

Automorphisms of p -compact groups and their root data

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We construct a model for the space of automorphisms of a connected p -compact group in terms of the space of automorphisms of its maximal torus normalizer and its root datum. As a consequence we show that any homomorphism to the outer automorphism group of a p -compact group can be lifted to a group action, analogous to a classical theorem of de Siebenthal for compact Lie groups. The model of this paper is used in a crucial way in our paper [2], where we prove the conjectured classification of 2-compact groups and determine their automorphism spaces.

55R35; 20G99, 22E15, 55P35

1 Introduction

A p -compact group is a homotopy theoretic version of a compact Lie group at a prime p , and consists of a triple $(BX, X, e: X \xrightarrow{\cong} \Omega BX)$, where BX is a pointed p -complete space, X is a space with finite \mathbb{F}_p -cohomology, and e is a homotopy equivalence between X and the based loop space on BX ; see the papers by Dwyer and Wilkerson [16; 14] and by the authors with Møller and Viruel [3]. The goal of this paper is to give an algebraic model, which we denote $B\text{aut}(\mathbf{D}_X)$, of $B\text{Aut}(BX)$, the classifying space of the topological monoid $\text{Aut}(BX)$ of self-homotopy equivalences of BX , for a connected p -compact group X . We construct a natural map $\Phi: B\text{Aut}(BX) \rightarrow B\text{aut}(\mathbf{D}_X)$ and prove here that Φ induces an isomorphism on π_i , $i > 1$ (Theorem D). It is proved to be an isomorphism also on π_1 in our sequel paper [2], where we prove the conjectured classification of 2-compact groups using this model. As a consequence of the properties of Φ proved in this paper, we deduce the following result, which can be seen as a generalization of a classical theorem of de Siebenthal [33, Chapitre I, Section 2, no. 2] for compact connected Lie groups (see also Bourbaki [8, Section 4, no. 10] and Theorem 6.1).

Let $\mathcal{Z}(X)$ denote the center of the p -compact group X , $\check{\mathcal{Z}}(X)$ its discrete approximation (see Dwyer and Wilkerson [17]), and $\text{Out}(BX)$ the group of components of $\text{Aut}(BX)$.

Theorem A For any connected p -compact group X , the fibration

$$B^2 \mathcal{Z}(X) \rightarrow B \operatorname{Aut}(BX) \rightarrow B \operatorname{Out}(BX)$$

has a section. In particular, for a discrete group Γ , any group homomorphism $\Gamma \rightarrow \operatorname{Out}(BX)$ lifts to a group action $B\Gamma \rightarrow B \operatorname{Aut}(BX)$, and if $\tilde{H}^*(\Gamma; \mathbb{Q}) = 0$, the set of liftings is a nontrivial $H^2(\Gamma; \check{\mathcal{Z}}(X))$ -torsor.

The theorem can fail without the assumption that X is connected, the smallest examples being $X = D_{16}$ or Q_{16} , the dihedral and quaternion groups of order 16. Like for Lie groups, the theorem can be used to construct group actions and hence to construct new p -compact groups or p -local finite groups from old, using homotopy fixed point methods; see eg Dwyer and Wilkerson [16] and Broto and Møller [10].

Constructing the model requires understanding the relationship between the automorphisms of the maximal torus normalizer \mathcal{N}_X and of the root datum \mathbf{D}_X . On the level of objects \mathbf{D}_X can be constructed from \mathcal{N}_X and vice versa: For compact Lie groups this was proved in an influential 1966 paper by Tits [35] and it was recently generalized to 2-compact groups by Dwyer and Wilkerson [19]; for p -compact groups, p odd, it follows trivially from a theorem of the first named author [1, Theorem 1.2]. On the level of automorphisms the problem is that the outer automorphism groups of \mathcal{N}_X and \mathbf{D}_X in general differ, which means that we have to introduce an extra ingredient: In Theorem C we prove that the outer automorphism group of \mathbf{D}_X naturally corresponds to a subgroup of the outer automorphism group of \mathcal{N}_X , which permutes certain “root subgroups”. This is the main technical result of the paper. Even in the case of compact Lie groups the result (stated as Theorem B) appears to be new, though many of the ingredients are hidden in Tits’ paper mentioned above.

We now embark on describing our results more precisely, for which we need some notation. For a principal ideal domain R , an R -root datum \mathbf{D} is defined to be a triple $(W, L, \{Rb_\sigma\})$, where L is a free R -module of finite rank, $W \subseteq \operatorname{Aut}_R(L)$ is a finite subgroup generated by reflections (ie, elements σ such that $1 - \sigma \in \operatorname{End}_R(L)$ has rank one), and $\{Rb_\sigma\}$ is a collection of rank one submodules of L , indexed by the reflections σ in W , satisfying

$$\operatorname{im}(1 - \sigma) \subseteq Rb_\sigma \text{ and } w(Rb_\sigma) = Rb_{w\sigma w^{-1}} \text{ for all } w \in W.$$

The element $b_\sigma \in L$, called the *coroot* corresponding to σ , is determined up to a unit in R . Together with σ it determines a *root* $\beta_\sigma: L \rightarrow R$ via the formula

$$\sigma(x) = x + \beta_\sigma(x)b_\sigma.$$

For $R = \mathbb{Z}$ there is a one-to-one correspondence between \mathbb{Z} -root data and classically defined root data, by associating $(L, \{\pm b_\sigma\}, L^*, \{\pm \beta_\sigma\})$ to $(W, L, \{\mathbb{Z}b_\sigma\})$; see [19, Proposition 2.16]. It is easy to see that we always have

$$Rb_\sigma \subseteq \ker(1 + \sigma + \cdots + \sigma^{|\sigma|-1}: L \rightarrow L),$$

so given σ the possibilities for Rb_σ are in bijection with the cyclic R -submodules of $H^1(\langle \sigma \rangle; L)$. In particular if $H^1(\langle \sigma \rangle; L) = 0$ for all reflections σ , as is the case for $R = \mathbb{Z}_p$, p odd, the root datum $\mathbf{D} = (W, L, \{Rb_\sigma\})$ is uniquely determined by the underlying reflection group (W, L) . In fact a coup d'œil at the classification of finite \mathbb{Z} - or \mathbb{Z}_2 -reflection groups [3, Theorems 11.1 and 11.5] reveals that also for $R = \mathbb{Z}$ or \mathbb{Z}_2 the root datum is determined by (W, L) , unless (W, L) contains direct factors isomorphic to $(W_{\text{SO}(2n+1)}, L_{\text{SO}(2n+1)}) \cong (W_{\text{Sp}(n)}, L_{\text{Sp}(n)})$ —these exceptions, however, cannot easily be ignored since $\text{SO}(3)$ and $\text{SU}(2)$ are ubiquitous. Lastly we remark that eg, for $R = \mathbb{Z}$ or \mathbb{Z}_p one can instead of the collection $\{Rb_\sigma\}$ equivalently consider their span, the *coroot lattice*, $L_0 = +_\sigma Rb_\sigma \subseteq L$, as is easily seen; this was the definition given in [3, Section 1], under the name “ R -reflection datum”.

Two R -root data $\mathbf{D} = (W, L, \{Rb_\sigma\})$ and $\mathbf{D}' = (W', L', \{Rb'_{\sigma'}\})$ are said to be isomorphic if there exists an isomorphism $\varphi: L \rightarrow L'$ such that $\varphi W \varphi^{-1} = W'$ and $\varphi(Rb_\sigma) = Rb'_{\varphi\sigma\varphi^{-1}}$. In particular the *automorphism group* of \mathbf{D} is given by $\text{Aut}(\mathbf{D}) = \{\varphi \in N_{\text{Aut}_R(L)}(W) \mid \varphi(Rb_\sigma) = Rb_{\varphi\sigma\varphi^{-1}}\}$ and we define the *outer automorphism group* as $\text{Out}(\mathbf{D}) = \text{Aut}(\mathbf{D})/W$.

Tits constructed in [35] for any \mathbb{Z} -root datum \mathbf{D} an extension

$$1 \rightarrow T \rightarrow \nu(\mathbf{D}) \rightarrow W \rightarrow 1,$$

where $T = L \otimes_{\mathbb{Z}} S^1$, such that $N_G(T) \cong \nu(\mathbf{D}_G)$ for any compact connected Lie group G with root datum \mathbf{D}_G (see Theorem 3.2(1)). The group $\nu(\mathbf{D})$ will be called the *maximal torus normalizer* associated to \mathbf{D} . Conversely the \mathbb{Z} -root datum \mathbf{D} can be recovered from $\nu(\mathbf{D})$; see Proposition 4.2. For each reflection $\sigma \in W$ one can algebraically construct a canonical rank one “root subgroup” $N_\sigma = \nu(\mathbf{D})_\sigma$ inside $N = \nu(\mathbf{D})$, and similarly there are root subgroups $N_G(T)_\sigma$ of $N_G(T)$. A more precise version of Tits' result is that there is an essentially *unique* isomorphism between $N_G(T)$ and $\nu(\mathbf{D})$ sending $N_G(T)_\sigma$ to $\nu(\mathbf{D})_\sigma$ (see Theorem 3.2(1)). Define $\text{Aut}(N, \{N_\sigma\}) = \{\varphi \in \text{Aut}(N) \mid \varphi(N_\sigma) = N_{\varphi\sigma\varphi^{-1}}\}$ and $\text{Out}(N, \{N_\sigma\}) = \text{Aut}(N, \{N_\sigma\})/\text{Inn}(N)$ (where Aut and Out here have their usual meanings for Lie groups!). As usual $H^1(W; T) = \text{Der}(W, T)/\text{PDer}(W, T)$, derivations modulo principal derivations, and $H^1(W; T)$ embeds naturally in $\text{Out}(N)$ via the homomorphism $[f] \mapsto [\varphi_f]$, where $\varphi_f \in \text{Aut}(N)$ is given by $\varphi_f(x) = f(\bar{x})x$. The following is our main result in the Lie group case.

Theorem B Let $\mathbf{D} = (W, L, \{\mathbb{Z}b_\sigma\})$ be a \mathbb{Z} -root datum with corresponding maximal torus normalizer $N = v(\mathbf{D})$. There is a short exact sequence

$$1 \rightarrow H^1(W; T) \rightarrow \text{Out}(N) \rightarrow \text{Out}(\mathbf{D}) \rightarrow 1,$$

which has a canonical splitting $s: \text{Out}(\mathbf{D}) \rightarrow \text{Out}(N)$ with image $\text{Out}(N, \{N_\sigma\})$.

Furthermore, if G is a compact connected Lie group with root datum \mathbf{D}_G , the splitting s fits into the following commutative diagram of isomorphisms:

$$\begin{array}{ccc} & \text{Out}(N, \{N_\sigma\}) & \\ \text{res} \nearrow & & \nwarrow \text{res} \\ \text{Out}(G) & \xrightarrow[\cong]{\text{res}} & \text{Out}(\mathbf{D}_G) \end{array}$$

A split exact sequence as above was first established by Hämmerli [21, Theorem 1.2], using the existence of an ambient Lie group and case-by-case arguments, but without identifying the image of the splitting, the key point for our purposes.

For p odd, define the *discrete maximal torus normalizer* $\check{v}(\mathbf{D})$ of a \mathbb{Z}_p -root datum \mathbf{D} to be the semidirect product $\check{T} \rtimes W$, where $\check{T} = L \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$. By a theorem of the first named author [1, Theorem 1.2], $\check{\mathcal{N}}_X \cong \check{v}(\mathbf{D}_X)$, for any connected p -compact group X , where \mathbf{D}_X is the unique \mathbb{Z}_p -root datum with underlying reflection group (W_X, L_X) .

For $p = 2$, Dwyer and Wilkerson recently constructed in [19] a group $\check{v}(\mathbf{D})$ (again called the discrete maximal torus normalizer) from a \mathbb{Z}_2 -root datum \mathbf{D} , generalizing Tits' construction, as the middle term in an extension

$$1 \rightarrow \check{T} \rightarrow \check{v}(\mathbf{D}) \rightarrow W \rightarrow 1,$$

and showed how to associate a \mathbb{Z}_2 -root datum \mathbf{D}_X to any connected 2-compact group X , such that $\check{\mathcal{N}}_X \cong \check{v}(\mathbf{D}_X)$. As for \mathbb{Z} -root data we construct in Section 3 algebraic root subgroups $\check{\mathcal{N}}_\sigma = \check{v}(\mathbf{D})_\sigma$ of $\check{\mathcal{N}} = \check{v}(\mathbf{D})$ for any \mathbb{Z}_p -root datum \mathbf{D} and similarly one can define subgroups $(\check{\mathcal{N}}_X)_\sigma$ of $\check{\mathcal{N}}_X$. We strengthen the result of Dwyer–Wilkerson in Section 3 (Theorem 3.2(2)), keeping track of root subgroups, and this sharpening is key to our results. At the same time we remove an unfortunate reliance in [19] on a classification of 2-compact groups of semisimple rank 2.

As above define

$$\text{Aut}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\}) = \{\varphi \in \text{Aut}(\check{\mathcal{N}}) \mid \varphi(\check{\mathcal{N}}_\sigma) = \check{\mathcal{N}}_{\varphi\sigma\varphi^{-1}}\}$$

and $\text{Out}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\}) = \text{Aut}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\}) / \text{Inn}(\check{\mathcal{N}})$. For p -compact groups, we prove the following analog of Theorem B.

Theorem C Let $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ be a \mathbb{Z}_p -root datum with corresponding discrete maximal torus normalizer $\check{\mathcal{N}} = \check{\mathfrak{v}}(\mathbf{D})$. There is a short exact sequence

$$1 \rightarrow H^1(W; \check{T}) \rightarrow \text{Out}(\check{\mathcal{N}}) \rightarrow \text{Out}(\mathbf{D}) \rightarrow 1,$$

which has a canonical splitting $s: \text{Out}(\mathbf{D}) \rightarrow \text{Out}(\check{\mathcal{N}})$ with image $\text{Out}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\})$.

Furthermore, if X is a connected p -compact group with discrete maximal torus normalizer $\check{\mathcal{N}} = \check{\mathcal{N}}_X$, the canonical homomorphism $\Phi: \text{Out}(BX) \rightarrow \text{Out}(\check{\mathcal{N}})$ has image contained in $\text{Out}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\})$.

Here Φ is π_1 of the ‘‘Adams–Mahmud’’ map $\Phi: B\text{Aut}(BX) \rightarrow B\text{Aut}(B\check{\mathcal{N}}_X)$ given by lifting self-equivalences of BX to self-equivalences of $B\check{\mathcal{N}}_X$; see [3, Lemma 4.1 and Proposition 5.1]. For p odd, Theorem C degenerates since in this case $H^1(W; \check{T}) = 0$ by [1, Theorem 3.3].

