$ [16] and the *generalized Conway knot* $K^{\mathrm{C}}_{m,n}$ (see [24]) are given by

$K^{\mathrm{KT}}_{m,n} =$ 

, $K^{\mathrm{C}}_{m,n} =$ 

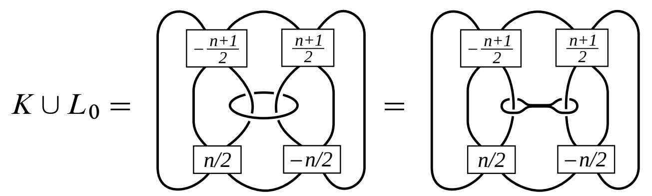
,

where a boxed "$k/2$" implies k half twists, as before. We show in Proposition 4.14 and Proposition 4.15, that the degree of the 2–loop polynomial for these knots are $2n-1$ and $4n-2$ respectively. This implies, by Theorem 4.7, that their genera are at least n and $2n-1$. Since there exist their Seifert surfaces of these genera, it follows that these are exactly their genera, as it has been shown in [8] geometrically, and in [38] by using the knot Floer homology.

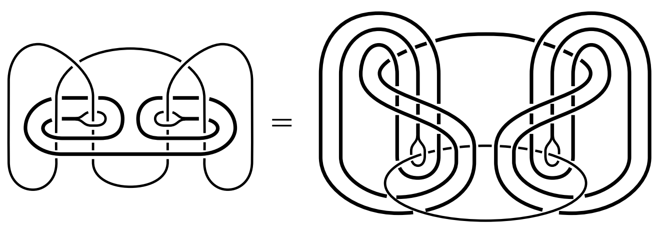
Proposition 4.14 *The t_1–degree of the 2–loop polynomial $\Theta_{K^{\mathrm{KT}}_{m,n}}(t_1,t_2)$ of the generalized Kinoshita–Terasaka knot $K^{\mathrm{KT}}_{m,n}$ is equal to $2n-1$.*

Proof For $n=2$, a concrete presentation of $\Theta_{K^{\mathrm{KT}}_{m,2}}(t_1,t_2)$ given in Proposition 2.4 shows that the proposition holds in this case.

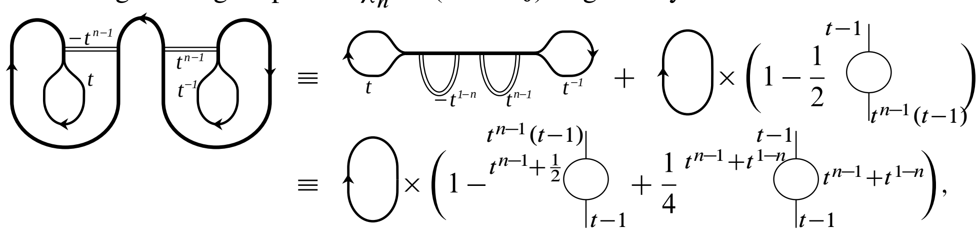
For general n, similarly as in Section 2.3, we put



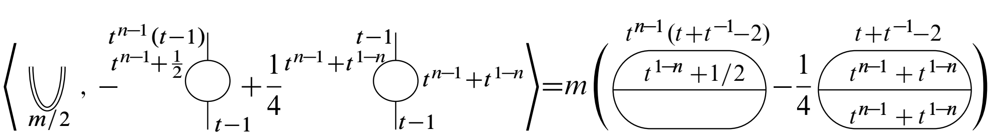
where we mean by a thick line 2 parallel copies of the line. For example, for $n=4$, it is isotopic to



By calculating its Kontsevich invariant in a similar way as in Section 2.3, it is shown that the highest-degree part of $\chi_{\hbar}^{-1}Z(K\cup L_0)$ is given by



where the first and second equivalences are obtained from Lemma 5.3 and Lemma 5.9 respectively. Hence, by a similar calculation as in Section 2.3, the highest-degree part of $Z^{(2\text{–loop})}(K^{\mathrm{KT}}_{m,n})$ is given by



Further, the highest-degree part of the 2–loop polynomial $\Theta_{K^{\mathrm{KT}}_{m,n}}(t_1,t_2)$ is presented, modulo the equivalence (5), by

$$12m\Big(t_1^{n-1}(t_1+t_1^{-1}-2)\big(t_2^{1-n}+\tfrac{1}{2}\big) \\ -\tfrac{1}{4}(t_1+t_1^{-1}-2)(t_2^{n-1}+t_2^{1-n})(t_1^{n-1}t_2^{n-1}+t_1^{1-n}t_2^{1-n})\Big). \tag{29}$$

As its reduction, the highest-degree part of the reduced 2–loop polynomial $\widehat{\Theta}_{K^{\mathrm{KT}}_{m,n}}(t)$ is presented by

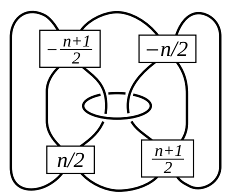
$$2m(t^{2n-2}+t^{2-2n}),$$

whose degree is equal to $2n-2$. This implies that the degree of (29) is at least $2n-1$. Further, the degree of the symmetrization of (29) is at most $2n-1$. Therefore, the degree of $\Theta_{K^{\mathrm{KT}}_{m,n}}(t_1,t_2)$ is equal to $2n-1$, as required. □

Proposition 4.15 *The t_1–degree of the 2–loop polynomial $\Theta_{K^{\mathrm{C}}_{m,n}}(t_1,t_2)$ of the generalized Conway knot $K^{\mathrm{C}}_{m,n}$ is equal to $4n-2$.*

Proof For $n=2$, a concrete presentation of $\Theta_{K^{\mathrm{KT}}_{m,2}}(t_1,t_2)$ given in Proposition 2.5 shows that the proposition holds in this case.

For general n, we put

$K \cup L_0 =$ 

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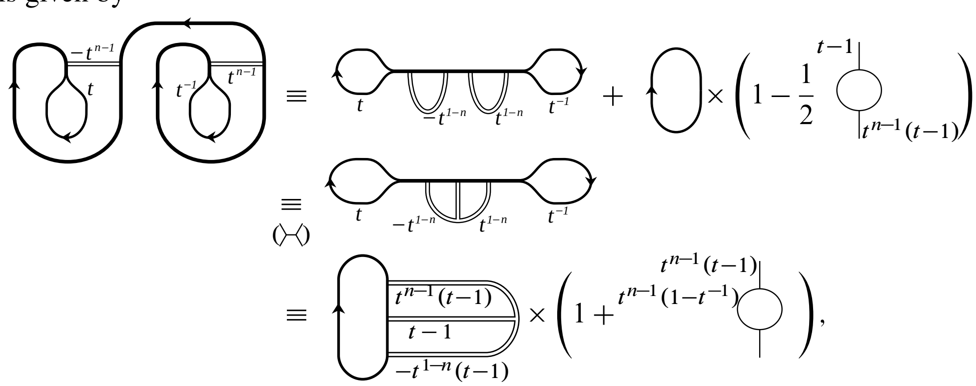
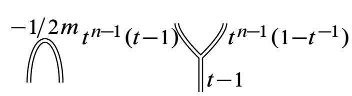
By calculating its Kontsevich invariant similarly to the proof of Proposition 4.14 following arguments in Section 2.4, it is shown that the highest-degree part of $\chi_\hbar^{-1}Z(K\cup L_0)$ is given by



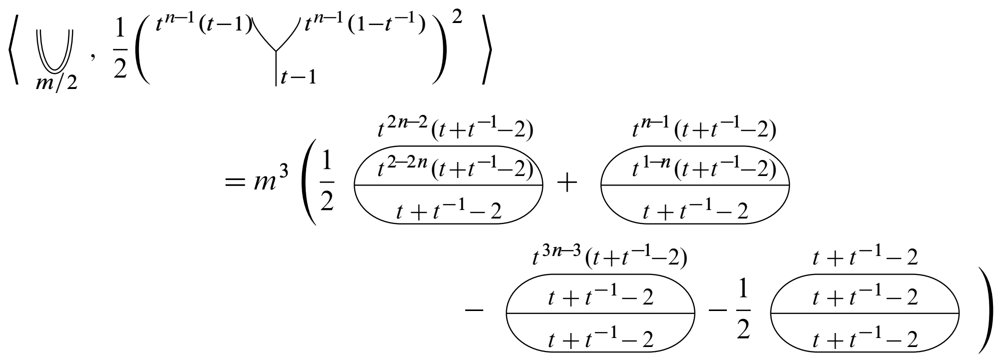
,

where the first and second equivalences are obtained from Lemma 5.3 and Lemma 5.10 respectively. Hence, the highest-degree part of $\chi^{-1}\check{Z}(K\cup L)$ is given by



.

Further, the highest-degree part of $Z^{(2\text{–loop})}(K^{\mathrm{C}}_m)$ is given by



Hence, the highest-degree part of the 2–loop polynomial $\Theta_{K_m^{\mathrm{C}}}(t_1, t_2)$ is presented, modulo the equivalence (5), by

$$12m^3(t_1+t_1^{-1}-2)(t_2+t_2^{-1}-2)(t_1t_2+t_1^{-1}t_2^{-1}-2)\\ \left(\tfrac{1}{2}t_1^{2n-2}t_2^{2-2n}+t_1^{n-1}t_2^{1-n}-t_1^{3n-3}-\tfrac{1}{2}\right).$$

Further, its symmetrization is presented by

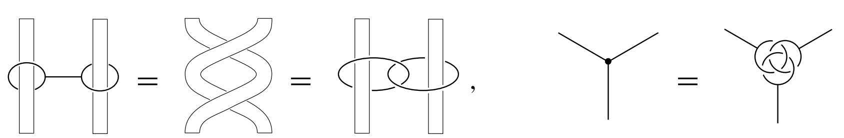
$$m^3(t_1^{n-1}t_2^{1-n}+t_1^{1-n}t_2^{n-1}-2)(t_1^{2n-2}t_2^{n-1}+t_1^{2-2n}t_2^{1-n}-2)\\ \times(t_1^{n-1}t_2^{2n-2}+t_1^{1-n}t_2^{2-2n}-2)(t_1+t_1^{-1}-2)(t_2+t_2^{-1}-2)(t_1t_2+t_1^{-1}t_2^{-1}-2),$$

whose t_1–degree is equal to $4n-2$. Therefore, the degree of $\Theta_{K_m^{\mathrm{C}}}(t_1, t_2)$ is equal to $4n-2$, as required. □

4.3 Clasper surgery formulas

In this section, we show surgery formulas for the 2–loop polynomial under surgery along some types of graph claspers, as consequences of calculations of the previous section.

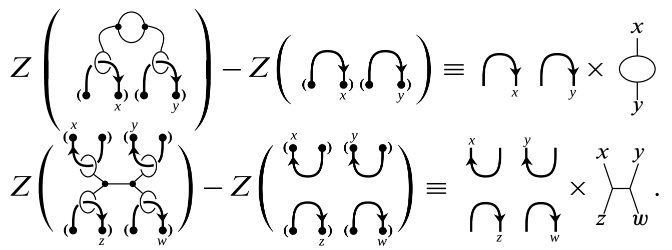
A *graph clasper* is an embedded graph in a knot complement, such as shown in Lemma 4.16, defined by



where the right-hand side of the first formula implies the result of surgery along the Hopf link. The embedded graph in the first picture is called a *clasper*, and a circle at an end of a clasper is called a *leaf*; see [12] for details of claspers.

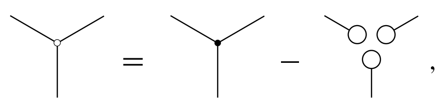
As a particular case of a modification of results in [20], we obtain the following lemma; see also [33] for calculations of the Kontsevich invariant for graph claspers.

Lemma 4.16 *We have that*



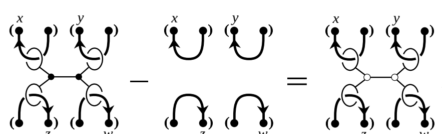
Sketch proof We show a sketch proof of the lemma; for details see [33].

Introducing a white trivalent vertex by

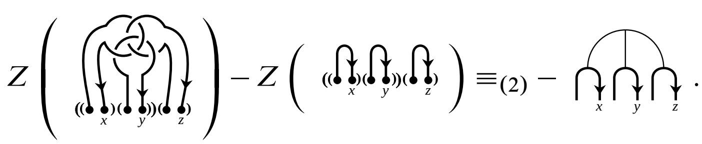


the left-hand side of the second formula of the lemma is equal to the Kontsevich invariant of

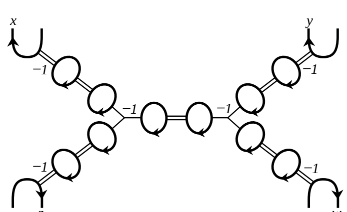
(30)



because, when we break one trivalent vertex of a graph clasper, the whole of the graph clasper vanishes. By Lemma 4.20 below, The Kontsevich invariant of a white vertex is presented by



Hence, by replacing each clasper in (30) with a Hopf link, the Kontsevich invariant of (30) is obtained from

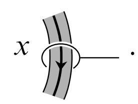


by the Aarhus integral, which gives the surgery along the Hopf links. This gives the right-hand side of the second formula of the lemma.

The first formula of the lemma is obtained in the same way. $\square$

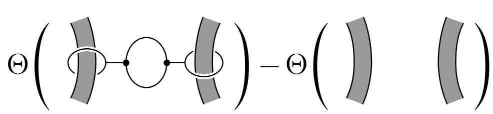
Let F be a Seifert surface of a knot K. The Seifert form $H_1(F) \otimes H_1(F) \to \mathbb{R}$ is defined by taking $a \otimes b$ to the linking number of a and b^+, where b^+ denotes the puss-off of b in the normal direction of F. It is presented by a Seifert matrix, fixing a basis of $H_1(F)$. We denote by e_x, e_y the vectors presenting cohomology classes $x, y \in H^1(F)$ for the basis. The scaler $e_x^T\,(t^{1/2}V - t^{-1/2}V^T)^{-1}\,e_y$ depends only on the Seifert form and $x, y \in H^1(F)$, independently of the choice of a basis of $H_1(F)$. The Alexander polynomial of the knot is given by $\Delta_K(t) = \det(t^{1/2}V - t^{-1/2}V^T)$.

A leaf of a clasper in the complement of a Seifert surface F of a knot is associated with a cohomology class in $H^1(F)$ counting cycles as



The following two propositions can alternatively be obtained from a surgery formula in [19].

Proposition 4.17 *Consider a graph clasper of the form in the following formula, embedded in the complement of a Seifert surface of a knot K. Let x, y be cohomology classes in $H^1(F)$ associated with the leaves of the graph clasper. Then, the change of the 2–loop polynomial of the knot by surgery along the graph clasper is presented, modulo the equivalence (5), by*



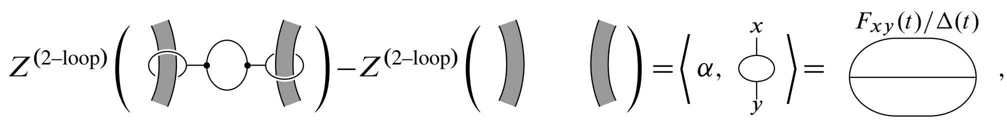
$$\sim\ 12\,F_{xy}(t_1)\,\Delta_K(t_2)\,\Delta_K(t_3),$$

where

$$\frac{F_{xy}(t)}{\Delta_K(t)} = -(t^{1/2} - t^{-1/2})\,e_x^T\,\big(t^{1/2}V - t^{-1/2}V^T\big)^{-1}\,e_y.$$

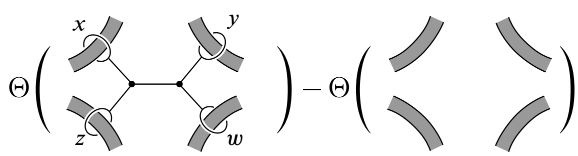
Proof Since the formula of the lemma is independent of a choice of a Seifert surface F and a basis of $H_1(F)$, it is sufficient to show the proposition for a particular choice of them. We assume that V is a Seifert matrix for a natural basis of $H_1(F)$ of a natural Seifert surface F of K_T in (27). We can also assume, without loss of generality, that

x and y are cohomology classes given by components X_i and X_j of T in (27). By Proposition 4.3,



where α is given in Proposition 4.1, and $F_{xy}(t)/\Delta(t)$ given above is equal to the one given in Lemma 4.2. The corresponding 2–loop polynomial gives the required formula. $\square$

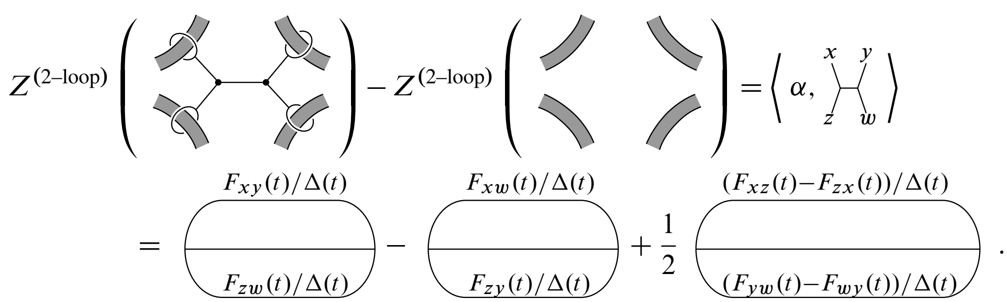
Proposition 4.18 *Consider a graph clasper of the form in the following formula, embedded in the complement of a Seifert surface of a knot K. Let x, y, z, w be cohomology classes in $H^1(F)$ associated with the leaves of the graph clasper as shown at the leaves. Then, the change of the 2–loop polynomial of the knot by surgery along the graph clasper is presented, modulo the equivalence (5), by*



$$\sim 12\big(F_{xy}(t_1)\, F_{zw}(t_2) - F_{xw}(t_1)\, F_{zy}(t_2) + \tfrac{1}{2}\big(F_{xz}(t_1) - F_{zx}(t_1)\big)\big(F_{yw}(t_2) - F_{wy}(t_2)\big)\big)\, \Delta_K(t_3),$$

where $F_{XY}(t)$ is the one given in Proposition 4.17.

Proof In the same way as the proof of Proposition 4.17,

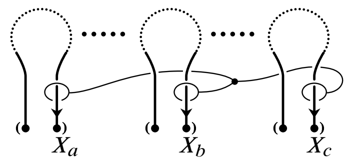


The corresponding 2–loop polynomial gives the required formula. $\square$

The following proposition can alternatively be obtained by using a surgery formula in [26].

