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A class of non-fillable contact structures

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A geometric obstruction, the so called “PS–structure”, for a contact structure to not being fillable has been found by Niederkrüger [9]. This generalizes somehow the concept of overtwisted structure to dimensions higher than 3. This paper elaborates on the theory showing a big number of closed contact manifolds having a "PS–structure". So, they are the first examples of non-fillable high dimensional closed contact manifolds. In particular we show that $S^3 \times \prod_j \Sigma_j$, for $g(\Sigma_j) \geq 2$, possesses this kind of contact structure and so any connected sum with those manifolds also does it.

57R17; 53D10

1 Introduction

Since the mid-eighties there have been two clear categories of 3–dimensional contact manifolds: overtwisted and tight. The first ones were introduced by Eliashberg, following Gromov ideas, and happened to satisfy a kind of h–principle making their study a mere homotopical question [2]. Tightness has been more evasive being a coarse classification a matter of only very recent research and being the study purely ”geometrical” with no hope of h–principle flexibility.

A key result to understand this overtwisted class is a corollary of Gromov's pseudo-holomorphic curves foundational paper [7]:

Theorem 1.1 *Every overtwisted manifold is non-fillable.*

Several definitions can be given for fillability. The most restrictive one is Stein-fillable. A contact manifold is Stein-fillable if it can be expressed as a level hypersurface for a proper plurisubharmonic function on a Stein manifold. It is very easy to check that locally, forgetting about the complex structure, a neighborhood of the manifold is symplectomorphic to a neighborhood of the 1–section of the symplectization $(C\times(0,\infty), d(e^t\alpha))$. This provides another definition. A contact manifold (C,α) is symplectically fillable if there exists a symplectic manifold (M,ω) with boundary ∂M such that a neighborhood U of the boundary is symplectomorphic

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to $(C\times(1-\epsilon,1],d(e^t\alpha))$. There is also a concept of weak filling where the contact manifold (C,α) is the boundary of a symplectic manifold (M,ω) satisfying that $\ker\alpha$ is a symplectic subspace for ω. This is usually referred to as α being dominated by ω. Again this is a weaker condition than being symplectically fillable. Also, in dimension 3, it has been proved that these inclusions are strict, so we have

$$\text{Stein-fillable} \subsetneq \text{Symplectically-fillable} \subsetneq \text{Weakly-fillable} \subsetneq \text{Tight}.$$

Theorem 1.1 works for weakly symplectic fillings. Finally the division *tight-overtwisted* does not translate in a completely faithful way to the fillings since some examples of tight manifolds not having fillings at all have been found . However, in most of the cases tight manifolds are fillable.

All the previous theory has been established only in dimension 3, since many of the examples of contact structures in higher dimension are expected to be Stein-fillable. The main reason is that a good generalization of the concept of overtwisted to higher dimensions has been hard to find in the last 20 years. M Gromov gave a natural candidate in his foundational article [7]. The first successful attempt to completely develop this idea has been Niederkrüger [9]. There, it has been shown that the existence of a special geometrical structure on a contact manifold automatically implies the non-fillability of it. This generalizes Theorem 1.1 and shows that the definition makes some sense. Let us show it.

Definition 1.2 A PS–structure in a contact manifold (C^{2n-1},α) is an embedding $\mathcal{P}(S)=D^2\times S\hookrightarrow C$, where S is a closed $(n-2)$–dimensional manifold satisfying that

- (i) $B=\{0\}\times S$ is tangent to the contact distribution $D=\ker\alpha$.
- (ii) $S_D=T\mathcal{P}(S)\cap D$ defines a singular distribution on $\mathcal{P}(S)$ which has a singular set defined by B.
- (iii) S_D is an integrable distribution on $\mathcal{P}(S)$, in fact a Legendrian foliation.
- (iv) The boundary $\partial\mathcal{P}(S)=S^1\times S$ is a closed leaf of the foliation S_D, and so it is a Legendrian submanifold.
- (v) B is a family of singularities of elliptic type, that is, any small 2–dimensional disk D_ϵ, transverse to B, has an induced distribution $D\cap TD_\epsilon$ with a singularity of elliptic type.
- (vi) The rest of the leaves of the foliation S_D are of the form $S\times(0,1)$ satisfying that their boundary is contained in $B\cup\partial\mathcal{P}(S)$

A manifold with a PS–structure is called PS–overtwisted.

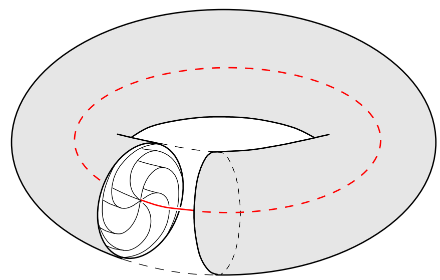

Figure 1: $\mathcal{P}(S^1)$ PS–structure in a 5–dimensional manifold

Recall that a 2–dimensional PS–structure is just an overtwisted disk. So far, it has been proved:

Theorem 1.3 (Niederkrüger [9, Theorem 1]) *A PS–overtwisted manifold does not admit a semi-positive symplectic filling.*

We mean by a semi-positive filling a weak symplectic filling given by a semi-positive symplectic manifold. This covers in particular Stein fillings and exact symplectic ones. It is expected to have a result covering all the possible fillings. The main obstruction is technical and is due to not have used ”virtual cycles” in the proof of Theorem 1.3. In the particular case of dimension 5 all the fillings can be made ”semi-positive” and so the Theorem works for any filling. We remark that in Gromov [7] some sketches of the proof of a result similar to Theorem 1.3 were already given. Moreover, Yuri Chekanov also has an alternative proof to Theorem 1.3.

This paper has the task of showing the existence of a very ample class of PS–overtwisted contact manifolds. The main result we will prove is

Theorem 1.4 *Assume that a contact manifold (M', α) admits a codimension 2 contact submanifold N, with trivial normal bundle, such that the restricted contact form is PS–overtwisted, then $M' \bigcup_N (N \times T^2)$ admits a PS–overtwisted contact structure.*

Later we will see that using Theorem 1.4 is simple to construct quite a lot families of high dimensional examples. For instance

Corollary 1.5 *The manifold $S^3 \times \prod_i \Sigma_i$, where Σ_i is a Riemann surface with genus at least 2, admits a PS–overtwisted contact structure.*

And so we immediately obtain:

Corollary 1.6 *The connected sum of any contact manifold with $S^3 \times \prod_i \Sigma_i$ admits a PS–overtwisted contact structure.*

In dimension 5, we obtain:

Corollary 1.7 *The manifold $M_o = (S^3 \times T^2) \bigcup_{S^3} S^5$ admits a PS–overtwisted contact structure.*

Recall that this manifold has homology $H_2(M_o) = H_3(M_o) = 0$ and $H_1(M_o) = H_4(M_o) = \mathbb{Z}^2$. So, it is not a sphere but it has a simple homology type. It is reasonable to conjecture:

Conjecture 1.8 *S^{2n+1} admits a PS–overtwisted structure and so any contact manifold admits an exotic PS–overtwisted structure.*

This conjecture has been already proved by Niederkrüger and van der Koert [10]. This partially recovers the familiar 3–dimensional picture in which for any contact structure, it is possible to construct an overtwisted one homotopically equivalent as a plane distribution. Recall that Corollary 1.7 gets close to this conjecture in dimension 5.

