

Harmonic sections in sphere bundles, normal neighborhoods of reduction loci, and instanton moduli spaces on definite 4-manifolds

ANDREI TELEMAN

In the first part of the paper we prove an existence theorem for gauge invariant L^2 -normal neighborhoods of the reduction loci in the space $\mathcal{A}_a(E)$ of oriented connections on a fixed Hermitian 2-bundle E . We use this to obtain results on the topology of the moduli space $\mathcal{B}_a(E)$ of (non-necessarily irreducible) oriented connections, and to study the Donaldson μ -classes globally around the reduction loci. In this part of the article we use essentially the concept of harmonic section in a sphere bundle with respect to an Euclidean connection.

Second, we concentrate on moduli spaces of instantons on definite 4-manifolds with arbitrary first Betti number. We prove strong generic regularity results which imply (for bundles with “odd” first Chern class) the existence of a *connected*, dense open set of “good” metrics for which all the reductions in the Uhlenbeck compactification of the moduli space are simultaneously regular. These results can be used to define new Donaldson type invariants for definite 4-manifolds. The idea behind this construction is to notice that, for a good metric g , the geometry of the instanton moduli spaces around the reduction loci is always the same, independently of the choice of g . The connectedness of the space of good metrics is important, in order to prove that no wall-crossing phenomena (jumps of invariants) occur. Moreover, we notice that, for low instanton numbers, the corresponding moduli spaces are a priori compact and contain no reductions at all so, in these cases, the existence of well-defined Donaldson type invariants is obvious. Note that, on the other hand, there seems to be very difficult to introduce well defined numerical Seiberg-Witten invariants for definite 4-manifolds. For instance, the construction proposed by Okonek and the author in [7] gives a \mathbb{Z} -valued function defined on a countable set of chambers.

The natural question is to decide whether these new Donaldson type invariants yield essentially new *differential* topological information on the base manifold, or have a purely topological nature.

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Introduction

The main goal of this article is to study moduli spaces of instantons over 4-manifolds with negative definite intersection form. The vanishing of b_+ has an important consequence on the geometry of the instanton moduli spaces: all line bundles admit ASD connections (with respect to any metric) hence, as soon as a rank 2-bundle E splits topologically, the corresponding instanton moduli space will always contain reductions. In other words, “one cannot get rid of reductions by perturbing the metric”. On the other hand, our main applications will concern 4-manifolds with $b_1 > 0$, and for such manifolds the spaces of reductions are positive dimensional. Therefore, it is very important to study carefully the *global* geometry of the moduli space of connections around the loci of reductions. This will be our first goal.

To be more precise, let E be a rank 2-Hermitian bundle on a 4-manifold M , and denote $D := \det(E)$, $d = c_1(D)$. Consider the affine space $\mathcal{A}_a(E)$ of connections A on E which induce a fixed connection a on D , and the moduli space $\mathcal{B}_a(E) = \mathcal{A}_a(E)/\mathcal{G}_E$, where \mathcal{G}_E is the gauge group $\Gamma(\mathrm{SU}(E))$.

Let $l \in H^2(X, \mathbb{Z})$ such that $l(d-l) = c_2(E)$ and consider the set $\lambda = \{l, d-l\}$ (which has either one or two elements). We denote by $\mathcal{A}_a^\lambda(E)$ the subspace of connections $A \in \mathcal{A}_a(E)$, which are simply reducible of type λ , that is, those which admit only two parallel line subbundles whose Chern classes are $l, d-l$. Such a connection will be called λ -reducible. $\mathcal{A}_a^\lambda(E)$ becomes a (locally closed) submanifold of the affine space $\mathcal{A}_a(E)$ (after suitable Sobolev completions). Our first problem is the construction of a gauge invariant L^2 -normal neighborhood of this submanifold. More precisely, we will show that, denoting by N^λ the L^2 -normal bundle of $\mathcal{A}_a^\lambda(E)$, the restriction of the natural map $\nu: N^\lambda \rightarrow \mathcal{A}_a(E)$ to a sufficiently small gauge invariant neighborhood \mathcal{U}^λ of the zero section is a diffeomorphism on its image. Moreover, the neighborhood \mathcal{U}^λ is defined by a inequality of the form $\|\alpha\|_{L^\infty} \leq \varepsilon(A)$ (on the fiber N_A), where the assignment $A \mapsto \varepsilon(A)$ is gauge invariant and continuous (with respect to a sufficiently fine Sobolev topology on the space of connections).

Although this statement is very natural, the proof is not easy. The difficulty comes from the fact that infinite dimensional manifolds are not locally compact. Even the fact the ν is injective on a neighborhood of the zero section is not trivial. The main difficulty is to characterize in a convenient way the connections which are “close” to the reducible locus $\mathcal{A}_a^\lambda(E)$, that is, those which are “almost” λ -reducible.

Our argument is based on the following idea: A reduction $A \in \mathcal{A}_a^\lambda(E)$ admits a parallel section in the sphere bundle $S(\mathrm{su}(E))$. A connection which is close to being reducible should admit an *energy minimizing harmonic section* in this sphere bundle. The precise

meanings of the words “energy” and “harmonic” are the following: we associate to any connection $A \in \mathcal{A}_a(E)$ the energy functional

$$E_A(u) := \|d_A(u)\|_{L^2}^2, \quad E_A: \Gamma(S(\text{su}(E))) \longrightarrow \mathbb{R}_{\geq 0},$$

and we agree to call the critical points of this functional A -harmonic sections. After proving these results about normal neighborhoods of reduction loci, we realized that our problem can be naturally generalized in the following way: For an Euclidean bundle F on an arbitrary compact Riemannian manifold, construct a gauge invariant L^2 -normal neighborhood of the space of Euclidean connections $d_A: A^0(F) \rightarrow A^1(F)$ with 1-dimensional kernel. Therefore, in Section 1 we will treat this more general problem, which is of independent interest; the results concerning reducible Hermitian connections will be easily deduced as applications, taking $F = \text{su}(E)$.

Section 2 deals with the first consequences of our existence results for gauge invariant L^2 -normal neighborhoods: a global description of the universal $\text{SO}(3)$ bundle, explicit formulae for the Donaldson μ -classes around the reduction loci and an explicit description of the homotopy type of the whole moduli space $\mathcal{B}_a(E)$. This space is much more complex than its open subspace $\mathcal{B}_a^*(E)$, which plays a central role in classical gauge theory and whose rational homotopy type has been described by Donaldson and Kronheimer [3]. Our description of $\mathcal{B}_a(E)$ yields an easy method to compute the cohomology of this space using the Mayer-Vietoris exact sequence.

Section 3 is dedicated to the geometry of the instanton moduli space around a reduction locus. Using our existence results for normal neighborhoods, we show that *in a neighborhood of the reduction locus associated with a fixed topological splitting of the bundle, the instanton moduli problem reduces to an abelian moduli problem*, which is very much similar to the Seiberg-Witten one. These abelian equations read

$$(1) \quad \begin{cases} F_b^+ - \frac{1}{2}F_a^+ & = (\alpha \wedge \bar{\alpha})^+, \\ (d_b^* \otimes_{a^\vee}, d_b^+ \otimes_{a^\vee})\alpha & = 0, \end{cases}$$

which are equations for a pair (b, α) , where b is a connection on a fixed Hermitian line bundle L and $\alpha \in A^1(L^{\otimes 2} \otimes D^\vee)$. This result provides simple descriptions of the linear spaces of harmonic spaces of the deformation elliptic complexes at the reductions. It is very important to have a *global* description of these linear spaces. Similar results are stated for the loci of *twisted reductions* (that is, instantons which are locally reducible, but globally irreducible).

Our most important technical results are obtained in Section 3.3: we prove strong generic regularity theorems at the reductions. Regularity at the reductions is an old, classical problem in gauge theory (see Freed and Uhlenbeck [4], Donaldson and

Kronheimer [3]) and one might wonder whether there are still unsolved questions on this problem. The point is that Freed–Uhlenbeck’s generic regularity result is not sufficient for our purposes. For our purposes, we need a *connected*, dense open set of good metrics (metrics for which all reductions in the Uhlenbeck compactification are regular). Our proof has two steps:

Step 1 Define a connected, dense open set of *admissible* C^r –metrics, for which the vanishing loci of the harmonic representatives of the classes

$$2l - d, \quad l \cdot (d - l) \leq c_2(E)$$

have good geometric properties (see Section A.2). Our admissibility condition is very natural: we require that the rank of the intrinsic derivative of these harmonic representatives at any vanishing point is at least 2. The difficulty is to check that this condition defines indeed a connected, dense set of metrics.

In particular, for our admissible metrics, the vanishing loci of all these harmonic representatives have Hausdorff dimension at most 2. Note that this holds for *any* C^∞ –metric by a result of Bär [1], but this result does not appear to generalize for C^r –metrics. For a metric for which this Hausdorff dimension bound holds, the statement in [4, Lemma 4.16] holds, making possible the second step. This stronger version of [4, Lemma 4.16] is proved in detail in Appendix A (as Corollary A.9).

Step 2 Regard the linear space formed by the *second* harmonic spaces at the reductions as the moduli space associated with an *abelian* moduli problem, and prove a transversality theorem for this moduli problem with respect to variations of g (in the space of admissible metrics). In other words, we will prove that the parameterized moduli space (obtained by letting the metric vary in the set of admissible metrics) is smooth away of the zero-section. Next we show that – for a bundle E with odd Chern class – the projection map from the \mathbb{C}^* –quotient of this parameterized moduli space (minus the zero-section) on the space of admissible metrics is Fredholm of *negative* index ≤ -2 , hence its image has connected complement.

Section 4 deals with applications of our results. First we prove a simple geometric property of a particular instanton moduli space on a 4–manifold which has the homology type of a class VII surface with $b_2 = 2$: the two circles of reductions belong always (for any metric!) to the same connected component. We continue with the construction of a new class of Donaldson invariants; the new invariants are defined in two situations:

First we show that the topological data can be chosen such that the rank 2–bundles E' with $c_2(E') \leq c_2(E)$ and $c_1(E') = c_1(E)$ are all topologically non-decomposable; in this case the Uhlenbeck compactification of $\mathcal{M}_a^{\text{ASD}}(E)$ contains no reduction, so Donaldson invariants can be defined in the usual way. In particular, in the case when

$4c_2(E) - c_1(E)^2 = 0$, one obtains a moduli space of $\text{SO}(3)$ -representations of $\pi_1(M)$, modulo $\text{SU}(2)$ -conjugation.

Second, one chooses the topological data such that the bundle E splits topologically, $4c_2(E) - c_1(E)^2 \leq 3$, and $c_1(E) \notin 2H^2(M, \mathbb{Z}) + \text{Tors}$. In this case $\mathcal{M}_a^{\text{ASD}}(E)$ is compact for any metric, and our results concerning generic regularity at the reductions apply. Therefore, for a generic metric, $\mathcal{M}_a^{\text{ASD}}(E)^*$ is a smooth manifold with a finite number of cylindrical ends towards the reduction loci. Every end is a cylinder on a projective bundle over a $b_1(M)$ -dimensional torus.

1 Harmonic sections, parallel sections

1.1 Harmonic sections in sphere bundles

Let (M, g) be a compact oriented Riemannian n -manifold and let F be a real rank r vector bundle on M endowed with an inner product, and denote by $S(F)$ the unit sphere bundle of F . Let A be an Euclidean connection on F . The energy functional on the space of sections $\Gamma(S(F))$ is defined by

$$E_A(u) = \|d_A(u)\|_{L^2}^2 = \langle d_A u, d_A u \rangle.$$

The critical points of this functional will be called A -harmonic sections.

Proposition 1.1 *A section $u \in \Gamma(S(F))$ is harmonic if and only if it satisfies the Euler-Lagrange equation*

$$d_A^* d_A u - |d_A u|^2 u = 0.$$

Proof The section $u \in \Gamma(S(F))$ is a critical point of E_A if and only if

$$\langle d_A^* d_A u, v \rangle_{L^2} = 0$$

for every $v \in T_u(\Gamma(S(F)))$. This happens if and only if there exists a real function φ such that $d_A^* d_A u = \varphi u$. On the other hand, the well-known identity

$$\frac{1}{2} \Delta |u|^2 = (d_A^* d_A u, u) - |d_A u|^2$$

shows that one must have $\varphi = |d_A u|^2$. □

For a fixed connection A the theory of A -harmonic sections is very much similar to the theory of sphere valued harmonic maps. In particular, one has a parabolic evolution equation given by the gradient flow of the functional E_A , and using this equation and its

convergence properties, one can study the existence of an A -harmonic representative in a given homotopy class of sections.

Two vectors a, b of an Euclidean vector space V define an endomorphism $a \wedge_V b \in \text{so}(V)$ given by $a \wedge_V b(h) = (a, h)b - (b, h)a$. The subspace

$$\{a \wedge_V b \mid b \in V\} \subset \text{so}(V)$$

coincides with the orthogonal complement $\text{so}(a^\perp)^\perp$ of $\text{so}(a^\perp)$ in $\text{so}(V)$, and the map $a^\perp \ni b \mapsto a \wedge_V b \in \text{so}(a^\perp)^\perp$ is an isomorphism. Similarly, for a section $u \in A^0(F)$ and a form $v \in A^1(F)$ we obtain a form $u \wedge_F v \in A^1(\text{so}(F))$.

Lemma 1.2 *Let $A \in \mathcal{A}(F)$, $u \in \Gamma(S(F))$ and $A_0 := A - u \wedge_F d_{A_0}u$. Then*

- (1) *The section u is A_0 -parallel.*
- (2) *A_0 is the unique connection making u parallel with $A - A_0 \in A^1(\text{so}(u^\perp)^\perp)$.*

Proof (1) It holds that

$$d_{A_0}u = d_{A_0}u - (u \wedge_F d_{A_0}u)(u) = d_{A_0}u - (u, u)d_{A_0}u + (u, d_{A_0}u)u = 0,$$

because, since $(u, u) \equiv 1$, one has $(u, d_{A_0}u) = 0$.

(2) If $A - A_0 \in A^1(\text{so}(u^\perp)^\perp)$, there exists a unique form $v \in A^1(u^\perp)$ such that $A - A_0 = u \wedge_F v$. The relation $d_{A_0}u = 0$ is equivalent to $d_{A_0}u = v$. □

Proposition 1.3 *Let $u \in \Gamma(S(F))$ and $A \in \mathcal{A}(F)$. The following conditions are equivalent*

- (1) *The section u is A -harmonic.*
- (2) *Putting $A_0 := A - u \wedge_F d_{A_0}u$, one has $d_{A_0}^*(A - A_0) = 0$.*

Proof We write locally $d_{A_0}u = \nabla_{e_i}^A u e^i$ with respect to a local orthonormal frame (e_i) . Therefore, taking into account that u is A_0 parallel (by Lemma 1.2) and $(u, \nabla_{e_i}^A u) = 0$, one has

$$\begin{aligned} d_{A_0}^*(A - A_0) &= (d_{A_0})^*(u \wedge_F d_{A_0}u) \\ &= u \wedge_F (d_{A_0})^* d_{A_0}u \\ &= u \wedge_F (d_A - u \wedge_F d_{A_0}u)^* d_{A_0}u \\ &= u \wedge_F [d_A^* d_{A_0}u + *(u \wedge_F d_{A_0}u) * d_{A_0}u] \\ &= u \wedge_F [d_A^* d_{A_0}u + *(u \wedge_F \nabla_i^A u e^i) * \nabla_j^A u e^j] \\ &= u \wedge_F [d_A^* d_{A_0}u - (\nabla_i^A u, \nabla_j^A u)u * (e^i \wedge *e^j)] \\ &= u \wedge_F [d_A^* d_{A_0}u - |d_{A_0}u|^2 u]. \end{aligned}$$

Since $d_A^*d_A u - |d_A u|^2 u \in A^0(u^\perp)$, the vanishing of $u \wedge_F [d_A^*d_A u - |d_A u|^2 u]$ is equivalent to the vanishing of $d_A^*d_A u - |d_A u|^2 u$. □

Using Lemma 1.2 and Proposition 1.3 we obtain:

Corollary 1.4 *Let $u \in \Gamma(S(F))$ be a A -harmonic section. Then there exists a unique connection $A_0 \in \mathcal{A}(F)$ with the properties:*

- (1) $A - A_0 \in A^1(\text{so}(u^\perp)^\perp)$.
- (2) A is in Coulomb gauge with respect to A_0 .
- (3) u is a A_0 -parallel.

1.2 A normal neighborhood of the locus of Euclidean connections with 1-dimensional kernel

We define the locally closed subspace

$$\mathcal{A}'(F) := \{A \in \mathcal{A}(F) \mid \dim(\ker(d_A)) = 1\} \subset \mathcal{A}(F).$$

Let $\mathcal{S}(F)$ be the space of trivial rank 1-subbundles of F . This space can be naturally identified with $\Gamma(S(F))/\{\pm 1\}$, and becomes a smooth manifold after suitable Sobolev completions. Two trivial rank 1-subbundles which are sufficiently C^0 -close have isomorphic complements, so they are conjugate modulo the action of the gauge group $\text{Aut}(F) = \Gamma(\text{SO}(F))$. In other words, the action of $\text{Aut}(F)$ on $\mathcal{S}(F)$ is locally transitive. One has an obvious surjective map

$$w: \mathcal{A}'(F) \rightarrow \mathcal{S}(F), \quad A \mapsto \langle \ker(d_A) \rangle,$$

where $\langle \ker(d_A) \rangle = \text{im}(\text{ev}: \ker(d_A) \times M \rightarrow F)$ stands for the subbundle generated by the line $\ker(d_A)$. We will use the subscript $(\cdot)_k$ (for $k \in \mathbb{N}$ sufficiently large) to denote Sobolev completion with respect to the L_k^2 -norm.

Proposition 1.5 *The subset $\mathcal{A}'(F)_k \subset \mathcal{A}(F)_k$ is a submanifold, and the surjection $w: \mathcal{A}'(F)_k \rightarrow \mathcal{S}(F)_{k+1}$ is a submersion.*

Proof We omit Sobolev indices to save on notations. Let $A \in \mathcal{A}'(F)$ and $\lambda = \langle \ker(d_A) \rangle \in \mathcal{S}(F)$. The fiber $w^{-1}(\lambda)$ is obviously an affine subspace of $\mathcal{A}(F)$ which can be identified with $\mathcal{A}(u^\perp)$. The stabilizer H of λ is a closed Lie subgroup H of the gauge group $\text{Aut}(F)$ whose Lie algebra \mathfrak{h} can be identified with $A^0(\text{so}(u^\perp))$, so it has a topological complement $\mathfrak{h}^\perp = A^0(\text{so}(u^\perp)^\perp)$. The restriction of the map $\mathfrak{h}^\perp \times w^{-1}(\lambda) \rightarrow \mathcal{A}(F)$ given by $(h, B) \mapsto \exp(h)(B)$ to a sufficiently small open

neighborhood of $(0, A)$ is an embedding which parameterizes a neighborhood of A in $\mathcal{A}'(F)$. This gives the submanifold structure of $\mathcal{A}'(F)$. Using the local transitivity of the gauge action on $\mathcal{S}(F)$, it follows that $h \mapsto \exp(h)(A)$ defines a local slice of w at A , proving that w is a submersion at A . \square

The main goal of this section is to prove that

Theorem 1.6 *The submanifold $\mathcal{A}'(F) \subset \mathcal{A}(F)$ has a tubular, gauge invariant, L^2 -normal neighborhood.*

The idea is very simple and natural: construct a neighborhood of $\mathcal{A}'(F)$ consisting of connections which admit an (up to sign) unique harmonic, energy minimizing section.