We now define the space $B\text{aut}(\mathbf{D})$, which provides an algebraic model for $B\text{Aut}(BX)$, depending only on $\mathbf{D} = \mathbf{D}_X$. For p odd, set $B\text{aut}(\mathbf{D}) = B\text{Aut}(B\check{\mathcal{N}}_X)$, and recall that in this case $\Phi: B\text{Aut}(BX) \rightarrow B\text{aut}(\mathbf{D})$ is a homotopy equivalence by [3, Theorem 1.4]. For $p = 2$ we need to modify $B\text{Aut}(B\check{\mathcal{N}}_X)$ to get a model, since the H^1 -term in Theorem C can be nonzero. Let Y be the covering space of $B\text{Aut}(B\check{\mathcal{N}}_X)$ corresponding to the subgroup $\text{Out}(\check{\mathcal{N}}_X, \{(\check{\mathcal{N}}_X)_\sigma\})$ of the fundamental group. The space $B\text{aut}(\mathbf{D})$ is obtained from Y by killing certain $\mathbb{Z}/2$ -summands of $\pi_2(Y)$, one for each direct factor of \mathbf{D} isomorphic to $\mathbf{D}_{\text{SO}(2n+1)\hat{2}}$; we refer to Section 5 for a precise description of this. With $\mathcal{Z}(\mathbf{D})$ denoting the center of \mathbf{D} , defined in Section 5, we prove in Section 6 that we can identify the homotopy type of $B\text{aut}(\mathbf{D})$ with $(B^2\mathcal{Z}(\mathbf{D}))_{h\text{Out}(\mathbf{D})}$.

Theorem D For a connected p -compact group X with root datum \mathbf{D}_X , the Adams–Mahmud map induces a natural map

$$\Phi: B\text{Aut}(BX) \rightarrow (B^2\mathcal{Z}(\mathbf{D}_X))_{h\text{Out}(\mathbf{D}_X)}$$

which is an isomorphism on π_i for $i > 1$ and is the canonical map $\Phi: \text{Out}(BX) \rightarrow \text{Out}(\mathbf{D}_X)$ on π_1 .

From this Theorem A follows easily. As mentioned, we prove in our sequel paper [2], as part of our inductive proof of the classification of 2-compact groups, that Φ also induces an isomorphism on π_1 for $p = 2$. For p odd, Φ induces an isomorphism on π_1 by our work with Møller and Viruel [3, Thm 1.1].

Organization of the paper In Section 2 we recall the reflection extension of Dwyer–Wilkerson, and establish some further properties, which we use in Section 3 to construct

the root subgroups $\nu(\mathbf{D})_\sigma$. In Section 4 we prove Theorem B and Theorem C, and in Section 5 we construct the space $B\text{aut}(\mathbf{D})$ which we use in Section 6 to prove Theorem A and Theorem D.

Notation For a $\mathbb{Z}\langle\sigma\rangle$ -module A we define $A^-(\sigma) = \ker(\sum_{i=0}^{|\sigma|-1} \sigma^i: A \rightarrow A)$ and $A^+(\sigma) = \ker(1 - \sigma: A \rightarrow A)$. If A is an abelian group, A_0 denotes the maximal divisible subgroup of A and ${}_2A = \ker(A \xrightarrow{-2} A)$. For a compact Lie group G , $\text{Aut}(G)$ denotes the group of Lie group automorphisms of G . If Y is a topological space, $\text{Aut}(Y)$ denotes the topological monoid of self-homotopy equivalences of Y . A \mathbb{Z}_p -root datum \mathbf{D} is said to be of *Coxeter type* if $\mathbf{D} \cong \mathbf{D}' \otimes_{\mathbb{Z}} \mathbb{Z}_p = (W, L \otimes_{\mathbb{Z}} \mathbb{Z}_p, \{\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}b_\sigma\})$ for a \mathbb{Z} -root datum $\mathbf{D}' = (W, L, \{\mathbb{Z}b_\sigma\})$. We call a \mathbb{Z}_p -root datum $\mathbf{D} = (W, L, \{\mathbb{Z}b_\sigma\})$ *exotic* if $L \otimes_{\mathbb{Z}} \mathbb{Q}$ is an irreducible $\mathbb{Q}_p[W]$ -module and \mathbf{D} is not of Coxeter type. A *subdatum* of a \mathbb{Z}_p -root datum $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ is a \mathbb{Z}_p -root datum of the form $(W', L, \{\mathbb{Z}_p b_\sigma\}_{\sigma \in \Sigma'})$ where (W', L) is a reflection subgroup of (W, L) and Σ' is the set of reflections in W' . We will freely use the terminology of p -compact groups, though we try to give concrete references for the facts we use—we refer the reader to [16; 14; 3] for background information.

Acknowledgements The second named author was partially supported by NSF grant DMS-0354633 and the Alfred P Sloan foundation.

2 The reflection extension

In this section we introduce certain subextensions $\rho(W)_\sigma$ of the *reflection extension* $\rho(W)$ of Dwyer–Wilkerson [19] (recalled below) for any reflection σ in a finite \mathbb{Q}_2 -reflection group W , and describe their basic properties (Proposition 2.3). The results generalize work of Tits [35] over \mathbb{Q} , and are key to constructing the root subgroups $\nu(\mathbf{D})_\sigma$ of the normalizer extension $\nu(\mathbf{D})$ in the next section.

We begin with some definitions and recollections. Let W be a finite \mathbb{Q}_2 -reflection group and let $\Sigma = \bigcup_i \Sigma_i$ denote the partition of the set of reflections Σ in W into conjugacy classes. Choose for each conjugacy class Σ_i an element $\tau_i \in \Sigma_i$. Since -1 is the only nontrivial element of finite order in \mathbb{Q}_2^\times we may find an element $a_i \neq 0$ in the \mathbb{Q}_2 -vector space underlying W such that $\tau_i(a_i) = -a_i$. Let $C_i = C_W(\tau_i)$ and C_i^\perp be the stabilizer of a_i in W . It is clear that we get a direct product decomposition $C_i = \langle \tau_i \rangle \times C_i^\perp \cong \mathbb{Z}/2 \times C_i^\perp$. Now consider the extension

$$1 \rightarrow \mathbb{Z} \xrightarrow{(\cdot 2, \cdot 0)} \mathbb{Z} \times C_i^\perp \rightarrow \mathbb{Z}/2 \times C_i^\perp \rightarrow 1,$$

where $\cdot 0$ denotes the trivial homomorphism. This extension produces an element in $H^2(C_i; \mathbb{Z})$. By Shapiro's lemma we have $H^2(C_i; \mathbb{Z}) \cong H^2(W; \mathbb{Z}[\Sigma_i])$ and hence we get an extension $\rho_i(W)$ of W by $\mathbb{Z}[\Sigma_i]$. Following Dwyer–Wilkerson [19] we define the *reflection extension* $\rho(W)$ of W as the sum of the extensions $\rho_i(W)$. Noting that $\bigoplus_i \mathbb{Z}[\Sigma_i] \cong \mathbb{Z}[\Sigma]$, we see that the reflection extension has the form $1 \rightarrow \mathbb{Z}[\Sigma] \rightarrow \rho(W) \xrightarrow{\pi} W \rightarrow 1$. It is clear that the extensions $\rho_i(W)$ are well-defined up to equivalence of extensions and hence $\rho(W)$ is also well-defined.

Lemma 2.1 *Let W_1 and W_2 be finite \mathbb{Q}_2 -reflection groups and set $W = W_1 \times W_2$. Then the reflection extension of W splits as a product*

$$\rho(W) \cong \rho(W_1) \times \rho(W_2).$$

Proof Let τ_i be a reflection in W_1 . For the construction of $\rho(W_1)$ we have to consider the extension

$$1 \rightarrow \mathbb{Z} \xrightarrow{(\cdot 2, \cdot 0)} \mathbb{Z} \times C_i^\perp \rightarrow \mathbb{Z}/2 \times C_i^\perp \rightarrow 1,$$

where $C_i = C_{W_1}(\tau_i)$. Since $C_W(\tau_i) = C_i \times W_2$ and $C_W(\tau_i)^\perp = C_i^\perp \times W_2$ it is clear that the associated sequence to be used in the construction of $\rho(W)$ equals

$$1 \rightarrow \mathbb{Z} \xrightarrow{(\cdot 2, \cdot 0)} \mathbb{Z} \times (C_i^\perp \times W_2) \rightarrow \mathbb{Z}/2 \times (C_i^\perp \times W_2) \rightarrow 1.$$

Via Shapiro's lemma the first extension corresponds to an extension

$$1 \rightarrow \mathbb{Z}[\Sigma_i] \rightarrow \rho_i(W_1) \rightarrow W_1 \rightarrow 1.$$

It is now clear that the associated sequence for W must be

$$1 \rightarrow \mathbb{Z}[\Sigma_i] \rightarrow \rho_i(W_1) \times W_2 \rightarrow W_1 \times W_2 \rightarrow 1.$$

From this it follows that the reflection extension for W must be the sum of the extensions $\rho(W_1) \times W_2$ and $W_1 \times \rho(W_2)$ which equals the extension $\rho(W_1) \times \rho(W_2)$ as claimed. \square

We also need the following lemma, which can be extracted from [19].

Lemma 2.2 [19, Proof of Lemma 10.1] *Let W be a finite \mathbb{Q}_2 -reflection group, and let W_1 be a reflection subgroup. Let Σ be the set of reflections in W and Σ_1 the set of reflections in W_1 . Then the pullback of the reflection extension*

$$1 \rightarrow \mathbb{Z}[\Sigma] \rightarrow \rho(W) \rightarrow W \rightarrow 1$$

along the inclusion $W_1 \rightarrow W$ is isomorphic to the sum of the reflection extension $1 \rightarrow \mathbb{Z}[\Sigma_1] \rightarrow \rho(W_1) \rightarrow W_1 \rightarrow 1$ and the semidirect product

$$1 \rightarrow \mathbb{Z}[\Sigma \setminus \Sigma_1] \rightarrow \mathbb{Z}[\Sigma \setminus \Sigma_1] \rtimes W_1 \rightarrow W_1 \rightarrow 1. \quad \square$$

We can now prove the main statement of this section, which generalizes [35, Proposition 2.10] to finite \mathbb{Q}_2 -reflection groups.

Proposition 2.3 *Let W be a finite \mathbb{Q}_2 -reflection group and $\rho(W)$ the associated reflection extension. For a reflection $\sigma \in W$, define*

$$C_\sigma = (1 - \sigma)\mathbb{Z}[\Sigma], \quad Q_\sigma = \{x \in \rho(W) \mid x^2 = \sigma \in \mathbb{Z}[\Sigma]\},$$

and $\rho(W)_\sigma = \langle Q_\sigma \rangle$. Then the following holds:

- (1) For $x \in \rho(W)$ with $w = \pi(x)$ we have $xC_\sigma x^{-1} = C_{w\sigma w^{-1}}$ and $xQ_\sigma x^{-1} = Q_{w\sigma w^{-1}}$.
- (2) $\rho(W)$ contains no involutions.
- (3) $\pi(Q_\sigma) = \{\sigma\}$.
- (4) C_σ is a subgroup of $\mathbb{Z}[\Sigma]$ and we have $C_\sigma = \ker(\mathbb{Z}[\Sigma] \xrightarrow{1+\sigma} \mathbb{Z}[\Sigma])$.
- (5) Q_σ is a coset of C_σ in $\rho(W)$; more precisely, for $x \in Q_\sigma$ we have $Q_\sigma = xC_\sigma = C_\sigma x$.
- (6) We have a short exact sequence $1 \rightarrow C_\sigma \oplus \mathbb{Z}\sigma \rightarrow \rho(W)_\sigma \xrightarrow{\pi} \langle \sigma \rangle \rightarrow 1$.

Proof Part (1) is obvious. By Lemma 2.1 we see that if $W = W_1 \times W_2$ is a product of two finite \mathbb{Q}_2 -reflection groups, then $\rho(W) = \rho(W_1) \times \rho(W_2)$. Thus W satisfies (2) if and only if W_1 and W_2 does.

Moreover, if $\sigma \in W_1$ is a reflection we get

$$Q_\sigma(W) = Q_\sigma(W_1) \times \{x \in \rho(W_2) \mid x^2 = 1\},$$

where $Q_\sigma(W)$ and $Q_\sigma(W_1)$ denotes Q_σ defined with respect to W and W_1 respectively. In particular W satisfies (3) for a fixed $\sigma \in W_1$ if and only if W_1 satisfies (3) for this σ and W_2 satisfies (2).

Since any finite \mathbb{Q}_2 -reflection group is a product of a Coxeter group and a number of copies of $W_{\text{DI}(4)}$ (cf Clark–Ewing [12]), the above remarks shows that it suffices to prove (2) and (3) in the Coxeter case and in the case of $W_{\text{DI}(4)}$.

We first deal with the Coxeter case. Here (3) holds by [35, Proposition 2.10(b)]. Now let W be any Coxeter group and let W' be a nontrivial Coxeter group. Choose any

reflection $\sigma \in W'$. Since (3) holds for the Coxeter group $W' \times W$ with respect to σ we obtain (2) for W by the above. This proves (2) and (3) in the Coxeter case.

In the case $W = W_{\text{DI}(4)}$, there is a reflection subgroup W_1 of W isomorphic to $W_{\text{Spin}(7)}$ (cf eg [19, Proof of Proposition 9.12, DI(4) case]). It is easily checked that any element $w \in W$ with $w^2 = 1$ is conjugate to an element in W_1 . Assume first that $x \in \rho(W)$ with $x^2 = 1$. Then $\pi(x) \in W$ satisfies $\pi(x)^2 = 1$ so that up to a conjugation in $\rho(W)$ we may assume $\pi(x) \in W_1$, ie, that x belongs to the pullback of $\rho(W)$ along the inclusion $W_1 \rightarrow W$. Under the isomorphism given by Lemma 2.2, the element x corresponds to a pair of elements (x_1, x_2) with $x_1 \in \rho(W_1)$ and $x_2 \in \mathbb{Z}[\Sigma \setminus \Sigma_1] \rtimes W_1$ having the same image under the projections to W_1 . Since $x^2 = 1$ we have $x_1^2 = 1$ and $x_2^2 = 1$. Since we have already established (2) for the Coxeter group W_1 we conclude that $x_1 = 1$. Hence x_2 projects to the identity in W_1 , so x_2 is contained in the subgroup $\mathbb{Z}[\Sigma \setminus \Sigma_1]$ of $\mathbb{Z}[\Sigma \setminus \Sigma_1] \rtimes W_1$. Since this subgroup contains no involutions we see that $x_2 = 1$. Therefore $x = 1$ which proves (2) for $W_{\text{DI}(4)}$.

To prove (3) for $W_{\text{DI}(4)}$, note first that by Lemma 2.2 $\rho(W_1)$ may be considered as a subgroup of the pullback of $\rho(W)$ along $W_1 \rightarrow W$ since this pullback is the sum of the reflection extension of W_1 and a semidirect product. Thus $\rho(W_1)$ may be viewed as a subgroup of $\rho(W)$. Since (3) holds for the Coxeter group W_1 we get $\sigma \in \pi(Q_\sigma)$ for $\sigma \in \Sigma_1$. Hence $\sigma \in \pi(Q_\sigma)$ for all $\sigma \in \Sigma$ by (1) as any reflection in W is conjugate to one in W_1 .

