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Geodesible contact structures on 3–manifolds

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In this paper, we study and almost completely classify contact structures on closed 3–manifolds which are totally geodesic for some Riemannian metric. Due to previously known results, this amounts to classifying contact structures on Seifert manifolds which are transverse to the fibers. Actually, we obtain the complete classification of contact structures with negative (maximal) twisting number (which includes the transverse ones) on Seifert manifolds whose base is not a sphere, as well as partial results in the spherical case.

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Introduction

A geodesible plane field on a manifold is a plane field ξ which is totally geodesic for some Riemannian metric g, which means that every geodesic of g that is somewhere tangent to ξ is everywhere tangent to ξ. If such a plane field is integrable then (locally) its leaves are totally geodesic submanifolds for g and this class of foliations has been much studied. Following a suggestion of É Ghys, we investigate here, by contrast, geodesible contact structures on 3–manifolds. The basic example of a totally geodesic contact structure is the standard contact structure on the sphere which, by definition, is the plane field orthogonal to the Hopf fibers for the canonical Riemannian metric. The goal of this paper is to give an almost complete classification of geodesible contact structures on closed 3–manifolds.

The existence of a geodesible plane field is a strong constraint on the topology of the underlying manifold. In dimension 3 for instance, a theorem due to Y Carrière shows that, up to diffeomorphism, a closed manifold equipped with a coorientable geodesible plane field is either a Seifert manifold with a plane field transverse to its fibers or a torus bundle over the circle which has linear monodromy $A \in SL_2(\mathbb{Z})$ satisfying $\mathrm{tr}(A) > 2$ and is endowed with a plane field transverse to the foliation spanned by an eigendirection of A (which is the strong stable or unstable foliation of the Anosov flow given by A); see Section 1. On torus bundles, contact structures are completely classified by E Giroux and K Honda in [16; 23]. This classification implies that a geodesible contact structure exists if and only if $\mathrm{tr}(A) > 2$ (as for any

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coorientable plane field) and that, in this case, it is unique up to isotopy. On Seifert manifolds, in contrast, the situation is much more delicate: currently there is no general classification of tight contact structures and our main task will be to classify those which are transverse to the fibers—since these are the geodesible ones, up to isotopy. In order to state the results, we define now the twisting number of a contact structure on a Seifert manifold.

Let V be a Seifert 3–manifold and $K \subset V$ a regular fiber, that is, a fiber admitting a trivialized neighborhood $D^2 \times \mathrm{S}^1 \supset \{0\} \times \mathrm{S}^1 = K$ in which all circles $\{\cdot\} \times \mathrm{S}^1$ are also Seifert fibers. Then K has a canonical (homotopy class of) normal framing given by this splitting. Now let ξ be a contact structure on V and L a Legendrian curve smoothly isotopic to K. Given a smooth isotopy φ taking K to L, let $t(L, \varphi)$ be the difference between the contact framing of L and the image by φ of the canonical framing of K. Then define the twisting number of L by $t(L) = \sup_\varphi t(L, \varphi)$ and the twisting number of ξ to be the supremum of the twisting numbers $t(L)$ for all Legendrian curves isotopic to regular fibers. One can prove that overtwisted contact structures have infinite twisting number.

Theorem A (Section 4 and Section 6) *Let V be a closed Seifert 3–manifold. A contact structure ξ on V is isotopic to a transverse one if and only if it is universally tight and has negative twisting number. On the other hand, any transverse contact structure ξ on V is symplectically fillable.*

It turns out that, when it exists, the (unique) geodesible contact structure on a torus bundle is also universally tight and symplectically fillable. Hence geodesibility is (apparently the first example of) a compatibility condition between a Riemannian metric and a contact structure which implies tightness (and symplectic fillability).

Note here that all our Seifert manifolds are oriented with oriented fibers, and that all contact structures we consider are positive for this orientation and (co) oriented unless explicitly stated otherwise.

The next step is to determine which Seifert manifolds admit transverse contact structures. For this we need a concrete description of Seifert manifolds. Start with a compact oriented surface R of genus g with $r+1$ boundary components, take $R \times \mathrm{S}^1$ with the product orientation and attach $r+1$ solid tori $W_0, \dots, W_r$ to its boundary, in order to get a closed manifold. The gluing is prescribed by integers $b, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$ as follows: a meridian disk of W_i is attached to a curve whose homology class is $-\alpha_i[\partial_i R] + \beta_i[\mathrm{S}^1]$, $0 \le i \le r$, where $(\alpha_0, \beta_0) = (1, b)$, $1 \le \beta_i < \alpha_i$ for $i > 0$ and $\partial_i R$ is the i–th boundary component of R with the induced orientation (see Section 1.3). The Seifert invariants of the resulting manifold are, by definition, $(g, b, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$.

Theorem B (Section 5) *Let V be a Seifert manifold with invariants*

$$(g, b, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$$

and let n be a positive integer. The following properties are equivalent:

(i) *V carries a transverse contact structure whose twisting number is at least $-n$;*

(ii) *V carries a contact structure whose twisting number is negative and is at least $-n$;*

(iii) *there exist integers $x_0, \dots, x_r$ such that $\sum x_i = 2-2g$ and $(x_i - 1)/n < \beta_i/\alpha_i$ for $0 \le i \le r$, with $\alpha_0 = 1$ and $\beta_0 = b$.*

Using the equivalence of (i) and (iii), one can recover the following criterion:

Corollary *A Seifert manifold with invariants*

$$(g, b, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$$

carries a transverse contact structure if and only if one of the following conditions holds:

- $-b - r \le 2g - 2$;
- $g = 0$, $r \le 2$ *and* $-b - \sum \beta_i/\alpha_i < 0$;
- $g = 0$, $r \ge 3$, $-b - r = -1$ *and there exist relatively prime integers $0 < a < m$ and a reordering of the pairs (α_i, β_i), $1 \le i \le r$, such that:*

$$\frac{\beta_1}{\alpha_1} > \frac{m-a}{m}, \quad \frac{\beta_2}{\alpha_2} > \frac{a}{m}, \quad \text{and} \quad \frac{\beta_i}{\alpha_i} > \frac{m-1}{m} \ \forall i \ge 3.$$

This corollary was first proved for circle bundles ($r = 0$) by E Giroux in [17] and A Sato and T Tsuboi in [39]. Then the case of general Seifert manifolds was treated by K Honda in [21] (almost) and by P Lisca and G Matić in [29]. The proof in [21] is direct and relies on subtle results on circle diffeomorphisms established in Eisenbud, Hirsch and Neumann [6], Jankins and Neumann [25; 26] and Naimi [34]. The proof in [29] combines the Eliashberg–Thurston perturbation theorem with the more analytical adjunction inequality in symplectic geometry. In the case $g = 0$, $r = 3$ and $b = -2$, a topological proof of Theorem B by P Ghiggini appeared in [11] while this paper was in preparation.