Let $A_0 \in \mathcal{A}'(F)$, and let u one of the two generators in $\Gamma(\mathcal{S}(F))$ of the line $\ker(d_{A_0})$. The splitting $F = \langle u \rangle \oplus u^\perp$ defines an A_0 -parallel splitting

$$\mathfrak{so}(F) = \mathfrak{so}(u^\perp) \oplus \mathfrak{so}(u^\perp)^\perp = \mathfrak{so}(u^\perp) \oplus u \wedge_F u^\perp,$$

which gives an L^2 -orthogonal decomposition

$$A^1(\mathfrak{so}(F)) =$$

$$A^1(\mathfrak{so}(u^\perp)) \oplus d_{A_0}[A^0(\mathfrak{so}(u^\perp)^\perp)] \oplus \ker[d_{A_0}^*: A^1((\mathfrak{so}(u^\perp)^\perp) \rightarrow A^0(\mathfrak{so}(u^\perp)^\perp))].$$

The geometric interpretations of the three factors in the decomposition above are the following: the space $A^1(\mathfrak{so}(u^\perp))$ is the tangent space of the fiber $w^{-1}(\langle u \rangle)$ at A , $n_{A_0} := (d_{A_0})[A^0(\mathfrak{so}(u^\perp)^\perp)]$ is the normal space at A_0 of this fiber in the submanifold $\mathcal{A}'(F)$, whereas the space

$$N_{A_0} := \ker[(d_{A_0})^*: A^1((\mathfrak{so}(u^\perp)^\perp) \rightarrow A^0(\mathfrak{so}(u^\perp)^\perp))]$$

is the normal space at A_0 of the submanifold $\mathcal{A}'(F)$ in the space of connections $\mathcal{A}(F)$.

Let $N \rightarrow \mathcal{A}'(F)$ be the normal vector bundle of the submanifold $\mathcal{A}'(F)$, whose fiber over $A_0 \in \mathcal{A}'(F)$ is just the space N_{A_0} defined above. One has a natural map $\nu: N \rightarrow \mathcal{A}(F)$ given by

$$\nu(A_0, \alpha) = A_0 + \alpha \quad \forall A_0 \in \mathcal{A}'(F), \quad \alpha \in N_{A_0},$$

which is obviously a local isomorphism at every point of the form $(A_0, 0)$. The map ν is equivariant with respect to the natural gauge actions on N , $\mathcal{A}'(F)$ and $\mathcal{A}(F)$. For $\zeta \in A^0(\mathfrak{so}(F)) = \text{Lie}(\text{Aut}(F))$ we denote by $\zeta^\#$ the tangent field (the infinitesimal transformation) of N associated with ζ and by $\zeta_\#$ the corresponding tangent field of $\mathcal{A}(F)$. The equivariance property of ν implies

$$(2) \quad d(\nu)(\zeta^\#_{(A_0, \alpha_0)}) = [\zeta_\#]_{A_0 + \alpha_0} = d_{A_0 + \alpha_0}(\zeta) = d_{A_0}(\zeta) + [\alpha_0, \zeta].$$

Lemma 1.7 *Let $A_0 \in \mathcal{A}'(F)$. There exists $\varepsilon(A_0) > 0$ such that the differential $d_{(A_0, \alpha_0)} \nu$ at (A_0, α_0) is an isomorphism for every $\alpha_0 \in N_{A_0}$ with $\|\alpha_0\|_{L^\infty} \leq \varepsilon(A_0)$. The assignment $A_0 \mapsto \varepsilon(A_0)$ can be chosen to be continuous and gauge invariant.*

Proof The submersion $w: \mathcal{A}'(F) \rightarrow \mathcal{S}(F)$ induces a submersion $\tilde{w}: N \rightarrow \mathcal{S}(F)$. Fix $(A_0, \alpha_0) \in N$ and choose $u_0 \in \ker(d_{A_0}) \cap \Gamma(\mathcal{S}(F))$. The subspace

$$\{\zeta_{(A_0, \alpha_0)}^\# \mid \zeta \in A^0(\text{so}(u_0^\perp)^\perp)\}$$

is a \tilde{w} -horizontal space at (A_0, α_0) , that is, a topological complement of the vertical tangent space $T_{(A_0, \alpha_0)}(\tilde{w}^{-1}(u_0))$. This complement is isomorphic with $A^0(\text{so}(u_0^\perp)^\perp)$.

A vertical tangent vector

$$v' \in T_{(A_0, \alpha_0)}(\tilde{w}^{-1}(u_0))$$

can be written as $v' = (b, \beta)$, where $b \in A^1(\text{so}(u_0^\perp))$ and $\beta \in A^1(\text{so}(u_0^\perp)^\perp)$ satisfy the equation

$$d_{A_0}^* \beta - *[b \wedge * \alpha_0] = 0.$$

(obtained by differentiating the relation $d_{A_0}^* \alpha_0 = 0$ in the direction $(\dot{A}_0, \dot{\alpha}_0) = (b, \beta)$). Therefore, one has an isomorphism

$$T_{(A_0, \alpha_0)}(N) = \{(b, \beta) \in A^1(\text{so}(u_0^\perp)) \oplus A^1(\text{so}(u_0^\perp)^\perp) \mid d_{A_0}^* \beta - *[b \wedge * \alpha_0] = 0\} \oplus A^0(\text{so}(u_0^\perp)^\perp).$$

Using (2) one obtains

$$d\nu(b, \beta, \zeta) = b + \beta + d_{A_0}(\zeta) + [\alpha_0, \zeta].$$

The statement follows now directly from Lemma 1.8 below. □

Lemma 1.8 *If $\|\alpha_0\|_{L^\infty}$ is sufficiently small, then the operator*

$$A^1(\text{so}(u_0^\perp))_k \oplus A^1(\text{so}(u_0^\perp)^\perp)_k \oplus A^0(\text{so}(u_0^\perp)^\perp)_{k+1} \xrightarrow{P} A^1(\text{so}(F))_k \oplus A^0(\text{so}(u_0^\perp)^\perp)_{k-1}$$

given by

$$(b, \beta, \zeta) \mapsto (b + \beta + d_{A_0}(\zeta) + [\alpha_0, \zeta], d_{A_0}^* \beta - *[b \wedge * \alpha_0])$$

is an isomorphism.

Proof We omit as usually the Sobolev indexes. Using the decomposition

$$A^1(\text{so}(F)) = A^1(\text{so}(u_0^\perp)) \oplus A^1(\text{so}(u_0^\perp)^\perp),$$

the operator P can be written as

$$P = \begin{bmatrix} \text{id} & 0 & [\alpha_0, \cdot] \\ 0 & \text{id} & d_{A_0} \\ - * [\cdot \wedge * \alpha_0] & d_{A_0}^* & 0 \end{bmatrix}$$

If $(b, \beta, \zeta) \in \ker(P)$, one obtains easily

$$\begin{aligned} (3) \quad & b = -[\alpha_0, \zeta], \\ & d_{A_0}^* \beta = -d_{A_0}^* d_{A_0} \zeta = *(b \wedge * \alpha_0) = *(-[\alpha_0, \zeta] \wedge * \alpha_0), \\ & \Delta_{A_0} \zeta = *[[\alpha_0, \zeta] \wedge * \alpha_0]. \end{aligned}$$

On the other hand, since u_0 is parallel and $\dim(\ker(d_{A_0})) = 1$, one has

$$\ker(d_{A_0}: A^0(\text{su}(u_0^\perp)^\perp) \rightarrow A^1(\text{su}(u_0^\perp)^\perp)) = u_0 \wedge_F \ker(d_{A_0}: A^0(u_0^\perp) \rightarrow A^1(u_0^\perp)) = \{0\}$$

so the minimal eigenvalue $\eta(A_0)$ of $\Delta_{A_0}: A^0(\text{su}(u_0^\perp)^\perp) \rightarrow A^0(\text{su}(u_0^\perp)^\perp)$ is positive. Therefore (3) implies $\zeta = 0$ (hence $\ker(P) = 0$) as soon as $\|\alpha_0\|_{L^\infty}^2$ is sufficiently small with respect to $\eta(A_0)$. For surjectivity, note that the equation

$$(4) \quad P(b, \beta, \zeta) = (c, \gamma, \eta)$$

becomes

$$b + [\alpha_0, \zeta] = c, \quad \beta + d_{A_0} \zeta = \gamma, \quad d_{A_0}^* \beta - *[b \wedge * \alpha_0] = \eta.$$

Consider first the weaker equation (for the single unknown ζ)

$$\Delta_{A_0} \zeta - *[[\alpha_0, \zeta] \wedge * \alpha_0] = -*[c \wedge * \alpha_0] - \eta + d_{A_0}^* \gamma,$$

which is uniquely solvable if $\|\alpha_0\|_{L^\infty}^2$ is sufficiently small, so one gets a unique solution $\zeta \in A^0(\text{so}(u_0^\perp)^\perp)_{k+1}$ for a triple (c, γ, η) of Sobolev type $(L_k^2, L_k^2, L_{k-1}^2)$. Then set $\beta := \gamma - d_{A_0} \zeta$, $b := c - [\alpha_0, \zeta]$, and we get a solution of the equation (4) of Sobolev type $(L_k^2, L_k^2, L_{k+1}^2)$. \square

Lemma 1.9 *Let $A_0 \in \mathcal{A}'(F)$ and $u_0 \in \ker(d_A) \cap \Gamma(S(F))$. There exists $\varepsilon(A_0) > 0$ such that for every $\alpha \in A^1(\text{so}(u_0^\perp)^\perp)$ with $d_{A_0}^*(\alpha) = 0$ and $\|\alpha\|_{L^\infty} \leq \varepsilon(A_0)$ the energy functional $E_{A_0+\alpha}$ on the space $\Gamma(S(F))$ obtains its absolute minimum at $\pm u_0$ and only at these sections. Moreover, the assignment $\mathcal{A}'(F) \ni A_0 \mapsto \varepsilon(A_0)$ can be chosen to be continuous and gauge invariant.*

Proof Consider a section $u \in \Gamma(S(F))$, put $v := u - u_0$, $v' := u + u_0$, and set $\alpha := u_0 \wedge_F a$, where $a \in A^1(u_0^\perp)$. This implies $|\alpha|^2 = 2|a|^2$. One has

$$\begin{aligned} \|d_{A_0+\alpha}u\|^2 - \|d_{A_0+\alpha}u_0\|^2 &= \|d_{A_0+\alpha}v\|^2 + 2\langle d_{A_0+\alpha}u_0, d_{A_0+\alpha}v \rangle = \\ &= \|d_{A_0}v\|^2 + \|\alpha(v)\|^2 + 2\langle d_{A_0}v, \alpha(v) \rangle + 2\langle \alpha(u_0), d_{A_0}v \rangle + 2\langle \alpha(u_0), \alpha(v) \rangle. \end{aligned}$$

But

$$d_{A_0}^*(\alpha(u_0)) = -*d_{A_0}(*\alpha(u_0)) = -*d_{A_0}(*\alpha)(u_0) \pm *(*\alpha \wedge d_{A_0}u_0) = 0$$

(because $d_{A_0}^*(\alpha) = 0$ and $d_{A_0}u_0 = 0$), so the term $2\langle \alpha(u_0), d_{A_0}v \rangle$ on the right vanishes. On the other hand it holds pointwise

$$\begin{aligned} \langle \alpha(u_0), \alpha(v) \rangle &= \langle (u_0 \wedge_F a)(u_0), (u_0 \wedge_F a)(v) \rangle = \\ &= \langle a, (u_0, v)a - (a, v)u_0 \rangle = |a|^2(u_0, v) = \frac{1}{2}|\alpha|^2(u_0, v), \\ 1 &= |u_0 + v|^2 = |u_0|^2, \\ 2(u_0, v) &= -|v|^2. \end{aligned}$$

Therefore

$$\begin{aligned} \|d_{A_0+\alpha}u\|^2 - \|d_{A_0+\alpha}u_0\|^2 &= \|d_{A_0}v\|^2 + \|\alpha(v)\|^2 + 2\langle d_{A_0}v, \alpha(v) \rangle - \frac{1}{2} \int_M |\alpha|^2 |v|^2 \\ &\geq \|d_{A_0}v\|^2 - c_1 \sup |\alpha| \|d_{A_0}v\|_{L^2} \|v\|_{L^2} - c_2 \sup |\alpha|^2 \|v\|_{L^2}^2 \end{aligned}$$

Since the same computation also applies to $-u_0$, we get

$$\begin{aligned} (5) \quad E_{A_0+\alpha}(u) - E_{A_0+\alpha}(u_0) &\geq \|d_{A_0}v'\|^2 - c_1 \sup |\alpha| \|d_{A_0}v'\| \|v'\| - c_2 \sup |\alpha|^2 \|v'\|^2 \\ E_{A_0+\alpha}(u) - E_{A_0+\alpha}(u_0) &\geq \|d_{A_0}v\|^2 - c_1 \sup |\alpha| \|d_{A_0}v\| \|v\| - c_2 \sup |\alpha|^2 \|v\|^2 \end{aligned}$$

Since $\|u\|_{L^2}^2 = \|u_0\|_{L^2}^2 = \text{Vol}(M)$, one has $(u - u_0) \perp_{L^2} (u + u_0)$, so the triangle $(u_0, u, -u_0)$ is L^2 -right at the vertex u , and one has

$$\sin^2 \angle(\mathbb{R}v, \mathbb{R}u_0) + \sin^2 \angle(\mathbb{R}v', \mathbb{R}u_0) = 1.$$

Therefore, either $\sin \angle(\mathbb{R}v, \mathbb{R}u_0) \geq \frac{1}{\sqrt{2}}$ or $\sin \angle(\mathbb{R}v', \mathbb{R}u_0) \geq \frac{1}{\sqrt{2}}$. Suppose we are in the first case.

We get the inequality

$$\|v\|_{L^2}^2 \leq 2 \|\text{pr}_{\mathbb{R}u_0^\perp} v\|^2 \leq \frac{2}{\lambda(A_0)} \|d_A(v)\|^2,$$

where $\text{pr}_{\mathbb{R}u_0^\perp}$ stands for the L^2 -orthogonal projection on the L^2 -orthogonal complement of the line $\mathbb{R}u_0 = \ker(d_{A_0})$ and $\lambda(A_0)$ is the first positive eigenvalue of $\Delta_{A_0}: A^0(F) \rightarrow A^0(F)$. In other words, $\frac{1}{\sqrt{\lambda(A_0)}}$ is the norm of the inverse

$$G_{A_0}: \text{im}(d_{A_0}) \rightarrow \ker(d_{A_0})^\perp$$

of d_{A_0} with respect to the L^2 -norms.

Using (5) we get an estimate of the form

$$E_{A_0+\alpha}(u) - E_{A_0+\alpha}(u_0) \geq \|d_{A_0}v\|^2 - c_1(A_0) \sup |\alpha| \|d_{A_0}v\|^2 - c_2(A_0) \sup |\alpha|^2 \|d_{A_0}v\|^2$$

for positive constants $c_1(A_0), c_2(A_0)$. Taking

$$\varepsilon(A_0) = \min \left(\frac{1}{4c_1(A_0)}, \sqrt{\frac{1}{2c_2(A_0)}} \right),$$

we get for $\sup |\alpha| \leq \varepsilon(A_0)$

$$E_{A_0+\alpha}(u) - E_{A_0+\alpha}(u_0) \geq \frac{1}{2} \|d_{A_0}v\|^2 \geq \frac{1}{4} \lambda(A_0) \|v\|^2,$$

which is strictly positive unless $v = 0$, that is, $u = u_0$. The same argument applies in the case $\sin \angle(\mathbb{R}v', \mathbb{R}u_0) \geq \frac{1}{\sqrt{2}}$ by replacing u with $-u_0$ and v with v' . \square

The inequality $\|\alpha\|_{L^\infty} < \varepsilon(A_0)$ as A_0 varies in $\mathcal{A}'(F)$ defines a gauge invariant neighborhood \mathcal{N} of the zero section in the normal bundle N of this submanifold.

Corollary 1.10 *The restriction of the natural map*

$$v: N \longrightarrow \mathcal{A}(F), \quad v(A_0, \alpha) := A_0 + \alpha$$

to \mathcal{N} is injective.

Proof Indeed, if $v(A_0, \alpha) = v(B_0, \beta) = A$, then one must have $\ker(d_{A_0}) = \ker(d_{B_0})$, because the absolute minimum of E_A on $\Gamma(S(F))$ is unique up to sign. By Corollary 1.4, we obtain $A_0 = B_0$, so finally $\alpha = \beta$. \square

Combining with Lemma 1.7, we get

Theorem 1.11 *There exists a gauge invariant neighborhood \mathcal{U} of the zero section in the bundle N which (after suitable Sobolev completions) is mapped diffeomorphically onto its image via the natural map v .*

Remark The map $\nu|_{\mathcal{U}}$ can be regarded as a system of “gauge equivariant polar coordinates” around the submanifold $\mathcal{A}'(F)$.

Remark In the finite dimensional framework, one can prove easily that in general, for any submanifold X of a Riemannian manifold Y there exists a neighborhood of the zero section in the normal bundle $N_{X/Y}$ which is mapped diffeomorphically via the exponential map onto a normal neighborhood of X . However, in the infinite dimensional framework, the problem is much more difficult. Theorem 1.11 solves this problem in the special case of the embedding $\mathcal{A}'(F) \subset \mathcal{A}(F)$.

1.3 Normal neighborhoods of the reduction loci in the space of Hermitian connections

Let E be a rank 2 Hermitian bundle over a 4 manifold M and denote $D := \det(E)$, $d := c_1(D)$. Consider the involution $i_d: x \mapsto d - x$ on $H^2(M, \mathbb{Z})$. A congruence class $\lambda \in H^2(M, \mathbb{Z})/\langle i_d \rangle$ will be called a topological decomposition of E if it coincides with the set of Chern classes of the terms of a splitting of E as direct sum of line bundles (that is, when $x(d - x) = c_2(E)$ for $x \in \lambda$).

Fix a connection $a \in \mathcal{A}(D)$ and denote by $\mathcal{A}_a(E)$ the affine space of connections on E inducing a on D . Our gauge group is the group $\mathcal{G}_E := \Gamma(\text{SU}(E))$ of determinant 1 unitary isomorphisms of E . A connection will be called *simply reducible* if $\dim(\ker(d_A: A^0(\text{su}(E)) \rightarrow A^1(\text{su}(E)))) = 1$. Such a connection admits precisely two parallel line subbundles (which, of course, might be isomorphic), and these subbundles give an A -parallel orthogonal splitting of E . If A admits a parallel line bundle L with $2c_1(L) \neq d$, then A is automatically simply reducible and L, L^\perp are the unique A -parallel line subbundles of E . In particular, if $d \notin 2H^2(M, \mathbb{Z})$, then any reducible connection on E is simply reducible. If we fix a line subbundle $L \hookrightarrow E$, there exists a natural bijection between the simply reducible connections on E for which L is parallel, and the subspace $\mathcal{A}^*(L) \subset \mathcal{A}(L)$ of abelian connections $b \in \mathcal{A}(L)$ for which $b \not\sim a \otimes b^\vee$. One has automatically $\mathcal{A}^*(L) = \mathcal{A}(L)$ when $2c_1(L) \neq d$, whereas $\mathcal{A}^*(L)$ is the complement of $2^{b_1(N)}$ gauge orbits in $\mathcal{A}(L)$ when $2c_1(L) = d$.

Let λ be a topological decomposition of E and denote by $\mathcal{A}_a^\lambda(E)$ the subspace of *simple* reducible connections $A \in \mathcal{A}_a(E)$ with the property that the set of the Chern classes of the two A -parallel line subbundles of E coincides with λ . Such a connection will be called λ -reducible. Denote also by $\Gamma^\lambda(S(\text{su}(E)))$ the set of sections $u \in \Gamma(S(\text{su}(E)))$ with the property that the set $c(u)$ of Chern classes of the eigen line subbundles of u coincides with λ . Putting

$$(6) \quad L_u := \ker\left(u - \frac{i}{\sqrt{2}}\text{id}_E\right)$$

these eigen line subbundles are $L_{\pm u}$. One has a natural surjection

$$w: \mathcal{A}_a^\lambda(E) \longrightarrow \Gamma^\lambda(S(\text{su}(E)))/\{\pm 1\}$$

which associates to every connection $A \in \mathcal{A}_a^\lambda(E)$ the unordered pair of sections of the sphere bundle $S(\text{su}(E))$ which are A -parallel.