Conversely, assume that $x \in Q_\sigma$ for some $\sigma \in \Sigma$. As above $\pi(x) \in W$ satisfies $\pi(x)^2 = 1$, so up to conjugation in $\rho(W)$ we may assume that $\pi(x) \in W_1$, ie that x belongs to the pullback of $\rho(W)$ along the inclusion $W_1 \rightarrow W$. As above x corresponds to a pair of elements (x_1, x_2) with $x_1 \in \rho(W_1)$ and $x_2 \in \mathbb{Z}[\Sigma \setminus \Sigma_1] \rtimes W_1$ having the same image under the projections to W_1 . Then $x^2 = \sigma \in \mathbb{Z}[\Sigma]$ corresponds to (x_1^2, x_2^2) . Hence if $\sigma \in \Sigma_1$ we have $x_1^2 = \sigma$ (and $x_2^2 = 1$) and if $\sigma \notin \Sigma_1$ we have $x_2^2 = \sigma$ (and $x_1^2 = 1$). However the second case cannot occur since no element in $\mathbb{Z}[\Sigma \setminus \Sigma_1] \rtimes W_1$ has square σ (the square of an element (x, w) in the semidirect product equals $(x + w \cdot x, w^2)$ and the image of $x + w \cdot x$ under the augmentation $\mathbb{Z}[\Sigma \setminus \Sigma_1] \rightarrow \mathbb{Z}$ is even). We thus conclude that $\sigma \in \Sigma_1$ and $x_1^2 = \sigma \in \Sigma_1$. Thus x_1 belongs to the subgroup $Q_\sigma(W_1)$ and hence x_1 projects to $\sigma \in W_1$ since (3) holds for W_1 . This shows that $\pi(x) = \sigma$ as desired. This proves (3) for $W_{\text{DI}(4)}$ and hence (2) and (3) holds in general.

Part (4) amounts to showing that $H^1(\langle \sigma \rangle; \mathbb{Z}[\Sigma]) = 0$. Since σ is an involution the $\langle \sigma \rangle$ -module $\mathbb{Z}[\Sigma]$ splits as a direct sum of \mathbb{Z} 's with trivial action and \mathbb{Z}^2 's with the action given by permuting the generators. Since $H^1(\langle \sigma \rangle; -) = 0$ in both cases, the claim follows.

To prove (5), consider $x \in Q_\sigma$. By (3) any element in Q_σ has the form xu for some $u \in \mathbb{Z}[\Sigma]$. The computation $(xu)^2 = x^2x^{-1}uxu = \sigma x^{-1}uxu$ shows that for $u \in \mathbb{Z}[\Sigma]$ we have $xu \in Q_\sigma$ if and only if $x^{-1}uxu = 1$. Since $\pi(x) = \sigma$ this is equivalent to $(1 + \sigma) \cdot u = 0$. By (4) this proves that $Q_\sigma = xC_\sigma$ and one gets $Q_\sigma = C_\sigma x$ analogously.

Finally, to see (6), note that $\rho(W)_\sigma \xrightarrow{\pi} \langle \sigma \rangle$ is surjective by (3). Fixing $x \in Q_\sigma$ we have $\rho(W)_\sigma = \langle x, C_\sigma \rangle$ by (5). Since $x^k C_\sigma = C_\sigma x^k$ for $k \in \mathbb{Z}$ by (5), any element in $\rho(W)_\sigma$ has the form $x^k c$ for some $k \in \mathbb{Z}$ and some $c \in C_\sigma$. As $x \notin \ker(\pi)$ and $x^2 = \sigma \in \ker(\pi)$ it follows that the kernel of $\rho(W)_\sigma \xrightarrow{\pi} \langle \sigma \rangle$ equals $\langle C_\sigma, \sigma \rangle$ proving the proposition since $\sigma \notin C_\sigma$. \square

3 The normalizer extension

In this section we define the normalizer extension and the associated root subgroups algebraically, using the results and notation of the previous section. We use this to give a strengthened version, needed for our purposes, of the result of Dwyer–Wilkerson [19, Proposition 1.10] on the maximal torus normalizer in a connected 2–compact group, where we also keep track of certain root subgroups defined below. This version is on a par with the corresponding result for compact Lie groups by Tits [35]. We furthermore circumvent the use of a classification of connected 2–compact groups of semisimple rank 2 used in [19, Proposition 9.15]. (The semisimple rank of a connected p –compact group is the rank of its universal cover.) For this refer to [4, Theorem 6.1] which collects a range of disparate results, including old classification results on finite H –spaces. We prefer to avoid this reliance, since we use the results of this paper in our general classification of 2–compact groups [2], and we are able to do this using a few low-dimensional group cohomology computations instead.

We start by giving an alternative definition of \mathbb{Z} – and \mathbb{Z}_p –root data in terms of *markings*. For a \mathbb{Z} –root datum $\mathbf{D} = (W, L, \{\mathbb{Z}b_\sigma\})$, the *marking* associated to the reflection $\sigma \in W$ is the element $h_\sigma = b_\sigma/2 \in T = L \otimes_{\mathbb{Z}} S^1$. Conversely,

$$\mathbb{Z}b_\sigma = \frac{2}{|h_\sigma|} \ker(L \xrightarrow{1+\sigma} L),$$

so one might as well define a \mathbb{Z} –root datum in terms of the h_σ instead of the $\mathbb{Z}b_\sigma$, and we will use these two viewpoints interchangeably without further comment. (The conditions on h_σ corresponding to the conditions on $\mathbb{Z}b_\sigma$ say that $h_\sigma \in T_0^-(\sigma)$, $h_\sigma^2 = 1$ and $h_\sigma \neq 1$ if σ acts nontrivially on ${}_2T$, cf [19, Definition 2.12]). This definition of the markings h_σ carries over verbatim to \mathbb{Z}_p –root data replacing T by $\check{T} = L \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$; see [19, Section 6] for the case $p = 2$, for $p > 2$ we simply have $h_\sigma = 1 \in \check{T}$.

Next we recall the construction of root data for compact Lie groups and p -compact groups: For a compact connected Lie group G with maximal torus T , maximal torus normalizer $N = N_G(T)$, Weyl group $W = N/T$ and integral lattice $L = \pi_1(T)$, we define the *topological root subgroup* as

$$N_\sigma = \{x \in C_N(T_0^+(\sigma)) \mid x \text{ is conjugate in } C_G(T_0^+(\sigma)) \text{ to some } y \in T_0^-(\sigma)\},$$

for any reflection $\sigma \in W$. This is a subgroup of N , $\{x^2 \mid x \in N_\sigma \setminus T_0^-(\sigma)\}$ consists of a single element h_σ (see [19, Lemma 5.2]), and $\mathbf{D}_G = (W, L, \{h_\sigma\})$ is a \mathbb{Z} -root datum, the root datum of G (see [19, Proposition 5.5]). One can check that

$$N_\sigma = C_N(T_0^+(\sigma)) \cap C_G(T_0^+(\sigma))'$$

(the prime denoting the commutator subgroup), the definition from [35, 4.1], and that $C_N(T_0^+(\sigma))$ is the pullback of N along the inclusion $\langle \sigma \rangle \rightarrow W$. Note that by definition, the topological root subgroups are well behaved under conjugation: For $x \in N$ with image $w \in W$ we have $xN_\sigma x^{-1} = N_{w\sigma w^{-1}}$ (cf [19, Proposition 5.5]).

For a connected 2-compact group X , we can use the above definition verbatim (replacing G by X , N by \check{N}_X and T by \check{T}) to define *topological root subgroups* $(\check{N}_X)_\sigma$ of X and a \mathbb{Z}_2 -root datum \mathbf{D}_X , the *root datum of X* . The verification that these have the desired properties is given in [19, Section 9] and uses that any connected 2-compact group of semisimple rank 1 is either $(\mathrm{SU}(2) \times (S^1)^n)_2^\wedge$, $(\mathrm{SO}(3) \times (S^1)^n)_2^\wedge$, or $(\mathrm{U}(2) \times (S^1)^n)_2^\wedge$ for some $n \geq 0$; this is essentially classical, and follows easily from [15]. As above we also have $x\check{N}_\sigma x^{-1} = \check{N}_{w\sigma w^{-1}}$ for $x \in \check{N}_X$ with image $w \in W_X$ [19, Proposition 9.10].

For a connected p -compact group X , p odd, the first-named author proved that the extension $1 \rightarrow \check{T} \rightarrow \check{N}_X \rightarrow W \rightarrow 1$ is split [1, Theorem 1.2], and the splitting is unique up to conjugation by an element in \check{T} [1, Theorem 3.3]. Hence we can define the *topological root subgroups* $(\check{N}_X)_\sigma$ of \check{N}_X as the images of the subgroups $\check{T}^-(\sigma) \rtimes \langle \sigma \rangle \subseteq \check{T} \rtimes W$ under the isomorphism $\check{N}_X \cong \check{T} \rtimes W$ of extensions; this is independent of the chosen isomorphism since two such isomorphisms differ by conjugation by an element in \check{T} . Note also that $\check{T}^-(\sigma) = \check{T}_0^-(\sigma)$ since $H^1(\langle \sigma \rangle; \check{T}) = 0$ for p odd. We also define the root datum \mathbf{D}_X of X as the unique \mathbb{Z}_p -root datum whose underlying reflection group is (W_X, L_X) .

We define, following Dwyer–Wilkerson [19, Section 3], the *normalizer extension*

$$1 \rightarrow T \rightarrow v(\mathbf{D}) \rightarrow W \rightarrow 1$$

of a \mathbb{Z} -root datum $\mathbf{D} = (W, L, \{\mathbb{Z}b_\sigma\})$ as the pushforward of the reflection extension $1 \rightarrow \mathbb{Z}[\Sigma] \rightarrow \rho(W) \rightarrow W \rightarrow 1$ constructed in Section 2, along the W -map $f: \mathbb{Z}[\Sigma] \rightarrow T$ given by $\sigma \mapsto h_\sigma$.

For a reflection $\sigma \in W$, define the *algebraic root subgroup* $\nu(\mathbf{D})_\sigma$ of $\nu(\mathbf{D})$ via the extension

$$1 \rightarrow T_0^-(\sigma) \rightarrow \nu(\mathbf{D})_\sigma \rightarrow \langle \sigma \rangle \rightarrow 1$$

obtained as the pushforward along $f|_{C_\sigma \oplus \mathbb{Z}\sigma}: C_\sigma \oplus \mathbb{Z}\sigma \rightarrow T_0^-(\sigma)$ of the extension $1 \rightarrow C_\sigma \oplus \mathbb{Z}\sigma \rightarrow \rho(W)_\sigma \xrightarrow{\pi} \langle \sigma \rangle \rightarrow 1$, constructed in Proposition 2.3(6). Note that this makes sense, since $f(C_\sigma \oplus \mathbb{Z}\sigma)$ is contained in $T_0^-(\sigma)$ because

$$f(C_\sigma) = f((1 - \sigma)\mathbb{Z}[\Sigma]) \subseteq (1 - \sigma)T = T_0^-(\sigma)$$

and $f(\sigma) = h_\sigma \in T_0^-(\sigma)$. Also note that by Proposition 2.3(1) we have $x\nu(\mathbf{D})_\sigma x^{-1} = \nu(\mathbf{D})_{w\sigma w^{-1}}$ for $x \in \nu(\mathbf{D})$ with image $w \in W$.

Similarly, for a \mathbb{Z}_2 -root datum $\mathbf{D} = (W, L, \{\mathbb{Z}_2 b_\sigma\})$ the *normalizer extension*

$$1 \rightarrow \check{T} \rightarrow \check{\nu}(\mathbf{D}) \rightarrow W \rightarrow 1$$

is defined to be the pushforward of $\rho(W)$ along $f: \mathbb{Z}[\Sigma] \rightarrow \check{T}$ defined by $\sigma \mapsto h_\sigma$, and again the *algebraic root subgroup* $\check{\nu}(\mathbf{D})_\sigma$ is defined via the extension

$$1 \rightarrow \check{T}_0^-(\sigma) \rightarrow \check{\nu}(\mathbf{D})_\sigma \rightarrow \langle \sigma \rangle \rightarrow 1$$

which is obtained as the pushforward of $1 \rightarrow C_\sigma \oplus \mathbb{Z}\sigma \rightarrow \rho(W)_\sigma \xrightarrow{\pi} \langle \sigma \rangle \rightarrow 1$ along $f|_{C_\sigma \oplus \mathbb{Z}\sigma}: C_\sigma \oplus \mathbb{Z}\sigma \rightarrow \check{T}_0^-(\sigma)$.

Recall that for a \mathbb{Z}_p -root datum $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$, p odd, we have defined the discrete maximal torus normalizer $\check{\nu}(\mathbf{D})$ as $\check{T} \rtimes W$. Similarly we define the *algebraic root subgroups* by $\check{\nu}(\mathbf{D})_\sigma = \check{T}_0^-(\sigma) \rtimes \langle \sigma \rangle$.

Remark 3.1 We remark that the name root subgroup for N_σ perhaps is a bit unfortunate: If $G(\mathbb{C})$ is a connected reductive algebraic group over \mathbb{C} and G is the corresponding maximal compact subgroup, then N_σ with respect to G is the maximal compact subgroup in the normalizer of $T(\mathbb{C})_0^-(\sigma)$ in $\langle U_\alpha, U_{-\alpha} \rangle$, where $\pm\alpha$ are the two roots associated to σ and U_α is what is usually called the root subgroup corresponding to α in the algebraic group $G(\mathbb{C})$. (See Borel [7, Theorem IV.13.18(4)(d)], Springer [34, Proposition 8.1.1(i)] or Humphreys [23, Theorem 26.3].)

We can now state an improved version of the main theorem of Dwyer–Wilkerson [19], which says that the algebraic and topological definitions of the maximal torus normalizer and the root subgroups coincide.

Theorem 3.2 We have the following identifications.

- (1) [35, Théorème 4.4] If G is a compact connected Lie group with root datum $\mathbf{D}_G = (W, L, \{\mathbb{Z}b_\sigma\})$, then there exists an isomorphism of extensions

$$\begin{array}{ccccccc} 1 & \longrightarrow & T & \longrightarrow & v(\mathbf{D}_G) & \longrightarrow & W \longrightarrow 1 \\ & & \parallel & & \downarrow \cong & & \parallel \\ 1 & \longrightarrow & T & \longrightarrow & N_G(T) & \longrightarrow & W \longrightarrow 1 \end{array}$$

taking $v(\mathbf{D}_G)_\sigma$ to $N_G(T)_\sigma$ for all σ , and this specifies the isomorphism uniquely up to conjugation by an element in T .

- (2) (compare [19, Proposition 1.10]) If X is a connected p -compact group with root datum $\mathbf{D}_X = (W, L, \{\mathbb{Z}_p b_\sigma\})$ and discrete maximal torus normalizer \check{N}_X , then the result of (1) continues to hold replacing G by X , T by \check{T} , $v(\mathbf{D}_G)$ by $\check{v}(\mathbf{D}_X)$ and $N_G(T)$ by \check{N}_X .

Before the proof we need to establish some properties of the algebraic root subgroups. The first lemma tells us how to recover the elements h_σ from the root subgroups.

Lemma 3.3 Let $\mathbf{D} = (W, L, \{\mathbb{Z}b_\sigma\})$ be a \mathbb{Z} -root datum with associated maximal torus normalizer $v(\mathbf{D})$ and root subgroups $v(\mathbf{D})_\sigma$. Then

$$\{x^2 \mid x \in v(\mathbf{D})_\sigma \setminus T_0^-(\sigma)\} = \{h_\sigma\}$$

for all reflections $\sigma \in W$. The analogous result holds for \mathbb{Z}_2 -root data.