Proposition 4.19 *Let T be a tangle as in (27), and let T^C be a tangle obtained from T by surgery along a graph clasper C as*

$$T^C =$$



Suppose that

$$\chi^{-1}Z(T) \equiv_{(2)} \prod_{1\le i\le 2g}{}_{\sqcup} \overset{n_i/2}{\underset{X_i\;X_i}{\bigcap}} \prod_{1\le i<j\le 2g}{}_{\sqcup} \overset{k_{ij}}{\underset{X_i\;X_j}{\bigcap}} \times \Big(1+\sum_{i,j,l}\kappa_{ijl} \overset{X_i\;X_l}{\underset{X_j}{\Upsilon}}\Big).$$

Then

$$Z^{(2\text{–loop})}(K_T^C) - Z^{(2\text{–loop})}(K_T) = \langle\, \alpha,\ \beta\,\rangle,$$

where

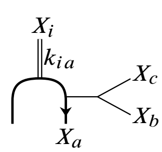
$$\beta = -\overset{X_a\;X_c}{\underset{X_b}{\Upsilon}} \times \bigg(-\frac{1}{2}\overset{X_a\;X_c}{\underset{X_b}{\Upsilon}} + \sum_{i,j,l}\kappa_{ijl}\overset{X_i\;X_l}{\underset{X_j}{\Upsilon}} + \frac{1}{2}\sum_{i=1,3,\cdots,2g-1}\Big(\overset{Z_i\;Z_i}{\underset{Z_{i+1}}{\Upsilon_t}} + \overset{Z_i}{\underset{Z_{i+1}\;Z_{i+1}}{\curlywedge_t}}\Big)\bigg)$$
$$+\frac{1}{2}\Big(\overset{X_a\;X_a}{\underset{X_b\;X_c}{\mathrm{H}}} + \overset{X_b\;X_b}{\underset{X_a\;X_c}{\mathrm{H}}} + \overset{X_c\;X_c}{\underset{X_a\;X_b}{\mathrm{H}}}\Big) + \frac{1}{2}\sum_{1\le i\le g}\Big(k_{ia}\overset{X_i\;X_c}{\underset{X_a\;X_b}{\mathrm{H}}} + k_{ib}\overset{X_i\;X_a}{\underset{X_b\;X_c}{\mathrm{H}}} + k_{ic}\overset{X_i\;X_b}{\underset{X_c\;X_a}{\mathrm{H}}}\Big).$$

Proof By Lemma 4.20 below, we have that

$$\chi^{-1}Z(T^C) - \chi^{-1}Z(T)$$
$$\equiv \prod_{1\le j\le 2g}{}_{\sqcup} \overset{n_i/2}{\underset{X_i\;X_i}{\bigcap}} \prod_{1\le i<j\le 2g}{}_{\sqcup} \overset{k_{ij}}{\underset{X_i\;X_j}{\bigcap}} \times \bigg(-\overset{X_a\;X_c}{\underset{X_b}{\Upsilon}} \times \Big(1-\frac{1}{2}\overset{X_a\;X_c}{\underset{X_b}{\Upsilon}} + \sum_{i,j,l}\kappa_{ijl}\overset{X_i\;X_l}{\underset{X_j}{\Upsilon}}\Big)$$
$$+\frac{1}{2}\Big(\overset{X_a\;X_a}{\underset{X_b\;X_c}{\mathrm{H}}} + \overset{X_b\;X_b}{\underset{X_a\;X_c}{\mathrm{H}}} + \overset{X_c\;X_c}{\underset{X_a\;X_b}{\mathrm{H}}}\Big) + \frac{1}{2}\sum_{1\le i\le g}\Big(k_{ia}\overset{X_i\;X_c}{\underset{X_a\;X_b}{\mathrm{H}}} + k_{ib}\overset{X_i\;X_a}{\underset{X_b\;X_c}{\mathrm{H}}} + k_{ic}\overset{X_i\;X_b}{\underset{X_c\;X_a}{\mathrm{H}}}\Big)\bigg),$$

since

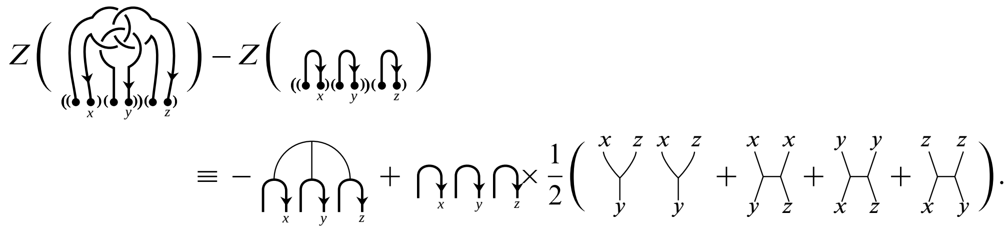
$$\chi^{-1}$$



$$= \overset{k_{ia}}{\underset{X_i\;X_a}{\bigcap}} \times \Big(\overset{X_a\;X_c}{\underset{X_b}{\Upsilon}} + \frac{k_{ia}}{2}\overset{X_i\;X_c}{\underset{X_a\;X_b}{\mathrm{H}}}\Big).$$

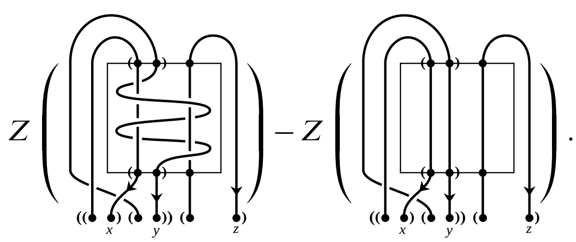
Hence, by Proposition 4.3, we obtain the required formula. □

Lemma 4.20 *We have that*

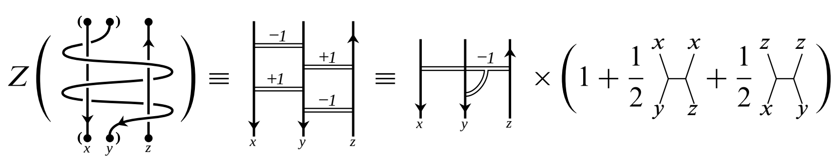


Proof The left-hand side of the required formula is presented by

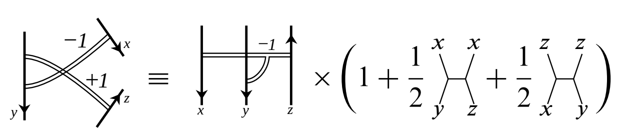
(31)



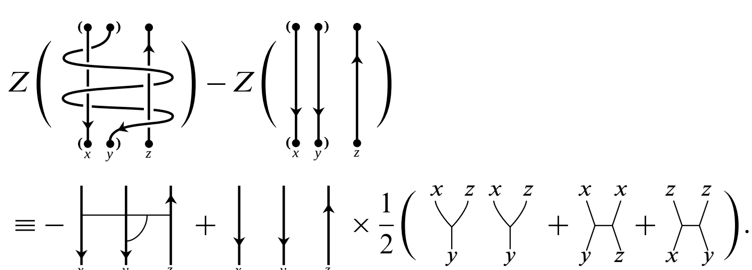
We calculate the difference of the part surrounded by the box. We have that



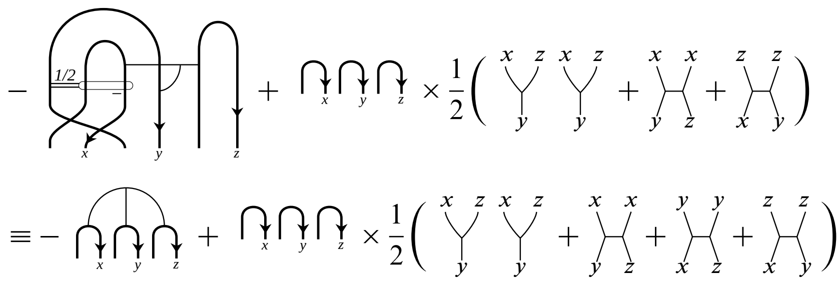
where the second equivalence is obtained from the following formula,



which is obtained by Lemma 5.6. Hence,

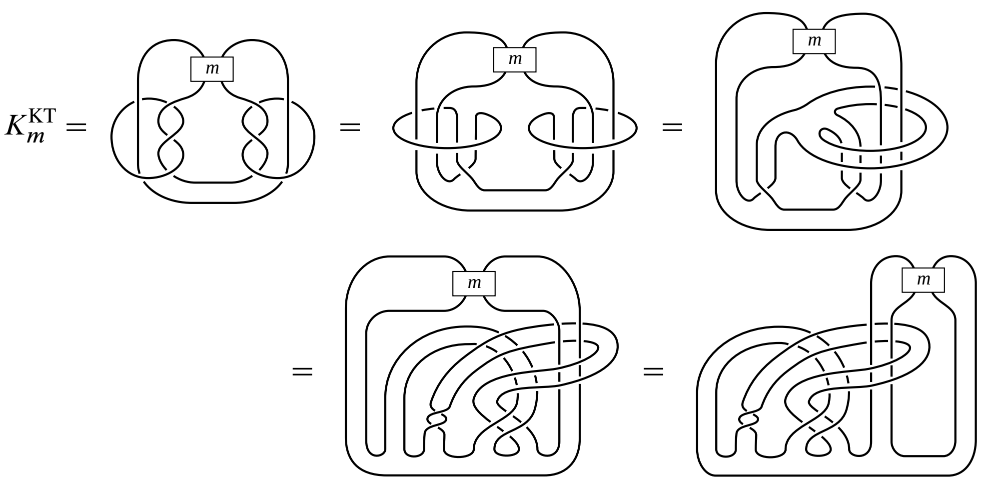


Therefore, (31) is calculated as follows,

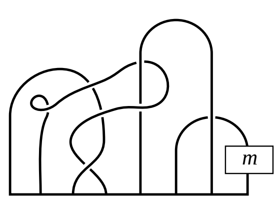


This gives the right-hand side of the required formula. □

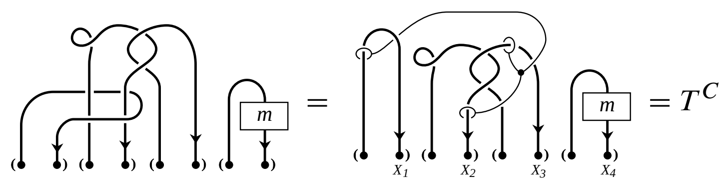
Example 4.21 Using Proposition 4.19, we calculate the 2–loop polynomial of the Kinoshita–Terasaka knot K_m^{KT} again, to verify Proposition 2.4, as follows. The Kinoshita–Terasaka knot is presented by



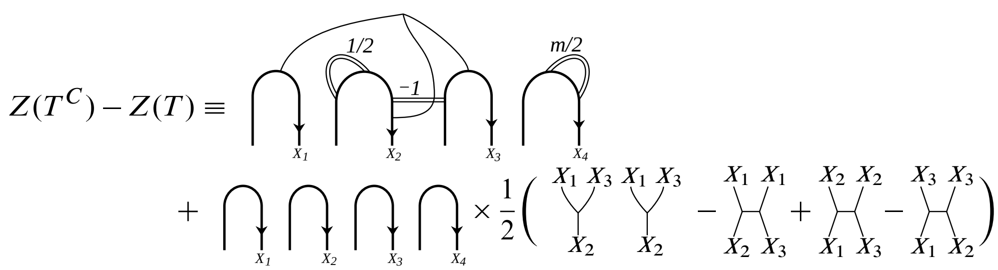
It is isotopic to the boundary of the ribbon graph given by the following knotted trivalent graph:



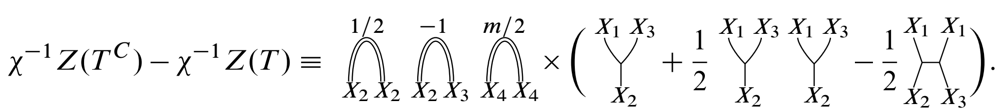
Hence, the Kinoshita–Terasaka knot is isotopic to K_{T^C} given in (27) for



where T denotes the tangle in the middle picture ignoring the graph clasper C, and T^C denotes the tangle obtained from T by surgery along C, noting that K_T is isotopic to the trivial knot. We have that



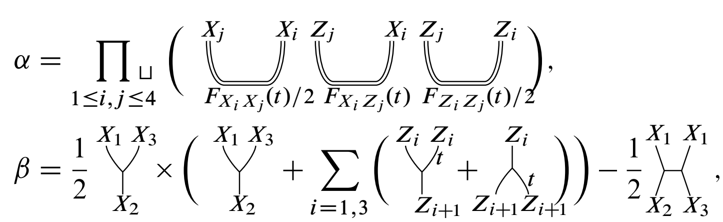
Hence,



By Proposition 4.19,

$$Z^{(2\text{–loop})}(K_m^{\mathrm{KT}}) = Z^{(2\text{–loop})}(K_T^C) - Z^{(2\text{–loop})}(K_T) = \langle\, \alpha,\ \beta\, \rangle,$$

where $\Delta_{K_T}(t) = \Delta_{K_T^C}(t) = 1$, and



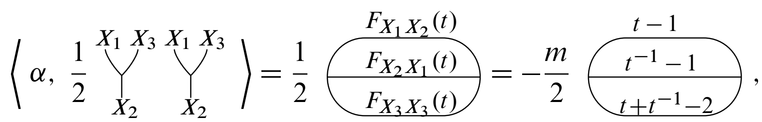
and $F_{PQ}(t)$'s are given by

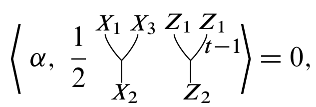
$$F_{X_i X_j}(t) = e_i^T \begin{pmatrix} \begin{matrix} -t+2-t^{-1} \\ +m(t^2-4t+6-4t^{-1}+t^{-2}) \end{matrix} & t-1 & m(t^2-3t+3-t^{-1}) & -t^2+2t-1 \\ -1+t^{-1} & 0 & 0 & 0 \\ m(-t+3-3t^{-1}+t^{-2}) & 0 & m(-t+2-t^{-1}) & t-1 \\ -1+2t^{-1}-t^{-2} & 0 & -1+t^{-1} & 0 \end{pmatrix} e_j\,,$$

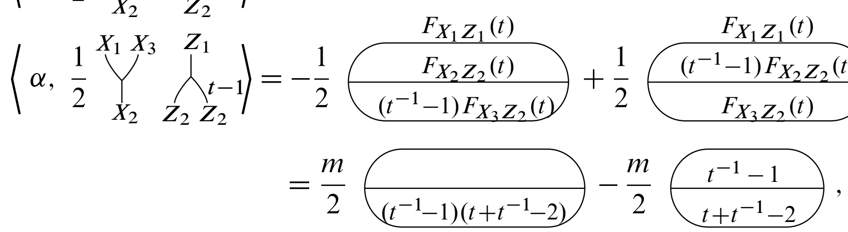
$$F_{X_i Z_j}(t) = e_i^T \begin{pmatrix} -1 & -t+1+m(t^2-3t+3-t^{-1}) & t-1 & m(t^2-2t+1) \\ 0 & -1 & 0 & 0 \\ 0 & m(-t+2-t^{-1}) & -1 & m(-t+1) \\ 0 & -1+t^{-1} & 0 & -1 \end{pmatrix} e_j \,,$$

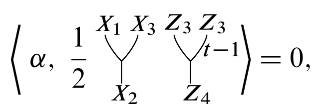
$$F_{Z_i Z_j}(t) = e_i^T \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1+m(-t+2-t^{-1}) & -1 & m(-t+1) \\ 0 & -1 & 0 & 0 \\ 0 & m(1-t^{-1}) & 0 & m \end{pmatrix} e_j \,.$$

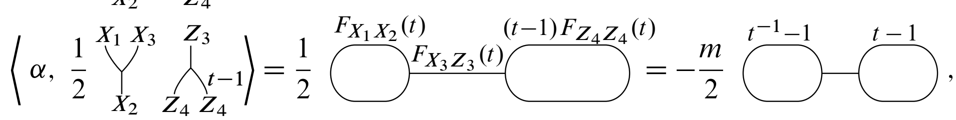
We have that

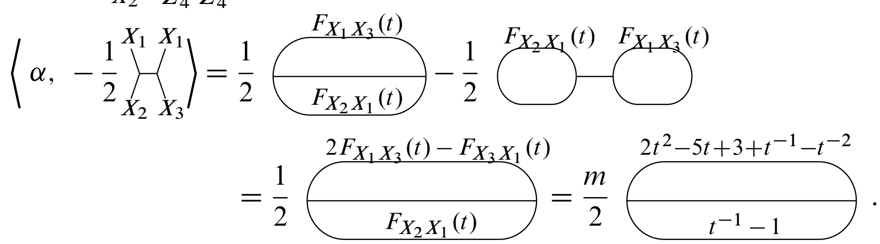




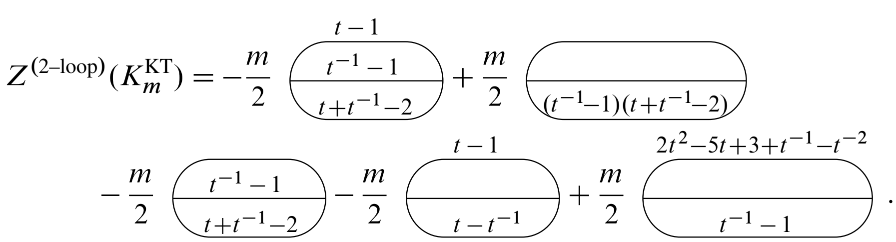








Hence,



Therefore, by definition, the 2–loop polynomial of the Kinoshita–Terasaka knot is presented by

$$\Theta_{K_m^{\mathrm{KT}}}(t_1, t_2) = m\,(2T_{1,0} - 2T_{2,0} - 2T_{2,1} + T_{3,1}),$$

as we showed in Proposition 2.4.

5 Calculation of Gaussian diagrams

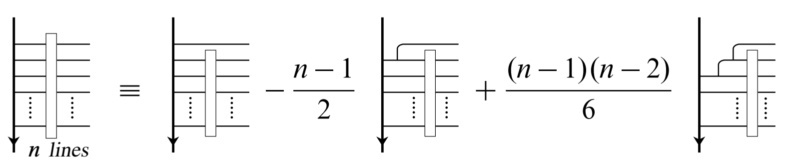
In this section, we develop methodology to calculate Gaussian diagrams. We prove basic formulas in Section 5.1, calculate the PBW isomorphism for Gaussian diagrams in Section 5.2, and show further formulas to calculate the 2–loop polynomial for knots of genus 1 and for knots of any genus in Section 5.3 and Section 5.4 respectively.

5.1 Basic formulas for Gaussian diagrams

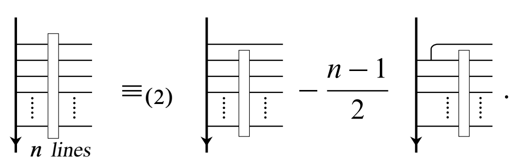
In this section, we show basic formulas to calculate Gaussian diagrams. Some of the formulas are useful when we move an exponential chord beyond other chords.

Recall that a box over parallel chords denotes the symmetrizer of the chords as in (3).