For sufficiently large Sobolev index k , the space $\Gamma(S(\text{su}(E)))_{k+1}$ becomes a Banach manifold and the subset $\Gamma^\lambda(S(\text{su}(E)))_{k+1}$ is open and closed in $\Gamma(S(\text{su}(E)))_{k+1}$, so it is a union of connected components. The gauge group $\mathcal{G}_{E,k+1}$ acts smoothly on $\Gamma(S(\text{su}(E)))_{k+1}$, leaving invariant $\Gamma^\lambda(S(\text{su}(E)))_{k+1}$; on $\Gamma^\lambda(S(\text{su}(E)))_{k+1}/\{\pm 1\}$ this gauge group acts transitively. The same arguments as in the proof of Proposition 1.5 show that

Proposition 1.12 *The subset $\mathcal{A}_a^\lambda(E)_k \subset \mathcal{A}_a(E)_k$ is a submanifold, which is a locally trivial affine bundle over $\Gamma^\lambda(S(\text{su}(E)))_{k+1}/\{\pm 1\}$. The fiber over a class $[u] \in \Gamma(S(\text{su}(E)))_{k+1}/\{\pm 1\}$ can be naturally identified with the space of abelian connections $\mathcal{A}^*(L_u)_k$.*

Fix a reducible connection $A \in \mathcal{A}_a^\lambda(E)$ with $w(A) = [u]$. Put $S_u := (L_u^{\otimes 2} \otimes D^\vee)$. The bundle $\text{su}(E)$ splits as an orthogonal sum of A -parallel summands

$$\text{su}(E) = (M \times \mathbb{R}u) \oplus S_u = \text{so}(u^\perp) \oplus \text{so}(u^\perp)^\perp,$$

and, as in Section 1.2, we obtain the following L^2 -orthogonal decomposition of the tangent space $T_A(\mathcal{A}_a(E)) = A^1(\text{su}(E))$ at A :

$$A^1(\text{su}(E)) = A^1(M, \mathbb{R})u \oplus A^1(S_u) = A^1(M, \mathbb{R})u \oplus d_A[A^0(S_u)] \oplus \ker[d_A^*: A^1(S_u) \rightarrow A^0(S_u)].$$

The first summand $A^1(M, \mathbb{R})u$ is the tangent space of the fiber $w^{-1}(u) \simeq \mathcal{A}^*(L_u)$ at A , $n_A := d_A[A^0(S_u)]$ is the normal space at A of this fiber in the submanifold $\mathcal{A}_a^\lambda(E)$, whereas the space $N_A := \ker[d_A^*: A^1(S_u) \rightarrow A^0(S_u)]$ is the normal space at A of $\mathcal{A}_a^\lambda(E)$ in the space of connections $\mathcal{A}_a(E)$.

We denote by $N^\lambda \rightarrow \mathcal{A}_a^\lambda(E)$ the L^2 -normal vector bundle of the submanifold $\mathcal{A}_a^\lambda(E)$, whose fiber over $A \in \mathcal{A}_a^\lambda(E)$ is N_A .

Using Theorem 1.11 we obtain the following important result, which gives an L^2 -normal neighborhood of the submanifold $\mathcal{A}_a^\lambda(E)$ of simple reductions of type λ , and a system of polar coordinates around this submanifold.

Theorem 1.13 *There exists a gauge invariant neighborhood \mathcal{U}^λ of the zero section in the normal bundle $N^\lambda \rightarrow \mathcal{A}_a^\lambda(E)$ which (after suitable Sobolev completions) is mapped diffeomorphically onto its image via the natural map*

$$v: N^\lambda \rightarrow \mathcal{A}_a(E), \quad (A_0, \alpha) \mapsto A_0 + \alpha.$$

The elements of the orthogonal slice $v(N_{A_0}^\lambda \cap \mathcal{U}^\lambda) \subset \mathcal{A}_a(E)$ through $A_0 \in \mathcal{A}_a^\lambda(E)$ are connections A for which the two elements of

$$\ker(d_{A_0}: A^0(\text{su}(E)) \rightarrow A^1(\text{su}(E))) \cap \Gamma(S(\text{su}(E)))$$

(which are A_0 -parallel) are the unique energy-minimizing A -harmonic sections in $\Gamma(S(\text{su}(E)))$.

From now on we will always assume that \mathcal{U}^λ is defined by an inequality of the form $\|\alpha\|_{L^\infty} \leq \varepsilon(A_0)$, where $A_0 \mapsto \varepsilon(A)$ is continuous and gauge invariant (see Lemma 1.7 and Lemma 1.9).

1.4 Twisted reductions

When the base manifold has nontrivial first homology group $H_1(M, \mathbb{Z})$, one also has to take into account the twisted reductions, that is, the connections which are irreducible but whose pull-back on a double cover of M become reducible. Although the stabilizer $\mathcal{G}_{E,A}$ of such a connection is just the center $\{\pm \text{id}_E\}$ of the gauge group, these loci of twisted connections and the geometry of the instanton moduli spaces around these loci must be studied in detail; the reason is simple: the classical transversality results with respect to metric variations (see Donaldson and Kronheimer [3]) fails not only at a reduction, but also at a twisted reduction, so it is not clear whether one can achieve regularity of an instanton moduli space at such a point by perturbing the metric.

Let $\rho: \pi_1(M, x_0) \rightarrow \{\pm 1\}$ be a group epimorphism and denote by $\pi_\rho: M_\rho \rightarrow M$ the double cover associated with $\ker(\rho)$. The tautological involution of M_ρ will be denoted by ι . A connection $A \in \mathcal{A}_a(E)$ will be called ρ -twisted reducible (or a ρ -twisted reduction) if it is irreducible, but its pullback $\pi_\rho^*(A) \in \mathcal{A}_{\pi_\rho^*(a)}(\pi_\rho^*(E))$ is reducible. For such a connection one has an orthogonal $\pi_\rho^*(A)$ -parallel splitting $\pi_\rho^*(E) = L' \oplus L''$ and an isomorphism $L'' \simeq \iota^*(L')$. L' and L'' are the unique $\pi_\rho^*(A)$ -parallel subbundles of E , because, if not, the $\text{SO}(3)$ -connection associated with $\pi_\rho^*(A)$ would be trivial. In this case A will be projectively flat and the holonomy of A acts on the projective line $\mathbb{P}(E_{x_0})$ by an involution. Therefore, A would admit (at least) two parallel line subbundles, contradicting the assumption that A was irreducible.

In other words $\pi_\rho^*(A)$ must be simply reducible of type $\lambda = (l, \iota^*(l))$, where l is a solution in $H^2(M_\rho, \mathbb{Z})$ of the system

$$(7) \quad l + \iota^*(l) = \pi_\rho^*(d), \quad l \cdot \iota^*(l) = 2c_2(E).$$

For a ρ -twisted reduction A the $\text{SO}(3)$ -bundle $\text{su}(E)$ has an A -parallel splitting

$$\text{su}(E) = R_\rho \oplus F,$$

where R_ρ is the Euclidean real line subbundle of $\text{su}(E)$ consisting of trace-free anti-Hermitian endomorphisms $u_x \in \text{su}(E_x)$ whose eigen lines are $L'_{\tilde{x}}, L''_{\tilde{x}} = L'_{\iota(\tilde{x})}$, where \tilde{x} is a lift of $x \in M$ in M_ρ , and L', L'' are the $\pi_\rho^*(A)$ -parallel line subbundles of $\pi_\rho^*(E)$. R_ρ is isomorphic with the non-orientable Euclidean real line bundle associated with the representation $\rho: \pi_1(M, x_0) \rightarrow O(1)$. The second term F is an $O(2)$ -bundle with $\det(F) \simeq R_\rho$, whose pull-back to M_ρ has two $\text{SO}(2) = \text{U}(1)$ -reductions, isomorphic with $[L']^\vee \otimes L''$ and $[L'']^\vee \otimes L'$.

We fix a topological decomposition $\lambda = \{l, \iota^*(l)\}$ of $\pi_\rho^*(E)$ and we denote by $\mathcal{A}_a^\lambda(E)$ the subspace of ρ -twisted reductions $A \in \mathcal{A}_a(E)$ with the property that $\pi_\rho^*(A)$ is simply reducible of type λ .

Denote by $\Gamma_\iota(S(\text{su}(\pi_\rho^*(E))))$ the set of sections u of the sphere bundle of $\text{su}(\pi_\rho^*(E))$ satisfying the property $\iota^*(u) = -u$ and denote by $\Gamma_\iota^\lambda(S(\text{su}(\pi_\rho^*(E))))$ the subset of $\Gamma_\iota(S(\text{su}(\pi_\rho^*(E))))$ consisting of sections u , such that the Chern classes of the eigen line sub-bundles of $\pi_\rho^*(u)$ are $l, \iota^*(l)$.

In the same way as in the case of non-twisted connections one gets a locally trivial, gauge equivariant fibration

$$w: \mathcal{A}_a^\lambda(E) \longrightarrow \Gamma_\iota^\lambda(S(\text{su}(\pi_\rho^*(E)))) / \{\pm 1\},$$

and the fiber over a class $[u]$ can be identified with the subspace

$$\mathcal{A}_\iota^*(L_{\pi_\rho^*(u)}) \subset \mathcal{A}(L_{\pi_\rho^*(u)})$$

of abelian connections $b \in \mathcal{A}(L_{\pi_\rho^*(u)})$ satisfying

$$b \otimes \iota^*(b) = \iota^*(a) \text{ (via the obvious isomorphism } L \otimes \iota^*(L) \simeq \pi_\rho^*(D) \text{)}, \quad b \not\equiv \iota^*(b).$$

The second condition is superfluous when $l \neq \iota^*(l)$. The space $A^1(\text{su}(E))$ splits as

$$A^1(\text{su}(E)) = A^1(R_\rho) \oplus d_A[A^0(F)] \oplus \ker[d_A^*: A^1(F) \rightarrow A^0(F)],$$

where the third term can be identified with the normal space N_A^λ at A of the submanifold $\mathcal{A}_a^\lambda(E)$ of ρ -twisted connections of type λ .

Using similar methods as in the proofs of Theorems 1.11, 1.13 one gets easily the following existence theorem for L^2 -normal neighborhoods of the loci of twisted reductions.

Theorem 1.14 *There exists a gauge invariant neighborhood \mathcal{U}^λ of the zero section in the normal bundle $N^\lambda \rightarrow \mathcal{A}_a^\lambda(E)$ which (after suitable Sobolev completions) is mapped diffeomorphically onto its image via the natural map*

$$v: N^\lambda \rightarrow \mathcal{A}_a(E), \quad (A_0, \alpha) \mapsto A_0 + \alpha.$$

The elements of the orthogonal slice $v(N_{A_0} \cap \mathcal{U}^\lambda) \subset \mathcal{A}_a(E)$ through $A_0 \in \mathcal{A}_a^\lambda(E)$ are connections $A \in \mathcal{A}_a(E)$ for which the two elements of the intersection

$$\ker(d_{\pi_\rho^*(A_0)}: A^0(\text{su}(\pi_\rho^*(E))) \rightarrow A^1(\text{su}(\pi_\rho^*(E)))) \cap \Gamma_\rho(S(\text{su}(\pi_\rho^*(E))))$$

(which are $\pi_\rho^*(A_0)$ -parallel) are the unique energy-minimizing $\pi_\rho^*(A)$ -harmonic sections in $\Gamma(S(\text{su}(\pi_\rho(E))))$.

As in the non-twisted case we will suppose that \mathcal{U}^λ is defined by an inequality of the form $\|\alpha\|_{L^\infty} \leq \varepsilon(A)$, where the assignment $\mathcal{A}_a^\lambda(E) \ni A \mapsto \varepsilon(A)$ is gauge invariant and continuous.

2 The Donaldson μ -classes around the reductions and the homotopy type of $\mathcal{B}_a(E)$

2.1 The universal bundle around reductions

The structure of the universal bundle around a single reduction is well-known. A complete description can be found in Donaldson and Kronheimer [3, pages 186–187]. However, for our purposes, this classical result is not sufficient, because we will need the structure of the universal bundle around positive dimensional subspaces of reductions.

We recall that the universal $\text{SO}(3)$ -bundle on $\mathcal{B}_a^*(E) \times M$ is defined as

$$\mathbb{F} := (\mathcal{A}_a^*(E) \times \text{su}(E)) / \bar{\mathcal{G}}_E,$$

where $\bar{\mathcal{G}}_E := \mathcal{G}_E / \{\pm 1\}$ acts in the natural way on both factors. Alternatively, one can let $\bar{\mathcal{G}}_E$ act from the right and define \mathbb{F} to be the bundle with fiber $\text{su}(E)$ over $\mathcal{B}_a^*(E)$ which is associated with the principal $\bar{\mathcal{G}}_E$ -bundle $\mathcal{A}_a^*(E) \rightarrow \mathcal{B}_a^*(E)$. Let $\lambda = \{l, d-l\}$ be a topological decomposition of E . We will assume for simplicity that $2l \neq d$ which assures that $\mathcal{A}(L) = \mathcal{A}^*(L)$ for every Hermitian line bundle of Chern class l ; in particular the fiber of the fibration $w: \mathcal{A}_a^\lambda(E) \rightarrow \Gamma^\lambda(S(\text{su}(E))) / \{\pm 1\}$ over $\{\pm u\}$ can

be identified with the affine space $\mathcal{A}(L_u)$ (see Section 1.3). We will identify $\mathcal{A}(L_u)$ with its image in $\mathcal{A}_a(E)$.

We will omit the upper script λ in the notations $N^\lambda, \mathcal{U}^\lambda$ introduced in the previous section and we denote by N^*, \mathcal{U}^* the complement of the zero section in N and \mathcal{U} (see Theorem 1.13). Replacing \mathcal{U} be a smaller gauge invariant neighborhood if necessary, we may assume that $v(\mathcal{U}^*) \subset \mathcal{A}_a^*(E)$. We put

$$\mathcal{V} := v(\mathcal{U})/\bar{\mathcal{G}}_E \subset \mathcal{B}_a(E), \quad \mathcal{V}^* := v(\mathcal{U}^*)/\bar{\mathcal{G}}_E \subset \mathcal{B}_a^*(E).$$

\mathcal{V} is a neighborhood of the moduli space of λ -reductions $\mathcal{B}_a^\lambda(E) := \mathcal{A}_a^\lambda(E)/\mathcal{G}_E$. Fix a section $u \in \Gamma^\lambda(S(\text{su}(E)))$ and denote by \mathcal{G}_u the subgroup of \mathcal{G}_E consisting of elements $g \in \mathcal{G}$ which leave L_u invariant. We have a natural isomorphism $\mathcal{G}_u \simeq \mathcal{G}$, where $\mathcal{G} := \mathcal{C}^\infty(M, S^1)$. Put

$$\bar{\mathcal{G}} := \mathcal{G}/\{\pm 1\}; \quad \bar{\mathcal{G}}_u := \mathcal{G}_u/\{\pm 1\} \subset \bar{\mathcal{G}}_E,$$

and denote by $\mathcal{G}^{x_0}, \mathcal{G}_E^{x_0}, \mathcal{G}_u^{x_0}$ the subgroups of \mathcal{G} (respectively $\mathcal{G}_E, \mathcal{G}_u$) of elements f with $f(x_0) = 1$. Note that these subgroups are mapped injectively into $\bar{\mathcal{G}}, \bar{\mathcal{G}}_E, \bar{\mathcal{G}}_u$, so we will use the same notations for the corresponding subgroups of these groups.

The main point which will be used in our computation is that the fixing of the section u defines a $\bar{\mathcal{G}}_u$ -reduction of the restriction of the principal $\bar{\mathcal{G}}_E$ -bundle $\mathcal{A}_a^*(E) \rightarrow \mathcal{B}_a^*(E)$ to the subspace $\mathcal{V}^* \subset \mathcal{B}_a^*(E)$.

Proposition 2.1 *Suppose $\lambda = \{l, d - l\}$ with $2l \neq d$. Let $u \in \Gamma^\lambda(S(\text{su}(E)))$ and let $N_u \subset \mathcal{A}(L_u) \times A^1(S_u)$ be the restriction of the normal bundle N to the fiber $w^{-1}([u]) \simeq \mathcal{A}(L_u)$. Then*

- (1) $v(N_u \cap \mathcal{U})$ is the submanifold of the normal neighborhood $v(\mathcal{U})$ consisting of connections for which $\pm u$ are harmonic and energy minimizing.
- (2) The embedding $v|_{N_u \cap \mathcal{U}}: N_u \cap \mathcal{U} \hookrightarrow \mathcal{A}_a(E)$ induces isomorphisms

$$\mathcal{V} \simeq (N_u \cap \mathcal{U})/\bar{\mathcal{G}}_u, \quad \mathcal{V}^* \simeq (N_u^* \cap \mathcal{U})/\bar{\mathcal{G}}_u.$$

- (3) The map v induces an isomorphism between \mathcal{V} and the cone bundle over the projectivization $\mathbb{P}(\bar{N}_u)$ of the vector bundle

$$\bar{N}_u := N_u/\mathcal{G}_u^{x_0}$$

over $\mathcal{B}(L_u) = \mathcal{A}(L_u)/\mathcal{G}_u^{x_0}$, and \mathcal{V}^* is identified with the complement of the vertex section in this cone bundle. In particular one has a homotopy equivalence

$$(8) \quad \mathcal{V}^* \stackrel{h}{\simeq} \mathbb{P}(\bar{N}_u) \subset \mathcal{A}(L_u) \times_{\mathcal{G}_u^{x_0}} \mathbb{P}(A^1(S_u)).$$

- (4) $\nu(N_u \cap \mathcal{U}^*)$ is a $\bar{\mathcal{G}}_u$ -reduction of the restriction of the principal $\bar{\mathcal{G}}_E$ -bundle $\mathcal{A}_a^*(E) \rightarrow \mathcal{B}_a^*(E)$ to \mathcal{V}^* .

Proof The first statement follows easily from Theorem 1.13. The second and the third statements are obvious. For the fourth, it suffices to prove that the $\bar{\mathcal{G}}_E$ -orbit of a point $\nu(t) \in \nu(\mathcal{U}^*)$ intersects $\nu(N_u \cap \mathcal{U}^*)$ along a $\bar{\mathcal{G}}_u$ -orbit; since ν is injective on \mathcal{U} , it suffices to show that the $\bar{\mathcal{G}}_E$ -orbit of $t = (b, \alpha) \in N^*$ intersects N_u^* along a $\bar{\mathcal{G}}_u$ -orbit. Since $\bar{\mathcal{G}}_E$ acts transitively on $\Gamma^\lambda(S(\text{su}(E)))/\{\pm 1\}$, we see that $t \cdot \bar{\mathcal{G}}_E \cap N_u^* \neq \emptyset$, so we can suppose $t \in N_u^*$. If an element $g \in \mathcal{G}_E$ maps t into N_u^* , then $g(b) \in \mathcal{A}(L_u) \subset \mathcal{A}_a(E)$, that is, u is both b -parallel and $g(b)$ -parallel. This implies $\text{ad}_g(u) = \pm u$. Since we assumed that $l \neq d - l$, we have $L_u \not\cong L_{-u}$, so necessarily $\text{ad}_g(u) = u$ that is, $g \in \mathcal{G}_u$. \square

Corollary 2.2 *The open subspace $\mathcal{V}^* \subset \mathcal{B}_a^*(E)$ has the homotopy type of the product $[S^1]^{b_1(M)} \times \mathbb{P}^\infty$.*

Proof Let g be a Riemannian metric on M . The moduli space $\mathcal{M}_g^{YM}(L_u)$ of Yang–Mills connections on L_u with respect to g is isomorphic to

$$iH^1(X, \mathbb{R})/2\pi iH^1(M, \mathbb{Z}) \simeq [S^1]^{b_1(M)}$$

and the inclusion $\mathcal{M}_g^{YM}(L_u) \hookrightarrow \mathcal{B}(L_u)$ is a homotopy equivalence. On the other hand, since $\mathcal{M}_g^{YM}(L_u)$ is compact, the restriction of the infinite rank vector bundle \bar{N}_u to this subspace is trivial. This completes the proof. \square

The decomposition

$$\text{su}(E) = [M \times (i\mathbb{R})] \oplus S_u$$

is \mathcal{G}_u -invariant. Therefore

Corollary 2.3 *The restriction $\mathbb{F}|_{\mathcal{V}^* \times M}$ decomposes as a direct sum*

$$\mathbb{F}|_{\mathcal{V}^* \times M} \simeq [(\mathcal{V}^* \times M) \times i\mathbb{R}] \oplus S_u,$$

where S_u is the complex line bundle on $\mathcal{V}^* \times M$ defined by

$$S_u := (\nu(N_u \cap \mathcal{U}^*) \times S_u) / \bar{\mathcal{G}}_u.$$

In particular $p_1(\mathbb{F}) = c_1(S_u)^2$.

