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A dual version of the ribbon graph decomposition of moduli space

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This note gives a construction of a dual version of the ribbon graph decomposition of the moduli spaces of Riemann surfaces.

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1 Introduction

The ribbon graph decomposition of moduli space is a non-compact orbi-cell complex homeomorphic to $\mathcal{M}_{g,n}\times\mathbb{R}^n_{>0}$. This was introduced by Harer–Mumford–Thurston [4] and Penner [9], and used to great effect by Kontsevich [5; 6] in his proof of the Witten conjecture and his construction of classes in moduli spaces associated to A_∞ algebras.

In this note, I discuss a dual version of the ribbon graph decomposition of the moduli spaces of Riemann surfaces with boundary and marked points, which I introduced in the unpublished preprint [1], and used in [2] to construct open-closed topological conformal field theories. This dual version of the ribbon graph decomposition is a compact orbi-cell complex with a natural weak homotopy equivalence to the moduli space. In the case when all of the marked points are on the boundary of the surface, the combinatorics of the cell complex is captured by ribbon graphs, as usual. In the general case, we find a variant of the ribbon graph complex.

The idea of the construction is as follows. We use certain partial compactifications $\overline{\mathcal{N}}_{g,h,r,s}$ of the moduli spaces $\mathcal{N}_{g,h,r,s}$ of Riemann surfaces of genus g with $h>0$ boundary components, r boundary marked points, and s internal marked points. The partial compactifications we use are closely related to the Deligne–Mumford spaces; we allow Riemann surfaces with a certain kind of singularity, namely nodes on the boundary. The moduli space $\overline{\mathcal{N}}_{g,h,r,s}$ is an orbifold with corners. The boundary is the locus of singular surfaces. Therefore the inclusion $\mathcal{N}_{g,h,r,s}\hookrightarrow\overline{\mathcal{N}}_{g,h,r,s}$ is a weak homotopy equivalence of orbispaces. Inside $\overline{\mathcal{N}}_{g,h,r,s}$ is a natural orbi-cell complex $D_{g,h,r,s}$, which is the locus where all the irreducible components of the surface are discs (with at most one internal marked point). We show that the map $D_{g,h,r,s}\hookrightarrow\overline{\mathcal{N}}_{g,h,r,s}$ is a weak homotopy equivalence. This shows that $D_{g,h,r,s}\simeq\mathcal{N}_{g,h,r,s}$, giving the

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desired cellular model for $\mathcal{N}_{g,h,r,s}$. When $s=0$, the combinatorics of the cell complex $D_{g,h,r,0}$ is governed by standard ribbon graphs. When $s>0$, we find a variant type of ribbon graph, which has two types of vertex.

In the standard approach, the non-compact orbi-cell decomposition of moduli space gives a chain model for the Borel–Moore homology, or equivalently the cohomology, of moduli space. The chains are given by ribbon graphs, and the differential is given by summing over ways of contracting an edge, to amalgamate two distinct vertices. The approach used here gives a compact orbi-cell complex, which gives a chain model for homology of moduli space. In the case when $s=0$, this is precisely the dual of the standard ribbon graph complex. That is, the chains are given by ribbon graphs, and the differential is given by summing over ways of splitting a vertex into two. When $s>0$, the complex $D_{g,h,r,s}$ is dual to a generalised ribbon graph complex considered by Penner, the complex of "quasi-filling arc families in a partially decorated bordered surface".

The main disadvantage of the approach described here, compared to the more traditional approach, is that we do not find a cell complex homeomorphic to moduli space, but instead a space homotopy equivalent.

On the other hand, one advantage of this approach is that the cell complex we find is manifestly compatible with the gluing maps between the moduli spaces we use. We use "open-string" type gluing; instead of gluing boundary components of surfaces, we glue together marked points on the boundary. This allows us to give [1; 2] a generators-and-relations description (up to homotopy) for the moduli spaces, considered as a PROP.

The ribbon graph cell complex $D_{g,h,r,s}$, together with the map (in the homotopy category) $D_{g,h,r,s} \to \mathcal{N}_{g,h,r,s}$, arises immediately from the geometry of the moduli spaces $\overline{\mathcal{N}}_{g,h,r,s}$. The only part of the construction that requires any work is showing that the inclusion $D_{g,h,r,s} \hookrightarrow \overline{\mathcal{N}}_{g,h,r,s}$ is a weak homotopy equivalence. However, the basic idea of the proof is very simple. The key point is to construct a deformation retraction of $\overline{\mathcal{N}}_{g,h,0,s}$ onto its boundary. This is achieved by using the exponential map (for the hyperbolic metric) to flow the boundary of a surface in $\mathcal{N}_{g,h,0,s}$ inwards until it is singular, which results in a surface in $\partial\overline{\mathcal{N}}_{g,h,0,s}$. From considering the fibration $\overline{\mathcal{N}}_{g,h,r,s} \to \overline{\mathcal{N}}_{g,h,0,s}$ given by forgetting the boundary marked points and stabilising, we deduce that the inclusion $\partial\overline{\mathcal{N}}_{g,h,r,s} \hookrightarrow \overline{\mathcal{N}}_{g,h,r,s}$ is also a weak equivalence. An inductive argument, using the fact that the boundary $\partial\overline{\mathcal{N}}_{g,h,r,s}$ of $\overline{\mathcal{N}}_{g,h,r,s}$ is a union of products of similar moduli spaces, allows us to show that the inclusion $D_{g,h,r,s} \hookrightarrow \partial\overline{\mathcal{N}}_{g,h,r,s}$ is a weak equivalence, from which the main result follows.

