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Automorphisms of 2–dimensional right-angled Artin groups

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We study the outer automorphism group of a right-angled Artin group A_Γ in the case where the defining graph Γ is connected and triangle-free. We give an algebraic description of $\mathrm{Out}(A_\Gamma)$ in terms of *maximal join* subgraphs in Γ and prove that the Tits' alternative holds for $\mathrm{Out}(A_\Gamma)$. We construct an analogue of *outer space* for $\mathrm{Out}(A_\Gamma)$ and prove that it is finite dimensional, contractible, and has a proper action of $\mathrm{Out}(A_\Gamma)$. We show that $\mathrm{Out}(A_\Gamma)$ has finite virtual cohomological dimension, give upper and lower bounds on this dimension and construct a spine for outer space realizing the most general upper bound.

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1 Introduction

A right-angled Artin group is a group given by a finite presentation whose only relations are commutators of the generators. These groups have nice algorithmic properties and act naturally on CAT(0) cube complexes. Also known as graph groups, they occur in many different mathematical contexts; for some particularly interesting examples we refer to the work of Bestvina and Brady [3] on finiteness properties of groups, Croke and Kleiner [10] on boundaries of CAT(0) spaces, and Abrams [1], Ghrist [12] and Ghrist and Peterson [13] on configuration spaces in robotics. For a general survey of right-angle Artin groups, see Charney [7].

A nice way to describe a right-angled Artin group is by means of a finite simplicial graph Γ. If V is the vertex set of Γ, then the group A_Γ is defined by the presentation

$$A_\Gamma = \langle V \mid vw = wv \text{ if } v \text{ and } w \text{ are connected by an edge in } \Gamma \rangle.$$

At the two extremes of this construction are the case of a graph with n vertices and no edges, in which case A_Γ is a free group of rank n, and that of a complete graph on n vertices, in which case A_Γ is a free abelian group of rank n. In general, right-angled Artin groups can be thought of as interpolating between these two extremes. Thus it seems reasonable to consider automorphism groups of right-angled Artin groups as

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interpolating between $\mathrm{Aut}(F_n)$, the automorphism group of a free group, and $GL_n(\mathbb{Z})$, the automorphism group of a free abelian group. The automorphism groups of free groups and of free abelian groups have been extensively studied but, beyond work of Servatius [20] and Laurence [18] on generating sets, there seems to be little known about the automorphism groups of general right-angled Artin groups.

In this paper we begin a systematic study of automorphism groups of right-angled Artin groups. We restrict our attention to the case that the defining graph Γ is connected and triangle-free or, equivalently, A_Γ is freely indecomposable and contains no abelian subgroup of rank greater than two. This class already contains many interesting Artin groups. If Γ is a tree, then A_Γ is the fundamental group of a graph manifold; if Γ contains minimal cycles of length at least five, then A_Γ contains hyperbolic surface groups. For further discussion of these groups and their subgroups, see Behrstock and Neumann [2] and Crisp, Sageev and Sapir [9].

A key example is when Γ is a complete bipartite graph, which we call a *join* since Γ is the simplicial join of two disjoint sets of vertices. The associated Artin group A_Γ is a product of two free groups. If neither of the free groups is cyclic, then the automorphism group of A_Γ is just the product of the automorphism groups of the two factors (or possibly index two in this product). If one of the free groups is cyclic, however, the automorphism group is much larger, containing in addition an infinite group generated by transvections from the cyclic factor to the other factor.

We will show that for any connected, triangle-free graph Γ, the maximal join subgraphs of Γ play a key structural role in the automorphism group. We do this by proving that the subgroups A_J generated by maximal joins J are preserved up to conjugacy (and up to diagram symmetry) by automorphisms of A_Γ. This gives rise to a homomorphism

$$\mathrm{Out}^0(A_\Gamma) \to \prod \mathrm{Out}(A_J),$$

where $\mathrm{Out}^0(A_\Gamma)$ is a finite index normal subgroup of $\mathrm{Out}(A_\Gamma)$ which avoids certain diagram symmetries. We use algebraic arguments to prove that the kernel of this homomorphism is a finitely generated free abelian group. Elements of this kernel commute with certain transvections, called “leaf transvections,” and adding them forms an even larger free abelian subgroup of $\mathrm{Out}(A_\Gamma)$. We show that this larger subgroup is the kernel of a homomorphism into a product of outer automorphism groups of free groups. We then derive the Tits alternative for $\mathrm{Out}(A_\Gamma)$ using the fact that the Tits alternative is known for outer automorphism groups of free groups.

The second part of the paper takes a geometric turn. The group $\mathrm{Out}(F_n)$ can be usefully represented as symmetries of a topological space known as *outer space*. This space, introduced by Culler and Vogtmann in [11], may be described as a space of actions of F_n on trees. Outer space has played a key role in the study of the groups $\mathrm{Out}(F_n)$ (see for example, the survey article [23]). For $GL(n,\mathbb{Z})$ the analogous "outer space" of actions of $\mathbb{Z}^n$ on $\mathbb{R}^n$ is the classical homogeneous space $SL(n,\mathbb{R})/SO(n,\mathbb{R})$. In Section 4 of this paper we construct an outer space $\mathcal{O}(A_\Gamma)$ for the right-angled Artin group A_Γ associated to any connected, triangle-free graph Γ. If Γ is a single join, $\mathcal{O}(A_\Gamma)$ consists of actions of A_Γ on products of trees $T \times T'$. In general, a point in outer space is a graph of such actions, parameterized by a collection of maximal joins in Γ. We prove that the space $\mathcal{O}(A_\Gamma)$ is finite dimensional, contractible, and has a proper action of $\mathrm{Out}^0(A_\Gamma)$.

In the last section we give upper and lower bounds on the virtual cohomological dimension of $\mathrm{Out}(A_\Gamma)$ and construct a spine for $\mathcal{O}(A_\Gamma)$, ie a simplicial equivariant deformation retract of $\mathcal{O}(A_\Gamma)$, which realizes the most general upper bound. A lower bound is given by the rank of any free abelian subgroup, such as the subgroup found in the first part of the paper. We show that this subgroup can be expanded even further to give a better lower bound. In some examples, the upper and lower bounds agree, giving the precise virtual cohomological dimension of $\mathrm{Out}(A_\Gamma)$.

2 Preliminaries

2.1 Special subgroups

A *simplicial graph* is a graph which is a simplicial complex, ie a graph with no loops or multiple edges. Vertices of valence one are called *leaves*, and all other vertices are *interior*. To each finite simplicial graph Γ we associate the *right-angled Artin group* A_Γ as described in the introduction.

A *special subgroup* of A_Γ is a subgroup generated by a subset of the vertices of Γ. If Θ is the full subgraph of Γ spanned by this subset, the special subgroup is naturally isomorphic to the Artin group A_Θ. In the discussion which follows, we will need to know the normalizers $N(A_\Theta)$, the centralizers $C(A_\Theta)$, and the centers $Z(A_\Theta)$ of special subgroups $A_\Theta \subset A_\Gamma$. To describe them, the following notation is useful.

Definition 2.1 Let Θ be a full subgraph of a simplicial graph Γ. Then $\Theta^\perp$ is the intersection of the (closed) stars of all the vertices in Θ:

$$\Theta^\perp = \bigcap_{v \in \Theta} \mathrm{st}(v)$$

Identifying the vertices of Γ with generators of A_Γ, one can also describe $\Theta^\perp$ as the subgraph of Γ spanned by the vertices which commute with every vertex in Θ. We remark that this notation differs from that of Godelle [14], who excludes points of Θ from $\Theta^{\perp}$.

Proposition 2.2 *For any graph Γ, and any special subgroups A_Θ and A_Λ of A_Γ:*

(1) *The normalizer, centralizer, and center of A_Θ are given by*

$$N(A_\Theta) = A_{\Theta\cup\Theta^\perp}, \quad C(A_\Theta) = A_{\Theta^\perp} \quad \text{and} \quad Z(A_\Theta) = A_{\Theta\cap\Theta^\perp}.$$

(2) *If $gA_\Theta g^{-1} \subseteq A_\Lambda$, then $\Theta \subseteq \Lambda$ and $g = g_1 g_2$ for some $g_1 \in N(A_\Lambda)$, $g_2 \in N(A_\Theta)$.*

Proof These statements are easily derived from work of Servatius and work of Godelle, as follows. In [20], Servatius proves that the centralizer of a single vertex v is the special subgroup generated by $\mathrm{st}(v)$ and hence for a set of vertices Θ, the centralizer, $C(A_\Theta)$, is generated by the intersection of these stars, which is exactly $\Theta^\perp$. It follows that the center of A_Θ is $A_\Theta \cap C(A_\Theta) = A_{\Theta\cap\Theta^\perp}$. In [14], Godelle considers normalizers and centralizers of special subgroups in a larger class of Artin groups, Artin groups of "FC type". He defines the quasi-centralizer, $QZ(A_\Theta)$, of a special subgroup to be the group of elements g which conjugate the *set* Θ to itself, and he proves that $N(A_\Theta) = A_\Theta \cdot QZ(A_\Theta)$. In the right-angled case, no two generators are conjugate, hence $QZ(A_\Theta) = C(A_\Theta) = A_{\Theta^\perp}$ and $N(A_\Theta) = A_{\Theta\cup\Theta^\perp}$.

Godelle also describes the set of elements which conjugate one special subgroup A_Θ into another A_Λ in terms of a category $\mathrm{Ribb}(V)$ whose objects are subsets of the generating set V and whose morphisms conjugate one subset of V into another. In the case of a right-angled Artin group, since no two generators are conjugate, there are no morphisms between distinct objects of $\mathrm{Ribb}(V)$ and the group of morphisms from an object Θ to itself is precisely the centralizer $C(A_\Theta) = A_{\Theta^\perp}$. Proposition 3.2 of [14] asserts that $gA_\Theta g^{-1} \subseteq A_\Lambda$ if and only if $g = g_1 g_2$ where $g_1 \in A_\Lambda$ and g_2 is a morphism in a certain subcategory of $\mathrm{Ribb}(V)$. In the right-angled case, it is straightforward to verify that such a morphism exists if and only if $\Theta \subseteq \Lambda$ and $g_2 \in A_{\Theta^\perp}$. (In Godelle's notation, he decomposes Θ into disjoint subsets $\Theta = \Theta_s \cup \Theta_{as}$ where Θ_s is the set of generators lying in the center of A_Θ. His theorem states that g_2 must commute with Θ_s and conjugate Θ_{as} to a set R with $R \cup \Theta_s \subseteq \Lambda$. In the right-angled case, this is possible only if $R = \Theta_{as}$ and g_2 also commutes with Θ_{as}.) Thus $g \in A_\Lambda A_{\Theta^\perp} \subseteq N(A_\Lambda) N(A_\Theta)$. □

2.2 Cube complexes

Associated to each right-angled Artin group A_Γ, there is a CAT(0) cube complex C_Γ on which A_Γ acts, constructed as follows. The 1–skeleton of C_Γ is the Cayley graph of A_Γ with generators the vertices V of Γ. There is a cube of dimension $k > 1$ glued in wherever possible, ie wherever the 1–skeleton of a cube exists in the Cayley graph. In the quotient by the action there is a k–dimensional torus for each complete subgraph of Γ with k vertices. The cube complex associated to a free group is simply the Cayley graph of the free group, ie a tree on which the free group acts freely. The cube complex associated to the complete graph on n vertices is the standard cubulation of $\mathbf{R}^n$. The cube complex associated to a join $U * W$ is a product $T_U \times T_W$, where T_U (resp T_W) is a tree on which the free group $F\langle U\rangle$ (resp $F\langle W\rangle$) acts freely with quotient a rose.

2.3 Generators for the automorphism group of a right-angled Artin group

A set of generators for $\mathrm{Aut}(A_\Gamma)$ was found by M Laurence [18], extending work of H Servatius [20]. There are five classes of generators:

1. Inner automorphisms
2. Inversions
3. Partial conjugations
4. Transvections
5. Symmetries

Inversions send a standard generator of A_Γ to its inverse.

A *partial conjugation* exists when removal of the (closed) star of some vertex v disconnects the graph Γ. In this case one obtains an automorphism by conjugating all of the generators in one of the components by v. (See example in Figure 1.)

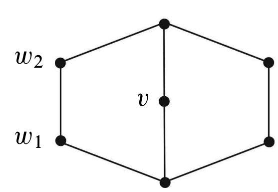

Figure 1: Graph with a partial conjugation $w_i \mapsto v^{-1} w_i v$

Transvections occur whenever there are vertices v and w such that $\mathrm{st}(v) \supset \mathrm{lk}(w)$; in this case the transvection sends $w \mapsto wv$. There are two essentially different types of transvections, depending on whether or not v and w commute:

1. *Type I transvections*: v and w are not connected by an edge.
2. *Type II transvections*: v and w are connected by an edge.

Figure 2: Graphs with transvections $w \mapsto wv$

(See examples in Figure 2.)

Finally, *symmetries* are induced by symmetries of the graph, and permute the generators.

We will be especially interested in the subgroup we obtain by leaving out the graph symmetries:

Definition 2.3 The subgroup of $\mathrm{Aut}(A_\Gamma)$ generated by inner automorphisms, inversions, partial conjugations and transvections is called the *pure automorphism group* and is denoted $\mathrm{Aut}^0(A_\Gamma)$. The image of $\mathrm{Aut}^0(A_\Gamma)$ in $\mathrm{Out}(A_\Gamma)$ is the group of *pure outer automorphisms* and is denoted $\mathrm{Out}^0(A_\Gamma)$.

The subgroups $\mathrm{Aut}^0(A_\Gamma)$ and $\mathrm{Out}^0(A_\Gamma)$ are easily seen to be normal and of finite index in $\mathrm{Aut}(A_\Gamma)$ and $\mathrm{Out}(A_\Gamma)$ respectively. We remark that if A_Γ is a free group or free abelian group, then $\mathrm{Aut}^0(A_\Gamma) = \mathrm{Aut}(A_\Gamma)$.