The cohomology algebra of \mathcal{V}^* can be easily described explicitly using Corollary 2.2. The construction below yields generators with explicit geometric interpretation.

Put $L = L_u$, $S = S_u = L^{\otimes 2} \otimes D^\vee$, $\mathcal{S} = \mathcal{S}_u$ to save on notations. The subbundle $N_u \subset \mathcal{A}(L) \times A^1(S)$ has an intrinsic interpretation in terms of L : it is just the bundle $\mathcal{K} = \ker \delta$ of kernels of the family of operators

$$\delta := (d_{b^{\otimes 2} \otimes a^\vee}^*)_{b \in \mathcal{A}(L)}$$

and $\mathcal{U}_u := \mathcal{U} \cap N_u$ is a $\bar{\mathcal{G}}$ -invariant neighborhood \mathcal{U}_u of the zero section in this bundle. The bundle \mathcal{K} descends to a bundle $\bar{\mathcal{K}} := \mathcal{K}/\mathcal{G}^{x_0}$ over $\mathcal{B}(L)$. Using the isomorphism $\mathcal{V}^* \simeq \mathcal{U}_u^*/\bar{\mathcal{G}}$ induced by ν , one can identify \mathcal{S} with the line bundle

$$(\mathcal{U}_u^* \times S)/\bar{\mathcal{G}} \longrightarrow (\mathcal{U}_u^*/\bar{\mathcal{G}}) \times M \subset (\mathcal{K}^*/\bar{\mathcal{G}}) \times M = (\bar{\mathcal{K}}^*/S^1) \times M.$$

Therefore, \mathcal{S} can be identified (via ν) to the restriction to $[\mathcal{U}_u^*/\bar{\mathcal{G}}] \times M$ of the line bundle

$$\mathbb{S} := (\mathcal{A}(L) \times (A^1(S) \setminus \{0\}) \times S)/\bar{\mathcal{G}}$$

on $\mathcal{B}_u^*(E) \times M$, where $\mathcal{B}_u^*(E)$ is the quotient

$$\mathcal{B}_u^*(E) := (\mathcal{A}(L) \times (A^1(S) \setminus \{0\}))/\bar{\mathcal{G}} = \mathcal{A}(L) \times_{\mathcal{G}^{x_0}} [\mathbb{R}_{>0} \times \mathbb{P}(A^1(S))],$$

which can be regarded as a locally trivial bundle over $\mathcal{B}(L)$ with fiber $\mathbb{R}_{>0} \times \mathbb{P}(A^1(S))$. The space $\mathcal{B}_u^*(E)$ is very much similar to the infinite dimensional gauge quotient of the space of irreducible configurations in Seiberg–Witten theory. More precisely, let in general V be a line bundle and W a complex vector bundle on M . The natural map

$$p_{V,W}: \mathcal{B}^*(V, W) := (\mathcal{A}(V) \times [A^0(W) \setminus \{0\}])/\mathcal{G} \longrightarrow \mathcal{B}(V)$$

is a locally trivial bundle over $\mathcal{B}(V)$ with fiber $\mathbb{R}_{>0} \times \mathbb{P}(A^0(W))$. On the product $\mathcal{B}(V) \times M$ one has a *tautological line bundle* \mathbb{V}_W defined by

$$\mathbb{V}_W := (\mathcal{A}(V) \times [A^0(W) \setminus \{0\}] \times V)/\mathcal{G}.$$

Lemma 2.4 *With the notations and assumptions above one has*

- (1) *There exists a natural isomorphism*

$$\delta_V: H_1(M, \mathbb{Z}) \rightarrow H^1(\mathcal{B}(V))$$

which induces an isomorphism $H^(\mathcal{B}^*(V), \mathbb{Z}) \simeq \Lambda^*(H_1(M, \mathbb{Z}))$.*

- (2) *The morphism $p_{V,W}^* \circ \delta_V$ induces an isomorphism*

$$H^*(\mathcal{B}^*(V, W), \mathbb{Z}) \simeq \Lambda^*(H_1(M, \mathbb{Z})) \otimes \mathbb{Z}[h_{VW}],$$

where h_{VW} is a degree 2-cohomology class defined as the Chern class of the principal S^1 -bundle

$$(\mathcal{A}(V) \times [A^0(W) \setminus \{0\}])/\mathcal{G}^{x_0} \longrightarrow (\mathcal{A}(V) \times [A^0(W) \setminus \{0\}])/\mathcal{G} = \mathcal{B}^*(V, W).$$

This class restricts to the canonical (tautological) class of the fibers of the map $p_{V,W}: \mathcal{B}^*(V, W) \rightarrow \mathcal{B}(V)$ (which are homotopically equivalent to $\mathbb{P}(A^0(W))$).

(3) The Chern class of the tautological line bundle \mathbb{V}_W on $\mathcal{B}^*(V, W) \times M$ is

$$c_1(\mathbb{V}_W) = p_1^*(h_{VW}) \otimes 1 + \delta_V + 1 \otimes p_2^*(c_1(V)).$$

Here we denoted by the same symbol δ_V the element in $H^1(\mathcal{B}(V)) \otimes H^1(M, \mathbb{Z})$ defined by this morphism, as well as the pullback of this element via the projection $\mathcal{B}^*(V, W) \times M \rightarrow \mathcal{B}(V) \times M$.

Proof The proof uses the same arguments which are used for the computation of the cohomology algebra of the moduli space of irreducible configurations in Seiberg–Witten theory (see for instance Okonek and Teleman [6]). \square

Corollary 2.5 *The line bundle \mathbb{S} can be identified with the pull-back of $\mathbb{S}_{\Lambda^1 \otimes S}$ via the composition*

$$\mathcal{B}_u^*(E) \longrightarrow \mathcal{B}^*(S, \Lambda^1 \otimes S)$$

induced by

$$\mathcal{A}(L) \times [A^1(S) \setminus \{0\}] \ni (b, \alpha) \mapsto (b^{\otimes 2} \otimes a^\vee, \alpha) \in \mathcal{A}(S) \times [A^1(S) \setminus \{0\}].$$

In particular one has

$$c_1(\mathbb{S}) = p_1^*(h_u) \otimes 1 + 2\delta_L + p_2^*(c_1(S)),$$

where h_u is the Chern class of the principal $\bar{\mathcal{G}}/\mathcal{G}^{x_0} = S^1$ -bundle

$$(\mathcal{A}(L) \times (A^1(S) \setminus \{0\})) / \mathcal{G}^{x_0} \longrightarrow (\mathcal{A}(L) \times (A^1(S) \setminus \{0\})) / \bar{\mathcal{G}} = \mathcal{B}_u^*(E).$$

Proof The map $(b, \alpha) \mapsto (b^{\otimes 2} \otimes a^\vee, \alpha)$ induces an isomorphism u which fits in the diagram

$$\begin{aligned} \mathcal{B}_u^*(E) &= (\mathcal{A}(L) \times [A^0(\Lambda^1 \otimes S) \setminus \{0\}]) / \bar{\mathcal{G}} \xrightarrow{u} (\mathcal{A}(S) \times [A^0(\Lambda^1 \otimes S) \setminus \{0\}]) / \mathcal{G}^2 \\ &\xrightarrow{v} (\mathcal{A}(S) \times [A^0(\Lambda^1 \otimes S) \setminus \{0\}]) / \mathcal{G} = \mathcal{B}^*(S, \Lambda^1 \otimes S), \end{aligned}$$

where v is the obvious epimorphism. The map

$$\mathcal{A}(L) \times [A^1(S) \setminus \{0\}] \times S \ni (b, \alpha, s) \mapsto (b^{\otimes 2} \otimes a^\vee, \alpha, s) \in \mathcal{A}(S) \times [A^1(S) \setminus \{0\}] \times S$$

induces an isomorphism $\mathbb{S} \simeq (v \circ u \times \text{id}_M)^*(\mathbb{S}_{\Lambda^1 \otimes S})$. It suffices to notice that

$$(v \circ u)^*(h_{S, \Lambda^1 \otimes S}) = h_u, \quad (v \circ u)^*(\delta_S) = 2\delta_L. \quad \square$$

Corollary 2.6 *Suppose that $b_+(M) = 0$. The restrictions of the Donaldson μ -classes to \mathcal{V}^* are given by the formulae*

$$\begin{aligned} \mu(\lambda) &= -\langle \delta_L \cup_M c_1(S), \lambda \rangle \in H^1(\mathcal{V}^*, \mathbb{Z}) & \forall \lambda \in H_3(M, \mathbb{Z}), \\ \mu(x) &= -\frac{1}{2} \langle c_1(S), x \rangle h_u \in H^2(\mathcal{V}^*, \mathbb{Q}) & \forall x \in H_2(X, \mathbb{Z}), \\ \mu(\sigma) &= -\delta_L(\sigma) \cup h_u \in H^3(\mathcal{V}^*, \mathbb{Z}) & \forall \sigma \in H_1(X, \mathbb{Z}), \\ \nu := \mu(*) &= -\frac{1}{4} h_u^2 \in H^4(\mathcal{V}^*, \mathbb{Q}). \end{aligned}$$

Proof Every monomial of the form $a \cup b$, with $a, b \in H^1(B, \mathbb{Z})$ satisfies the relation $(a \cup b)^2 = 0$ so, since we supposed $b_+(M) = 0$, one gets easily that $a \cup b \in \text{Tors}(H^2(M, \mathbb{Z}))$. Therefore

$$\text{im}(\cup: H^1(M, \mathbb{Z}) \otimes H^1(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z})) \subset \text{Tors}(H^2(M, \mathbb{Z})).$$

This implies $\delta_L^2 = 0$. Therefore

$$\begin{aligned} -\frac{1}{4} p_1(\mathbb{F}|_{\mathcal{V}^*}) &= -\frac{1}{4} c_1(S)^2 \\ &= -\frac{1}{4} [p_1^*(h_u^2) \otimes 1 + 4p_1^*(h_u) \cup \delta_L + 2p_1^*(h_u) \cup p_2^*(c_1(S)) \\ &\quad + 4\delta_L \cup p_2^*(c_1(S)) + p_2^*(c_1(S)^2)]. \end{aligned}$$

Using Donaldson’s formula $\mu(x) = -\frac{1}{4} p_1(\mathbb{F})/x$ for $x \in H_*(X, \mathbb{Z})$, one gets easily the claimed formulae. □

2.2 The topology of the moduli space $\mathcal{B}_a(E)$

Describing the weak homotopy type of the moduli space $\mathcal{B}_a^*(E)$ of *irreducible* connections is a well-known classical problem in gauge theory. This problem is treated in detail in Donaldson and Kronheimer [3], where the authors also compute the rational cohomology algebra of this space. Surprisingly, describing the weak homotopy type of the whole moduli space $\mathcal{B}_a(E)$ of connections is a delicate problem, which, to our knowledge, cannot be solved with similar methods.

Our result concerning the existence of L^2 -normal neighborhoods of the reduction loci, gives a solution to this problem. Suppose for simplicity that $d := c_1(D)$ is not divisible by 2 in $H^2(M, \mathbb{Z})$, so that any reducible connection $A \in \mathcal{A}_a(L)$ is simply reducible. Let Λ_E be the set topological decomposition of E , that is, the set of unordered pairs $\lambda = \{l, d - l\}$ with $l(d - l) = c_2(E)$.

Using the notations and conventions of Section 2.1, we get the following decomposition of the space $\mathcal{B}_a(E)$ as a fibered sum

$$(9) \quad \begin{array}{ccc} & \mathcal{B}_a(E) & \\ & \nearrow & \nwarrow \\ \mathcal{B}_a^*(E) & & [\coprod_{\lambda \in \Lambda_E} \mathcal{V}^\lambda] \\ & \nwarrow & \nearrow \\ & [\coprod_{\lambda \in \Lambda_E} \mathcal{V}^{\lambda^*}] & \end{array}$$

where $\mathcal{V}^\lambda := \mathcal{U}^\lambda / \mathcal{G}_E$ is the gauge quotient of the gauge invariant L^2 -normal neighborhood \mathcal{U}^λ of the reduction locus $\mathcal{A}_a^\lambda(E)$ constructed in Section 1.3. In this diagram the homotopy types of the terms are known (see Corollary 2.2). Choosing a Hermitian line bundle L with $c_1(L) \in \lambda$, one has

$$\mathcal{V}^\lambda \overset{h}{\simeq} \mathcal{B}(L) \overset{h}{\simeq} [S^1]^{b_1(M)}, \quad \mathcal{V}^{\lambda^*} \overset{h}{\simeq} [S^1]^{b_1(M)} \times \mathbb{P}^\infty,$$

this description determines the homotopy type of the space $\mathcal{B}_a(E)$. In particular one can compute the cohomology of this space using the Mayer–Vietoris exact sequence.

Example 1 Suppose that $H_1(M, \mathbb{Z}) \simeq \mathbb{Z}$, $b_2(M) = b_2^-(M) = 2$. The intersection form of M is diagonalizable over \mathbb{Z} (since $b_2(M) = 2$, this follows from the classification of unimodular symmetric bilinear forms, but can also be obtained as a special case of Donaldson’s first theorem). Therefore, one can find a basis (e_1, e_2) in $H^2(M, \mathbb{Z})$ such that $e_i^2 = -1$, $e_1 \cup e_2 = 0$. It follows that $d := e_1 + e_2$ is an integral lift of the Stiefel–Whitney class $w_2(M)$. Let now E be a rank 2–Hermitian bundle with $c_1(E) = d$, $c_2(E) = 0$ and put again $D = \det(E)$.

Our problem is to compute the degree k -cohomology of the space $\mathcal{B}_a(E)$, for $1 \leq k \leq 4$ using the Mayer–Vietoris sequence applied to the decomposition (9). The set Λ_E has two elements:

$$\lambda_0 := \{0, d\}, \quad \lambda_1 := \{e_1, e_2\}.$$

Denote by $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_0^*, \mathcal{V}_1^*$ the corresponding subspaces of $\mathcal{B}_a(E)$, and put $\mathcal{V} := \mathcal{V}_0 \cup \mathcal{V}_1, \mathcal{V}^* := \mathcal{V}_0^* \cup \mathcal{V}_1^*$. We get exact sequences

$$\begin{aligned} \dots \longrightarrow H^{i-1}(\mathcal{B}_a^*(E)) \oplus H^{i-1}(\mathcal{V}) &\longrightarrow H^{i-1}(\mathcal{V}^*) \longrightarrow H^i(\mathcal{B}_a(E)) \\ &\longrightarrow H^i(\mathcal{B}_a^*(E)) \oplus H^i(\mathcal{V}) \longrightarrow H^i(\mathcal{V}^*) \longrightarrow \dots \end{aligned}$$

Using the standard description of the cohomology of $\mathcal{B}_a^*(E)$ (see Donaldson and Kronheimer [3]), we obtain

$$H^1(\mathcal{B}_a^*(E)) = \langle \mu(\lambda_0) \rangle_{\mathbb{Q}} \simeq \mathbb{Q}, \quad H^2(\mathcal{B}_a^*(E)) = \langle \mu(f_1), \mu(f_2) \rangle_{\mathbb{Q}} \simeq \mathbb{Q}^2,$$

where λ_0 is a generator of $H_3(M, \mathbb{Z})$ and $f_i := \text{PD}(e_i)$. Since we assumed $b_1(X) = 1$, we obtain easily using Poincaré duality that

$$\text{im}(\cup: H^1(M) \otimes H^2(M) \rightarrow H^3(M)) = 0$$

hence, by Corollary 2.6, we obtain $\mu(\lambda_0)|_{\mathcal{V}^*} = 0$. Note also that the restriction morphism $H^i(\mathcal{V}) \rightarrow H^i(\mathcal{V}^*)$ is an isomorphism for $i = 0, 1$ and is injective for all i . The exact sequence above for $i = 1$ yields

$$H^1(\mathcal{B}_a(E)) = \ker(H^1(\mathcal{B}_a^*(E)) \oplus H^1(\mathcal{V}) \rightarrow H^1(\mathcal{V}^*)) = \langle (\mu(\lambda_0), 0) \rangle_{\mathbb{Q}} \simeq \mathbb{Q}.$$

Therefore, the cohomology class defined by the Chern–Simons functional [3] associated with a hypersurface representing λ_0 extends to the whole moduli space $\mathcal{B}_a(E)$. This is a general phenomenon. For $i = 2$, we obtain

$$H^2(\mathcal{B}_a(E)) = \ker(H^2(\mathcal{B}_a^*(E)) \rightarrow H^2(\mathcal{V}^*)).$$

On the other hand by Corollary 2.6

$$\mu(f_i)|_{\mathcal{V}_0^*} = \frac{1}{2} \langle e_1 + e_2, f_i \rangle h_{u_0}, \quad \mu(f_i)|_{\mathcal{V}_1^*} = \frac{1}{2} \langle e_2 - e_1, f_i \rangle h_{u_1},$$

where $u_i \in \Gamma(S(\text{su}(E)))$ are chosen such that $c_1(L_{u_0}) = 0$, $c_1(L_{u_1}) = e_1$ (so L_{u_i} is a line bundle representing the topological decomposition λ_i). Therefore $H^2(\mathcal{B}_a(E), \mathbb{Q}) = 0$. Taking into account that the restriction morphism $H^2(\mathcal{B}_a^*(E)) \rightarrow H^2(\mathcal{V}^*)$ is surjective and $H^3(\mathcal{V}) = 0$, the same exact sequence for $i = 3$ yields:

$$H^3(\mathcal{B}_a(E)) = \ker[H^3(\mathcal{B}_a^*(E)) \rightarrow H^3(\mathcal{V}^*) = H^3(\mathcal{V}_0^*) \oplus H^3(\mathcal{V}_1^*)].$$

The space $H^3(\mathcal{V}_i^*)$ is generated by $\delta_{L_i}(\sigma_0) \cup h_{u_i}$, where σ_0 is a generator of $H_1(M, \mathbb{Z})$, whereas

$$H^3(\mathcal{B}_a^*(E)) = \langle \mu(\sigma_0), \mu(\lambda_0)\mu(f_1), \mu(\lambda_0)\mu(f_2) \rangle_{\mathbb{Q}}.$$

By Corollary 2.6 one has $\mu(\sigma_0)|_{\mathcal{V}_i^*} = -\delta_{L_i}(\sigma_0)h_{u_i}$ and we have seen that $\mu(\lambda_0)|_{\mathcal{V}^*} = 0$. This shows that

$$H^3(\mathcal{B}_a(E)) = \langle \mu(\lambda_0)\mu(f_1), \mu(\lambda_0)\mu(f_2) \rangle_{\mathbb{Q}} \simeq \mathbb{Q}^2.$$

The cohomology space $H^4(\mathcal{B}_a(E))$ fits in the exact sequence

$$0 \longrightarrow \frac{H^3(\mathcal{V}_0^*) \oplus H^3(\mathcal{V}_1^*)}{\langle (\delta_{L_0}(\sigma_0)h_{u_0}, \delta_{L_1}(\sigma_0)h_{u_1}) \rangle} \longrightarrow H^4(\mathcal{B}_a(E)) \longrightarrow H^4(\mathcal{B}_a^*(E)) \longrightarrow H^4(\mathcal{V}^*).$$

The space $H^4(\mathcal{B}_a^*(E))$ is freely generated by the five classes

$$\mu(*), \quad \mu(f_1)^2, \quad \mu(f_2)^2, \quad \mu(f_1)\mu(f_2), \quad \mu(\lambda_0)\mu(\sigma_0),$$

whereas $H^4(\mathcal{V}^*)$ is freely generated by $h_{u_0}^2$ and $h_{u_1}^2$. This shows that

$$\ker(H^4(\mathcal{B}_a^*(E)) \rightarrow H^4(\mathcal{V}^*)) = \langle \mu(*) + \mu(f_1)^2, \mu(*) + \mu(f_2)^2, \mu(\lambda_0)\mu(\sigma_0) \rangle_{\mathbb{Q}} \simeq \mathbb{Q}^3.$$

The quotient on the left in the exact sequence above is 1-dimensional, so $H^4(\mathcal{B}_a(E)) \simeq \mathbb{Q}^4$.