1.1 Acknowledgements

I'd like to thank Mohammed Abouzaid and the referee for their comments.

2 Moduli of Riemann surfaces with boundary

2.1 Riemann surfaces with nodal boundary

A connected Riemann surface of genus g with $h > 0$ boundary components has the following equivalent descriptions.

(1) A compact connected ringed space Σ, isomorphic as a topological space to a genus g surface with h boundary components, and locally isomorphic to $\{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\}$, with its sheaf of holomorphic functions.

(2) A smooth, proper, connected, complex algebraic curve C of genus $2g-1+h$, with a real structure, such that $C \setminus C(\mathbb{R})$ has precisely two components, and $C(\mathbb{R})$ consists of h disjoint circles; together with a choice of a component of $C \setminus C(\mathbb{R})$.

(3) Suppose $2g-2+h > 0$. Then, a Riemann surface with boundary is equivalently a 2–dimensional connected compact oriented C^∞ manifold Σ with boundary, of genus g with h boundary components, together with a metric of constant curvature -1 such that the boundary is geodesic.

(2) and (3) can be shown to be equivalent (when $2g-2+h > 0$) as follows. Given Σ, C is obtained by gluing Σ and $\overline{\Sigma}$ along their boundary. The real structure on C is arises from the anti-holomorphic involution which is the identity on $\partial\Sigma$ and interchanges Σ and $\overline{\Sigma}$. We denote by $C(\mathbb{R})$ the set of fixed points of the anti-holomorphic involution.

Conversely, given C, Σ is the closure of the chosen component of $C \setminus C(\mathbb{R})$ in C. The hyperbolic metric on Σ is the restriction of the unique complete hyperbolic metric on C compatible with the complex structure.

I will also need Riemann surfaces with nodes on the boundary. A connected Riemann surface with nodal boundary has the following equivalent descriptions.

(1) A compact connected ringed space Σ, locally isomorphic to the ringed space

$$\{(z, w) \in \mathbb{C} \times \mathbb{C} \mid zw = 0, \operatorname{Im} z \geq 0, \operatorname{Im} w \geq 0\}$$

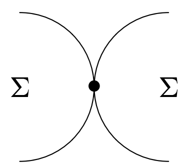
with its sheaf of germs of holomorphic functions. This sheaf is defined to be the inverse image of the sheaf of germs of holomorphic functions on $\mathbb{C} \times \mathbb{C}$.

(2) A proper connected complex algebraic curve C, with at most nodal singularities, and a real structure. The real structure on each connected component C_0 of the normalization $\widetilde{C}$ of C must be of the form (2) above; we also require a choice of component of $C_0 \setminus C_0(\mathbb{R})$. All the nodes of C are required to be real, that is in $C(\mathbb{R})$.
We will let $\Sigma \subset C$ be the closure of the chosen components of $C \setminus C(\mathbb{R})$.

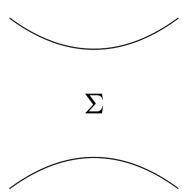
(3) A compact possibly disconnected Riemann surface with boundary $\widetilde{\Sigma}$, together with an unordered finite collection of disjoint points in $\partial\widetilde{\Sigma}$, arranged into unordered pairs. Σ is the space obtained from $\widetilde{\Sigma}$ by identifying each pair of points on $\partial\widetilde{\Sigma}$.

As before, to go from the first description to the second, form the double of the surface Σ, which is an algebraic curve with a real structure.

Near a node, Σ looks like



The number of boundary components of Σ can be defined as follows. $\partial\Sigma$ will be a union of circles, glued together at points as above. Define a smoothing of $\partial\Sigma$, by replacing each node as above by



The number of boundary components of Σ is defined to be the number of connected components of this smoothing.

Σ has genus g if it has h boundary components and the genus of the nodal algebraic curve $C = \Sigma_{\cup \partial\Sigma} \overline{\Sigma}$ is $2g - 1 + h$.

We are also interested in surfaces Σ with marked points. These can be of two types: on the boundary of Σ, or else in the interior $\Sigma \setminus \partial\Sigma$. These marked points must be distinct from the nodes and each other. When Σ has marked points, the double C of Σ is an algebraic curve with marked points, distinct from the nodes. C has a real structure, and some of the marked points are in $C(\mathbb{R})$, and some are in $C \setminus C(\mathbb{R})$. We say that Σ is stable if the double C is, that is if C has only finitely many automorphisms.