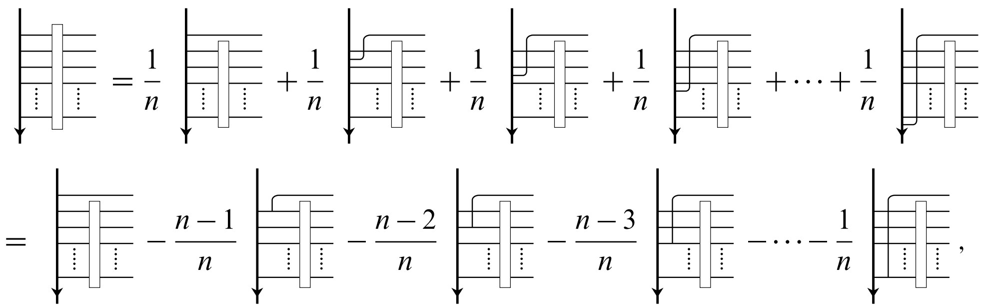
Lemma 5.1



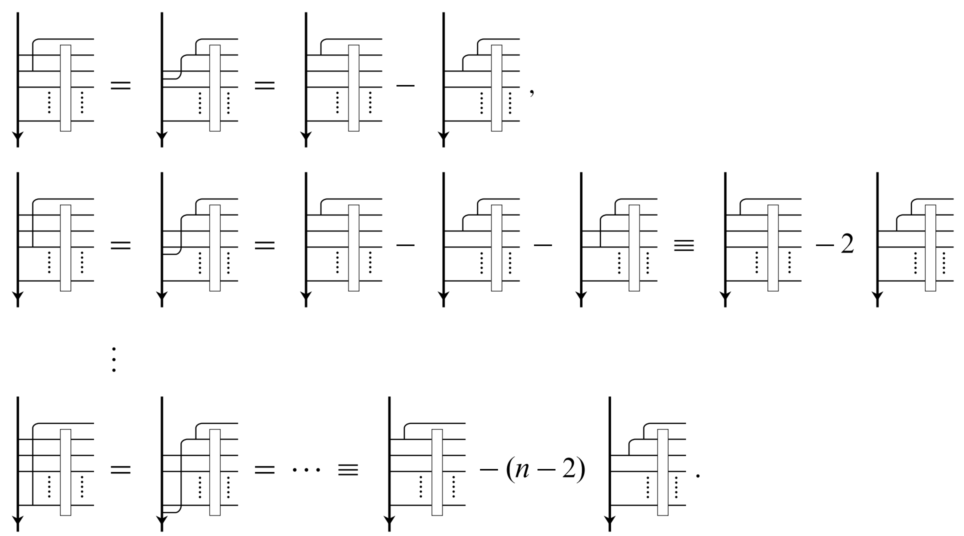
In particular,



Proof When the left-hand side of the following formula is a part of a diagram, we can calculate the part as follows,



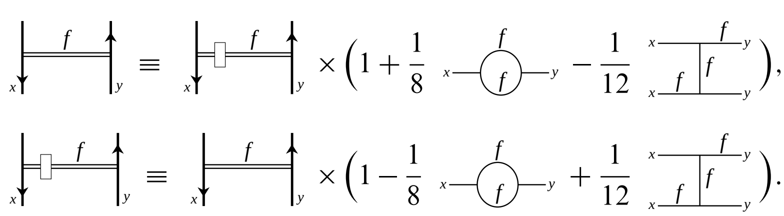
where n is the number of chords shown in the left-hand side. Diagrams in the right-hand side are further calculated as follows,



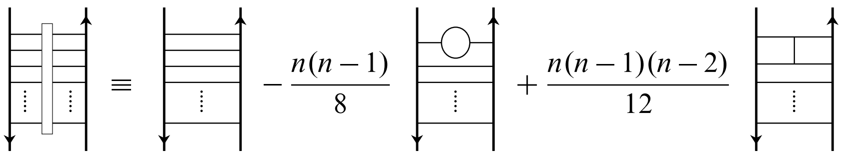
Hence, we obtain the first formula of the lemma.

The second formula of the lemma is a reduction of the first formula. □

Lemma 5.2 *For a power series f,*

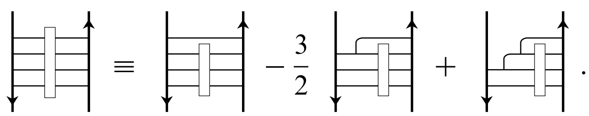


Proof The first formula of the lemma is easily obtained from the second formula. The second formula is reduced to its expansion,

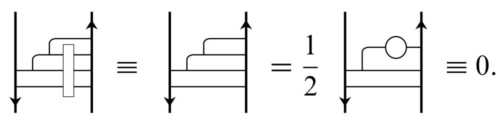
(32) 

where the diagrams are of degree n. We show this formula by induction on n.

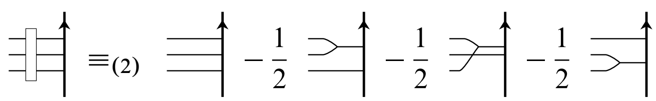
For simplicity, we calculate the case $n=4$; the general case can be calculated in the same way. By Lemma 5.1, we have that

(33) 

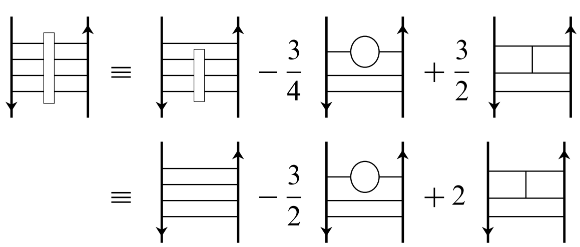
The last diagram vanishes because



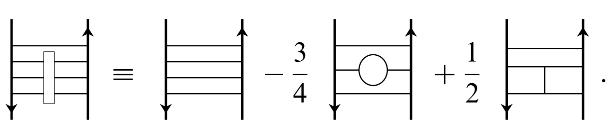
By substituting the following relation



into the second last diagram of (33), we have that

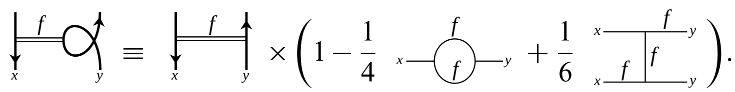


where the second equivalence is obtained from the assumption of induction,

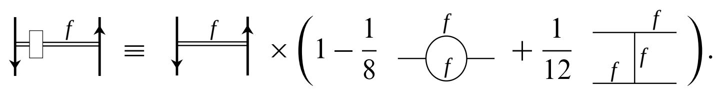


Hence, we obtain (32) for $n=4$. □

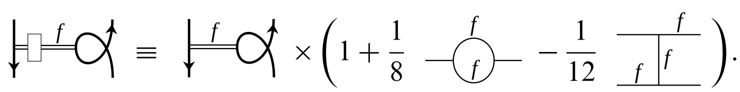
Lemma 5.3 *For a power series f,*



Proof By Lemma 5.2,

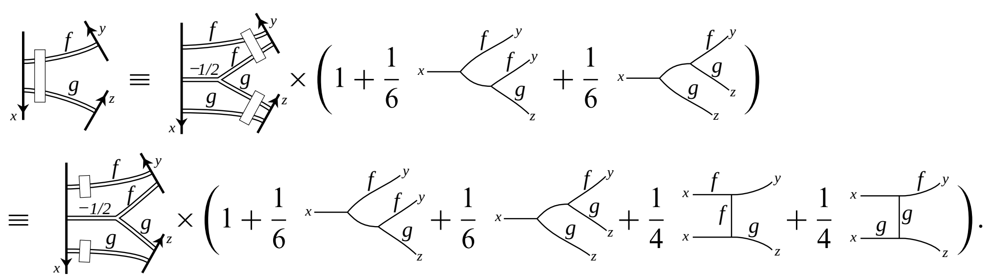


By applying the antipode on the right string and replacing f with $-f$, we have that

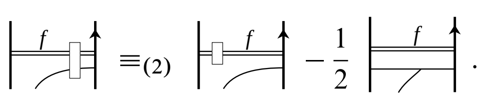


Since the diagrams in the left-hand sides of the above two formulas are equal, we obtain the required formula. □

Lemma 5.4 *For power series f and g,*

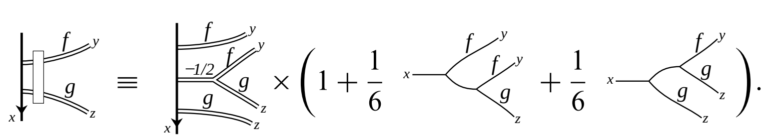


Proof The second equivalence of the lemma is easily obtained by using the relation,

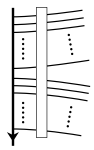
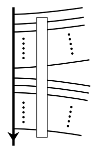
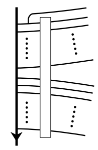
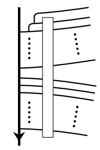


The first equivalence is rewritten as Lemma 5.5 below. □

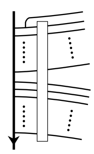
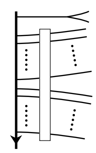
Lemma 5.5 *For power series f and g,*

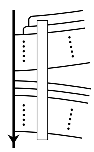
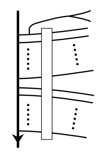


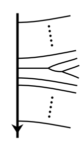
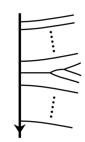
Proof For simplicity, we omit f, g, x, y, z in diagrams of the proof. By Lemma 5.1,

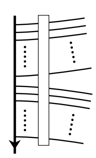
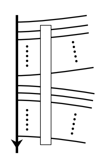
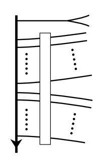
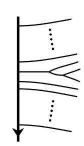
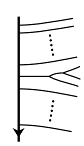
 $\equiv$  $-\dfrac{n+m-1}{2}$  $+\dfrac{(n+m-1)(n+m-2)}{6}$  ,

where n (resp. m) is the number of chords whose right ends are labeled by y (resp. z) in the diagram of the left-hand side. The last two terms are calculated as follows,

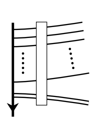
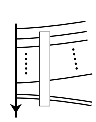
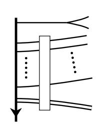
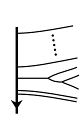
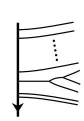
$-\dfrac{n+m-1}{2}$  $=-\dfrac{m}{2}$  ,

$\dfrac{(n+m-1)(n+m-2)}{6}$  $=\dfrac{m(n+m-2)}{6}$ 

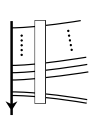
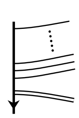
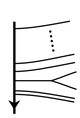
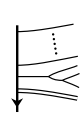
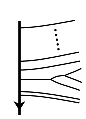
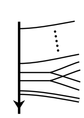
$=-\dfrac{m(n-1)}{6}$  $+\dfrac{m(m-1)}{6}$  .

Hence,  $\equiv$  $-\dfrac{m}{2}$  $-\dfrac{m(n-1)}{6}$  $+\dfrac{m(m-1)}{6}$  .

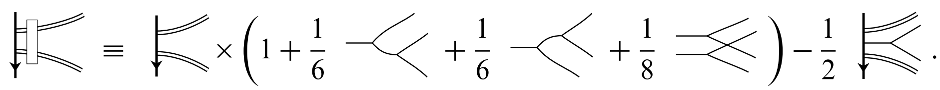
This formula implies the mth part of the expansion of the following formula,

 $\equiv$  $-\dfrac{1}{2}$  $-\dfrac{n-1}{6}$  $+\dfrac{1}{6}$  ,

where n is the number of chords whose right ends are labeled by y (relatively upward ends) in the diagram of the left-hand side. From this formula, we obtain the following formula by induction on n,

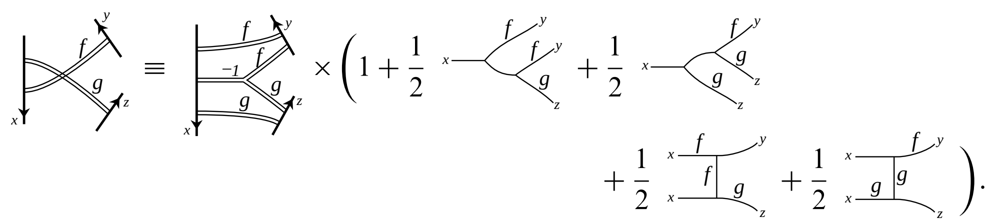
 $\equiv$  $-\dfrac{n}{2}$  $+\dfrac{n(n-1)}{6}$  $+\dfrac{n}{6}$  $+\dfrac{n(n-1)}{8}$  .

This formula implies the nth part of the expansion of the upper double lines of the following formula,

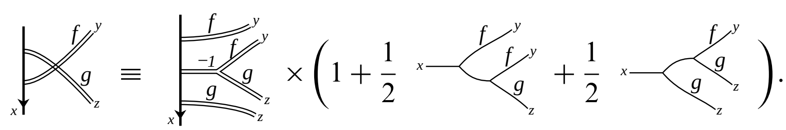


This implies the required formula. $\square$

Lemma 5.6 *For power series* f *and* g*,*

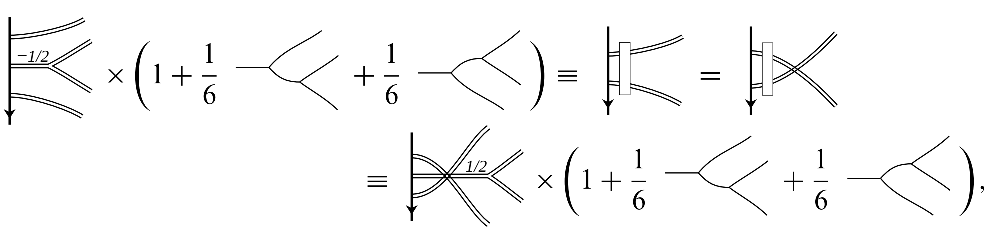


Proof The lemma is easily obtained from the following formula,

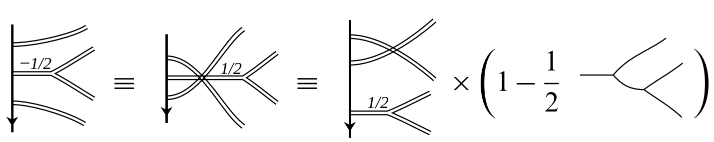


We show this formula, omitting f, g, x, y, z for simplicity.

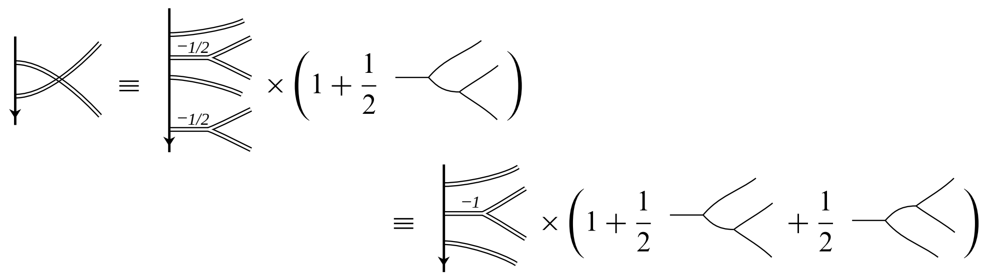
By applying Lemma 5.5 twice, we have that



hence

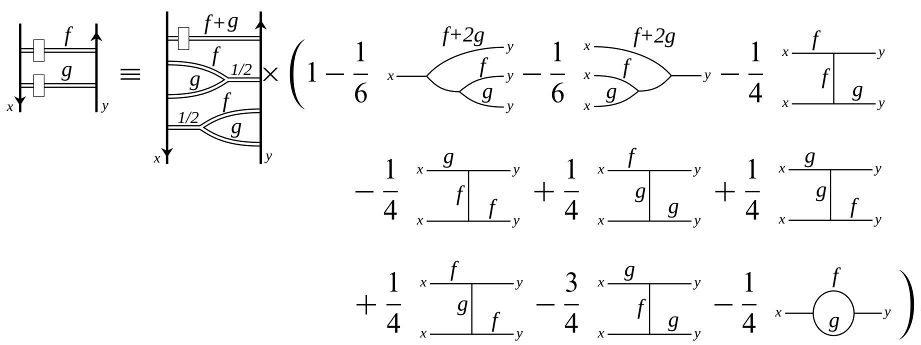


It follows that

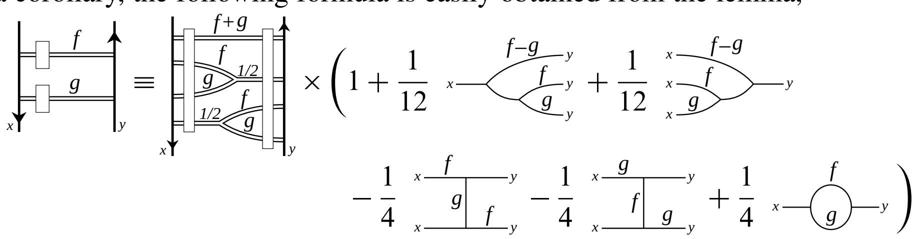


as required. □

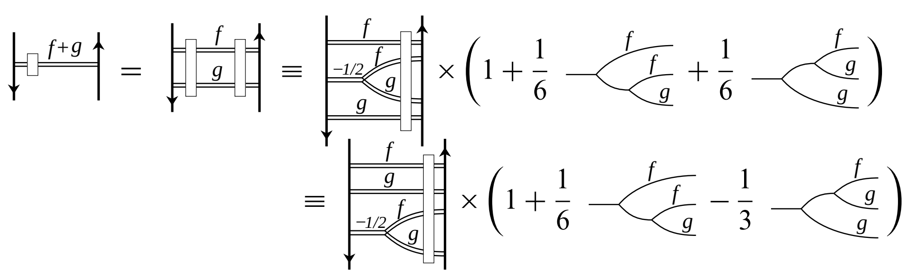
Lemma 5.7 *For power series f and g,*



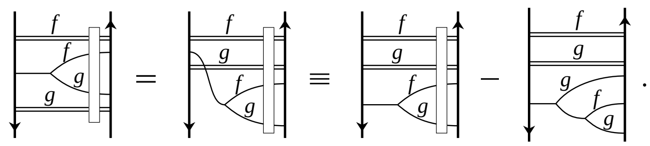
As a corollary, the following formula is easily obtained from the lemma,



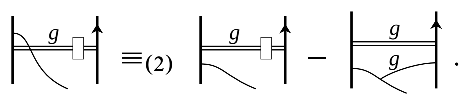
Proof of Lemma 5.7 For simplicity, we omit x, y in diagrams of the proof. By applying Lemma 5.4 to the left box of the second diagram of the following formula, we have that

(34) 

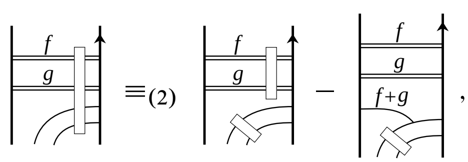
where the last equivalence is obtained from the following formula,



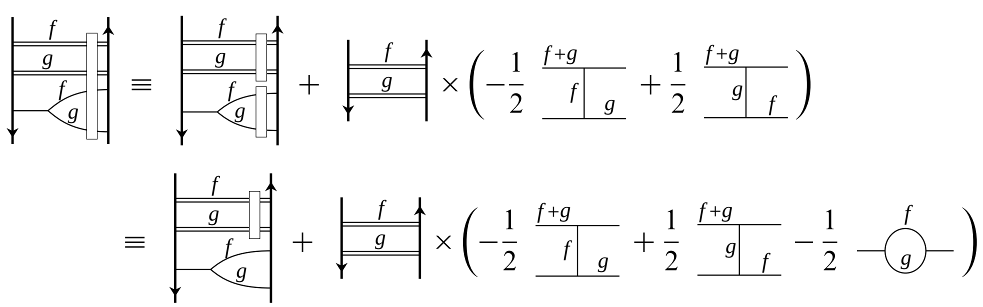
Here, the equivalence is obtained from the relation,