Finally observe that no examples of non-fillable contact structures had been found in dimension higher than 3. All the previous results are a huge source of such structures.

In Section 2 we define a notion of parallel transport for contact fibrations which is a key step in our proof. The relation between a contact fibration and the group of Hamiltonian contactomorphisms is important for our purposes and it is studied in Section 3. Next, we review some old and new constructions of contact structures in Section 4. Section 5 just places together all the previous results to prove the main theorem. Finally some Corollaries of the main theorem are obtained in Section 6. In particular we give examples of high dimensional manifolds which admit contact structures both fillable and non-fillable lying in the same homotopy class of hyperplane fields.

The important objects in the proofs will be the contact structures and the contact forms used all along the way will be auxiliary. This can be a bit confusing. Sometimes we mention that the contact forms are isotopic, this really means that the associated contact structures are isotopic; because they are the ones to which Gray's stability theorem applies.

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2 Contact connections

2.1 Review of the symplectic case

We briefly review a classical construction in symplectic geometry as a motivation for our proofs.

Definition 2.1 Let $\pi\colon M \to B$ be a fibration such that M admits a symplectic structure ω satisfying the fibers $F = \pi^{-1}(b)$ of π are symplectic submanifolds, then the distribution $H = (TF)^{\perp\omega}$ is called the symplectic connection associated to the symplectic fibration (M, B, π, ω).

The main result is

Proposition 2.2 *Let $\pi\colon M \to B$ be a symplectic fibration with associated symplectic connection H. For each $p_0 \in M$, H allows us to lift any path $\gamma\colon [0,1] \to B$, such that $\pi(p_0) = \gamma(0)$, to a unique path $\widetilde{\gamma}_{p_0}\colon [0,1] \to M$ satisfying:*

- $\widetilde{\gamma}_{p_0}(0) = p_0$
- $\pi \circ \widetilde{\gamma}_{p_0} = \gamma$
- $\widetilde{\gamma}'_{p_0}(t) \in H_{\widetilde{\gamma}_{p_0}(t)}$

Moreover, for γ immersed, the induced map

$$\begin{aligned} m\colon F_0 = \pi^{-1}(\gamma(0)) &\to F_1 = \pi^{-1}(\gamma(1)) \\ p &\to \widetilde{\gamma}_p(1) \end{aligned}$$

is a symplectomorphism.

Proof The first part of the statement, as usual for any connection on a fibration, consists of solving the equation:

$$\begin{aligned} \pi_*(\widetilde{\gamma}'_{p_0}(t)) &= \gamma'(t) \\ \widetilde{\gamma}'_{p_0}(t) &\in H \end{aligned}$$

From $\widetilde{\gamma}'$ we obtain $\widetilde{\gamma}$ as a consequence of the theorem of existence and uniqueness of solutions of ODEs.

For the second part we define the submanifold with boundary $P = \pi^{-1}(\gamma([0,1]))$ that is an immersed submanifold whenever γ is immersed. Define a vector field X in P as

$$\begin{aligned} d\pi(X(p)) &= \gamma_*\left(\tfrac{d}{dt}\right), \\ X(p) &\in H(p). \end{aligned}$$

This vector field preserves the fibers $F_t = \pi^{-1}(\gamma(t))$. We want to show that this field infinitesimally preserves the symplectic form, this is

$$L_X \omega_{|P} = 0. \tag{1}$$

Using the Cartan formula we obtain

$$L_X \omega_{|P} = d i_X \omega_{|P} + i_X d\omega_{|P} = d i_X \omega_{|P}.$$

Now recall that $X \in H$ and H is symplectically orthogonal to TF_t, so $i_X \omega_{|P}(v) = 0$, $\forall v \in TF_t$. Finally we have $i_X \omega(X) = 0$ and therefore $d i_X \omega_{|P} = 0$ as we wanted to show. So, being true formula (1), we obtain that m is a symplectomorphism. □

Remark 2.3 We need to assume that the vector fields are complete in the previous proposition. It will always be the case in the applications since we will suppose that the symplectic form is far from being degenerate, that is, $\omega^n > \delta > 0$ for a fixed constant δ at all the points where the fibration is defined. This is a sufficient condition to make the connection complete, meaning by this that the lift of a complete field on the base is complete.

2.2 The contact case

We want to adapt the previous language to the contact situation. We denote $D = \ker \alpha$.

Definition 2.4 Let $\pi\colon C \to B$ be a fibration such that C admits a contact structure α satisfying the fibers $F = \pi^{-1}(b)$ of π are contact submanifolds, then the distribution $H = (TF \cap D)^{\perp d\alpha} \subset D$ is called the contact connection associated to the contact fibration (C, B, π, α).

Recall that D is transverse to the fibers if they are contact and so the previous definition makes sense. Again we have that the natural monodromy preserves the contact structure on the fibres:

Proposition 2.5 *Let $\pi\colon C \to B$ be a contact fibration with associated contact connection H. For each $p_0 \in C$, H allows us to lift any path $\gamma\colon [0,1] \to B$, such that $\pi(p_0) = \gamma(0)$, to a unique path $\widetilde{\gamma}_{p_0}\colon [0,1] \to M$ satisfying:*

- $\widetilde{\gamma}_{p_0}(0) = p_0$
- $\pi \circ \widetilde{\gamma}_{p_0} = \gamma$
- $\widetilde{\gamma}'_{p_0}(t) \in H_{\widetilde{\gamma}_{p_0}(t)}$

Moreover, for γ immersed, the induced map

$$m\colon F_0 = \pi^{-1}(\gamma(0)) \to F_1 = \pi^{-1}(\gamma(1))$$
$$p \to \widetilde{\gamma}_p(1)$$

is a contactomorphism.

Proof Again the first part of the statement is the usual lift defined for any connection on a fibration.