Remark A similar method can be used to compute the cohomology of the pair $(\mathcal{B}_a^*(E), \mathcal{V}^*)$, where $\mathcal{V} = \coprod_{[L] \in R_E} \mathcal{V}^L$ is a normal neighborhood of the reducible locus in $\mathcal{B}_a(E)$.

Example 2 Let M be a 4-manifold with the topological properties considered in the example above. The exact sequence of the pair $(\mathcal{B}_a^*(E), \mathcal{V}^*)$

$$\rightarrow H^{i-1}(\mathcal{B}_a^*(E)) \rightarrow H^{i-1}(\mathcal{V}^*) \rightarrow H^i(\mathcal{B}_a^*(E), \mathcal{V}^*) \rightarrow H^i(\mathcal{B}_a^*(E)) \rightarrow H^i(\mathcal{V}^*) \rightarrow$$

written for $i = 4$ shows that the natural morphism $H^4(\mathcal{B}_a^*(E), \mathcal{V}^*) \rightarrow H^4(\mathcal{B}_a(E))$ (induced by the restriction morphism $H^*(\mathcal{B}_a(E), \mathcal{V}) \rightarrow H^*(\mathcal{B}_a(E))$ and the excision isomorphism $H^*(\mathcal{B}_a(E), \mathcal{V}) \rightarrow H^*(\mathcal{B}_a^*(E), \mathcal{V}^*)$) is an isomorphism.

3 The instanton moduli space around the reductions

We denote by $\mathcal{M}_a^{\text{ASD}}(E) \subset \mathcal{B}_a(E)$ the moduli of projectively ASD a -oriented connections in E , that is, the moduli space

$$\mathcal{M}_a^{\text{ASD}}(E) := \mathcal{A}_a^{\text{ASD}}(E) / \mathcal{G}_E, \quad \mathcal{A}_a^{\text{ASD}}(E) := \{A \in \mathcal{A}_a(E) \mid (F_A^0)^+ = 0\}.$$

In the first subsection we will study the intersection of this moduli space with a normal neighborhood \mathcal{V}^λ of the reduction locus $\mathcal{B}_a^\lambda(E) := \mathcal{A}_a^\lambda(E) / \mathcal{G}_E$. We will see that, in a neighborhood of a reduction locus, the instanton moduli problem is equivalent to an *abelian* moduli problem, which is very much similar to the Seiberg–Witten moduli problem. We will denote by $\mathcal{M}_a^\lambda(E)$ the subspace $\mathcal{M}_a^\lambda(E) := \mathcal{M}_a^{\text{ASD}}(E) \cap \mathcal{B}_a^\lambda(E)$ of λ -reducible instantons.

3.1 An abelian gauge theoretical problem

Let L be a Hermitian line bundle which is isomorphic to a line subbundle of E , and put $S := L^{\otimes 2} \otimes D^\vee$. Consider the moduli space $\mathcal{M}_a(L)$ of solutions of the system

$$(10) \quad \begin{cases} F_b^+ - \frac{1}{2}F_a^+ & = (\alpha \wedge \bar{\alpha})^+ \\ (d_{b^{\otimes 2} \otimes a^\vee}^*, d_{b^{\otimes 2} \otimes a^\vee}^+) \alpha & = 0 \end{cases}$$

for pairs $(b, \alpha) \in \mathcal{A}(L) \times A^1(S)$, modulo the abelian gauge group $\mathcal{G} = \mathcal{C}^\infty(M, S^1)$. This system is very much similar to the Seiberg–Witten system; indeed, the left hand operator in the first equation is elliptic and can be written as coupled Dirac operator. The main difference is that in general there is no a priori bound for the α -component on the space of solutions of this system and, in general, the moduli space is not compact.

We denote by $\mathcal{M}_a^{\text{red}}(L)$, $\mathcal{M}_a^*(L)$ the subspaces of reducible (respectively irreducible) solutions. As in Seiberg–Witten theory “reducible pair” means “pair with trivial α -component”. Therefore one has a natural identification

$$\mathcal{M}_a^{\text{red}}(L) \simeq T_a(L) := \mathcal{A}_a(L) / \mathcal{G}, \text{ where } \mathcal{A}_a(L) := \{b \in \mathcal{A}(L) \mid F_b^+ = \frac{1}{2}F_a^+\},$$

The space $T_a(L)$ is either empty, or a $b_1(M)$ -dimensional torus (when the harmonic representative of $c_1(S) = 2c_1(L) - c_1(D)$ is ASD).

Put $l := c_1(L)$ and $\lambda = \{l, d - l\}$ and fix an isomorphism $E = L \oplus (D \otimes L^\vee)$. The map

$$\psi: (b, \alpha) \mapsto A_{b,\alpha} := \begin{pmatrix} d_b & \alpha \\ -\bar{\alpha} & d_{a \otimes b^\vee} \end{pmatrix} \in \mathcal{A}_a(E)$$

descends to a map $\mathcal{M}_a(L) \rightarrow \mathcal{M}_a^{\text{ASD}}(E)$. The image of this map is the subspace consisting of those instantons which can be brought in Coulomb gauge with respect to a λ -reducible connection. Suppose again that $2l \neq d$ (such that any reduction having a parallel line subbundle of Chern class l is simple), and consider the continuous, gauge invariant function $\varepsilon: \mathcal{A}_a^\lambda(E) \rightarrow \mathbb{R}_{>0}$ defining the normal neighborhood \mathcal{U}^λ of $\mathcal{A}_a^\lambda(E)$ (see Section 1.3). For $b \in \mathcal{A}(L)$ put

$$\varepsilon(b) := \varepsilon(A_b), \text{ where } A_b := b \oplus (a \otimes b^\vee) \in \mathcal{A}_a^\lambda(E),$$

and denote \mathcal{W}^λ the \mathcal{G} -invariant subspace of the configuration space $\mathcal{A}(L) \times A^1(S)$ defined by the inequality $\|\alpha\|_{L^\infty} \leq \varepsilon(b)$, and by \mathcal{Z}^λ its \mathcal{G} -quotient.

Proposition 3.1 *The restriction of the instanton moduli problem to the normal neighborhood $\nu(\mathcal{U}^\lambda)$ of $\mathcal{A}_a^\lambda(E)$ and the restriction of the abelian moduli problem (10) to the neighborhood \mathcal{W}^λ of $\mathcal{A}(L) \times \{0\}$ are equivalent moduli problems.*

In particular, the map induced by ψ applies $\mathcal{M}_a(L) \cap \mathcal{Z}^\lambda$ isomorphically onto the neighborhood $\mathcal{M}_a^{\text{ASD}}(E) \cap \mathcal{V}^\lambda$ of the subspace $\mathcal{M}_a^\lambda(E)$ of λ -reducible instantons, and induces an isomorphism $\mathcal{M}_a^{\text{red}}(L) \simeq \mathcal{M}_a^\lambda(E)$.

Proof This follows directly from (2) in Proposition 2.1. □

Corollary 3.2 Let $b \in \mathcal{A}_a(L)$. The deformation elliptic complex \mathcal{C}_{A_b} at the corresponding reduction A_b splits as a direct sum $\mathcal{C}_{A_b} = \mathcal{C}_0^+(M) \oplus \mathcal{C}_{b^{\otimes 2} \otimes a^\vee}^+(S)$, where $\mathcal{C}_0^+(M)$ is the standard d^+ -elliptic complex for $i\mathbb{R}$ -valued forms and $\mathcal{C}_{b^{\otimes 2} \otimes a^\vee}^+(S)$ is the d^+ -elliptic complex for S -valued forms associated with the connection $b^{\otimes 2} \otimes a^\vee$.

Remark The complex index of the elliptic complex $\mathcal{C}_{b^{\otimes 2} \otimes a^\vee}^+(S)$ is given by

$$\text{ind}_{\mathbb{C}}(\mathcal{C}^+(S)) = c_1^2(S) + (b_+(M) - b_1(M) + 1).$$

3.2 A twisted abelian gauge theoretical problem

Let $\rho: \pi_1(M, x_0) \rightarrow \mathbb{Z}_2$ be a group epimorphism and consider the associated objects $\pi_\rho: M_\rho \rightarrow M$, $\iota: M_\rho \rightarrow M_\rho$ introduced in Section 1.4.

Let L be a Hermitian line bundle on M_ρ whose Chern class l is a solution of the system (7) and put $\lambda = \{l, \iota^*(l)\}$. In this section we will assume that $l \neq \iota^*(l)$. We fix an isomorphism $\pi_\rho^*(E) = L \oplus \iota^*(L)$. We denote by $\mathcal{A}_\iota(L)$ the subspace of $\mathcal{A}(L)$ consisting of connections $b \in \mathcal{A}(L)$ such that $b \otimes \iota^*(b) = \pi_\rho^*(a)$. The natural gauge group acting on this space of connections is

$$\mathcal{G}_\iota := \{f \in \mathcal{C}^\infty(M_\rho, S^1) \mid \iota^*(f) = \bar{f}\}.$$

The Lie algebra of this group can be identified with $A^0(R_\rho)$, where R_ρ is the real line bundle associated with the representation ρ . One has a natural embedding $\mathcal{G}_\iota \rightarrow \mathcal{G}_{\pi_\rho^*(E)}$ factorizes through an embedding $\mathcal{G}_\iota \hookrightarrow \mathcal{G}_E$.

One has a \mathcal{G}_ι -equivariant map $\mathcal{A}_\iota(L) \rightarrow \mathcal{A}_a^\lambda(E)$ given by $b \mapsto A_b$, where A_b is the unique ρ -twisted connection of type λ whose pull-back to M_ρ is $b \oplus \iota^*(b)$.

Put $S = L \otimes \iota^*(L)^\vee = L^{\otimes 2} \otimes \pi_\rho^*(D)$. This Hermitian line bundle comes with a tautological isomorphism $\iota^*(S) = S^\vee = \bar{S}$. We introduce the spaces of ι -twisted S -valued forms by

$$A_\iota^k(S) := \{\alpha \in A^k(S) \mid \iota^*(\alpha) = -\bar{\alpha}\}.$$

Note that, for a connection $b \in \mathcal{A}_l(L)$, one has $F_b - \frac{1}{2}\pi_\rho^*(F_a) \in A_l^2(S)$. Endow M_ρ with the pull-back metric $\pi_\rho^*(g)$. Our abelian moduli problem is now

$$(11) \quad \begin{cases} F_b^+ - \frac{1}{2}\pi_\rho^*(F_a)^+ & = (\alpha \wedge \bar{\alpha})^+ \\ (d_{b \otimes 2 \otimes \pi_\rho^*(a)^\vee}^*, d_{b \otimes 2 \otimes \pi_\rho^*(a)^\vee}^+) \alpha & = 0. \end{cases}$$

for pairs $(b, \alpha) \in \mathcal{A}_l(L) \times A_l^1(S)$, modulo the abelian gauge group \mathcal{G}_l .

The stabilizer of *any* pair $(b, \alpha) \in \mathcal{A}_l(L) \times A_l^1(S)$ is $\{\pm 1\} \subset \mathcal{G}_l$, so, in fact, this twisted abelian moduli problem has no reductions at all. However, the moduli space $\mathcal{M}_{a,l}(L)$ of solutions has two distinguished gauge invariant subspaces $\mathcal{M}_{a,l}^{\text{red}}(L)$, $\mathcal{M}_{a,l}^*(L)$ consisting of classes of solutions with vanishing (respectively, non-vanishing) α -component. As in the previous section we obtain a map $\psi: \mathcal{M}_{a,l}(L) \rightarrow \mathcal{M}_a^{\text{ASD}}(E)$ which applies isomorphically $\mathcal{M}_{a,l}^{\text{red}}(L)$ onto the space $\mathcal{M}_a^\lambda(E)$ of ρ -twisted reducible instantons of type λ .

The space $\mathcal{M}_{a,l}^{\text{red}}(L)$ has a simple geometric interpretation: one has an obvious identification

$$\mathcal{M}_{a,l}^{\text{red}}(L) \simeq (\mathcal{A}_{\pi_\rho^*(a)}(L) \cap \mathcal{A}_l(L)) / \mathcal{G}_l \subset \mathcal{M}_{\pi_\rho^*(a)}^{\text{red}}(L)$$

which shows that $\mathcal{M}_{a,l}^{\text{red}}(L)$ is the subspace of $\mathcal{M}_{\pi_\rho^*(a)}^{\text{red}}(L)$ defined by the equation $[b \otimes \iota^*(b)] = [a]$. If the harmonic representative of $c_1(S)$ is not ASD (with respect to the pull-back metric), the space $\mathcal{M}_{a,l}^{\text{red}}(L)$ will be empty. When this representative is ASD, $\mathcal{M}_{\pi_\rho^*(a)}^{\text{red}}(L)$ will be a (non-empty!) subtorus of the torus $\mathcal{M}_{\pi_\rho^*(a)}^{\text{red}}(L)$; this subtorus is (non-canonically) isomorphic to the quotient

$$iH^1(M_\rho, \mathbb{R})_l / 2\pi iH^1(M_\rho, \mathbb{Z})_l,$$

where the symbol $(-)_l$ means ι -twisted, that is, the subspace of $(-)$ consisting of solutions of the equation $\iota^*(x) = -x$. In particular, if $\iota^*: H^1(M_\rho, \mathbb{Z}) \rightarrow H^1(M_\rho, \mathbb{Z})$ is the identity and $b_+(M_\rho) = 0$, this space consists of a single point.

Consider the continuous, gauge invariant function $\varepsilon: \mathcal{A}_a^\lambda(E) \rightarrow \mathbb{R}_{>0}$ defining the normal neighborhood \mathcal{U}^λ (see Section 1.4) of $\mathcal{A}_a^\lambda(E)$ and, for $b \in \mathcal{A}(L)$ put $\varepsilon(b) := \varepsilon(A_b)$. As in the previous section we introduce the space $\mathcal{W}^\lambda \subset \mathcal{A}_l(L) \times A_l^1(S)$ defined by the inequality $\|\alpha\|_{L^\infty} < \varepsilon(b)$ and its gauge quotient $\mathcal{Z}^\lambda = \mathcal{W}^\lambda / \mathcal{G}_l$. We obtain:

Proposition 3.3 *The restriction of the instanton moduli problem to the normal neighborhood $\nu(\mathcal{U}^\lambda)$ of the space of ρ -twisted, type λ -reductions $\mathcal{A}_a^\lambda(E)$ and the restriction of the abelian moduli problem (11) to the neighborhood \mathcal{W}^λ of $\mathcal{A}(L) \times \{0\}$ are equivalent moduli problems.*

In particular, the map induced by ψ applies $\mathcal{M}_{a,t}(L) \cap \mathcal{Z}^\lambda$ isomorphically onto the neighborhood $\mathcal{M}_a^{\text{ASD}}(E) \cap \mathcal{V}^\lambda$ of the subspace $\mathcal{M}_a^\lambda(E)$ of ρ -twisted, λ -reducible instantons, and induces an isomorphism $\mathcal{M}_{a,t}^{\text{red}}(L) \simeq \mathcal{M}_a^\lambda(E)$.

Corollary 3.4 Let $b \in \mathcal{A}_{\pi_\rho^*(a)}(L) \cap \mathcal{A}_t(L)$. The deformation complex \mathcal{C}_{A_b} at the corresponding ρ -twisted reduction A_b splits as a direct sum $\mathcal{C}_{A_b} = \mathcal{C}_{0,t}^+(M_\rho) \oplus \mathcal{C}_{b \otimes \iota^*(b)^\vee, t}^+(S)$, where $\mathcal{C}_{0,t}^+(M_\rho)$ is the d^+ -elliptic complex

$$0 \longrightarrow iA_t^0(M_\rho) \longrightarrow iA_t^1(M_\rho) \xrightarrow{d^+} iA_{+,t}^2(M_\rho) \longrightarrow 0$$

of imaginary ι -twisted forms, and $\mathcal{C}_{b \otimes \iota^*(b)^\vee, t}^+(S)$ is the d^+ -elliptic complex of ι -twisted S -valued forms associated with the connection $b \otimes \iota^*(b)^\vee$.

The indices of the two elliptic complexes can be computed easily: The dimension h_t^k of the k th harmonic space of $\mathcal{C}_{0,t}^+(M_\rho)$ is

$$h_t^0 = 0, \quad h_t^1 = b_1(M_\rho) - b_1(M), \quad h_t^2 = b_+(M_\rho) - b_+(M),$$

Taking into account that $\text{ind}(\mathcal{C}_0^+(M_\rho)) = 2\text{ind}(\mathcal{C}_0^+(M))$, we get

$$\text{ind}(\mathcal{C}_{0,t}^+(M_\rho)) = b_+(M_\rho) - b_1(M_\rho) - (b_+(M) - b_1(M)) = b_+(M) - b_1(M) + 1.$$

For the complex $\mathcal{C}_{b \otimes \iota^*(b)^\vee, t}^+(S)$, note that the complex $\mathcal{C}_{b \otimes \iota^*(b)^\vee}^+(S)$ splits as a direct sum

$$\mathcal{C}_{b \otimes \iota^*(b)^\vee}^+(S) = \mathcal{C}_{b \otimes \iota^*(b)^\vee, 0}^+(S) \oplus \mathcal{C}_{b \otimes \iota^*(b)^\vee, t}^+(S)$$

where $\mathcal{C}_{b \otimes \iota^*(b)^\vee, 0}^+(S)$ is the d^+ -complex of S -valued forms α on M_ρ satisfying the relation $\iota^*(\alpha) = \bar{\alpha}$. Multiplication by i defines a real isomorphism $\mathcal{C}_{b \otimes \iota^*(b)^\vee, 0}^+(S) \simeq \mathcal{C}_{b \otimes \iota^*(b)^\vee, t}^+(S)$. Therefore

$$\text{ind}[\mathcal{C}_{b \otimes \iota^*(b)^\vee, t}^+(S)] = \text{ind}_{\mathbb{C}} \mathcal{C}^+(S) = c_1^2(S) + 2(b_+(M) - b_1(M) + 1).$$

3.3 Generic regularity at the reductions

The purpose of this section is to prove a strong generic regularity result for reducible instantons. We agree to call *regular* any solution $A \in \mathcal{A}_a^{\text{ASD}}(E)$ (irreducible or reducible) with $\mathbb{H}_A^2 = 0$. Our result allows to prove that, under certain cohomological conditions on our data, there exists a *connected*, dense, open set of metrics for which no irregular reduction appears in the moduli space. This will allow us in the next section to introduce Donaldson type invariants for definite manifolds, even in the cases when non-empty reduction loci are present in the moduli space.

Let M be a 4-manifold with $b_+(M) = 0$, and E a Hermitian bundle of rank 2 on M ; put as usual $D := \det(E)$, $d := c_1(D)$, $c = c_2(E)$. Let $\lambda = \{l, d - l\}$ be a topological decomposition of E with $2l \neq d$. The second cohomology of the deformation elliptic complex of a λ -reducible instanton reduces to the second cohomology of the elliptic complex $\mathcal{C}_{b \otimes 2 \otimes a^\vee}^+(S)$, where $S = L^{\otimes 2} \otimes D^\vee$.

Denote by $\mathcal{M}et$ ($\mathcal{M}et^r$) the space of smooth (respectively class C^r) Riemannian metrics on M , where $r \gg k$. Our first result is a transversality theorem (with respect to variations of the metric g) for the complement $\mathcal{H}_g(S)^*$ of the zero section in the complex linear space

$$\mathcal{H}_g(S) := \coprod_{[\sigma] \in \mathcal{M}^{ASDg}(S)} [\mathbb{H}^0(\mathcal{C}_\sigma^{+g}(S)) \oplus \mathbb{H}^2(\mathcal{C}_\sigma^{+g}(S))]$$

over the torus $\mathcal{M}^{ASDg}(S)$. Unfortunately this transversality result can be proved only for metrics having the following property

H(S): *The vanishing locus of the g_0 harmonic representative of $c_1^{DR}(S)$ has Hausdorff dimension ≤ 2 .*

This condition is satisfied by any C^∞ -metric by a result of Bär [1], and any $c_1(S)$ -admissible metric $g \in \mathcal{M}et_{\geq 2}^r(c_1^{DR}(S))$ (see Section A.2 in Appendix A).