The main tool in our work is a set of normal forms which we now describe. Take a Seifert manifold $V = (R \times S^1) \cup W_0 \cup \dots \cup W_r$ and fix a complex structure J on R

which defines the orientation of R. For each nonsingular 1–form λ on R, the Pfaff equation

$$\cos(n\theta)\,\lambda + \sin(n\theta)\,\lambda \circ J = 0$$

defines a contact structure on $R \times S^1$ and any contact structure ξ on V which extends this one will be denoted by

$$\xi = \xi(\lambda, n, \xi_0, \dots, \xi_r) \qquad \text{where} \quad \xi_i = \xi_{|W_i}, \quad 0 \le i \le r\,.$$

If, in addition, the integer $-n$ is equal to the twisting number $t(\xi)$ then we say that ξ is in normal form.

The starting point of our study is the observation that any contact structure on V with negative twisting number is isotopic to one which is in normal form. In particular, the integers x_i in condition (iii) of the existence criterion are the indices of λ along the boundary components of R. The collection of these indices is called the multi-index of the normal form.

Normal forms and their multi-indices will be our main tools to classify contact structures with negative twisting numbers. The following theorem determines the possible negative twisting numbers for a contact structure on a given Seifert manifold provided the base surface has positive genus.

Theorem C (Section 7) *Let V be a Seifert manifold with invariants*

$$(g, b, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)), \quad g > 0,$$

and let $\xi = \xi(\lambda, n, \xi_0, \dots, \xi_r)$ be a contact structure on V with multi-index $(x_0, \dots, x_r)$. Then ξ is in normal form—that is, $t(\xi) = -n$—if and only if, for $0 \le i \le n$, the following conditions hold:

- *the contact structure $\xi_i = \xi_{|W_i}$ is tight;*
- *$(x_i - 1)/n < \beta_i/\alpha_i$ and the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, (α_i, β_i), $(n, x_i - 1)$ does not contain any integer point whose abscissa is less than n except possibly its vertices—here again, $\alpha_0 = 1$ and $\beta_0 = b$.*

Note that the second condition is automatically fulfilled when $n = 1$ and can be also expressed in arithmetic terms using the continued fraction expansions of the β_i/α_i's; see Section 3.3. When $g = 0$, $t(\xi) = -n$ implies the above conditions but the converse is not known.

To classify contact structures with negative twisting number on a given Seifert manifold, it remains to understand when two contact structures in normal form with the same twisting number are isotopic. Here two cases appear which require different approaches.

Recall that a tight contact structure ξ on a solid torus W whose boundary is ξ–convex has a relative Euler class in $H^2(W, \partial W)$: it is the obstruction to extending inside W a nonsingular vector field on ∂W which is tangent to $\xi_{|\partial W}$—tightness implies the existence of such a vector field.

Theorem D (Section 8) *Let V be a Seifert manifold with invariants*

$$(g, b, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$$

and assume that $-b - r < 2g - 2$.

Every contact structure on V with twisting number -1 has a normal form whose multi-index is $(2 - 2g - r, 1, \dots, 1)$ and has a Stein filling whose underlying smooth manifold depends only on V.

Two contact structures $\xi(\lambda, 1, \xi_0, \dots, \xi_r)$ and $\xi(\lambda', 1, \xi'_0, \dots, \xi'_r)$ in normal form with the above multi-index are isotopic if and only if each ξ_i has the same relative Euler class as ξ'_i.

Among all isotopy classes of contact structures with twisting number -1, exactly two contain transverse contact structures—and only one if we consider nonoriented contact structures.

In the $g = 0$ case, tight contact structures on these manifolds were classified by H Wu in [43] (using a similar strategy to distinguish isotopy classes). The case of circle bundles ($r = 0$) was also previously treated by E Giroux in [17] but without the precise counting of isotopy classes and by K Honda in [23] but with some gap in this counting. To prove here the second part of the theorem, we use a result of P Lisca and G Matić [28]: the contact structures are distinguished by the first Chern classes of their Stein fillings.

In the next theorem, we denote by $\lceil x \rceil$ the smallest integer which is not less than x. We also call R–class of a contact structure in normal form $\xi(\lambda, n, \xi_0, \dots, \xi_r)$ the homotopy class of λ among nonsingular 1–forms on R.

Theorem E (Section 8) *Let V be a Seifert manifold with invariants*

$$(g, b, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)), \qquad g > 0,$$

let n be a positive integer and assume that $-b - r = 2g - 2$ or $n > 1$.

Every contact structure on V with twisting number $-n$ has a normal form and any such normal form has multi-index $(nb, \lceil n\beta_1/\alpha_1 \rceil, \dots, \lceil n\beta_r/\alpha_r \rceil)$.

Two contact structures $\xi(\lambda, n, \xi_0, \dots, \xi_r)$ and $\xi(\lambda', n, \xi'_0, \dots, \xi'_r)$ in normal form are isotopic if and only if they have the same R–class and each ξ_i has the same relative Euler class as ξ'_i.

Among all isotopy classes of contact structures with twisting number $-n$ and a fixed R–class, exactly one if $n\beta_i \equiv 1 \mod \alpha_i$ for all i and two otherwise contain transverse contact structures—and only one in all cases if we consider nonoriented contact structures.

The case of circle bundles ($r = 0$) was treated by E Giroux and K Honda in [17; 23]. On the other hand, in the case $g = 1$ and $r = 1$, tight contact structures on V were classified by P Ghiggini in [13] but without the twisting number computation and without determining which contact structures are universally tight or transverse.

A key step in the proof of the above theorem is to show that, given a contact structure with twisting number $-n$ under our hypotheses, two Legendrian curves which are smoothly isotopic to the regular fibers and have twisting number $-n$ are Legendrian isotopic. An analogous statement was obtained by J Etnyre and K Honda in [10] for Seifert structures on S^3—but the result was a corollary of the classification—and by P Ghiggini in [14] for Seifert structures on T^3.

According to our existence criterion, the two theorems above classify all contact structures with negative twisting number on any Seifert manifold whose base is a surface of positive genus.

