Small values of the Lusternik–Schnirelmann category for manifolds

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We prove that manifolds of Lusternik–Schnirelmann category 2 necessarily have free fundamental group. We thus settle a 1992 conjecture of Gomez-Larrañaga and Gonzalez-Acuña by generalizing their result in dimension 3 to all higher dimensions. We also obtain some general results on the relations between the fundamental group of a closed manifold M, the dimension of M and the Lusternik–Schnirelmann category of M, and we relate the latter to the systolic category of M.

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1 Introduction

We follow the normalization of the Lusternik–Schnirelmann category (LS category) used in the recent monograph of Cornea, Lupton, Oprea and Tanré [7] (see Section 3 for a definition). We will denote the invariant $\mathrm{cat}_{\mathrm{LS}}$. Spaces satisfying $\mathrm{cat}_{\mathrm{LS}} = 0$ are contractible, while a closed manifold satisfying $\mathrm{cat}_{\mathrm{LS}} = 1$ is homotopy equivalent (and hence homeomorphic) to a sphere.

The characterization of closed manifolds of LS category 2 was initiated in 1992 by J Gomez-Larrañaga and F Gonzalez-Acuña [14] (see also Oprea and Rudyak [28]), who proved the following result on closed manifolds M of dimension 3: the fundamental group of M is free and nontrivial if and only if its LS category is 2. Furthermore, they conjectured that the fundamental group of every closed n–manifold, $n \geq 3$, of LS category 2 is necessarily free [14, Remark, p 797]. Our interest in this natural problem was also stimulated in part by our recent work on the comparison of the LS category and the systolic category [24; 23; 22], which was inspired, in turn, by M Gromov's systolic inequalities [15; 16; 17; 18].

In the present text we prove this 1992 conjecture. Recall that all closed surfaces different from S^2 are of LS category 2.

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1.1 Theorem *A closed connected manifold of LS category* 2 *either is a surface or has free fundamental group.*

1.2 Corollary *Every manifold* $M^n, n \geq 3$, *with nonfree fundamental group satisfies* $\operatorname{cat}_{\mathrm{LS}}(M) \geq 3$.

We found that there is no restriction on the fundamental group for closed manifolds of LS category 3. In particular we proved the following.

1.3 Theorem *Given a finitely presented group* π *and nonnegative integers* k, l, *there exists a closed manifold* M *such that* $\pi_1(M) = \pi$, *while* $\operatorname{cat}_{\mathrm{LS}} M = 3 + k$ *and* $\dim M = 5 + 2k + l$. *Furthermore, if* π *is not free, then* M *can be chosen* 4*–dimensional with* $\operatorname{cat}_{\mathrm{LS}} M = 3$.

Thus, there is no restriction on the fundamental group of manifolds of LS category 3 and higher.

The above results lead to the following questions:

1.4 Question If a 4–dimensional CW–complex X has free fundamental group, then we have the bound $\operatorname{cat}_{\mathrm{LS}} X \leq 3$. Is the stronger bound $\operatorname{cat}_{\mathrm{LS}} X \leq 2$ necessarily satisfied?

We prove the inequality $\operatorname{cat}_{\mathrm{LS}} M \leq n - 2$ for connected n–manifolds with free fundamental group and $n > 4$; see Proposition 4.4. In [34], J Strom proved a stronger inequality $\operatorname{cat}_{\mathrm{LS}} X \leq \frac{2}{3} \dim X$ for an arbitrary CW–space X. Later, it was proved by the first author [8] that if the fundamental group is free, then the bound

$$\operatorname{cat}_{\mathrm{LS}} X \leq \frac{1}{2} \dim X + 1 \tag{1–1}$$

is satisfied by every CW–complex X.

The above Question 1.4 has an affirmative answer when M is a closed orientable manifold, in view of a theorem due to J A Hillman [21] and T Matumoto and K Katanaga [27] which states that a closed 4–dimensional manifold with free fundamental group has a CW–decomposition in which the three-skeleton has the homotopy type of a wedge of spheres.

1.5 Question Is it true that $\operatorname{cat}_{\mathrm{LS}}(M \setminus \{\mathrm{pt}\}) = 1$ for any closed manifold M with $\operatorname{cat}_{\mathrm{LS}} M = 2$? This is proved in [14] for the case $\dim M = 3$. A direct proof would imply the main theorem trivially.

1.6 Question Given integers m and n, describe the fundamental groups of closed manifolds M with $\dim M = n$ and $\operatorname{cat}_{\mathrm{LS}} M = m$.

Note that in the case $m = n$, the fundamental group of M is of cohomological dimension $\geq n$; see eg Theorem 5.4 of Berstein and Švarc. Thus, we can ask when the converse holds.

1.7 Question Given a finitely presented group π and an integer $n \geq 4$ such that $H^n(\pi) \neq 0$, when can one find a closed manifold M satisfying $\pi_1(M) = \pi$ and $\dim M = \operatorname{cat}_{\mathrm{LS}} M = n$? Note that Proposition 5.12 shows that such a manifold M does not always exist.