For the second part we mimic the proof of Proposition 2.2 defining the submanifold with boundary $P = \pi^{-1}(\gamma([0,1]))$. It is an immersed submanifold whenever γ is immersed. Define a vector field X in P as

$$d\pi(X(p)) = \gamma_*\left(\tfrac{d}{dt}\right),$$
$$X(p) \in H(p).$$

This vector field preserves the fibers of $F_t = \pi^{-1}(\gamma(t))$. We want to show that this field infinitesimally preserves the contact structure, that is equivalent to

$$L_X \alpha_{|P} = f\alpha_{|P}, \tag{2}$$

for some function f. We claim that f has to be chosen to be $f = d\alpha(X, R_{F_t})$, where R_{F_t} is the Reeb vector field associated to the fiber F_t. Using the Cartan formula we obtain:

$$L_X\alpha_{|P} = di_X\alpha_{|P} + i_X d\alpha_{|P} = i_X d\alpha_{|P},$$

because $X \in D = \ker\alpha$. Now recall that $X \in H$ and H is symplectically orthogonal to $TF_t \cap D$, so $(i_X d\alpha_{|P})(v) = 0$, $\forall v \in TF_t \cap D$. We also have $i_X d\alpha(X) = 0$. So we have that the equation (2) is true for all $v \in D \cap TP$. To finish we check it against R_{F_t} and we obtain

$$(L_X\alpha)(R_{F_t}) = d\alpha(X, R_{F_t}) = d\alpha(X, R_{F_t})\alpha(R_{F_t}) = f\alpha(R_{F_t}).$$

Therefore the formula (2) is true and m is a contactomorphism. □

Again a condition similar to the one in Remark 2.3 must be assumed.

Remark 2.6

(i) The contact connection depends on the choice of contact form and so different contact forms supported on the same contact structure give rise to different monodromies. Monodromies are isotopic inside the contactomorphism group for homotopic contact structures.

(ii) Let us analyze the most simple case. Choose a contact manifold (C_0, α_0) and define on $C_0 \times D^2$ the standard product structure, which in polar coordinates is written as

$$\alpha = \alpha_0 + r^2 d\theta. \tag{3}$$

It is simple to check that the Reeb vector field R_0 restricts to the Reeb vector field of each fiber. Therefore $f = 0$ in formula (2) and so the parallel transport also preserves the contact form itself (not just its kernel). If we take a path $\gamma\colon [0, 2\pi] \to D^2$ defined as $\theta \to (r(\theta), \theta)$ we obtain that

$$\gamma' = \frac{\partial r}{\partial \theta}\frac{\partial}{\partial r} + \frac{\partial}{\partial \theta},$$

and the lift of this field is

$$X = \frac{\partial r}{\partial \theta}\frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} - r^2 R_0.$$

Thus, the monodromy at time θ is generated (in the vertical direction) by the vector field $-r^2 R_0$. Therefore, after integration, the monodromy is given by the flow of the Reeb vector field in the fiber at time

$$t = -\int_0^{2\pi} r^2(\theta) d\theta = -2Area(\gamma),$$

where $Area(\gamma)$ is the area of the domain bounded by the closed curve γ.

2.3 Parallel transport of PS–structures

Our next task is to understand how a PS–structure behaves under parallel transport. We just want to check two elementary facts.

Lemma 2.7 *Let $\pi\colon (C, \alpha) \to B$ be a contact fibration with associated contact connection H. Let N be an isotropic submanifold of F_b for some $b \in B$ and $\gamma\colon [0,1] \to B$ an immersed path such that $\gamma(0) = b$. Denote m_t the parallel transport induced by γ between F_0 and F_t, then $m(N) = \bigsqcup_{t=0}^{1} m_t(N)$ is an immersed isotropic submanifold of C.*

Proof The new manifold $m(N)$ admits the tangent decomposition

$$Tm(N)_p = (m_t)_*(TN) \bigoplus \langle \widetilde{\gamma}'(t) \rangle.$$

m_t is a contactomorphism and therefore $(m_t)_*(TN) \subset D$. Moreover, the parallel transport is defined by trajectories following H that is a subspace of $D = \ker \alpha$ and so $\widetilde{\gamma}'(t) \in D$. Thus, $Tm(N) \subset D$ and so $m(N)$ is an isotropic submanifold. □

Remark 2.8

(i) The new submanifold is embedded if γ is embedded or if the contactomorphism m_t separates N from itself in all the crossing fibers.

(ii) If N is Legendrian and $\dim B = 2$ then $m(N)$ is Legendrian, just because of the dimensions.

Lemma 2.9 *Let $\pi\colon (C,\alpha) \to B$ be a contact fibration with contact connection H and assume that $\dim B = 2$. Let $\mathcal{P}(S)$ be a PS–structure in F_0 and $\gamma\colon S^1 \to B$ a closed immersed (resp. embedded) path starting at b_0. Assume that the monodromy $m_1\colon F_0 \to F_1 = F_0$ restricts to the identity on the PS–structure, then $m(\mathcal{P}(S)) = \bigsqcup m_t(\mathcal{P}(S))$ is an immersed (resp. embedded) PS–structure with core $S \times S^1$ in C.*

Proof Define $D_0 = D \cap TF_0$. Taking into account that the parallel transport follows D, it is easy to check that the parallel transport of the distribution, indeed foliation, $D_{\mathcal{P}(S)} = D_0 \cap T\mathcal{P}(S)$ generates the distribution $D_{m(\mathcal{P}(S))} = D \cap Tm(\mathcal{P}(S))$. Recall that given a flow ϕ_t and an integrable distribution $\mathcal{D}$ the object $\phi_t(\mathcal{D})$ is still an integrable distribution, moreover $\bigsqcup \phi_t(\mathcal{D})$ is still integrable. Since the parallel transport is a flow we have that $D_{m(\mathcal{P}(S))}$ is a foliation (the distribution remains integrable), still by isotropic leaves because of Lemma 2.7.

On the other hand the core S of $\mathcal{P}(S)$ generates by parallel transport the core $S \times S^1$ of the new PS–structure. Finally the boundary transports to create a new Legendrian boundary, again thanks to Lemma 2.7. □

The previous results show that it is very simple to create immersed PS–structures. In fact, it can be done in a very local picture.

Proposition 2.10 *Let (C,α) be a contact manifold and $N \subset C$ a PS–overtwisted codimension 2 contact submanifold of it with trivial symplectic normal bundle. Then there exists an immersed PS–structure of C on an arbitrary small neighborhood of N.*

Proof As a consequence of the Gray's stability theorem, there exists a neighborhood U of N and a contactomorphism $\phi\colon U \to N \times D^2$, satisfying that the target space is equipped with the standard contact distribution defined as in formula (3). We fix this standard form in the fibration to define a contact connection H.