Let us suppose that Σ is smooth, has non-empty boundary, and has (r,s) boundary and internal marked points. Then Σ is unstable if and only if it is a disc and $r+2s \leq 2$ or it is an annulus and $r=s=0$. More generally, let Σ be a singular surface. Let $\widetilde{\Sigma}$ be its normalisation, which is obtained by pulling apart all the nodes of Σ. Each node of Σ gives two extra boundary marked points on $\widetilde{\Sigma}$. Then Σ is unstable if and only if one of the connected components of $\widetilde{\Sigma}$ is.

2.2 Moduli spaces of surfaces with boundary

For integers $g,r,s \geq 0, h > 0$, let $\overline{\mathcal{N}}_{g,h,r,s}$ be the moduli space of stable Riemann surfaces Σ of genus g with boundary, possibly with nodes on the boundary, with h boundary components, with r marked points on the boundary $\partial\Sigma$, and s marked points in the interior $\Sigma \setminus \partial\Sigma$. All of the marked points are required to be distinct from the nodes and each other.

This moduli space is non-empty except for the cases when $g=0$, $h=1$ and $r+2s<3$, or $g=0$, $h=2$ and $r=s=0$.

Let $\mathcal{N}_{g,h,r,s} \subset \overline{\mathcal{N}}_{g,h,r,s}$ be the locus of non-singular Riemann surfaces (with boundary).

The moduli spaces $\overline{\mathcal{N}}_{g,h,r,s}$ are open subsets of those constructed by Liu in [7]. Note that in Liu's work, Riemann surfaces are allowed to have nodes in the interior as well as on the boundary, whereas the surfaces we use are not allowed to have nodes in the interior. Also Liu allows the length of boundary components to shrink to zero, turning boundary components into punctures. For us, boundary components always have positive length. Similar moduli spaces were also considered by Fukaya et al [3]. The simplest way to construct these moduli spaces is to realise that they are very closely related to the real points of the Deligne–Mumford moduli spaces $\overline{\mathcal{M}}_{2g-1+h,r+2s}$.

Lemma 2.1 *$\overline{\mathcal{N}}_{g,h,r,s}$ is an orbifold with corners of dimension $6g-6+3h+r+2s$. The interior of $\overline{\mathcal{N}}_{g,h,r,s}$ is $\mathcal{N}_{g,h,r,s}$.*

Recall that an orbifold with corners is, by definition, an orbi-space locally modelled on $\mathbb{R}^k_{\geq 0}$. The reason we get corners is that there is only one way to smooth any node. If our surface has k nodes then it is in a point in the moduli space locally modelled on the origin in $\mathbb{R}^k_{\geq 0}$.

Let $D_{g,h,r,s} \subset \overline{\mathcal{N}}_{g,h,r,s}$ be the locus consisting of those surfaces whose irreducible components are all discs with at most one interior marked point. By "irreducible component" I mean connected component of the normalisation of the surface.

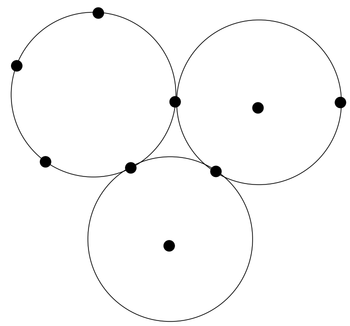


Figure 1: A point in $D_{0,2,4,2}$ given by three discs glued together at the three nodes.

The main result of this note is the following Theorem.

Theorem 2.2 *The inclusion $D_{g,h,r,s} \hookrightarrow \overline{\mathcal{N}}_{g,h,r,s}$ is a weak homotopy equivalence of orbispaces.*

The notion of weak homotopy equivalence between orbispaces is briefly discussed in the appendix. The proof of this theorem will be given in the next section.

Proposition 2.3 *Stratify the space $D_{g,h,r,s}$ by saying Σ, Σ' are in the same stratum if there exists a homeomorphism $\Sigma \to \Sigma'$ preserving the marked points and orientation (but not necessarily respecting the holomorphic structure).*

Then this stratification is a decomposition of the compact orbi-space $D_{g,h,r,s}$ into orbi-cells. When $r > 0$, $D_{g,h,r,s}$ is an ordinary space instead of an orbi-space, and the stratification gives a cell decomposition in the usual sense.

We will show this by labelling the possible topological types of surface in $D_{g,h,r,s}$ by a kind of ribbon graph.

Let $\Gamma_{g,h,r,s}$ denote the set of isomorphism classes of connected graphs γ, with the following extra data and conditions.

(1) γ has r ordered external edges (or tails).

(2) For each vertex $v \in V(\gamma)$, the set of germs of edges at v is cyclically ordered.

(3) The set of vertices $V(\gamma)$ is split into $V_0(\gamma) \amalg V_1(\gamma)$, and there is given an isomorphism $V_1(\gamma) \cong \{1, \dots, s\}$.

(4) All vertices in $V_0(\gamma)$ are at least trivalent, and all vertices in $V_1(\gamma)$ are at least one valent.