Throughout the paper, we assume the reader is familiar with the theory of ξ–convex surfaces developed by E Giroux in [15] but we will briefly recall in Section 2 and Section 3 the results we need about the classification of tight contact structures on toric annuli and solid tori.

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1 Geodesible plane fields

Totally geodesic contact structures were considered by R Lutz and T Hangan for constant curvature Riemannian metrics in [20] and by R Lutz alone with extra relations between a contact structure and a arbitrary metric in [31]. Here by contrast we consider contact structures which are totally geodesic for some arbitrary Riemannian metric with no extra condition. Besides the theory of ξ–convex surfaces—which was not available to Lutz—what makes this generality possible is the relation between geodesible plane fields and a certain class of 1–dimensional foliations. This relation gives the following topological characterization due to Y Carrière [3].

Proposition 1.1 *A coorientable plane field ξ on a closed orientable 3–manifold V is geodesible if and only if the pair (V, ξ) is isomorphic to one of the following:*

- *a Seifert manifold endowed with a plane field transverse to its fibers;*
- *a hyperbolic torus bundle T^3_A with monodromy A—where $A \in SL_2(\mathbb{Z})$ and $\mathrm{tr}(A) > 2$—equipped with a plane field transverse to the foliation spanned by one of the eigendirections of A.*

Given a geodesible plane field ξ this relation also explains how the metrics g such that ξ is totally geodesic for g look like. In the last subsection we define our notation about Seifert manifolds.

1.1 Totally geodesic plane fields and Riemannian foliations

Throughout this section, $\mathfrak{F}$ will be a 1–dimensional foliation on a 3–manifold and F the corresponding line field.

Definition 1.2 A codimension 2 foliation is *Riemannian* if it admits transverse disks equipped with Riemannian metrics such that each leaf meets at least one disk and holonomy maps are isometries.

This definition and the following proposition essentially go back to Reinhart [37]. They can be generalized to arbitrary dimensions and codimensions.

Proposition 1.3 *A plane field on a 3–manifold is geodesible if and only if it is transverse to some 1–dimensional Riemannian foliation.*

Proof First suppose that a plane field ξ on a 3–manifold M is transverse to a Riemannian foliation $\mathfrak{F}$. We equip ξ with the Riemannian metric g pulled back from the transverse disks by holonomy and then extend this metric to TM such that ξ is orthogonal to F and F is given an arbitrary metric. We now show that ξ is totally geodesic for g.

It is sufficient to prove that for every $(x, v) \in \xi$ there exists a small curve γ tangent to ξ, distance minimizing and satisfying $\gamma(0) = x$ and $\gamma'(0) = v$. Let T be a transverse disk intersecting the leaf through x and π the projection from a neighborhood of x to T along the leaves. Let $\overline{\gamma}$ be a small geodesic in T starting at $\pi(x)$ with velocity vector $\pi_* v$ and γ be the unique lift of $\overline{\gamma}$ to M which is tangent to ξ and set $y = \gamma(t)$ for some small t. For every curve α with $\alpha(0) = x$ and $\alpha(t) = y$ we denote by α'_ξ the orthogonal projection of α' onto ξ. The curve γ is not longer than α because

$$\int |\alpha'| \geq \int |\alpha'_\xi| = \int |(\pi \circ \alpha)'| = l(\pi \circ \alpha) \geq l(\overline{\gamma}) = \int |\overline{\gamma}'| = \int |\gamma'|$$

where l is the length function on T and we used that $\pi_*\colon \xi \to TT$ is isometric by construction and $\overline{\gamma}$ is (locally) minimizing.

Conversely, suppose that ξ is totally geodesic for some metric g. We show that $\xi^\perp$ integrates to a Riemannian foliation $\mathfrak{F}$.

We take any system of transverse disks for $\mathfrak{F}$ intersecting all leaves. Each point x on a leaf intersecting a transverse disk T at a point y defines a metric on $T_y T$ by pushing $g|_\xi$ using the infinitesimal holonomy. We have to show that this metric is independent of x on a given leaf. It is sufficient to prove that if x and x' are on the same plaque L then the infinitesimal holonomy is an isometry from ξ_x to $\xi_{x'}$.

The geodesics orthogonal to the plaques are tangent to ξ so all the nearby plaques are contained in tubes around L according to the (generalized) Gauss lemma. This implies that $\mathfrak{F}$ can be locally parametrized by a flow which preserves the tubes and the fibration in disks of a tubular neighborhood of L. This flow then preserves the distance to L and so the infinitesimal holonomy from ξ_x to $\xi_{x'}$ is an isometry. □

1.2 The Carrière classification

If a 1–dimensional foliation is directed by a Killing vector field for some Riemannian metric g then the induced metrics on a suitable system of transverse disks intersecting all leaves are invariant by holonomy so this foliation is Riemannian. Such a foliation is called isometric.

Example 1.4 Any foliation given by a locally free action of S^1 is Riemannian since any Riemannian metric can be averaged to give an invariant metric.

This gives a canonical Riemannian foliation on every Seifert manifold where the leaves are the fibers. Amongst such manifolds, lens spaces (including S^3 and $S^2 \times S^1$) also have deformed versions of the preceding example which are still isometric foliations:

Example 1.5 We view S^3 sitting in $\mathbb{C}^2$ as

$$\mathrm{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,;\, |z_1|^2 + |z_2|^2 = 1\}.$$

For every pair (λ, μ) of real numbers there is a flow $\phi_t(z_1, z_2) = (e^{i\lambda t} z_1, e^{i\mu t} z_2)$ which defines a Riemannian foliation. If λ is a rational multiple of μ then we get an example of the preceding type. The lens space $L_{p,q}$ is the quotient of S^3 by $(z_1, z_2) \mapsto (e^{2i\pi/p} z_1, e^{2i\pi q/p} z_2)$ and ϕ_t clearly induces a Riemannian foliation on the quotient. These foliations are always C^∞ close to a Seifert foliation.

The deformed example on $\mathrm{S}^2 \times \mathrm{S}^1$ is:

Example 1.6 Let f be a rotation of S^2 with irrational angle. The foliation of $\mathrm{S}^2 \times \mathbb{R}$ by $\{\cdot\} \times \mathbb{R}$ induces an isometric foliation of

$$\frac{\mathrm{S}^2 \times \mathbb{R}}{(x,t) \sim (f(x), t+1)} \simeq \mathrm{S}^2 \times \mathrm{S}^1$$

which is $\mathcal{C}^\infty$–close to a Seifert fibration.