Proof If $x_1, x_2 \in v(\mathbf{D})_\sigma \setminus T_0^-(\sigma)$, then $x_2 = tx_1$ for some $t \in T_0^-(\sigma)$ and hence $x_2^2 = (tx_1tx_1^{-1})x_1^2 = (t\sigma(t))x_1^2 = x_1^2$. On the other hand we have a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & C_\sigma \oplus \mathbb{Z}\sigma & \longrightarrow & \rho(W)_\sigma & \longrightarrow & \langle \sigma \rangle \longrightarrow 1 \\ & & \downarrow f & & \downarrow & & \parallel \\ 1 & \longrightarrow & T_0^-(\sigma) & \longrightarrow & v(\mathbf{D})_\sigma & \longrightarrow & \langle \sigma \rangle \longrightarrow 1 \end{array}$$

in which $Q_\sigma \subseteq \rho(W)_\sigma = \langle Q_\sigma \rangle$ is mapped into $v(\mathbf{D})_\sigma \setminus T_0^-(\sigma)$ by Proposition 2.3(3). Since $x^2 = \sigma$ for any $x \in Q_\sigma$ and $f(\sigma) = h_\sigma$ this proves the claim. Obviously the proof carries over verbatim to \mathbb{Z}_2 -root data. \square

We now enumerate the subgroups of $v(\mathbf{D})$ which behave like the root subgroups $v(\mathbf{D})_\sigma$ in the sense of Lemma 3.3.

Lemma 3.4 Let $\mathbf{D} = (W, L, \{\mathbb{Z}b_\sigma\})$ be a \mathbb{Z} -root datum with corresponding maximal torus normalizer $N = v(\mathbf{D})$. Let $\sigma \in W$ be a reflection, and let $N(\sigma)$ be the preimage under the projection $N \rightarrow W$ of the subgroup $\langle \sigma \rangle$. Denote by \mathcal{X} the set of subgroups H of N which sit in an exact sequence

$$(3-1) \quad 1 \rightarrow T_0^-(\sigma) \rightarrow H \rightarrow \langle \sigma \rangle \rightarrow 1$$

and satisfy $\{x^2 \mid x \in H \setminus T_0^-(\sigma)\} = \{h_\sigma\}$. Then \mathcal{X} is a nontrivial $H^1(\langle \sigma \rangle; T)$ -torsor with the action induced by the obvious action of $\text{Der}(\langle \sigma \rangle, T) \subseteq \text{Aut}(N(\sigma))$ on \mathcal{X} .

The corresponding result for \mathbb{Z}_2 -root data is also valid.

Proof We already know that $N_\sigma \in \mathcal{X}$ by Lemma 3.3 so \mathcal{X} is nonempty. Pick $x_0 \in N_\sigma \setminus T_0^-(\sigma)$; we have $N_\sigma = \langle T_0^-(\sigma), x_0 \rangle$. It is clear that any subgroup H of N which sits in an exact sequence of the form (3-1) must have the form $H = \langle T_0^-(\sigma), tx_0 \rangle$ for some $t \in T$ and the computation $(tx_0)^2 = (tx_0tx_0^{-1})x_0^2 = t\sigma(t)h_\sigma$ shows that $\langle T_0^-(\sigma), tx_0 \rangle \in \mathcal{X}$ if and only if $t \in T^-(\sigma)$.

This shows that we have a transitive action of $T^-(\sigma)$ on \mathcal{X} given by $(t, \langle T_0^-(\sigma), x \rangle) \mapsto \langle T_0^-(\sigma), tx \rangle$. In addition it is easily seen that under the identification $T^-(\sigma) = \text{Der}(\langle \sigma \rangle, T)$ this action corresponds to the natural action of $\text{Der}(\langle \sigma \rangle, T) \subseteq \text{Aut}(N(\sigma))$ on \mathcal{X} . Since the stabilizer of a point in \mathcal{X} equals $T_0^-(\sigma) = \text{PDer}(\langle \sigma \rangle, T)$ we see that the induced action of $H^1(\langle \sigma \rangle; T)$ on \mathcal{X} is free and transitive, ie \mathcal{X} is an $H^1(\langle \sigma \rangle; T)$ -torsor. Clearly this argument also works for \mathbb{Z}_2 -root data. \square

Next we have the following proposition, which will be used again in Section 4, in the proofs of Theorem B and Theorem C.

Proposition 3.5 Let $\mathbf{D} = (W, L, \{\mathbb{Z}b_\sigma\})$ be a \mathbb{Z} -root datum with corresponding maximal torus normalizer $N = v(\mathbf{D})$. Then $\text{Der}(W, T) \cap \text{Aut}(N, \{N_\sigma\}) = \text{PDer}(W, T)$. Similarly, if $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ is a \mathbb{Z}_p -root datum with corresponding discrete maximal torus normalizer $\check{N} = \check{v}(\mathbf{D})$ then $\text{Der}(W, \check{T}) \cap \text{Aut}(\check{N}, \{\check{N}_\sigma\}) = \text{PDer}(W, \check{T})$.

The proof of this proposition relies on the following lemma (compare [29, Proposition 6.4], [22, Proof of Lemma 2.5(i)] and [26, Proof of Proposition 3.5]).

Lemma 3.6 Let (W, L) be a finite \mathbb{Z} -reflection group and let S be a system of simple reflections. Then the homomorphism

$$T \rightarrow \prod_{\sigma \in S} T_0^-(\sigma), \quad t \mapsto \left(t\sigma(t)^{-1} \right)_{\sigma \in S}$$

is surjective. The same result holds (replacing T by \check{T}) when (W, L) is a finite \mathbb{Z}_p -reflection group of Coxeter type (ie, $(W, L) \cong (W_1, L_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ for a \mathbb{Z} -reflection group (W_1, L_1)).

Proof Let $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. It suffices to prove that the homomorphism $\Psi: V \rightarrow \prod_{\sigma \in S} V^-(\sigma)$ given by $\Psi(x) = (x - \sigma(x))_{\sigma \in S}$ is surjective. The kernel of Ψ obviously equals V^W and since $\dim V - \dim V^W = |S|$ and $\dim V^-(\sigma) = 1$ it follows that Ψ is surjective. The second statement follows from the first. \square

Proof of Proposition 3.5 Assume first that $\mathbf{D} = (W, L, \{\mathbb{Z}b_{\sigma}\})$ is a \mathbb{Z} -root datum. Then $f \in \text{Der}(W, T)$ corresponds to the automorphism φ of N given by $\varphi(x) = f(\bar{x})x$. Since φ induces the identity on W it follows that $\varphi \in \text{Aut}(N, \{N_{\sigma}\})$ if and only if $\varphi(N_{\sigma}) = N_{\sigma}$ for all $\sigma \in \Sigma$. By the properties of N_{σ} this is again equivalent to having $f(\sigma) \in T_0^-(\sigma)$ for all σ . Hence we get $\text{PDer}(W, T) \subseteq \text{Der}(W, T) \cap \text{Aut}(N, \{N_{\sigma}\})$ since if f is a principal derivation given by $f(w) = (1 - w) \cdot t$ for some $t \in T$ we have $f(\sigma) = (1 - \sigma) \cdot t \in (1 - \sigma)T = T_0^-(\sigma)$. To prove the reverse inclusion we have to prove that any $f \in \text{Der}(W, T)$ with $f(\sigma) \in T_0^-(\sigma)$ for all σ is a principal derivation. This follows directly from Lemma 3.6.

Now let $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_{\sigma}\})$ be a \mathbb{Z}_p -root datum. As above it suffices to prove that any $f \in \text{Der}(W, \check{T})$ with $f(\sigma) \in \check{T}_0^-(\sigma)$ for all $\sigma \in \Sigma$ is principal. For p odd, this follows from the fact that $H^1(W; \check{T}) = 0$ by [1, Theorem 3.3]. Now assume $p = 2$. If \mathbf{D} is of Coxeter type the claim follows directly from Lemma 3.6. In the general case we can write \mathbf{D} as a product of a \mathbb{Z}_2 -root datum of Coxeter type and a number of copies of $\mathbf{D}_{\text{DI}(4)}$, cf [19, Proposition 7.4]. Since $\mathbf{D}_{\text{DI}(4)}$ contains a subdatum isomorphic to $\mathbf{D}_{\text{Spin}(7)\hat{2}}$ (cf [19, Proof of Proposition 9.12, DI(4) case]) and $[W_{\text{DI}(4)}: W_{\text{Spin}(7)\hat{2}}] = 7$, it follows that \mathbf{D} contains a subdatum $(W_1, L, \{\mathbb{Z}_2 b_{\sigma}\})$ of Coxeter type such that $[W: W_1]$ is odd. From the above it now follows that $[f] = 0 \in H^1(W_1; \check{T})$. By a transfer argument the restriction homomorphism $H^1(W; \check{T}) \rightarrow H^1(W_1; \check{T})$ is injective proving the claim in the general case. \square

For a \mathbb{Z}_p -root datum $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_{\sigma}\})$ we define the *singular set*

$$(3-2) \quad S(\sigma) = \langle \check{T}_0^+(\sigma), h_{\sigma} \rangle.$$

One easily sees that this agrees with the definition in [17, Definition 7.3] (cf the proof of Proposition 4.2 below), and that $S(\sigma) = \ker(\beta_{\sigma} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^{\infty}: \check{T} \rightarrow \mathbb{Z}/p^{\infty})$, which is analogous to the standard Lie theoretical definition.

For the proof of Theorem 3.2, we need the following result, which essentially summarizes elements of [19, Sections 9–10].

Lemma 3.7 *Let X be a connected 2–compact group with maximal discrete torus \check{T} and root datum $\mathbf{D}_X = (W, L, \{\mathbb{Z}_2 b_\sigma\}_{\sigma \in \Sigma})$. Let A be a subgroup of \check{T} , and let $Y = \mathcal{C}_X(A)_1$. Then W_Y is a \mathbb{Z}_2 –reflection group generated by the reflections $\Sigma_Y = \{\sigma \in \Sigma \mid A \subseteq S(\sigma)\}$ and \mathbf{D}_Y identifies with the subdatum $(W_Y, L, \{\mathbb{Z}_2 b_\sigma\}_{\sigma \in \Sigma_Y})$ of \mathbf{D}_X . Furthermore \check{N}_Y is the pullback of \check{N}_X and $\check{v}(\mathbf{D}_Y)$ is the pullback of $\check{v}(\mathbf{D}_X)$ along the inclusion $W_Y \rightarrow W_X$.*

Proof The statement about the set of reflection Σ_Y in W_Y and the result about normalizers follows from [17, Theorem 7.6(2)]. The fact that \mathbf{D}_Y is a subdatum of \mathbf{D}_X now follows by definition of the root datum. The last statement follows easily from Lemma 2.2, cf [19, Lemma 10.1]. \square

Finally we need the following calculation.

Lemma 3.8 *We have $H^1(W_{\text{DI}(4)}; \check{T}_{\text{DI}(4)}) = \mathbb{Z}/2$ and $H^1(\langle \sigma \rangle; \check{T}_{\text{DI}(4)}) = \mathbb{Z}/2$ for any reflection $\sigma \in W_{\text{DI}(4)}$.*

Proof Let $W = W_{\text{DI}(4)}$ and $\check{T} = \check{T}_{\text{DI}(4)}$ for short. Any reflection $\sigma \in W$ acts nontrivially on ${}_2\check{T}$, so up to conjugation the action of σ is given by $(t_1, t_2, t_3) \mapsto (t_2, t_1, t_3)$. It now follows directly that $H^1(\langle \sigma \rangle; \check{T}) \cong \check{T}^-(\sigma) / \check{T}_0^-(\sigma) = \mathbb{Z}/2$.

To see the first claim note that $W \cong Z(W) \times \text{GL}_3(\mathbb{F}_2)$, where $Z(W) = \langle -1 \rangle$. The associated Lyndon–Hochschild–Serre spectral sequence for computing $H^*(W; \check{T})$ has E_2 –term

$$E_2^{s,t} = H^s(\text{GL}_3(\mathbb{F}_2); H^t(\langle -1 \rangle; \check{T})).$$

For t odd we have $E_2^{s,t} = 0$ since $H^t(\langle -1 \rangle; \check{T}) = 0$. For position reasons we thus get $H^1(W; \check{T}) \cong E_2^{1,0}$. Since $H^0(\langle -1 \rangle; \check{T}) \cong (\mathbb{F}_2)^3$ with the natural action of $\text{GL}_3(\mathbb{F}_2)$, we get $H^1(W; \check{T}) \cong H^1(\text{GL}_3(\mathbb{F}_2); (\mathbb{F}_2)^3)$. It is well known that $H^1(\text{GL}_3(\mathbb{F}_2); (\mathbb{F}_2)^3) = \mathbb{Z}/2$; probably the easiest way to see this directly is to observe that for P_{triv} , the projective cover of the trivial $\mathbb{F}_2[\text{GL}_3(\mathbb{F}_2)]$ –module, the second radical (Loewy) layer $\text{rad}(P_{\text{triv}}) / \text{rad}^2(P_{\text{triv}})$ equals the standard representation $(\mathbb{F}_2)^3$ plus its dual by [6, page 216] (or a direct calculation), so the result follows from the minimal resolution; see also [32, Proposition 4(a)], [28, Table C] or [5, Table I]. \square

Proof of Theorem 3.2 In the Lie group case (1), the existence of the isomorphism follows from [35, Théorème 4.4] after translating the definitions (isomorphism in [35, Théorème 4.4] means “isomorphism in the category \mathcal{N} ” defined in [35, 3.1] which exactly means sending $\nu(\mathbf{D})_\sigma$ to $N_G(T)_\sigma$; Tits states his results for reductive algebraic groups, and these translate into results for compact Lie groups via the usual method).

A different exposition of this is given in [19, Proof of Theorem 5.7] noting that, in the notation of [19], q_i is sent to x_i and hence $\nu(\mathbf{D})_{\sigma_i} = \langle T_0^-(\sigma_i), q_i \rangle$ is sent to $N_G(T)_{\sigma_i} = \langle T_0^-(\sigma_i), x_i \rangle$ for all simple reflections σ_i , so $\nu(\mathbf{D})_{\sigma}$ is sent to $N_G(T)_{\sigma}$ for all σ since both the topological and the algebraic subgroups are well behaved under conjugation. The uniqueness follows directly from Proposition 3.5.

We now prove part (2). Assume first that $p > 2$. In this case the result follows immediately from the definitions above: There is an isomorphism of extensions $\check{\mathcal{N}}_X \cong \check{T} \rtimes W = \check{\nu}(\mathbf{D}_X)$ by [1, Theorem 1.2] and by definition the topological root subgroups $(\check{\mathcal{N}}_X)_{\sigma}$ are the images of the algebraic root subgroups $\check{\nu}(\mathbf{D}_X)_{\sigma} = \check{T}^-(\sigma) \rtimes \langle \sigma \rangle$ under this isomorphism. Since $H^1(W; \check{T}) = 0$ [1, Theorem 3.3], uniqueness also follows.

Now assume that $p = 2$. Since the uniqueness part follows directly from Proposition 3.5, we only need to prove the existence of the isomorphism. First note that if X is the 2-completion of a compact connected Lie group then the result follows from part (1) since in this case the definitions and constructions for 2-compact groups agree in the obvious way with those for compact Lie groups (cf [19, Lemmas 8.1 and 9.17]). In particular the result holds when X has semisimple rank 1, by the classification mentioned in the beginning of this section. The general case can be reduced to the case where X has semisimple rank 2. We therefore first explain this case (where we use some group cohomology calculations, but essentially avoid classification results for 2-compact groups) and then explain how the general case follows from this.