A related numerical invariant called the *systolic category* can be thought of as a Riemannian analogue of the LS category [22]. In [9] we apply Corollary 1.2 to prove that the systolic category of a 4–manifold is a lower bound for its LS category.

1.8 Theorem *Every closed orientable* 4*–manifold* M *satisfies the inequality*

$$\operatorname{cat}_{\mathrm{sys}}(M) \leq \operatorname{cat}_{\mathrm{LS}}(M).$$

In particular, this inequality implies that if a 4–manifold M has a free fundamental group then $\operatorname{cat}_{\mathrm{sys}}(M) = \operatorname{cat}_{\mathrm{LS}}(M)$. In a related development in systolic topology, an intriguing model for BS^3 built out of BS^1 was used in Bangert, Katz, Shnider and Weinberger [2] and Katz and Shnider [26] to prove that the symmetric metric of the quaternionic projective space, contrary to expectation, is *not* its systolically optimal metric.

The proof of the main theorem proceeds roughly as follows. If the group $\pi := \pi_1(M)$ is not free, then by a result of J Stallings and R Swan, the group π is of cohomological dimension at least 2. We then show that π carries a suitable nontrivial 2–dimensional cohomology class u with twisted coefficients, and of category weight 2. Viewing M as a subspace of $K(\pi,1)$ that contains the 2–skeleton $K(\pi,1)^{(2)}$, and keeping in mind the fact that the 2–skeleton carries the fundamental group, we conclude that the restriction (pullback) of u to M is nonzero and also has category weight 2. By Poincaré duality with twisted coefficients, one can find a complementary $(n-2)$–dimensional cohomology class. By a category weight version of the cuplength argument, we therefore obtain a lower bound of 3 for $\operatorname{cat}_{\mathrm{LS}} M$.

In Section 2, we review the material on local coefficient systems, a twisted version of Poincaré duality and 2–dimensional cohomology of nonfree groups. In Section 3, we review the notion of category weight. In Section 4, we prove our main result, Theorem 1.1. In Section 5 we prove Theorem 1.3.

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2 Cohomology with local coefficients

A *local coefficient system* $\mathcal{A}$ on a path connected CW–space X is a functor from the fundamental groupoid $\Gamma(X)$ of X, to the category of abelian groups. See Hatcher [20] or Whitehead [37] for the definition and properties of local coefficient systems.

In other words, an abelian group $\mathcal{A}_x$ is assigned to each point $x \in X$, and for each path α joining x to y, an isomorphism $\alpha^*\colon \mathcal{A}_y \to \mathcal{A}_x$ is given. Furthermore, paths that are homotopic are required to yield the same isomorphism.

Given a map $f\colon Y \to X$ and a local coefficient system $\mathcal{A}$ on X, we define a local coefficient system on Y, denoted $f^*\mathcal{A}$, as follows. The map f yields a functor $\Gamma(f)\colon \Gamma(Y) \to \Gamma(X)$, and we define $f^*\mathcal{A}$ to be the functor $\mathcal{A} \circ \Gamma(f)$. Given a pair of coefficient systems $\mathcal{A}$ and $\mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is defined by setting $(\mathcal{A} \otimes \mathcal{B})_x = \mathcal{A}_x \otimes \mathcal{B}_x$.

2.1 Example A useful example of a local coefficient system is given by the following construction. Given a fiber bundle $p\colon E \to X$ over X, set $F_x = p^{-1}(x)$. Then the family $\{H_k(F_x)\}$ can be regarded a local coefficient system; see Whitehead [37, Example 3, Chapter VI, Section 1]. An important special case is that of an n–manifold M and spherical tangent bundle $p\colon E \to M$ with fiber S^{n-1}, yielding a local coefficient system $\mathcal{O}$ with $\mathcal{O}_x = H_{n-1}(S_x^{n-1}) \cong \mathbb{Z}$. This local system is called the *orientation sheaf* of M.

2.2 Remark Let $\pi = \pi_1(X)$, and let $\mathbb{Z}[\pi]$ be the group ring of π. Note that all the groups A_x are isomorphic to a fixed group A. We will refer to A as a *stalk* of $\mathcal{A}$. There is a bijection between local coefficients on X and $\mathbb{Z}[\pi]$–modules [31, Chapter 1, Exercises F]. If $\mathcal{A}$ is a local coefficient system with stalk A, then the natural action of the fundamental group on A turns A into a $\mathbb{Z}[\pi]$–module. Conversely, given a $\mathbb{Z}[\pi]$–module A, one can construct a local coefficient system $\mathcal{L}(A)$ such that induced $\mathbb{Z}[\pi]$–module structure on A coincides with the given one, cf [20].

We recall the definition of the (co)homology groups with local coefficients via modules [20]:

$$H^k(X;\mathcal{A}) \cong H^k(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X}), A), \delta) \tag{2–1}$$

$$H_k(X;\mathcal{A}) \cong H_k(A \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}), 1 \otimes \partial). \tag{2–2}$$

Here $(C_*(\widetilde{X}), \partial)$ is the chain complex of the universal cover $\widetilde{X}$ of X, A is the stalk of the local coefficient system $\mathcal{A}$, and δ is the coboundary operator. Note that in the tensor product we used the right $\mathbb{Z}[\pi]$ module structure on A defined via the standard rule $ag = g^{-1}a$, for $a \in A, g \in \pi$.