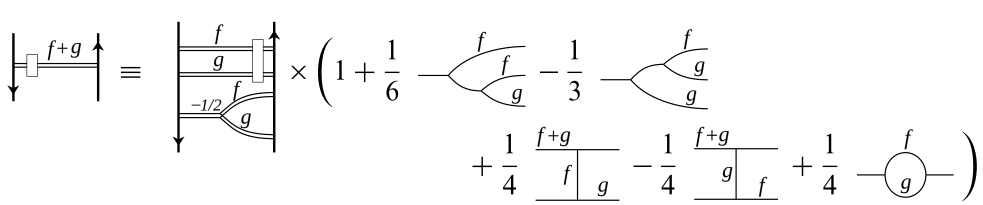
Since



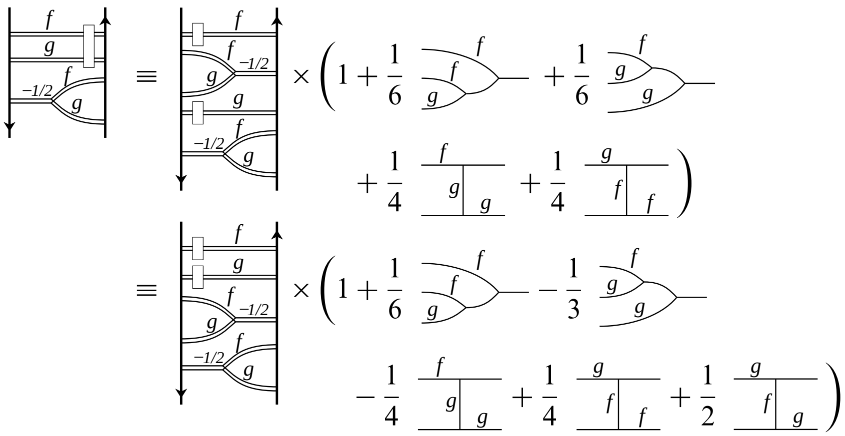
we have that



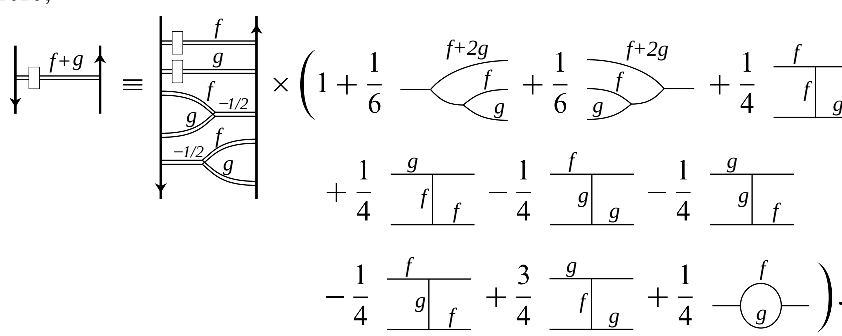
Hence, from (34), we have that



By applying Lemma 5.4 again, we have that

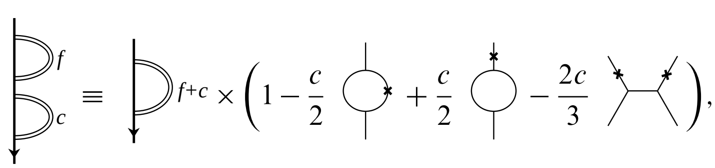


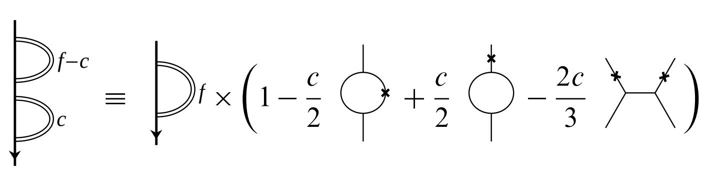
Therefore,



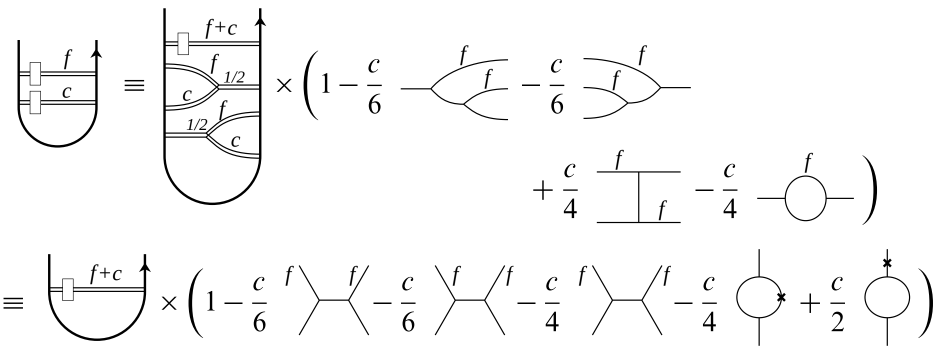
Hence, we obtain the formula of the lemma. □

Lemma 5.8 *The following formulas hold for a power series f and a scalar c, under the notation (13),*



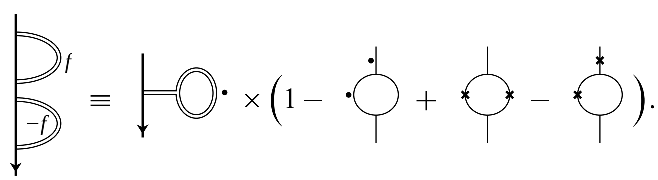


Proof By Lemma 5.7,

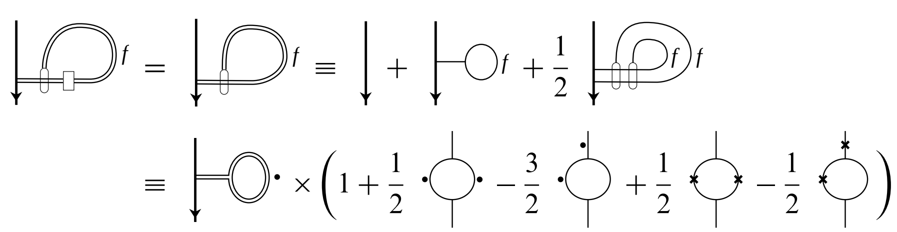


By removing the boxes by Lemma 5.2, we obtain the first formula of the lemma. The second formula is obtained from the first one by replacing f with $f-c$. □

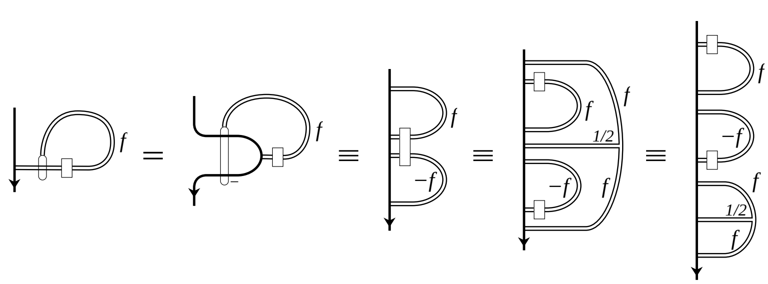
Lemma 5.9 *The following formula holds for a power series f, under the notation* (13),

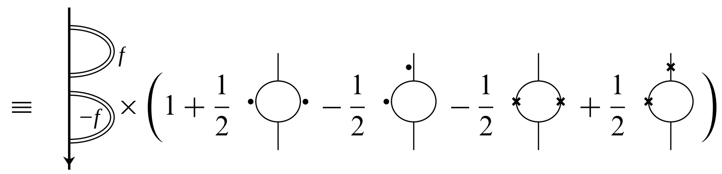


Proof We have that

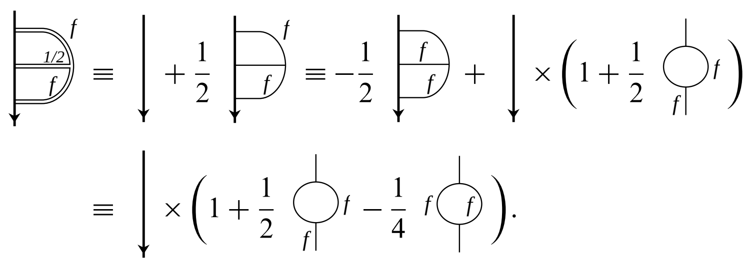


On the other hand, by Lemma 5.5, we have that



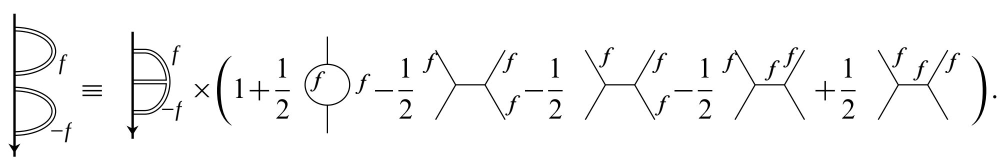


where we obtain the last equivalence from Lemma 5.2 and the following formula,

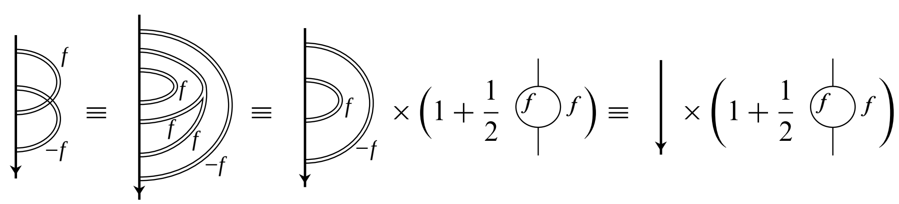


Hence, we obtain the required formula. □

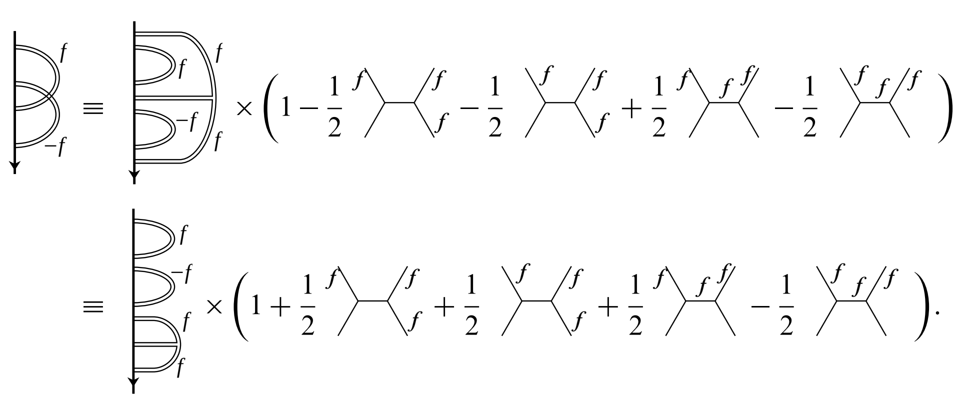
Lemma 5.10 *For a power series f,*



Proof By Lemma 5.6,



By applying Lemma 5.6 again in another way,

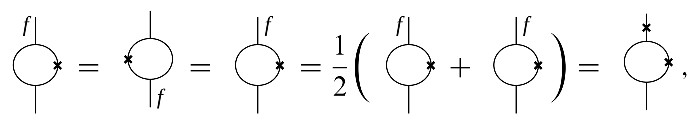


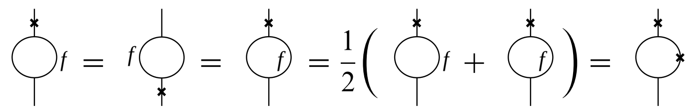
The lemma follows from the above formulas. □

5.2 Calculation of the inverse image of Gaussian diagrams by the PBW isomorphism

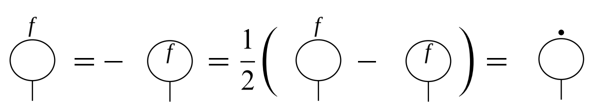
In this section, we calculate the inverse image of an exponential chord by the PBW isomorphism. This result is used, in Section 2, to calculate the rational version of the Aarhus integral, which constructs the 2–loop polynomial of knots from their surgery presentations.

Recall the markings introduced in (13). For example, the following diagrams are calculated as follows,

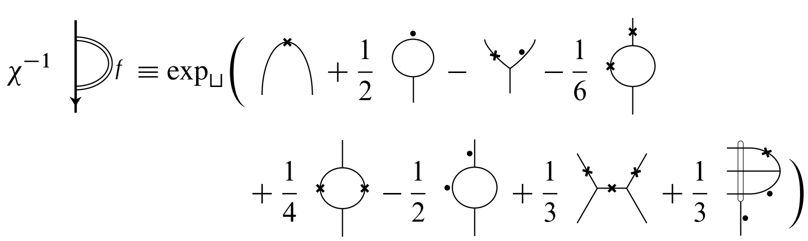
(35) 

(36) 

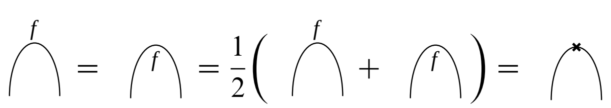
Further, when the left-hand side of the following formula is a part of a diagram, the part is calculated as follows,

(37) 

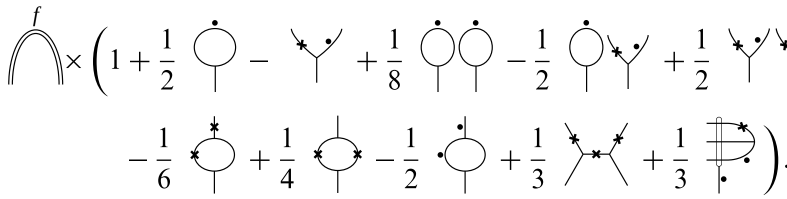
Lemma 5.11 *Under the notation (13),*



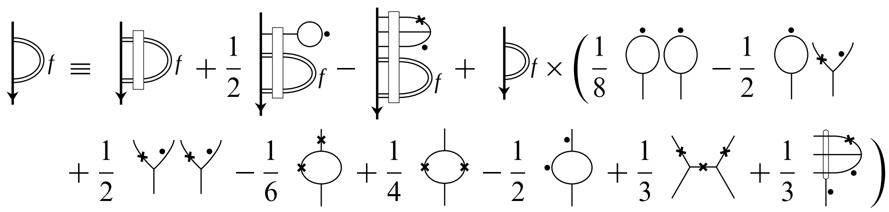
Proof Since



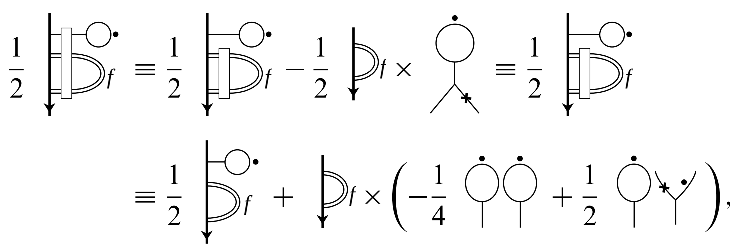
the right-hand side of the formula of the lemma is equivalent to



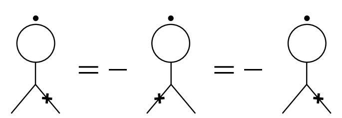
Hence, the lemma is reduced to the following formula,

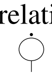
(38) 

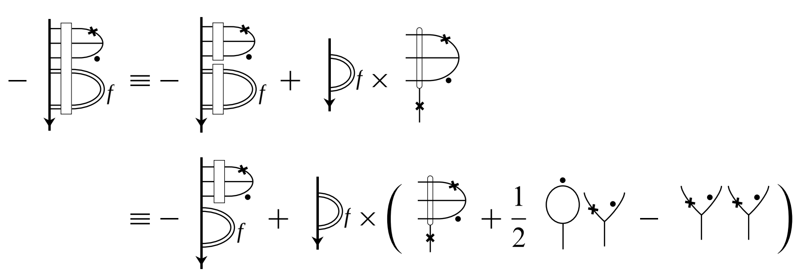
The second term of the right-hand side is calculated as follows,



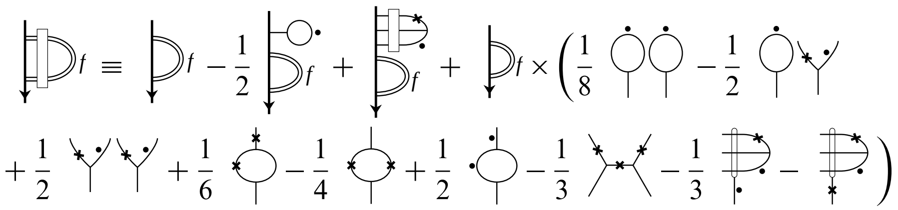
where the last equivalence is obtained by Lemma 5.12, and the second last diagram of the first line vanishes because



Here, the first equality is derived from the AS relation and the second equality is derived from the fact that the marking can move over  by the IHX relation. The third term of the right-hand side of (38) is similarly calculated as follows,

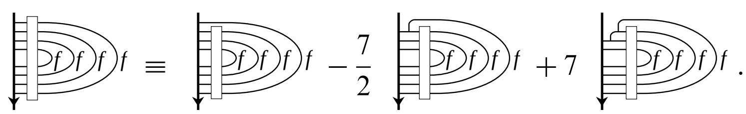


Hence, from (38), the lemma is reduced to the following formula,

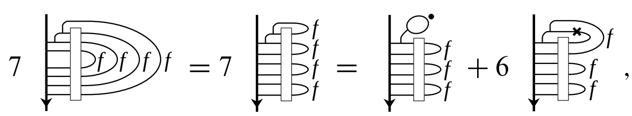
(39) 

We show the nth part of the expansion of this formula by induction on n.

