Essential curves in handlebodies and topological contractions

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If $X$ is a compact set, a topological contraction is a self-embedding $f$ such that the intersection of the successive images $f^k(X)$, $k \geq 0$, consists of one point. In dimension 3, we prove that there are smooth topological contractions of the handlebodies of genus $\geq 2$ whose image is essential.

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1 Introduction

For a compact set $X$ and a topological embedding $f : X \to X$, we shall say that $f$ is a topological contraction if $\bigcap_{k \geq 0} f^k(X)$ consists of one point. We shall show that such a contraction can be very complicated when $X$ is a 3–dimensional handlebody. Namely, we have the following result for which some more classical definitions will be recalled thereafter.

Theorem A There exists a North–South diffeomorphism $f$ of the 3–sphere $S^3$ and a Heegaard decomposition $S^3 = P_- \cup P_+$ of genus $g \geq 2$ with the following properties:

1. $f| P_+$ is a topological contraction;
2. $f(P_+)$ is essential in $P_+$.

We shall limit ourselves to $g = 2$, since the generalization will be clear. We recall that a 3–dimensional handlebody of genus 2 is diffeomorphic to the regular neighborhood $P$ of the planar figure eight $\Gamma$. A compression disk of $P$ is a smooth embedded disk in $P$ whose boundary lies in $\partial P$ in which it is not homotopic to a point. Among the compression disks are the meridian disks $\pi^{-1}(x)$, where $x$ is a regular point in $\Gamma$ and $\pi : P \to \Gamma$ is the regular neighborhood projection (that is, a submersion over the smooth part of $\Gamma$). A subset $X$ of $P$ is said to be essential in $P$ if it intersects every compression disk.$^2$

$^1$Any point other than the center of the figure eight.

$^2$This definition goes back to Rolfsen’s book [2, p 110].
A diffeomorphism $f$ of $S^3$ is a *North–South diffeomorphism* if it has two fixed points only, one source $\alpha \in P_-$ and one sink $\omega \in P_+$, every other orbit going from $\alpha$ to $\omega$.

A *Heegaard splitting* of $S^3$ is made of an embedded surface dividing $S^3$ into two handlebodies. According to a famous theorem of F Waldhausen such a decomposition is unique up to isotopy [3]. It is not hard to prove that the phenomenon mentioned in Theorem A does not happen with a Heegaard splitting of genus 1: if $T$ is a solid torus and $f$ is a topological contraction of $T$, then there is a compression disk of $T$ avoiding $f(T)$.

The example which we are going to construct for proving Theorem A is based on the next theorem, for which some more notation is introduced. Let $\Gamma_0 \subset \Gamma$ be a simple closed curve. There exists a solid torus $T \subset \mathbb{R}^3$ which contains $P$ and which is a tubular neighborhood of $\Gamma_0$. Let $i_0: P \to T$ be this inclusion. We say that a simple curve is unknotted in $T$ if it bounds an embedded disk in $T$.

**Theorem B** There exists an essential simple curve $C$ in $P$ such that $i_0(C)$ is unknotted in $T$.

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## 2 Essential curves

Our candidate for $C$ in Theorem B is pictured in Figure 1.

It is clear that $i_0(C)$ is unknotted in $T$ (or, equivalently, in the complement of the vertical axis which is drawn in Figure 1). A way of proving that $C$ is essential in $P$ is to prove the following lemma (actually equivalent as the referee noticed$^4$).

**Lemma 1** Let $p: \widetilde{P} \to P$ be the universal cover of $P$ and let $\widetilde{C}$ be the preimage $p^{-1}(C)$. Then $\widetilde{C}$ is essential in $\widetilde{P}$.

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$^4$ According to the Loop theorem and Dehn’s lemma, $C$ essential in $P$ implies $\widetilde{C}$ essential in $\widetilde{P}$.
Figure 1

Proof We have the following description of $\tilde{P}$: it is a 3–ball with a Cantor set $E$ removed from its bounding 2–sphere\(^5\). This Cantor set is the set of ends of $\tilde{P}$. A simple curve in $\partial \tilde{P}$ is not homotopic to zero if it divides $E$ into two nonempty parts. We get a fundamental domain $F$ for the action of $\pi_1(P)$ on $\tilde{P}$ by cutting $P$ along two non-parallel meridian disks $D_0$ and $D_1$. Figure 2 shows what the pair $(F, C \cap F)$ looks like: $F$ is a 3–ball whose boundary consists of the union $\partial_1 F$ of four disks $d_0, d_0', d_1, d_1'$ and a punctured sphere $\partial_0 F$, where $p(d_0) = p(d_0') = D_0$ and $p(d_1) = p(d_1') = D_1$; $\tilde{C} \cap F$ is made of four strands with end points in $\partial_1 F$ and pairwise linked as it is shown.