Assume that X has semisimple rank 2. Since $\mathbf{D}_{\mathrm{DI}(4)}$ has semisimple rank 3, [19, Proposition 7.4] shows that $\mathbf{D}_X \cong \mathbf{D}_G \otimes_{\mathbb{Z}} \mathbb{Z}_2$ for some compact connected Lie group G of semisimple rank 2. In particular G has Cartan type $A_1 A_1$, A_2 , $B_2 = C_2$ or G_2 .

If G has Cartan type $A_1 A_1$, then $X' = X/\mathcal{Z}(X)$ is a connected center-free 2-compact group of rank 2 and $(W_{X'}, L_{X'} \otimes \mathbb{Q})$ is reducible. Hence by [18, Theorem 1.3] X' splits as a product of two center-free 2-compact groups of rank 1. Hence by the classification of 2-compact groups of rank 1, $X' \cong \mathrm{SO}(3)_{\hat{2}} \times \mathrm{SO}(3)_{\hat{2}}$, and consequently X is the 2-completion of a compact connected Lie group, and the result follows.

In the case where G has Cartan type A_2 , B_2 or G_2 , an inspection of the classification of \mathbb{Z} -root data reveals that $\mathbf{D}_X \cong \mathbf{D}_G \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is isomorphic to $\mathbf{D}_{H \times (S^1)^n} \otimes_{\mathbb{Z}} \mathbb{Z}_2$, $n \geq 0$, for $H = \mathrm{SU}(3)$, $\mathrm{Spin}(5)$, $\mathrm{SO}(5)$, $(\mathrm{Spin}(5) \times S^1)/C_2$ (with C_2 the unique “diagonally” embedded central subgroup of order 2 in $\mathrm{Spin}(5) \times S^1$), or G_2 . Since $\mathbf{D}_X \cong \mathbf{D}_{H_{\hat{2}}} \times \mathbf{D}_{((S^1)^n)_{\hat{2}}}$, [18, Theorem 1.4] implies that $X \cong X' \times ((S^1)^n)_{\hat{2}}$ for a connected 2-compact group X' with $\mathbf{D}_{X'} \cong \mathbf{D}_H \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Hence, to prove part (2) for X , it suffices to prove it for X' . We now do this by going through the five possibilities for H one by one:

For $H = \mathrm{SU}(3)$ and G_2 we have $H^2(W_{X'}; \check{T}_{X'}) = 0$, so the extensions $\check{\nu}(\mathbf{D}_{X'})$ and $\check{\mathcal{N}}_{X'}$ are isomorphic. Moreover, as $H^1(\langle \sigma \rangle; \check{T}_{X'}) = 0$ for any reflection $\sigma \in W_{X'}$, Lemma 3.4 shows that any isomorphism between the two extensions will preserve root subgroups.

In the case $H = \mathrm{Spin}(5)$, there is, up to conjugation, a unique noncentral element of order two $\nu: B\mathbb{Z}/2 \rightarrow BX'$ in X' . By the description of the root datum of a centralizer, Lemma 3.7, $Y = \mathcal{C}_{X'}(\nu)$ is a connected 2–compact group with $\mathbf{D}_Y \cong \mathbf{D}_{(\mathrm{SU}(2) \times \mathrm{SU}(2))\hat{2}}$. In particular the extensions $\check{\nu}(\mathbf{D}_Y)$ and $\check{\mathcal{N}}_Y$ are isomorphic by the result in the A_1A_1 case, and moreover a computation shows that the restriction $H^2(W_{X'}; \check{T}) \cong (\mathbb{Z}/2)^2 \rightarrow H^2(W_Y; \check{T}) \cong (\mathbb{Z}/2)^4$ is injective. Combining this with Lemma 3.7 shows that the extensions $\check{\nu}(\mathbf{D}_{X'})$ and $\check{\mathcal{N}}_{X'}$ are isomorphic. We now argue that this isomorphism can be taken to preserve the root subgroups: Let $f \in \mathrm{Der}(W_{X'}, \check{T})$. By the proof of Proposition 3.5 it follows that if $f|_{\langle \sigma \rangle} \in \mathrm{PDer}(\langle \sigma \rangle, \check{T})$ for all reflections σ , then $f \in \mathrm{PDer}(W_{X'}, \check{T})$. In other words the intersection of the kernels of the restrictions, $\bigcap_{\sigma} \ker(H^1(W_{X'}; \check{T}) \rightarrow H^1(\langle \sigma \rangle; \check{T}))$, equals 0, where the intersection runs over all reflections σ . However the kernel of the restriction only depends on the conjugacy class of σ (cf eg, [11, Exercise III.9.1]), so letting σ_1 and σ_2 denote representatives for the two conjugacy classes of reflections in $W_{X'}$, we have

$$\ker(H^1(W_{X'}; \check{T}) \rightarrow H^1(\langle \sigma_1 \rangle; \check{T})) \cap \ker(H^1(W_{X'}; \check{T}) \rightarrow H^1(\langle \sigma_2 \rangle; \check{T})) = 0.$$

A direct computation shows that (up to exchange of σ_1 and σ_2), we have $H^1(W_{X'}; \check{T}) = \mathbb{Z}/2$, $H^1(\langle \sigma_1 \rangle; \check{T}) = \mathbb{Z}/2$ and $H^1(\langle \sigma_2 \rangle; \check{T}) = 0$, so we conclude that the restriction $H^1(W_{X'}; \check{T}) \rightarrow H^1(\langle \sigma_1 \rangle; \check{T})$ is surjective. Hence, by Lemma 3.4, $H^1(W; \check{T})$ acts transitively on the subgroups of $\check{\nu}(\mathbf{D}_{X'})$ which look like $\check{\nu}(\mathbf{D}_{X'})_{\sigma_1}$. Therefore any isomorphism between $\check{\nu}(\mathbf{D}_{X'})$ and $\check{\mathcal{N}}_{X'}$ can be modified by an element of $\mathrm{Der}(W_{X'}, \check{T}) \subseteq \mathrm{Aut}(\check{\nu}(\mathbf{D}_{X'}))$ in such a way that $\check{\nu}(\mathbf{D}_{X'})_{\sigma_1}$ is sent to $(\check{\mathcal{N}}_{X'})_{\sigma_1}$. Moreover as $H^1(\langle \sigma_2 \rangle; \check{T}) = 0$, Lemma 3.4 shows that the modified isomorphism automatically sends $\check{\nu}(\mathbf{D}_{X'})_{\sigma_2}$ to $(\check{\mathcal{N}}_{X'})_{\sigma_2}$. Since any reflection is conjugate to σ_1 or σ_2 and the root subgroups are well behaved under conjugation, the modified isomorphism sends every root subgroup of $\check{\nu}(\mathbf{D}_{X'})$ to the corresponding root subgroup of $\check{\mathcal{N}}_{X'}$ as desired.

When $H = \mathrm{SO}(5)$, $W_{X'}$ has two conjugacy classes of reflections represented by σ_1 and σ_2 say. Moreover X' has two conjugacy classes of elements of order two, $\nu_1, \nu_2: B\mathbb{Z}/2 \rightarrow BX'$, and we find that (up to permutation) $Y_1 = \mathcal{C}_{X'}(\nu_1)_1$ has root datum isomorphic to $\mathbf{D}_{\mathrm{SO}(4)\hat{2}}$ and that $Y_2 = \mathcal{C}_{X'}(\nu_2)_1$ satisfies $\mathbf{D}_{Y_2} \cong \mathbf{D}_{(\mathrm{SO}(3) \times S^1)\hat{2}}$. Hence the extensions $\check{\nu}(\mathbf{D}_{Y_i})$ and $\check{\mathcal{N}}_{Y_i}$ are isomorphic for $i = 1$ and 2 by the result in the A_1A_1 case and the semisimple rank one case. Furthermore a calculation shows

that the homomorphism

$$H^2(W_{X'}; \check{T}) \cong (\mathbb{Z}/2)^2 \rightarrow H^2(W_{Y_1}; \check{T}) \times H^2(W_{Y_2}; \check{T}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$$

given by restrictions is an isomorphism, so by Lemma 3.7 the extensions $\check{\nu}(\mathbf{D}_{X'})$ and $\check{\mathcal{N}}_{X'}$ are isomorphic. Since the Weyl group action is the same as in the case $H = \text{Spin}(5)$, the argument given in that case shows that we can modify the isomorphism between $\check{\nu}(\mathbf{D}_{X'})$ and $\check{\mathcal{N}}_{X'}$ by an element in $\text{Der}(W_{X'}, \check{T}) \subseteq \text{Aut}(\check{\nu}(\mathbf{D}_{X'}))$ to obtain an isomorphism which preserves root subgroups.

Finally, when $H = (\text{Spin}(5) \times S^1)/C_2$, there is an element of order two $\nu: B\mathbb{Z}/2 \rightarrow BX'$ such that $Y = C_{X'}(\nu)$ is a connected 2-compact group with $\mathbf{D}_Y \cong \mathbf{D}_{H'\hat{2}}$, where $H' = (\text{SU}(2) \times \text{SU}(2) \times S^1)/C_2$, where C_2 is the unique “diagonally” embedded central subgroup of order 2 in $\text{SU}(2) \times \text{SU}(2) \times S^1$. In particular $\check{\nu}(\mathbf{D}_Y)$ and $\check{\mathcal{N}}_Y$ are isomorphic by the above and since a computation shows that the restriction

$$H^2(W_{X'}; \check{T}) \cong \mathbb{Z}/2 \rightarrow H^2(W_Y; \check{T}) \cong (\mathbb{Z}/2)^2$$

is injective we conclude that $\check{\nu}(\mathbf{D}_{X'})$ and $\check{\mathcal{N}}_{X'}$ are isomorphic. Let σ_1 and σ_2 denote representatives for the two conjugacy classes of reflections in $W_{X'}$. The computations $H^1(W_{X'}; \check{T}) = (\mathbb{Z}/2)^2$, $H^1(\langle \sigma_1 \rangle; \check{T}) = \mathbb{Z}/2$ and $H^1(\langle \sigma_2 \rangle; \check{T}) = \mathbb{Z}/2$ combined with the argument in the case $H = \text{Spin}(5)$ implies that the restriction

$$H^1(W_{X'}; \check{T}) \rightarrow H^1(\langle \sigma_1 \rangle; \check{T}) \times H^1(\langle \sigma_2 \rangle; \check{T})$$

is an isomorphism, so as above we can choose an isomorphism between $\check{\nu}(\mathbf{D}_{X'})$ and $\check{\mathcal{N}}_{X'}$ which preserves root subgroups. This finishes the proof of part (2) in the case where X has semisimple rank at most 2.

We now proceed to give the proof in the general case. We follow [19, Proof of Proposition 9.12] closely, but since extra arguments are needed we choose to continue in some detail. By [18, Theorem 6.1] and [19, Proposition 7.4] we have $X \cong X_1 \times X_2$ where \mathbf{D}_{X_1} is of Coxeter type and \mathbf{D}_{X_2} is a product of a number of copies of $\mathbf{D}_{\text{DI}(4)}$, and it is hence enough to treat these two cases separately.

If \mathbf{D}_X is of Coxeter type, we let $\{\sigma_i\}_{i \in I}$ be a set of simple reflections. Then by [35, Section 2] (cf also [19, 4.6 and Definition 6.15]) $\check{\nu}(\mathbf{D}_X)$ is generated by \check{T} and symbols q_i , $i \in I$, subject to the relations $q_i^2 = h_{\sigma_i}$, $q_i t q_i^{-1} = \sigma_i(t)$ for $t \in \check{T}$ and

$$\underbrace{q_i q_j \dots}_{m_{ij} \text{ terms}} = \underbrace{q_j q_i \dots}_{m_{ij} \text{ terms}}$$

for $i \neq j$, where m_{ij} denotes the order of $\sigma_i \sigma_j$. Choose elements $x_i \in (\check{\mathcal{N}}_{X'})_{\sigma_i} \setminus \check{T}_0^-(\sigma_i)$ for each $i \in I$. As in [19, Proof of Proposition 9.12, Coxeter case] we see that the

elements x_i satisfies the first two relations and that it suffices to check the third relation for the subgroup $Y = \mathcal{C}_X(\check{T}_0^{(\sigma_i, \sigma_j)})$ of X , which has semisimple rank 2. In this case there is a compact connected Lie group G with $\mathbf{D}_Y \cong \mathbf{D}_{G\hat{2}}$. By the above, part (2) of the theorem holds for Y , so there is an isomorphism between the extensions $\check{\mathcal{N}}_Y$ and $\check{\mathcal{N}}_{G\hat{2}}$ which preserves root subgroups. Hence it suffices to check the third relation for $G\hat{2}$ where it follows from the classical result of Tits (cf [19, Theorem 5.8 and Lemma 9.17]). Hence there is an isomorphism $\check{\nu}(\mathbf{D}_X) \xrightarrow{\cong} \check{\mathcal{N}}_X$ induced by $q_i \mapsto x_i$ which clearly preserves root subgroups. This proves the result in the Coxeter case.

Finally assume that $\mathbf{D}_X \cong \mathbf{D}_{\text{DI}(4)}$. In this case there is a unique element of order two $\nu: B\mathbb{Z}/2 \rightarrow BX$ and $Y = \mathcal{C}_X(\nu)$ satisfies $\mathbf{D}_Y \cong \mathbf{D}_{\text{Spin}(7)\hat{2}}$. Hence there is an isomorphism between the extensions $\check{\nu}(\mathbf{D}_Y)$ and $\check{\mathcal{N}}_Y$ by the result in the Coxeter case. As $[W_X:W_Y] = 7$ is odd, the restriction $H^2(W_X; \check{T}) \rightarrow H^2(W_Y; \check{T})$ is injective, so Lemma 3.7 shows that $\check{\nu}(\mathbf{D}_X)$ and $\check{\mathcal{N}}_X$ are isomorphic. Let $\sigma \in W = W_{\text{DI}(4)}$ be a reflection. Since all reflections in W are conjugate, the argument given above shows that the restriction $H^1(W; \check{T}) \rightarrow H^1(\langle \sigma \rangle; \check{T})$ is injective. Now $H^1(W; \check{T}) \cong \mathbb{Z}/2$ and $H^1(\langle \sigma \rangle; \check{T}) \cong \mathbb{Z}/2$ by Lemma 3.8, so the restriction is an isomorphism. As above this implies that we can modify the isomorphism between $\check{\nu}(\mathbf{D}_X)$ and $\check{\mathcal{N}}_X$ to an isomorphism which sends root subgroups to root subgroups as desired. \square

4 Proof of Theorem B and Theorem C

In this section we prove Theorem B and Theorem C using the material from the previous sections. First we prove the following result which says that if φ is an automorphism of W which sends reflections to reflections, then the class in $H^2(W; \mathbb{Z}[\Sigma])$ corresponding to the reflection extension $\rho(W)$ is fixed under $(\varphi, \varphi^{-1})^*: H^2(W; \mathbb{Z}[\Sigma]) \rightarrow H^2(W; \mathbb{Z}[\Sigma])$, where φ^{-1} induces a map $\mathbb{Z}[\Sigma] \rightarrow \mathbb{Z}[\Sigma]$ by assumption.