The following examples are more exotic since they are not isometric foliations.

Example 1.7 For every matrix $A \in \mathrm{SL}_2(\mathbb{Z})$, if $\mathrm{tr}(A) > 2$ then A has two positive eigenvalues $\lambda, 1/\lambda$. The eigendirection corresponding to λ defines a foliation on $T^2 \times \mathbb{R}$ which induces a 1–dimensional Riemannian foliation on T^3_A; see [3]. Remark that if $\mathrm{tr}(A) < -2$ then one gets a nonorientable foliation so we are not interested in this case because we only consider coorientable contact structures.

Proof of Proposition 1.1 According to Proposition 1.3, a plane field is geodesible if and only if it is transverse to a Riemannian foliation. According to Carrière's classification in [3], any closed oriented 3–manifold equipped with a 1–dimensional Riemannian foliation is diffeomorphic to one of the following:

- the torus T^3 with a linear foliation;
- a lens space with a foliation of Example 1.5;
- the product $\mathrm{S}^2 \times \mathrm{S}^1$ with a foliation of Example 1.6;
- a Seifert manifold with its fibration (see Example 1.4);
- a torus bundle T^3_A, $\mathrm{tr}(A) > 2$, with the foliation of Example 1.7.

Because transversality is an open condition, a plane field is transverse to one of the first three types of foliations if and only if it is transverse to a foliation of the fourth type. □

1.3 Seifert manifolds

We recall the definitions of Seifert manifolds and their invariants to fix conventions for notation and orientations. We then consider two classes of examples: the Seifert structures of S^3 and the bundles of cooriented contact elements of 2–dimensional orbifolds.

Conventions for Seifert invariants vary greatly from papers to papers. For instance, here we use the same one as P Lisca and G Matić in [29] but H Wu uses a different one in [43]. To go from one convention to another it suffices to use the normalization $1 \leq \beta_i < \alpha_i$ and the fact that the rational and integer Euler numbers—denoted by e and e_0 respectively in the following—are the same with any convention. In particular, β_i/α_i in one convention can become $1 - \beta_i/\alpha_i$ in another one but this can be checked using e and e_0. In the notation $M(r_1, \dots, r_k)$ used for instance in [43], $e = \sum r_i$. A useful exercise in conversion is to compare the conventions in the discussion following the proof of Theorem C at the end of Section 7 to the conventions in the cited papers.

Let B be a closed oriented 2–orbifold of genus g with r elliptic points $f_1, \dots, f_r$ of order $\alpha_1, \dots, \alpha_r$ and no other exceptional point. Let f_0 be a regular point of B and $D_0, \dots, D_r$ be pairwise disjoint closed disks such that f_i is in D_i for every i. Denote by R the smooth surface $B \setminus \bigcup \mathring{D}_i$. In the following, boundary components of R will always be oriented as boundary components of the D_i's.

Let V' be $R \times S^1$, fix an orientation for the S^1 factor and use the product orientation on V'. The first homology group of the boundary components of V' have a basis (S_i, F_i) where S_i is the homology class of $\partial D_i \times \{\cdot\}$ and F_i is the homology class of $\{\cdot\} \times S^1$ (S stands for section and F for fiber). We orient these boundary components by imposing the intersection number $S_i \cdot F_i = +1$.

The Seifert manifold V with invariants $(g, b, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$ is, by definition, the manifold obtained by filling each boundary torus of V' with a solid torus W_i which has a meridian whose homology class is $M_i = \alpha_i S_i + \beta_i F_i$ with $(\alpha_0, \beta_0) = (1, b)$ and $0 < \beta_i < \alpha_i$ for $i > 0$.

Each solid torus W_i is identified with $W(\alpha_i, \beta_i)$ where $W(\alpha, \beta)$ is the quotient of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times [0, 1]$ by the relation $(x, y, t) \sim (x', y', t')$ if $\alpha(y' - y) = \beta(x' - x)$ and $t = t' = 0$. The solid torus $W(\alpha, \beta)$ is foliated by the images of the circles $\{\cdot\} \times \mathbb{R}/\mathbb{Z} \times \{\cdot\}$. This foliation is called the standard Seifert fibration on $W(\alpha, \beta)$.

These foliations inside the W_i's extend the foliation of V' by S^1 to a foliation of V such that B is the orbit space. The leaves of this foliation are called the fibers of V. Smooth points of B are called regular fibers whereas elliptic points are called exceptional fibers.

A closed subset of a Seifert manifold is said to be vertical if it is isotopic to a union of regular fibers.

The rational Euler number of V is $e(V) = -\sum_{i=0}^{r} \beta_i/\alpha_i$ and the integer Euler number is $e_0(V) = -b - r$.

Example 1.8 (Seifert structures on S^3) Let (L_0, L_1) be two fibers of the Hopf fibration. One can see S^3 as a quotient of $T^2 \times [0,1]$ such that the projection map p is a diffeomorphism from $T^2 \times (0,1)$ to $S^3 \setminus (L_0 \cup L_1)$ and collapses T_0 to L_0 and T_1 to L_1—where we set $T_t = T^2 \times \{t\}$. For each point $x \in L_0$ (resp. $x \in L_1$) $p^{-1}(x)$ is a circle $\{pt\} \times \mathbb{R}/\mathbb{Z} \subset T_0$ (resp. $\mathbb{R}/\mathbb{Z} \times \{pt\} \subset T_1$). Any foliation of $T^2 \times (0,1)$ by circles not homologous to the above circles project to a foliation of $S^3 \setminus (L_0 \cup L_1)$ which, together with L_0 and L_1 gives a Seifert structure on S^3. These are the only Seifert structure on S^3; see eg Scott [41].

Example 1.9 (Cooriented contact elements bundles) Seifert manifolds are the total spaces of circle bundles over 2–orbifolds in the sense of Thurston. One type of such bundles is especially interesting for us, the cooriented contact elements bundle ST^*B of an orbifold B where we orient the fibers unlike the natural orientation. It has Seifert invariants

$$(g, 2-2g-r, (\alpha_1, 1), \dots, (\alpha_r, 1))$$

and rational Euler number $e(ST^*B) = -\chi(B)$ where the orbifold Euler number of B is

$$\chi(B) = \chi(R) + \sum_{i=0}^{r} \frac{1}{\alpha_i} = 2 - 2g - r + \sum_{i=1}^{r} \frac{1}{\alpha_i}.$$

As in the special case of smooth surfaces, ST^*B carries a canonical (positive) contact structure denoted by ξ_B.