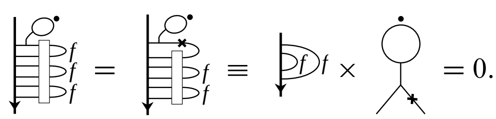
In the following of this proof, for simplicity, we calculate the 4th part of the expansion of (39); the general case can be calculated in the same way. By Lemma 5.1, we have that

(40) 

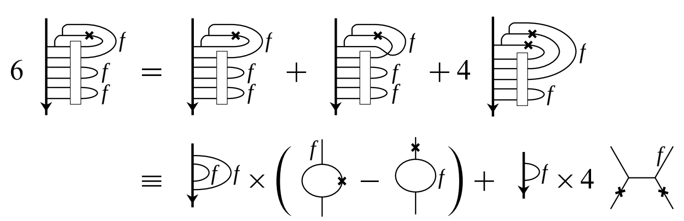
The last term is calculated as follows,

(41) 

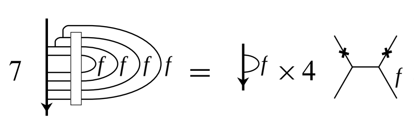
where the second last diagram is obtained by using the relation (37); this diagram vanishes as follows,



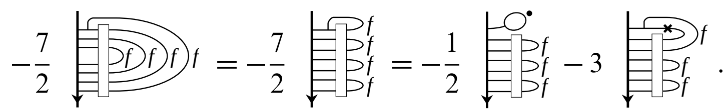
The last term of (41) is calculated as follows,



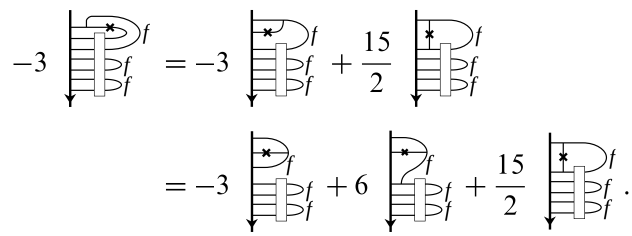
The first term of the last line vanishes by (35) and (36). Hence, from (41), the last term of (40) is written

(42) 

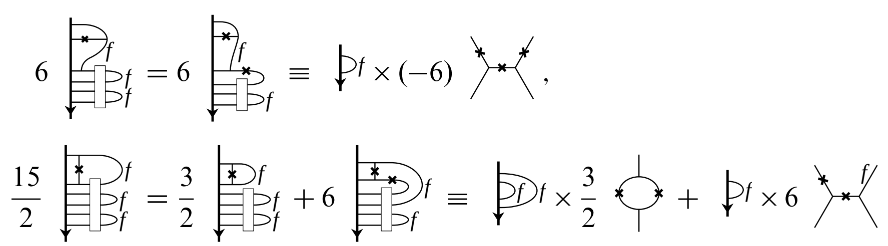
The second last term of (40) is calculated as follows,

(43) 

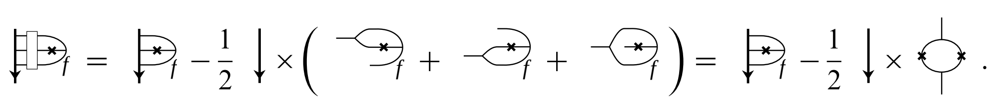
By applying Lemma 5.1 twice to the last term, we have that



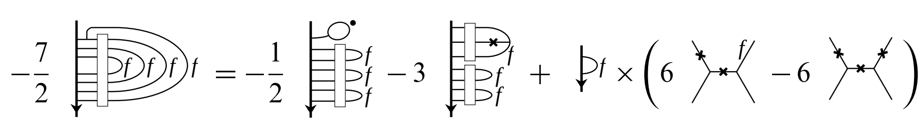
The last two terms are calculated respectively as follows,



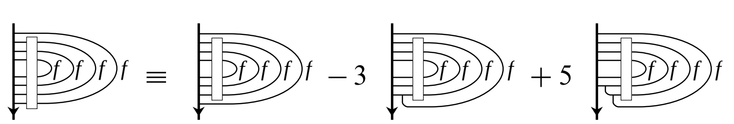
Further,



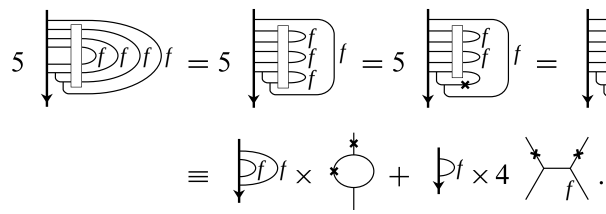
Hence, from (43), the second last term of (40) is written

(44) 

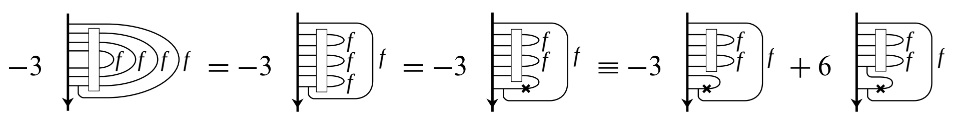
We calculate the first term of the right-hand side of (40) by using Lemma 5.1 as follows,

(45) 

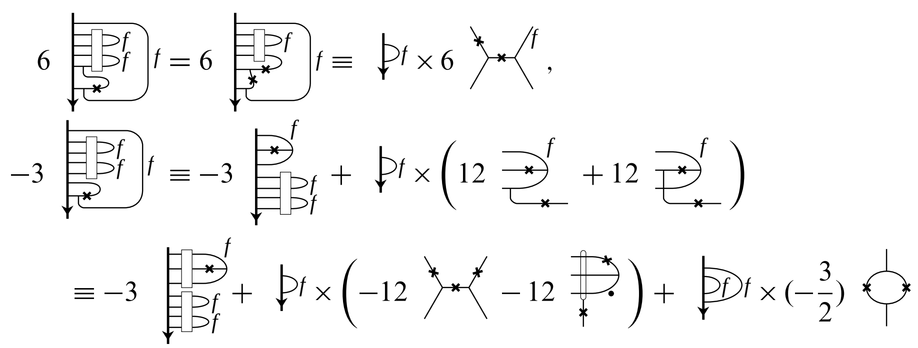
The last term is calculated as follows,



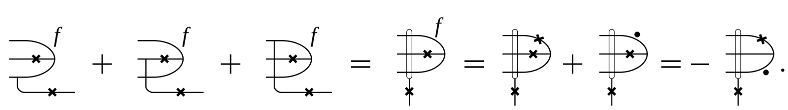
The second last term of (45) is calculated as follows,



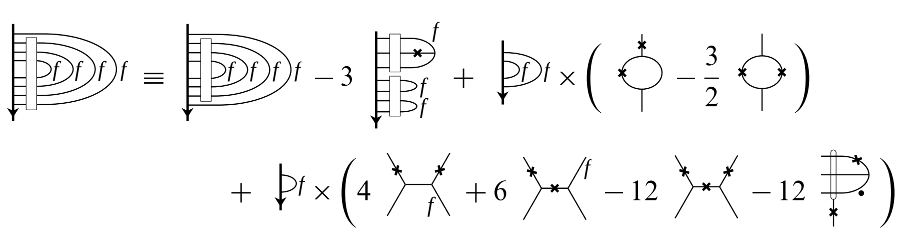
where the last equivalence is derived from Lemma 5.1. The last two terms are calculated as follows,



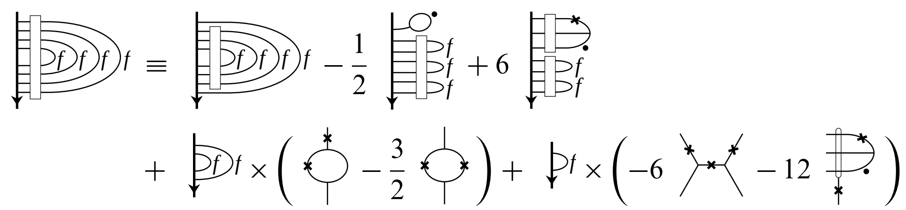
where the last equivalence is obtained from the relation,



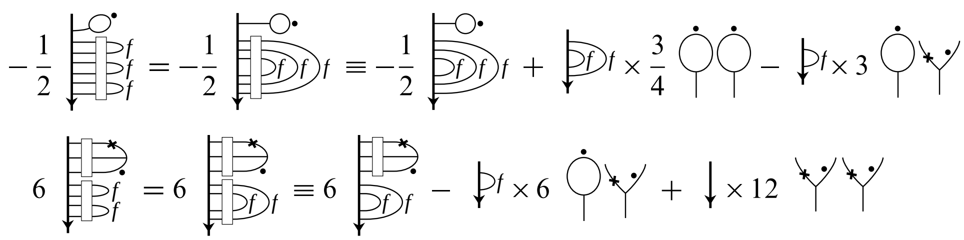
Hence, from (45), the first term of the right-hand side of (40) is written



Therefore, from (42), (44), and the above formula, (40) is rewritten

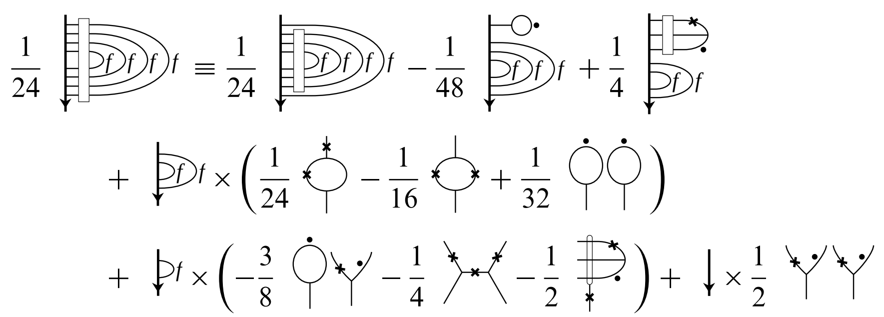


The last two diagrams of the first line are calculated by (47) and (48) as follows,

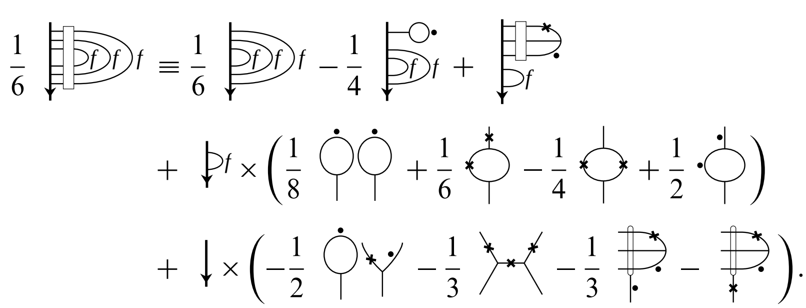


Hence,

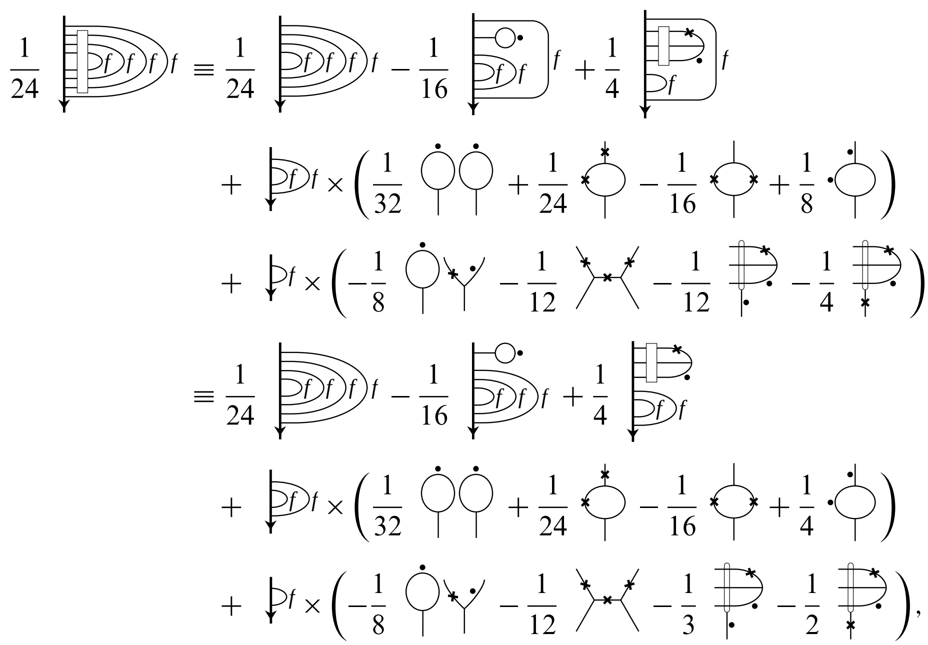
(46)



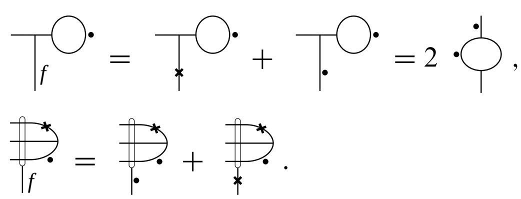
By the assumption of induction,



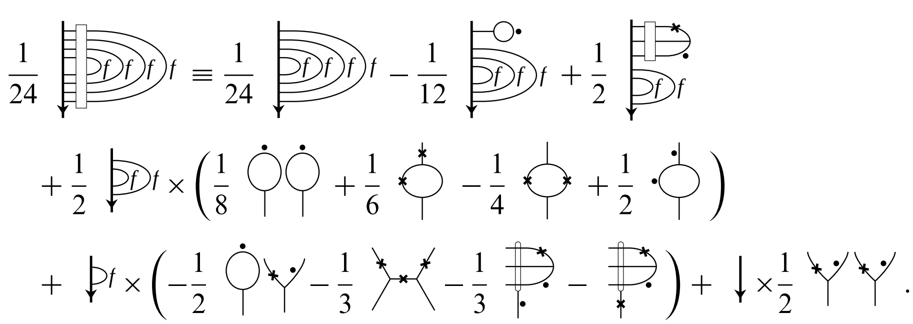
Substituting this formula into the first term of the right-hand side of (46), we calculate the term as follows,



where the last equivalence is obtained by using the following relations,

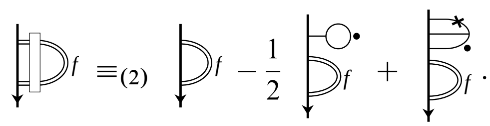


Therefore, from (46),

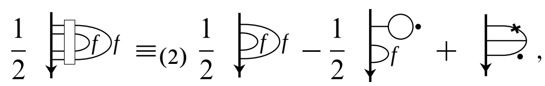


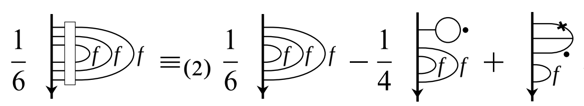
This is exactly the 4th part of the expansion of (39), as required. □

Lemma 5.12 *Under the notation (13),*



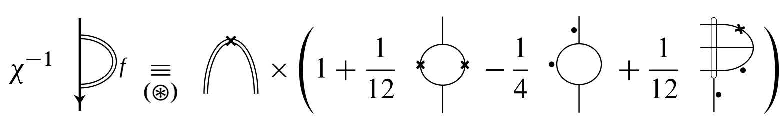
In particular, as the second and third parts of the expansion of the formula of the lemma,

(47) 

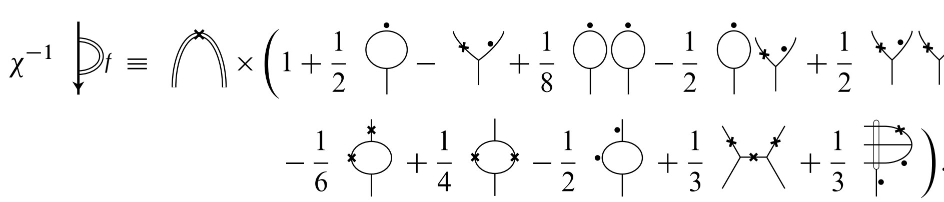
(48) 

Proof of Lemma 5.12 The formula of the lemma is a reduction of (39). In fact, it can be proved in the same way as the proof of (39), but the proof is far easier than the proof of (39), which we show in the proof of Lemma 5.11. A detailed proof is left to the reader. □

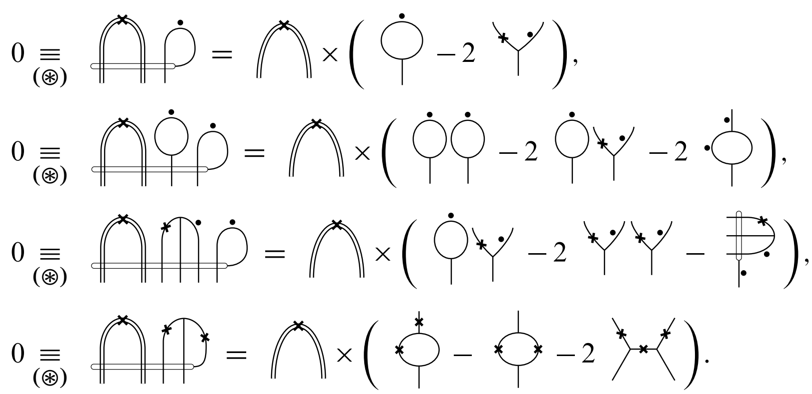
Lemma 5.13 *Under the notation (13),*



Proof By Lemma 5.11,

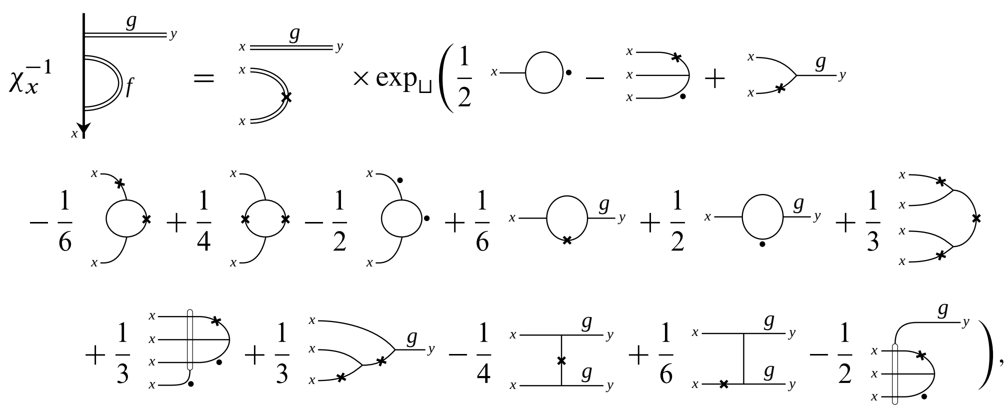


The required formula is obtained from this formula by using the following link relations,

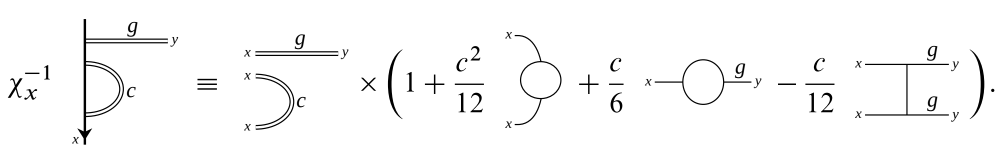


□

Lemma 5.14 *Under the notation (13),*

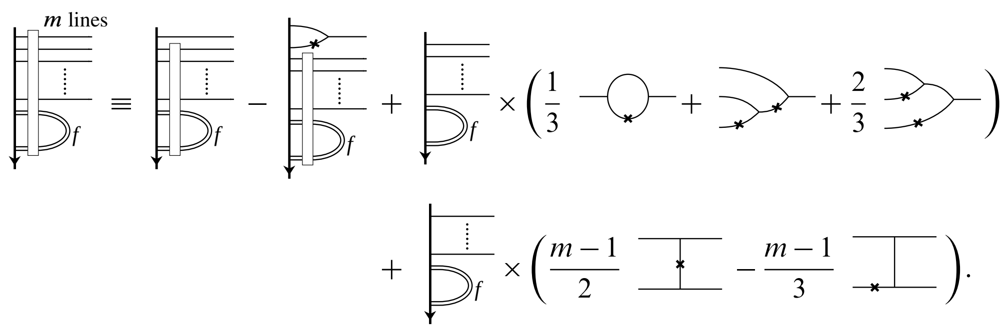


for power series f and g. In particular, when f is a scalar c,

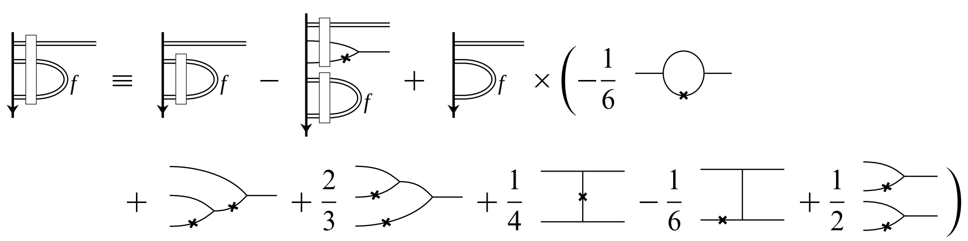


Proof The second formula of the lemma is easily obtained from the first formula. In the following of this proof, we show the first formula.