Lemma 4.1 *Let W be a finite \mathbb{Q}_2 -reflection group with associated reflection extension $1 \rightarrow \mathbb{Z}[\Sigma] \xrightarrow{\iota} \rho(W) \xrightarrow{\pi} W \rightarrow 1$. For any $\varphi \in \text{Aut}(W)$ with $\varphi(\Sigma) = \Sigma$, there is an automorphism ψ of $\rho(W)$ fitting into the commutative diagram:*

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \mathbb{Z}[\Sigma] & \xrightarrow{\iota} & \rho(W) & \xrightarrow{\pi} & W \longrightarrow 1 \\
 & & \parallel & & \downarrow \psi & & \parallel \\
 1 & \longrightarrow & \mathbb{Z}[\Sigma] & \xrightarrow{\iota \circ \varphi} & \rho(W) & \xrightarrow{\varphi^{-1} \circ \pi} & W \longrightarrow 1
 \end{array}$$

Moreover ψ is unique up to conjugation by an element in $\mathbb{Z}[\Sigma]$.

Proof Define a category \mathcal{D} as follows (cf [11, Chapter III, Section 8]): The objects are pairs (G, M) , where G is a group and M is a G -module, and a morphism from (G, M) to (G', M') is a pair of maps $(\alpha: G \rightarrow G', f: M' \rightarrow M)$ where α is a group homomorphism and f is a G -module homomorphism, ie $f(\alpha(g) \cdot m') = g \cdot f(m')$ for $g \in G$ and $m' \in M'$. On the level of cohomology this induces a homomorphism $(\alpha, f)^*: H^*(G'; M') \rightarrow H^*(G; M)$, which in degree 2 takes the equivalence class of an extension $0 \rightarrow M' \rightarrow E' \rightarrow G' \rightarrow 1$ to the equivalence class of the extension $0 \rightarrow M \rightarrow E \rightarrow G \rightarrow 1$ obtained by taking the pullback along $\alpha: G \rightarrow G'$ followed by the pushforward along $f: M' \rightarrow M$.

Now let $k \in H^2(W; \mathbb{Z}[\Sigma])$ denote the class of the reflection extension. The class of the extension

$$0 \rightarrow \mathbb{Z}[\Sigma] \xrightarrow{\iota \circ \varphi} \rho(W) \xrightarrow{\varphi^{-1} \circ \pi} W \rightarrow 1$$

is then given by $(\varphi, \varphi^{-1})^*(k)$ and the first part of the proposition is that this equals k , since in that case (φ, φ^{-1}) induces an automorphism $\psi: \rho(W) \rightarrow \rho(W)$ as wanted.

Let $\Sigma = \bigcup_i \Sigma_i$ denote the partition of the set of reflections Σ in W into conjugacy classes. By construction $k \in H^2(W; \mathbb{Z}[\Sigma])$ is given by a sum of elements $k_i \in H^2(W; \mathbb{Z}[\Sigma_i])$. The map (φ, φ^{-1}) also induces homomorphisms $H^2(W; \mathbb{Z}[\Sigma_i]) \rightarrow H^2(W; \mathbb{Z}[\varphi^{-1}(\Sigma_i)])$ and we prove below that $(\varphi, \varphi^{-1})^*(k_i) = k_{i'}$ where $\Sigma_{i'} = \varphi^{-1}(\Sigma_i)$. Thus $(\varphi, \varphi^{-1})^*$ permutes the k_i 's and hence fixes k .

To see this claim, let $\tau_i \in \Sigma_i$ and define C_i and C_i^\perp as in Section 2. Moreover, let $\text{incl}_i: C_i \rightarrow W$ denote the inclusion, $p_i: C_i = \langle \tau_i \rangle \times C_i^\perp \rightarrow \langle \tau_i \rangle$ the projection and $f_i: \mathbb{Z}[W/C_i] \rightarrow \mathbb{Z}$ the C_i -module homomorphism given by

$$f_i(wC_i) = \begin{cases} 1 & \text{if } w \in C_i, \\ 0 & \text{if } w \notin C_i. \end{cases}$$

By [11, Exercise III.8.2] the isomorphism $H^2(W; \mathbb{Z}[W/C_i]) \xrightarrow{\cong} H^2(C_i; \mathbb{Z})$ from Shapiro's lemma is induced by (incl_i, f_i) (see also [19, Lemma 4.8]). Choosing $\tau_{i'} = \varphi^{-1}(\tau_i)$ as the representative from $\Sigma_{i'} = \varphi^{-1}(\Sigma_i)$, the claim that $(\varphi, \varphi^{-1})^*(k_i) = k_{i'}$ then follows from the commutativity of the diagram

$$\begin{array}{ccccc} H^2(\langle \tau_i \rangle; \mathbb{Z}) & \xrightarrow{p_i^*} & H^2(C_i; \mathbb{Z}) & \xleftarrow[\cong]{(\text{incl}_i, f_i)^*} & H^2(W; \mathbb{Z}[W/C_i]) & \xlongequal{\quad} & H^2(W; \mathbb{Z}[\Sigma_i]) \\ \varphi^* \downarrow & & & & & & \downarrow (\varphi, \varphi^{-1})^* \\ H^2(\langle \tau_{i'} \rangle; \mathbb{Z}) & \xrightarrow{p_{i'}^*} & H^2(C_{i'}; \mathbb{Z}) & \xleftarrow[\cong]{(\text{incl}_{i'}, f_{i'})^*} & H^2(W; \mathbb{Z}[W/C_{i'}]) & \xlongequal{\quad} & H^2(W; \mathbb{Z}[\Sigma_{i'}]) \end{array}$$

which is easily established by checking that the corresponding diagram in \mathcal{D} commutes. To see the second part, note first that

$$H^1(W; \mathbb{Z}[\Sigma]) = \bigoplus_i H^1(W; \mathbb{Z}[\Sigma_i]) \cong \bigoplus_i H^1(C_i; \mathbb{Z}) = 0.$$

It is clear that ψ is unique up to an automorphism of $\rho(W)$ inducing the identity on $\mathbb{Z}[\Sigma]$ and on W . If χ is any such automorphism, we get a well-defined derivation $f: W \rightarrow \mathbb{Z}[\Sigma]$ given by $f(w) = \chi(\tilde{w})\tilde{w}^{-1}$, where $\tilde{w} \in \rho(W)$ is any lift of $w \in W$. Since $H^1(W; \mathbb{Z}[\Sigma]) = 0$, this must be a principal derivation which means that χ must be conjugation by an element in $\mathbb{Z}[\Sigma]$, so ψ is unique up to conjugation by an element in $\mathbb{Z}[\Sigma]$. \square

The next proposition, which we will need later on, shows how to recover the root datum \mathbf{D} from the isomorphism type of the associated maximal torus normalizer $\nu(\mathbf{D})$, detailing a remark of Dwyer–Wilkerson [19, Remark 5.6]. A result of this type was first proved via different arguments by Curtis–Wiederhold–Williams [13], for G a compact connected Lie group with finite center, and by Osse [31] and Notbohm [30] for G an arbitrary compact connected Lie group.

First note that T is characterized as the largest divisible subgroup in $\nu(\mathbf{D})$, and therefore the extension $1 \rightarrow T \rightarrow \nu(\mathbf{D}) \rightarrow W \rightarrow 1$ can be recovered, so the only issue is how to recover the markings h_σ .

Proposition 4.2 [19, Remark 5.6] *Let $\mathbf{D} = (W, L, \{\mathbb{Z}b_\sigma\})$ be a \mathbb{Z} -root datum and let $N = \nu(\mathbf{D})$ be the associated maximal torus normalizer. Let $\sigma \in W$ be a reflection and let $N(\sigma)$ denote the pullback of $1 \rightarrow T \rightarrow N \rightarrow W \rightarrow 1$ along the inclusion $\langle \sigma \rangle \rightarrow W$. Then the element h_σ is the unique element in $T_0^-(\sigma) \cap \{x^2 \mid x \in N(\sigma) \setminus T\}$ which is a marking for the reflection σ on T (ie an element $x \in T_0^-(\sigma)$ such that $x^2 = 1$ and $x \neq 1$ if σ acts nontrivially on ${}_2T$, cf [19, Definition 2.12]).*

For \mathbb{Z}_2 -root data the corresponding result (obtained by replacing T by \check{T} and $\nu(\mathbf{D})$ by $\check{\nu}(\mathbf{D})$) also holds.

Proof By Lemma 3.3, h_σ lies in $T_0^-(\sigma) \cap \{x^2 \mid x \in N(\sigma) \setminus T\}$. We need to see that there are no other elements in this set which are markings for σ . This is clear when σ acts nontrivially on ${}_2T$, since h_σ is then by definition the unique marking for σ in $T_0^-(\sigma)$. In general let $x \in N(\sigma) \setminus T$ with $x^2 = h_\sigma$. For $t \in T$ we have $(tx)^2 = (txtx^{-1})x^2 = t\sigma(t)h_\sigma$ proving that $\{x^2 \mid x \in N(\sigma) \setminus T\} = h_\sigma \cdot (1 + \sigma)(T) = h_\sigma T_0^+(\sigma)$. Hence our claim is that h_σ is the only marking for σ in $T_0^-(\sigma) \cap h_\sigma T_0^+(\sigma) = h_\sigma(T_0^-(\sigma) \cap T_0^+(\sigma))$. If σ acts trivially on ${}_2T$, the action of σ is (up to an automorphism of T) given by $(t_1, t_2, \dots, t_n) \mapsto (t_1^{-1}, t_2, \dots, t_n)$ so $T_0^-(\sigma) \cap T_0^+(\sigma) = 1$

proving the claim in this case also. This proves the first part of the proposition. The argument for \mathbb{Z}_2 -root data is identical. \square

We are now ready to prove Theorem B.

Proof of Theorem B We divide the proof of the first part into several steps.

Step 1 The restriction homomorphism $\text{Aut}(N) \rightarrow N_{\text{Aut}(T)}(W)$ has image contained in $\text{Aut}(\mathbf{D})$: We just need to see that for any $\varphi \in \text{Aut}(N)$ we have $\varphi(h_\sigma) = h_{\varphi\sigma\varphi^{-1}}$. However this follows from the description of h_σ in terms of N as given in Proposition 4.2 above.

Step 2 The restriction $\text{Aut}(N) \rightarrow \text{Aut}(\mathbf{D})$ is surjective: Clearly any automorphism $\varphi \in \text{Aut}(\mathbf{D})$ induces an automorphism of W preserving Σ , which we (by abuse of notation) also denote φ . By Lemma 4.1 there thus exists an automorphism ψ of $\rho(W)$ satisfying $\pi \circ \psi = \varphi \circ \pi$ and $\psi \circ \iota = \iota \circ \varphi$. The first condition shows that (φ, ψ) defines an automorphism of $T \rtimes \rho(W)$ and the second that the subgroup $\langle (h_\sigma, \sigma^{-1}) \mid \sigma \in \Sigma \rangle$ is preserved by this automorphism (here σ^{-1} denotes the inverse of $\sigma \in \mathbb{Z}[\Sigma] \subseteq \rho(W)$). Hence (φ, ψ) gives a well-defined automorphism of $N = (T \rtimes \rho(W)) / \langle (h_\sigma, \sigma^{-1}) \mid \sigma \in \Sigma \rangle$ which obviously restricts to φ on T . Picking a ψ for each $\varphi \in \text{Aut}(\mathbf{D})$ gives an explicit set theoretical splitting $s: \text{Aut}(\mathbf{D}) \rightarrow \text{Aut}(N)$. (Note that s depends on the choices of ψ .)

Step 3 The kernel of the restriction homomorphism $\text{Aut}(N) \rightarrow \text{Aut}(\mathbf{D})$ equals the group of derivations $\text{Der}(W, T)$: First note that we may view $\text{Der}(W, T)$ as a subgroup of $\text{Aut}(N)$ by sending $f \in \text{Der}(W, T)$ to $\varphi_f \in \text{Aut}(N)$ given by $\varphi_f(x) = f(\bar{x})x$. Likewise if $\varphi \in \text{Aut}(N)$ restricts to the identity on T then we may define $f \in \text{Der}(W, T)$ by $f(w) = \varphi(\tilde{w})\tilde{w}^{-1}$, where $\tilde{w} \in N$ is a lift of $w \in W$, and with this definition $\varphi_f = \varphi$.

The three claims above show that we have a short exact sequence

$$(4-1) \quad 1 \rightarrow \text{Der}(W, T) \rightarrow \text{Aut}(N) \rightarrow \text{Aut}(\mathbf{D}) \rightarrow 1.$$

Now consider the subgroup $\text{Inn}(N)$ of $\text{Aut}(N)$. The image under the restriction homomorphism $\text{Inn}(N) \rightarrow N_{\text{Aut}(T)}(W)$ obviously equals W . Moreover it is easily seen that under the identification from Step 3, the kernel equals the group $\text{PDer}(W, T)$ of principal derivations. Hence (4-1) has the exact subsequence

$$1 \rightarrow \text{PDer}(W, T) \rightarrow \text{Inn}(N) \rightarrow W \rightarrow 1$$

and the quotient exact sequence is the exact sequence in the theorem.

Step 4 The composition $\text{Aut}(\mathbf{D}) \xrightarrow{s} \text{Aut}(N) \rightarrow \text{Out}(N)$ is a well-defined homomorphism: From the construction of s and Lemma 4.1 it follows that the value of $s(\varphi)$ is well-defined up to an automorphism which on $T \rtimes \rho(W)$ has the form (id_T, c_a) where c_a denotes conjugation in $\rho(W)$ by an element $a \in \mathbb{Z}[\Sigma]$. Obviously this agrees with conjugation by the element $(1, a) \in T \rtimes \rho(W)$. Thus the composition $\text{Aut}(\mathbf{D}) \xrightarrow{s} \text{Aut}(N) \rightarrow \text{Out}(N)$ is independent of the choices involved and by construction it is clear that the composition is a homomorphism.

Step 5 The homomorphism $\text{Aut}(\mathbf{D}) \rightarrow \text{Out}(N)$ from Step 4 factors through $\text{Out}(\mathbf{D})$: We have to see that s sends elements of W to elements of $\text{Inn}(N)$. For $w \in W$ choose $x \in \rho(W)$ with $\pi(x) = w$. It is easily checked that we can take $\psi = c_x$ (ie, conjugation by x) in the definition of $s(w)$. Then $s(w)$ is induced by the automorphism of $T \rtimes \rho(W)$ given by $(t \mapsto w \cdot t, c_x)$ and since this automorphism agrees with conjugation by $(1, x) \in T \rtimes \rho(W)$ the claim follows.

Step 6 The values of s belong to $\text{Aut}(N, \{N_\sigma\})$: Let $\varphi \in \text{Aut}(\mathbf{D})$ and choose $\psi \in \text{Aut}(\rho(W))$ as in Step 2. Since ψ stabilizes $\mathbb{Z}[\Sigma]$ as a set and agrees with φ on this subgroup, it follows that $\psi(Q_\sigma) = Q_{\varphi\sigma\varphi^{-1}}$. Obviously $\varphi(T_0^-(\sigma)) = T_0^-(\varphi\sigma\varphi^{-1})$ and hence $s(N_\sigma) = N_{\varphi\sigma\varphi^{-1}}$ by definition.

Step 7 By now we know that the splitting $s: \text{Out}(\mathbf{D}) \rightarrow \text{Out}(N)$ takes values in $\text{Out}(N, \{N_\sigma\})$. It then follows from Proposition 3.5 that the image of s equals $\text{Out}(N, \{N_\sigma\})$ as claimed. This finishes the proof of the first part of the theorem.