3 Maximal joins

3.1 Restriction to connected, two-dimensional right-angled Artin groups

If Γ is disconnected, then A_Γ is a free product of the groups associated to the components of Γ. Guirardel and Levitt [16] have constructed a type of outer space for a free product with at least one noncyclic factor, which can be used to reduce the problem of understanding the outer automorphism group to understanding the outer automorphism groups of the free factors. Therefore, in this paper we will consider only connected graphs Γ.

We will further restrict ourselves to the case that Γ has no triangles. In this case, the associated cube complex C_Γ is 2–dimensional so we call these *two-dimensional* right-angled Artin groups. To avoid technicalities, we also assume that Γ has at least 2 edges.

Type II transvections are severely limited in the two-dimensional case. Since v and w are connected by an edge, the vertex w must actually be a terminal vertex, ie a leaf:

if there were another vertex $u \neq v$ connected to w, then the condition $\mathrm{st}(v) \supset \mathrm{lk}(w)$ would imply that u, v and w form a triangle in the graph (see Figure 2). For this reason, we call Type II transvections *leaf transvections*.

If Γ is triangle-free and Θ is a subgraph with at least one edge, then $\Theta^\perp \subset \Theta$, so by Proposition 2.2,

$$N(A_\Theta) = A_\Theta \quad \text{and} \quad C(A_\Theta) = Z(A_\Theta) = A_{\Theta^\perp}.$$

If Θ contains two nonadjacent edges, then the latter groups are trivial.

Key example A join $\Gamma = U * W$, has no triangles. As we noted above, the associated right-angled Artin group A_Γ is $F\langle U\rangle \times F\langle W\rangle$. It is easy to deduce the structure of the automorphism group from Laurence's generators. If U and W each contain at least two elements then every automorphism preserves the two factors (or possibly switches them if $|U| = |W|$). Thus $\mathrm{Out}(A_\Gamma)$ contains $\mathrm{Out}(F\langle U\rangle) \times \mathrm{Out}(F\langle W\rangle)$ as a subgroup of index at most 2. If $U = \{u\}$ and $|W| = \ell \geq 2$, then $A_\Gamma = \mathbb{Z} \times F\langle W\rangle$ with the center $\mathbb{Z}$ generated by u, and the elements of W are all leaves. Any automorphism of A_Γ must preserve the center and hence induces an automorphism of $F\langle W\rangle$, as well as an automorphism of the center $\mathbb{Z}$. The map $\mathrm{Out}(A_\Gamma) \to \mathrm{Out}(\mathbb{Z}) \times \mathrm{Out}(F\langle W\rangle)$ splits and its kernel is the group generated by leaf transvections. The leaf transvections commute, so $\mathrm{Out}(A_\Gamma) \cong \mathbb{Z}^\ell \rtimes (\mathbb{Z}/2 \times \mathrm{Out}(F\langle W\rangle))$.

We assume for the rest of this paper that Γ is connected and triangle-free. In addition, we assume that Γ contains at least two edges.

3.2 Restricting automorphisms to joins

A connected, triangle-free graph Γ can be covered by subgraphs which are joins. For example, to each interior vertex v we can associate the join $J_v = L_v * L_v^\perp$, where $L_v = \mathrm{lk}(v)$. Note that $L_v^\perp$ always contains v. If $L_v^\perp = \{v\}$, then $J_v = \mathrm{st}(v)$; in this case we say that v is a *cyclic vertex* since $F(L_v^\perp) \cong \mathbb{Z}$ is cyclic.

Lemma 3.1 *If v and w are interior vertices joined by an edge of Γ, then $L_v^\perp \subseteq L_w$, so we have:*

$$\begin{array}{rccc} J_v = & L_v & * & L_v^\perp \\ & \cup & & \cap \\ J_w = & L_w^\perp & * & L_w \end{array}$$

In particular, $J_w \cap J_v = L_w^\perp * L_v^\perp$.

We remark that the J_v is not properly contained in any other join subgraph of Γ, ie J_v is a *maximal join* in Γ. The following proposition shows that the special subgroups A_J associated to maximal join subgraphs J of Γ are preserved up to conjugacy by pure automorphisms:

Proposition 3.2 *Let $\phi \in \mathrm{Aut}^0(A_\Gamma)$ be a pure automorphism of A_Γ and let $J = U * W$ be a maximal join in Γ. Then ϕ maps $A_J = F\langle U\rangle \times F\langle W\rangle$ to a conjugate of itself. Moreover, if U contains no leaves, then ϕ preserves the factor $F\langle U\rangle$ up to conjugacy.*

Proof It suffices to verify the proposition for the generators of $\mathrm{Aut}^0(A_\Gamma)$.

Inner automorphisms These obviously send each A_J to a conjugate of itself.

Inversions An inversion sends each A_J to itself and preserves the factors.

Partial conjugations If ϕ is a partial conjugation by a vertex v, we claim that ϕ either fixes all of A_J or conjugates all of A_J by v. Suppose first that v is not in $U * W$. The link of v cannot contain vertices of both U and W, since there would then be a triangle in Γ. Furthermore, $\mathrm{lk}(v)$ cannot contain all of U since then adding it to W would make a larger join, contradicting maximality. Therefore the subgraph of $U * W$ spanned by vertices not in $\mathrm{lk}(v)$ is still connected, so ϕ has the same effect (either trivial or conjugation by v) on generators corresponding to all vertices in $U * W$. Next, suppose that v is actually in $U * W$, say $v \in U$. The resulting partial conjugation restricted to A_{U*W} is an internal automorphism of A_{U*W} which may conjugate some generators of $F\langle U\rangle$ by v, but has no effect on generators of $F\langle W\rangle$.

Transvections We claim that a transvection either fixes A_J or acts as an internal automorphism of A_J. If ϕ is a transvection sending $s \to sv$, then ϕ is the identity on A_{U*W} unless $s \in U * W$, say $s \in U$. If ϕ is not a leaf transvection, then the condition $\mathrm{st}(v) \supset \mathrm{lk}(s)$ and maximality imply that v is also in U, so the restriction of ϕ is an internal automorphism of A_{U*W} preserving the factor $F\langle U\rangle$ and fixing $F\langle W\rangle$. If ϕ is a leaf transvection, then $W = \{v\}$ and $A_{U*W} = F\langle U\rangle \times \mathbb{Z}$. In this case, ϕ fixes the (central) $\mathbb{Z}$ factor and multiplies s by the generator of $\mathbb{Z}$. □

This proposition has two easy corollaries. First, let $\mathrm{Sym}(\Gamma)$ denote the group of diagram symmetries of Γ and $\mathrm{Sym}^0(\Gamma) = \mathrm{Sym}(\Gamma) \cap \mathrm{Aut}^0(\Gamma)$. Clearly $\mathrm{Aut}(A_\Gamma)/\mathrm{Aut}^0(A_\Gamma) \cong \mathrm{Out}(A_\Gamma)/\mathrm{Out}^0(A_\Gamma) \cong \mathrm{Sym}(\Gamma)/\mathrm{Sym}^0(\Gamma)$. Denote this quotient group by $Q(\Gamma)$.

Corollary 3.3 *The quotient maps from $\mathrm{Aut}(A_\Gamma)$, $\mathrm{Out}(A_\Gamma)$, and $\mathrm{Sym}(\Gamma)$ to $Q(\Gamma)$ split.*

Proof It suffices to define a splitting of the projection $\mathrm{Sym}(\Gamma) \to Q(\Gamma)$. Composing with the inclusion of $\mathrm{Sym}(\Gamma)$ into $\mathrm{Aut}(A_\Gamma)$ or $\mathrm{Out}(A_\Gamma)$, gives a splitting in the other two cases.

We first characterize elements of $\mathrm{Sym}^0(\Gamma)$ as those graph symmetries which only permute vertices with the same link. Define an equivalence relation on the vertices of Γ by $v \sim w$ if $\mathrm{lk}(v) = \mathrm{lk}(w)$. The elements of an equivalence class $[v]$ generate a free subgroup, and any automorphism of this subgroup extends (via the identity) to an automorphism of the whole Artin group A_Γ. (These are the automorphisms generated by inversions and transvections involving only elements of $[v]$.) Since $\mathrm{Aut}^0 = \mathrm{Aut}$ for a free group, any permutation of $[v]$ can be realized by an element of $\mathrm{Sym}^0(\Gamma)$ which is the identity outside $[v]$. Composing these gives an automorphism in $\mathrm{Sym}^0(\Gamma)$ realizing any permutation of the elements of each equivalence class.

Conversely, if a graph symmetry ϕ is in $\mathrm{Sym}^0(\Gamma)$ we claim that it acts by permuting the elements of each equivalence class $[v]$. To see this, note that for any nonleaf vertex v, it follows from Proposition 3.2 that ϕ preserves J_v and $L_v^\perp$, and hence it must also preserve L_v. Since any graph symmetry takes links to links, ϕ permutes $[v]$. Moreover, if L_v contains a leaf w, then ϕ permutes $[w]$, the set of all leaves in L_v.

Now choose an ordering on the vertices in each equivalence class. Then we can define a splitting of $\mathrm{Sym}(\Gamma) \to Q(\Gamma)$ by mapping a coset to the unique element of the coset which is order preserving on every equivalence class. □

The second corollary of Proposition 3.2 will be crucial for our analysis.

Corollary 3.4 *For every maximal join $J \subset \Gamma$, there is a restriction homomorphism $R_J\colon \mathrm{Out}^0(A_\Gamma) \to \mathrm{Out}(A_J)$.*

Proof Fix J. Then for any element of $\mathrm{Out}^0(A_\Gamma)$, there is a representative $\phi \in \mathrm{Aut}^0(A_\Gamma)$ which maps A_J to itself. Any two such representatives differ by conjugation by an element of the normalizer of A_J. But the normalizer $N(A_J)$ is equal to A_J, so the restriction of ϕ to A_J is a well-defined element of $\mathrm{Out}(A_J)$. □

Let $\mathcal{M}$ be the set of all maximal join subgraphs of Γ. We can put all of the homomorphisms R_J for $J \in M$ together to obtain a homomorphism

$$R = \prod_{J \in \mathcal{M}} R_J\colon \mathrm{Out}^0(A_\Gamma) \to \prod_{J \in \mathcal{M}} \mathrm{Out}(A_J).$$

To understand $\mathrm{Out}^0(A_\Gamma)$, then, we would like to understand the image and kernel of this homomorphism. But, first we note that there is a lot of redundant information in the set of all maximal joins used to define R; for example, the maximal joins of the form J_v already cover Γ.

Even the covering of Γ by the maximal joins J_v is inefficient. If v and w have the same link, then $J_v = J_w$ so we don't need them both. With this in mind, we now specify a subgraph Γ_0 of Γ which will turn out to contain all of the information we need. The key idea is that of vertex equivalence, which we already encountered in the proof of Corollary 3.3.

Definition 3.5 Vertices v and w of Γ are called *equivalent* if they have the same link, ie $L_v = L_w$. Equivalence classes of vertices are partially ordered by the relation $[v] \leq [w]$ if $L_v \subseteq L_w$.

To define Γ_0, we choose a vertex in each maximal equivalence class and let Γ_0 be the full subgraph of Γ spanned by these vertices. In the special case that Γ is a star $\{v\} * W$, set $\Gamma_0 = \{v\}$. Up to isomorphism, Γ_0 is independent of the choice of representatives. We denote by V_0 the set of vertices in Γ_0.

Examples 3.6 (i) If Γ is a tree then Γ_0 is the subtree spanned by the vertices which are not leaves.

(ii) If Γ is the graph in Figure 3, then Γ_0 is the single edge spanned by v and w.

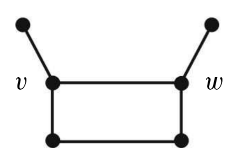

Figure 3: $\Gamma_0 = [v, w]$

Whether we are working with vertices of Γ or Γ_0, the notation L_v will always refer to the link of v in the original graph Γ. Likewise, a vertex is considered a leaf if it is a leaf of the full graph Γ.

Recall that a vertex v of Γ is called a *cyclic vertex* if the associated maximal join J_v is equal to $\mathrm{st}(v)$. The following lemma specifies the properties of Γ_0 which we will need.

Lemma 3.7 *Let Γ_0 be defined as above.*

(1) Γ_0 *is a connected subgraph of* Γ.

(2) *The vertex set* V_0 *of* Γ_0 *contains every cyclic vertex and no leaves of* Γ.

(3) *Every vertex of* Γ *lies in* L_w *for at least one* $w \in V_0$ *and lies in* $L_w^{\perp}$ *for at most one* $w \in V_0$.

Proof (1) Let v, w be two vertices in Γ_0 and let

$$v = v_0, v_1, \dots v_k = w$$

be an edgepath in Γ connecting v to w. If v_1 does not lie in Γ_0, then there is a vertex $v_1' \in \Gamma_0$ with $L_{v_1} \subseteq L_{v_1'}$. Replacing v_1 by v_1' gives another edgepath in Γ from v to w whose first edge lies in Γ_0. The first statement of the lemma now follows by induction on k.

(2) Note that $[v] \subseteq L_v^{\perp}$ for any v, and $[v] = L_v^{\perp}$ if and only if $[v]$ is maximal. Equality holds when v is cyclic, since in this case $L_v^{\perp} = \{v\}$. If v is a leaf then v is connected to some interior vertex w by an edge (recall that we have assumed that the diameter of Γ is at least two). If $\Gamma = \mathrm{st}(w)$ we have defined $\Gamma_0 = \{w\}$ so $v \notin V_0$. If Γ is not a star, w must be connected to some other interior vertex u. It follows that $L_v = \{w\} \subsetneq L_u$, hence $v \notin V_0$.