(5) The Euler characteristic of γ is $2-2g-h$.

(6) As γ is a ribbon graph, we can as usual talk about boundary components of γ. There are h unordered boundary components.

Let $\Sigma \in D_{g,h,r,s}$. Associate to Σ a graph $\gamma(\Sigma) \in \Gamma_{g,h,r,s}$. There is one vertex of $\gamma(\Sigma)$ for each irreducible component of Σ, an edge for each node, and an external edge for each marked point on $\partial\Sigma$. As the irreducible components of Σ are all discs, the set of nodes and marked points on the boundary of each irreducible component has a natural cyclic ordering, coming from the orientation on the boundary of a disc. Thus the graph associated to Σ has the structure of a ribbon graph. There are s ordered internal marked points on Σ, with at most one on a given irreducible component. A vertex of $\gamma(\Sigma)$ is in $V_0(\gamma(\Sigma))$ if it doesn't contain an internal marked point, and it is in $V_1(\gamma(\Sigma))$ if it does. The isomorphism $V_1(\gamma(\Sigma)) \cong \{1,\dots,s\}$ is given by the ordering of the internal marked points of Σ.

Lemma 2.4 *For any graph $\gamma \in \Gamma_{g,h,r,s}$, the space of $\Sigma \in D_{g,h,r,s}$ with $\gamma(\Sigma) = \gamma$ is an orbi-cell.*

Proof Let $\gamma \in \Gamma_{g,h,r,s}$, and let $v \in V(\gamma)$. Let $E(v)$ be the set of germs of edges at v. To give a surface $\Sigma \in D_{g,h,r,s}$ with $\gamma(\Sigma) = \gamma$ amounts to giving, for each vertex $v \in V(\gamma)$, a disc D with distinct points on ∂D labelled by $E(v)$, in a way compatible with the cyclic order on $E(v)$; and in addition, if $v \in V_1(\gamma)$, a point in the interior of D. If we change this data by an automorphism of γ, then we get an isomorphic surface.

For $v \in V(\gamma)$, let $\widetilde{X}(v)$ be the set of injective maps $f\colon E(v) \to S^1$, such that the cyclic order induced on the image of f by the orientation of S^1 coincides with the given cyclic order on $E(v)$.

If $v \in V_0(\gamma)$ let $X(v) = \widetilde{X}(v)/PSL_2(\mathbb{R})$, where $PSL_2(\mathbb{R})$ acts on S^1 by Möbius transformations. If $v \in V_1(\gamma)$ let $X(v) = \widetilde{X}(v)/S^1$. Note that the spaces $X(v)$ are cells.

Then the orbi-space of surfaces $\Sigma \in D_{g,h,r,s}$ with $\gamma(\Sigma) = \gamma$ can be identified with the orbicell

$$\left(\prod_{v \in V(\gamma)} X(v)\right) / \operatorname{Aut}(\gamma)$$

where $\operatorname{Aut}(\gamma)$ is the group of automorphisms of γ preserving all the labellings. □

This completes the proof of Proposition 2.3, except for the clause about the case when $r > 0$. This follows from Lemma 3.9 below, which shows that when $r > 0$ surfaces in $\overline{\mathcal{N}}_{g,h,r,s}$ have no non-trivial automorphisms.

One can use this orbi-cell complex to give a chain model for the rational homology of the moduli spaces $\overline{\mathcal{N}}_{g,h,r,s} \simeq \mathcal{N}_{g,h,r,s}$. A basis for the chain complex is given by ribbon graphs γ in $\Gamma_{g,h,r,s}$ together with an orientation on the corresponding orbi-cell. An orientation can be given by choosing an ordering of the set of vertices of γ, and at each vertex an ordering of the set of germs of edges. It is not difficult to calculate how changing this ordering changes the orientation. The boundary in this chain complex is given by summing over all ways of splitting a vertex in V_0 into two vertices in V_0, and splitting a vertex in V_1 into a vertex in V_1 and a vertex in V_0. In the case $s = 0$, this recovers the usual ribbon graph model for homology of moduli spaces. When $s > 0$, this chain complex is combinatorially distinct.

When $r > 0$, as $D_{g,h,r,s}$ is in this case an ordinary cell complex and not an orbi-cell complex, we find a complex computing the integral homology of moduli space.

Instead of working with an explicit chain complex, I prefer to think of this result as giving (in the case $s = 0$) a generators-and-relations description for the PROP controlling open topological conformal field theory, see [1; 2]. When $s > 0$ the algebraic statement corresponding to this cell decomposition is a generators-and-relations description for a certain natural module over this PROP.

3 Proof of main theorem

The key point is Proposition 3.3, which shows that the orbi-space $\overline{\mathcal{N}}_{g,h,0,s}$ deformation retracts onto its boundary $\partial\overline{\mathcal{N}}_{g,h,0,s}$, except for the cases when $(g,h) = (0,1)$ and $s \in \{0,1\}$.

The first step in the proof is the following lemma.