The space $\mathcal{H}_g(S)^*$ can be identified with the \mathcal{G}_{x_0} -quotient of the space of solutions of the system

$$(12) \quad \begin{cases} F_\sigma^{+g} & = 0 \\ d_\sigma \zeta + d_\sigma^* \eta & = 0 \end{cases}$$

for triples $(\sigma, \zeta, \eta) \in \mathcal{A}(S) \times [(A^0(S) \oplus A_{+g}^2(S)) \setminus \{0\}]$.

Regard (12) as an equation for systems (g, σ, ζ, η) , where $g \in \mathcal{M}et^r$, $\sigma \in \mathcal{A}(S)_k$, and $(\zeta, \eta) \in [(A^0(S)_k \oplus A_{+g}^2(S)_k) \setminus \{0\}]$. Therefore our configuration space is now

$$\mathcal{A}_k^* := \mathcal{A}(S)_k \times [A^0(S)_k \oplus \mathcal{A}_+^2(S)_k]^*$$

where $\mathcal{A}^0(S)_k$ is the trivial bundle $\mathcal{M}et^r \times \mathcal{A}^0(S)_k$ over $\mathcal{M}et^r$, $\mathcal{A}_+^2(S)_k$ is the bundle of S -valued selfdual forms over $\mathcal{M}et^r$ (see Section A.1 in the Appendix), and the symbol $[\cdot]^*$ on the right stands for the complement of the zero-section. Regarding \mathcal{A}_k^* as a locally trivial Banach bundle over the Banach manifold $\mathcal{M}et^r$, and denoting by $p: \mathcal{A}_k^* \rightarrow \mathcal{M}et^r$ the obvious projection, we see that the left hand terms of the equations (12) define sections u, v in the bundles $p^*(i[\mathcal{A}_+^2]_{k-1})$ and $\mathcal{A}_k^* \times A^1(S)_{k-1}$ over \mathcal{A}_k^* .

Theorem 3.5 *The section (u, v) in the bundle $[p^*(i[\mathcal{A}_+^2]_{k-1})] \oplus A^1(S)_{k-1}$ over \mathcal{A}_k is submersive at any solution $(g_0, \sigma_0, \zeta_0, \eta_0)$ where g_0 has the property $\mathbf{H}(S)$ and $(\zeta_0, \eta_0) \neq 0$.*

Proof Note first that, under our assumptions, the connection σ cannot admit nontrivial parallel sections, so applying $d_{\sigma_0}^*$ to the second equation, we get $\zeta_0 = 0$. The hypothesis gives $\eta_0 \neq 0$. Use the metric g_0 as a background metric to parameterize the manifold Met^r and to trivialize the bundles $i[\mathcal{A}_+^2]_{k-1}, \mathcal{A}_+^2(S)_k$ over this manifold (see Section A.1). The system (12) is equivalent with

$$(13) \quad \begin{cases} [(h^{-1})^*(F_\sigma)]^+ & = 0 \\ d_\sigma \zeta - *_{g_h} d_\sigma [h^*(\eta)] & = 0 \end{cases}$$

where now the upper script $+$ is used for the selfdual projection with respect to g_0 , and $\eta \in A_+^2(S)_k$. The left hand terms of the equations define a smooth map

$$\Gamma(\text{Sym}^+(T_M, g_0))^r \times \mathcal{A}(S)_k \times [(A^0(S)_k \times A_+^2(S)_k) \setminus \{0\}] \xrightarrow{(U,V)} i[\mathcal{A}_+^2]_{k-1} \times A^1(S)_{k-1}.$$

Our task is to prove that the differential of this map at $(\text{id}, \sigma_0, 0, \eta_0)$ is surjective. Let $(\alpha, \beta) \in i[\mathcal{A}_+^2]_{k-1} \times A^1(S)_{k-1}$ be a pair which is L^2 -orthogonal to the range of this differential. Using variations of the variables ζ and η (for $h = \text{id}$) we get

$$(14) \quad d_{\sigma_0}^* \beta = 0, \quad d_{\sigma_0}^+ \beta = 0.$$

Using the notations of Section A.1 in Appendix, one has

$$\frac{\partial U}{\partial h} \Big|_{(\text{id}, \sigma_0, 0, \eta_0)} (\chi) = -m_-^+(\chi)(F_{\sigma_0}), \quad \frac{\partial V}{\partial h} \Big|_{(\text{id}, \sigma_0, 0, \eta_0)} (\chi) = - * d_{\sigma_0}(m(\chi)\eta_0).$$

for any g_0 -symmetric endomorphism χ . For the second formula we used the fact that $d_{\sigma_0}\eta_0 = 0$ (recall that $\zeta_0 = 0$ and $*\eta_0 = \eta_0$), so the term containing the differential of $*_{g_h}$ with respect to h vanishes. Therefore, using variations $\chi \in A^0(\text{Sym}(T_M, g_0))^r$ of h with the property

$$m_-^+(\chi)(F_{\sigma_0}) = 0,$$

and noting that $m_+^-(\chi) = [m_-^+(\chi)]^*$ (see Section A.1), we obtain for any such χ

$$(15) \quad 0 = \langle *d_{\sigma_0}(m(\chi)\eta_0), \beta \rangle_{L^2} = -\langle m(\chi)\eta_0, *d_{\sigma_0}\beta \rangle_{L^2} = \langle m_+^-(\chi)\eta_0, d_{\sigma_0}\beta \rangle_{L^2} = \langle [m_-^+(\chi)]^*\eta_0, d_{\sigma_0}\beta \rangle_{L^2}.$$

Here we used the fact that the 2-form $d_{\sigma_0}\beta$ is ASD. In all these formulae $\langle \cdot, \cdot \rangle$ stands for the real (Euclidean) inner product. By Lemma A.1 in Appendix we see that any homomorphism $m \in A^0(\text{Hom}(\Lambda_-^2, \Lambda_+^2))^r$ can be written as $m_-^+(\chi)$ for a certain

symmetric endomorphism χ . Therefore (15) holds for any such section m for which $m(F_{\sigma_0}) = 0$. Now regard β as an element in $A^0(\text{Hom}_{\mathbb{R}}(S_{\mathbb{R}}^{\vee}, \Lambda_{-}^2))_{k-1}$, η_0 as an element in $A^0(\text{Hom}_{\mathbb{R}}(S_{\mathbb{R}}^{\vee}, \Lambda_{+}^2))_k$, and denote by $[\eta_0]_{\mathbb{R}}^* \in A^0(\text{Hom}_{\mathbb{R}}(\Lambda_{+}^2, S_{\mathbb{R}}^{\vee}))_k$ its adjoint with respect to the obvious real inner products. Changing the position of η_0 in (15) we obtain

$$\langle m^*, d_{\sigma_0}\beta \circ [\eta_0]_{\mathbb{R}}^* \rangle_{L^2} = 0,$$

for every $m \in A^0(\text{Hom}(\Lambda_{-}^2, \Lambda_{+}^2))^r$ for which $m(F_{\sigma_0}) = 0$. The condition $m(F_{\sigma_0}) = 0$ is equivalent to the condition $\text{im}(m^*) \subset F_{\sigma_0}^{\perp}$. Let U be the complement of the vanishing locus of F_{σ_0} . We conclude that $d_{\sigma_0}\beta \circ [\eta_0]_{\mathbb{R}}^*|_U$ is L^2 -orthogonal on the whole space of compactly supported $F_{\sigma_0}^{\perp}$ -valued bundle homomorphisms $\Lambda_{+}^2|_U \rightarrow \Lambda_{-}^2|_U$. Therefore $d_{\sigma_0}\beta \circ [\eta_0]_{\mathbb{R}}^*|_U$ takes values in the real line bundle generated by F_{σ_0} . This implies that either there exists a non-empty open subset $V \subset U$ on which η_0 has (real) rank at most 1, or $d_{\sigma_0}\beta|_U$ takes values in real line bundle generated by $F_{\sigma_0}|_U$. In the first case we obtain $\eta_0 = 0$ by Proposition A.10 in the Appendix. This contradicts the assumption. In the second case one gets $\beta = 0$ by Corollary A.9 (which applies because g_0 has the property $\mathbf{H}(S)$). Finally, using variations of σ and the assumption $b_{+}(M) = 0$, we obtain $\alpha = 0$. \square

Denote by $\text{Met}_{\text{bad}}^r(\lambda)$ the subspace of metrics for which there exists a non-regular λ -reducible instanton. Let $U \subset \text{Met}^r$ be any open subset of metrics satisfying the property $\mathbf{H}(S)$.

Theorem 3.6 *Suppose that $(2l-d)^2 < 0$. Then $\text{Met}_{\text{bad}}^r(\lambda) \cap U$ is closed and nowhere dense in U and the natural morphism*

$$\pi_i(U \setminus \text{Met}_{\text{bad}}^r(\lambda)) \rightarrow \pi_i(U)$$

is bijective for any i in the range $0 \leq i \leq -2(2l-d)^2 + b_1(M) - 2$ and surjective for $i = -2(2l-d)^2 + b_1(M) - 1$.

Proof By Theorem 3.5 it follows that the section (u, v) is transversal at any solution with metric component in U , so the vanishing locus $Z(u, v) \cap p^{-1}(U)$ is a smooth Banach manifold over U . The gauge quotient

$$\mathcal{H}(S)^*|_U := (Z(u, v) \cap p^{-1}(U)) / \mathcal{G}_{k+1, x_0}$$

will also be a smooth manifold, and the natural projection $\mathcal{H}(S)^*|_U \rightarrow \text{Met}^r$ is Fredholm of real index $2[c_1(S)^2 + (1 - b_1(M))] + b_1(M) = 2c_1(S)^2 - b_1(M) + 2$. One has a natural \mathbb{C}^* -action on $\mathcal{H}(S)^*$, and the projection

$$\mathcal{H}(S)^* / \mathbb{C}^* \longrightarrow \text{Met}^r$$

will be Fredholm of real index $2c_1(S)^2 - b_1(M)$. It suffices to apply Lemma A.7 in the Appendix. □

The same arguments can be used to prove regularity at a locus of twisted reductions (see Sections 1.4 and 3.2). However, there is an important detail here which should be taken into account carefully: in general, for an epimorphism $\pi(M) \rightarrow \mathbb{Z}_2$, the condition $b_+(M) = 0$ does *not* imply $b_+(M_\rho) = 0$. When $b_+(M_\rho) = 0$, one has $h_t^2 = 0$, $h_t^1 = b_1(M) - 1$ (see Section 3.2), and one can obtain generic regularity at the ρ -twisted reduction in the same way as for non-twisted reductions, by extending our proofs to the twisted case. The main difference is that the “normal” elliptic complex $\mathcal{C}_{b \otimes i^*(b)^\vee, i}^+$ has no complex structure.

In the twisted case, one has a Hermitian line bundle S on M_ρ which comes with an isomorphism $i^*(S) \simeq \bar{S}$; and for any metric g on M the $\pi_\rho(g)$ -harmonic representative of $c_1(S) = l - i^*(l)$ is ρ -equivariant, so the condition $\mathbf{H}(S)$ has sense for g . Put $\lambda := \{l, i^*(l)\}$ and denote by $\text{Met}_{\text{bad}}^r(\lambda)$ the space of \mathcal{C}^r -metrics for which there exists a non-regular ρ -twisted reducible instanton of type λ . The result for the twisted case is

Proposition 3.7 *Suppose that $b_+(M) = b_+(M_\rho) = 0$ and let $l \in H^2(M_\rho, \mathbb{Z})$ a solution of the system (7) with $(l - i^*(l))^2 < 0$. Put $\lambda := \{l, i^*(l)\}$ and let $U \subset \text{Met}^r$ an open set of metrics having the property $\mathbf{H}(S)$. Then $\text{Met}_{\text{bad}}^r(\lambda) \cap U$ is closed and the natural morphism*

$$\pi_i(U \setminus \text{Met}_{\text{bad}}^r(\lambda)) \rightarrow \pi_i(U)$$

is bijective for any i in the range $0 \leq i \leq -(l - i^(l))^2 + b_1(M) - 2$, and surjective for $i = -(l - i^*(l))^2 + b_1(M) - 1$.*

Remark Note that $-(l - i^*(l))^2 = 2[4c_2(E) - c_1(E)^2]$, hence under our assumptions one has $-(l - i^*(l))^2 \geq 2$.

Corollary 3.8 *Let M be a 4-manifold and E a Hermitian rank 2 bundle on M .*

(1) *Suppose that*

$$(16) \quad b_+(M) = 0 \text{ and } c_1(E) \notin 2H^2(M, \mathbb{Z}) + \text{Tors.}$$

There exists a connected, dense, open subset $\text{Met}_{\text{good}}^r(E) \subset \text{Met}^r$ such that, for any $g \in \text{Met}_{\text{good}}^r(E)$, the reductions in the Uhlenbeck compactification of the moduli space $\mathcal{M}_{a,g}^{\text{ASD}}(E)$ of g -instantons are all regular.

(2) *Suppose that (16) holds and for every epimorphism $\rho: \pi_1(M, x_0) \rightarrow \mathbb{Z}_2$*

$$(17) \quad b_+(M_\rho) = 0 \text{ and } \pi_\rho^*(c_1(E)) \notin 2H^2(M_\rho, \mathbb{Z}) + \text{Tors.}$$

There exists a connected, dense, open subset $\text{Met}_{\text{vgood}}^r(E) \subset \text{Met}^r$ such that, for any $g \in \text{Met}_{\text{vgood}}^r(E)$, the reductions and the twisted reductions in the Uhlenbeck compactification of the moduli space $\mathcal{M}_{a,g}^{\text{ASD}}(E)$ of g -instantons are all regular.

Proof (1) Denote by $\Lambda(c)$ the finite set of unordered pairs $\lambda = \{l, d-l\}$ satisfying $l \cdot (d-l) \leq c$. For $c = c_2(E)$, this set is the set of all topological decompositions of all bundles E' which must be considered in the construction of the Uhlenbeck compactification of $\mathcal{M}_a^{\text{ASD}}(E)$. Put

$$\text{Met}_{\text{good}}^r(E) := \text{Met}_{\text{adm}}^r(c) \setminus \left[\bigcup_{\lambda \in \Lambda(c)} \text{Met}_{\text{bad}}^r(\lambda) \right]$$

(see Section A.2 in the Appendix).

(2) One may define $\text{Met}_{\text{vgood}}^r(E)$ in a similar way by replacing $\text{Met}_{\text{adm}}^r(c)$ with $\text{Met}_{\text{tadm}}^r(c)$ (see Section A.2) and removing the bad loci associated with all twisted and non-twisted reductions of bundles E' with $\det(E') = \det(E)$, $c_2(E') \leq c_2(E)$. \square

Remark Suppose that $b_+(M) = 0$, $b_2(M) > 0$, and $b_+(M_\rho) = 0$ for any epimorphism $\rho: \pi_1(M, x_0) \rightarrow \mathbb{Z}_2$. Let d be an integral lift of $w_2(M)$. Then $d \notin 2H^2(M, \mathbb{Z}) + \text{Tors}$ and $\pi_\rho^*(d) \notin 2H^2(M_\rho, \mathbb{Z}) + \text{Tors}(H^2(M_\rho, \mathbb{Z}))$ for every epimorphism ρ , so Corollary 3.8 (2) applies for any bundle E with $c_1(E) = d$.

Proof By Donaldson first theorem, the intersection form on $H^2(M, \mathbb{Z})/\text{Tors}$ is trivial over \mathbb{Z} . Choosing an orthonormal basis $(e_i)_{1 \leq i \leq b_2(M)}$ in this lattice, one obtains $d \cdot e_i \equiv e_i^2 \equiv -1 \pmod{2}$, so d cannot be divisible by 2 in $H^2(M, \mathbb{Z})/\text{Tors}$.

On the other hand, the class $\pi_\rho^*(d)$ is an integral lift of $\pi_\rho^*(w_2(M)) = w_2(M_\rho)$. Since $b_+(M_\rho) = b_+(M) = 0$, one gets easily (comparing the signatures and the Euler characteristics of the two manifolds) that $b_1(M_\rho) = 2b_1(M) - 1$ and $b_2(M_\rho) = 2b_2(M) > 0$. Therefore the same argument applies for M_ρ , proving that $\pi_\rho^*(d)$ cannot be divisible by 2 in $H^2(M_\rho, \mathbb{Z})/\text{Tors}$. \square

Remark Similar generic regularity results can be obtained using abstract perturbations of the ASD-equations around the reduction loci (see [3, page 156]). However, since in our general framework the reductions are not necessarily isolated points in the moduli space, this method is more complicated than in the classical case. Moreover, for our purposes (see Section 4) one must check that the perturbed moduli space still has a natural compactification, and that the ‘‘cobordism type’’ of this compactification is well defined.

Combining Corollary 3.8 with the classical transversality theorem for irreducible instantons (see Donaldson and Kronheimer [3], Freed and Uhlenbeck [4]) one shows that

Lemma 3.9 *The set $\text{Met}_{\text{wgood}}^r(E) \subset \text{Met}_{\text{vgood}}^r(E)$ of metrics for which $\mathcal{M}_{a,g}^{\text{ASD}}(E)$ contains only regular solutions is dense of the second Baire category. This set is also open (but in general non-connected!) when $\Delta(E) := 4c_2(E) - c_1(E)^2 \leq 3$.*

Proof The condition $\Delta(E) \leq 3$ implies that the projection $\tilde{\mathcal{M}}_a^{\text{ASD}}(E) \rightarrow \text{Met}^r$ of the parameterized instanton moduli space on the space of metrics is proper. The openness of $\text{Met}_{\text{wgood}}^r(E)$ in $\text{Met}_{\text{vgood}}^r(E)$ follows by elliptic semicontinuity. \square

4 Applications

4.1 Geometric properties of instanton moduli spaces on manifolds with $b_+ = 0$

We will see that using our regularity results combined with the topological results obtained in Sections 2.1 and 2.2 one can obtain important information about the geometry of the ASD moduli spaces.

The purpose of this section is not to give an exhaustive list of all possible applications of this type, but only to illustrate the method with an explicit example, which came to my attention when I began to work on the classification of class VII surfaces with $b_2 = 2$ [12].

Let M be a 4-manifold with the topological properties considered in the examples studied in Section 2.2: $H_1(M, \mathbb{Z}) \simeq \mathbb{Z}$, $b_2(M) = b_-(M) = 2$. Consider again a rank 2-Hermitian bundle E on M with $c_1(E) = d = e_1 + e_2$ (where (e_1, e_2) is an orthonormal basis of $H^2(M, \mathbb{Z}) \simeq \mathbb{Z}^{\oplus 2}$) and $c_2(E) = 0$. As in Section 2.2 put $\lambda_0 := \{0, d\}$, $\lambda_1 := \{e_1, e_2\}$ and note that $\Lambda_E = \{\lambda_0, \lambda_1\}$. Let L_0, L_1 be Hermitian line bundles with Chern classes $c_1(L_0) = 0$, $c_1(L_1) = e_1$, and put $S_i := L_i^{\otimes 2} \otimes D^\vee$.

The expected dimension of the instanton moduli space $\mathcal{M}_a^{\text{ASD}}(E)$ is 4 and, since $\Delta(E) := 4c_2(E) - c_1(E)^2 < 4$, this moduli space is compact. $\mathcal{M}_a^{\text{ASD}}(E)$ contains two circles of reductions $\mathcal{M}_a^{\lambda_0}(E)$ and $\mathcal{M}_a^{\lambda_1}(E)$. An interesting application of our results is the following:

Theorem 4.1 *For every Riemannian metric g on M and abelian connection $a \in \mathcal{A}(\det(E))$, the two circles of reductions $\mathcal{M}_a^{\lambda_0}(E)$, $\mathcal{M}_a^{\lambda_1}(E)$ belong to the same connected component of the moduli space $\mathcal{M}_a^{\text{ASD}}(E)$.*

Proof For a metric $g \in \text{Met}_{\text{good}}^r(E)$ the moduli space $\mathcal{M}_a^{\text{ASD}}(E)$ contains only regular solutions. Regularity at the reductions implies that the linear spaces

$$\mathcal{H}_i^1 := \bigcup_{[b] \in T_a(L)} \mathbb{H}^1(\mathcal{C}_{b \otimes 2 \otimes a^\vee}^+(S_i)) \rightarrow T_a(L) \simeq \mathcal{M}_a^{\lambda_i}(E)$$

are rank 2 complex vector bundles (see Section 3.1). $\mathcal{M}_a^{\lambda_i}(E)$ has a neighborhood v_i which can be identified with the S^1 -quotient of an S^1 -invariant neighborhood of the zero section of \mathcal{H}_i^1 . Let σ be a generator of $H_1(M, \mathbb{Z})$. By Corollary 2.6 we see that the restriction of the Donaldson class $\mu(\sigma)$ to the boundary $\partial(v_i)$ coincides (up to sign) with the fundamental class of this 3-manifold. More precisely, if we choose an orientation of the moduli space as in Donaldson and Kronheimer [3, page 283], and we endow $\partial(v_i)$ with the boundary orientations, we see that $\mu(\sigma)|_{\partial(v_0)} = \pm[\partial(v_0)]$ and $\mu(\sigma)|_{\partial(v_1)} = \mp[\partial(v_1)]$. Therefore $\partial(v_i)$ cannot be homologically trivial in $\mathcal{B}_a^*(E)$. This shows that the two boundaries (hence also the corresponding circles) belong to the same connected component.