Recall that for CW–complexes X, there is a natural bijection between equivalence classes of local coefficient systems and locally constant sheaves on X. One can therefore define (co)homology with local coefficients as the corresponding sheaf cohomology as in Bredon [5]. In particular, we refer to [5] for the definition of the cup product

$$\cup\colon H^i(X;\mathcal{A}) \otimes H^j(X;\mathcal{B}) \to H^{i+j}(X;\mathcal{A}\otimes\mathcal{B})$$

and the cap product

$$\cap\colon H_i(X;\mathcal{A}) \otimes H^j(X;\mathcal{B}) \to H_{i-j}(X;\mathcal{A}\otimes\mathcal{B}).$$

A nice exposition of the cup and the cap products in a slightly different setting can be found in Brown [6]. In particular, we have the cap product

$$H_k(X;\mathcal{A}) \otimes H^k(X;\mathcal{B}) \to H_0(X;\mathcal{A}\otimes\mathcal{B}) \cong A \otimes_{\mathbb{Z}[\pi]} B.$$

2.3 Proposition *Given an integer $k \geq 0$, there exists a local coefficient system $\mathcal{B}$ and a class $v \in H^k(X;\mathcal{B})$ such that, for every local coefficient system $\mathcal{A}$ and nonzero class $a \in H_k(X;\mathcal{A})$, we have $a \cap v \neq 0$.*

Proof Throughout the proof $\otimes$ denotes $\otimes_{\mathbb{Z}[\pi]}$. We convert the stalk of $\mathcal{A}$ into a right $\mathbb{Z}[\pi]$–module A as above. Below we use the isomorphisms (2–1) and (2–2). Consider the chain $\mathbb{Z}[\pi]$–complex:

$$\dots \longrightarrow C_{k+1}(\widetilde{X}) \xrightarrow{\partial_{k+1}} C_k(\widetilde{X}) \xrightarrow{\partial_k} C_{k-1}(\widetilde{X}) \longrightarrow \dots$$

For the given k, we set $B := C_k(\widetilde{X}) / \operatorname{Im} \partial_{k+1}$. Let $\mathcal{B}$ be the corresponding local system on X. Thus, we obtain the exact sequence of $\mathbb{Z}[\pi]$–modules

$$C_{k+1}(\widetilde{X}) \xrightarrow{\partial_{k+1}} C_k(\widetilde{X}) \xrightarrow{f} B \to 0.$$

Note that the epimorphism f can be regarded as a k–cocycle with values in $\mathcal{B}$, since $\delta f(x) = f\partial_{k+1}(x) = 0$. Let $v := [f] \in H^k(X;\mathcal{B})$ be the cohomology class of f. Now we prove that

$$a \cap [f] \neq 0.$$

Since the tensor product is right exact, we obtain the diagram

$$\begin{array}{ccccccc} A\otimes C_{k+1}(\widetilde{X}) & \xrightarrow{1\otimes\partial_{k+1}} & A\otimes C_k(\widetilde{X}) & \xrightarrow{1\otimes f} & A\otimes B & \longrightarrow & 0 \\ & & & & g\big\downarrow & & \\ & & & & A\otimes C_{k-1}(\widetilde{X}) & & \end{array}$$

where the row is exact. The composition

$$A\otimes C_k(\widetilde{X}) \xrightarrow{1\otimes f} A\otimes B \xrightarrow{g} A\otimes C_{k-1}(\widetilde{X})$$

coincides with $1\otimes\partial_k$. We represent the class a by a cycle

$$z \in A\otimes C_k(\widetilde{X}).$$

Since $z \notin \mathrm{Im}(1\otimes\partial_{k+1})$, we conclude that

$$(1\otimes f)(z) \neq 0 \in A\otimes B = H_0(X;\mathcal{A}\otimes\mathcal{B}).$$

Thus, for the cohomology class v of f we have $a\cap v \neq 0$. □

Every closed connected n–manifold M satisfies $H_n(M;\mathcal{O}) \cong \mathbb{Z}$. A generator (one of two) of this group is called the *fundamental class* of M and is denoted by $[M]$.

One has the following generalization of the Poincaré duality isomorphism.

2.4 Theorem [5, Corollary 10.2] *The homomorphism*

$$\Delta\colon H^i(M;\mathcal{A}) \to H_{n-i}(M;\mathcal{O}\otimes\mathcal{A})$$

defined by setting $\Delta(a) = [M]\cap a$*, is an isomorphism.*

In fact, in [5] there is the sheaf $\mathcal{O}^{-1}$ at the right, but for manifolds we have $\mathcal{O} = \mathcal{O}^{-1}$.

Given a group π and a $\mathbb{Z}[\pi]$–module A, we denote by $H^*(\pi;A)$ the cohomology of the group π with coefficients in A; see eg Brown [6]. Recall that $H^i(\pi;A) = H^i(K(\pi,1);\mathcal{L}(A))$; see Remark 2.2.