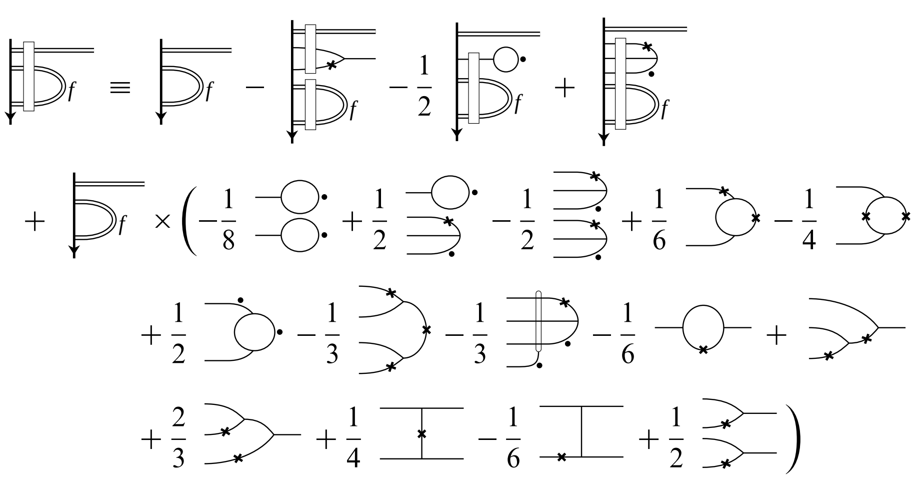
In a similar way as the proof of Lemma 5.2, we can show that

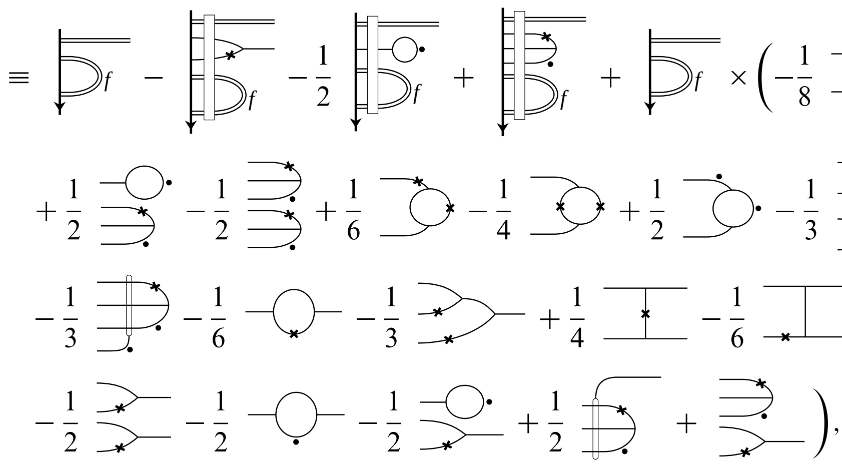


By this formula we can show the mth part of the expansion of the following formula by induction on n,

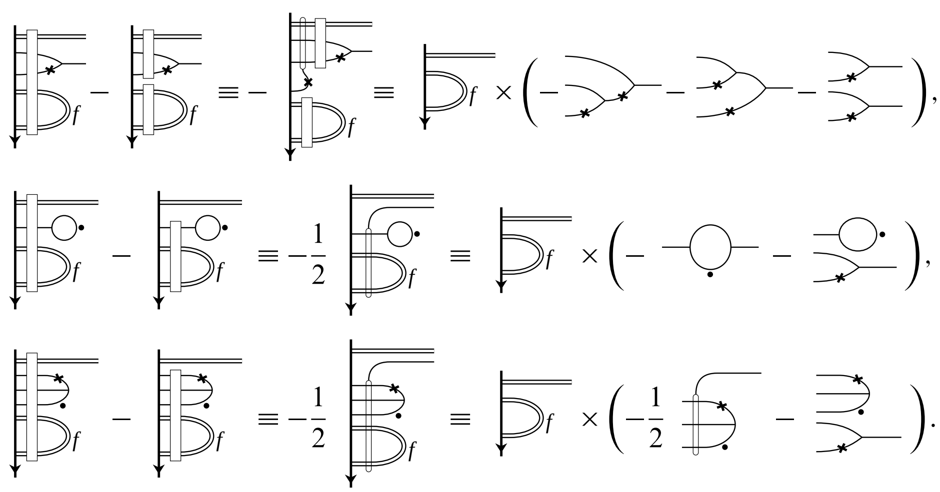


Further, applying (38) to the first diagram of the right-hand side, we have that





where the second equivalence is obtained from the following relations,

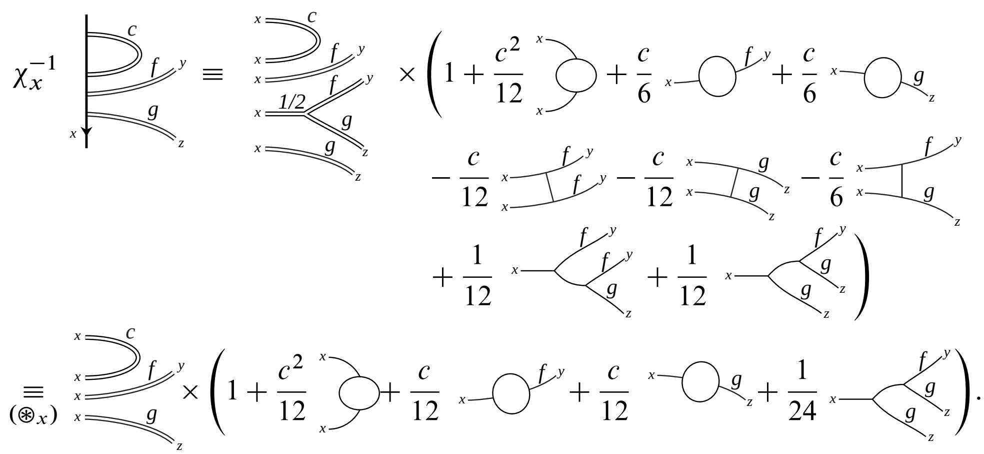


Therefore, we obtain the first formula of the lemma. □

5.3 Formulas for Gaussian diagrams used in Section 3.4

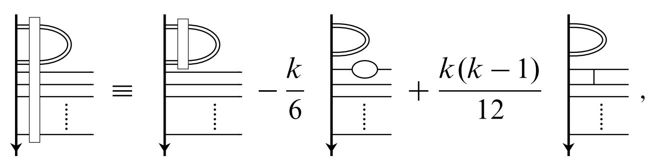
In this section, we show formulas for Gaussian diagrams which are used in Section 3.4 to calculate the 2–loop polynomial for knots of genus 1.

Lemma 5.15 *For power series f, g and a scalar c,*

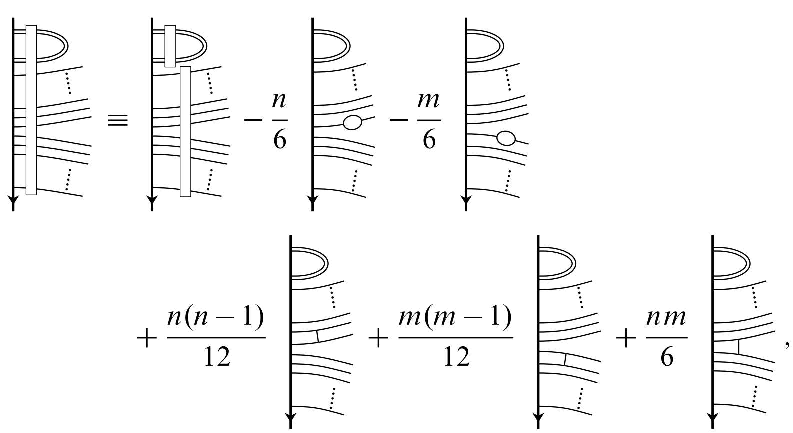


Proof For simplicity, we omit f, g, c in formulas of the proof.

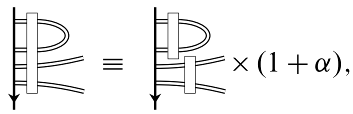
In a similar way as in the proof of Lemma 5.14, we have that



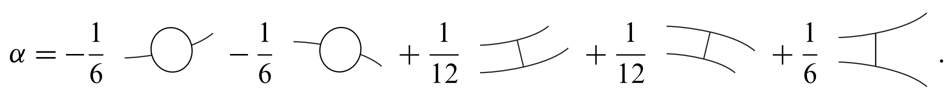
where k is the number of horizontal chords in the diagram of the left-hand side. By splitting the chords into two parts, we have that



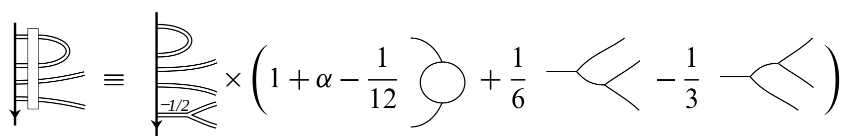
where n (resp. m) is the number of horizontal chords whose right ends are labeled by y (resp. z). Hence,



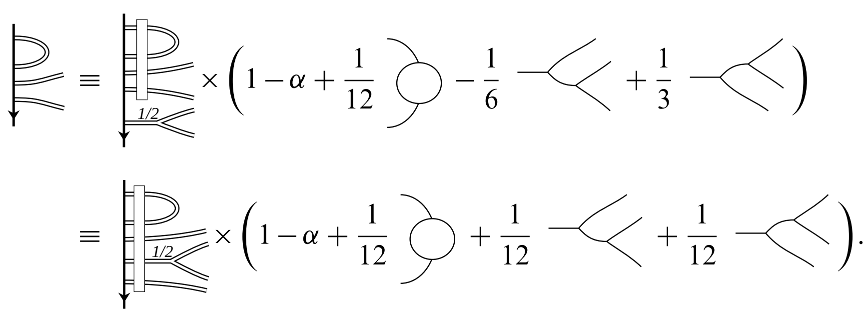
where



Therefore, by Lemma 5.11 and Lemma 5.16,

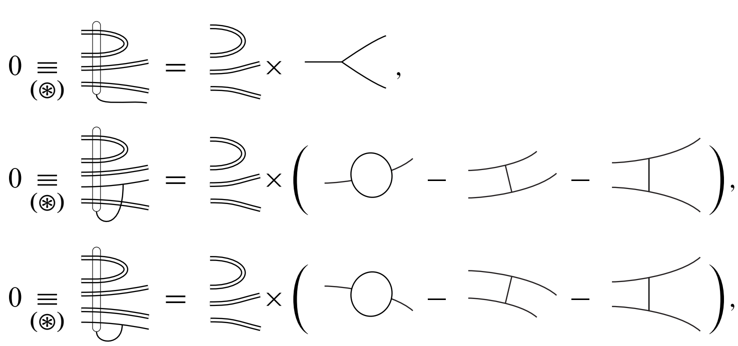


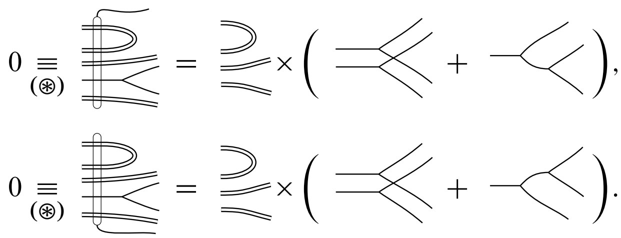
It follows that



Hence, we obtain the first equivalence of the required formula.

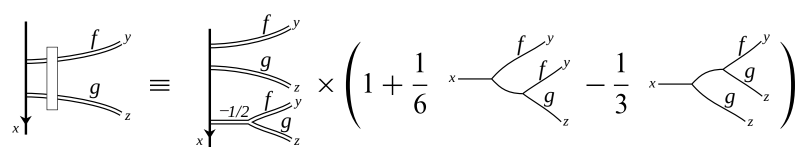
The second equivalence of the required formula is obtained from the following link relations,



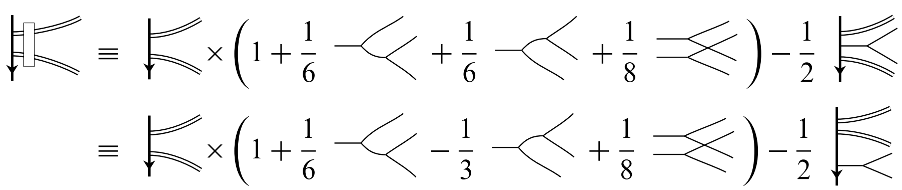


□

Lemma 5.16 *For power series f and g,*

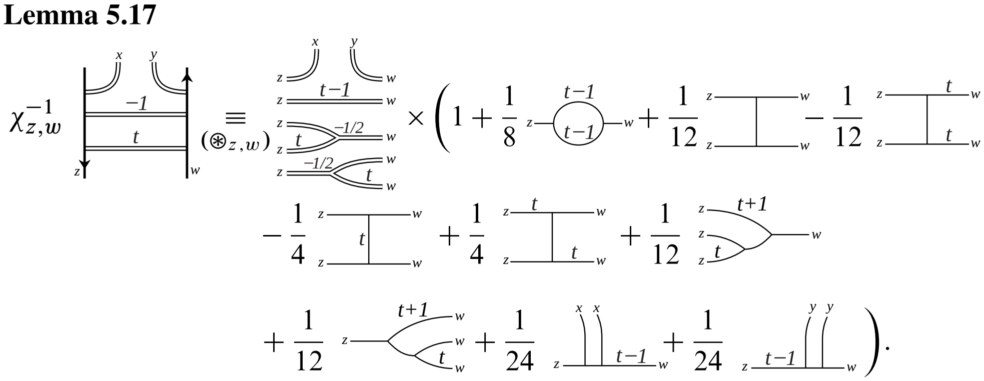


Proof By Lemma 5.5,

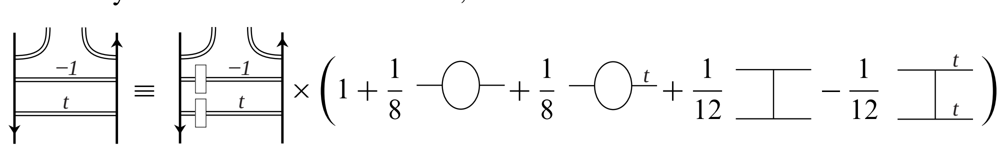


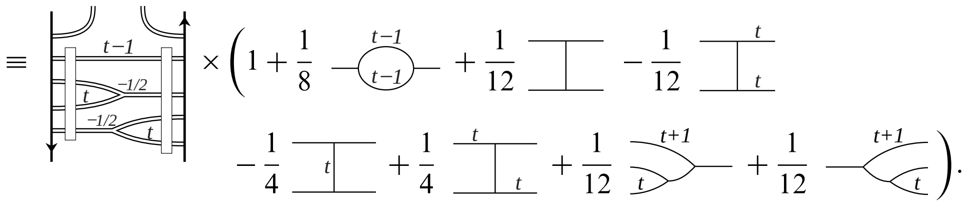
This implies the required formula. □

Lemma 5.17

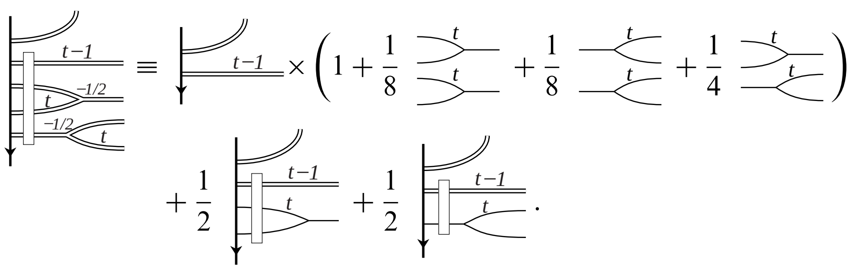


Proof By Lemma 5.2 and Lemma 5.7,

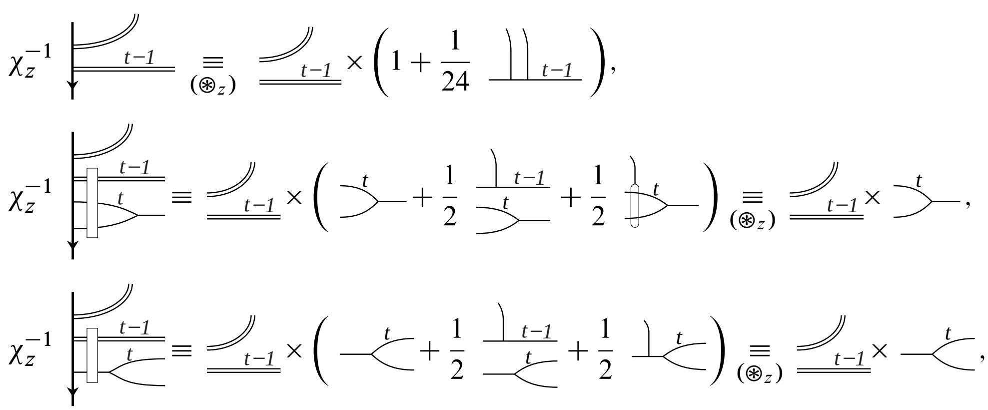




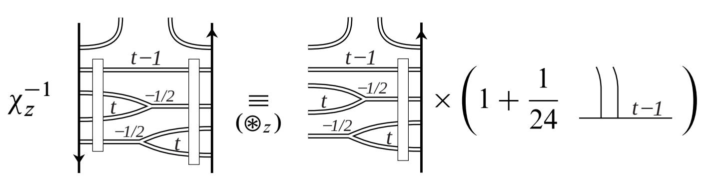
A left part of the first diagram of the right-hand side is calculated as follows,