2 Contact structures on toric annuli

Here we recall some facts about tight contact structures on toric annuli which are needed in this text. All the results here are contained in Giroux [16] or follow directly from results therein (see also Honda [22] for a different approach to these questions). The main source of variations lies in boundary conditions since those most frequently used in [16] are not convenient for our purposes.

In the following, each time we consider a torus bundle, the fiber of a point t will be denoted by T_t and T will denote a torus.

A suspension σ on a torus T—that is, a nonsingular foliation admitting a simple closed curve intersecting all leaves transversely—determines a line $D(\sigma)$ in $H_1(T,\mathbb{R})$ called the asymptotic direction of σ and defined in [40]: the real cycles $[L_t]/t$ where L_t is an orbit of length t closed by a minimizing geodesic segment have a common limit up to sign in $H_1(T,\mathbb{R})$ when t goes to infinity and the line $D(\sigma)$ spanned by this limit is independent of the starting point and Riemannian metric. If σ is linear then $D(\sigma)$ is the direction of σ and if σ has a periodic orbit then $D(\sigma)$ is generated by its homology class for any orientation. We say that $D(\sigma)$ is rational if it contains a nonzero point of $H_1(T,\mathbb{Z})$.

If ξ is a (positive) contact structure on $T\times I$—where I is an interval—such that each ξT_t is a suspension then the function

$$\begin{pmatrix} I & \to & P(H_1(T,\mathbb{R}))\simeq \mathbb{R}/\pi\mathbb{Z} \\ t & \mapsto & D(\xi T_t) \end{pmatrix}$$

is nonincreasing for the orientation of $H_1(T,\mathbb{R})$ inherited from the orientation of T via the intersection form.

Remark 2.1 If there exists t such that $D(\xi T_t)$ is not rational or ξT_t is conjugated to the suspension of a rotation then $t\mapsto D(\xi T_t)$ is nonconstant.

When a ξ–convex torus T in a contact manifold has a Legendrian fibration over the circle then we say that this fibration is a ruling of T or that T is ruled by this fibration. In this case ξT has smooth singularity circles and $D(\xi T)$ is defined to be the line in $H_1(T)$ spanned by the homology class of a singularity circle with any orientation.

The characteristic foliation of a torus is said to be admissible if it is linear or ruled with two circles of singularities or a suspension divided by two curves.

Tight contact structures on toric annuli, solid tori, lens spaces and torus bundles with admissible boundary are made of contact structures printing suspensions on each torus and orbit flips which we now define (an explicit model is given in [16, Section 1.F]).

Definition 2.2 A contact structure ξ on $T\times[0,1]$ has an orbit flip with homology class $\pm d$ if all the ξT_t are divided by $2k$ circles with total homology class $\pm 2d$ when they are all given the same orientation, ξT_t is a suspension with $2k$ periodic orbits for every $t\neq 1/2$, $\xi T_{1/2}$ is ruled and the periodic orbits of ξT_0 and ξT_1 have opposite orientations.

Note that, in general, the homology class of an orbit flip is defined up to sign but in our study of contact structures on Seifert manifolds one of the two possible classes will intersect the fiber class positively and we will call it *the* homology class of the orbit flip.

Definition 2.3 A contact structure on $T \times [0,1]$ is in normal form with flip locus $\{\pm d_1, \dots, \pm d_n\}$ if ξT_0 and ξT_1 are admissible foliations and if there exists a sequence $t_1 < \dots < t_n \in (0,1)$ such that:

- $\xi_{|T\times(0,t_1)}$ and $\xi_{|T\times(t_n,1)}$ print a suspension on each T_t;
- for every i, $\xi_{|T\times(t_i,t_{i+1})}$ prints a suspension on each T_t with nonconstant asymptotic direction;
- for every i there exists a neighborhood J of t_i such that $\xi_{|T\times J}$ has an orbit flip with homology class $\pm d_i$.

The following definitions are useful to describe the links between tight contact structures on toric annuli and the geometry of $H_1(T,\mathbb{R})$, its integral lattice $H_1(T,\mathbb{Z})$ and the intersection form on them.

Definition 2.4 Let σ_0 and σ_1 be admissible foliations on T. The Giroux cone $\mathcal{C}(\sigma_0,\sigma_1)$ of (σ_0,σ_1) is the cone without vertex bordered on the left by $D(\sigma_0)$ and on the right by $D(\sigma_1)$. The boundary lines—deprived of 0—are in the cone if and only if the corresponding foliations have a dividing set.

Let E be in each connected component of $\mathcal{C}$ the convex hull of $\mathcal{C}(\sigma_0,\sigma_1) \cap H_1(T,\mathbb{Z})$. An edge of a part of a lattice is a maximal subset of aligned points. The Giroux polygon $\mathcal{P}(\sigma_0,\sigma_1)$ of (σ_0,σ_1) is the set of integral homology classes which belong to a finite length edge of ∂E—or is the intersection of two infinite edges—ordered by the intersection form from right to left. We denote by $\partial\mathcal{P}$ the (possibly empty) set of its extremal points.

If ξ is a contact structure on $T\times[0,1]$ we also define $\mathcal{C}(\xi) = \mathcal{C}(\xi T_0, \xi T_1)$ and similarly for $\mathcal{P}$. See Figure 1 for an example where σ_0 has no dividing set.

These cones naturally have two connected components and everything is symmetric with respect to the origin but in the context of Seifert manifolds we will always consider the component made of homology classes intersecting the fiber class positively and call *this component* the Giroux cone.

The following lemma is a special case of [16, Lemma 3.34] and can also be proved by exhibiting an annulus transverse to the T_t's and not satisfying the Giroux criterion [17, Theorem 4.5 a)]. We call a element of $H_1(T,\mathbb{Z})$ simple if it has a simple curve representative or, equivalently, if it is not a multiple of an other homology class.