Let $\mathrm{cd}(\pi)$ denote the cohomological dimension of π over $\mathbb{Z}$, ie the largest m such that there exists an $\mathbb{Z}[\pi]$–module A with $H^m(\pi;A) \neq 0$.

2.5 Theorem [32; 35] *If* $\operatorname{cd}\pi \le 1$ *then* π *is a free group.*

We will need the following known fact from the cohomology theory of groups.

2.6 Lemma *If* π *be a group with* $\operatorname{cd}\pi = q \ge 2$. *Then* $H^2(\pi; A) \ne 0$ *for some* $\mathbb{Z}[\pi]$*–module* A.

Proof We use the fact that cohomology of the group π with coefficients in an injective $\mathbb{Z}[\pi]$–module are trivial and the fact that every $\mathbb{Z}[\pi]$–module A' can be imbedded into an injective $\mathbb{Z}[\pi]$–module J [6]. Let $0 \to A' \to J \to A'' \to 0$ be an exact sequence of $\mathbb{Z}[\pi]$–modules with J injective. Then by the coefficients long exact sequence $H^k(\pi; A') = H^{k-1}(\pi; A'')$ for $k > 1$. Since $H^q(\pi; B) \ne 0$ for some B, the proof can be completed by an obvious induction. □

3 Category weight and lower bounds for $\operatorname{cat}_{\mathrm{LS}}$

In this section, we review the notion of category weight and its relation to the Lusternik–Schnirelmann category.

3.1 Definition [4; 12; 13] Let $f\colon X \to Y$ be a map of (locally contractible) CW-spaces. The *Lusternik–Schnirelmann category of* f, denoted $\operatorname{cat}_{\mathrm{LS}}(f)$, is defined to be the minimal integer k such that there exists an open covering $\{U_0, \dots, U_k\}$ of X with the property that each of the restrictions $f|A_i\colon A_i \to Y$, $i = 0, 1, \dots, k$ is null-homotopic.

The *Lusternik–Schnirelmann category* $\operatorname{cat}_{\mathrm{LS}} X$ *of a space* X is defined as the category $\operatorname{cat}_{\mathrm{LS}}(1_X)$ of the identity map.

3.2 Definition The *category weight* $\operatorname{wgt}(u)$ of a nonzero cohomology class $u \in H^*(X; \mathcal{A})$ is defined as follows:

$$\operatorname{wgt}(u) \ge k \iff \{\varphi^*(u) = 0 \text{ for every } \varphi\colon F \to X \text{ with } \operatorname{cat}_{\mathrm{LS}}(\varphi) < k\}.$$

3.3 Remark E Fadell and S Husseini [11] originally proposed the notion of category weight. In fact, they considered an invariant similar to the wgt of 3.2 (denoted in [11] by cwgt), but where the defining maps $\varphi\colon F \to X$ were required to be inclusions rather than general maps. As a consequence, cwgt is not a homotopy invariant, and thus a delicate quantity in homotopy calculations. Yu Rudyak [29; 30] and J Strom [33] proposed a homotopy invariant version of category weight as defined in Definition 3.2.

3.4 Proposition [29; 33] *Category weight has the following properties.*

(1) $1 \le \mathrm{wgt}(u) \le \mathrm{cat}_{\mathrm{LS}}(X)$, *for all* $u \in \widetilde{H}^*(X;\mathcal{A}), u \neq 0$.

(2) *For every* $f\colon Y \to X$ *and* $u \in H^*(X;\mathcal{A})$ *with* $f^*(u) \neq 0$ *we have* $\mathrm{cat}_{\mathrm{LS}}(f) \ge \mathrm{wgt}(u)$ *and* $\mathrm{wgt}(f^*(u)) \ge \mathrm{wgt}(u)$.

(3) *For* $u \in H^*(X;\mathcal{A})$ *and* $v \in H^*(X;\mathcal{B})$ *we have*

$$\mathrm{wgt}(u \cup v) \ge \mathrm{wgt}(u) + \mathrm{wgt}(v).$$

(4) *For every* $u \in H^s(K(\pi,1);\mathcal{A})$, $u \neq 0$, *we have* $\mathrm{wgt}(u) \ge s$.

Proof See Cornea et al [7, Section 2.7 and Proposition 8.22]. The proofs in loc. cit. can be easily adapted to local coefficient systems. □

4 Manifolds of LS category 2

In this section we prove that the fundamental group of a closed connected manifold of LS category 2 is free.

4.1 Theorem *Let* M *be a closed connected manifold of dimension at least* 3*. If the group* $\pi := \pi_1(M)$ *is not free, then* $\mathrm{cat}_{\mathrm{LS}} M \ge 3$.