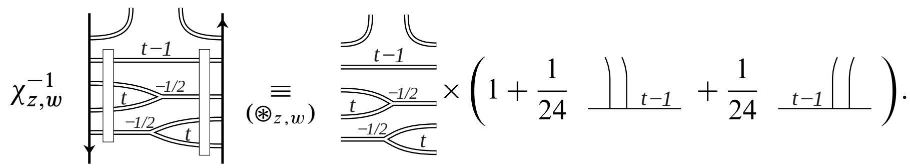
The map χ_z^{-1} takes diagrams of the right-hand side as follows,



where the first formula is obtained from Lemma 5.18 below, and the second and third formulas are obtained by calculating them directly. Hence,

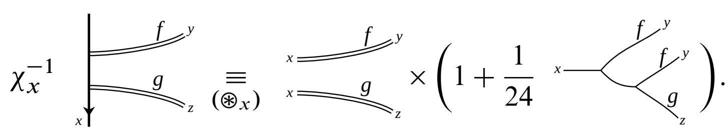


Further, in the same way, we have that



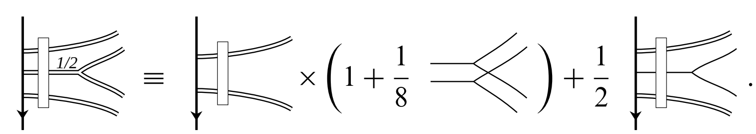
The required formula follows from the above formula and the first formula of the proof. □

Lemma 5.18 *For power series f and g,*

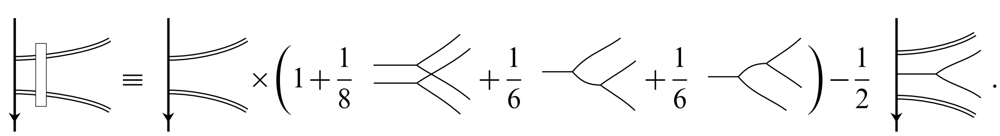


Proof By definition,

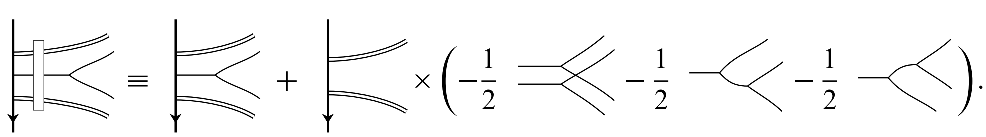
(49)



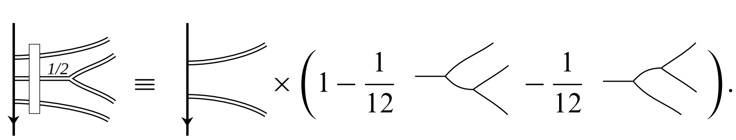
The first diagram of the right-hand side is calculated, by Lemma 5.4, as follows,



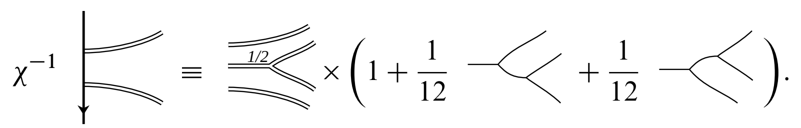
Further, the last diagram of (49) is directly calculated as follows,



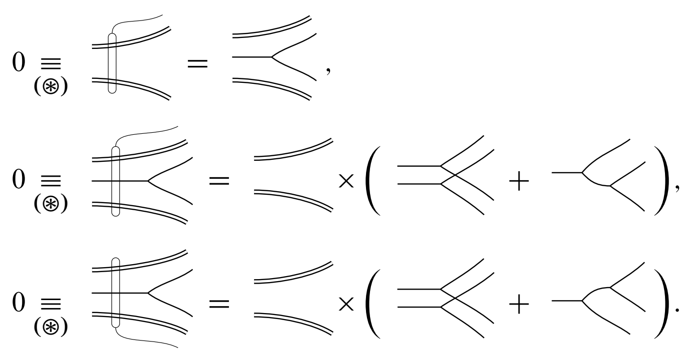
Hence, (49) is rewritten



Therefore, from the definition of χ,

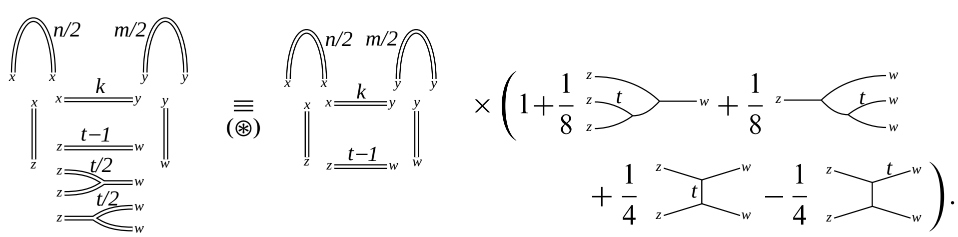


The required formula follows from this formula by using the following link relations,

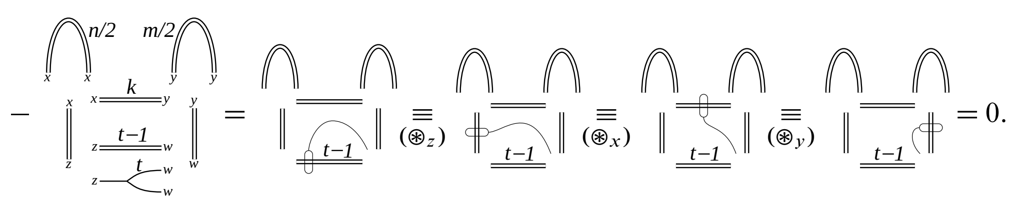


□

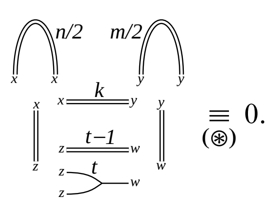
Lemma 5.19



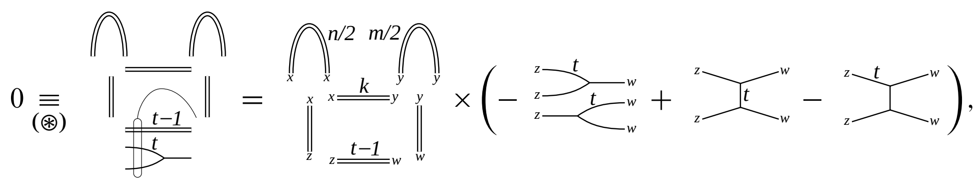
Proof For simplicity, we omit n, m, k in some diagrams in the proof. We have that

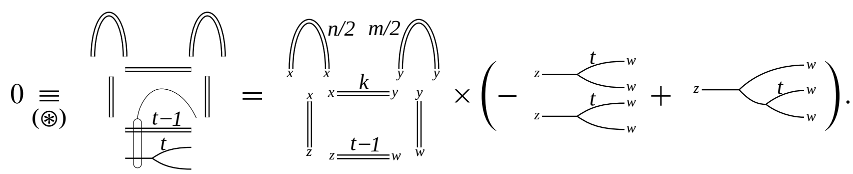


In the same way,

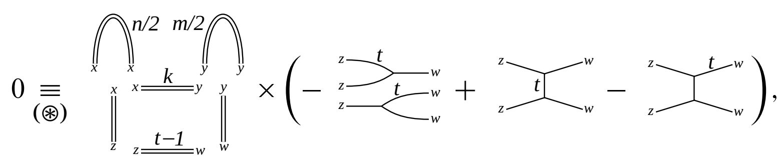


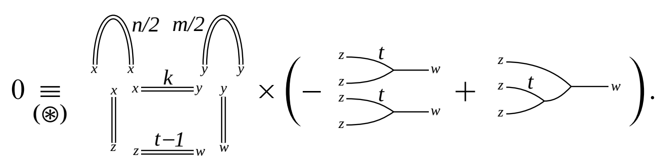
Similarly,





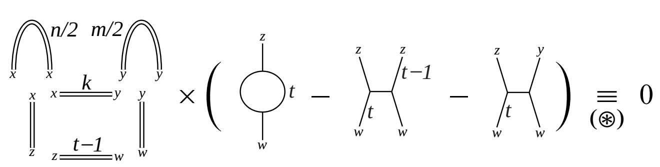
Further, in the same way,



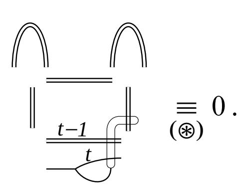


The required formula follows from the above relations. □

Lemma 5.20



Proof The lemma is obtained from the following link relation,

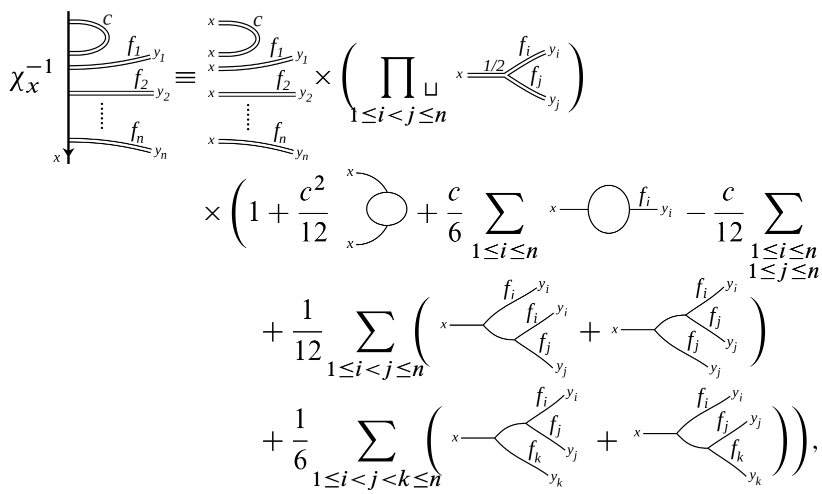


□

5.4 Formulas for Gaussian diagrams used in Section 4

In this section, we show formulas for Gaussian diagrams which are used in Section 4 to calculate the 2–loop polynomial for knots of any genus.

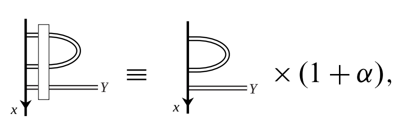
Lemma 5.21 *For a scalar c and power series $f_1, f_2, \cdots, f_n$,*



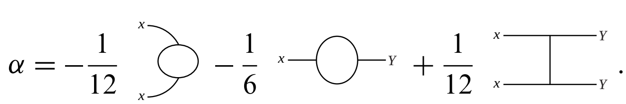
where $\prod_\sqcup$ denotes the product with respect to the disjoint-union product.

Proof For simplicity, we omit $f_1, f_2, \cdots, f_n$ in diagrams of the proof.

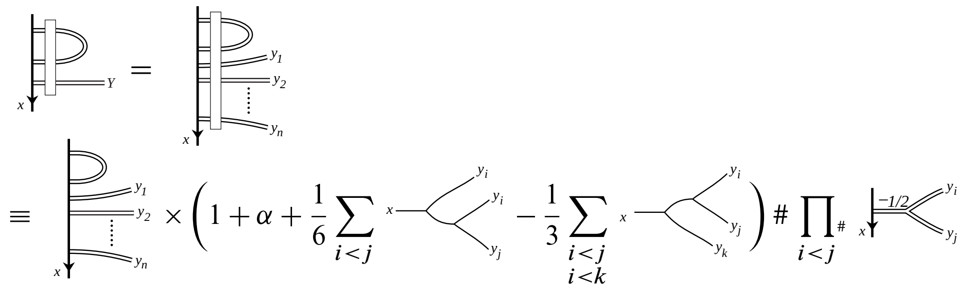
In a similar way as in the proof of Lemma 5.15, we have that



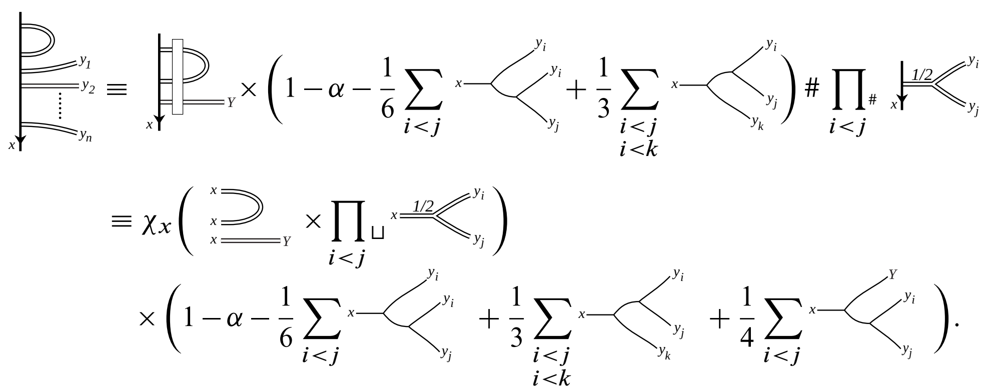
where we put $Y = y_1 + y_2 + \cdots + y_n$ and



Therefore, by Lemma 5.22 below,

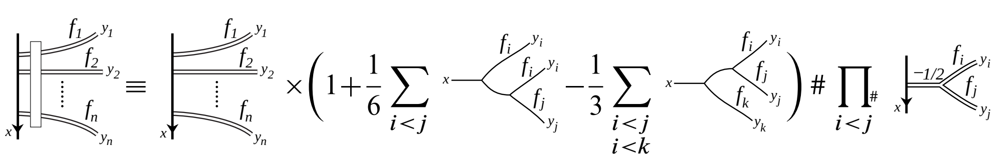


where $\prod_\#$ denotes the product with respect to the connected-sum product and the last term is connect-summed to the bottom of the first diagram of the last line. Hence,



This implies the required formula. □

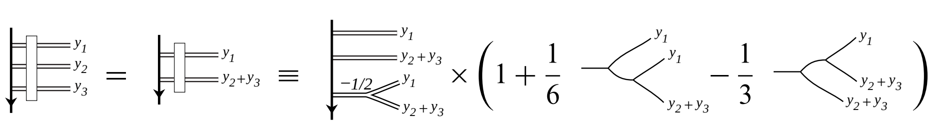
Lemma 5.22 *For power series $f_1, f_2, \cdots, f_n$,*



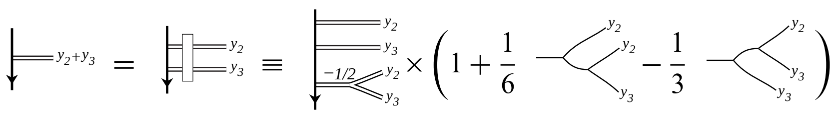
where $\prod_\#$ denotes the product with respect to the connected-sum product.

Proof For simplicity, we omit $x, f_1, f_2, \cdots, f_n$.

By Lemma 5.16,



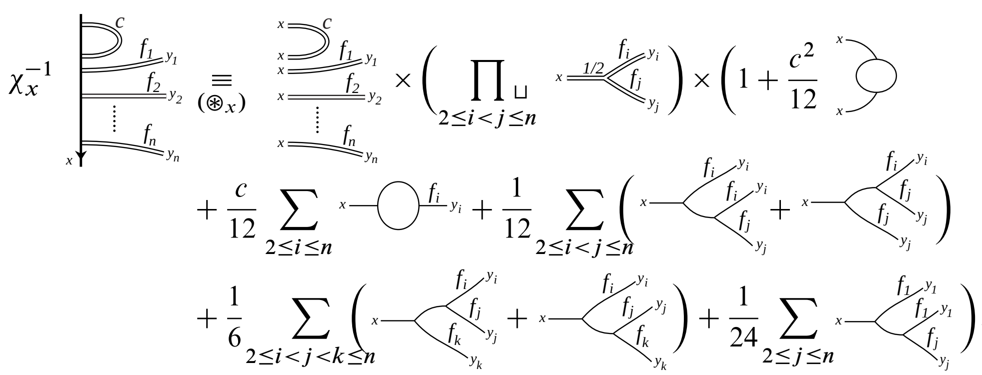
By applying Lemma 5.16 again,



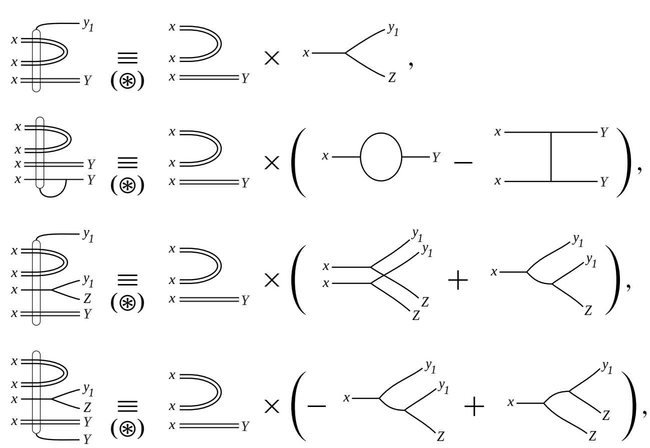
From the above two formulas, we obtain the required formula for $n = 3$.