Lemma 2.5 *If a contact structure on a toric annulus has an orbit flip whose homology class is nonsimple and not on the boundary of the corresponding Giroux cone then it is overtwisted.*

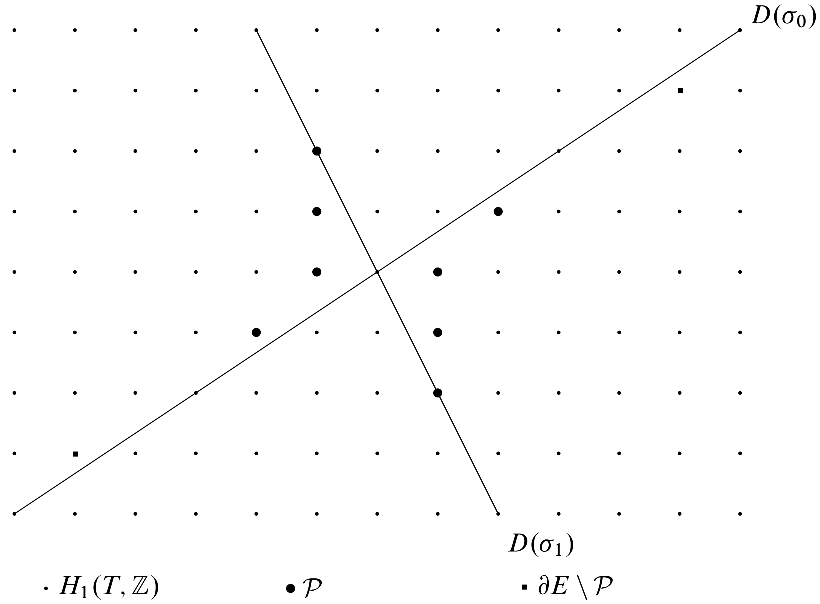

Figure 1: A Giroux cone and its polygon

For all contact structures printing admissible foliations on the boundary there is a well defined relative Euler class once σ_0 is oriented since such an orientation fixes a (co)orientation for any (positive) contact structure printing σ_0 on T_0. We denote by $\mathrm{Tight}_0(\sigma_0,\sigma_1)$ the set of tight contact structures with zero torsion printing σ_0 and σ_1 on the boundary components. In the following, a ∂–isotopy is an isotopy relative to the boundary.

Theorem 2.6 (E Giroux [16]) *Let σ_0 and σ_1 be admissible foliations on T with σ_0 oriented and $\mathcal{P} := \mathcal{P}(\sigma_0,\sigma_1)$.*

a) *Every contact structure in $\mathrm{Tight}_0(\sigma_0,\sigma_1)$ is ∂–isotopic to a contact structure in normal form whose flip locus is in $\mathcal{P}$.*
b) *A contact structure in normal form on $T\times[0,1]$ which prints σ_0 on T_0 and σ_1 on T_1 is tight if and only if its flip locus is contained in $\mathcal{P}$. It is universally tight if and only if its flip locus is contained in $\partial\mathcal{P}$.*
c) *If $D(\sigma_0)\neq D(\sigma_1)$ then two contact structures in $\mathrm{Tight}_0(\sigma_0,\sigma_1)$ are ∂–isotopic if and only if they have the same relative Euler class.*
d) *If two contact structures on $T\times[0,1]$ print on each T_t suspensions whose asymptotic directions do not cover the whole projective line and if they coincide on the boundary then they are ∂–isotopic.*

e) *Any tight contact structure with* ξ*–convex boundary on* $T\times[0,1]$ *is isotopic to a contact structure with* $T_{1/3}$ *(resp.* $T_{2/3}$*) divided by two curves parallel to the dividing set of* T_0 *(resp.* T_1*).*

In the preceding theorem, the correspondence with [16] is: a) and b) are in Proposition 1.8, c) is in Theorem 1.5, d) is Theorem 3.3 and e) follows from Proposition 3.22.

If a contact structure on $T\times[0,1]$ is in normal form on $T\times[0,1/2]$ and $T\times[1/2,1]$ with flip loci R and R' then the flexibility lemma and Theorem 2.6 d) can be used to prove that ξ has a normal form with flip locus $(R\cup R')\setminus(R\cap R')$ so Theorem 2.6 b) can be used to get gluing results. This strategy is used in Section 3.2.

Finite covers We now turn to finite covering maps between toric annuli. If ρ is such a map and d is a homology class represented by a finite collection C of embedded oriented circles in T then $\rho^{-1}(C)$ is a finite collection of oriented embedded circles whose total homology class will be denoted by $\rho^* d$.

If ξ is a contact structure in normal form on $T\times[0,1]$ with flip locus $\{\pm d_1,\dots,\pm d_n\}$ and ρ is a finite covering map preserving the product structure of $T\times[0,1]$ then $\rho^*\xi$ has orbit flips with homology classes $\pm\rho^* d_1,\dots,\pm\rho^* d_n$ so we can use Lemma 2.5 to detect overtwisted covers.

This fact is used in Section 6 but we can already give a general corollary: using Theorem 2.6 and Lemma 2.5 we answer the question of [23, page 97]. Given a virtually overtwisted contact structure ξ on $T^2\times[0,1]$ with each boundary component divided by two curves, Honda asks which covering spaces $\mathbb{R}^2/(m\mathbb{Z}\times n\mathbb{Z})\times[0,1]$ are overtwisted. Denote by $\xi(m,n)$ the lifted contact structure. We now explain how to compute n_0, m_0 such that $\xi(m,n)$ is overtwisted whenever $n\geq n_0$ and $m\geq m_0$. This will leave a finite number of $\xi(m,n)$ which have to be analyzed directly, using Theorem 2.6 b).

The crucial point is that $H_1(T^2)$ has a preferred basis and a Euclidean structure in addition to its lattice and intersection form. According to Theorem 2.6 a) and b), ξ is isotopic to a contact structure in normal form whose flip locus contains a class $d\in\mathcal{P}\setminus\partial\mathcal{P}$. Let L_h and L_v be affine lines containing d and directed by the two basis vectors.

Here we assume that $\partial\mathcal{C}$ has no component parallel to the axes (the special case we neglect can be dealt with using the same methods). One can then check that there exist $a\in\partial\mathcal{C}\cap L_h$ and $b\in\partial\mathcal{C}\cap L_v$ such that the line (ab) divides $H_1(T^2,\mathbb{R})$ into two open half-planes, one containing 0 and the other one containing d. Let l_h (resp. l_v) be the distance between d and a (resp. b) and set $n_0=\lfloor 1/l_h\rfloor+1$ and $m_0=\lfloor 1/l_v\rfloor+1$. If $n\geq n_0$ and $m\geq m_0$ then the lifted Giroux cone contains integer points $a'\in\rho^*[ad]$

and $b' \in \rho^*[bd]$ proving that $\rho^* d$ is neither on the Giroux polygon of $\xi(m,n)$ nor on the boundary of its Giroux cone so that $\xi(m,n)$ is overtwisted according to Lemma 2.5.