Proof By Theorem 2.5 and Lemma 2.6, there a local coefficient system $\mathcal{A}$ on $K(\pi,1)$ such that $H^2(K(\pi,1);\mathcal{A}) \neq 0$. Choose a nonzero element $u \in H^2(K(\pi,1);\mathcal{A})$. Let $f\colon M \to K(\pi,1)$ be the map that induces an isomorphism of fundamental groups, and let $i : K \to M$ be the inclusion of the 2–skeleton. (If M is not triangulable, we take i to be any map of a 2–polyhedron that induces an isomorphism of fundamental groups.) Then

$$(fi)^*\colon H^2(K(\pi,1);\mathcal{A}) \to H^2(K;(fi)^*\mathcal{A})$$

is a monomorphism. In particular, we have $f^*u \neq 0$ in $H^2(M;(f)^*\mathcal{A})$. Now consider the class

$$a = [M] \cap f^*u \in H_{n-2}(M;\mathcal{O}^{-1} \otimes f^*\mathcal{A}),$$

where $n = \dim M$. Then $a \neq 0$ by Theorem 2.4. Hence, by Proposition 2.3, there exists a class $v \in H^{n-2}(M;\mathcal{B})$ such that $a \cap v \neq 0$. We claim that $f^*u \cup v \neq 0$. Indeed, one has

$$[M] \cap (f^*u \cup v) = ([M] \cap f^*u) \cap v = a \cap v \neq 0.$$

Now, $\mathrm{wgt}\, f^*u \ge 2$ by Proposition 3.4, items (2) and (4). Furthermore, $\mathrm{wgt}(v) \ge 1$ by Proposition 3.4, item (1). We therefore obtain the lower bound $\mathrm{wgt}(f^*u \cup v) \ge 3$ by Proposition 3.4, item (3). Since $f^*u \cup v \neq 0$, we conclude that $\mathrm{cat}_{\mathrm{LS}} M \ge 3$ by Proposition 3.4, item (1). □

4.2 Corollary *If $M^n, n \geq 3$ is a closed manifold with* $\operatorname{cat}_{\mathrm{LS}} M \leq 2$*, then $\pi_1(M)$ is a free group.*

4.3 Remark An alternative approach to Theorem 4.1 would be using the Berstein–Švarc class $\mathfrak{b} \in H^1(\pi; I(\pi))$ where $I(\pi)$ is the augmentation ideal of π. If $\operatorname{cd}(\pi) \geq 2$ then $\mathfrak{b}^2 \neq 0$ by [10] (see also Theorem 5.4). In particular, $H^2(\pi; I(\pi) \otimes I(\pi)) \neq 0$, and we obtain an alternative proof of Lemma 2.6.

The following Proposition is a special case of [8, Corollary 4.2]. Here we give a relatively simple geometric proof.

4.4 Proposition *Let M be a closed connected n–dimensional PL manifold, $n > 4$, with free fundamental group. Then* $\operatorname{cat}_{\mathrm{LS}} M \leq n-2$*.*

Proof If X is a 2–dimensional (connected) CW–complex with free fundamental group then $\operatorname{cat}_{\mathrm{LS}} X \leq 1$; see eg Katz, Rudyak and Sabourau [25, Theorem 12.1]. Hence, if Y is a k–dimensional complex with free fundamental group then $\operatorname{cat}_{\mathrm{LS}} Y \leq k-1$ for $k > 2$. Now, let K be a triangulation of M, and let L be its dual triangulation. Then $M \setminus L^{(l)}$ is homotopy equivalent to $K^{(k)}$ whenever $k + l + 1 = n$. Hence,

$$\operatorname{cat}_{\mathrm{LS}} M \leq \operatorname{cat}_{\mathrm{LS}} K^{(k)} + \operatorname{cat}_{\mathrm{LS}} L^{(l)} + 1.$$

Since $\pi_1(K)$ and $\pi_1(L)$ are free, we conclude that $\operatorname{cat}_{\mathrm{LS}} K^{(k)} \leq k-1$ and $\operatorname{cat}_{\mathrm{LS}} L^l \leq l-1$ for $k, l > 1$. Thus $\operatorname{cat}_{\mathrm{LS}} M \leq k-1+l-1+1 = n-2$. □

5 Manifolds of higher LS category

Gromov [17, 4.40] called a polyhedron X *n–essential* if there is no map $f\colon X \to K(\pi, 1)^{(n-1)}$ to the $(n-1)$–dimensional skeleton of an Eilenberg–MacLane complex that induces an isomorphism of the fundamental groups. We extend his definition as follows.

5.1 Definition A CW–space X is called *strictly k–essential*, $k > 1$ if for every CW–complex structure on X there is no map between the skeleta $f\colon X^{(k)} \to K(\pi, 1)^{(k-1)}$ that induces an isomorphism of the fundamental groups.

Clearly, a strictly n–essential space is Gromov n–essential, while the converse is false. Furthermore, an n–dimensional polyhedron is strictly n–essential if it is Gromov n–essential.

5.2 Theorem *Let M be a closed strictly k–essential manifold. If its dimension satisfies* $\dim M \geq k+1$, *then its LS category also satisfies* $\mathrm{cat}_{\mathrm{LS}} M \geq k+1$.

Proof We first consider the case $k=2$. If $\mathrm{cat}_{\mathrm{LS}} M \leq 2$, then, by Theorem 4.1, $\pi_1(M)$ is free. Hence there is a map $f\colon M \to \vee S^1$ that induces an isomorphism of the fundamental groups, and M is not strictly 2–essential.