(3) Since Γ is connected, every vertex v lies in the link of some other vertex and hence lies in the link of some maximal vertex. Since $L_w^{\perp} = [w]$ for every vertex w in Γ_0, v lies in at most one such $L_w^{\perp}$. $\square$

3.3 The kernel of the restriction and projection homomorphisms

We are interested in determining the kernel of the map R constructed from the restriction homomorphisms $R_J\colon \mathrm{Out}^0(A_\Gamma) \to \mathrm{Out}(A_J)$. We first consider the kernel of the analogous map R_0 defined by looking only at the R_J for maximal joins $J = J_v$, for $v \in V_0$:

$$R_0 = \prod_{v \in V_0} R_{J_v}\colon \mathrm{Out}^0(A_\Gamma) \to \prod_{v \in V_0} \mathrm{Out}(A_{J_v})$$

This kernel consists of outer automorphisms such that any representative in $\mathrm{Aut}^0(A_\Gamma)$ acts by conjugation on each A_{J_v}, for $v \in \Gamma_0$.

For any $\phi \in \mathrm{Out}^0(A_\Gamma)$ and any interior vertex v, we can choose a representative automorphism ϕ_v such that $\phi_v(A_{J_v}) = A_{J_v}$. Since $L_v^{\perp}$ contains no leaves, ϕ_v must also preserve $A_{L_v^{\perp}}$ by Proposition 3.2. If v and w are interior vertices connected by an edge, then the representatives ϕ_v and ϕ_w are related as follows.

Lemma 3.8 *Suppose v, w are interior vertices connected by an edge in Γ, and $\phi \in \mathrm{Out}^0(A_\Gamma)$ is represented by automorphisms ϕ_v and ϕ_w with $\phi_v(A_{J_v}) = A_{J_v}$ and $\phi_w(A_{J_w}) = A_{J_w}$. Then there exists $g_v \in A_{J_v}$ and $g_w \in A_{J_w}$ such that $c(g_v) \circ \phi_v = c(g_w) \circ \phi_w$, where $c(g)$ denotes conjugation by g.*

Proof Since ϕ_v and ϕ_w represent the same element in $\mathrm{Out}^0(A_\Gamma)$, $\phi_w \circ \phi_v^{-1} = c(g)$ for some $g \in A_\Gamma$. Since ϕ_v preserves $A_{L_v^\perp}$, and $A_{L_v^\perp} \subset A_{J_w}$, we have $gA_{L_v^\perp}g^{-1} \subset A_{J_w}$. By Proposition 2.2 (2), we must then have $g = g_1 g_2$ with $g_1 \in N(A_{J_w}) = A_{J_w}$ and $g_2 \in N(A_{L_v^\perp}) = A_{J_v}$. Taking $g_w = g_1^{-1}$ and $g_v = g_2$ gives the desired formula. □

A vertex v is a *separating vertex* if $\Gamma - \{v\}$ is disconnected. It is easy to see that separating vertices are cyclic. It is also easy to see that conjugating any component of $\Gamma - \{v\}$ gives an element of the kernel of R. We remark, however, that a component of $\Gamma - \{v\}$ may contain more than one component of $\Gamma - \mathrm{st}(v)$, so not every partial conjugation by v lies in this kernel. For example, in the graph in Figure 4, $\Gamma - \{v\}$ has two components while $\Gamma - \mathrm{st}(v)$ has five, and the partial conjugation of u by v restricts nontrivially in $\mathrm{Out}(A_{J_w})$.

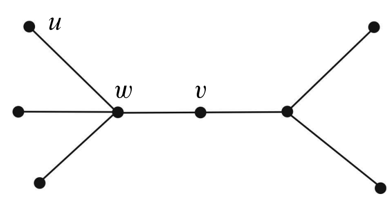

Figure 4: $\Gamma - \mathrm{st}(v)$ has more components than $\Gamma - \{v\}$.

In the rest of the paper, we will need to count several things associated to a vertex of Γ, so we now establish some notation.

Notation Let v be a vertex of Γ and Γ_0 the graph defined above.

(1) The valence, or degree, of v in Γ is denoted $\delta(v)$.
(2) If $v \in \Gamma_0$, the valence of v in Γ_0 is denoted $\delta_0(v)$.
(3) The number of connected components of the complement $\Gamma - \{v\}$ is denoted $\delta_C(v)$.
(4) The number of leaves attached by an edge to v is denoted $\ell(v)$.

Proposition 3.9 *The kernel K_0 of the homomorphism R_0 is a finitely generated, free abelian group, generated by conjugations by separating vertices v on nonleaf components of $\Gamma - \{v\}$. If Γ is a star, the kernel is trivial; otherwise it has rank $\sum_{v \in V_0} (\delta_C(v) - \ell(v) - 1)$.*

Proof If Γ is the star of v then $J_v = \Gamma$ and R_0 is injective. So assume that Γ is not a star. Note that in this case, A_Γ has trivial center.

Let $r(v) = \delta_C(v) - \ell(v)$. We will prove the theorem by defining a homomorphism

$$\mu = \prod_{v \in V_0} \mu_v \colon K_0 \to \prod_{v \in V_0} \mathbb{Z}^{r(v)}/\Delta \cong \prod_{v \in V_0} \mathbb{Z}^{r(v)-1},$$

where Δ is the diagonal subgroup of $\mathbb{Z}^{r(v)}$, and showing that μ is an isomorphism.

Let ϕ be an element of K_0. For each $u \in V_0$ choose a representative $\phi_u \in \mathrm{Aut}^0(A_\Gamma)$ which acts as the identity on vertices of J_u. If v and w in Γ_0 are connected by an edge, then $\phi_v \phi_w^{-1}$ is the identity on vertices of $J_v \cap J_w$. Now $\phi_v \phi_w^{-1}$ is inner, so it is conjugation by a (unique) element $g_{v,w}$ of the centralizer $C(A_{J_v \cap J_w})$. By Proposition 2.2 this centralizer is the special subgroup associated to $(J_v \cap J_w)^\perp$. Since Γ has no triangles, $(J_v \cap J_w)^\perp = (L_v^\perp * L_w^\perp)^\perp$ is either the edge spanned by v and w (if both v and w are cyclic), the vertex v (if v is cyclic but w is not), the vertex w (if w is cyclic but v is not), or empty (if neither v nor w is cyclic). Thus $g_{v,w} = w^{-m} v^n$ with n (resp. m) equal zero if v (resp. w) is not cyclic, and the equation of Lemma 3.8 reduces to

$$c(v^n) \circ \phi_v = c(w^m) \circ \phi_w.$$

We are now ready to define $\mu_v \colon K_0 \to \mathbb{Z}^{r(v)}/\Delta$. Let $C_1, \dots, C_{r(v)}$ be the nonleaf components of $\Gamma - \{v\}$. In each C_i, choose a vertex w_i adjacent to v which is also in Γ_0. We have $c(v^{n_i}) \circ \phi_v = c(w_i^{m_i}) \circ \phi_{w_i}$ for unique integers n_i and m_i and we define $\mu_v(\phi) = [(n_1, \dots, n_{r(v)})]$.

We have made several choices, and we must show that $\mu_v(\phi)$ is independent of these choices. The integers n_i depend, a priori, on our choice of representatives ϕ_v and ϕ_{w_i}. But ϕ_v is unique up to conjugation by an element of A_Γ centralizing J_v, namely a power of v, and similarly for ϕ_{w_i}. It follows that the choice of ϕ_{w_i} has no effect on n_i whereas the choice of a different representative for ϕ_v will change all of the integers n_i by the same amount. Therefore the class of $(n_1, \dots, n_{r(v)})$ modulo the diagonal Δ is independent of ϕ_v and the ϕ_{w_i}.

We now show that $\mu_v(\phi)$ is independent of the choice of the w_i. Let w_i' be a different choice, and connect w_i to w_i' by a simple path in C_i, ie a path which goes through each vertex at most once. In particular, each vertex on the path is an interior vertex.

We first claim that the restriction of ϕ to $\mathrm{Out}(J_u)$ is trivial for every interior vertex of Γ, not just for those in Γ_0. Suppose u is an interior vertex of $\Gamma - \Gamma_0$. By Lemma 3.7, u is adjacent to some $w \in \Gamma_0$, so $L_u^\perp \subset L_w$. By definition of Γ_0, $L_u \subset L_v$ for some $v \in \Gamma_0$, so $J_u \subset L_v \cup L_w$. Since v and w are in Γ_0 and connected by an edge, we can find n and m with $c(v^n) \circ \phi_v = c(w^m) \circ \phi_w$. Setting $\phi_u = c(v^n) \circ \phi_v$ we have that ϕ_u acts trivially on every vertex of J_u.

For any two interior vertices u, u' of Γ, we define $g_{u,u'}$ to be the (unique) element of A_Γ such that $\phi_u \circ \phi_{u'}^{-1}$ is conjugation by $g_{u,u'}$. If u and u' are connected by an edge, then in light of the previous paragraph, the same argument used for the case $v, w \in \Gamma_0$ applies to show that $g_{u,u'} = u^{-n}(u')^m$ for some integers m and n.

We now return to the simple path in C_i joining w_i and w_i'. For each edge $[u, u']$ of this path, $g_{u,u'}$ is a word which does not involve v. Observe that $g_{w,w'}$ is the product of the $g_{u,u'}$, so that $g_{w,w'}$ does not involve v. It follows that the powers of v in g_{v,w_i} and $g_{v,w_i'}$ are the same, showing that $\mu(\phi)$ is well-defined.

It is straightforward to verify that μ_v is a homomorphism.

To see that μ_v is surjective, take any $r(v)$–tuple of integers $(n_1, \dots, n_r)$. The product ϕ of partial conjugations of C_i by $v_i^{n_i}$ satisfies $\mu_v(\phi) = (n_1, \dots, n_r)$.

Finally, we show that $\mu = \prod v \in V_0 \mu_v$ is injective. Suppose ϕ lies in the kernel of μ. Then we can choose a representative ϕ_v such that for any w adjacent to v, $g_{v,w}$ is just a power of w. The same reasoning applied to w implies that $g_{w,v}$ is just a power of v. But $g_{v,w} = g_{w,v}^{-1}$ so we must have $g_{v,w} = 1$. It follows that for any adjacent pair of vertices, $\phi_v = \phi_w$. Since Γ is connected, this gives a representative of ϕ which acts trivially on the join of every vertex; in other words, ϕ is trivial in $\mathrm{Out}^0(A_\Gamma)$. □

Remark 3.10 Since the generators of K_0 given by Proposition 3.9 restrict to inner automorphisms on *every* join, it follows from the theorem that the homomorphism R, which was defined over all maximal joins J instead of just the joins J_v with $v \in V_0$, has the same kernel as R_0.

One advantage of restricting attention to the joins J_v for $v \in V_0$ is that we can further define a *projection homomorphism*, as follows. Since vertices of V_0 are interior, $L_v^\perp$ contains no leaves, hence by Proposition 3.2, every $\phi \in \mathrm{Out}^0(A_\Gamma)$ has a representative ϕ_v which preserves both A_{J_v} and $A_{L_v^\perp}$. Thus ϕ_v descends to an automorphism $\overline{\phi}_v$ of $A_{L_v} = A_{J_v}/A_{L_v^\perp}$. This gives rise to a homomorphism $P_v : \mathrm{Out}^0(A_\Gamma) \to \mathrm{Out}(F\langle L_v\rangle)$. Let P be the product homomorphism

$$P = \prod_{v \in V_0} P_v \colon \mathrm{Out}^0(A_\Gamma) \to \prod \mathrm{Out}(F\langle L_v\rangle).$$

Recall that $\delta_C(v)$ denotes the number of connected components of $\Gamma - \{v\}$.

Proposition 3.11 *The kernel K_P of P is a free abelian group, generated by K_0 and the set of leaf transvections. If Γ is a star $\{v\} * W$ then K_P has rank $|W|$; otherwise, it has rank $\sum_{v\in V_0}(\delta_C(v)-1)$.*

Proof Let ℓ be the number of leaves in Γ. It is clear that leaf transvections are contained in K_P and that they generate a free abelian group of rank ℓ. It is also easy to see that leaf transvections commute with the generators of K_0 since if $u \mapsto uv$ is a leaf transvection, then u and v are connected by an edge and hence belong to the same component of $\Gamma - \{w\}$ for any $w \in \Gamma_0$. Together with K_0, the leaf transvections thus generate a free abelian subgroup of the specified rank, by Proposition 3.9.

It remains only to show that this subgroup is all of K_P. Consider an element ϕ in K_P. For $v \in V_0$, let ϕ_v denote a representative automorphism which preserves A_{J_v} and hence also $A_{L_v^\perp}$. Then for $w \in L_v$, we have $\phi_v(w) = wg$ for some $g \in A_{L_v^\perp}$ (up to conjugation by an element of A_{J_v}). Since w, and hence $\phi_v(w)$, commutes with $A_{L_v^\perp}$, g must lie in the center of $A_{L_v^\perp}$. If v is not cyclic, the center is trivial, so $g = 1$. If v is cyclic, then $g = v^k$. Any automorphism preserve centralizers, so if $k \neq 0$, $\phi_v(C(w)) = C(wv^k) = \langle w, v\rangle$, so $C(w)$ is free abelian of rank 2. This implies that w is a leaf. We conclude that ϕ_v acts as the identity on nonleaf elements of L_v and as leaf transvections on leaf elements. It follows that there exists a product of leaf transvections θ such that $\phi \circ \theta$ lies in K_0. □

We conclude this section with an easy consequence of Proposition 3.11. We say that a group G satisfies the *Tits alternative* if every subgroup of G either contains a nonabelian free group or is virtually solvable. Tits [22] proved that all finitely generated linear groups satisfy the Tits alternative and Bestvina, Feighn and Handel [4; 5] proved that $\mathrm{Out}(F_n)$ does likewise.