3 Contact structures on solid tori

In this section we explain how the results of the previous section extend to results on solid tori (here again, everything comes from [16]) and gather some results about contact structures with negative twisting number on solid tori which directly use the classification of tight contact structures.

3.1 Classification results

Let W be an oriented solid torus with a meridian class $M \in H(\partial W)$, σ be an oriented admissible foliation on ∂W and denote by $\mathrm{Tight}(\sigma)$ the set of tight contact structures on W printing σ on the boundary. We see W as $\mathrm{S}^1 \cup (T \times (0,1])$ which is a quotient of $T \times [0,1]$. A contact structure on W which is transverse to S^1 can be lifted to $T \times [0,1]$ and is said to be in normal form if this lift $\tilde{\xi}$ is in normal form. Using this construction $\tilde{\xi}$ prints on T_0 a linear foliation σ_M whose direction is spanned by M in $H_1(\partial W)$. The flip locus of ξ is then defined as the flip locus of $\tilde{\xi}$. We denote by L_M the set of integer homology classes in $\mathcal{C}$ whose intersection with M is ± 1.

Theorem 3.1 (E Giroux [16, Lemma 4.2]) *Let W be a solid torus, σ an oriented admissible foliation on ∂W, σ_M a linear foliation coming from meridian disks and $\mathcal{P}$ the Giroux polygon $\mathcal{P}(\sigma_M, \sigma)$.*

a) *Every $\xi \in \mathrm{Tight}(\sigma)$ is ∂–isotopic to a contact structure in normal form whose flip locus is in $\mathcal{P}$.*
b) *A contact structure ξ in normal form on W and printing σ on the boundary is tight if and only if its flip locus is contained in $\mathcal{P} \cup L_M$. It is universally tight if and only if its flip locus is contained in $\partial\mathcal{P} \cup L_M$.*
c) *Two contact structures in $\mathrm{Tight}(\sigma)$ are ∂–isotopic if and only if they have the same relative Euler class.*

Corollary 3.2 *Let W be a solid torus, σ an oriented admissible foliation on ∂W and $\xi \in \mathrm{Tight}(\sigma)$. Denote by σ_M a linear foliation on ∂W spanned by meridian disks. There exists a boundary-parallel torus divided by two curves with homology class $d \in H_1(\partial W, \mathbb{Z})$ if and only of d is a simple class in $\mathcal{C}(\sigma_M, \sigma)$.*

When σ is an admissible foliation with dividing curves, there is only a finite number of ∂–isotopy classes of tight contact structures in $\mathrm{Tight}(\sigma)$ which can be computed using the above theorem.

Corollary 3.3 *With the same notation as above, if σ is a ξ–convex admissible foliation then $\mathcal{P}$ has a finite number of edges $e_1, \dots, e_k$ and*

$$\mathrm{Card}\big(\mathrm{Tight}(\sigma)\big) = \prod_{i=1}^{k} \mathrm{Card}(e_i).$$

For example, in Figure 1 with $\sigma_0 = \sigma_M$ and $\sigma_1 = \sigma$, we have six tight contact structures. Note that the $\mathrm{Card}(e_i)$ in the preceding corollary can be computed using the continued fraction expansion of the slope of $D(\sigma)$ in a suitable basis of $H_1(T, \mathbb{R})$; see [16, Section 1.G].

We now apply this classification to Seifert fibered solid tori:

Lemma 3.4 *Let $W = W(\alpha, \beta)$ be a solid torus with a standard Seifert fibration (see Section 1.3) and denote by M a meridian class in $H_1(\partial W)$. Let ξ be a contact structure on W such that $\xi\partial W$ is ruled by vertical curves with two dividing circles. Let d be the homology class of these circles oriented such that $d \cdot M > 0$.*

Such a ξ is universally tight if and only if it is ∂–isotopic to a transverse contact structure in the interior of W.

If $d \cdot M = 1$ then there is only one ∂–isotopy class of tight contact structures which coincide with ξ on ∂W. This class contains universally tight contact structures tangent to the fibers as well as contact structures positively and negatively transverse (in the interior of W).

If $d \cdot M > 1$ then there are exactly two ∂–isotopy classes of universally tight contact structures which coincide with ξ on ∂W. They contain contact structures either positively or negatively transverse but not both and no tangent contact structure.

Proof A contact structure in normal form is isotopic through contact structures in normal form to a transverse contact structure if and only if its flip locus is empty. This potentially leaves two isotopy classes of contact structures ξ' depending on the orientation of the suspensions $\xi' T_t$ near the boundary. Theorem 3.1 c) and a Euler class computation give one or two isotopy classes depending on the intersection number $d \cdot M$ as announced.

Since any tangent contact structure can be perturbed into a positively or negatively transverse contact structure in the interior of W, it remains only to prove that, when $d \cdot M = 1$, there is a tangent contact structure which coincides with ξ on ∂W.

Let (S, F) be a base of $H_1(\partial W)$ such that F is the fiber class and $M = \alpha S + \beta F$. Because $d \cdot M = 1$, we can write d as $nS + qF$ with $n\beta - q\alpha = 1$ so that $W(\alpha, \beta)$ can be seen as

$$\big(D^2 \times \mathbb{R}\big) \big/ \big((z, t) \sim (e^{2i\pi n/\alpha} z, t + 1)\big)$$

because the meridian in the above model has intersection α and $-\beta$ respectively with the fibers induced by the $\mathbb{R}$ factor and the section induced by $s \mapsto \big(e^{-2i\pi qs}, -\beta s\big)$. The map

$$\begin{pmatrix} W(\alpha,\beta) & \to & W(\alpha,1) \simeq ST^*D \\ (z,t) & \mapsto & (z, nt) \end{pmatrix}$$

where D is the base of W is a n–fold fibered covering map. Seen as a covering map from W to ST^*D, it can be chosen to extend the one over ∂D associated to ξ by (the oriented version of) [17, Proposition 3.3]—which is stated in a slightly generalized form as Proposition 8.9 below. The pullback of the canonical contact structure of ST^*D by this covering map coincides with ξ on ∂W and is tangent to the fibers. □

3.2 A gluing lemma

Lemma 3.5 *Let $W(\alpha, \beta) = T^2 \times [0, 1]/\sim$ be a solid torus with a standard Seifert fibration and ξ a contact structure on W such that $T_{1/2}$ and T_1 are divided by two curves intersecting each fiber only once and the fiber class is not in the Giroux cone of ξ. If $\xi_{|T^2\times[0,1/2]/\sim}$ and $\xi_{|T^2\times[1/2,1]/\sim}$ are tight then ξ is tight.*

Proof We set $\xi_1 = \xi_{|T^2\times[0,1/2]/\sim}$, $\xi_2 = \xi_{|T^2\times[1/2,1]/\sim}$ and we denote by $\mathcal{P}$, $\mathcal{P}_1$ and $\mathcal{P}_2$ the Giroux polygons associated to ξ, ξ_1 and ξ_2.

