

Erratum to “Stabilization for the automorphisms of free groups with boundaries”

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We correct the proof of Theorem 4.1 in our paper “[Stabilization for the automorphisms of free groups with boundaries](#)”.

[20F28](#); [57M07](#)

The purpose of this note is to point out a gap in an argument in our paper [5] and explain how to fill it (or bypass it).

The gap in [5] occurs in Section 4 in the proof of assertions (A) and (B), at the point where there are diagram chasing arguments in two commutative diagrams (displayed on page 1333 for case (A)). In each diagram the groups G_n in the two rows are isomorphic but not identical. If we denote by G_n and G'_n these two isomorphic groups, the first diagram chase needs the composition $H_i(G_n) \rightarrow H_i(G_{n+1}) \rightarrow H_i(G_{n+1}, G'_n)$ to be trivial, which is the case because G_n and G'_n are conjugate in G_{n+1} . The second diagram chase needs the composition $H_{i+1}(G_n, G_{n-1}) \rightarrow H_i(G_{n-1}) \rightarrow H_i(G'_n)$ to be trivial, but there is no *a priori* reason for this to be true, although it is true and follows *a posteriori* from Theorem 2 below. An analogous diagram chase is used in the proof of assertion (C), but in that case there is an isomorphism $G_n \rightarrow G'_n$ commuting with the inclusion of G_{n-1} , so that the diagram chase is correct. For (A) and (B), there is an isomorphism $G_n \rightarrow G'_n$ that commutes with the inclusion of a subgroup H of G_{n-1} , with $H_*(H) \cong H_*(G_{n-1})$ in a range given by Theorem 2.

Recall from [5] that $M_{n,k}^s = N \# (\#_n S^1 \times S^2) \# (\#_k S^1 \times D^2) \# (\#_s D^3)$, where N is a fixed compact connected oriented 3-manifold, and that $A_{n,k}^s$ denotes the quotient of the mapping class group $\pi_0 \text{Diff}(M_{n,k}^s \text{ rel } \partial M_{n,k}^s)$ by twists along spheres embedded in M . Assertion (A) says that the map $\alpha_i: H_i(A_{n,k}^{s+2}) \rightarrow H_i(A_{n+1,k}^{s+1})$, induced by identifying discs in the last two boundary spheres of M , is surjective when $n \geq 3i$ and an isomorphism when $n \geq 3i + 2$. Assertion (B) says that the same is true for the map $\beta_i: H_i(A_{n,k}^{s+2}) \rightarrow H_i(A_{n+1,k}^s)$, where β_i is induced by identifying the last two boundary spheres of M . Assertions (A) and (B) are used in [5] to show (1) and (3) in Theorem 4.1, namely that the stabilization maps $\phi_i: H_i(A_{n,k}^s) \rightarrow H_i(A_{n+1,k}^s)$, induced by gluing

a twice punctured $S^1 \times S^2$ along a boundary sphere, and $\mu_i: H_i(A_{n,k}^s) \rightarrow H_i(A_{n,k}^{s+1})$, induced by gluing a 3-punctured sphere along one of its boundary components, are isomorphisms when $n \geq 3i + 3$. To fix the gap described above, it is sufficient to know – independently – that $\mu_i: H_i(A_{n-1,k}^{s+1}) \rightarrow H_i(A_{n-1,k}^{s+2})$ is surjective when $n-1 \geq 3i + 1$, but the new argument actually proves simultaneously that ϕ_i and μ_i are isomorphisms in the above range, avoiding completely the two commutative diagrams mentioned above, so that the point becomes moot. The proof, sketched below, uses two simplicial complexes built from a sphere complex studied in [5]. The main ingredient of the proof is the high connectivity of these complexes, which is deduced from the connectivity of the earlier sphere complex by a combinatorial argument. This last argument is applied in a more general context in our paper [4] and hence not repeated here. We actually obtain in this way a better stability range, namely an isomorphism when $n \geq 2i + 2$ for both ϕ_i and μ_i .