Theorem 3.12 *If Γ is connected and triangle-free, then $\mathrm{Out}(A_\Gamma)$ satisfies the Tits alternative.*

Proof It is an easy exercise to check that the property that a group satisfies the Tits alternative is preserved under direct products and abelian extensions. Since $\mathrm{Out}(F\langle L_v\rangle)$ satisfies the Tits alternative by Bestvina–Feign–Handel, it follows that the image of P also does. Since the kernel of P is abelian, we conclude that $\mathrm{Out}^0(A_\Gamma)$, and hence also $\mathrm{Out}(A_\Gamma)$, satisfies the Tits alternative. □

Remark 3.13 Many of the results in this section generalize to higher dimensional right-angled Artin groups. The details will appear in a subsequent paper.

4 Outer space

In this section we introduce "outer space" for a right-angled Artin group A_Γ. We continue to assume that Γ is a connected, triangle-free graph and has diameter ≥ 2.

Let $\mathcal{O}(F)$ denote the unreduced, unprojectivized version of Culler and Vogtmann's outer space for a free group F [11]. This space can be described as the space of minimal, free, isometric actions of F on simplicial trees. (The terms "unreduced" and "unprojectivized" specify that quotient graphs may have separating edges, and that we are not considering homothetic actions to be equal.) Our initial approach to constructing outer space for A_Γ was to consider minimal, free, isometric actions of A_Γ on CAT(0) 2–complexes. In the case of a single join, these turn out to be products of trees. More generally, (under mild hypotheses) such a 2–complex is a union of geodesic subspaces which are products of trees. However, the interaction between these subspaces proved difficult to control and we ultimately found that it was easier to work directly with the tree-products.

4.1 Outer space for a join

Let us examine more closely the case when $\Gamma = U * W$ is a single join. Suppose that $A_\Gamma = F\langle U\rangle \times F\langle W\rangle$ acts freely and cocompactly by isometries on a piecewise Euclidean CAT(0) 2–complex X with no proper invariant subspace.

If neither U nor W is a singleton, then $A_\Gamma = F\langle U\rangle \times F\langle W\rangle$ has trivial center, so the splitting theorem for CAT(0) spaces [6, Theorem 6.21] says that X splits as a product of two one-dimensional CAT(0) complexes (ie trees) $T_U \times T_W$, and the action of A_Γ is *orthogonal*, ie it is the product of the actions of $F\langle U\rangle$ on T_U and $F\langle W\rangle$ on T_W. Twisting an action by an element of $\mathrm{Out}(A_\Gamma)$, which contains $\mathrm{Out}(F\langle U\rangle)\times\mathrm{Out}(F\langle W\rangle)$ as a subgroup of index at most two, preserves the product structure. Thus it makes sense to take as our outer space the product of the Culler–Vogtmann outer spaces $\mathcal{O}(F\langle U\rangle)\times\mathcal{O}(F\langle W\rangle)$.

If $U=\{v\}$ is a single vertex then $A_\Gamma = \mathbb{Z}\times F\langle W\rangle$ where $\mathbb{Z}$ is generated by v. In this case, X is equal to the min set for v, so that X splits as a product $\alpha_v \times T_W$, where α_v is an axis for v and T_W is a tree [6, Theorem 6.8]. We identify the axis α_v with a real affine line, with v acting by translation in the positive direction by an amount $t_v > 0$. The tree T_W has a free $F\langle W\rangle$–action induced by the projection $A_\Gamma \to F\langle W\rangle$, but the action of A_Γ on X need not be an orthogonal action. Recall that $\mathrm{Out}(A_\Gamma)\cong \mathbb{Z}^\ell \rtimes (\mathbb{Z}/2\times \mathrm{Out}(F\langle W\rangle))$ where $\ell = |W|$ and $\mathbb{Z}^\ell$ is generated by leaf transvections. Twisting an orthogonal action by a leaf transvection $w\mapsto wv$ results in an action which is no longer orthogonal. Instead, w now acts as translation in a diagonal direction on the plane in $\alpha_v \times T_W$ spanned by α_v and the axis for w in T_W, ie the translation vector has a nontrivial α_v–component. More generally, for any free minimal action of A_Γ on a CAT(0) 2–complex $X = \alpha_v \times T_W$, the generator v acts only in the α_v–direction,

$$v \cdot (r, x) = (r + t_v, x),$$

while an element $w_i \in W$ has a "skewing constant" $\lambda(w_i)$, ie w_i acts by

$$w_i \cdot (r, x) = (r + \lambda(w_i), w_i \cdot x).$$

A free action of A_Γ on X is thus determined by an $F\langle W\rangle$–tree T_W, the translation length t_v and an ℓ–tuple of real numbers $(\lambda(w_1), \dots, \lambda(w_\ell))$. So it is reasonable in this case, to take for outer space the product $\mathcal{O}(F\langle W\rangle) \times \mathbb{R}_{>0} \times \mathbb{R}^\ell$.

4.2 Tree spaces

Our analysis of the join case motivates the following definition.

Let $J = U * W$ be a join in Γ. An *admissible tree-space* X_J for J is a product of two simplicial, metric trees, T_U and T_W with free, minimal, isometric actions of $F\langle U\rangle$ and $F\langle W\rangle$ respectively, and an action of $A_J = F\langle U\rangle \times F\langle W\rangle$ on $X_J = T_U \times T_W$ of the following type.

(1) If J contains no leaves, then the action is the product of the given actions,

$$(g_1, g_2) \cdot (x_1, x_2) = (g_1 \cdot x_1, g_2 \cdot x_2).$$

(2) Suppose J contains leaf vertices, say in W. This forces $U = \{v\}$, so $A_J = \langle v\rangle \times F\langle W\rangle$, $T_U = \mathbb{R}$ and v acts on T_U as translation by some positive real number t_v. Then there exists a homomorphism $\lambda\colon F\langle W\rangle \to \mathbb{R}$ which is zero on nonleaf vertices of W, such that

$$(v^n, g) \cdot (r, x) = (r + nt_v + \lambda(g),\ g \cdot x).$$

We remark that the definition of admissible depends not only on the join J, but on the graph Γ as well since Γ determines which vertices are considered as leaves. Since Γ is fixed throughout, this should not cause any confusion.

A point in outer space for A_Γ will be a collection of admissible tree-spaces satisfying certain compatibility conditions. Recall that to each interior vertex of Γ we have associated a maximal join $J_v = L_v * L_v^\perp$. If e is an edge from v to w, set $J_e = J_v \cap J_w = L_v^\perp * L_w^\perp$. If e lies in Γ_0, then J_e contains no leaves.

Definition 4.1 A *graph of tree-spaces* $\mathfrak{X} = \{X_v, X_e, i_{e,v}\}$ for A_Γ consists of the following data:

(1) for each vertex $v \in \Gamma_0$, an admissible tree-space X_v for J_v

(2) for each edge $e \in \Gamma_0$ with vertices v and w, an admissible tree space X_e for J_e and a pair of A_{J_e}–equivariant isometric embeddings:

$$X_v \xleftarrow{i_{e,v}} X_e \xrightarrow{i_{e,w}} X_w$$

We define *outer space* for A_Γ to be the set

$$\mathcal{O}(A_\Gamma) = \{\mathfrak{X} \mid \mathfrak{X} \text{ is a graph of tree-spaces for } A_\Gamma\}/\sim$$

where $\sim$ is the equivalence relation induced by replacing any X_v (respectively X_e) with an equivariantly isometric space X'_v (respectively X'_e), and composing the associated connecting maps $i_{e,v}$, by the equivariant isometry. A natural topology for $\mathcal{O}(A_\Gamma)$ will be described in Section 4.4.

Since J_e contains no leaves, the group $A_{J_e} = F\langle L_v^\perp\rangle \times F\langle L_w^\perp\rangle$ acts orthogonally on X_e and the maps $i_{e,v}$ and $i_{e,w}$ split as products. Write $X_v = T_v \times T_v^\perp$, where T_v is an $F\langle L_v\rangle$–tree, and $T_v^\perp$ is an $F\langle L_v^\perp\rangle$–tree. Then, up to equivariant isometry, we may assume that $X_e = T_v^\perp \times T_w^\perp$, and that $i_{e,v} = (i_1, i_2)$ is the identity on the first factor while $i_{e,w} = (j_1, j_2)$ is the identity on the second. Denote the embedding $i_2\colon T_w^\perp \to T_v$ by $i(w,v)$. The *image* of $i(w,v)$ is uniquely determined, namely it is the minimal $F\langle L_w^\perp\rangle$–invariant subtree of T_v. However, the *map* $i(w,v)$ is not necessarily unique. It is unique if w is not cyclic, since in this case $T_w^\perp$ has no nontrivial equivariant isometries. But if w is cyclic then $T_w^\perp$ is a real line and the possible equivariant inclusions i_2 are parameterized by $\mathbb{R}$.

4.3 Basepoints

To keep track of the inclusions $i(w,v)$, it will be convenient to introduce basepoints. These will also play a crucial role in the proof of the contractibility of $\mathcal{O}(A_\Gamma)$.

Basepoints for free actions of free groups Let $F\langle S\rangle$ be a finitely generated free group with a specified basis S of cardinality at least 2, and let T be a metric tree with a minimal, free, isometric action of $F\langle S\rangle$. Each generator $s \in S$ preserves a unique line $\alpha(s)$ in T called the axis for s. Orient the axis so that s acts as translation in the positive direction. We choose a base point $b(s)$ on $\alpha(s)$ as follows.

For each generator $t \neq s$, the set of points on $\alpha(s)$ of minimal distance from $\alpha(t)$ is a closed connected interval (possibly a single point). Define the projection $p(t,s)$ of $\alpha(t)$ on $\alpha(s)$ to be the initial point of this interval, and the basepoint $b(s)$ to be the minimum of these projections, with respect to the ordering given by the orientation of $\alpha(s)$. (See Figure 5). The basepoints $b(s)$ will be called the *unrestricted basepoints* of the action.

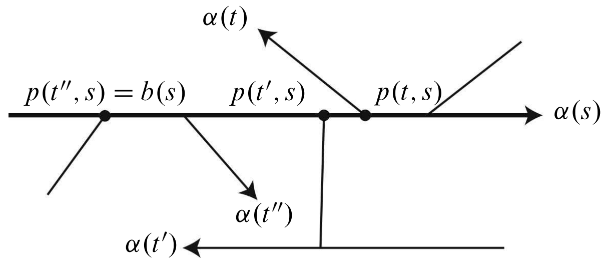

Figure 5: Projections and basepoint on the axis $\alpha(s)$

Basepoints in $\mathfrak{X}$ Suppose now that we have a graph $\mathfrak{X}$ of tree-spaces for A_Γ. Assume Γ is not a star, so Γ_0 contains at least two vertices. If $L_w^\perp$ contains at least two vertices, then for each $u \in L_w^\perp$ we let $b(u)$ be the unrestricted basepoint on the axis $\alpha(u)$ for the action of $F\langle L_w^\perp \rangle$ on $T_w^\perp$.

If w is cyclic the axis $\alpha(w)$ is the *entire* tree $T_w^\perp$, in which case we cannot use this method to choose a basepoint. If v is adjacent to w in Γ_0, we have an equivariant isometry $i(w,v)\colon T_w^\perp \to T_v$. In T_v, the element w has an axis $\alpha_v(w)$ with an unrestricted basepoint $b_v(w)$. We can use $i(w,v)$ to pull this back to a point $c_v(w)$ on $\alpha(w)$.

In this way, we get a point $c_v(w)$ on $\alpha(w)$ for each v adjacent to w in Γ_0. We take the minimum $b(w)$ of these points as a basepoint for $\alpha(w)$. (See Figure 6.)

4.4 Topology on outer space

We will define the topology on $\mathcal{O}(A_\Gamma)$ by embedding $\mathcal{O}(A_\Gamma)$ in a product of topological spaces and giving it the resulting subspace topology.

In the case that Γ is a star $\{v\} * W$, we have seen that $\mathcal{O}(A_\Gamma)$ can be identified with $\mathcal{O}(F\langle W\rangle) \times \mathbb{R}_{>0} \times \mathbb{R}^\ell$, where $\ell = |W|$. We take this to be a homeomorphism.

For any graph Γ, recall that V_{cyc} denotes the set of cyclic vertices and $\delta_0(v)$ the valence of v in Γ_0.

Proposition 4.2 *Let Γ be a graph which is not a star. Let ℓ be the number of leaves of Γ, and let $k = \sum_{v \in V_{\text{cyc}}} \delta_0(v)$. Then there is an injective map*

$$\mathcal{O}(A_\Gamma) \hookrightarrow \left(\prod_{v \in \Gamma_0} \mathcal{O}(F\langle L_v \rangle) \right) \times \mathbb{R}^k \times \mathbb{R}^\ell.$$

The image of this map is of the form $Y \times Q \times \mathbb{R}^\ell$, where Y is a subspace of $\prod_{v \in V_0} \mathcal{O}(F\langle L_v\rangle)$ and Q is a piecewise linear subspace of dimension $\sum_{v \in V_{\text{cyc}}} (\delta_0(v) - 1)$.