Lemma 3.1 *There is a map $\overline{\mathcal{N}}_{g,h,r+1,s} \to \overline{\mathcal{N}}_{g,h,r,s}$, which forgets the last point and stabilises. This is a locally trivial fibration in the orbispace sense.*

Proof Recall that there is a map

$$\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

$2g - 2 + n > 0$, given by forgetting the $(n+1)$th marked point, and contracting any resulting unstable components. This morphism can be identified with the universal curve on the stack $\overline{\mathcal{M}}_{g,n}$.

There is an induced map

$$\pi\colon \overline{\mathcal{N}}_{g,h,r+1,s} \to \overline{\mathcal{N}}_{g,h,r,s}$$

for (g,h,r,s) stable, that is $4g-4+2h+r+2s>0$. This map removes the $(r+1)$th marked point on the surface. If this leaves the surface with a disc with two special points, where a special point is a marked point or a node, then we contract that disc.

We need to show that this is a topologically locally trivial map, in the orbifold sense. Let $\Sigma \in \overline{\mathcal{N}}_{g,h,r,s}$, and let us pick one of the boundary components of Σ, which we denote by $\partial_0\Sigma$. (Recall the definition of boundary components of a singular surface). Let us consider adding on a marked point to $\partial_0\Sigma$, near a node n of $\partial_0\Sigma$. Let $U \subset \partial_0\Sigma$ be a neighbourhood of a node. Let $V \subset \overline{\mathcal{N}} r, h, r+1, s$ be the space of those surfaces Σ' where the extra marked point on Σ' occurs in the neighbourhood U of the node n. Then, I claim the map $V \to U$ is a homeomorphism. It is clear we can add on a unique marked point p to a smooth point of U. If p approaches the node, then we bubble off a disc, which is inserted into $\partial_0\Sigma$ at this node. This disc has two nodes and one marked point, p. There is one way to glue on such a disc, so we can instead think of adding on a marked point at the node. We get the same configuration if p approaches from the other side. Therefore the space V of possible ways of adding on a marked point is homeomorphic to U.

The same behaviour occurs if we looked at a smooth region of a boundary component. This makes it clear that the map is a locally trivial fibration. □

Lemma 3.2 *If the map $\partial\overline{\mathcal{N}}_{g,h,r,s} \hookrightarrow \overline{\mathcal{N}}_{g,h,r,s}$ is a weak homotopy equivalence, then so is the map $\partial\overline{\mathcal{N}}_{g,h,r+1,s} \hookrightarrow \overline{\mathcal{N}}_{g,h,r+1,s}$.*

Proof Consider the following fibre square.

$$\begin{array}{ccc} \partial\overline{\mathcal{N}}_{g,h,r+1,s} & \longrightarrow & \overline{\mathcal{N}}_{g,h,r+1,s} \\ \downarrow & & \downarrow \\ \partial\overline{\mathcal{N}}_{g,h,r,s} & \longrightarrow & \overline{\mathcal{N}}_{g,h,r,s} \end{array}$$

The vertical arrows are the maps which forget the last marked point. As the vertical arrows are locally trivial fibrations, and in particular the fibres are all the same, the result follows. □

The key step in the proof is the following proposition.

Proposition 3.3 *For all (g,h) with $2g-2+h+s>0$, the inclusion*

$$\partial\overline{\mathcal{N}}_{g,h,0,s} \hookrightarrow \overline{\mathcal{N}}_{g,h,0,s}$$

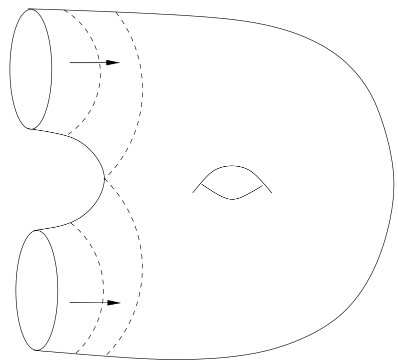


Figure 2: By moving the boundaries of the surface inwards, we find a singular surface.

is a homotopy equivalence of orbi-spaces.

Proof Let $\Sigma \in \mathcal{N}_{g,h,0,s}$. Note that Σ is a smooth surface. Let $p_1, \dots, p_s \in \Sigma \setminus \{\partial \Sigma\}$ be the marked points, and let

$$\Sigma_0 = \Sigma \setminus \{p_1, \dots, p_s\}.$$

Then Σ_0 has a unique hyperbolic metric, compatible with the conformal structure, such that the boundary is geodesic and such that the double of Σ_0 is geodesically complete. To construct this metric, one uses the double C_0 of Σ_0, defined by

$$C_0 = \Sigma_0 \amalg_{\partial \Sigma_0} \overline{\Sigma}_0.$$

This double C_0 has a unique complete hyperbolic metric, and the anti-holomorphic involution on C_0 is an isometry. It follows that the fixed points of the involution are geodesic, so that the restriction of this metric to Σ_0 has geodesic boundary.

