Essential curves in handlebodies and topological contractions

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If X is a compact set, a *topological contraction* is a self-embedding f such that the intersection of the successive images $f^k(X)$, $k > 0$, consists of one point. In dimension 3, we prove that there are smooth topological contractions of the handlebodies of genus ≥ 2 whose image is essential.

57M25; 37D15

1 Introduction

For a compact set X and a topological embedding $f\colon X \to X$, we shall say that f is a *topological contraction* if $\bigcap_{k\geq 0} f^k(X)$ consists of one point. We shall show that such a contraction can be very complicated when X is a 3–dimensional handlebody. Namely, we have the following result for which some more classical definitions will be recalled thereafter.

Theorem A *There exists a North–South diffeomorphism f of the 3–sphere S^3 and a Heegaard decomposition $S^3 = P_- \cup P_+$ of genus $g \geq 2$ with the following properties:*

(1) *$f|P_+$ is a topological contraction;*

(2) *$f(P_+)$ is essential in P_+.*

We shall limit ourselves to $g = 2$, since the generalization will be clear. We recall that a 3–dimensional *handlebody* of genus 2 is diffeomorphic to the regular neighborhood P in $\mathbb{R}^3$ of the planar figure eight Γ. A *compression disk* of P is a smooth embedded disk in P whose boundary lies in ∂P in which it is not homotopic to a point. Among the compression disks are the *meridian* disks $\pi^{-1}(x)$, where x is a regular point[1] in Γ and $\pi\colon P \to \Gamma$ is the regular neighborhood projection (that is, a submersion over the smooth part of Γ). A subset X of P is said to be *essential* in P if it intersects every compression disk[2] .

[1] Any point other than the center of the figure eight.
[2] This definition goes back to Rolfsen's book [2, p 110].

Published: 12 May 2008 DOI: 10.2140/gt.2008.12.981

A diffeomorphism f of S^3 is a *North–South diffeomorphism* if it has two fixed points only, one source $\alpha \in P_-$ and one sink $\omega \in P_+$, every other orbit going from α to ω.

A *Heegaard splitting* of S^3 is made of an embedded surface dividing S^3 into two handlebodies. According to a famous theorem of F Waldhausen such a decomposition is unique up to isotopy [3]. It is not hard to prove that the phenomenon mentioned in Theorem A does not happen with a Heegaard splitting of genus 1: if T is a solid torus and f is a topological contraction of T, then there is a compression disk of T avoiding $f(T)$.

The example which we are going to construct for proving Theorem A is based on the next theorem, for which some more notation is introduced. Let $\Gamma_0 \subset \Gamma$ be a simple closed curve. There exists a solid torus $T \subset \mathbb{R}^3$ which contains P and which is a tubular neighborhood of Γ_0. Let $i_0\colon P \to T$ be this inclusion. We say that a simple curve is unknotted in T if it bounds an embedded disk in T.

Theorem B *There exists an essential simple curve C in P such that $i_0(C)$ is unknotted in T.*

The second author is grateful to Sylvain Gervais and Nathan Habegger for interesting conversations on link invariants, in particular on Milnor's invariants [1]. He is also grateful to the organizers[3] of the conference in Toulouse in memory of Heiner Zieschang (*Braids, groups and manifolds*, Sept 2007), who offered him the opportunity to give a short talk on the subject of this paper. We also thank the anonymous referee for very valuable comments.

2 Essential curves

Our candidate for C in Theorem B is pictured in Figure 1.

It is clear that $i_0(C)$ is unknotted in T (or, equivalently, in the complement of the vertical axis which is drawn in Figure 1). A way of proving that C is essential in P is to prove the following lemma (actually equivalent as the referee noticed[4]).

Lemma 1 *Let $p\colon \widetilde{P} \to P$ be the universal cover of P and let $\widetilde{C}$ be the preimage $p^{-1}(C)$. Then $\widetilde{C}$ is essential in $\widetilde{P}$.*

[3] Michel Boileau, Thomas Fiedler, John Guaschi and Claude Hayat

[4] According to the Loop theorem and Dehn's lemma, C essential in P implies $\widetilde{C}$ essential in $\widetilde{P}$.

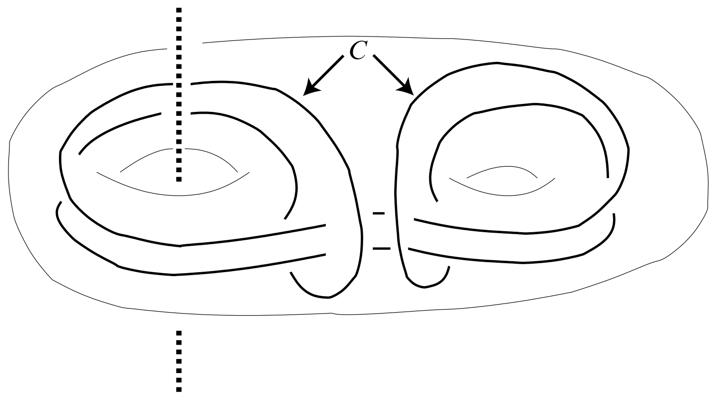

Figure 1

Proof We have the following description of $\widetilde{P}$: it is a 3–ball with a Cantor set E removed from its bounding 2–sphere[5]. This Cantor set is the set of ends of $\widetilde{P}$. A simple curve in $\partial\widetilde{P}$ is not homotopic to zero if it divides E into two nonempty parts. We get a fundamental domain F for the action of $\pi_1(P)$ on $\widetilde{P}$ by cutting P along two non-parallel meridian disks D_0 and D_1. Figure 2 shows what the pair $(F, \widetilde{C}\cap F)$ looks like: F is a 3–ball whose boundary consists of the union $\partial_1 F$ of four disks d_0, d_0', d_1, d_1' and a punctured sphere $\partial_0 F$, where $p(d_0) = p(d_0') = D_0$ and $p(d_1) = p(d_1') = D_1$; $\widetilde{C}\cap F$ is made of four strands with end points in $\partial_1 F$ and pairwise linked as it is shown.

