Essential curves in handlebodies and topological contractions

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If \( X \) is a compact set, a topological contraction is a self-embedding \( f \) such that the intersection of the successive images \( f^k(X) \), \( k > 0 \), consists of one point. In dimension 3, we prove that there are smooth topological contractions of the handlebodies of genus \( \geq 2 \) whose image is essential.

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1 Introduction

For a compact set \( X \) and a topological embedding \( f : X \to X \), we shall say that \( f \) is a topological contraction if \( \bigcap_{k \geq 0} f^k(X) \) consists of one point. We shall show that such a contraction can be very complicated when \( X \) is a 3–dimensional handlebody. Namely, we have the following result for which some more classical definitions will be recalled thereafter.

**Theorem A** There exists a North–South diffeomorphism \( f \) of the 3–sphere \( S^3 \) and a Heegaard decomposition \( S^3 = P_+ \cup P_- \) of genus \( g \geq 2 \) with the following properties:

(1) \( f|P_+ \) is a topological contraction;

(2) \( f(P_-) \) is essential in \( P_+ \).

We shall limit ourselves to \( g = 2 \), since the generalization will be clear. We recall that a 3–dimensional handlebody of genus 2 is diffeomorphic to the regular neighborhood \( P \) in \( \mathbb{R}^3 \) of the planar figure eight \( \Gamma \). A compression disk of \( P \) is a smooth embedded disk in \( P \) whose boundary lies in \( \partial P \) in which it is not homotopic to a point. Among the compression disks are the meridian disks \( \pi^{-1}(x) \), where \( x \) is a regular point\(^1\) in \( \Gamma \) and \( \pi : P \to \Gamma \) is the regular neighborhood projection (that is, a submersion over the smooth part of \( \Gamma \)). A subset \( X \) of \( P \) is said to be essential in \( P \) if it intersects every compression disk\(^2\).

\(^1\) Any point other than the center of the figure eight.

\(^2\) This definition goes back to Rolfsen’s book [2, p 110].
A diffeomorphism $f$ of $S^3$ is a *North–South diffeomorphism* if it has two fixed points only, one source $\alpha \in P_-$ and one sink $\omega \in P_+$, every other orbit going from $\alpha$ to $\omega$.

A *Heegaard splitting* of $S^3$ is made of an embedded surface dividing $S^3$ into two handlebodies. According to a famous theorem of F. Waldhausen such a decomposition is unique up to isotopy [3]. It is not hard to prove that the phenomenon mentioned in Theorem A does not happen with a Heegaard splitting of genus $1$: if $T$ is a solid torus and $f$ is a topological contraction of $T$, then there is a compression disk of $T$ avoiding $f(T)$.

The example which we are going to construct for proving Theorem A is based on the next theorem, for which some more notation is introduced. Let $\Gamma_0 \subset \Gamma$ be a simple closed curve. There exists a solid torus $T \subset \mathbb{R}^3$ which contains $P$ and which is a tubular neighborhood of $\Gamma_0$. Let $i_0: P \to T$ be this inclusion. We say that a simple curve is unknotted in $T$ if it bounds an embedded disk in $T$.

**Theorem B** There exists an essential simple curve $C$ in $P$ such that $i_0(C)$ is unknotted in $T$.

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## 2 Essential curves

Our candidate for $C$ in Theorem B is pictured in Figure 1.

It is clear that $i_0(C)$ is unknotted in $T$ (or, equivalently, in the complement of the vertical axis which is drawn in Figure 1). A way of proving that $C$ is essential in $P$ is to prove the following lemma (actually equivalent as the referee noticed).

**Lemma 1** Let $p: \tilde{P} \to P$ be the universal cover of $P$ and let $\tilde{C}$ be the preimage $p^{-1}(C)$. Then $\tilde{C}$ is essential in $\tilde{P}$.

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3 Michel Boileau, Thomas Fiedler, John Guaschi and Claude Hayat

4 According to the Loop theorem and Dehn’s lemma, $C$ essential in $P$ implies $\tilde{C}$ essential in $\tilde{P}$.

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Proof We have the following description of \( \tilde{P} \): it is a 3–ball with a Cantor set \( E \) removed from its bounding 2–sphere. This Cantor set is the set of ends of \( \tilde{P} \). A simple curve in \( \partial\tilde{P} \) is not homotopic to zero if it divides \( E \) into two nonempty parts. We get a fundamental domain \( F \) for the action of \( \pi_1(P) \) on \( \tilde{P} \) by cutting \( P \) along two non-parallel meridian disks \( D_0 \) and \( D_1 \). Figure 2 shows what the pair \((F, \tilde{C} \cap F)\) looks like: \( F \) is a 3–ball whose boundary consists of the union \( \partial_1 F \) of four disks \( d_0, d'_0, d_1, d'_1 \) and a punctured sphere \( \partial_0 F \), where \( p(d_0) = p(d'_0) = D_0 \) and \( p(d_1) = p(d'_1) = D_1 \); \( \tilde{C} \cap F \) is made of four strands with end points in \( \partial_1 F \) and pairwise linked as it is shown.