Now assume $k \geq 3$. Let $K = K(\pi_1(M), 1)$. Consider a map

$$f\colon M^{(k-1)} \to K^{(k-1)}$$

such that the restriction $f|_{M^{(2)}}$ is the identity homeomorphism of the 2–skeleta $M^{(2)}$ and $K^{(2)}$. We consider the problem of extension of f to M.

We claim that the first obstruction $o(f) \in H^k(M; E)$ (taken with coefficients in a local system E with the stalk $\pi_{k-1}(K^{(k-1)})$) to the extension is not equal to zero.

Indeed, if $o(f) = 0$, then there exists a map $\bar{f}\colon M^{(k)} \to K^{(k-1)}$ which coincides with f on the $(k-2)$–skeleton. The map

$$\bar{f}_*\colon \pi_1(M^{(k)}) \to \pi_1(K^{(k-1)})$$

can be viewed as an endomorphism of $\pi_1(M)$ that is identical on generators, and therefore $\bar{f}_*$ is an isomorphism. Hence M is not strictly k–essential.

Consider the commutative diagram

$$\begin{array}{ccccc} M^{(k-1)} & \xrightarrow{f} & K^{(k-1)} & \xrightarrow{\mathrm{id}} & K^{(k-1)} \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow & & \\ M & \xrightarrow{\tilde{f}} & K & & \end{array}$$

where i and j are the inclusions of the skeleta. Let α be the first obstruction to the extension of id to a map $K \to K^{(k-1)}$. By commutativity of the above diagram, we have $o(f) = \tilde{f}^*(\alpha)$. Now, asserting as in the proof of Theorem 4.1, we get that $\tilde{f}^*(\alpha) \cup v \neq 0$ for some v with $\dim v = \dim M - k$, and $\mathrm{wgt}\, f^*\alpha = k$. Since $\dim M > k$, we conclude that $\dim v \geq 1$ and thus $\mathrm{cat}_{\mathrm{LS}} M \geq k+1$. □

5.3 Remark If a closed manifold M^n is n–essential then $\mathrm{cat}_{\mathrm{LS}} M = n$; see eg the paper by the second and third authors [24] and the book by the second author [22, Theorem 12.5.2].

The following theorem for $n \geq 3$ was proven by Berstein [3, Theorem A] and Švarc [36, Theorem 20]; see also Cornea et al [7, Proposition 2.51]. The case $n=2$ was proved in Dranishnikov and Rudyak [10].

5.4 Theorem *If* $\dim X = \mathrm{cat}_{\mathrm{LS}}\, X = n$, *then* $u_X^n \neq 0$ *where* u_X *is the image* $j^*(\mathfrak{b}) \in H^1(X; I(\pi))$, $j\colon X \to K(\pi,1)$ *induces an isomorphism of the fundamental groups, and* $\mathfrak{b} \in H^1(\pi, I(\pi))$ *is the Berstein–Švarc class. (For the case* $n = \infty$ *this means that* $u^k \neq 0$ *for all* k.)

5.5 Proposition *For every nonfree finitely presented group* π, *there exists a closed* 4*–dimensional manifold* M *with fundamental group* π *and* $\mathrm{cat}_{\mathrm{LS}}\, M = 3$.

Proof Let K be a 2–skeleton of $K(\pi,1)$. Take an embedding of K in $\mathbb{R}^5$ and let $M = \partial N$ be the boundary of the regular neighborhood N of this skeleton. Then there is a retraction $N \to K$, and, clearly, the map $f\colon M \subset N \to K$ induces an isomorphism of fundamental groups. Now, let $u_M \in H^1(M; I(\pi))$ be the class described in the Theorem 5.4. Then $u_M = f^* u_K$, and hence $u_M^4 = 0$. Therefore $\mathrm{cat}_{\mathrm{LS}}\, M < 4$ by Theorem 5.4, and thus $\mathrm{cat}_{\mathrm{LS}}\, M = 3$. □

Let M_f be the mapping cylinder of $f\colon X \to Y$. We use the notation $\pi_*(f) = \pi_*(M_f, X)$. Then $\pi_i(f) = 0$ for $i \leq n$ amounts to saying that it induces isomorphisms $f_*\colon \pi_i(X_1) \to \pi_i(Y_1)$ for $i \leq n$ and an epimorphism in dimension $n+1$. Similar notation $H_*(f) = H_*(M_f, X)$ we use for homology.

5.6 Lemma *Let* $f_j\colon X_j \to Y_j$ *be a family of maps of CW–spaces such that* $H_i(f_j) = 0$ *for* $i \leq n_j$. *Then* $H_i(f_1 \wedge \cdots \wedge f_s) = 0$ *for* $i \leq \min\{n_j\}$.