Fix a curve $\gamma\colon S^1 \to D^2$ satisfying:

(i) $\gamma(0) = 0$

(ii) $\gamma(p) = -\gamma(p+\pi)$ (radial symmetry)

(iii) γ is immersed, with a unique self-intersection point $\gamma(0) = \gamma(\pi) = 0 \in D^2$

Because of (ii) and (iii), the domain bounded by γ has total area 0. Therefore the monodromy map, according to Remark 2.6 (iii), is the identity. Now we use Lemma 2.9 to construct an immersed PS–structure $\mathcal{P}(S \times S^1)$ in $N \times D^2$ out of the one contained in the fiber $N \times \{0\}$. Using ϕ^* we get a PS–structure in a neighborhood of N. Recall that γ can be chosen arbitrarily close to $\{0\} \in D^2$, thus the PS–structure can be constructed arbitrarily close to N. □

Remark 2.11 The PS–structure would be embedded if the monodromy separates the initial PS–structure from itself at the crossing fibers. For instance, assume that we are in the 5 dimensional case, therefore the fiber is 3 dimensional and the initial PS–structure is a classical overtwisted disk. In our previous lemma there is only one crossing point at time π. Recall that the monodromy at that time is given by the flow of the Reeb vector field at a time t_0. This time t_0 is given by minus twice the area of the domain bounded by the closed curve $\gamma\colon [0,\pi] \to D^2$. So any negative value is possible (up to a limit given by half of the area of D^2). Now we choose as α_0, among all the possible options, one extending the canonical contact form of the canonical neighborhood of the overtwisted disk given by

$$\alpha_{ot} = \cos(r) dz + r \sin(r) d\theta.$$

Recall that the overtwisted disk is given by a slight perturbation of $D(2\pi) \times \{0\}$. The Reeb vector field is

$$R = \frac{\sin r + r \cos r}{r + \sin r \cos r} \frac{\partial}{\partial z} + \frac{\sin r}{r + \sin r \cos r} \frac{\partial}{\partial \theta}$$

and it is radially invariant. Moreover its z–component is positive at $r = 0$ and negative at $r = 2\pi$. This implies that the image of the disk through the flow at any time intersects the initial disk along a circle. It is impossible to separate the disk using the monodromy from itself. One can check that any other choice of contact form offers the same result as there is a topological obstruction to separate the disk from itself using the Reeb flow for small times.

The previous result is not unexpected. Also in dimension 3, it is simple to build immersed overtwisted disks for any contact 3–dimensional manifold: those disks exist inside the standard contact ball. The problems arise when one tries to construct an embedded one. To do that with a PS–structure we need to add some new ideas and to change a bit the topological picture.

3 Contact Hamiltonians as contact monodromies

We are going to show how to produce an arbitrary Hamiltonian contactomorphism by parallel transport on a cleverly chosen contact fibration. Again the ideas of these results come from the symplectic fibrations case. However, we do not review that theory because all our results are clear enough without referring to the symplectic counterparts.

3.1 Review of basic definitions and results

Definition 3.1 The Hamiltonian vector field X associated to the Hamiltonian function $H\colon C \to \mathbb{R}$ is the unique vector field defined by the pair of equations:

$$i_X \alpha = H \tag{4}$$

$$i_X d\alpha = (d_R H)\alpha - dH \tag{5}$$

A computation shows that for any Hamiltonian field $L_X \alpha = (d_R H)\alpha$, therefore the field preserves the contact structure. Moreover:

Theorem 3.2 *Any contactomorphism connected to the identity, in the contactomorphism group, is the flow of a time dependent Hamiltonian vector field.*

Proof If the contactomorphism is connected to the identity, it is generated by a time-dependent vector field X_t. Choose $H_t = i_{X_t}\alpha$ and check that the associated Hamiltonian contactomorphism coincides with X_t. □

Lemma 3.3 *A contactomorphism C^2–close to the identity (inside the contactomorphisms group $\mathrm{Cont}(C)$) can be generated by a C^1–small time dependent Hamiltonian function.*

Proof Take $\phi_t\colon C \to C$ a contactomorphism. By hypothesis, it can be generated by a vector field $\{X_t\}_{t=0}^1$ satisfying

$$\left(\frac{\partial \phi_t(p)}{\partial t}\right)_{t=t_0} = X_t(\phi_t(p)).$$

Thus a bound in the C^2–norm of ϕ_t provides a bound in the C^1–norm of X_t. Therefore the equation (4) provides also bounds for the C^1–norm of H_t. Being precise we obtain that $||X_t||_{C^2} \leq \delta$ implies that $||H_t||_{C^1} \leq M\delta$, and M is a constant that only depends on the contact form α. □

3.2 Hamiltonians as monodromies

The following result clarifies a general geometric picture, though we restrict our proof to the very particular case we will need

Proposition 3.4 *Let $\pi\colon (C,\alpha_0)\times D^2(\delta)\to D^2(\delta)$ be the product contact fibration. For each α_0 and $\delta_0>0$, there exists a small enough constant $\epsilon>0$ such that for any $\delta>\delta_0$ and for any Hamiltonian function $\{g_t\}_{t=0}^1\colon C\to\mathbb{R}$ satisfying*

$$|g|\le\epsilon, |dg|\le\epsilon,$$

where t is considered as another coordinate for the understanding of the second equation, there is a contact form $\overline{\alpha}$ isotopic to the initial one, through contact structures, such that the path (given in polar coordinates)

$$\begin{aligned}\gamma\colon [0,\delta] &\to D^2(\delta)\\ t &\mapsto \left(\tfrac{1}{2}\left(t+\tfrac{1}{2}\delta\right),0\right)\end{aligned}$$

has as associated monodromy the contactomorphism generated by the Hamiltonian $\{g_t\}$. Moreover the deformation of the contact form can be supported in the set $[\delta/4,3\delta/4]\times[-\pi/4,\pi/4]$ (this neighborhood is written in polar coordinates).

Proof Let ϕ_t be the contact flow associated to the Hamiltonian g_t. Recall that by performing a reparametrization $t=t(s)$ (not necessarily a diffeomorphism) we obtain $\widehat{\phi}_s=\phi_{t(s)}$. Just choose the reparametrization in such a way that $\widehat{\phi}_s=\phi_0=id$ for $s\in[0,\beta]$ and $\widehat{\phi}_s=\phi_1$ for $s\in[1-\beta,1]$ for a small $\beta>0$. The Hamiltonian generating function changes to $\widehat{g}_s=\frac{dt}{ds}g_{t(s)}$. Taking β small enough this just increases the derivatives of $\widehat{g}_s$ slightly. Therefore we may assume that g_t satisfies that $g_t=0$ for $t\in[0,\beta]\cup[1-\beta,1]$ at the cost of decreasing slightly the constant ϵ in our statement. This allows us to extend the definition of g_t as $\{g_t\}_{t=-\infty}^{\infty}\colon C\to\mathbb{R}$ with $g_t=0$ for $t\notin(\beta,1-\beta)$.