One can show easily that (i) \( \partial_1 F \setminus \tilde{C} \) is incompressible and boundary incompressible in \( F \setminus \tilde{C} \), and (ii) \( \partial_0 F \) is incompressible in \( F \setminus \tilde{C} \). Now suppose on the contrary that \( \tilde{C} \) is not essential and consider a compression disk \( \Delta \) of \( \tilde{P} \) avoiding \( \tilde{C} \). We take \( \Delta \) to be transversal to \( \tilde{D} := p^{-1}(D_0 \cup D_1) \). A standard innermost circle/arc argument, using (i), shows that we may assume that \( \Delta \) is contained in \( F \setminus \tilde{C} \), contradicting (ii).

Remarks 1 (1) Globally \( \tilde{C} \) looks like an infinite Borromean chain: any finite number of components is unlinked. We would like to know whether there exists a topological algebraic tool proving that \( C \) is essential in \( P \).

(2) Our referee proposed another example where \( [C] = aba^{-1}b^{-1} \) with respect to the obvious basis \( a, b \) of \( \pi_1(P) \). In this case \( C \) is essential in \( P \) by an algebraic argument: the quotient \( \pi_1(P)/[[C]] \) is isomorphic to \( \mathbb{Z} \times \mathbb{Z} \). If \( C \) were inessential in \( P \) this quotient would be isomorphic to \( \mathbb{Z} \ast \mathbb{Z}_n \) for some \( n \geq 0 \).

\(^5\)Take the universal cover of \( \Gamma \) properly embedded in the hyperbolic plane and take a 3–dimensional thickening of it.
3 Proof of Theorem A

We recall the embedding $i_0: P \to \text{int } T$. We start with a curve $C$ in $P$ which meets the conclusion of Theorem B. We equip it with its 0–normal framing (a section of this framing is not linked with $C$ in $\mathbb{R}^3$) and we choose an embedding $j_0: T \to P$ whose image is a tubular neighborhood of $C$. Let $B$ be a small ball in $\text{int } T$. As $C$ is unknotted in $T$, there is an ambient isotopy, supported in $\text{int } T$, deforming $i_0$ to $i_1: P \to \text{int } T$ such that $i_1 \circ j_0(T)$ is a standard small solid torus in $B$. One half of the desired Heegaard splitting of genus 2 will be given by $P_C \cup i_1$. At the present time $f$ is only defined on $T$ by $f := i_1 \circ j_0: T \to \text{int } T$. If we compose $i_1$ with a sufficiently strong metric contraction of $B$ into itself (with respect to some metric), then $f$ is a metric contraction. Hence $\bigcap_{k>0} f^k(T)$ consists of one point.

Choose a round ball $B'$ containing $T$ in its interior. Since $f|T$ is isotopic to the inclusion $T \to \mathbb{R}^3$, $f$ extends as a diffeomorphism $B' \to B$, and further as a diffeomorphism $S^3 \to S^3$. We are free to choose $f: S^3 \setminus B' \to S^3 \setminus B$ as we like. Let $B''$ be the closure of $S^3 \setminus B'$ and $\varphi: S^3 \to S^3$ be a diffeomorphism which is the identity on $B'$ and a strong metric contraction on a ball containing $f^{-1}(B'')$. If we replace $f$ by $f \circ \varphi^{-1}$ (without changing the notation), then $f^{-1}|B''$ becomes a metric contraction and the intersection $\bigcap_{k>0} f^{-k}(S^3 \setminus B')$ consists of one point. We now have a North–South diffeomorphism $f$ of $S^3$ which induces a topological contraction of $T$. Since $f(T) \subset \text{int } P_+ \subset P_+ \subset \text{int } T$, $f$ also induces a topological contraction of $P_+$.

It remains to prove that $f(P_+)$ is essential in $P_+$. We know that $i_1(C)$ is essential in $P_+$. As a consequence, any compression disk $\Delta$ of $P_+$ intersects $f(T)$. We can
take $\Delta$ to be transversal to $f(\partial T)$ such that no intersection curve is null-homotopic in $f(\partial T)$. Let $\gamma$ be an intersection curve which is innermost in $\Delta$ and let $\delta$ be the disk that $\gamma$ bounds in $\Delta$.

**Lemma 2** We have $\delta \subset f(T)$.

**Proof** If not, we have $\delta \subset P_+ \setminus f(\text{int} T)$ and the simple curve $\gamma$ in $f(\partial T)$ is unlinked with the core $i_1(C)$. Therefore, up to isotopy in $f(\partial T)$, it is a section of the $0$–framing. In that case, $i_1(C)$ itself bounds an embedded disk in $P_+$. This is impossible, as $i_1(C)$ is essential in $P_+$.

Therefore $\delta$ is a compression disk of the solid torus $f(T)$. But $P_+ = i_1(P)$, like $P$ itself, is essential in $T$. Hence $f(P_+)$ is essential in $f(T)$ and $\delta$ must intersect $f(P_+)$.

**References**


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