Let V be the unit inward pointing normal vector field on $\partial \Sigma = \partial \Sigma_0$. Using the geodesic flow on Σ_0, we can flow $\partial \Sigma$ inwards, as in Figure 2. That is, for each $s \in \mathbb{R}_{\geq 0}$ in some neighbourhood of 0, we have an exponential map

$$\exp_s \colon \partial \Sigma \to \Sigma_0$$

by flowing the boundary in for a time s along the geodesic flow. As the double C_0 of Σ_0 is complete, this exists for all time as a map to C_0.

Definition 3.4 *Let $T(\Sigma) \in \mathbb{R}_{>0}$ be the smallest number such that the subset*

$$\exp_{T(\Sigma)}(\partial \Sigma) \subset \Sigma_0$$

is singular.

Lemma 3.5 *The only singularities of* $\exp_{T(\Sigma)}(\partial\Sigma)$ *are nodes. In other words, for each* $x \in \Sigma_0$, $\exp^{-1}_{T(\Sigma)}(x)$ *consists of at most two points in* $\partial\Sigma$.

Proof

Suppose this is not the case. Let $y_1, y_2, y_3 \in \partial\Sigma$ be three distinct points such that the points $\exp_{T(\Sigma)}(y_i) \in \Sigma_0$ coincide. Let $x = \exp_{T(\Sigma)}(y_i)$. Let $U_i \subset \partial\Sigma$ be small neighbourhoods of y_i in $\partial\Sigma$. Note that the derivative of $\exp_{T(\Sigma)}\colon \partial\Sigma \to \Sigma$ never vanishes. We can assume that the U_i are such that $\exp_{T(\Sigma)}\colon U_i \to \Sigma$ is injective.

Consider the three unparameterised C^∞ paths $\gamma_i = \exp_{T(\Sigma)}(U_i) \subset \Sigma$. These intersect only at x. Suppose γ_1, γ_2 intersect transversely at x. Then, for some ϵ sufficiently small, $\exp_{T(\Sigma)-\epsilon}(U_1)$ and $\exp_{T(\Sigma)-\epsilon}(U_2)$ also intersect transversely, which contradicts the fact that $T(\Sigma)$ is the smallest time at which $\exp_{T(\Sigma)}(\partial\Sigma)$ has a singularity.

Therefore, the tangent vectors to all the paths $\gamma_1, \gamma_2, \gamma_3$ at x all lie on the same line. This implies that their normal vectors do also. These normal vectors are non-zero; at least two of them coincide up to rescaling by a positive real number. Suppose γ_1, γ_2 have this property. The geodesic $\exp_s(y_i)$ for $0 \le s \le T(\Sigma)$ is normal to γ_i. Since $\exp_s(y_1)$ and $\exp_s(y_2)$ point in the same direction at x, and both are unit speed geodesics, it follows that they coincide. Thus, $y_1 = y_2$, which is a contradiction. □

Lemma 3.6

(1) $T(\Sigma)$ *is half the length of the shortest smooth geodesic arc in* Σ_0 *which meets the boundary at both ends at right angles.*

(2) $T(\Sigma)$ *is one quarter of the length of the shortest smooth closed geodesic in the double* C_0 *of* Σ_0 *whose free homotopy class is non-trivial and invariant under the anti-holomorphic involution of* C_0.

Proof For part (1), at the first time there is a singularity, there are two points $y_1, y_2 \in \partial(\Sigma_0)$ such that $\exp_{T(\Sigma)}(y_1) = \exp_{T(\sigma)}(y_2)$. Let us call this point x. The proof of the previous lemma implies that the two tangent vectors $\exp_{T(\Sigma)-\epsilon}(y_1)$ and $\exp_{T(\Sigma)-\epsilon}(y_2)$ at x are opposite. This implies that the path

$$\exp_s(y_1) \text{ if } s \le T(\Sigma)$$
$$\exp_{2T(\Sigma)-s}(y_2) \text{ if } s \ge T(\Sigma)$$

is a smooth geodesic arc of length $2T(\Sigma)$. Thus, if ϕ is the shortest such geodesic arc, $2T(\Sigma) \le l(\phi)$. Conversely, if y_1, y_2 are the end points of ϕ, then $\exp_{l(\phi)/2}(y_1) = \exp_{l(\phi)/2}(y_2)$ so that $2l(\phi) = 2T(\Sigma)$.

To prove part (2), note that if γ is such a geodesic on C_0, then $\gamma = \overline{\gamma}$, as there is a unique smooth closed geodesic in each free homotopy class. This implies that γ meets $\partial\Sigma_0$ precisely twice, and each time at right angles, so that γ is the double of a geodesic arc in Σ_0 which meets the boundary at right angles. □

For each $0 \le t \le T(\Sigma)$, define a surface

$$\Sigma(t) \stackrel{\text{def}}{=} \Sigma \setminus \cup_{s<t} \exp_s(\partial\Sigma).$$

For $0 \le t < T(\Sigma)$, $\Sigma(t)$ is in $\mathcal{N}_{g,h,0}$. The surface $\Sigma(T(\Sigma))$ is in $\partial\overline{\mathcal{N}}_{g,h,0}$. To see this, observe that the previous lemma implies $\Sigma(T(\Sigma))$ has only nodal singularities. Also there are no unstable components, simply because there are no hyperbolic polygons with ≤ 2 sides.