Proof Note that

$$M(f_1 \wedge \cdots \wedge f_s) \cong Y_1 \wedge \cdots \wedge Y_s \cong M(f_1) \wedge \cdots \wedge M(f_s).$$

Now, by using the Künneth formula and considering the homology exact sequence of the pair $(M(f_1) \wedge \cdots \wedge M(f_s), X_1 \wedge \cdots \wedge X_s)$, we obtain the result. □

5.7 Proposition *Let* $f_j\colon X_j \to Y_j$, $3 \leq j \leq s$ *be a family of maps of CW–spaces such that* $\pi_i(f_j) = 0$ *for* $i \leq n_j$, $n_j \geq 1$. *Then the joins satisfy*

$$\pi_k(f_1 * f_2 * \cdots * f_s) = 0$$

for $k \leq \min\{n_j\} + s - 1$.

Proof By the version of the Relative Hurewicz Theorem for non–simply connected X_j [20, Theorem 4.37], we obtain $H_i(f_j) = 0$ for $i \leq n_j$. By Lemma 5.6 we obtain that $H_k(f_1 \wedge \cdots \wedge f_s) = 0$ for $k \leq \min\{n_j\}$. Since the join $A_1 * \cdots * A_s$ is homotopy equivalent to the iterated suspension $\Sigma^{s-1}(A_1 \wedge \cdots \wedge A_s)$ over the smash product, we conclude that $H_k(f_1 * \cdots * f_s) = 0$ for $k \le \min\{n_j\} + s - 1$. Since $X_1 * \cdots * X_s$ is simply connected for $s \ge 3$, by the standard Relative Hurewicz Theorem we obtain that $\pi_k(f_1 * \cdots * f_s) = 0$ for $k \le \min\{n_j\} + s - 1$. □

Given two maps $f\colon Y_1 \to X$ and $g\colon Y_2 \to X$, we set

$$Z = \{(y_1, y_2, t) \in Y_1 * Y_2 \mid f(y_1) = g(y_2)\}$$

and define the *fiberwise join*, or *join over* X of f and g as the map:

$$f *_X g\colon Z \to X, \quad (f *_X g)(y_1, y_2, t) = f(y_1)$$

Let $p_0^X\colon PX \to X$ be the Serre path fibration. This means that PX is the space of paths on X that start at the base point of the pointed space X, and $p_0(\alpha) = \alpha(1)$. We denote by $p_n^X\colon G_n(X) \to X$ the n–fold fiberwise join of p_0.

The proof of the following theorem can be found in [7].

5.8 Theorem (Ganea, Švarc) *For a CW–space X, $\mathrm{cat}_{\mathrm{LS}}(X) \le n$ if and only if there exists a section of $p_n\colon G_n(X) \to X$.*

5.9 Proposition *The connected sum $S^k \times S^l \# \cdots \# S^k \times S^l$ is a space of LS category 2.*

Proof This can be deduced from a general result of K Hardy [19] because the connected sum of two manifolds can be regarded as the double mapping cylinder. Alternatively, one can note that, after removing a point, the manifold on hand is homotopy equivalent to the wedge of spheres. □

5.10 Theorem *For every finitely presented group π and $n \ge 5$, there is a closed n–manifold M of LS category 3 with $\pi_1(M) = \pi$.*

Proof If the group π is the free group of rank s, we let M' be the k–fold connected sum $S^1 \times S^2 \# \cdots \# S^1 \times S^2$. Then M' is a closed 3–manifold of LS category 2 with $\pi_1(M') = F_s$. Then the product manifold $M = M' \times S^{n-3}$ has cuplength 3 and is therefore the desired manifold.

Now assume that the group π is not free. We fix a presentation of π with s generators and r relators. Let M' be the k–fold connected sum $S^1 \times S^{n-1} \# \cdots \# S^1 \times S^{n-1}$. Then M' is a closed n–manifold of the category 2 with $\pi_1(M') = F_s$. For every relator w we fix a nicely imbedded circle $S^1_w \subset M'$ such that $S^{-1}_w \cap S^{-1}_v = \varnothing$ for $w \ne v$. Then we perform the surgery on these circles to obtain a manifold M.

Clearly, $\pi_1(M)=\pi$. We show that $\mathrm{cat}_{\mathrm{LS}}(M)\le 3$, and so $\mathrm{cat}_{\mathrm{LS}}\, M=3$ by Theorem 4.1.

As usual, the surgery process yields an $(n+1)$–manifold X with $\partial X = M\sqcup M'$. Here X is the space obtained from $M'\times I$ by attaching handles $D^2\times D^{n-1}$ of index 2 to $M'\times 1$ along the above circles. We note that $\mathrm{cat}_{\mathrm{LS}}(X)\le 3$.

On the other hand, by duality, X can be obtained from $M\times I$ by attaching handles of index $n-1$ to the boundary component of $M\times I$. In particular, the inclusion $f\colon M\to X$ induces an isomorphism of the homotopy groups of dimension $\le n-3$ and an epimorphism in dimension $n-2$. Hence the map

$$\Omega f\colon \Omega M\to \Omega X$$

induces isomorphisms in dimensions $\le n-4$ and an epimorphism in dimension $n-3$. Thus, $\pi_i(\Omega f)=0$ for $i\le n-3$.