The required formula for any n is obtained similarly by induction on n $\square$

Lemma 5.23 *For a scalar c and power series $f_1, f_2, \cdots, f_n$,*

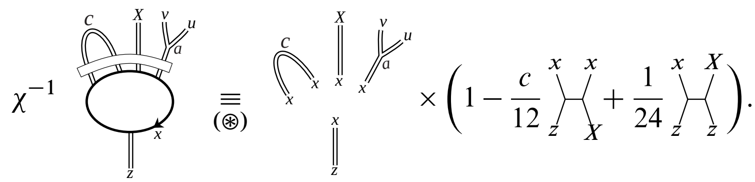


Proof The lemma is obtained from Lemma 5.21 by using the following link relations,

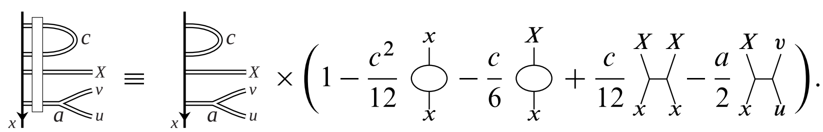


where we put $Y = y_1 + y_2 + \cdots + y_n$ and $Z = y_2 + \cdots + y_n$. $\square$

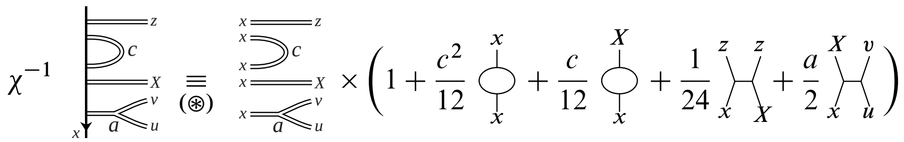
Lemma 5.24 *For scalars a and c,*



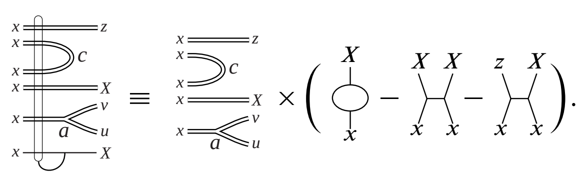
Proof Similarly as in the proof of Lemma 5.15, we have that



Further, by modifying Lemma 5.23, we have that

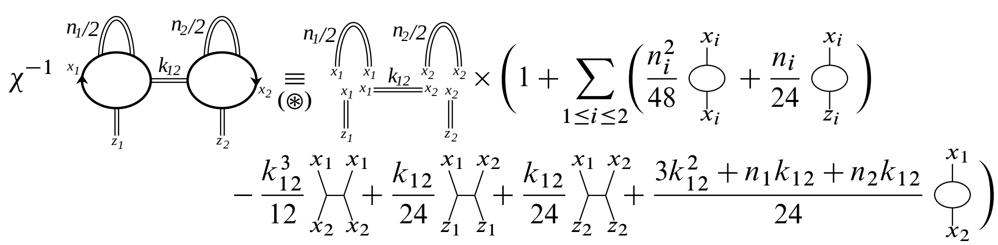


We obtain the required formula from the above two formulas, using the following link relation,

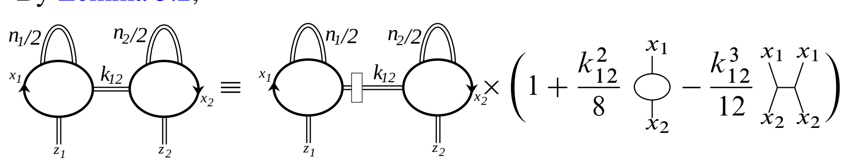


□

Lemma 5.25 *For scalars n_1, n_2, k_{12},*

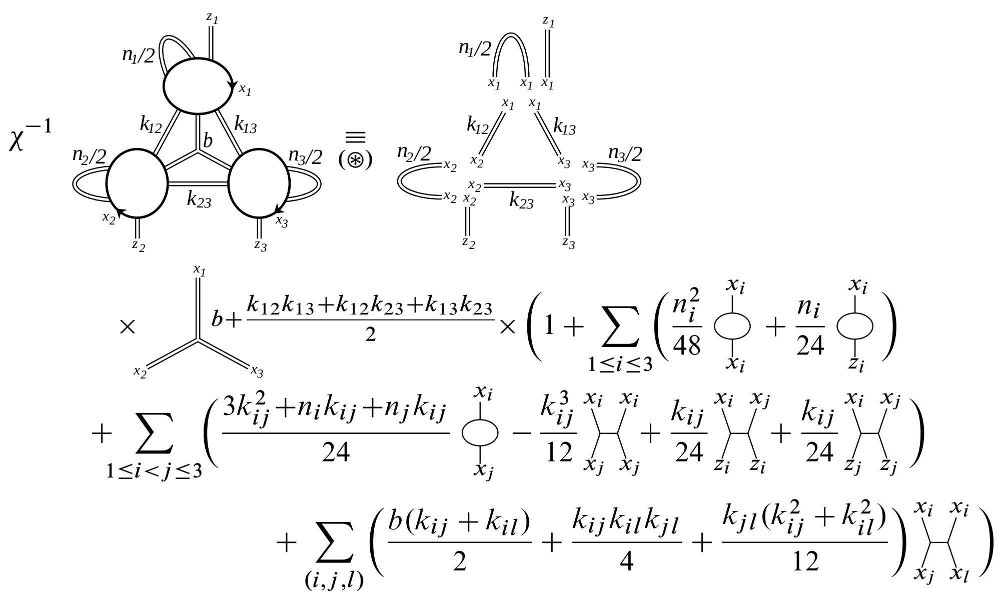


Proof By Lemma 5.2,



By applying Lemma 5.15 to each solid circle of the first diagram of the right-hand side, we obtain the required formula. □

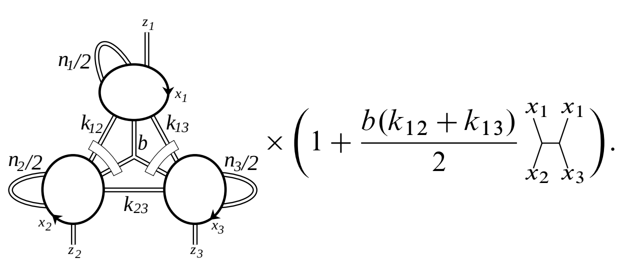
Lemma 5.26 *For scalars n_i, k_{ij}, b,*



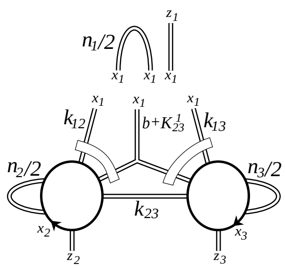
where the last sum is taken over $(i,j,l)=(1,2,3),\ (2,1,3),\ (3,1,2)$.

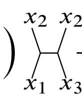
Proof We obtain the terms of the right-hand side of the required formula labeled by at most 2 colors in the same form as in Lemma 5.25. It is sufficient to calculate the terms labeled by 3 colors. In this proof we consider the equivalence modulo the diagrams with 2 trivalent vertices and at most 2 colors.

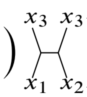
The diagram of the left-hand side of the required formula is equivalent to



By applying Lemma 5.27 to the solid circle labeled by x_1, the first diagram is taken by $\chi_{x_1}^{-1}$ (modulo the equivalence) to

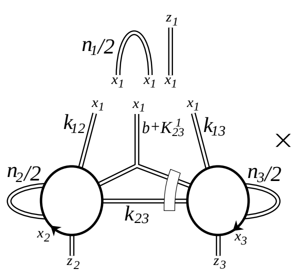


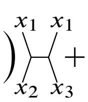
$$\times\Big(1+\Big(\frac{k_{12}^2k_{13}}{12}+\frac{bk_{12}}{2}\Big)$$


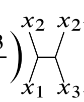
$$+\Big(\frac{k_{12}k_{13}^2}{12}+\frac{bk_{13}}{2}\Big)$$


$$\Big),$$

where we put $K^i_{jl}=k_{ij}k_{il}/2$. Further, the first diagram is equivalent to

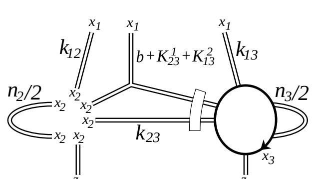


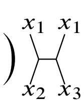
$$\times\Big(1-\frac{k_{12}+k_{13}}{2}\Big(b+\frac{k_{12}k_{13}}{2}\Big)$$


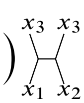
$$+\frac{k_{23}}{2}\Big(b+\frac{k_{12}k_{13}}{2}\Big)$$


$$\Big).$$

By applying Lemma 5.27 to the solid circle labeled by x_2, the lower part of the first diagram is taken by $\chi_{x_2}^{-1}$ (modulo the equivalence) to

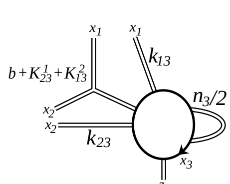


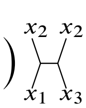
$$\times\Big(1+\Big(\frac{k_{12}^2k_{23}}{12}+\frac{k_{12}}{2}\Big(b+\frac{k_{12}k_{13}}{2}\Big)\Big)$$


$$+\Big(\frac{k_{12}k_{23}^2}{12}+\frac{k_{23}}{2}\Big(b+\frac{k_{12}k_{13}}{2}\Big)\Big)$$


$$\Big).$$

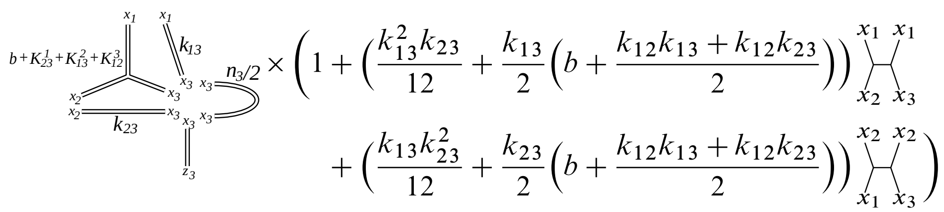
Further, the right part of the first diagram is equivalent to



$$\times\Big(1-\frac{k_{23}}{2}\Big(b+\frac{k_{12}(k_{13}+k_{23})}{2}\Big)$$


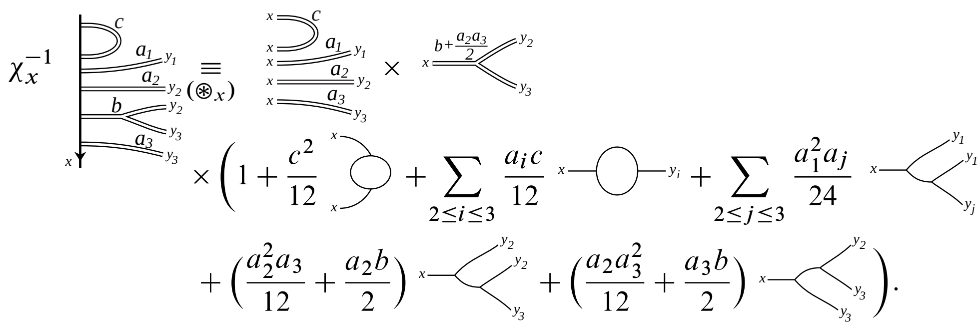
$$\Big).$$

By applying Lemma 5.27 to the solid circle labeled by x_3, the first diagram is taken by $\chi_{x_3}^{-1}$ (modulo the equivalence) to

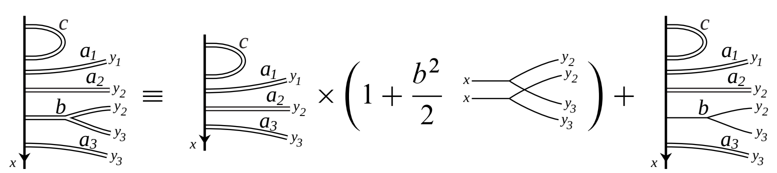


By summing the diagrams in the right part of the above formulas, we obtain the terms of the required formula colored by 3 colors, completing the proof. $\square$

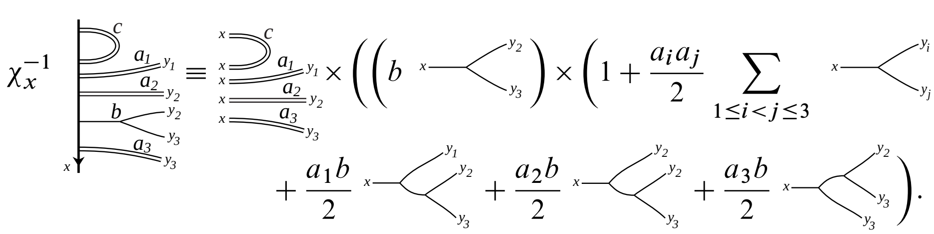
Lemma 5.27 *For scalars* a_1, a_2, a_3, b, c,



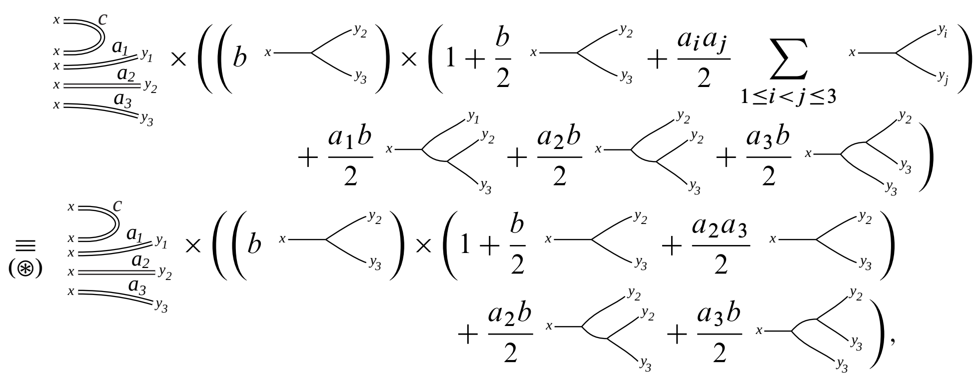
Proof By definition,



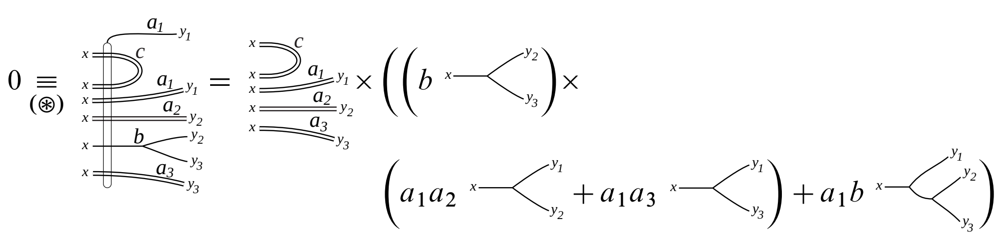
In a similar way as Lemma 5.21, we have that



The error term between the required formula and a particular case of Lemma 5.23 is

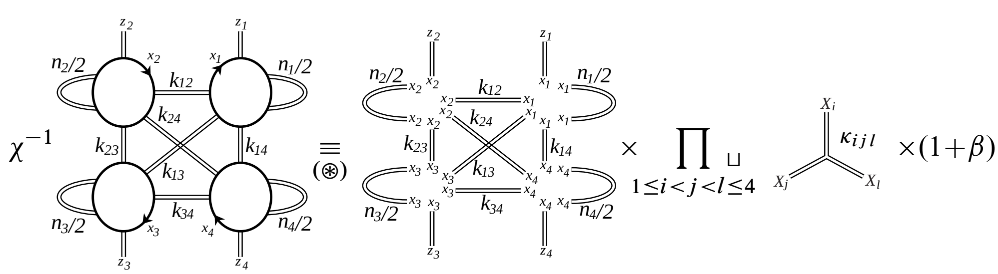


where the relation is obtained from the following link relation,

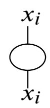
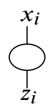


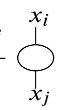
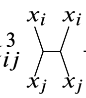
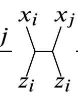
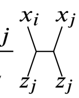
Hence, from a particular case of Lemma 5.23 and the above mentioned error term, we obtain the required formula. □

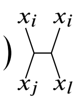
Lemma 5.28 *For scalars n_i, k_{ij} (with $k_{ij} = k_{ji}$),*

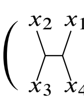
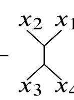


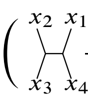
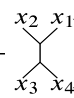
where we put $\kappa_{ijl} = (k_{ij}k_{il} + k_{ij}k_{jl} + k_{il}k_{jl})/2$ *and*

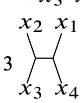
$12\beta = \sum_{1\le i\le 4}\Big(\frac{n_i^2}{4}$  $+ \frac{n_i}{2}$  $\Big)$

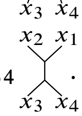
$+ \sum_{1\le i<j\le 4}\Big(\frac{3k_{ij}^2 + n_ik_{ij} + n_jk_{ij}}{2}$  $- k_{ij}^3$  $+ \frac{k_{ij}}{2}$  $+ \frac{k_{ij}}{2}$  $\Big)$

$+ \sum_{1\le i\le 4}\sum_{\substack{j<l\\ j,l\ne i}}\big(3k_{ij}k_{il}k_{jl} + k_{jl}(k_{ij}^2 + k_{il}^2)\big)$ 

$+ 2(k_{12}k_{13}k_{14} + k_{21}k_{23}k_{24} + k_{31}k_{32}k_{34} + k_{41}k_{42}k_{43})\Big($  $+$  $\Big)$

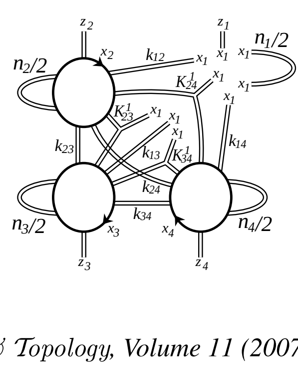
$+ 3(k_{12} + k_{34} - k_{14} - k_{23})k_{13}k_{24}\Big($  $-$  $\Big)$

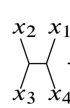
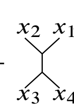
$+ 3(k_{12} + k_{13} + k_{24} + k_{34})k_{14}k_{23}$ 

$+ 3(k_{13} + k_{14} + k_{23} + k_{24})k_{12}k_{34}$  .

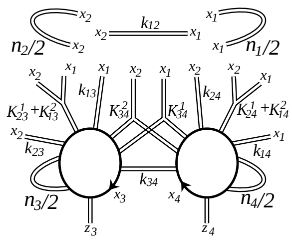
Proof We obtain the terms of the right-hand side of the required formula labeled by at most 3 colors in the same form as in Lemma 5.26. It is sufficient to calculate the terms labeled by 4 colors, similarly as the proof of Lemma 5.26. In this proof we consider the equivalence modulo the diagrams with 2 trivalent vertices and at most 3 colors.

The diagram of the left-hand side of the required formula is taken by $\chi_{x_1}^{-1}$ (modulo the equivalence) to



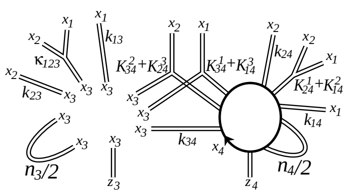
$\times\Big(1 + \frac{k_{12}k_{13}k_{14}}{6}\Big($  $+$  $\Big)\Big),$

where we put $K^i_{jl}=k_{ij}k_{il}/2$ as in the proof of Lemma 5.26. The first diagram is taken by $\chi^{-1}_{x_2}$ (modulo the equivalence) to



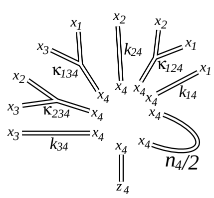
$$\times\left(1+\frac{k_{21}k_{23}k_{24}}{6}\left(\quad + \quad \right) + \frac{K^1_{24}k_{23}}{2} \quad + \frac{K^1_{23}k_{24}}{2} \quad \right).$$

Further, the first diagram is taken by $\chi^{-1}_{x_3}$ (modulo the equivalence) to



$$\times\left(1+\frac{k_{31}k_{32}k_{34}}{6}\left(\quad + \quad \right) + \frac{K^1_{34}k_{23}}{2} \quad + \frac{(K^1_{23}+K^2_{13})k_{34}}{2} \quad - \frac{K^2_{34}k_{13}}{2} \quad \right).$$

Furthermore, the first diagram is taken by $\chi^{-1}_{x_4}$ (modulo the equivalence) to



$$\times\left(1+\frac{k_{41}k_{42}k_{43}}{6}\left(\quad + \quad \right) + \frac{(K^2_{34}+K^3_{24})k_{14}}{2} \quad + \frac{(K^1_{24}+K^2_{14})k_{34}}{2} \quad + \frac{(-K^1_{34}+K^3_{14})k_{24}}{2} \quad \right).$$

By summing the diagrams in the right part of the above formulas, we obtain the terms of the required formula colored by 4 colors, completing the proof. □

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