Now define a map of orbispaces

$$\Phi\colon \mathcal{N}_{g,h,0} \times [0,1] \to \overline{\mathcal{N}}_{g,h,0}$$
$$\Phi(\Sigma, t) = \Sigma(tT(\Sigma)).$$

Lemma 3.7 *Φ extends continuously to a map $\overline{\mathcal{N}}_{g,h,0} \times [0,1] \to \overline{\mathcal{N}}_{g,h,0}$, by defining $\Phi(\Sigma, t) = \Sigma$ for $\Sigma \in \partial\overline{\mathcal{N}}_{g,h,0}$.*

Proof We have to show that the extension of Φ so defined is continuous. Let $\Sigma_i \in \mathcal{N}_{g,h,0}$ for $i \in \mathbb{Z}_{>0}$ be a sequence of surfaces converging to $\Sigma \in \partial\overline{\mathcal{N}}_{g,h,I}$, and let $t_i \in [0,1]$ be any sequence. We need to show that

$$\lim_{i\to\infty} \Phi(\Sigma_i, t_i) = \Sigma.$$

I claim that $T(\Sigma_i) \to 0$ as $i \to \infty$. For simplicity, we will prove this in the case when the limiting surface Σ has only a single node. Let $C_i \in \mathcal{M}_{g,n}$ be the double of $\Sigma_i \setminus$ marked points. Then, for $i \gg 0$, C_i has a unique shortest closed geodesic in the homotopy class corresponding to the node in the limiting surface $C \in \overline{\mathcal{M}}_{g,n}$. Lemma 3.6 says that $T(\Sigma_i)$ is one quarter of the length of this geodesic in C_i, and it is a standard fact that the length of this geodesic converges to zero.

Now, since $T(\Sigma_i) \to 0$, it follows that for any sequence $t_i \in [0,1]$ the sequence

$$\Phi(\Sigma_i, t_i) = \Sigma_i(t_i T(\Sigma_i))$$

has the same limit as Σ_i. That is, $\Phi(\Sigma_i, t_i)$ is obtained from Σ_i by moving the boundary inwards by $t_i T(\Sigma_i)$, and this tends to zero. Therefore $\lim \Phi(\Sigma_i, t_i) = \Sigma$. □

This completes the proof of Proposition 3.3. As Φ gives a deformation retraction of the orbispace $\overline{\mathcal{N}}_{g,h,0,s}$ onto its boundary $\partial\overline{\mathcal{N}}_{g,h,0,s}$. □

We are nearly finished with the proof of Theorem 2.2.

Lemma 3.8 *For all $g \geq 0, h \geq 1, r \geq 0, s \geq 0$, with*

$$(g,h,r,s) \neq (0,1,r,0)$$
$$(g,h,r,s) \neq (0,1,r,1)$$
$$(g,h,r,s) \neq (0,2,0,0)$$

the inclusion

$$\partial\overline{\mathcal{N}}_{g,h,r} \hookrightarrow \overline{\mathcal{N}}_{g,h,r}$$

is a weak homotopy equivalence.

Proof This follows from Proposition 3.3 and Lemma 3.2, except for the case when $(g,h,r,s) = (0,2,1,0)$. In this case, it is easy to see that $\overline{\mathcal{N}}_{0,2,1,0} = \mathbb{R}_{\geq 0}$, so $\partial\overline{\mathcal{N}}_{0,2,1,0} \hookrightarrow \overline{\mathcal{N}}_{0,2,1,0}$ is obviously a homotopy equivalence. □

Lemma 3.9 *Let $\Sigma \in \overline{\mathcal{N}}_{g,h,r,s}$, where $r \geq 1$. Then Σ has no non-trivial automorphisms fixing each of the marked points.*

Proof First suppose Σ is smooth. Put the hyperbolic metric on $\Sigma_0 = \Sigma \setminus \{p_1, \dots, p_s\}$. Any automorphism of Σ preserving the marked points induces an isometry of Σ_0, fixing all the marked points on $\partial\Sigma$. Since the automorphism must act as the identity on the tangent space to each marked point on $\partial\Sigma$, it must be the identity on a neighbourhood of each marked point. Since the automorphism is analytic, it must be the identity everywhere.

Now suppose Σ is singular. Let $p \in \partial\Sigma$ be a marked point. Let ϕ be an automorphism of Σ. Then ϕ is the identity on the irreducible component containing p. Suppose n is a node which joins this component of Σ to some other component. Then $\phi(n) = n$, which implies that ϕ is the identity on the other component at this node. Repeating this argument we see ϕ is the identity everywhere. □

Finally, we can finish the proof of the theorem.