One can show easily that (i) $\partial_1 F \setminus \tilde{C}$ is incompressible and boundary incompressible in $F \setminus \tilde{C}$, and (ii) $\partial_0 F$ is incompressible in $F \setminus \tilde{C}$. Now suppose on the contrary that $\tilde{C}$ is not essential and consider a compression disk $\mathcal{D}$ of $\tilde{C} \cap F$ avoiding $\tilde{C}$. We take $\Delta$ to be transversal to $\tilde{D} := p^{-1}(D_0 \cup D_1)$. A standard innermost circle/arc argument, using (i), shows that we may assume that $\Delta$ is contained in $F \setminus \tilde{C}$, contradicting (ii).

Remarks 1 (1) Globally $\tilde{C}$ looks like an infinite Borromean chain: any finite number of components is unlinked. We would like to know whether there exists a topological algebraic tool proving that $C$ is essential in $P$.

(2) Our referee proposed another example where $[C] = aba^{-1}b^{-1}$ with respect to the obvious basis $a, b$ of $\pi_1(P)$. In this case $C$ is essential in $P$ by an algebraic argument: the quotient $\pi_1(P)/[[C]]$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. If $C$ were inessential in $P$ this quotient would be isomorphic to $\mathbb{Z} \times \mathbb{Z}_n$ for some $n \geq 0$.

\(^5\)Take the universal cover of $\Gamma$ properly embedded in the hyperbolic plane and take a 3–dimensional thickening of it.
Proof of Theorem A

We recall the embedding $i_0: P \to \text{int } T$. We start with a curve $C$ in $P$ which meets the conclusion of Theorem B. We equip it with its 0–normal framing (a section of this framing is not linked with $C$ in $\mathbb{R}^3$) and we choose an embedding $j_0: T \to P$ whose image is a tubular neighborhood of $C$. Let $B$ be a small ball in $\text{int } T$. As $C$ is unknotted in $T$, there is an ambient isotopy, supported in $\text{int } T$, deforming $i_0$ to $i_1: P \to \text{int } T$ such that $i_1 \circ j_0(T)$ is a standard small solid torus in $B$. One half of the desired Heegaard splitting of genus 2 will be given by $P \cup i_1(P)$. At the present time $f$ is only defined on $T$ by $f := i_1 \circ j_0: T \to \text{int } T$. If we compose $i_1$ with a sufficiently strong metric contraction of $B$ into itself (with respect to some metric), then $f$ is a metric contraction. Hence $\bigcap_{k>0} f^k(T)$ consists of one point.

Choose a round ball $B'$ containing $T$ in its interior. Since $f|T$ is isotopic to the inclusion $T \hookrightarrow \mathbb{R}^3$, $f$ extends as a diffeomorphism $B' \to B$, and further as a diffeomorphism $S^3 \to S^3$. We are free to choose $f: S^3 \setminus B' \to S^3 \setminus B$ as we like. Let $B''$ be the closure of $S^3 \setminus B'$ and $\varphi: S^3 \to S^3$ be a diffeomorphism which is the identity on $B'$ and a strong metric contraction on a ball containing $f^{-1}(B'')$. If we replace $f$ by $f \circ \varphi^{-1}$ (without changing the notation), then $f^{-1}|B''$ becomes a metric contraction and the intersection $\bigcap_{k>0} f^{-k}(S^3 \setminus B')$ consists of one point. We now have a North–South diffeomorphism $f$ of $S^3$ which induces a topological contraction of $T$. Since $f(T) \subset \text{int } P_+ \subset P_+ \subset \text{int } T$, $f$ also induces a topological contraction of $P_+$.

It remains to prove that $f(P_+)$ is essential in $P_+$. We know that $i_1(C)$ is essential in $P_+$. As a consequence, any compression disk $\Delta$ of $P_+$ intersects $f(T)$. We can
take $\Delta$ to be transversal to $f(\partial T)$ such that no intersection curve is null-homotopic in $f(\partial T)$. Let $\gamma$ be an intersection curve which is innermost in $\Delta$ and let $\delta$ be the disk that $\gamma$ bounds in $\Delta$.

**Lemma 2** We have $\delta \subset f(T)$.

**Proof** If not, we have $\delta \subset P_+ \setminus f(\text{int } T)$ and the simple curve $\gamma$ in $f(\partial T)$ is unlinked with the core $i_1(C)$. Therefore, up to isotopy in $f(\partial T)$, it is a section of the 0–framing. In that case, $i_1(C)$ itself bounds an embedded disk in $P_+$. This is impossible, as $i_1(C)$ is essential in $P_+$. □

Therefore $\delta$ is a compression disk of the solid torus $f(T)$. But $P_+ = i_1(P)$, like $P$ itself, is essential in $T$. Hence $f(P_+)$ is essential in $f(T)$ and $\delta$ must intersect $f(P_+)$.

**References**


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