In order to prove the bound $\mathrm{cat}_{\mathrm{LS}}\, M\le 3$, it suffices to show that the Ganea–Švarc fibration $p_3\colon G_3(M)\to M$ has a section. Consider the commutative diagram

$$\begin{array}{ccccc} G_3M & \xrightarrow{q} & Z & \xrightarrow{f'} & G_3(X) \\ {\scriptstyle p^3_M}\big\downarrow & & {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle p^X_3} \\ M & = & M & \xrightarrow{f} & X \end{array}$$

where the right-hand square is the pullback diagram and $f'q=G_3(f)$. Note that q is uniquely determined. Since $\mathrm{cat}_{\mathrm{LS}}(X)\le 3$, by Theorem 5.8 there is a section $s\colon X\to G_3(X)$. It defines a section $s'\colon M\to Z$ of p'. It then suffices to show that the map $s'\colon M\to Z$ admits a homotopy lifting $h\colon M\to G_3M$ with respect to q, ie the map h with $qh\cong s'$. Indeed, we have

$$p^3_M h = p'qh \cong p's' = 1_M$$

and so h is a homotopy section of p^M_3. Since the latter is a Serre fibration, the homotopy lifting property yields an actual section.

Let F_1 and F_2 be the fibers of fibrations p^M_3 and p', respectively. Consider the commutative diagram generated by the homotopy exact sequences of the Serre fibrations p^M_3 and p':

$$\begin{array}{ccccccccc} \pi_i(F_1) & \longrightarrow & \pi_i(G_3(M)) & \xrightarrow{(p^M_3)_*} & \pi_i(M) & \longrightarrow & \pi_{i-1}(F_1) & \longrightarrow & \cdots \\ \big\downarrow{\scriptstyle \phi_*} & & \big\downarrow{\scriptstyle q_*} & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \phi_*} & & \\ \pi_i(F_2) & \longrightarrow & \pi_i(Z) & \xrightarrow{(p')_*} & \pi_i(M) & \longrightarrow & \pi_{i-1}(F_2) & \longrightarrow & \cdots \end{array}$$

Note that we have

$$\phi = \Omega(f) * \Omega(f) * \Omega(f) * \Omega(f).$$

By Proposition 5.7 and since $\pi_i(\Omega f) = 0$ for $i \le n-3$, we conclude that $\pi_i(\phi) = 0$ for $i \le n-3+3 = n$. Hence ϕ induces an isomorphism of the homotopy groups of dimensions $\le n-1$ and an epimorphism in dimension n. By the Five Lemma we obtain that q_* is an isomorphism in dimensions $\le n-1$ and an epimorphism in dimension n. Hence the homotopy fiber of q is $(n-1)$–connected. Since $\dim M = n$, the map s' admits a homotopy lifting $h\colon M \to G_3(M)$. □

5.11 Corollary *Given a finitely presented group π and nonnegative integer numbers k, l there exists a closed manifold M such that $\pi_1(M) = \pi$, while $\operatorname{cat}_{\mathrm{LS}} M = 3+k$ and $\dim M = 5+2k+l$.*

Proof By Theorem 5.10, there exists a manifold N such that $\pi_1(N) = \pi$, $\operatorname{cat}_{\mathrm{LS}} N = 3$ and $\dim N = 5+l$. Moreover, this manifold N possesses a detecting element, ie a cohomology class whose category weight is equal to $\operatorname{cat}_{\mathrm{LS}} N = 3$. For π free this follows since the cuplength of N is equal to 3, for other groups we have the detecting element $f^*u \cup v$ constructed in the proof of Theorem 4.1. If a space X possesses a detecting element then, for every $m > 0$, we have $\operatorname{cat}_{\mathrm{LS}}(X \times S^m) = \operatorname{cat}_{\mathrm{LS}} X + 1$ and $X \times S^m$ possesses a detecting element [30]. Now, the manifold $M := N \times (S^2)^k$ is the desired manifold. □

Generally, we have a question about relations between the category, the dimension, and the fundamental group of a closed manifold. The following proposition shows that the situation quite intricate.

5.12 Proposition *Let p be an odd prime. Then there exists a closed $(2n+1)$–manifold with $\operatorname{cat}_{\mathrm{LS}} M = \dim M$ and $\pi_1(M) = \mathbb{Z}_p$, but there are no closed $2n$–manifolds with $\operatorname{cat}_{\mathrm{LS}} M = \dim M$ and $\pi_1(M) = \mathbb{Z}_p$.*

Proof An example of $(2n+1)$–manifold is the quotient space $S^{2n+1}/\mathbb{Z}_p$ with respect to a free $\mathbb{Z}_p$–action on S^{2n+1}. Now, given a $2n$–manifold with $\pi_1(M) = \mathbb{Z}_p$, consider a map $f\colon M \to K(\mathbb{Z}_p, 1)$ that induces an isomorphism of fundamental groups. Since $H_{2n}(K(\mathbb{Z}_p, 1)) = 0$, it follows from the obstruction theory and Poincaré duality that f can be deformed into the $(2n-1)$–skeleton of $K(\mathbb{Z}_p, 1)$, cf [1, Section 8]. Hence, M is not $2n$–essential, and thus $\operatorname{cat}_{\mathrm{LS}} M < 2n$ [24]. □

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