Let us start with the case $\delta=2$, we solve the general case later. Choose a cut-off function $\xi\colon\mathbb{R}\to\mathbb{R}$ satisfying:

(i) $\xi(t)=1$ for $t\in(-\pi/8,\pi/8)$

(ii) $\xi(t)=0$ for $|t|\ge\pi/4$

(iii) $0\le\xi(t)\le1$, for all $t\in\mathbb{R}$

(iv) $|\xi'(t)|\le4$

Define $\tilde{g}(r,\theta,p) = \xi(\theta)\cdot g_{r-1/2}(p)$ which is globally defined in $C\times D^2(2)$. Now we can define

$$\overline{\alpha} = \alpha_0 + r^2 d\theta - \tilde{g} dr.$$

It is obvious that, for ϵ small enough, this is a contact form isotopic to the standard contact form α, just by linear interpolation.

The exterior differential of $\overline{\alpha}$ is

$$d\overline{\alpha} = d\alpha_0 + 2r dr \wedge d\theta - d\tilde{g} \wedge dr.$$

We impose the conditions of Proposition 2.5 to lift the vector $\frac{\partial}{\partial r}$. This implies that $X = \frac{\partial}{\partial r} + v$, where $v \in TC$. Moreover we also have:

- $i_X \overline{\alpha} = 0$, which translates to

$$i_v \alpha_0 = \tilde{g}. \tag{6}$$

- $i_X d\overline{\alpha}(w) = 0$, for all $w \in D_0$, which translates to

$$i_v d\alpha_0(w) + d_w \tilde{g} = 0, \forall w \in D_0. \tag{7}$$

We can extend the equation (7) from D_0 to TC obtaining

$$i_v d\alpha_0 = s\alpha_0 - d\tilde{g}, \tag{8}$$

where $d\tilde{g}$ is the exterior differential of $\tilde{g}$ restricted to TC. We substitute the Reeb vector field on equation (8) to obtain $s = d_{R_0}\tilde{g}$. Thus, equations (6) and (8) are the equations for v to be the Hamiltonian vector field associated to the Hamiltonian function $\tilde{g}(r,\cdot,\cdot)$ for $r\in[1/2,3/2]$. We are restricted to the path, in polar coordinates $[1/2,3/2]\times\{0\}$ and therefore we have that $\tilde{g}(r,0,\cdot) = g_{r-1/2}$ that is what we wanted.

Now we go to the case of a general $\delta > 0$. We can perform a change of parameter in the associated contact flow ϕ_t to $\hat{\phi}_s = \phi_{s/\delta}$, for this we change the generating Hamiltonians to $\hat{g}_s = (g_{s/\delta})/\delta$. We can follow the previous proof from that point since the new Hamiltonian functions $\hat{g}_s$ are now supported for $s\in[0,\delta]$. At the end of the day we obtain that the new contact form $\overline{\alpha}$ generates the required contactomorphism ϕ_1 as the monodromy generated over the path $[1/2\delta, 3/2\delta]\times\{0\}$. The only remark to be made is that the Hamiltonians $\hat{g}$ have as C^1–norm the C^1–norm of g multiplied by δ^{-1}. Therefore the constant $\epsilon > 0$ to be chosen to make $\overline{\alpha}$ isotopic to α depends on δ as announced in the statement of the Proposition. □

4 Review of some constructions

We need to review three constructions in order to complete our insight of the problem.

4.1 Contact structures on $C \times T^2$

We briefly review the idea of the proof of the main result in Bourgeois [1]. We need to introduce:

Definition 4.1 An open book associated to a manifold M is a pair (ϕ, N) such that:

- N is a codimension 2 submanifold with trivial normal bundle.
- $\phi\colon M - N \to S^1$ is a regular fibration.

Definition 4.2 A compatible open book (ϕ, N) of a contact manifold (C, α) is a topological open book satisfying

- The fibers of ϕ are symplectic with respect to the restriction of the closed 2–form $d\alpha$.
- The submanifold N is contact.
- The Reeb vector field is tangent to N.

Theorem 4.3 (Giroux [6]) *Any cooriented contact manifold (C, D) supports a compatible open book for some contact form α associated to D.*

Now the result we want to review is

Theorem 4.4 (Bourgeois [1, Theorem 1]) *Let (C, α) a contact manifold. There is a contact structure on $C \times T^2$ which is T^2–invariant.*

Proof F Bourgeois just takes an open book decomposition compatible with (C, α) as constructed in Theorem 4.3. Assume that $\dim C = 2n - 3$. To prove Theorem 4.4 we take a compatible open book (ϕ, N) and $\phi\colon N \to S^1 \subset \mathbb{C}$ is multiplied by a function ρ constructed as a distance to N (constant outside a neighborhood of N). So we get a new map $\Phi = \rho \cdot \phi = (\Phi_1, \Phi_2)$. Then the contact form in the product is just

$$\alpha_\epsilon = \epsilon(\Phi_1 d\theta_1 + \Phi_2 d\theta_2) + \beta, \tag{9}$$

where (θ_1, θ_2) are linear coordinates of the torus and β is the contact form on C. It is a simple computation to write down

$$\begin{aligned}\alpha_\epsilon \wedge (d\alpha_\epsilon)^{n-1} = {}& -\epsilon^2 (n-1)(d\beta)^{n-2} \wedge \rho^2 d\phi \wedge d\theta_1 \wedge d\theta_2 \\ & - \epsilon^2 (n-1)(n-2)\beta \wedge (d\beta)^{n-3} \wedge \rho d\rho \wedge d\phi \wedge d\theta_1 \wedge d\theta_2.\end{aligned}$$

The two expressions are equal or greater than zero and one is strictly bigger than zero away from N and the other one in a neighborhood of it. In the original paper $\epsilon = 1$, but realize that for any $\epsilon \neq 0$ positive or negative the statement is still true. This will be important for our purposes. □

We can easily check that

Lemma 4.5 *Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the standard 2–torus. There is a constant $M > 0$ such that for any path $\gamma\colon [0,\delta] \to T^2$, parametrized by the arc-length satisfying $|\gamma''| < 1$, the monodromy generated by γ, for the connection associated to α_ϵ, is a contactomorphism generated by a Hamiltonian function $(H_t)_{t=0}^1$ satisfying*

$$|H_t| \leq M|\epsilon|\delta, |dH_t| \leq M|\epsilon|\delta.$$

Proof It is a straightforward computation. Just take into account that the constant M depends on the norms of α, Φ_1, Φ_2 and some of their derivatives. □

4.2 Connected sum of contact fibrations

We briefly recall a result of Geiges [5]. We keep the proof, but we adapt the details to our language. The result is

Theorem 4.6 *The fibered sum of contact fibrations over surfaces, satisfying*

- *the fibers of each fibration are contactomorphic,*
- *the normal bundles have opposite orientations,*

is a contact fibration.