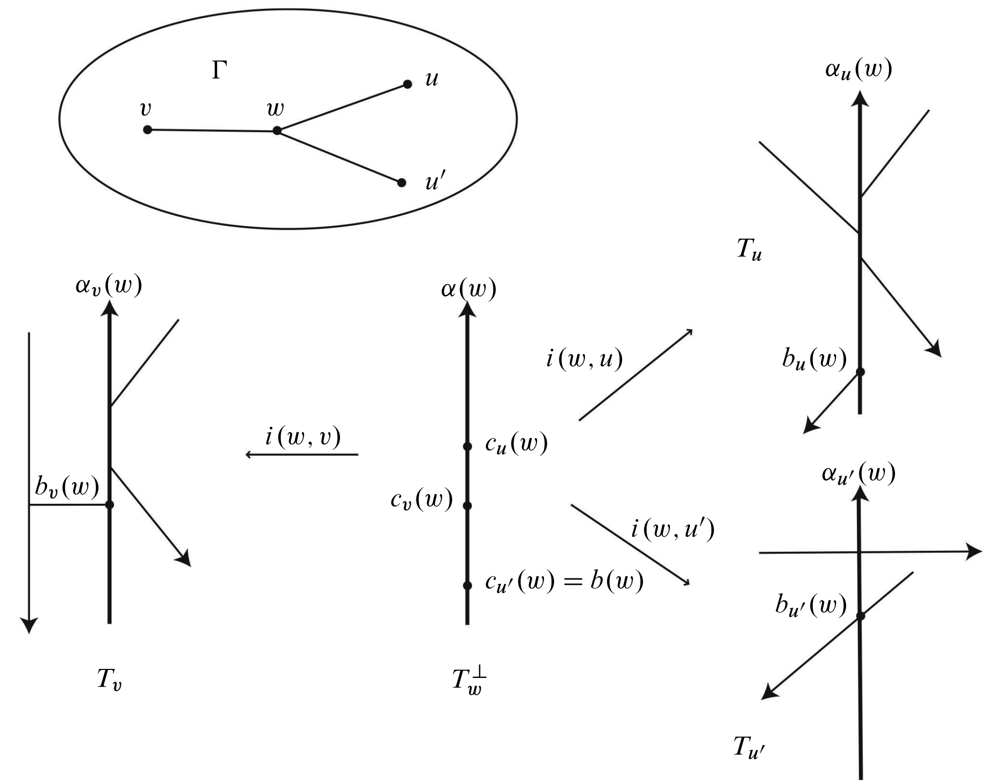

Figure 6: Basepoint on the axis for w in $T_w^\perp$

Proof Let $\mathfrak{X}$ be an element of $\mathcal{O}(A_\Gamma)$. The vertex spaces X_v are determined by free group actions on two trees, $T_v^\perp$ and T_v, together with a real number $\lambda(w)$ for each leaf vertex $w \in L_v$. Since Γ is not a star and Γ_0 is connected, there is at least one vertex w of L_v which is also in Γ_0. The tree $T_v^\perp$ is equivariantly isometric to the minimal $F\langle L_v^\perp \rangle$ subtree of T_w for this w. Thus the adjacency relations of Γ, the trees T_v for $v \in V_0$ and the real numbers $\lambda(w)$ for w a leaf completely determine the vertex spaces X_v. As noted above, the edge spaces are of the form $X_e = T_u^\perp \times T_v^\perp$ with orthogonal action of $F\langle L_u^\perp \rangle \times F\langle L_v^\perp \rangle$, so that they too are determined by the actions on the trees T_v.

It remains to account for the connecting maps between edge spaces and vertex spaces. If w is not cyclic, then for any vertex $v \in L_w \cap \Gamma_0$, there is a unique equivariant map from $T_w^\perp$ into T_v, so the connecting map $X_e \to X_v$ is uniquely determined. If w is cyclic, then $T_w^\perp$ is a copy of the real line, and an equivariant embedding of $T_w^\perp \to T_v$ is determined by the position of the image of $b(w)$ on $\alpha_v(w)$ ie by the difference

$\Delta_v(w) = c_v(w) - b(w)$, where the points $c_v(w)$ and $b(w)$ are as defined in Section 4.3 and illustrated in Figure 6.

Associated to $\mathfrak{X}$ we have a point $T_v \in \mathcal{O}(F\langle L_v\rangle)$ for each $v \in V_0$, a real number $\lambda(w)$ for each leaf w, and a $\delta_0(w)$–tuple of real numbers $(\Delta_v(w))$, for each $w \in V_{\mathrm{cyc}}$. This data completely determines $\mathfrak{X}$. Thus, we have an injective map

$$f\colon \mathcal{O}_\Gamma \to \left(\prod_{v\in V_0} \mathcal{O}(F\langle L_v\rangle)\right) \times \mathbb{R}^k \times \mathbb{R}^\ell.$$

Now consider the image of f. There are no restrictions at all on the numbers $\lambda(w)$; given any $\mathfrak{X}$ in $\mathcal{O}(A_\Gamma)$, arbitrarily changing $\lambda(w)$ for any leaf w gives rise to another valid graph of tree-spaces. As for the numbers $\Delta_v(w) = c_v(w) - b(w)$, since $b(w)$ was defined to be the infimum of the $c_v(w)$, we must have $\Delta_v(w) \geq 0$ for all $v \in L_w \cap V_0$, and at least one of these must equal 0. There are no other restrictions; the points $c_v(w)$ can be varied independently of each other by changing a single embedding $i(w, v)$.

Let $Q_w \subset \mathbb{R}^{\delta_0(w)}$ be defined by

$$Q_w = \{(r_1, \dots r_{\delta_0(w)}) \mid r_i \geq 0 \text{ for all } i \text{ and } r_j = 0 \text{ for some } j\}.$$

In other words, Q_w is the boundary of the positive orthant of $\mathbb{R}^{\delta_0(w)}$. So the image of f is of the form $Y \times \prod_{w\in V_{\mathrm{cyc}}} Q_w \times \mathbb{R}^\ell$ where Y is a subspace of $\prod_{v\in V_0} \mathcal{O}(F\langle L_v\rangle)$. □

Remark 4.3 For future reference, we remark that Q_w can also be identified with $\mathbb{R}^{\delta_0(w)}/\mathbb{R}$ (where $\mathbb{R}$ acts as diagonal translation) with coordinates given by the $c_v(w)$'s.

Example 4.4 If $\Gamma = U * W$ is a join with $|U|, |W| \neq 1$, then Γ_0 consists of a single edge e joining a pair of vertices $u \in U$ and $w \in W$. In this case, $J_e = J_u = J_w = \Gamma$ and a point in $\mathcal{O}(A_\Gamma)$ is determined by a single tree space for Γ (since there is a unique equivariant isometry of such a tree-space). In this case, $\ell = \delta_0(u) - 1 = \delta_0(w) - 1 = 0$ and Y is all of $\mathcal{O}(F\langle U\rangle) \times \mathcal{O}(F\langle W\rangle)$. Thus, the definition of outer space for a join agrees with that proposed in Section 4.1.

From now on, $\mathcal{O}(A_\Gamma)$ will be viewed as a topological space with the subspace topology induced by the embedding f described above. Each space $\mathcal{O}(F)$ is endowed with the equivariant Gromov–Hausdorff topology. In this topology, a neighborhood basis of an action of F on a tree T is given by the sets $N(X, H, \epsilon)$, where $X = \{x_1, \dots, x_N\}$ is a finite set of points in T and $H = \{g_1, \dots, g_M\}$ is a finite set of elements of F. An action of F on T' is in this neighborhood if there is a subset $X' = \{x'_1, \dots, x'_N\}$ of T' such that for all i, j, k,

$$|d_T(x_i, g_j x_k) - d_{T'}(x'_i, g_j x'_k)| < \epsilon.$$

5 Contractibility

In this section we prove the following theorem:

Theorem 5.1 *For any connected, triangle-free graph Γ, the space $\mathcal{O}(A_\Gamma)$ is contractible.*

The proof follows ideas of Skora [21] and Guirardel and Levitt [16; 15] on "unfolding" trees. In their work, however, the unfolding of a tree was defined with respect to a single basepoint. In our case we will need to preserve several basepoints.

We first consider a single action. Let $F\langle S\rangle$ be a free group with a preferred generating set S and let $T \in \mathcal{O}(F\langle S\rangle)$ be an $F\langle S\rangle$–tree.

Proposition 5.2 *Let $P = \{p(t,s)\}$ be any set of projections, with $s,t \in S$. The convex hull $B(P,T)$ of P in T depends continuously on T.*

Proof If T' is any other $F\langle S\rangle$–tree, let $P' = \{p'(t,s)\}$ be the corresponding projections in T'. We must show: given $\epsilon > 0$ and $T \in \mathcal{O}(F\langle S\rangle)$, there is a neighborhood N of T such that $B(P',T')$ is ϵ–close to $B(P,T)$ for T' in N.

To show $B(P',T')$ is ϵ–close to $B(P,T)$ in the Gromov–Hausdorff topology, we need to take an arbitrary finite set of points X in $B(P,T)$ and find corresponding points X' in $B(P',T')$ such that the distances between points in X' are within ϵ of the distances between the corresponding points in X.

For T' close to T in the equivariant Gromov–Hausdorff topology, we can find points X' in T' with the required properties, but that is not good enough; we need the points X' to be in $B(P',T')$. We can fix this by projecting each point $x' \in X'$ onto $B(P',T')$, but we need to be sure that this projection is sufficiently close to x'.

Each projection $p = p(t,s)$ is uniquely determined by the following set of equations [16]:

$$\begin{aligned} d(p,sp) &= \ell(s) \\ d(p,tp) &= \ell(t) + 2D \\ d(s^{-1}p, t^{-1}p) &= \ell(s) + \ell(t) + 2D \\ d(s^{-1}p, tp) &= \ell(s) + \ell(t) + 2D \end{aligned} \tag{1}$$

where D is the distance from $\alpha(s)$ to $\alpha(t)$. If T' is in the $N(\{p\},\{s,t,st^{-1},st\},\epsilon)$–neighborhood of T, then we can find q' in T' satisfying equations (1) up to ϵ. By

Paulin [19], the corresponding lengths $\ell'(s), \ell'(t)$ and D' are within ϵ of $\ell(s), \ell(t)$ and D, so that q' satisfies the analogous equations $(1)'$ up to 4ϵ. The projection $p' = p'(t,s)$ in T' satisfies the equations $(1)'$ exactly. It is an easy exercise to verify that this implies that $d(p', q') < 3\epsilon/2$.

Now let $Y = X \cup P$, and let $H = S \cup S^{-1}$. If T' is in the $N(Y, H, \epsilon)$–neighborhood of T, then we can find $q'(t,s)$ and X' in T' with $d(q'(t,s), p'(t,s)) < 3\epsilon/2$, $|d(x_1', x_2') - d(x_1, x_2)| < \epsilon$ for all $x_1, x_2 \in X$, and $|d(x, p(t,s)) - d(x', q'(t,s))| < \epsilon$ for all $x \in X$. We claim that the projection of x' onto $B(P', T')$ is within 9ϵ of x' for each $x' \in X'$.

Each point x of $B(P,T)$ is determined by its distances to the points $p(s,t)$ (x is on some straight arc between $p_1 = p(t_1, s_1)$ and $p_2 = p(t_2, s_2)$; both the fact that it lies on this arc and its position on the arc are determined by its distances to p_1 and p_2).

If x is on $[p_1, p_2]$ then

$$\begin{aligned} d(p_1', x') + d(x', p_2') &\le d(p_1', q_1') + d(q_1', x') + d(x', q_2') + d(q_2', p_2') \\ &\le d(p_1, x) + d(x, p_2) + 3\epsilon/2 + 3\epsilon/2 + 2\epsilon \\ &= d(p_1, p_2) + 5\epsilon \end{aligned}$$

and
$$\begin{aligned} d(p_1', p_2') &\ge d(q_1', q_2') - d(p_1', q_1') - d(p_2', q_2') \\ &\ge d(p_1, p_2) - \epsilon - 3\epsilon \\ &= d(p_1, p_2) - 4\epsilon. \end{aligned}$$

For any three points in a tree, we have $d(a, [b,c]) = \frac{1}{2}(d(b,a) + d(a,c) - d(b,c))$, so

$$d(x', [p_1', p_2']) = \frac{d(p_1', x') + d(x', p_2') - d(p_1', p_2')}{2} \le \frac{9\epsilon}{2}.$$

Since $[p_1', p_2'] \subset B(P', T')$, the projection of x' onto $B(P', T')$ is within $9\epsilon/2$ of x', and we may replace x' by this projection. □

Now let P be a set of projections with at least one projection on each axis $\alpha(s)$, for $s \in S$. We use the convex hull $B(P,T)$ to define a new $F\langle S\rangle$–tree $T_0(P,T)$ as follows. For each $s \in S$, let $b(s)$ be the minimum of the projections $p(t,s)$ in P (where $\alpha(s)$ is oriented in the direction of translation by s.) Form a labeled graph $R(P,T)$ by attaching an oriented circle to $b(s)$ labeled s, whose length is the translation length of s acting on T. We call $R(P,T)$ a "stemmed rose." The subgraph $B(P,T)$ is the "stem" and the circles are the "petals". Note that for any point $x \in B(P,T)$, there is a canonical identification of $\pi_1(R, x)$ with $F\langle S\rangle$. Lifting to the universal cover of $R(P,T)$ defines an action $T_0(P,T)$ of $F\langle S\rangle$ on a tree.

Lemma 5.3 *The action $T_0(P,T)$ depends continuously on T.*

Proof By Proposition 5.2 the subtree $B(P,T)$ and basepoints $b(s)$ depend continuously on T, and by [19] the lengths $\ell(s)$ for $s \in S$ depend continuously on T. Together, this data completely determines $T_0(P,T)$. □

Now fix a copy of $B(P,T)$ in $T_0(P,T)$. Then there is a unique equivariant map $f_T\colon T_0(P,T) \to T$ which is the identity on $B(P,T)$. This map is a "folding" in the sense of Skora: for any segment $[x,y]$ on $T_0(P,T)$, there exist a nontrivial initial segment $[x,z] \subset [x,y]$, $x \neq z$, such that f_T restricted to $[x,z]$ is an isometry. We call such a folding map a morphism. Following [16], we can use this morphism to define a path from T to $T_0(P,T)$ in $\mathcal{O}(F\langle S\rangle)$ as follows. For $r \in [0,\infty)$, let $T_r(P,T)$ be the quotient of $T_0(P,T)$ by the equivalence relation $x \sim_r y$ if $f_T(x) = f_T(y)$ and $f_T[x,y]$ is contained in the ball of radius r around $f_T(x)$. Since f_T is a morphism, this gives $T_0(P,T)$ for $r=0$. By [21], $T_r(P,T)$ is a tree for all r, the action of $F\langle S\rangle$ on $T_0(P,T)$ descends to an action on $T_r(P,T)$ and for r sufficiently large, $T_r(P,T) = T$. Furthermore, the path $p_T\colon [0,1] \to \mathcal{O}(F\langle S\rangle)$ defined by $p_T(0) = T$ and $p_T(r) = T_{(1-r)/r}(P,T)$ for $r > 0$ is continuous.