Let $M = M_{n,k}^s$ be as above and let x_0, x_1 be two points in two boundary spheres $\partial_0 M, \partial_1 M$ of M , where we allow the possibility $x_0 = x_1$ with $\partial_0 M = \partial_1 M$. Recall from [5] that $\mathcal{S}_c(M)$ denotes the simplicial complex of isotopy classes of non-separating sphere systems in M . We use here an enhanced version of $\mathcal{S}_c(M)$: Let $X^A(M, x_0, x_1)$ be the simplicial complex whose vertices are pairs (S, a) , where S is in $\mathcal{S}_c(M)$ and a is the isotopy class of an arc from x_0 to x_1 intersecting S transversely in exactly one point, with a choice of orientation of the arc if $x_0 = x_1$. A p -simplex of $X^A(M, x_0, x_1)$ is a collection $\langle (S_0, a_0), \dots, (S_p, a_p) \rangle$ of such pairs that are disjoint except for the endpoints of the arcs, and such that the spheres $\langle S_0, \dots, S_p \rangle$ form a p -simplex of $\mathcal{S}_c(M)$.

There is a map $X^A(M, x_0, x_1) \rightarrow \mathcal{S}_c(M)$ which forgets the arcs, and we think of $X^A(M, x_0, x_1)$ as the complex $\mathcal{S}_c(M)$ with arcs labeling its vertices. In [4], we show how the connectivity of $X^A(M, x_0, x_1)$ can be deduced from that of $\mathcal{S}_c(M)$, established in [5, Proposition 3.2], using a combinatorial argument [4, Theorem 3.8].

Theorem 1 [4, Proposition 4.4] $X^A(M_{n,k}^s, x_0, x_1)$ is $(\frac{n-3}{2})$ -connected.

The group $A_{n,k}^s$ acts on the complex $X^A(M_{n,k}^s, x_0, x_1)$ since twists along spheres act trivially on embedded spheres and arcs that meet these spheres in one point transversely. The stabilizer of a vertex is isomorphic to $A_{n-1,k}^s$ both in the case that $x_0 = x_1$ and in the case that x_0 and x_1 lie on different boundary components of M , but in the first case the inclusion of the stabilizer into $A_{n,k}^s$ induces the stabilization ϕ_i described above, while in the second case it is the map $\psi: A_{n-1,k}^s \rightarrow A_{n,k}^s$ induced by gluing a 4-punctured sphere along two of its boundary spheres. The spectral sequences for the action of $A_{n,k}^s$ on $X^A(M_{n,k}^s, x_0, x_1)$ in each of the two cases yield the following stability.

Theorem 2 [4, Theorem 6.1] *The maps $\phi_i, \psi_i: H_i(A_{n,k}^s) \rightarrow H_i(A_{n+1,k}^s)$ are isomorphisms when $n \geq 2i + 2$ and surjections when $n \geq 2i + 1$ (with $s \geq 1$ for ϕ and $s \geq 2$ for ψ).*

(In [4], the two maps are denoted α and β .) Note now that $\phi = \eta\mu$ and $\psi = \mu\eta$, for η the map obtained by gluing a 3-punctured sphere along two of its boundary spheres. It follows from the theorem that both μ and η are isomorphisms in the same range. To show that μ is surjective in that range, which is enough to fix the gap, it suffices to consider only the map ψ , that is, the case where x_0 and x_1 lie on two different boundary components.

An analogous gap occurred in Hatcher–Vogtmann [2], in the proof of homological stability for $\text{Out}(F_n)$, which is the group $A_{n,0}^0$ with $N = S^3$ in our notation. The paper Hatcher–Vogtmann–Wahl [3] bypasses the gap by showing that the complex $X^A(M_{n,0}^s, x_0, x_1)$ (denoted Z_n in that paper) with x_0 and x_1 on different boundary components of $M_{n,0}^s$, becomes contractible when n goes to infinity, which is enough in that case by the main result of Hatcher–Vogtmann [1]. The present erratum gives an alternative argument to fill in the gap of [2], which does not depend on [1].

References

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