Proof Denote $C_j \to B_j$ the contact fibrations with contact forms α_j ($j = 1, 2$). Denote by $F_0^j = \pi_j^{-1}(b_j)$ the pair of fibers we are going to identify. Moreover, take the connection H_j on the fiber F_0^j as a geometric realization of the normal bundle. We may assume that the fibration can be trivialized to the standard product picture given by

$$\alpha_j = \alpha_0 + \rho(-1)^j r^2 d\theta, \tag{10}$$

where α_0 is the contact form on the fiber and (r,θ) are polar coordinates on the disk $D_j^2(0,1)$. This is always possible after small perturbation of the contact form, provided the constant $\rho > 0$ is chosen to be small enough (recall that it is usual to assume the radius small enough, but this is equivalent after scaling to fix the radius to 1 and to introduce ρ). Now we embed $D_j^2(0,1) \subset \mathbb{R}^2 \times \{(-1)^j\} \subset \mathbb{R}^3$. Moreover we construct a hypersurface S on $(\mathbb{R}^+ - \{0\}) \times \mathbb{R}$ (with coordinates (r,z)) defined by the zero set of a function $F\colon (\mathbb{R}^+ - \{0\}) \times \mathbb{R} \to \mathbb{R}$ satisfying that its zero set contains the hyperplanes $\{z = \pm 1\}$ for $r \geq 1/2$. Moreover we impose that F is transverse to zero and

$$\begin{cases} \frac{\partial F}{\partial r} \geq 0 \\ \frac{\partial F}{\partial r} > 0, z = 0, \\ \left(\frac{\partial F}{\partial z}\right) \cdot z \leq 0. \end{cases} \tag{11}$$

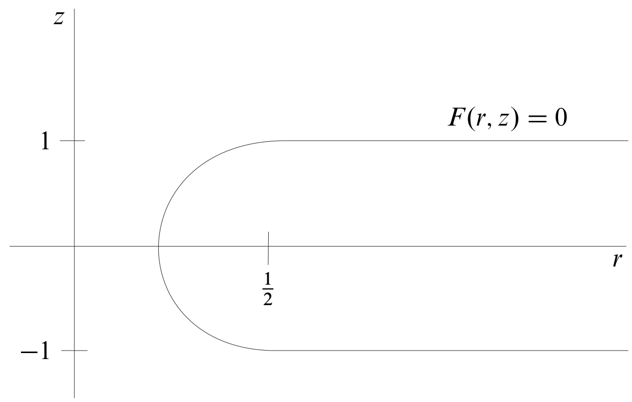

Figure 2: Hypersurface $S \subset \mathbb{R}^3$

We define a real hypersurface H inside $\mathbb{R}^3$ by the zero set of the function $G(r, \theta, z) = F(r, z) = 0$, which is obviously rotationally symmetric. Now the result just states that the form $\widehat{\alpha}$ on $F \times \mathbb{R}^3$ given by $\alpha_0 + z\rho r^2 d\theta$ restricted to $F \times H$ is a contact form which restricts to the contact forms α_j on the hypersurfaces $P_{\pm} = \{z = \pm(-1)^{j+1}\}$. Check Geiges [5] for the computation that shows that $\widehat{\alpha}$ is contact when restricted to $F \times H$. So, the contact fibration over the annulus $C(\rho) = H \cap \{r \leq 1\}$ acts as a normal model to construct a contact connection on the connected sum. Moreover the C^3–norm of G can be bounded independently of ρ. This implies that the C^2–norms of the new $\widehat{\alpha}$ can be chosen independently of ρ. □

We can also control the monodromy of the gluing region:

Lemma 4.7 *The contact connection arising from the connected sum construction of two contact fibrations satisfy that the holonomy map of a radial path from the interior boundary to the exterior boundary of the annulus $C(\rho)$ can be generated by a Hamiltonian $\{H_t\}_{t=0}^1$ satisfying $|H_t|_{C^1} \leq M\rho$.*

Proof To prove it we just have to check that the contact connection is bounded by a multiple of ρ. This is obvious from the construction. □

The result really holds for any number of derivatives, but first order bounds are enough for our purposes.

4.3 An overtwisted 3–sphere on an exotic S^5

We are going to define an exotic contact structure on S^5 such that it contains a contact 3–sphere with an overtwisted disk. K. Niederkruger suggested this result to the author and offered a proof, different from the one presented here.

Theorem 4.8 *There is an exotic contact structure on S^5 for which there is a contact sphere $S^3 \subset S^5$ whose contact structure, defined by restriction, is the standard overtwisted one.*

Proof We need to introduce some classical results contained in Milnor [8]:

Theorem 4.9 *Take $f\colon U \subset (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ a holomorphic polynomial with an isolated singularity at the origin. Given any $\epsilon > 0$ sufficiently small then:*

- (i) *Define $V = f^{-1}(0)$. $K_\epsilon = V \cap S^{2n+1}(\epsilon)$ is a smooth submanifold of dimension $2n-1$ embedded as a submanifold of the $2n+1$ sphere $S^{2n+1}(\epsilon)$.*
- (ii) *$\phi_f\colon \frac{f}{|f|} : S^{2n+1}(\epsilon) - K_\epsilon \to S^1$ is a smooth open book.*

This fibration is called the Milnor fibration of the isolated complex singularity. Moreover there is another canonical fibration

Theorem 4.10 *Take $f\colon U \subset (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ a holomorphic polynomial with an isolated singularity at the origin. Given any $\epsilon > 0$ sufficiently small, choose a constant $\delta > 0$ small enough with respect to ϵ such that all the fibers $f^{-1}(t)$ for $|t| \le \delta$ meet $S^{2n+1}(\epsilon)$ transversally. Let $C_\delta = \partial D_\delta \subset \mathbb{C}$ the circle of radius δ centered at 0, and set $N(\epsilon, \delta) = f^{-1}(C_\delta) \cap B_{2n+2}(\epsilon)$. Then*

$$f_{|N(\epsilon,\delta)}\colon N(\epsilon, \delta) \to C_\delta \cong S^1$$

is a C^∞ fiber bundle. Moreover $f_{|N(\epsilon,\delta)}$ and ϕ are equivalent fiber bundles.