Lemma 3.10 *The inclusion $D_{g,h,r,s} \hookrightarrow \overline{\mathcal{N}}_{g,h,r,s}$ is a weak homotopy equivalence of orbispaces.*

Proof By induction, suppose we have proved the result for all moduli spaces of lower dimension.

For $k \geq 1$, let $\partial_k \overline{\mathcal{N}}_{g,h,r,s}$ be the space of surfaces $\Sigma \in \partial \overline{\mathcal{N}}_{g,h,r,s}$, equipped with a map from the set $\{1, 2, \dots k\}$ to the set of nodes on Σ. Lemma 3.9 implies that $\partial_k \overline{\mathcal{N}}_{g,h,r,s}$ is an ordinary topological space, and not just an orbi-space. The spaces $\partial_k \overline{\mathcal{N}}_{g,h,r,s}$ are the $k-1$ simplices of a simplicial space. The face maps are the maps which forget a node. This simplicial space is the one obtained by iterated fibre products of the map $\partial_1 \overline{\mathcal{N}}_{g,h,r,s} \to \partial \overline{\mathcal{N}}_{g,h,r,s}$. Therefore the topological realisation $\left|\partial_* \overline{\mathcal{N}}_{g,h,r,s}\right|$ of this simplicial space is weakly equivalent to $\partial \overline{\mathcal{N}}_{g,h,r,s}$.

Similarly, let $\partial_k D_{g,h,r,s}$ be the space of surfaces in $D_{g,h,r,s}$ with a map from the set $\{1, 2, \dots k\}$ to the set of nodes on the surface. The spaces $\partial_k D_{g,h,r,s}$ form a simplicial space, and $\left|\partial_* D_{g,h,r,s}\right|$ is weakly equivalent to $D_{g,h,r,s}$.

There is a map of simplicial spaces $\partial_* D_{g,h,r,s} \to \partial_* \overline{\mathcal{N}}_{g,h,r,s}$. By induction, we know the maps $\partial_k D_{g,h,r,s} \to \partial_k \overline{\mathcal{N}}_{g,h,r,s}$ are weak equivalences. It follows that the associated map $\left|\partial_* D_{g,h,r,s}\right| \to \left|\partial_* \overline{\mathcal{N}}_{g,h,r,s}\right|$ on the realisations of our simplicial spaces is a weak equivalence.

The diagram

$$\begin{array}{ccc} \left|\partial_* D_{g,h,r,s}\right| & \longrightarrow & \left|\partial_* \overline{\mathcal{N}}_{g,h,r,s}\right| \\ \downarrow & & \downarrow \\ D_{g,h,r,s} & \longrightarrow & \partial \overline{\mathcal{N}}_{g,h,r,s} \end{array}$$

commutes, and the vertical arrows and top horizontal arrows are weak equivalences. It follows that the map $D_{g,h,r,s} \to \partial \overline{\mathcal{N}}_{g,h,r,s}$ is a weak equivalence, which implies that $D_{g,h,r,s} \to \overline{\mathcal{N}}_{g,h,r,s}$ is a weak equivalence. □

Appendix A Orbispaces

We recall briefly some definitions from the theory of topological stacks. See Noohi [8] for details. We use the word orbispace to refer to a weak topological Deligne–Mumford stack in the sense of [8]. An orbispace is a category fibred in groupoids over the category Top of compactly generated Hausdorff topological spaces, satisfying a descent (or sheaf) condition, and a representability condition. The Grothendieck topology on the category Top is that where the covering maps are the usual open coverings. The representability condition is that there exists a surjective map from an ordinary space which is a local homeomorphism.

Let X be an orbispace, and let $U \to X$ be a surjective local homeomorphism from a space U. We can form a simplicial space by taking iterated fibre products of U over X. The n simplices are $(U/X)^{n+1}$, the face maps are projections and the degeneracy maps are diagonals. Denote by $\mathfrak{N}^{\Delta}(U/X)$ this simplicial space, and by $\mathfrak{N}(U/X)$ its geometric realisation. The weak homotopy type of $\mathfrak{N}(U/X)$ is called the weak homotopy type of X. This is independent of the choice of U.

Suppose $f\colon Y \to X$ is a representable map of orbispaces. This means that all of the fibres are ordinary spaces. Pick a surjective local homeomorphism $U \to X$ as above. Then the map $Y \times_X U \to Y$ has the same property; in particular $Y \times_X U$ is an ordinary space. There is a map $\mathfrak{N}(Y \times_X U/Y) \to \mathfrak{N}(U/X)$. If this is a weak homotopy equivalence then we say that the map $f\colon Y \to X$ is a weak homotopy equivalence. This definition can be extended to non-representable maps by refining the cover $Y \times_X U \to Y$.

To see that this is the correct notion of weak homotopy type of an orbispace, observe that if G is a discrete group, acting on a space X, and we form the orbispace quotient X/G, then the map $X \to X/G$ is a local homeomorphism, and $\mathfrak{N}(X/(X/G))$ is one of the standard models for the homotopy quotient of X by G.

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