Assume finally that G is a compact connected Lie group with root datum \mathbf{D}_G . By Theorem 3.2(1) we can fix an identification of $N_G(T)$ with $N = \nu(\mathbf{D}_G)$ in such a way that the subgroups $N_G(T)_\sigma$ and N_σ correspond to each other.

Let $\text{Aut}(G, T)$ denote the subgroup of $\text{Aut}(G)$ which sends T to T . The composition

$$\text{Aut}(G, T) \xrightarrow{\text{res}} \text{Aut}(N) \rightarrow \text{Out}(N)$$

obviously factors through $\text{Out}(G)$. Moreover by definition of the subgroups $N_G(T)_\sigma$ (cf Section 3 and [19, Definition 5.1]) it is clear that this composition takes values in $\text{Out}(N, \{N_\sigma\})$. The homomorphism $\text{Aut}(G, T) \rightarrow \text{Out}(G)$ is surjective since all maximal tori in G are conjugate. Thus we have a well-defined homomorphism $\text{Out}(G) \rightarrow \text{Out}(N, \{N_\sigma\})$. Similarly using the definition of the elements h_σ (cf Section 3 and [19, Definition 5.3]), we get a well-defined restriction homomorphism $\text{Out}(G) \rightarrow \text{Out}(\mathbf{D}_G)$. It is now clear that the diagram

$$\begin{array}{ccc} & \text{Out}(N, \{N_\sigma\}) & \\ \text{res} \nearrow & & \searrow \text{res} \\ \text{Out}(G) & \xrightarrow{\text{res}} & \text{Out}(\mathbf{D}_G) \end{array}$$

commutes. The homomorphism $\text{Out}(N, \{N_\sigma\}) \rightarrow \text{Out}(\mathbf{D}_G)$ is an isomorphism by the above and it is a fundamental theorem in Lie theory that $\text{Out}(G) \rightarrow \text{Out}(\mathbf{D}_G)$ is an isomorphism (cf eg [8, Section 4, no. 10, Proposition 18]). This proves the theorem. \square

Remark 4.3 Note that eg, for $G = F_4$ the sequence $1 \rightarrow \text{Der}(W, T) \rightarrow \text{Aut}(N) \rightarrow \text{Aut}(\mathbf{D}) \rightarrow 1$ is *not* split: By [22] $H^1(W_{F_4}; T) = 0$ and clearly $\text{Out}(\mathbf{D}) = 1$ (cf [24, Section 12.2, Table 1]). Since $T^W = 0$, the sequence identifies with $1 \rightarrow T \rightarrow N_{F_4}(T) \rightarrow W_{F_4} \rightarrow 1$, which is not split by [13, Section 4, Application I].

Lemma 4.4 Let \mathbf{D} be a \mathbb{Z}_p -root datum, p odd, and set $\check{\mathcal{N}} = \check{\nu}(\mathbf{D})$. Then

$$\text{Aut}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\}) = \text{Aut}(\check{\mathcal{N}}).$$

Proof Recall that $\check{\mathcal{N}} = \check{T} \rtimes W$ by definition. Let $\varphi \in \text{Aut}(\check{\mathcal{N}})$. Since \check{T} is a characteristic subgroup of $\check{\mathcal{N}}$, φ induces an automorphism $\bar{\varphi}: W \rightarrow W$. Defining $\varphi': \check{\mathcal{N}} \rightarrow \check{\mathcal{N}}$ by $\varphi'(tw) = \varphi(t)\bar{\varphi}(w)$ one easily checks that $\varphi' \in \text{Aut}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\})$. Thus it suffices to prove that $\psi = \varphi(\varphi')^{-1} \in \text{Aut}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\})$. Since ψ is the identity on \check{T} it follows that $f(w) = \psi(w)w^{-1}$ defines an element in $\text{Der}(W, \check{T})$. Now $H^1((\sigma); \check{T}) = 0$ so $f(\sigma) \in (1 - \sigma)(\check{T}) = \check{T}_0^-(\sigma)$ and hence $\psi \in \text{Aut}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\})$ as desired. \square

Proof of Theorem C First consider the case $p = 2$. The proof of the first part of Theorem B carries over verbatim to give a proof of the first part of Theorem C. So to finish the proof in this case we just need to see that $\Phi: \pi_0(\text{Aut}(BX)) \rightarrow \text{Out}(\check{\mathcal{N}})$ has image contained in $\text{Out}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\})$, which basically follows from the definitions: For any homotopy equivalence $\varphi: BX \rightarrow BX$ there exists by [3, Lemma 4.1 and Proposition 5.1] an automorphism $\check{\varphi}: \check{\mathcal{N}} \rightarrow \check{\mathcal{N}}$, making the diagram

$$\begin{array}{ccc} B\check{\mathcal{N}} & \xrightarrow{B\check{\varphi}} & B\check{\mathcal{N}} \\ \downarrow i & & \downarrow i \\ BX & \xrightarrow{\varphi} & BX \end{array}$$

homotopy commute, and $\check{\varphi}$ depends only on the homotopy class of φ , up to an inner automorphism of $\check{\mathcal{N}}$. Let $\check{\mathcal{N}}(\sigma) = C_{\check{\mathcal{N}}}(\check{T}_0^+(\sigma))$ and

$$BX(\sigma) = BC_X(\check{T}_0^+(\sigma)) = \text{map}(B\check{T}_0^+(\sigma), BX)_i,$$

where $i: B\check{T}_0^+(\sigma) \rightarrow BX$ is the inclusion. By definition $\check{\varphi}(\check{\mathcal{N}}(\sigma)) = \check{\mathcal{N}}(\check{\varphi}\sigma\check{\varphi}^{-1})$ and the diagram

$$\begin{array}{ccccc} B\check{\mathcal{N}}(\sigma) & \longrightarrow & BX(\sigma) & \longrightarrow & BX \\ B\check{\varphi} \downarrow & & f \mapsto \varphi \circ f \circ B\check{\varphi}^{-1} \downarrow & & \varphi \downarrow \\ B\check{\mathcal{N}}(\check{\varphi}\sigma\check{\varphi}^{-1}) & \longrightarrow & BX(\check{\varphi}\sigma\check{\varphi}^{-1}) & \longrightarrow & BX \end{array}$$

commutes up to homotopy. This, together with the definition of $\check{\mathcal{N}}_\sigma$ (see Section 3 and [19, Definition 9.5]) makes it clear that $\check{\varphi}(\check{\mathcal{N}}_\sigma) = \check{\mathcal{N}}_{\check{\varphi}\sigma\check{\varphi}^{-1}}$, which is what we needed to prove.

Next consider the case p odd. In this case $\check{\nu}(\mathbf{D}) = \check{\mathcal{N}} = \check{T} \rtimes W$ and, eg by [3, Proposition 5.2] there is an exact sequence $1 \rightarrow H^1(W; \check{T}) \rightarrow \text{Out}(\check{\mathcal{N}}) \rightarrow N_{\text{Aut}(\check{T})}(W)/W \rightarrow 1$. Since the reflections have order prime to p the coroots are given by

$$\mathbb{Z}_p b_\sigma = \text{im}(L \xrightarrow{1-\sigma} L),$$

so $\text{Aut}(\mathbf{D}) = N_{\text{Aut}(\check{T})}(W)$. This proves the existence of the short exact sequence in the theorem. By [1, Theorem 3.3] we have $H^1(W; \check{T}) = 0$ so the sequence is canonically split. The theorem now follows from Lemma 4.4. \square

Remark 4.5 It is easy to see that for a \mathbb{Z} -root datum \mathbf{D} one has an exact sequence

$$1 \rightarrow \text{Der}(W, T) \rightarrow \text{Aut}(\nu(\mathbf{D})) \rightarrow {}^\nu N_{\text{Aut}(T)}(W) \rightarrow 1,$$

where ${}^\nu N_{\text{Aut}(T)}(W)$ denotes the subgroup of $N_{\text{Aut}(T)}(W)$ which fixes the extension class ν of the extension $1 \rightarrow T \rightarrow \nu(\mathbf{D}) \rightarrow W \rightarrow 1$ (cf [3, Section 5] for details). Hence Theorem B identifies $\text{Aut}(\mathbf{D})$ with ${}^\nu N_{\text{Aut}(T)}(W)$ as subgroups of $\text{Aut}(T)$. Similarly Theorem C shows that, for a \mathbb{Z}_p -root datum \mathbf{D} , $\text{Aut}(\mathbf{D})$ equals ${}^\nu N_{\text{Aut}(\check{T})}(W)$ as subgroups of $\text{Aut}(\check{T})$.

5 Construction of $B \text{aut}(\mathbf{D})$

The goal of this section is, for a \mathbb{Z}_p -root datum \mathbf{D} , to introduce the space $B \text{aut}(\mathbf{D})$ and a refined Adams–Mahmud map $\Phi: B \text{Aut}(BX) \rightarrow B \text{aut}(\mathbf{D})$ modifying the Adams–Mahmud map $B \text{Aut}(BX) \rightarrow B \text{Aut}(B\check{\mathcal{N}}_X)$ constructed in [3, Lemma 3.1].

As mentioned in the introduction, if p is odd, we just set $B \text{aut}(\mathbf{D}) = B \text{Aut}(B\check{\mathcal{N}}_{\mathbf{D}})$, where $B\check{\mathcal{N}}_{\mathbf{D}} = (B^2 L)_{hW}$, and let $\Phi: B \text{Aut}(BX) \rightarrow B \text{Aut}(B\check{\mathcal{N}}_X) \simeq B \text{aut}(\mathbf{D})$ be the standard Adams–Mahmud map. For p odd we also define $B \check{\text{aut}}(\mathbf{D}) = B \text{Aut}(B\check{\nu}(\mathbf{D}))$ so that we have a canonical map $B \check{\text{aut}}(\mathbf{D}) \rightarrow B \text{aut}(\mathbf{D})$ which is a partial \mathbb{F}_p -completion,

leaving the fundamental group unchanged, cf [9, Chapter VII, 6.8]. We now embark on giving the definitions for $p = 2$.

For any p , define the *discrete center* of \mathbf{D} as

$$\check{Z}(\mathbf{D}) = \bigcap_{\sigma} S(\sigma),$$

where $S(\sigma)$ is the singular set corresponding to σ , cf (3–2). In particular [17, Theorem 7.5] shows that $\check{Z}(X) \cong \check{Z}(\mathbf{D}_X)$ for a connected p -compact group X . The space $B^2\check{Z}(\mathbf{D})$ is defined as the \mathbb{F}_p -completion of $B^2\check{Z}(\mathbf{D})$, and the *center* $Z(\mathbf{D})$ is defined as the double loop space of this space.

Lemma 5.1 *If a reflection σ in a \mathbb{Z}_p -root datum $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ satisfies $S(\sigma) \neq \check{T}^+(\sigma)$ then $p = 2$ and σ is a reflection in a direct factor of \mathbf{D} isomorphic to $\mathbf{D}_{\text{SO}(2n+1)} \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Furthermore*

$$Z(\check{\nu}(\mathbf{D})) = \check{T}^W = \check{Z}(\mathbf{D}) \oplus V$$

as $\text{Out}(\mathbf{D})$ - and as $\text{Out}(\check{\nu}(\mathbf{D}))$ -modules, where $V = 0$ unless $p = 2$ in which case $V \cong (\mathbb{Z}/2)^s$ where s is the number of direct factors of \mathbf{D} isomorphic to $\mathbf{D}_{\text{SO}(2n+1)} \otimes_{\mathbb{Z}} \mathbb{Z}_2$.

Proof If p is odd, then by [17, Remark 7.7] $S(\sigma) = \check{T}^+(\sigma)$ and by [17, Theorem 7.5] $Z(\check{\nu}(\mathbf{D})) = \check{T}^W = \check{Z}(\mathbf{D})$. Hence we can assume that $p = 2$. Write $\mathbf{D} = \mathbf{D}_1 \times \mathbf{D}_2$, where $\mathbf{D}_2 = (W_2, L_2, \{\mathbb{Z}_2 b_\sigma\})$ is a direct product of copies of $\mathbf{D}_{\text{SO}(2n+1)\hat{2}}$, and $\mathbf{D}_1 = (W_1, L_1, \{\mathbb{Z}_2 b_\sigma\})$ does not contain any direct factors isomorphic to $\mathbf{D}_{\text{SO}(2n+1)\hat{2}}$.

By [19, Proposition 7.4] or [3, Theorem 11.1], we can write \mathbf{D}_1 as a direct product of a \mathbb{Z}_2 -root datum of Coxeter type and a number of copies of $\mathbf{D}_{\text{DI}(4)}$. Since $\check{T}_0^+(\sigma) = S(\sigma) = \check{T}^+(\sigma)$ for any reflection $\sigma \in W_{\text{DI}(4)}$ and $Z(\check{\nu}(\mathbf{D}_{\text{DI}(4)})) = \check{T}^{W_{\text{DI}(4)}} = 0$, we can furthermore assume that \mathbf{D}_1 is of Coxeter type. Hence, by [29, Proposition 3.1(ii)] (see also [27, Proposition 3.2(vi)] or [31, Section 4]), $S(\sigma) = T_0^+(\sigma)$ for all reflections $\sigma \in W_1$. In particular $\check{T}_1^{W_1} = \check{Z}(\mathbf{D}_1)$ whereas $\check{Z}(\mathbf{D}_2) = 0$ and $\check{T}_2^{W_2} \cong (\mathbb{Z}/2)^s$, where s is the number of direct factors in \mathbf{D}_2 .

By combining Remark 4.5 with [3, Proposition 5.4] we see that $\text{Out}(\mathbf{D}) = \text{Out}(\mathbf{D}_1) \times \text{Out}(\mathbf{D}_2)$ so $\check{T}^W = \check{T}_1^{W_1} \times \check{T}_2^{W_2} = \check{Z}(\mathbf{D}) \times V$ splits as an $\text{Out}(\mathbf{D})$ -module in the manner indicated in the lemma. Since the $\text{Out}(\check{\nu}(\mathbf{D}))$ -action factors through the $\text{Out}(\mathbf{D})$ -action, the above splitting is also a splitting as $\text{Out}(\check{\nu}(\mathbf{D}))$ -modules. \square

From now on, let \mathbf{D} be a \mathbb{Z}_2 -root datum and let $\check{\mathcal{N}} = \check{\nu}(\mathbf{D})$ be the associated discrete maximal torus normalizer. Let $B\check{\mathcal{N}} = B\mathcal{N}_{\mathbf{D}}$ denote the fiber-wise \mathbb{F}_2 -completion of $B\check{\mathcal{N}}$ with respect to the fibration $B\check{\mathcal{N}} \rightarrow BW$ [9, Chapter I, Section 8]. Define \check{Y} to

be the covering space of $BAut(B\check{\mathcal{N}})$ corresponding to the subgroup $Out(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\})$ of $Out(\check{\mathcal{N}}) = \pi_1(BAut(B\check{\mathcal{N}}))$.