To complete the proof for an arbitrary metric g , we use the density of the space $\text{Met}_{\text{good}}^r(E)$ and note that if the reduction circles $\mathcal{M}_a^{\lambda_i}(E)$ belonged to different connected components, the same would happen for any metric g' sufficiently close to g . □

Example Consider the 4-manifold $M = (S^1 \times S^3) \# \overline{\mathbb{P}^2} \# \overline{\mathbb{P}^2}$. This manifold has the differentiable type of a Hopf surface blown up at two points. It is convenient to endow M with the complex structure of a *minimal* class VII surface with $b_2 = 2$. Choosing the Gauduchon metric in a convenient way and using the Kobayashi–Hitchin correspondence to identify instantons with polystable bundles, one obtains (see [12]):

$$\mathcal{M}_a^{\text{ASD}}(E) \simeq S^4,$$

so (despite the presence of the reductions) the moduli space gets an obvious smooth structure on the moduli space. The two reduction circles $\mathcal{M}_a^{\lambda_i}(E)$ are smoothly embedded in the sphere.

4.2 New Donaldson invariants

In this section we introduce a new class of Donaldson type invariants, which are defined for definite 4-manifolds. Note that Donaldson type invariants for $\mathbb{Z}[\mathbb{Z}]$ homology $S^1 \times S^3$ -manifolds have already been considered by Ruberman and Savaliev [9] and by Furuta and Ohta [5]. Moreover, Ruberman and Savaliev introduced recently a new differential topological invariant – based on the Dirac equation – for a special class of 4-manifolds with $b_2 = 0$ and $b_1 = 1$ [10].

4.2.1 Low energy Donaldson invariants and Casson type invariants Let M be a negative definite 4-manifold, and let $(e_1, \dots, e_{b_2(M)})$ be an orthonormal basis in $H^2(M, \mathbb{Z})/\text{Tors}$. Let $d \in H^2(M, \mathbb{Z})$ be a lift of $e_1 + \dots + e_{b_2(M)}$ and denote by \bar{d} its image in $H^2(X, \mathbb{Z}_2)$. Note that one must have $\bar{d} = w_2(M)$, when $H^2(M, \mathbb{Z})$ is torsion free.

Let E be a Hermitian 2-bundle on M with $c_1(E) = d$ and put as usual $D := \det(E)$. If $\{l, d-l\}$ is a topological decomposition of E , then, writing $l = \sum l_i e_i$, with $l_i \in \mathbb{Z}$, one gets

$$c_2(E) = \sum l_i(l_i - 1) \geq 0.$$

Therefore, for $c_2(E) < 0$, the bundle E admits no topological decomposition. On the other hand the expected dimension of the Donaldson moduli space $\mathcal{M}_a^{\text{ASD}}(E)$ is

$$\delta = 2(4c_2(E) + b_2(M)) + 3(b_1(M) - 1).$$

If $c_2(E)$ is chosen such that $\Delta(E) = 4c_2(E) + b_2(M) \in \{0, 1, 2, 3\}$ the corresponding moduli space will be a priori compact (that is, compact independently of the metric). When $b_2(M) \geq 4$, the corresponding values of $c_2(E)$ are negative. When $b_1(M) \geq 1$ the corresponding expected dimension will be non-negative. Therefore:

Remark Suppose that $b_1(M) \geq 1$ and $b_2(M) \geq 4$, and choose $c_2(E) := -\lfloor \frac{b_2(M)}{4} \rfloor$. The corresponding moduli space will be a priori compact, of non-negative expected dimension $2\Delta(E) + 3(b_1 - 1)$ and will contain no reduction.

In other words, for this special value of $c_2(E)$, one can define very easily Donaldson type invariants by evaluating products of classes of the form $\mu(h)$ on the virtual fundamental class of the moduli space (see Brussee [2]). In this case, one does not really need regular moduli spaces, because the formalism of virtual fundamental classes gives directly a well defined homology class in the space $\mathcal{B}_a^*(E)$.

A very interesting case is when $b_2(M) \geq 4$ is divisible by 4. In this case this special value of $c_2(E)$ is $-\frac{b_2(M)}{4}$ and the corresponding discriminant $\Delta(E)$ vanishes. Therefore, in this case $\mathcal{M}^{\text{ASD}}(E)$ coincides with the moduli space of $PU(2)$ -representations of $\pi_1(M, x_0)$ with fixed Stiefel-Whitney class \bar{d} , modulo $SU(2)$ -conjugation. The invariants associated with such a moduli space should be called four-dimensional Casson type invariants. They should be regarded as an extension of the similar $SU(2)$ -invariant defined for $\mathbb{Z}[\mathbb{Z}]$ -homology $S^1 \times S^3$ -manifolds (see Furuta and Ohta [5], Ruberman and Savaliev [9]) to our new class of homology types. Note that, because of the absence of reductions in our moduli space, the definition of the invariant in our case is much easier.

Note the following simple vanishing result, which shows that, if non-trivial, this Casson type invariants can be regarded as obstructions to the representability of the basis elements e_i by 2-spheres.

Remark Suppose that $b_2(M)$ is divisible by 4 and one of the basis elements e_i is represented by a continuous map $S^2 \rightarrow M$. Then the moduli space associated with the Chern class $c_2(E) = -\frac{b_2(M)}{4}$ is empty. In particular, the corresponding Casson type invariants vanish.

Proof If $f: S \rightarrow M$ represents e_i (for an oriented closed surface S), one has

$$\langle f^*(\bar{d}), [S] \rangle = \langle \bar{d}, f_*([S]) \rangle = \langle d, f_*([S]) \rangle \bmod 2 = \langle d, e_i \rangle \bmod 2 = 1.$$

Let $x_0 \in S$, and $\rho: \pi_1(M, f(x_0)) \rightarrow PU(2)$ a representation of Stiefel–Whitney class \bar{d} . The composition $\pi_1(S, x_0) \rightarrow \pi_1(M, f(x_0)) \rightarrow PU(2)$ will be a representation with Stiefel–Whitney class $f^*(\bar{d}) \neq 0$, so S cannot be simply connected. \square

Interestingly, one has:

Remark There exist definite negative 4-manifolds, with the property that no element $e \in H^2(M, \mathbb{Z})/\text{Tors}$ with $e^2 = -1$ can be represented by a continuous map $f: S^2 \rightarrow M$.

Indeed, it suffices to consider a fake projective plane (see for instance Prasad and Yeung [8]) with reversed orientation. Since the universal cover of such a 4-manifold is the complex 2-ball, we see that the generator of its homology cannot be represented by a continuous map $f: S^2 \rightarrow M$.

Consider now the case

$$c_2(E) \in \left(-\left[\frac{b_2(M)}{4} \right], -1 \right].$$

In this range, one loses “a priori compactness”, but has moduli spaces with no reductions in their Uhlenbeck compactifications. In this range, one uses Donaldson’s method [3] to define the invariants geometrically: one uses metrics for which all strata are regular, constructs distinguished cycles representing the μ -classes (and which extends to the Uhlenbeck compactification) and defines the invariants by intersecting the moduli space with systems of such cycles (which can be chosen so that they intersect transversally in the main stratum).

4.2.2 Invariants associated with classes in $H^*(\mathcal{B}^*, \mathcal{V}^*)$ and invariants defined using the cobordism type of the moduli space We illustrate these types of invariants in the concrete situation considered in Sections 2.2 and 4.1: a negative definite 4–manifold with $H_1(M, \mathbb{Z}) \simeq \mathbb{Z}$, $b_2(M) = b_-(M) = 2$ endowed with a rank 2–Hermitian bundle E on M with $c_1(E) = d = e_1 + e_2$ (where (e_1, e_2) is an orthonormal basis of $H^2(M, \mathbb{Z}) \simeq \mathbb{Z}^{\oplus 2}$) and $c_2(E) = 0$.

The moduli space is a priori compact, but it always contains two circles of reductions. We have two ways to define invariants in this situation:

(1) Use a generic metric in the sense of Donaldson and Kronheimer [3], for which the irreducible part of the moduli space is regular. Regard the (oriented) moduli space $\mathcal{M}_a^{\text{ASD}}(E)^*$ of irreducible instantons as a cycle in the relative homology $H_4(\mathcal{B}^*, \mathcal{V}^*)$. On the other hand, we have seen in Section 2.2 that the relative rational cohomology $H^4(\mathcal{B}^*, \mathcal{V}^*) \simeq \mathbb{Q}^4$ and that this group fits in a short exact sequence

$$(18) \quad 0 \longrightarrow \frac{H^3(\mathcal{V}_0^*) \oplus H^3(\mathcal{V}_1^*)}{\langle (\delta_{L_0}(\sigma_0)h_{u_0}, \delta_{L_1}(\sigma_0)h_{u_1}) \rangle} \longrightarrow H^4(\mathcal{B}^*, \mathcal{V}^*) \\ \longrightarrow \langle \mu(*) + \mu(f_1)^2, \mu(*) + \mu(f_2)^2, \mu(\lambda_0)\mu(\sigma_0) \rangle_{\mathbb{Q}} \rightarrow 0.$$

Evaluating classes in $H^4(\mathcal{B}^*, \mathcal{V}^*)$ on the relative homology class given by $\mathcal{M}_a^{\text{ASD}}(E)^*$, one gets well-defined invariants. Note however that, since the exact sequence (18) does not split canonically, one cannot parameterize this set of invariants in an obvious way.

(2) For a metric $g \in \text{Met}_{\text{vgood}}^r(E)$ all solutions (including the reductions) in the moduli space are regular (see Lemma 3.9). The main observation here is that each reduction circle $\mathcal{M}_a^{\lambda_i}(E)$ has a neighborhood isomorphic to the S^1 –quotients of a neighborhood of the zero section in a rank 2 complex bundle \mathcal{H}_i^1 over $\mathcal{M}_a^{\lambda_i}(E)$ (see Section 4.1). But such a quotient is a locally trivial K –bundle over a circle, where K is the cone over $\mathbb{P}^1 \simeq S^2$, so it has a natural manifold structure. Therefore, for $g \in \text{Met}_{\text{vgood}}^r(E)$, $\mathcal{M}_a^{\text{ASD}}(E)$ is a compact 4–manifold, which can be oriented as in classical Donaldson theory [3, page 283].

The signature of this 4–manifold will be an invariant $\theta(M)$ of the base 4–manifold (endowed with the usual orientation data). Indeed, the main point here is that the set $\text{Met}_{\text{vgood}}^r(E)$ is connected. For two choices $g_0, g_1 \in \text{Met}_{\text{vgood}}^r(E)$, consider a path $\gamma: [0, 1] \rightarrow \text{Met}_{\text{vgood}}^r(E)$ connecting these metrics. A generic deformation (with fixed ends) of γ will define a cobordism between the moduli spaces associated with g_i . Note that *the cobordism constructed in this way is always trivial around the reductions.*

Remark In the example given in Section 4.1, one has $\mathcal{M}_a^{\text{ASD}}(E) \simeq S^4$, so the θ -invariant vanishes.

This shows that, if non-trivial, this invariant can also be regarded as an obstruction to the representability of the elements of the orthonormal basis $\{e_1, e_2\}$ by embedded spheres.

Appendix A

A.1 Metric-dependent spaces of selfdual and anti-selfdual forms

Let M a compact oriented connected 4-manifold and E a vector bundle. For every Riemannian metric g on M one has two associated spaces of E -valued (anti)selfdual forms $A_{\pm}^2(E)$. It is convenient to complete the space Met of metrics with respect to the C^r topology and the spaces $A_{\pm}^2(E)$ with respect to a Sobolev norm L_k^2 (where $r \gg k$). In this way one gets Banach vector bundles $[A_{\pm}^2(E)]_k$ on the Banach manifold Met^r of C^r -metrics.

One can trivialize globally these bundles in the following way. Fix a C^r -metric g_0 . The space Met^r can be identified with the space of positive g_0 -symmetric automorphisms of the tangent bundle T_M via the diffeomorphism $h \mapsto g_h := h^*(g_0)$. We get homeomorphisms

$$\Gamma(\text{Sym}^+(T_M, g_0))^r \times [A_{\pm}^2(E)]_k \xrightarrow{\cong} [A_{\pm}^2(E)]_k$$

given by $(h, \eta) \mapsto (g_h, h^*(\eta))$. It is important to notice that homeomorphisms associated with different metrics g_0 are pairwise differentiable compatible. Therefore one can use these homeomorphisms to define structures of Banach manifolds on the total spaces $[A_{\pm}^2(E)]_k$.

A positive symmetric automorphism $h \in \Gamma(\text{Sym}^+(T_M, g_0))^r$ defines a class C^r positive g_0 -symmetric automorphism $\Lambda^2 h$ of the bundle Λ_M^2 , given by $\lambda \mapsto h^*(\lambda)$. Using the g_0 -orthogonal decomposition $\Lambda_M^2 = \Lambda_{+g_0}^2 \oplus \Lambda_{-g_0}^2$, the automorphism $\Lambda^2 h$ can be written as

$$\Lambda^2 h = \begin{pmatrix} \mu_+^+(h) & \mu_+^-(h) \\ \mu_-^+(h) & \mu_-^-(h) \end{pmatrix},$$

where $\mu_{\pm}^{\pm}(h)$ are symmetric endomorphisms of $\Lambda_{\pm g_0}^2$ and $\mu_{\mp}^{\pm}(h): \Lambda_{\mp g_0}^2 \rightarrow \Lambda_{\pm g_0}^2$ have the property $\mu_-^+(h)^* = \mu_+^-(h)$. The tangent space of $\Gamma(\text{Sym}^+(T_M, g_0))^r$ at id

can be identified with $A^0(\text{Sym}(T_M, g_0))^r$. For a symmetric endomorphism χ we put

$$m(\chi) := \left. \frac{\partial \mu(h)}{\partial h} \right|_{\text{id}} (\chi), \quad m_{\pm}^{\pm}(\chi) := \left. \frac{\partial \mu_{\pm}^{\pm}(h)}{\partial h} \right|_{\text{id}} (\chi), \quad m_{\mp}^{\pm}(\chi) := \left. \frac{\partial \mu_{\mp}^{\pm}(h)}{\partial h} \right|_{\text{id}} (\chi).$$

Since $m(\chi)$ is symmetric, one has $m_{\pm}^{\pm}(\chi)^* = m_{\mp}^{\mp}(\chi)$.

Lemma A.1 *The map $\chi \mapsto m_{\mp}^{\pm}(\chi)$ defines an isomorphism*

$$A^0(\text{Sym}_0(T_M, g_0))^r \simeq A^0(\text{Hom}(\Lambda_{\mp}^2, \Lambda_{\pm}^2))^r,$$

where $\text{Sym}_0(T_M, g_0)$ denotes the bundle of trace free g_0 -symmetric endomorphisms of T_M .

This can be checked easily by diagonalizing χ in an orthonormal frame.

The space $A^0(\text{Sym}(T_M, g_0))^r$ can be regarded as the space of infinitesimal variations of C^r -metrics, whereas $A^0(\text{Sym}_0(T_M, g_0))^r$ is the space of infinitesimal variations of C^r -conformal structures.

A.2 Admissible metrics

Consider an oriented compact connected manifold M and a cohomology class $l \in H_{\text{DR}}^p(M)$, where $1 \leq p \leq n - 1$. For a Riemannian metric g on M we will denote by l_g the unique g -harmonic representative of l .

Let M a 4-manifold. The family of vector spaces

$$(\Lambda_{\pm, g, x}^2)_{g \in \text{Met}^r, x \in M}$$

defines a bundle Ω^{\pm} on the product space $\text{Met}^r \times M$. Suppose now that $b_+(M) = 0$. In this case any harmonic 2-form is ASD. For a de Rham 2-cohomology class l , the assignment $\text{Met}^r \times M \ni (g, x) \mapsto l_g(x)$ defines a universal $C^{r-\epsilon}$ section λ in the bundle Ω^- over $\text{Met}^r \times M$.

Lemma A.2 *Suppose that $b_+(M) = 0$ and $l \in H_{\text{DR}}^2(M, \mathbb{R}) \setminus \{0\}$. The associated universal section λ is submersive at every vanishing point.*

Proof Let (g_0, x_0) be a vanishing point of λ . We use the metric g_0 to simultaneously parameterize the space of metrics (using positive g_0 -symmetric endomorphisms) and to trivialize the bundles Ω^{\pm} . In this way, our section gives rise to a map

$$\Gamma(\text{Sym}^+(T_M, g_0))^r \times M \ni (h, x) \mapsto \omega_h(x) \in \Lambda_{-, x}^2.$$

where $\omega_h := [\mu(h)^{-1}]^*(l_{g_h}) \in A^2_-(M)$. Put $\omega_0 := \omega_{\text{id}} = l_{g_0}$. Writing $l_{g_h} = \omega_0 + d\alpha_h$ (with $\alpha_h \perp_{L^2} Z^1$), one gets for the derivative $\dot{\alpha}(\chi) := \left. \frac{d\alpha}{dh} \right|_{\text{id}}(\chi)$ the equation

$$p^+ [-m(\chi)^*(\omega_0) + d\dot{\alpha}(\chi)] = 0,$$

which gives the solution

$$\dot{\alpha}(\chi) = G^+(m(\chi)^{\pm}(\omega_0)),$$

where G^+ is the $[Z^1]^{\perp}$ -valued inverse of the operators d^+ . Here $Z^1 \subset A^1(M)$ denotes as usually the space of closed 1-forms. Therefore

$$\left. \frac{d\omega_h}{dh} \right|_{\text{id}}(\chi) = dG^+(m^{\pm}(\chi)(\omega_0)) - m(\chi)(\omega_0) = d^-G^+(m^{\pm}(\chi)(\omega_0)) - m^{\pm}(\chi)(\omega_0).$$

Therefore

$$\left. \frac{\partial \omega_h(x)}{\partial h} \right|_{(\text{id}, x_0)}(\chi) = d^-G^+(m^{\pm}(\chi)(\omega_0))(x_0),$$

because $\omega_0(x_0) = 0$. Suppose now that $v \in [\Lambda_{x_0}^-]$ is orthogonal on the range of this partial derivative.

The vector v defines a Dirac type ASD degree 2-current v_{x_0} . Since any element of $m \in A^0(\text{Hom}(\Lambda^2_-, \Lambda^2_+))$ has the form $m^{\pm}(\chi)$, we obtain that for any such m

$$(19) \quad 0 = \langle v, d^-G^+(m(\omega_0))(x_0) \rangle = \langle v_{x_0}, d^-G^+(m(\omega_0)) \rangle.$$

The assignment

$$\psi \mapsto \langle v_{x_0}, d^-G^+(\psi^+) \rangle$$

defines a self-dual degree 2-current, and the identity (19) implies that this current vanishes on $M \setminus Z(\omega_0)$. This current has the following important properties:

- (1) One has $d(\delta) = d(v_{x_0})$.
- (2) δ is supported at x_0 .

(1) Indeed, to prove the first claim, consider a test 1-form α , and compute

$$\langle \delta, d\alpha \rangle = \langle v_{x_0}, d^-G^+d^+\alpha \rangle = \langle v_{x_0}, d^-pr_{Z^1 \perp}(\alpha) \rangle = \langle v_{x_0}, d^-\alpha \rangle = \langle v_{x_0}, d\alpha \rangle.$$

(2) Using the fact that d is overdetermined elliptic on $A^2_+(M)$, it follows from the first statement that $\delta|_{M \setminus \{x_0\}}$ is smooth. On the other hand, by (19), δ vanishes on the dense open set $M \setminus Z(\omega_0)$. Therefore δ must vanish everywhere on $M \setminus \{x_0\}$.