This second fibration is called the Milnor tube. This theorem automatically shows that whenever the singularity is a Morse one, the monodromy is a generalized positive Dehn twist (see Seidel [11]).

Now we need the inverse of Theorem 4.3 that is much easier to prove. It reads as

Proposition 4.11 (Giroux [6]) *Given an open book (ϕ, N) satisfying the leaves are Stein and the monodromy is generated by a symplectomorphism, then there is a contact structure compatible with the open book.*

Assuming all the previous results, take the function $g(z_1, z_2, z_3) = z_1\overline{z}_2 + z_1 z_3 + \overline{z}_2 z_3$. We want to check if it has an associated Milnor fibration, in spite of not being holomorphic. We define

$$e\colon \mathbb{C}^3 = \mathbb{C}\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}\times\mathbb{C}\times\mathbb{C} = \mathbb{C}^3$$
$$(z_1, z_2, z_3) \to (z_1, \overline{z}_2, z_3).$$

We have that $g\circ e$ is a holomorphic polynomial with a Morse singularity at the origin, therefore Theorems 4.9 and 4.10 apply and we obtain that $\phi_{g\circ e}$ is an open book. Moreover recall that the page of the open book $\phi^{-1}_{g\circ e}(-1)$ is symplectic and the monodromy is a generalized Dehn twist, and so it can be represented by a symplectomorphism. Therefore the open book defines a contact structure in S^5, the standard one. Now we want to study ϕ_g. The link (binding) is exactly the same as for $\phi_{g\circ e}$. The page is defined as $e(\phi^{-1}_{g\circ e}(-1))$. It is diffeomorphic to the page $\phi^{-1}_{g\circ e}(-1)$ and so it admits a symplectic structure. Theorem 4.10 allows us to compute the monodromy (just reflecting through e) and it is obtained a generalized Dehn twist along a Lagrangian sphere as in the holomorphic case. The only difference is that the orientation of the twist is reversed. Recall that a reverse generalized Dehn twist admits a representation by a symplectomorphism. This proves that ϕ_g is an open book that supports a contact structure, thanks to Proposition 4.11 (as we will later see, it is not equivalent to the standard one).

Next we restrict our construction to $\mathbb{C}^2 \subset \mathbb{C}^3$. We have that $f_{|\mathbb{C}^2} = z_1\overline{z}_2$. It is well known that the associated Milnor fibration for that case is the standard fibration for the complementary of the negative Hopf link (see Etnyre [4]). This open book produces the standard overtwisted structure in S^3. Recall that the pages of this open book can be understood as symplectic submanifolds of the pages of the one generated by g. Moreover the generalized Dehn twist can be arranged to preserve the page of S^3 when restricted to it. This makes the standard overtwisted S^3 a contact submanifold of S^5. □

5 Proof of the main results

5.1 Proof of Theorem 1.4

By Gray's stability theorem there is a neighborhood of (N, α_{ot}) contactomorhpic to $(N \times D^2(\delta), \alpha_{ot} + r^2 d\theta)$ for a fixed $\delta > 0$. Now we recall Theorem 4.4 to construct a contact structure α_+ on $N \times T^2$, where N is equipped with the PS–overtwisted structure. Moreover in the equation (9) defining the contact structure we take ϵ to be negative and small enough. The precise value of the constant ϵ will be adjusted later.

Recall Lemma 2.9, this allows us to create an immersed PS–structure by parallel transport over a path $\gamma\colon S^1 \to D^2(\delta)$. Let us describe the first half of the path, being the other half defined by symmetry with respect to the origin. The path is defined in $D^2(\delta) \subset \mathbb{R}^2$ as the union of

(i) $[0, 3\delta/4] \times \{0\}$,

(ii) A path, in polar coordinates

$$\begin{aligned}\gamma\colon [0,\pi/2] &\to \mathbb{R}^+ \times [0,2\pi]\\ \theta &\mapsto (r(\theta),\theta),\end{aligned}$$

satisfying

(a) $r(0) = r(\pi/2) = 3\delta/4$,

(b) $3\delta/4 < r(\theta) < \delta$, for $\delta \in (0,\pi/2)$,

(iii) $\{0\} \times [0, 3\delta/4]$.

If we remove a disk, of radius ρ, $D^2(\rho) \subset D^2(\delta/4)$ and glue back $T^2 - D^2(\rho)$, we can remove the self-intersection in the path γ as shown in the Figure 3.

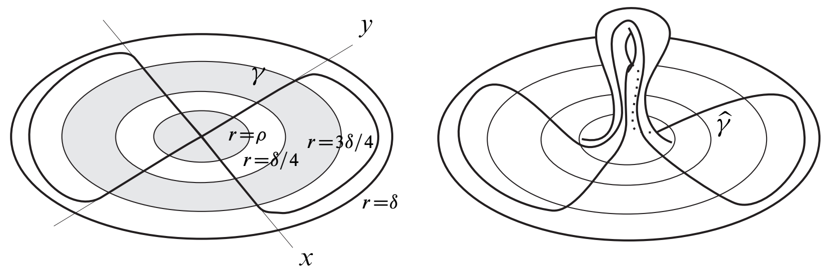

Figure 3: Gluing the torus in the basis

producing a new path $\hat{\gamma}$.

Now we glue the two contact fibrations $M' \supset N \times D^2(\delta) \to D^2(\delta)$ and $N \times T^2 \to T^2$, following the gluing of the basis. They satisfy the conditions to apply Theorem 4.6. This produces a new contact structure on $M' \bigcup_N (N \times T^2)$. Moreover we obtain a bound in the norms of the contact connection monodromy along $\hat{\gamma}$ in two regions:

- The gluing area, that is a ring $(-\rho,\rho) \times S^1$, in which we know that a radial path has a monodromy with a generating Hamiltonian C^1–bounded by a multiple of ρ because of Lemma 4.7.