Lemma 5.4 *The paths p_T define a deformation retraction of $\mathcal{O}(F\langle S\rangle)$ onto the subspace consisting of universal covers of stemmed roses marked by the generators S, which is contractible.*

Proof By Lemma 5.3, $T_0(P,T)$ depends continuously on T. This implies that the morphisms $f_T\colon T_0(P,T) \to T$ depend continuously on T and then, by Skora's argument (see Proposition 3.4 of [16]), that the folding paths stay close, ie $T_r(P,T)$ is close to $T_r(P',T')$ for all r. Therefore these paths give a deformation retraction of all of $\mathcal{O}(F\langle S\rangle)$ to the subspace of actions covering "stemmed roses" with petals marked by the generators S. Contracting all stems to points defines a further deformation onto actions covering the standard rose with lengths on its edges. Such an action is determined by the lengths of the generators, so this space is a product of positive rays, which is contractible. □

We are now ready to produce a contraction of $\mathcal{O}(A_\Gamma)$.

By definition of the topology, $\mathcal{O}(A_\Gamma)$ is homeomorphic to the product $Y \times Q \times \mathbb{R}^\ell$ from Proposition 4.2. The space $Q \subset \mathbb{R}^k$ is clearly contractible, so it remains only to show that $Y \times \{0\}$ is contractible where 0 is the origin in $Q \times \mathbb{R}^\ell \subset \mathbb{R}^{k+\ell}$. A point (T_v) in $\prod_{v \in V_0} \mathcal{O}(F\langle L_v\rangle)$ lies in Y if and only if it satisfies certain compatibility conditions.

Namely for each $w \in V_0$, the minimal $F\langle L_w^\perp\rangle$–trees in T_v must be equivariantly isometric for all $v \in L_w \cap V_0$.

By Proposition 4.2, a point $\mathfrak{X}$ in $\mathcal{O}(A_\Gamma)$ is determined by:

- for each $v \in \Gamma_0$, a tree T_v with $F\langle L_v\rangle$–action and a tree $T_v^\perp$ with $F\langle L_v^\perp\rangle$–action
- for each edge $[v,w]$ in Γ_0 an $F\langle L_w^\perp\rangle$–equivariant isometry $i(w,v)$ from $T_w^\perp$ into T_v
- for each leaf w of Γ_0, a real number $\lambda(w)$

If w is cyclic, the isometry $i(w,v)$ is determined by a real number $\Delta_v(w)$. If w is not cyclic, there is a unique equivariant isometry into T_v. We are assuming $\mathfrak{X}$ is in $Y \times \{0\}$, so $\Delta_v(w) = 0$ for all v, w and $\lambda(w) = 0$ for all leaves w.

Given a point $\mathfrak{X}$ in $Y \times \{0\}$, we want to produce a new point $\mathfrak{X}_0$ in $Y \times \{0\}$ and a morphism $f_\mathfrak{X}\colon \mathfrak{X}_0 \to \mathfrak{X}$. We will show $f_\mathfrak{X}$ depends continuously on $\mathfrak{X}$, and that the resulting paths $p_\mathfrak{X}$ define a deformation retraction of $Y \times \{0\}$ to a contractible subspace.

Definition of $\mathfrak{X}_0$ Let $v \in V_0$. We set $(T_0)_v = T_0(P_v, T_v)$, where P_v is chosen as follows. If $u \in L_v$ is equivalent to some noncyclic vertex $w \in \Gamma_0$, include all projections $p(t,u)$ for $t \in [w], t \neq u$. For all other $u \in L_v$, take all projections $p(t,u)$, for $t \in L_v, t \neq u$.

If v is not cyclic, set $(T_0)_v^\perp = T_0(P_v^\perp, T_v^\perp)$, where $P_v^\perp$ consists of all projections $p(t,u)$ for $t, u \in [v], t \neq u$. If v is cyclic then $(T_0)_v^\perp$ is a linear tree, and we fix a basepoint $b(v)$ arbitrarily.

If $[v,w]$ is an edge of Γ_0, the image of $T_v^\perp$ in T_w under $i = i(v,w)$ is the unique $L_v^\perp$–invariant subtree of T_v. The image of each projection $p(t,u)$ in $P_v^\perp$ is the analogous projection in T_v, so the image of $B(P_v^\perp, T_v^\perp)$ is contained in $B(P_w, T_w)$, is isometric to the intersection of $B(P_v, T_v)$ with $i(T_v^\perp)$ and has the same basepoints. We can therefore construct an isometric embedding $i_0(v,w)\colon T_0(P_v^\perp, T_v^\perp) \to T_0(P_w, T_w)$. If v is cyclic, we send the basepoint $b(v)$ of $(T_0)_v^\perp$ to the basepoint $b_w(v)$ on $\alpha_w(v)$.

The actions $(T_0)_v$ and $(T_0)_v^\perp$ and isometries $i_0(v,w)$ form a compatible system, giving a point $\mathfrak{X}_0$ of $\mathcal{O}(A_\Gamma)$. The real numbers $\Delta_w(v)$ for this point are all zero, so in fact $\mathfrak{X}_0$ is in $Y \times \{0\}$.

Morphisms $f_{\mathfrak{X}}$ and paths $p_{\mathfrak{X}}$ For each $v \in V_0$, we have morphisms

$$f_v \colon T_0(P_v, T_v) \to T_v \quad \text{and} \quad f_v^{\perp} \colon T_0(P_v^{\perp}, T_v^{\perp}) \to T_v^{\perp}$$

as defined above, as well as folding paths p_v from $T_0(P_v, T_v)$ to T_v and $p_v^{\perp}$ from $T_0(P_v^{\perp}, T_v^{\perp})$ to $T_v^{\perp}$. All of these objects depend continuously on $\mathfrak{X}$ by Lemma 5.3.

We next observe that if v and w are connected by an edge, the morphism $f_w^{\perp}$ is the restriction of f_v to $T_0(P_w^{\perp}, T_w^{\perp})$. Thus points in $T_0(P_w^{\perp}, T_w^{\perp})$ are identified under $f_w^{\perp}$ if and only if they are identified under f_v. Since the image of $T_w^{\perp}$ is a geodesic subspace of T_v, the folding process produces equivariant isometries $i_r(w,v) \colon p_w^{\perp}(r) \to p_v(r)$ for all r, with $\Delta_v(w) = 0$ for all cyclic vertices w.

Thus the paths p_v and $p_v^{\perp}$ form a compatible system $p_{\mathfrak{X}}$ of paths in $Y \times \{0\}$ and give a deformation retraction of $Y \times \{0\}$ onto the subspace whose vertex actions are of the form $T_0(P_v, T_v)$, ie with quotient a stemmed rose whose petals are marked by the generators L_v. Simultaneously contracting all stems to a point gives a further deformation retraction onto the subspace $Y_0 \times \{0\}$ of $Y \times \{0\}$ whose trees are universal covers of roses (without stems) marked by the generators L_v. These roses are uniquely defined by the translation lengths of the generators, so $Y_0 \times \{0\}$ is homeomorphic to a product of positive real rays. It follows that $Y \times \{0\}$ is contractible. This completes the proof of Theorem 5.1.

6 The action of the pure outer automorphism group

In this section we show that there is a proper (right) action of $\mathrm{Out}^0(A_\Gamma)$ on $\mathcal{O}(A_\Gamma)$. We continue to assume that Γ is connected, triangle-free, and contains more than one edge.

Let ϕ be an element of $\mathrm{Out}^0(A_\Gamma)$. Recall from Lemma 3.8 that if v and w are connected by an edge e in Γ_0 and ϕ_v, ϕ_w are representatives of $\phi \in \mathrm{Out}^0(A_\Gamma)$ preserving A_{J_v} and A_{J_w} respectively, then there exists $g_v \in A_{J_v}$ and $g_w \in A_{J_w}$ such that $c(g_v) \circ \phi_v = c(g_w) \circ \phi_w$. Setting $\phi_e = c(g_v) \circ \phi_v = c(g_w) \circ \phi_w$ gives a representative of ϕ which preserves both A_{J_v} and A_{J_w}, and hence also A_{J_e}. The automorphism ϕ_e is unique up to conjugation by an element of A_{J_e}.

Now let $\mathfrak{X} = \{X_v, X_e, i_{e,v}\}$ be an element of $\mathcal{O}(A_\Gamma)$ where $i_{e,v}$ denotes the isometric embedding $X_e \to X_v$. Let $X_v^{\phi_v}$ denote the space X_v with the A_{J_v} action twisted by ϕ_v. Notice that translation by g_v is an equivariant isometry $t(g_v) \colon X_v^{\phi_v} \to X_v^{\phi_e}$ (where the translation is taken with respect to the original action on X_v). Hence

$$X_e^{\phi_e} \xrightarrow{i_{e,v}} X_v^{\phi_e} \xrightarrow{t(g_v)^{-1}} X_v^{\phi_v}$$

is an equivariant embedding. We define

$$\mathfrak{X}\cdot\phi = \{X_v^{\phi_v}, X_e^{\phi_e}, t(g_v)^{-1} i_{e,v}\}.$$

It is straightforward to check that this is independent of the choices of ϕ_v and ϕ_e: any other choice gives an equivariantly isometric graph of tree-spaces.

To see that this defines an action, we must verify that if ρ is another element of $\mathrm{Out}^0(A_\Gamma)$, then $(\mathfrak{X}\cdot\phi)\cdot\rho = \mathfrak{X}\cdot(\phi\rho)$. Suppose

$$\phi_e = c(g_v)\circ\phi_v = c(g_w)\circ\phi_w \quad \text{and} \quad \rho_e = c(h_v)\circ\rho_v = c(h_w)\circ\rho_w.$$

Then $(\mathfrak{X}\cdot\phi)\cdot\rho = \{(X_v^{\phi_v})^{\rho_v}, (X_e^{\phi_e})^{\rho_e}, t'(h_v)^{-1}t(g_v)^{-1} i_{e,v}\}$ where t' denotes translation with respect to the (twisted) action on $X_v^{\phi_v}$.

To compute $\mathfrak{X}\cdot(\phi\rho)$, note that since ϕ_v and ρ_v preserve A_{J_v}, so does their composite, so without loss of generality, we may choose $(\phi\rho)_v = \phi_v\rho_v$. Observe also that

$$\phi_e\rho_e = c(g_v)\phi_v c(h_v)\rho_v = c(g_v\phi_v(h_v))\phi_v\rho_v = c(k_v)(\phi\rho)_v$$

where $k_v = g_v\phi_v(h_v)$ and likewise

$$\phi_e\rho_e = c(g_w)\phi_w c(h_w)\rho_w = c(g_w\phi_v(h_w))\phi_w\rho_w = c(k_w)(\phi\rho)_w.$$

It follows that we can take $(\phi\rho)_e = \phi_e\rho_e$ and that $t(k_v)^{-1} = t'(h_v)^{-1}t(g_v)^{-1}$ which completes the argument.

Theorem 6.1 *The action of* $\mathrm{Out}^0(A_\Gamma)$ *on* $\mathcal{O}(A_\Gamma)$ *defined above is proper.*

Proof We must show that for any compact set $C \subset \mathcal{O}(A_\Gamma)$,

$$S(C) = \{\phi \in \mathrm{Out}^0(A_\Gamma) \mid C \cap C\phi \neq \varnothing\}$$

is finite. Consider first the case in which Γ is a single join $\Gamma = U * W$. If $F\langle U\rangle$ and $F\langle W\rangle$ are both nonabelian, then $\mathcal{O}(A_\Gamma) = \mathcal{O}(F\langle U\rangle)\times\mathcal{O}(F\langle W\rangle)$ and the action is proper by [11]. If $U = \{v\}$, then $\mathcal{O}(A_\Gamma) \cong \mathbb{R}^\ell \times \mathbb{R}_{>0} \times \mathcal{O}(F\langle W\rangle)$ where the $\mathbb{R}^\ell$ keeps track of the skewing homomorphism $\lambda\colon F\langle W\rangle \to \mathbb{R}$ and $\mathbb{R}_{>0}$ records the translation length of v. Thus, we can specify an element of $\mathcal{O}(A_\Gamma)$ by a triple (λ, t_v, T_W). The group $\mathrm{Out}(A_\Gamma)$ decomposes as a semi-direct product, $\mathrm{Out}(A_\Gamma) \cong \mathbb{Z}^l \rtimes (\mathbb{Z}/2 \times \mathrm{Out}(F\langle W\rangle))$. Denoting an element of $\mathrm{Out}(A_\Gamma)$ by a triple $(\mathbf{z}, \epsilon, \phi)$ according to this decomposition, the action of $\mathrm{Out}(A_\Gamma)$ on $\mathcal{O}(A_\Gamma)$ is given by

$$(\lambda, t_v, T_W)\cdot(\mathbf{z}, \epsilon, \phi) = (\epsilon(\lambda + t_v\mathbf{z}), t_v, T_W^{\phi}).$$

In any compact set in $\mathcal{O}(A_\Gamma)$, the translation lengths t_v are bounded away from zero. Since the action of $\mathrm{Out}(F\langle W\rangle)$ on $\mathcal{O}(F\langle W\rangle)$ is proper, it is now easy to see that the action of $\mathrm{Out}(A_\Gamma)$ on $\mathcal{O}(A_\Gamma)$ is proper.