One can show easily that (i) $\partial_1 F \setminus \widetilde{C}$ is incompressible and boundary incompressible in $F\setminus\widetilde{C}$, and (ii) $\partial_0 F$ is incompressible in $F\setminus\widetilde{C}$. Now suppose on the contrary that $\widetilde{C}$ is not essential and consider a compression disk Δ of $\widetilde{P}$ avoiding $\widetilde{C}$. We take Δ to be transversal to $\widetilde{D} := p^{-1}(D_0\cup D_1)$. A standard innermost circle/arc argument, using (i), shows that we may assume that Δ is contained in $F\setminus\widetilde{C}$, contradicting (ii). □

Remarks 1 (1) Globally $\widetilde{C}$ looks like an *infinite Borromean* chain: any finite number of components is unlinked. We would like to know whether there exists a topological algebraic tool proving that C is essential in P.

(2) Our referee proposed another example where $[C] = aba^{-1}b^{-1}$ with respect to the obvious basis a, b of $\pi_1(P)$. In this case C is essential in P by an algebraic argument: the quotient $\pi_1(P)/\langle[C]\rangle$ is isomorphic to $\mathbb{Z}\times\mathbb{Z}$. If C were inessential in P this quotient would be isomorphic to $\mathbb{Z}*\mathbb{Z}_n$ for some $n\geq 0$.

[5] Take the universal cover of Γ properly embedded in the hyperbolic plane and take a 3–dimensional thickening of it.

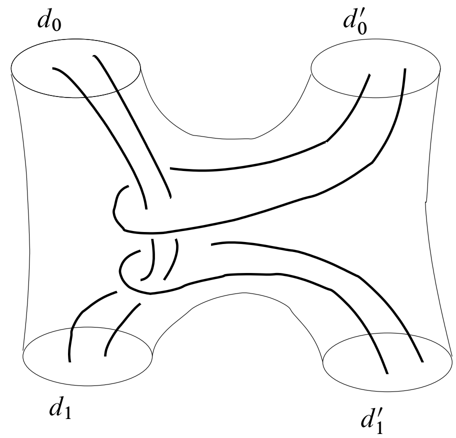

Figure 2

3 Proof of Theorem A

We recall the embedding $i_0\colon P \to \operatorname{int} T$. We start with a curve C in P which meets the conclusion of Theorem B. We equip it with its 0–normal framing (a section of this framing is not linked with C in $\mathbb{R}^3$) and we choose an embedding $j_0\colon T \to P$ whose image is a tubular neighborhood of C. Let B be a small ball in $\operatorname{int} T$. As C is unknotted in T, there is an ambient isotopy, supported in $\operatorname{int} T$, deforming i_0 to $i_1\colon P \to \operatorname{int} T$ such that $i_1 \circ j_0(T)$ is a standard small solid torus in B. One half of the desired Heegaard splitting of genus 2 will be given by $P_+ := i_1(P)$. At the present time f is only defined on T by $f := i_1 \circ j_0\colon T \to \operatorname{int} T$. If we compose i_1 with a sufficiently strong metric contraction of B into itself (with respect to some metric), then f is a metric contraction. Hence $\bigcap_{k>0} f^k(T)$ consists of one point.

Choose a round ball B' containing T in its interior. Since $f|T$ is isotopic to the inclusion $T \hookrightarrow \mathbb{R}^3$, f extends as a diffeomorphism $B' \to B$, and further as a diffeomorphism $S^3 \to S^3$. We are free to choose $f\colon S^3 \setminus B' \to S^3 \setminus B$ as we like. Let B" be the closure of $S^3 \setminus B'$ and $\varphi\colon S^3 \to S^3$ be a diffeomorphism which is the identity on B' and a strong metric contraction on a ball containing $f^{-1}(B")$. If we replace f by $f \circ \varphi^{-1}$ (without changing the notation), then $f^{-1}|B$" becomes a metric contraction and the intersection $\bigcap_{k>0} f^{-k}(S^3 \setminus B')$ consists of one point. We now have a North–South diffeomorphism f of S^3 which induces a topological contraction of T. Since $f(T) \subset \operatorname{int} P_+ \subset P_+ \subset \operatorname{int} T$, f also induces a topological contraction of P_+.

It remains to prove that $f(P_+)$ is essential in P_+. We know that $i_1(C)$ is essential in P_+. As a consequence, any compression disk Δ of P_+ intersects $f(T)$. We can take Δ to be transversal to $f(\partial T)$ such that no intersection curve is null-homotopic in $f(\partial T)$. Let γ be an intersection curve which is *innermost* in Δ and let δ be the disk that γ bounds in Δ.

Lemma 2 *We have* $\delta \subset f(T)$.

Proof If not, we have $\delta \subset P_+ \setminus f(\operatorname{int} T)$ and the simple curve γ in $f(\partial T)$ is unlinked with the core $i_1(C)$. Therefore, up to isotopy in $f(\partial T)$, it is a section of the 0–framing. In that case, $i_1(C)$ itself bounds an embedded disk in P_+. This is impossible, as $i_1(C)$ is essential in P_+. □

Therefore δ is a compression disk of the solid torus $f(T)$. But $P_+ = i_1(P)$, like P itself, is essential in T. Hence $f(P_+)$ is essential in $f(T)$ and δ must intersect $f(P_+)$. □

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Proposed: Cameron Gordon Received: 13 November 2007
Seconded: Martin Bridson, Walter Neumann Accepted: 8 February 2008