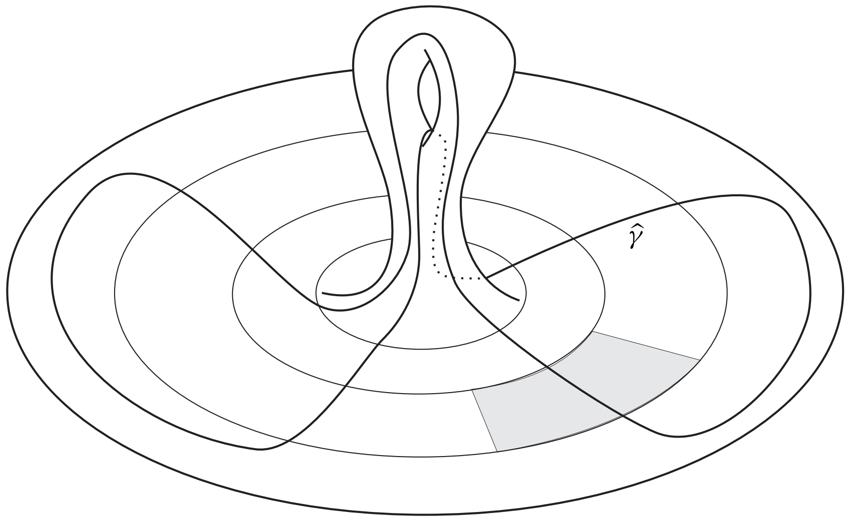

Figure 4: Deformation region is depicted in gray

- The torus in which we have a bound given by Lemma 4.5. We know that the C^1–norm of the Hamiltonian function generating the monodromy is bounded by $C|\epsilon|\delta$.

So we have that $\widehat{\gamma}$ produces a parallel transport in the complementary of $\gamma(S^1) \cap D^2(\delta)$ with generating Hamiltonian functions ($\widehat{\gamma}$ is cut in two disconnected paths in that region) bounded by $C \cdot (\rho + |\epsilon|)$. Therefore we have that the closed path γ has a monodromy generated by a Hamiltonian $\{H_t\}_{t=1}^{1}$ whose C^1–norm is bounded by $C(\rho + |\epsilon|)$. Recall that we are free to choose ϵ and ρ as small as we wish. Therefore we are in the hypothesis of Proposition 3.4 and so a perturbation of the contact fibration on a small region, as shown in Figure 4, provides the identity in the monodromy along $\widehat{\gamma}$. □

5.2 Proof of Corollaries 1.5, 1.6 and 1.7

We start with the 5–dimensional case. Assume that we want to chek that $S^3 \times \Sigma$ is PS–overtwisted. We take $g(\Sigma) - 1$ contact fibrations $S^3_{ot} \times T^2$ over the torus constructed using Theorem 4.4. If we fiber-glue them, using Theorem 4.6 we obtain a fibration $M' = S^3_{ot} \times \Sigma'$, where the genus of Σ' is $g(\Sigma) - 1$. Be aware of the orientations of the normal bundles to perform the gluings: it is needed to choose the constant ϵ in formula (9) positive in one case and negative in the rest. This places M' in the hypothesis of Theorem 1.4. So we obtain a PS–overtwisted contact structure in $M' \bigcup_{S^3_{ot}} (S^3_{ot} \times T^2) = S^3_{ot} \times \Sigma$.

Now, we proceed by induction in the dimension. We just assume that we have been able to find a contact manifold on $M_i = S^3 \times \prod_i \Sigma_i$ with a PS–structure. We want to construct a PS–structure in $M_i \times \Sigma$, with $g(\Sigma) \geq 2$. We use Theorems 4.4 and 4.6 to produce a contact structure in $M_i \times \Sigma'$, where $g(\Sigma') = g(\Sigma) - 1$. Now we are in the hypothesis of Theorem 1.4 and we get a PS–overtwisted contact structure in $M_i \times \Sigma$.

To prove Corollary 1.7, we start with the exotic contact structure lying in S^5 found in Theorem 4.8. There is an overtwisted $S^3 \subset S^5$ whose normal bundle is trivial, so we are in the hypothesis of Theorem 1.4 and we obtain that $S^5 \bigcup_{S^3} (S^3 \times T^2) = M_o$ admits a PS–overtwisted structure.

Finally Corollary 1.6 is a consequence of the connected sum theorem for contact manifolds.

6 Final remarks

The main results of this paper allow us to show some behaviors, expected to happen, in higher dimensional manifolds. A contact form induces a hyperplane distribution (the contact distribution) and a homotopy class of almost-complex structures in the distribution. The second may change with the signum chosen for the contact form. So the homotopy class of this pair is an invariant up to choice of orientation of the contact structure. We want to build examples of manifolds in which two contact structures share this invariant and one of them is not fillable.

A first example is

Corollary 6.1 *The manifold $S^3 \times \Sigma$, for $g(\Sigma) \geq 2$, admits two contact structures in the same homotopy class of hyperplane distributions with associated almost complex structure. The first one being holomorphically fillable, the second one not being fillable at all.*

Proof Recall that $S^3 \times \Sigma = \partial(B^4 \times \Sigma)$. Moreover $B^4 \times \Sigma$ is a complex, and so almost-complex, manifold and admits a 2–dimensional skeleton defined by the descending disks of the Morse function $f(z_1, z_2, z_3) = |(z_1, z_2)|^2 + f_0(z_3)$, where f_0 is the standard Morse function in Σ. Therefore, we are in the hypothesis of Eliashberg's Stein characterization theorem [3]. Recall that that result produces a Stein structure on $B^4 \times \Sigma$ such that the almost-complex structure is homotopical to the initial one. This implies that $S^3 \times \Sigma$ admits a Stein fillable contact structure in the homotopy class of the hyperplane distribution $D_0 \times T\Sigma \subset T(S^3 \times \Sigma)$, where D_0 is the standard contact form in S^3.

On the other hand the standard structure in S^3 lies in the same homotopy class that the standard overtwisted structure D_{ot}. Then, Corollary 1.5 produces a non-fillable contact structure $\hat{\alpha}$ in $S^3 \times \Sigma$. Its associated contact structure $\ker \hat{\alpha}$ is clearly homotopic to $D_{ot} \times \Sigma$, and therefore homotopic to the Stein-fillable one previously constructed. □

Recall that, in dimension 5 every filling is automatically semipositive, and so Theorem 1.3 restricts all the cases.

It is possible to copy the very same argument in dimension 7. We obtain:

Corollary 6.2 *The manifold $S^3 \times \Sigma_1 \times \Sigma_2$ admits two contact structures in the same homotopy class of hyperplane distributions with associated almost complex structure for $g(\Sigma_i) \geq 2$. The first one being holomorphically fillable, the second one being not semipositive-fillable at all.*

Proof Recall that $S^3 \times \Sigma_1 \times \Sigma_2 = \partial(B^4 \times \Sigma_1 \times \Sigma_2)$. From that point we just follow the argument used in Corollary 6.1 to conclude the result. □

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