Now let Γ be arbitrary, and let R_0 be the restriction homomorphism from Proposition 3.9. We first show that the kernel K_0 of R_0 acts properly on $\mathcal{O}(A_\Gamma)$. An element $\phi \in K_0$ acting on $\mathfrak{X}$ fixes all vertex spaces X_v and all edge spaces X_e and acts only on the connecting maps $i_{e,v}$, or equivalently on the factor Q in the product decomposition of $\mathcal{O}(A_\Gamma)$. By Remark 4.3, $Q = \prod_{w \in V_{\mathrm{cyc}}} Q_w$, where Q_w can be identified with $\mathbb{R}^{\delta_0(w)}/\mathbb{R}$ with coordinates given by the basepoints $c_v(w)$ for $v \in L_w \cap \Gamma_0$. We claim that the action of K_0 on the Q_w factor is given by the homomorphism $\mu_w \colon K_0 \to \mathbb{Z}^{\delta_0(w)}/\mathbb{Z}$ described in the proof of Proposition 3.9, with $m \in \mathbb{Z}$ acting on $c_v(w)$ as translation by w^m. The product $\mu = \prod \mu_w$ is an isomorphism, so it follows easily from the claim that the action of K_0 is proper.

To prove the claim, recall from the proof of Proposition 3.9, that for $\phi \in K_0$, and $v \in L_w \cap \Gamma_0$, we can write $c(v^n) \circ \phi_v = c(w^m) \circ \phi_w$ for some n, m. The v–factor of $\mu_w(\phi)$ is defined to be $-m$. Rewriting this equation as

$$c(w^{-m}) \circ \phi_v = c(v^{-n}) \circ \phi_w$$

we see that we can choose ϕ_e to be $c(w^{-m}) \circ \phi_v$. Then ϕ takes the connecting map $i_{e,v} \colon X_e \to X_v$ to

$$t(w^m)\, i_{e,v} = i_{e,v}\, t(w^m) \colon T_v^\perp \times T_w^\perp \to T_v^\perp \times T_v$$

and similarly for $i_{e,w}$. Note that $i_{e,v}\, t(w^m)$ is the identity on the first factor (assuming that was the case for $i_{e,v}$) and $i_{e,w} t(v^n)$ is the identity on the second. (This was the reason for our particular choice of ϕ_e.) Thus, the basepoint $c_v(w)$ in $\mathfrak{X} \cdot \phi$ is defined to be the inverse image of the natural basepoint $b_v(w) \in T_v$ under this connecting map. It is the w^{-m} translate of the basepoint $c_v(w)$ in $\mathfrak{X}$. This proves the claim.

To show that the whole group $\mathrm{Out}^0(A_\Gamma)$ acts properly, note that since the action of $\mathrm{Out}(A_{J_v})$ on $\mathcal{O}(A_{J_v})$ is proper for each J_v, for any compact C the image of $S(C)$ under R_0 is finite. Thus $S(C)$ is contained in a finite set of right cosets $K_0\phi_1, \dots K_0\phi_m$. Let $C_i = C \cup C\phi_i^{-1}$. If $\rho \in K_0$ is such that $C\rho\phi_i \cap C \neq \varnothing$, then $C_i\rho \cap C_i \neq \varnothing$, so by the paragraph above, there are only finitely many such ρ for each ϕ_i. We conclude that $S(C)$ is finite. □

7 Virtual cohomological dimension

Since $\mathrm{Out}(A_\Gamma)$ has torsion, its cohomological dimension is infinite. However, it follows easily from our results that $\mathrm{Out}(A_\Gamma)$ has torsion-free subgroups of finite index, so that its virtual cohomological dimension (vcd) is defined and finite. In this section we find upper and lower bounds on this vcd.

7.1 The projection homomorphism

We will use the projection homomorphism

$$P = \prod_{v \in V_0} P_v \colon \mathrm{Out}^0(A_\Gamma) \to \prod \mathrm{Out}(F\langle L_v \rangle)$$

defined in Section 3.3. Recall that K_P denotes the kernel of P and let $\mathrm{Im}(P)$ be the image.

Proposition 7.1 *The outer automorphism group of a two-dimensional right-angled Artin group has torsion-free subgroups of finite index.*

Proof Since the outer automorphism group of a free group is virtually torsion-free, for each $v \in V_0$, we may choose a torsion-free subgroup H_v of finite index in $\mathrm{Out}(F\langle L_v \rangle)$. By Proposition 3.11 the kernel K_P of P is free abelian, so that the preimage H_0 of $\prod_v H_v$ in $\mathrm{Out}^0(A_\Gamma)$ is also torsion-free of finite index. Since $\mathrm{Out}^0(A_\Gamma)$ has finite index in $\mathrm{Out}(A_\Gamma)$, this shows that $\mathrm{Out}(A_\Gamma)$ itself is virtually torsion-free. □

An application of the Hochschild–Serre spectral sequence now gives an upper bound for the virtual cohomological dimension of $\mathrm{Out}(A_\Gamma)$.

Theorem 7.2 *The virtual cohomological dimension of* $\mathrm{Out}(A_\Gamma)$ *satisfies*

$$\mathrm{vcd}(\mathrm{Out}(A_\Gamma)) \le \mathrm{rank}(K_P) + \mathrm{vcd}(\mathrm{Im}(P)) \le \sum_{v \in V_0} (\delta_C(v) - 1) \ + \sum_{v \in V_0} (2\delta(v) - 3).$$

Proof Consider the exact sequence

$$1 \to K_P \to \mathrm{Out}^0(A_\Gamma) \to \mathrm{Im}(P) \to 1.$$

Restricting this exact sequence to a torsion-free, finite index subgroup of $\mathrm{Out}(A_\Gamma)$, it follows that for any coefficient module, the $E^2_{p,q}$–term of the associated Hochschild–Serre spectral sequence is zero for $p > \mathrm{vcd}(\mathrm{Im}(P))$ or $q > \mathrm{rank}(K_P)$. The rank of K_P is $\sum_{v \in V_0} (\delta_C(v) - 1)$ by Proposition 3.11. Since $\mathrm{Im}(P)$ is a subgroup of $\prod \mathrm{Out}(F\langle L_v \rangle)$ and the vcd of $\mathrm{Out}(F_n)$ is equal to $2n-3$ [11], the vcd of $\mathrm{Im}(P)$ is at most $\sum_{v \in V_0} (2\delta(v) - 3)$. □

If the graph Γ_0 contains a vertex v which has no leaves attached and is contained in no squares, then the only generators of $\mathrm{Out}^0(A_\Gamma)$ which affect vertices in L_v are inversions and partial conjugations...there are no transvections onto vertices in L_v. Therefore the image of $\mathrm{Out}^0(A_\Gamma)$ in $\mathrm{Out}(F\langle L_v\rangle)$ is contained in the subgroup $P\Sigma(L_v)$ generated by *pure symmetric automorphisms*, ie automorphisms which send each generator to a conjugate of itself. By a result of Collins [8], the subgroup $P\Sigma(L_v)$ has vcd equal to $\delta(v)-2$. Thus we can improve the upper bound of Theorem 7.2 as follows.

Corollary 7.3 *Let W_0 be the vertices of V_0 which either have leaves attached or are contained in a square with v. The virtual cohomological dimension of* $\mathrm{Out}(A_\Gamma)$ *satisfies*

$$\mathrm{vcd}(\mathrm{Out}(A_\Gamma)) \le \sum_{v\in V_0} (\delta_C(v)+\delta(v)-3) + \sum_{v\in W_0} (\delta(v)-1).$$

In particular, if Γ has no leaves, triangles or squares, then the virtual cohomological dimension of $\mathrm{Out}(A_\Gamma)$ *satisfies*

$$\mathrm{vcd}(\mathrm{Out}(A_\Gamma)) \le \sum_{v\in V_0} (\delta(v)+\delta_C(v)-3).$$

7.2 Free abelian subgroups

The rank of a free abelian subgroup of a group gives a lower bound on its virtual cohomological dimension. We have already exhibited a free abelian subgroup K_P of $\mathrm{Out}(A_\Gamma)$, generated by leaf transvections and partial conjugations, but in general this is not the largest one can find. In this section we exhibit a subgroup which often properly contains K_P. We begin by identifying three subgroups of $\mathrm{Aut}(A_\Gamma)$.

(1) *The subgroup A*: Recall that Γ_0 is a subgraph of Γ with one vertex in each maximal equivalence class of vertices, and that the partial order on vertices is given by $v\le w$ if $\mathrm{lk}(v)\subseteq \mathrm{lk}(w)$. For each vertex v which is not in the vertex set V_0 of Γ_0, choose a vertex $w\in V_0$ with $v\le w$, and let A be the free abelian subgroup of $\mathrm{Aut}(A_\Gamma)$ generated by the left and right transvections $\lambda\colon v\mapsto wv$ and $\rho\colon v\mapsto vw$. The rank of A is $2|V\setminus V_0|$

(2) *The subgroup L*: Let L denote the free abelian subgroup of $\mathrm{Aut}(A_\Gamma)$ generated by leaf transvections. Then L has rank ℓ, the number of leaves of Γ.

(3) *The subgroup C*: Let C denote the subgroup of $\mathrm{Aut}(A_\Gamma)$ generated by partial conjugations by a vertex v of one component of $\Gamma-\{v\}$. Since this is trivial when a component has only one vertex (which is therefore a leaf), C has $\sum_{v\in V_0}$ $\delta_C(v)-\ell(v)$ generators.

The image of C in $\mathrm{Out}(A_\Gamma)$ is free abelian since any two generators have representatives which act on disjoint subsets of the vertices of Γ, and hence commute. It is easy to check that all generators of the subgroups A and L commute and generate a free abelian subgroup of $\mathrm{Aut}(A_\Gamma)$, and that every generator of C also commutes with A and L. We let G denote the (free abelian) image of the subgroup generated by A, L and C in $\mathrm{Out}(A_\Gamma)$, and will now compute the rank of G. The image of L in G is isomorphic to L and does not intersect the image of the subgroup generated by A and C. The subgroups A and C, on the other hand may intersect nontrivially and may contain inner automorphisms. We introduce the following terminology to keep track of the possibilities:

Notation A component of $\Gamma - \{v\}$ is a *leaf component* if it contains only one vertex. It is a *twig* if it is not a leaf but is contained in the ball of radius 2 about v, and a *branch* if it is neither a leaf nor a twig. Note that if Γ is a pentagon, the (unique) component of $\Gamma - \{v\}$ is a branch, since points on the interior of the edge opposite v have distance more than 2 from v. The number of twigs at v will be denoted $\tau(v)$.

Theorem 7.4 *If Γ is not a star, the subgroup G of $\mathrm{Out}(A_\Gamma)$ generated by the images of A, L and C is free abelian of rank*

$$2|V \setminus V_0| + \sum_{v \in V_0} (\delta_C(v) - \tau(v) - 1)$$

Proof The subgroup of $\mathrm{Aut}(A_\Gamma)$ generated by L and A has rank $\ell + 2|V \setminus V_0|$, where $\ell = \sum_{v \in V_0} \ell(v)$ is the total number of leaves in Γ. If v is a separating vertex (which is necessarily in V_0), then partial conjugation of a leaf component by v is trivial, and partial conjugation of a twig by v is contained in A. Partial conjugation of a branch by v is not contained in A. However, the subgroup generated by A and *all* partial conjugations of branches at v contains the inner automorphism associated to v. Thus when we pass to $\mathrm{Out}(A_\Gamma)$, partial conjugations at v contribute only $\beta(v)-1$ generators of G which are independent of A and L. $\square$

Theorem 7.4 and Corollary 7.3 are summarized in the following corollary.

Corollary 7.5 *The virtual cohomological dimension of $\mathrm{Out}(A_\Gamma)$ satisfies*

$$\begin{aligned} 2|V| + \sum_{v \in V_0} (\delta_C(v) - \tau(v) - 3) &\le \mathrm{vcd}(\mathrm{Out}(A_\Gamma)) \\ &\le \sum_{v \in V_0} (\delta_C(v) + \delta(v) - 3) + \sum_{v \in W_0} (\delta(v) - 1). \end{aligned}$$

7.3 Examples

Example 7.6 Consider the tree Γ in Figure 7 consisting of one interior edge with n leaves attached at one vertex v and m leaves attached at the other vertex w. The subtree Γ_0 is the single interior edge. We have $\delta_C(v)=\delta(v)=n+1$, and $\delta_C(w)=\delta(w)=m+1$ and $\tau(v)=\tau(w)=1$ so the left-hand side of the formula in Corollary 7.5 is

$$2(m+n+2)+(m-3)+(n-3)=3m+3n-2.$$

We have $W_0=V_0$, so the right-hand side is $(2m-1)+(2n-1)+m+n=3m+3n-2$. Thus, in this example, the upper and lower bounds agree giving a precise computation, $\operatorname{vcd}(\operatorname{Out}(A_\Gamma))=3(n+m)-2$.

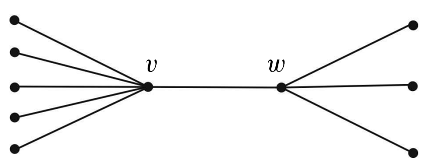

Figure 7

Example 7.7 More generally, suppose that Γ is an arbitrary tree. Then V_0 is the set of nonleaf vertices of Γ, $\delta(v)=\delta_C(v)$ for all v, and there is one twig for each univalent vertex of Γ_0. Let e be the number of edges in Γ, ℓ the number of leaves and ℓ_0 the number of leaves in Γ_0. A simple exercise shows that $e-1=\sum_{v\in V_0}(\delta(v)-1)$; using this, the formulas in Corollary 7.5 become:

$$e-1+2\ell-\ell_0\leq \operatorname{vcd}(\operatorname{Out}(A_\Gamma))\leq e+\ell-3+\sum_{v\in W_0}(\delta(v)-1).$$

Example 7.8 Consider the case of a single join $\Gamma=V*W$ with $V=\{v_1,\ldots,v_n\}$ and $W=\{w_1,\ldots w_n\}$, $n,m\geq 2$. Then Γ_0 consists of a single edge from, say, v_1 to w_1. The subgroup K_P is trivial, so Theorem 7.2 gives an upper bound of $(2n-3)+(2m-3)$ on the vcd. For the lower bound, we note $\delta_C(v_1)=\delta_C(w_1)=\tau(v_1)=\tau(w_1)=1$, so the lower bound is $2(m+n)-6$, matching the upper bound.