The statement follows now from the following Lemma. □

Lemma A.3 Let u be a Λ^2 -valued distribution on \mathbb{R}^4 which is supported at the origin and is closed. If u^- has order 0, then $u = 0$.

Proof If $du = 0$ and u^- has order 0, then u^+ must also have order 0, because, if not, taking the sum u_k^+ of all terms of highest order $k \geq 1$ in the decomposition of u^+ as sum of partial derivatives of Dirac type distributions, one would have $du_k^+ = 0$. Therefore u_k^+ would be singular harmonic self-dual form, which is impossible.

Therefore, u is an order zero Dirac type distribution, so it can be written as $u = \sum_{i < j} a_{ij} \delta_0 e^i \wedge e^j$, where δ_0 is the standard Dirac distribution. We get

$$du = \sum_{i < j < k} \left(a_{jk} \frac{\partial}{\partial x_i} |_0 - a_{ik} \frac{\partial}{\partial x_j} |_0 + a_{ij} \frac{\partial}{\partial x_k} |_0 \right) e^i \wedge e^j \wedge e^k,$$

and the relation $du = 0$ implies obviously $a_{ij} = 0$ for all $i < j$. □

Definition A.4 Let M be a 4-manifold with $b_+(M) = 0$ and $l \in H_{\text{DR}}^2(M) \setminus \{0\}$. A metric $g \in \text{Met}^r$ will be called

- (1) strictly l -admissible, if l_g is submersive (transversal to the zero section) at any vanishing point.
- (2) l -admissible, if the intrinsic derivative of the section $l_g \in \Gamma(\Lambda_{-,g}^2)$ at any vanishing point has rank at least 2.

Denote by $\text{Met}_3^r(l)$ and $\text{Met}_{\geq 2}^r(l)$ the space of (strictly) admissible C^r -metrics.

Proposition A.5 (1) Suppose that the regularity class r is sufficiently large. The space Met_3^r of l -strictly admissible metrics is open and dense in Met^r .

- (2) The space $\text{Met}_{\geq 2}^r$ of l -admissible metrics is open, dense and path connected in Met^r .

Proof The openness of the two sets is obvious taking into account the compactness of the manifold, the continuity of the map $g \mapsto l_g$ with respect to the C^1 topology on the space of sections, and the fact that, in general, for any bundle E and fixed base point x , the condition

$$s(x) = 0, \quad \text{rk}(D_x(s)) \leq k$$

is closed with respect to the C^1 -topology on the space of sections in E .

To prove the density of the set of admissible metrics, note that the vanishing locus $Z(\lambda)$ of the universal section λ is a smooth codimension 3-submanifold of $\text{Met}^r \times M$, and

the natural map $q: Z(\lambda) \rightarrow \text{Met}^r$ is Fredholm of index 1. $\text{Met}_3^r(l)$ is just the set of regular values of q . Put $Z := Z(\lambda)$.

Therefore the larger set $\text{Met}_{\geq 2}^r(l)$ will also be dense. In order to prove the fact that this set is also path connected, consider – for two given metrics $g_0, g_1 \in \text{Met}_{\geq 2}^r(l)$ – a differentiable path $\gamma: [0, 1] \rightarrow \text{Met}^r$ joining them.

Consider that map

$$G: \Gamma(SL(T_X))^r \times [0, 1] \longrightarrow \text{Met}^r$$

defined by $G(h, t) = h^*(\gamma(t))$. Since the partial derivative $\frac{\partial G}{\partial h}$ alone is surjective at any point, we conclude that the map G is transversal to q , so the fibered product $P := \{(h, t, z) \mid G(h, t) = q(z)\}$ is a smooth submanifold of $\Gamma(SL(T_X))^r \times [0, 1] \times Z$. The natural map $P \rightarrow \Gamma(SL(T_X))^r$ is proper and Fredholm of index 2. By the Sard–Smale theorem [11], in any neighborhood of id one can find a regular value h_0 of this map. Therefore, for such h_0

$$P_0 := \{(t, z) \mid G(h_0, t, z) = q(z)\}$$

is a smooth 2–dimensional submanifold of $[0, 1] \times Z$. Put $g'_t := G(h_0, t)$. The fiber over $t \in [0, 1]$ is contained in $\{t\} \times \{g'_t\} \times M$ and is identified with the vanishing locus $Z(l_{g'_t})$ of the section $l_{g'_t}$ under the projection on the third factor.

We claim that the path $G(h_0, \cdot)$ takes values in $\text{Met}_{\geq 2}^r(l)$. Indeed, the Zariski tangent space of $Z(l_{g'_t})$ at a point x is just the intersection of the tangent space of P_0 at (t, g'_t, x) with the tangent space of the fiber $\{t\} \times \{g'_t\} \times M \subset P_0$ over t . But the tangent space of P_0 at any point is 2–dimensional, so the Zariski tangent space of $Z(l_{g'_t})$ at x has dimension 1 or 2.

In order to complete the proof, it suffices to join g_0 to g'_0 and g_1 to g'_1 with paths in $\text{Met}_{\geq 2}^r(l)$. By the openness property of this space, it follows easily that (if h_0 is sufficiently close to id) the metrics $G(\text{id} + s(h_0 - \text{id}), 0)$, $G(\text{id} + s(h_0 - \text{id}), 1)$, $s \in [0, 1]$ remain in $\text{Met}_{\geq 2}^r(l)$ for every $s \in [0, 1]$. \square

Note that we see no reason why the space $\text{Met}_3^r(l)$ should be connected. Indeed, suppose that for two metrics $g_0, g_1 \in \text{Met}_3^r(l)$, the corresponding vanishing loci (which are finite unions of pairwise disjoint embedded circles) have different number of connected components. Then there is certainly no way to join the two metrics by a path in $\text{Met}_3^r(l)$.

Remark If $g \in \text{Met}_{\geq 2}^r(l)$, then any point $x \in Z(l_g)$ has a neighborhood $U_x \subset M$ such that $Z(l_g) \cap U_x$ is contained in a closed submanifold N_x of U_x of dimension 1 or 2. In particular the vanishing locus $Z(l_g)$ of the g –harmonic ASD form l_g has Hausdorff dimension ≤ 2 .

Proof Indeed, consider a smooth map $\mathbb{R}^n \supset V \xrightarrow{f} \mathbb{R}^m$ whose rank at $0 \in V$ is k and let $E := \text{im}(d_0(f))$. Then the composition $p_E \circ f$ is a submersion at 0, so its restriction on a sufficiently small neighborhood U of 0 will be a submersion. Note that $Z(f) \subset Z(p_E \circ f)$ and that $Z(p_E \circ f) \cap U = Z((p_E \circ f)|_U)$ is a codimension k submanifold of U . \square

The results above can be extended for twisted de Rham cohomology classes: Let $\rho: \pi_1(M) \rightarrow \mathbb{Z}_2$ be an epimorphism, and suppose that $b_+(M_\rho) = 0$ where $\pi_\rho: M_\rho \rightarrow M$ is the corresponding double cover of M . Let $l \in H_{\text{DR}}^2(M_\rho) \setminus \{0\}$ be a de Rham cohomology class with the property that $\iota^*(l) = -l$, where ι stands for the tautological involution of M_ρ . Then, for every metric g on M , the $\pi_\rho^*(g)$ -harmonic representative l_g of l is an ASD form on M_ρ satisfying the identity $\iota^*(l_g) = -l_g$. In other words, l_g is a ρ -twisted ASD form on M . At any point $x \in M$, l_g is defined up to sign. Therefore, the vanishing locus $Z(l_g) \subset M$ and the rank of the intrinsic derivative at a vanishing point are well-defined objects. In particular one can associate to l the sets of metrics $\text{Met}_3^r(l)$, $\text{Met}_{\geq 2}^r(l)$ as in the non-twisted case, and these sets also have the properties stated in Proposition A.5.

Proposition A.6 Let M be a 4-manifold and $d \in H^2(M, \mathbb{Z})$

(1) Suppose that

$$(20) \quad b_+(M) = 0 \text{ and } d \notin 2H^2(M, \mathbb{Z}) + \text{Tors.}$$

Then, for every $c \in H^4(M, \mathbb{Z})$ the set

$$\text{Met}_{\text{adm}}^r(c) := \bigcap_{\substack{l \in H^2(M, \mathbb{Z}) \\ l \cdot (d-l) \leq c}} \text{Met}_{\geq 2}^r(2l - d)$$

is open, dense and connected in Met^r .

(2) Suppose that (20) holds and, for every epimorphism $\rho: \pi_1(M, x_0) \rightarrow \mathbb{Z}_2$

$$(21) \quad b_+(M_\rho) = 0 \text{ and } \pi_\rho^*(d) \notin 2H^2(M_\rho, \mathbb{Z}) + \text{Tors.}$$

Then the set

$$\text{Met}_{\text{adm}}^r(c) := \text{Met}_{\text{adm}}^r(c) \cap \left[\bigcap_{\substack{\pi_1(M, x_0) \xrightarrow{\rho} \mathbb{Z}_2, l \in H^2(M_\rho, \mathbb{Z}) \\ l + \iota^*(l) = \pi_\rho^*(d), l \cdot \iota^*(l) \leq \pi_\rho^*(c)}} \text{Met}_{\geq 2}^r(l - \iota^*(l)) \right]$$

is open, dense and connected in Met^r .

Proof The important point here is that, since the intersection form of M is negative definite, the set of terms in the first intersection is finite. For the second intersection note first that ρ varies in a finite set (isomorphic to the set of index 2 subgroups of $H_1(M, \mathbb{Z})$) and, under our assumption, for any fixed ρ , there are only finitely many possibilities for l . To complete the proofs it suffices to note that the two properties in Proposition A.5 have been obtained by applying the Sard–Smale theorem to certain proper Fredholm maps, and using the fact that the set of regular values of such a map is open and dense. A finite intersection of such sets will also be open and dense. \square

A.3 Analytic results

We begin with the following easy result concerning the range of a proper Fredholm map $f: V \rightarrow W$ of negative index j .

Lemma A.7 *Let V, W be separable Banach manifolds and $f: V \rightarrow W$ a proper, smooth, Fredholm map of negative index $j \leq -1$. Then*

- (1) *The range $f(V)$ of f is closed and nowhere dense,*
- (2) *When $j \leq -2$ the natural map $\pi_i(W \setminus f(V)) \rightarrow \pi_i(W)$ is an isomorphism for $0 \leq i \leq -j - 2$ and is surjective for $i = -j - 1$.*

Note that we do not assume that W is connected. The second statement claims that $W_0 \setminus f(V)$ is connected for every connected component W_0 of W , and that the map $\pi_i(W_0 \setminus f(V), x_0) \rightarrow \pi_i(W_0, x_0)$ is an isomorphism (respectively surjective) for every $x_0 \in W_0 \setminus f(V)$ and $0 \leq i \leq -j - 2$ ($i = -j - 1$).

Proof The first statement follows directly from the Sard–Smale theorem. Since $\text{ind}(f) < 0$ a point $w \in W$ is a regular value for f if and only if $w \notin f(V)$. The second statement is proved using standard transversality arguments (see Donaldson and Kronheimer [3, Section 4.3.2]). We illustrate this method in the case $i = 0$. For this case we have to prove that $\pi_0(W \setminus f(V)) \rightarrow \pi_0(W)$ is injective (the surjectivity follows from (1)). Consider two points $w_0, w_1 \in W \setminus f(V)$ belonging to the same component W_0 of W and connect them with a smooth path $\gamma: [0, 1] \rightarrow W_0$. Using [3, Proposition 4.3.10] we obtain a smooth path $\gamma': [0, 1] \rightarrow W_0$ with $\gamma'|_{\{0,1\}} = \gamma|_{\{0,1\}}$ arbitrary close to γ which is transversal to f . Since $\text{ind}(f) \leq -2$, this transversality condition implies $\text{im}(\gamma') \cap f(V) = \emptyset$. \square

The following proposition plays an important role in the proof of our regularity results. This will allow us to refine the Freed–Uhlenbeck theorems [4] and to estimate the codimension of the spaces of bad metrics.

Proposition A.8 *Let M be an n -dimensional compact manifold, E a real rank r vector bundle and ∇ a linear connection on E . Let $U \subset M$ be an open set whose complement Σ has Hausdorff dimension $d \leq n - 2$. Let $\alpha \in A^1(E)$ and $\varphi \in \Gamma(U, E)$ such that $\nabla\varphi = \alpha|_U$. Then φ extends smoothly to a section $\psi \in \Gamma(M, E)$ satisfying $\nabla\psi = \alpha$.*

The idea of the proof is to use the (classical) methods of characteristics to solve the first order equation $\nabla f = \alpha$: we integrate the family of ordinary equations obtained by restricting our equation to an $(n - 1)$ -dimensional family of embedded paths which define a local foliation. We choose the paths such that all starting points belong to U , and we use the values of φ at these points as initial conditions. The condition on the Hausdorff dimension of Σ implies that a dense family of paths do not meet the set Σ (where φ is not defined), so on these paths the sections f and φ coincide. In this way one checks that the section f obtained by pathwise integration agrees with φ on the intersections of their domains, providing a proper extension of φ .

Proof For every smooth path $\gamma: (-1, 1) \rightarrow M$, consider the connection $\gamma^*(\nabla)$ on the bundle $\gamma^*(E)$ on $(-1, 1)$ and the affine ordinary differential equation

$$(22) \quad \gamma^*(\nabla)g = \gamma^*(\alpha)$$

for sections g in the bundle $\gamma^*(E)$. Using the general theory of ordinary differential equations, one gets, for every $e \in E_{\gamma(0)}$, a unique solution

$$g_{\gamma,e} \in \Gamma((-1, 1), \gamma^*(E))$$

of the equation (22) satisfying the initial condition $g_{\gamma,e}(0) = e$. This solution depends differentiably on the pair (γ, e) , where $e \in E_{\gamma(0)}$. Our hypothesis $\nabla\varphi = \alpha|_U$ implies $\varphi \circ \gamma = g_{\gamma,\varphi(\gamma(0))}$ for every smooth path $\gamma: (-1, 1) \rightarrow U$.

Let $\psi \in \Gamma(V, E)$ be a maximal element of the ordered set of extensions of φ defined on open subsets of M . The existence of such a maximal element follows by the Zorn lemma. Since U is dense in M (so also in V) one has

$$(23) \quad \nabla\psi = \alpha|_V$$

which implies

$$(24) \quad \psi \circ \gamma = g_{\gamma,\psi(\gamma(0))}$$

for every smooth path $\gamma: (-1, 1) \rightarrow V$. The complement $\Sigma' = M \setminus V$ will also have Hausdorff dimension $\leq n - 2$.

We claim that $V = M$. Suppose not, and let $x_0 \in M \setminus V$. Denoting by B^{n-1} the standard ball in \mathbb{R}^{n-1} consider a local parametrization

$$h: B^{n-1} \times (-1, 1) \xrightarrow{\cong} W \subset M$$

of M such that

$$h(B^{n-1} \times \{0\}) \subset V, \quad h(0, \frac{1}{2}) = x_0.$$

The idea is to extend ψ on $V \cup W$ using the solutions $g_{h_x, \psi(h(x,0))}$, $x \in B^{n-1}$, where h_x denotes the path $t \mapsto h(x, t)$.

So put

$$f(x, t) = g_{h_x, \psi(h(x,0))}(t), \quad \eta := f \circ h^{-1}.$$

We claim that $\eta|_{W \cap V} = \psi|_{W \cap V}$, which will complete the proof, because this would yield a proper extension of ψ , contradicting its maximality.

Our claim is equivalent to

$$f|_{h^{-1}(V)} = \psi \circ h|_{h^{-1}(V)}.$$

The two functions coincide on $[B^{n-1} \setminus \text{pr}_1(h^{-1}(\Sigma'))] \times (-1, 1)$, because, for any $x \in [B^{n-1} \setminus \text{pr}_1(h^{-1}(\Sigma'))]$ the corresponding path h_x is entirely contained in V so, for such x , both sections

$$t \mapsto f(x, t), \quad t \mapsto \psi(h(x, t))$$

coincide with $g_{h_x, \psi(h(x,0))}$, by (24) and the definition of f . It suffices to notice that, by our hypothesis, the set $h^{-1}(\Sigma')$ is of Hausdorff dimension at most $n - 2$, so its projection on the $(n - 1)$ dimensional ball B^{n-1} is also of Hausdorff dimension at most $n - 2$. Therefore, this projection cannot contain any non-empty open set, so its complement in B^{n-1} is dense, so $[B^{n-1} \setminus \text{pr}_1(h^{-1}(\Sigma'))] \times (-1, 1)$ is dense in $B^{n-1} \times (-1, 1)$, so the two functions coincide everywhere. \square

The following corollary shows that the statement of [4, Lemma 4.16] is true as soon as the vanishing of the ASD curvature has Hausdorff dimension ≤ 2 . In particular this statement is true for C^∞ -metrics by the results of Bär [1] and for admissible metrics.

Corollary A.9 *Let (M, g) be a compact oriented Riemannian 4-manifold endowed with a C^r -metric g , and S a Hermitian line bundle on M endowed with a non-flat ASD connection σ . Suppose that the vanishing of the curvature has Hausdorff dimension ≤ 2 . Let $\beta \in A^1(S)$ such that*

$$(1) \quad d_\sigma^* \beta = 0, \quad d_\sigma^+ \beta = 0,$$

- (2) The anti-selfdual S -valued form $d_\sigma \beta$ is a tensor multiple of F_σ at any point $x \in M$ for which $F_{\sigma,x} \neq 0$.

Then $\beta = 0$.

Proof Let U be the complement of the vanishing locus Σ of F_σ . By assumption we can write $d_\sigma \beta|_U = F_\sigma \otimes \zeta$, for a section $\zeta \in \Gamma(U, S)$. We get

$$[F_\sigma|_U] \wedge (\beta|_U - d_\sigma \zeta) = d_\sigma d_\sigma (\beta|_U - d_\sigma \zeta) = d_\sigma (d_\sigma \beta|_U - F_\sigma \otimes \zeta) = 0,$$

hence $\beta|_U - d_\sigma \zeta = 0$, since the wedge product with a non-trivial (real or purely imaginary) ASD form is invertible on 1-forms. By Proposition A.8 and the assumption on $Z(F_\sigma)$, the section ζ extends smoothly to a section ξ on M satisfying $d_\sigma \xi = \beta$. Since we assumed $d_\sigma^* \beta = 0$, we get immediately $\beta = 0$. \square

Proposition A.10 Let S be a Hermitian line bundle on an oriented Riemannian 4-manifold (M, g) and σ a Hermitian connection on S . Let $\eta \in A_+^2(S)$ with $d_\sigma \eta = 0$. Suppose that on an open set $U \subset M$, the form η (regarded as section in $\Lambda_+^2 \otimes S$) has real rank 1. Then $F_\sigma|_U = 0$.

Proof Supposing that U is simply connected, we can write $\eta = \omega \otimes \zeta$, where $\omega \in A_+^2(U)$ is a real selfdual form, and $\zeta \in \Gamma(U, S)$. By assumption, both ω and ζ are nowhere vanishing on U . Since $d_\sigma \eta = 0$, we get

$$d\omega \otimes \zeta + \omega \wedge d_\sigma \zeta = 0,$$

hence $\omega \wedge (\theta \otimes \zeta + d_\sigma \zeta) = 0$, where θ is the real form on U defined by $d\omega = \omega \wedge \theta$. We get

$$d_\sigma \zeta = -\theta \otimes \zeta, \quad F_\sigma \otimes \zeta = -d\theta \otimes \zeta + \theta \wedge d_\sigma \zeta = -d\theta \otimes \zeta - (\theta \wedge \theta) \otimes \zeta = -d\theta \otimes \zeta.$$

This yields $F_\sigma = -d\theta$, in which the left hand term is purely imaginary and the right hand term is real. \square

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LATP, CMI, Université de Provence
39 Rue F Joliot-Curie, 13453 Marseilles Cedex 13, France

teleman@cmi.univ-mrs.fr

<http://www.cmi.univ-mrs.fr/~teleman/>

Proposed: Simon Donaldson
Seconded: Ron Stern, Gang Tian

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