By Lemma 5.1, $\pi_2(\check{Y}) \cong \check{Z}(\mathbf{D}) \oplus V$ as $\pi_1(\check{Y})$ -modules, since by Theorem C, $\pi_1(\check{Y}) = Out(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\}) \xrightarrow{\cong} Out(\mathbf{D})$. Define $B\check{aut}(\mathbf{D})$ to be the space obtained from \check{Y} by first attaching 3-cells to kill exactly $V \subseteq \pi_2(\check{Y})$, which we can do since V is a $\pi_1(\check{Y})$ -invariant subgroup of $\pi_2(\check{Y})$, and then taking the second Postnikov section, killing all homotopy groups in dimensions three and higher. By construction $B\check{aut}(\mathbf{D})$ satisfies

$$\pi_i(B\check{aut}(\mathbf{D})) = \begin{cases} Out(\mathbf{D}) & \text{for } i = 1, \\ \check{Z}(\mathbf{D}) & \text{for } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

We define Y to be the covering space of $BAut(B\mathcal{N})$ with respect to the subgroup $Out(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\})$ of $\pi_1(BAut(B\mathcal{N})) \xleftarrow{\cong} Out(\check{\mathcal{N}})$, so that we have a canonical map $\check{Y} \rightarrow Y$. Define $Baut(\mathbf{D})$ to be the partial \mathbb{F}_2 -completion of $B\check{aut}(\mathbf{D})$ leaving the fundamental group unchanged. By construction $Baut(\mathbf{D})$ has universal cover homotopy equivalent to $B^2Z(\mathbf{D})$ and in particular $Baut(\mathbf{D})$ only has homotopy groups in dimensions 1, 2, and 3.

Since the maps $\check{Y} \rightarrow Y$ and $B\check{aut}(\mathbf{D}) \rightarrow Baut(\mathbf{D})$ are partial \mathbb{F}_2 -completions, the map $\check{Y} \rightarrow B\check{aut}(\mathbf{D})$ induces a map $Y \rightarrow Baut(\mathbf{D})$, well defined up to homotopy, by the universal property of the partial \mathbb{F}_2 -completion. On homotopy groups this map just kills the summand V in $\pi_2(Y)$.

By Theorem C the image of the map $\Phi: \pi_1(BAut(BX)) \rightarrow \pi_1(BAut(B\mathcal{N}))$ is contained in $Out(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\})$, hence it lifts uniquely to a basepoint-preserving map $BAut(BX) \rightarrow Y$. We define

$$\Phi: BAut(BX) \rightarrow Baut(\mathbf{D})$$

to be the composite $BAut(BX) \rightarrow Y \rightarrow Baut(\mathbf{D})$.

We note the following result, which we will need in the proof of Theorem D.

Proposition 5.2 *If $\mathbf{D} = \mathbf{D}_1 \times \mathbf{D}_2$ is a product of \mathbb{Z}_p -root data such that \mathbf{D}_1 and \mathbf{D}_2 have no direct factors in common, then we have a canonical map $Baut(\mathbf{D}_1) \times Baut(\mathbf{D}_2) \rightarrow Baut(\mathbf{D})$ which is a homotopy equivalence.*

Proof There is a natural map $BAut(B\mathcal{N}_{\mathbf{D}_1}) \times BAut(B\mathcal{N}_{\mathbf{D}_2}) \rightarrow BAut(B\mathcal{N}_{\mathbf{D}})$ which by [3, Proposition 5.4] is a homotopy equivalence for p odd. Assume that $p = 2$.

Here, by Remark 4.5 and [3, Proposition 5.4], $\text{Out}(\mathbf{D}_1) \times \text{Out}(\mathbf{D}_2) \xrightarrow{\cong} \text{Out}(\mathbf{D})$, and hence we have an isomorphism

$$(5-1) \quad \text{Out}(\check{\nu}(\mathbf{D}_1), \{\check{\nu}(\mathbf{D}_1)_\sigma\}) \times \text{Out}(\check{\nu}(\mathbf{D}_2), \{\check{\nu}(\mathbf{D}_2)_\sigma\}) \xrightarrow{\cong} \text{Out}(\check{\nu}(\mathbf{D}), \{\check{\nu}(\mathbf{D})_\sigma\}).$$

Passing to covers of the natural map above with respect to the isomorphic subgroups of the fundamental groups given by the isomorphism (5-1) produces a canonical map $Y_{\mathbf{D}_1} \times Y_{\mathbf{D}_2} \rightarrow Y_{\mathbf{D}}$ which by construction is a homotopy equivalence, since it induces an isomorphism on all homotopy groups. Killing the $\mathbb{Z}/2$ summands in π_2 corresponding to direct factors isomorphic to $\mathbf{D}_{\text{SO}(2n+1)\hat{2}}$ produces a homotopy equivalence $B\text{aut}(\mathbf{D}_1) \times B\text{aut}(\mathbf{D}_2) \rightarrow B\text{aut}(\mathbf{D})$ as wanted. \square

6 Proof of Theorem A and Theorem D

Theorem A and Theorem D will follow easily from the results of the previous section, combined with the following theorem, which is an analog of a classical theorem for Lie groups by de Siebenthal [33, Chapitre I, Section 2, no. 2].

Theorem 6.1

- (1) Let \mathbf{D} be a \mathbb{Z} -root datum and let $N = \nu(\mathbf{D})$. Then the exact sequence

$$1 \rightarrow \text{Inn}(N) \rightarrow \text{Aut}(N, \{N_\sigma\}) \rightarrow \text{Out}(N, \{N_\sigma\}) \rightarrow 1$$

is split.

- (2) Let \mathbf{D} be a \mathbb{Z}_2 -root datum of Coxeter type. Then the same statement holds with $N = \nu(\mathbf{D})$ replaced by $\check{\mathcal{N}} = \check{\nu}(\mathbf{D})$.
- (3) Let \mathbf{D} be a \mathbb{Z}_p -root datum of Coxeter type, p odd, with associated discrete maximal torus normalizer $\check{\mathcal{N}} = \check{\nu}(\mathbf{D})$. Then the exact sequence

$$1 \rightarrow \text{Inn}(\check{\mathcal{N}}) \rightarrow \text{Aut}(\check{\mathcal{N}}) \rightarrow \text{Out}(\check{\mathcal{N}}) \rightarrow 1$$

is split.

Proof (1) We define a splitting $s: \text{Out}(N, \{N_\sigma\}) \rightarrow \text{Aut}(N, \{N_\sigma\})$ as follows. First fix a simple system of roots B and let S be the corresponding set of simple reflections. We also fix a set of elements $\{x_\sigma\}_{\sigma \in S}$ with $x_\sigma \in N_\sigma \setminus T_0^-(\sigma)$. Let $[\varphi] \in \text{Out}(N, \{N_\sigma\})$ be an element represented by an automorphism $\varphi \in \text{Aut}(N, \{N_\sigma\})$. Since the action of W on the set of simple systems of roots is simply transitive (eg see [25, Theorem 1.8]), we can find a unique element $w \in W$ with $\varphi(B) = w(B)$. By composing φ with an appropriate inner automorphism of N we thus see that there exists a representative

φ' of $[\varphi]$ with $\varphi'(B) = B$. Moreover φ' is unique up to multiplication by an inner automorphism given by conjugation by an element in T .

Since $\varphi'(S) = S$ and $\varphi' \in \text{Aut}(N, \{N_\sigma\})$ we have $\varphi'(x_\sigma) \in N_{\varphi'(\sigma)} \setminus T_0^-(\varphi'(\sigma))$. Hence $\varphi'(x_\sigma) = t_{\varphi'(\sigma)} \cdot x_{\varphi'(\sigma)}$ for certain elements $t_\sigma \in T_0^-(\sigma)$, $\sigma \in S$, depending on φ' . Let $\varphi'' = c_t \circ \varphi'$, where $c_t \in \text{Inn}(N)$ denotes conjugation by an element $t \in T$. We now have

$$\varphi''(x_\sigma) = t \cdot t_{\varphi'(\sigma)} \cdot x_{\varphi'(\sigma)} \cdot t^{-1} = t_{\varphi'(\sigma)} \cdot t \cdot (\varphi'(\sigma)(t))^{-1} \cdot x_{\varphi'(\sigma)}.$$

Since $t_{\varphi'(\sigma)} \in T_0^-(\varphi'(\sigma))$ for $\sigma \in S$ it now follows from Lemma 3.6 that we can find $t \in T$ such that $\varphi''(x_\sigma) = x_{\varphi'(\sigma)}$. Moreover it is clear that such a $t \in T$ is unique up to multiplication by an element in $T^W = Z(N)$. We conclude that any element $[\varphi] \in \text{Out}(N, \{N_\sigma\})$ has a unique representative $\varphi'' \in \text{Aut}(N, \{N_\sigma\})$ with $\varphi''(B) = B$ and $\varphi''(x_\sigma) = x_{\varphi''(\sigma)}$. Hence the assignment $s([\varphi]) = \varphi''$ will be a group homomorphism and define the desired splitting.

(2) and (3) When \mathbf{D} is a \mathbb{Z}_p -root datum of Coxeter type the proof above goes through verbatim to show that the short exact sequence $1 \rightarrow \text{Inn}(\check{\mathcal{N}}) \rightarrow \text{Aut}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\}) \rightarrow \text{Out}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\}) \rightarrow 1$ is split. This proves the result since $\text{Aut}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\}) = \text{Aut}(\check{\mathcal{N}})$ for p odd by Lemma 4.4. □

Proof of Theorem D Write $\mathbf{D} = \mathbf{D}_X = (W, L, \{\mathbb{Z}_p b_\sigma\})$. We constructed the map $\Phi: B\text{Aut}(BX) \rightarrow B\text{aut}(\mathbf{D})$ in Section 5, and from the construction it is clear that Φ induces an isomorphism on π_i for $i > 1$, and that $B\text{aut}(\mathbf{D})$ is the total space of a fibration $B^2\mathcal{Z}(\mathbf{D}) \rightarrow B\text{aut}(\mathbf{D}) \rightarrow B\text{Out}(\mathbf{D})$. The remaining claim is that this fibration is split, which we now prove. The space $B\text{aut}(\mathbf{D})$ is by definition the partial \mathbb{F}_p -completion of the total space of the fibration

$$(6-1) \quad B^2\check{\mathcal{Z}}(\mathbf{D}) \rightarrow B\check{\text{aut}}(\mathbf{D}) \rightarrow B\text{Out}(\mathbf{D}),$$

and it is hence enough to see that this fibration splits. By [3, Theorem 11.1] (cf also [19, Proposition 1.12]) we may write $\mathbf{D} = \mathbf{D}_1 \times \mathbf{D}_2$, where \mathbf{D}_1 is of Coxeter type and \mathbf{D}_2 is a direct product of exotic \mathbb{Z}_p -root data. Since \mathbf{D}_1 and \mathbf{D}_2 have no direct factors in common, it follows from Proposition 5.2 combined with Remark 4.5 and [3, Proposition 5.4] that the fibration (6-1) is the product of the corresponding fibrations for \mathbf{D}_1 and \mathbf{D}_2 . By [3, Theorem 11.1] we have $\check{\mathcal{Z}}(\mathbf{D}_2) = 0$, so it suffices to prove that (6-1) splits when \mathbf{D} is of Coxeter type.

To prove this consider the composite

$$B\text{Out}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\}) \rightarrow B\text{Aut}(\check{\mathcal{N}}, \{\check{\mathcal{N}}_\sigma\}) \rightarrow \check{Y} \rightarrow B\check{\text{aut}}(\mathbf{D}).$$

Here the first map is induced by the splitting constructed in Theorem 6.1, the second map is the obvious map sending an automorphism to the induced self-homotopy equivalence of classifying spaces which factors through \check{Y} , and the third map is the map constructed in Section 5. By construction this is a section to the map $B \check{\text{aut}}(\mathbf{D}) \rightarrow B \text{Out}(\mathbf{D}) \simeq B \text{Out}(\mathcal{N}, \{\check{\mathcal{N}}_\sigma\})$ as desired. \square

Proof of Theorem A We continue with the notation of the previous proof. By construction the square

$$(6-2) \quad \begin{array}{ccc} B \text{Aut}(BX) & \longrightarrow & B \text{aut}(\mathbf{D}) \\ \downarrow & & \downarrow \\ B \text{Out}(BX) & \longrightarrow & B \text{Out}(\mathbf{D}) \end{array}$$

is a homotopy pullback square, so by the universal property of the homotopy pullback, the fibration $B^2 \mathcal{Z}(X) \rightarrow B \text{Aut}(BX) \rightarrow B \text{Out}(BX)$ has a section since $B^2 \mathcal{Z}(\mathbf{D}) \rightarrow B \text{aut}(\mathbf{D}) \rightarrow B \text{Out}(\mathbf{D})$ has one.

The fact that any homomorphism $\Gamma \rightarrow \text{Out}(BX)$ lifts to an action follows directly from the splitting. To see the parametrization in the case where $\check{H}^*(\Gamma; \mathbb{Q}) = 0$, note that, by the above pullback square (6-2), liftings to $B \text{Aut}(BX)$ agree with lifts:

$$\begin{array}{ccc} & & B \text{aut}(\mathbf{D}) \\ & \nearrow \text{dotted} & \downarrow \\ B \Gamma & \longrightarrow & B \text{Out}(\mathbf{D}) \end{array}$$

But since $B \Gamma$ is assumed to be rationally trivial, the fibration sequence

$$B^2(L^W \otimes \mathbb{Q}) \rightarrow B \check{\text{aut}}(\mathbf{D}) \rightarrow B \text{aut}(\mathbf{D})$$

shows that these lifts agree with lifts in the same diagram, but with $B \text{aut}(\mathbf{D})$ replaced by $B \check{\text{aut}}(\mathbf{D})$. Because (6-1) is split, obstruction theory now directly implies that the set of lifts is a nontrivial $H^2(\Gamma; \check{\mathcal{Z}}(\mathbf{D}))$ -torsor. \square

Remark 6.2 Theorem A implies that the second k -invariant of $B \text{Aut}(BX)$ vanishes, since $P_2(B \text{Aut}(BX)) \rightarrow P_1(B \text{Aut}(BX))$ has a section, where $P_n(\cdot)$ denotes the n -th Postnikov piece. The only possible nonzero k -invariant of $B \text{Aut}(BX)$ is therefore the third, which however need not vanish. This can occur for all p (as well as for compact Lie groups). The third k -invariant vanishes if and only if $P_3 B \text{Aut}(BX) \rightarrow P_2 B \text{Aut}(BX)$ has a section, which is equivalent to $B \mathcal{Z}(X)$ being a product of Eilenberg–Mac Lane spaces *as an* $\text{Out}(BX)$ -space, which need not be

the case. (By $\text{Out}(G)$ -spaces we mean the model category of spaces with an $\text{Out}(G)$ -action, where the weak equivalences are the $\text{Out}(G)$ -equivariant maps which are nonequivariant homotopy equivalences; see eg [20, Chapter VI, Section 4].) The point is that while $\check{Z}(X) \rightarrow \check{Z}(X)/C$ always has a section, where C is the largest divisible subgroup of $\check{Z}(X)$, it need *not* have an $\text{Out}(BX)$ -equivariant section. A concrete example is given by $G = (S^1 \times \text{SU}(n) \times \text{SU}(n))/\Delta$, where Δ is the “diagonal” central subgroup generated by $(\zeta, \zeta I, \zeta I)$, $\zeta = e^{2\pi i/n}$ —we leave the details to the reader.

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Received: 11 January 2007

Seconded: Paul Goerss, Ralph Cohen

Revised: 1 April 2008