Example 7.9 Suppose Γ is the graph in Figure 8, with $n>1$ vertices in V and $m>1$ vertices in W. For $v\in V$ we have $\tau(v)=1$, $\delta(v)=m$ and $\delta_C(v)=1$, while for each $w\in W$ we have $\tau(w)=0$, $\delta(w)=n+1$ and $\delta_C(w)=2$. Thus the rank of G is equal

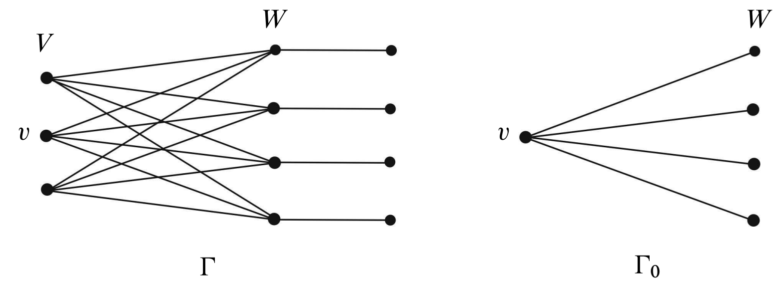

Figure 8

to $2(n+2m)+(-1)+(-m) = 2n+3m-3$, giving a lower bound on the vcd, and the upper bound on the vcd is equal to $2mn+2m-3$ since $W_0 = V_0$. So we obtain

$$3m+2n-3 \leq \mathrm{vcd}(\mathrm{Out}(A_\Gamma)) \leq 2mn+2m-3.$$

Thus the gap between the upper and lower bounds grows rapidly with m and n.

Example 7.10 When the rank of G is equal to the vcd, as in Example 7.6 above, it follows that G is a maximal rank abelian subgroup in $\mathrm{Out}(A_\Gamma)$. However, this is not always the case. For example, suppose Γ contains a vertex v such that $\Gamma - \{v\}$ includes a unique leaf w and a large number of nonleaf components. Then the generating set for G contains 3 transvections onto w (one leaf-transvection and two nonleaf transvections). In place of these 3 transvections, one could take all of the partial conjugations by w of a nonleaf component of $\Gamma - \{v\}$. This makes sense since the nonleaf components of $\Gamma - \{v\}$ are exactly the components of $\Gamma - \mathrm{st}(w)$. One can check that these partial conjugations by w commute with all of the other generators of G, giving a larger rank abelian subgroup. It would be interesting to determine the maximal rank of an abelian subgroup in $\mathrm{Out}(A_\Gamma)$ and whether that rank is always equal to the virtual cohomological dimension.

7.4 A spine for outer space

The dimension of outer space $\mathcal{O}(A_\Gamma)$ is in general much larger than the virtual cohomological dimension of $\mathrm{Out}(A_\Gamma)$. In the case of a free group F, the outer space $\mathcal{O}(F)$ contains an equivariant deformation retract, called the *spine*, with dimension equal to the vcd of $\mathrm{Out}(F)$. In this section we produce a similar spine of $\mathcal{O}(A_\Gamma)$. The dimension of this spine is at least as small as the upper bound on the vcd obtained in Theorem 7.2, and in several of the examples given in the previous section its dimension is equal to the exact vcd of $\mathrm{Out}(A_\Gamma)$.

We begin recalling the construction of the spine of outer space for a free group. Since we have not projectivized $\mathcal{O}(F)$, it decomposes as a union of open cubes in a cubical complex. To see this, we view points in $\mathcal{O}(F)$ as marked, metric graphs, ie metric graphs with an isomorphism (determined up to conjugacy) from F to the fundamental group of the graph. If T is a metric tree with an F–action, then the graph T/F has a natural marking, and the open cube containing this point is parameterized by varying the lengths of edges of this graph between 0 and infinity. Some faces of this cube lie in outer space, others do not. In particular, if a face contains a graph with an edge of infinite length, then that face does not lie in $\mathcal{O}(F)$.

Remark 7.11 Though it plays no role in what follows, we note that the cube complex obtained by including all faces of all cubes is topologically a cone, with cone point the point at which all edges have length zero. The link of this cone point is the usual simplicial closure of projectivized outer space.

Let $\bar{C}$ denote the closure of the open cube C inside of $\mathcal{O}(F)$. For two open cubes C_1, C_2 in $\mathcal{O}(F)$, say $C_1 < C_2$ if C_1 is a face of $\bar{C}_2$. The *spine* of $\mathcal{O}(F)$, denoted $Z(F)$, is the simplicial complex whose vertices are labeled by the open cubes in $\mathcal{O}(F)$ and whose simplices correspond to totally ordered sets of these cubes. Identifying a vertex v_C in $Z(F)$ with the barycenter of the cube C, we can view $Z(F)$ as a subspace of $\mathcal{O}(F)$. Each open cube C in $\mathcal{O}(F)$ deformation retracts onto the star of v_C in $Z(F)$ and these retracts fit together to give a retraction of $\mathcal{O}(F)$ onto $Z(F)$. Since the action of $\mathrm{Out}(F)$ on $\mathcal{O}(F)$ maps open cubes to open cubes and preserves the partial order, there is an induced action of $Z(F)$.

Now let A_Γ be an arbitrary right-angled Artin group. Recall from Proposition 4.2 that $\mathcal{O}(A_\Gamma)$ decomposes as a product $\mathbb{R}^\ell \times Q \times Y$ where Y is a subspace of the product $\prod_{v \in V_0} \mathcal{O}(F\langle L_v \rangle)$ of outer spaces for the free groups $F\langle L_v \rangle$. Since a product of cubes is cube, this product of outer spaces is as a union of open cubes, where a cube $C = \prod C_v$ corresponds to a specified marked graph for each $F\langle L_v \rangle$, and the edge lengths give coordinates for the cubes.

Lemma 7.12 *The intersection of each closed cube $\bar{C} = \prod \bar{C}_v$ with Y is a convex cell.*

Proof A collection of trees $\{T_v\}$ lies in Y if and only if the minimal $F\langle L_w^\perp \rangle$–subtrees are equivariantly isometric in all T_v with $v \in L_w$. By [17], each of these subtrees is uniquely determined by the translation lengths of a finite set of elements of $F\langle L_w^\perp \rangle$. For each T_v, these translation lengths are given by a linear combination of the edge lengths of the graph $T_v/F\langle L_v \rangle$. Since the edge lengths give coordinates for the cube $\bar{C}$, the intersection of Y with $\bar{C}$ is given by a finite set of linear equalities. □

It follows from the proof above that if C_1 is a face of $\bar{C}_2$ and their intersection with Y is nonempty, then $Y \cap C_1$ is a face of $Y \cap \bar{C}_2$.

Definition 7.13 We define the *spine of* Y to be the geometric realization of the poset of cells $Y \cap C$ partially ordered by the face relation. We denote this spine by $Z(A_\Gamma)$.

Proposition 7.14 *The action of* $\mathrm{Out}^0(A_\Gamma)$ *on* $\mathcal{O}(A_\Gamma)$ *descends to a proper action of* $\mathrm{Im}(P)$ *on* Y*. With respect to this action,* $Z(A_\Gamma)$ *can be identified with a piecewise linear* $\mathrm{Im}(P)$*–invariant subspace of* Y*. This subspace is a deformation retract of* Y *hence, in particular,* $Z(A_\Gamma)$ *is contractible.*

Proof Let $\mathfrak{X} = \{X_v, X_e, i_{e,v}\}$ be a point in $\mathcal{O}(A_\Gamma)$ and $\mathfrak{X} \cdot \phi = \{X_v^{\phi_v}, X_e^{\phi_e}, t(g_v)^{-1} i_{e,v}\}$ its translate by ϕ. Recall that X_v is a product of trees $T_v \times T_v^\perp$ with an action of $F\langle L_v\rangle \times F(L_v^\perp)$. Though this action is not necessarily a product action, it projects to an action of $F\langle L_v\rangle$ on T_v. The projection of $\mathfrak{X}$ on Y is given by the resulting set of actions $\{T_v\}$. The twisted tree-space $X_v^{\phi_v}$ is a product of the same two underlying trees with the action twisted by ϕ_v. The new action of $F\langle L_v\rangle$ on T_v depends only on the projection of ϕ_v to $\mathrm{Out}(F\langle L_v\rangle)$. The first statement of the lemma follows.

The action of $\mathrm{Out}(F\langle L_v\rangle)$ on $\mathcal{O}(F\langle L_v\rangle)$ is cellular and the stabilizer of any cell is finite. Hence the same is true of the action of $\mathrm{Im}(P)$ on $\prod_{v \in V_0} \mathcal{O}(F\langle L_v\rangle)$. By the discussion above, $\mathrm{Im}(P)$ preserves Y and hence it takes open cells $C \cap Y$ to open cells and preserves the face relation.

Let p be a point of $C \cap Y$. The orbit of p intersects $C \cap Y$ in a finite set of points. Since $C \cap Y$ is convex, the barycenter $\bar{p}$ is a point of $C \cap Y$ which is invariant under the stabilizer of $C \cap Y$, and the entire orbit of $\bar{p}$ intersects each cell $C' \cap Y$ in at most point. It follows that we can chose one point x_C of each cell $C \cap Y$ such that the set of points $\{x_C\}$ is $\mathrm{Im}(P)$–invariant. Now identify these points with the vertices of $Z(A_\Gamma)$ in the obvious way. Then for any simplex σ of $Z(A_\Gamma)$, the vertices of σ lie in the closure of a single cell $C \cap Y$ and their linear span forms a simplex in Y. The resulting simplicial complex is isomorphic to $Z(A_\Gamma)$. As in the case of the spine for a free group, retracting each cell $C \cap Y$ linearly onto the star of the vertex x_C gives a deformation retraction of Y onto $Z(A_\Gamma)$. □

Proposition 7.15 *The virtual cohomological dimension of* $\mathrm{Out}(A_\Gamma)$ *satisfies*

$$\mathrm{vcd}(\mathrm{Out}(A_\Gamma)) \le \mathrm{rank}(K_P) + \dim Z(A_\Gamma)$$

Proof It follows immediately from Proposition 7.14 that $\mathrm{Im}(P)$ has vcd bounded by the dimension of $Z(A_\Gamma)$. The result now follows from Theorem 7.2. □

Proposition 7.16 *The dimension of $Z(A_\Gamma)$ is at most $\sum_v 2\delta(v) - 3$.*

Proof A cube $C = \prod C_v$ has dimension at least $\sum_v \delta(v)$ and at most $\sum_v 3\delta(v) - 3$ since each cube C_v has dimension at least $\delta(v)$ and at most $3\delta(v) - 3$. Therefore the longest possible chain of inclusions of cells $Y \cap C$ is $(\sum_v 2\delta(v) - 3) + 1$, so that the dimension of $Z(\Gamma)$ is at most $\sum_v 2\delta(v) - 3$. □

In fact, $Z(A_\Gamma)$ is naturally isomorphic to a subcomplex of a simplicial subdivision of the product $\prod_v Z(F\langle L_v\rangle)$ of spines for the outer spaces associated to the vertices v of Γ_0. If the links of vertices of Γ_0 have large overlap, as in Example 7.9, $Z(A_\Gamma)$ will be much smaller than the full product, though it can be shown in this example that they have the same dimension.

On the other hand, in the case that Γ is a tree, we claim that $Z(A_\Gamma) = \prod Z(F\langle L_v\rangle)$. To verify this claim, we must show that Y intersects every open cell in $\prod \mathcal{O}(F\langle L_v\rangle)$. For Γ a tree, two links L_v and L_w intersect either in exactly one point (if v, w are distance 2 apart) or not at all. If $L_v \cap L_w = \{u\}$, we will say that T_v and T_w are *compatible* if the translation lengths of u in T_v and T_w agree. A point in Y is a V_0–tuple (T_v) of compatible trees.

A cell in $\mathcal{O}(F\langle L_v\rangle)$ is invariant under scaling, ie if T_v lies in an open cell C_v, then so does the tree obtained by scaling the metric on T_v by any $r > 0$. Thus it suffices to show that any V_0–tuple of trees (T_v) can be made compatible by rescaling. To do this, fix a pair of adjacent vertices v, u in Γ_0. Every vertex in Γ_0 is even distance from exactly one of these two vertices. If w is distance $2n$ from v, then there is a unique sequence of vertices $v = w_0, w_1, \dots w_n = w$ such that the link of w_{i-1} intersects the link of w_i in a vertex. Starting with T_v, we can inductively scale each T_{w_i} to be compatible with the previous one. Similarly, for vertices at distance $2n$ from u. The resulting collection of trees defines a point in Y.

Note that once the metrics on T_u and T_v are fixed, the scaling on the remaining trees is uniquely determined. Thus, modulo scaling the two base trees, a point in Y corresponds to a point in the product of the *projectivized* outer spaces $\overline{\mathcal{O}}(F\langle L_v\rangle)$. Summarizing, we have shown the following:

Corollary 7.17 *If Γ is a tree, then*

$$Y \cong \mathbb{R}^2 \times \prod \overline{\mathcal{O}}(F\langle L_v\rangle) \quad \text{and} \quad Z(A_\Gamma) \cong \prod Z(F\langle L_v\rangle),$$

where the products are taken over the nonleaf vertices in Γ.

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