According to Theorems 3.1 a) and 2.6 a), we can assume that ξ_1 and ξ_2 are in normal form and that their flip loci are contained in $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. The crucial fact we have to prove is that $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ because of the assumption on the dividing curves; see Figure 2. This will implies that ξ has a normal form whose flip locus is in $\mathcal{P}$ and so is tight according to Theorem 3.1 b).

Denote by F and M the fiber and meridian classes oriented such that $M \cdot F > 0$ and denote by d and d' the homology classes of each dividing curve of $T_{1/2}$ and T_1 respectively, oriented such that their intersection with F is $+1$. We now concentrate without further notice on the connected component of the Giroux cone of ξ which is contained in the half-plane $\mathcal{H} = \{h \cdot F > 0\}$ and the polygons inside it. Because

$d \cdot F = d' \cdot F = 1$ and $d' \cdot d \geq 0$, there exists $k \geq 0$ such that $d = d' + kF$ and, for every $x \in \mathcal{H}$, the signed area of the triangle $d'xd$ is

$$(x - d') \cdot (d - d') = (x - d') \cdot kF = k(x \cdot F - 1) \geq 0.$$

This proves that $\mathcal{P}_2 = [d', d]$ and that $[d', d] \subset \mathcal{P}$.

In particular, d is in $\mathcal{P}$ and this is exactly what is needed in order to get $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Indeed, if we denote by E, E_1 and E_2 the convex hulls of the integer points of the corresponding cones, what we want is $E = E_1 \cup E_2$ so we need to prove that $E' := E_1 \cup E_2$ is already convex. Denote by Δ the half-line from 0 containing d. Since E_1 and E_2 are convex, we only have to check that, for every $x \in E_1$ and $y \in E_2$, the point $z := [x, y] \cap \Delta$ is in E'. Since d is in $\mathcal{P}$, $E \cap \Delta = E' \cap \Delta$ and, by convexity of E, $z \in E$ so we are done. □

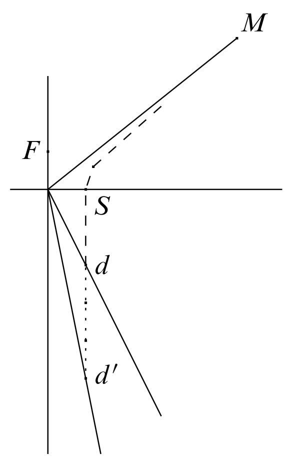

Figure 2: Giroux polygons for the gluing lemma

3.3 Local twisting number calculation

We first recall some arithmetic. If x is a real number and $a > 0$ and b are relatively prime integers then we say that b/a is a best lower approximation for x if it is maximum among rational numbers smaller than x whose denominator is not larger than a. The best lower approximations of x can be read from its continued fraction expansion:

$$x = [a_0; a_1, \dots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}}$$

for a unique sequence of integers a_i such that $a_i > 0$ for $i > 0$ and the last nonzero a_i, $i > 0$ (if it exists), is greater than 1.

The convergent of order k of x is $[a_0; a_1, \dots, a_k]$ and the intermediate fractions of order k, $k \geq 2$ of x are the $[a_0, a_1, \dots, a_{k-1}, a]$ with $1 \leq a < a_k$.

The best lower approximations of x are exactly the convergents and intermediate fractions of even order of x with the possible exception of x itself.

Proposition 3.6 *Let W be a solid torus with a standard Seifert fibration. Suppose (S, F) is a basis of $H_1(\partial W, \mathbb{Z})$ such that F is the homology class of the fiber and the meridian circles in ∂W—oriented such that they intersect F positively—have homology class $M = \alpha S + \beta F$ with $1 \leq \beta < \alpha$ if $\alpha > 1$.*

If ξ is a tight contact structure on W such that ∂W is divided by curves which can be oriented to have total homology class $d = 2nS + 2(x-1)F$, $n > 0$ then the following statements are equivalent:

- $t(\xi) = -n$;
- $(x-1)/n < \beta/\alpha$ *and the triangle in* $\mathbb{R}^2$ *with vertices* $(0,0)$, (α, β), $(n, x-1)$ *does not contain any integer point whose abscissa is less than* n *except possibly its vertices—in particular* $d/2$ *is simple;*
- *either* $n = 1$ *and* $x - 1 < \beta/\alpha$ *or the following conditions hold:*
  (i) $x = \lceil n\beta/\alpha \rceil$;
  (ii) $n\beta \equiv 1 \mod \alpha$ *or* $(x-1)/n$ *is a best lower approximation of* β/α.

Proof Let $\mathcal{C}$ denote the Giroux cone of ξ in $H_1(\partial W, \mathbb{R})$.

According to Theorem 2.6 e), Corollary 3.2 and the flexibility lemma, $t(\xi) = -n$ if and only if $\mathcal{C}$ does not contain any point $kS + lF$ with $k < n$—in particular $d/2$ is simple. This proves the equivalence of the first two points. In addition, the second part of the second point is clearly always satisfied when $n = 1$ so we now suppose that $n > 1$ and we explain the equivalence of the second and third conditions.

Suppose that the second condition is satisfied. Set $q = \lceil n\beta/\alpha \rceil - 1$ so that q is the greatest integer which is smaller than $n\beta/\alpha$.

Recall that, according to Pick's formula [36] (see eg Aigner and Ziegler [1] for a proof in English), any polygon with integer vertices in $\mathbb{R}^2$ has area

$$n_{\text{int}} + \frac{1}{2} n_{\text{bd}} - 1$$

where n_{int} is the number of interior integer points and n_{bd} is the number of boundary integer points.

The point with abscissa n which is the closest to the left boundary of $\mathcal{C}$ is (n,q) by definition. Also the triangle with vertices $(0,0)$, (n,q) and $(n,q-1)$ has area $n/2 > 1/2$ so it contains integer points in addition to its vertices by Pick's formula. Such points necessarily have abscissa less than n so they cannot be in $\mathcal{C}$. This implies that $\mathcal{C}$ cannot contain $(n,q-1)$; see the left-hand side of Figure 3. We conclude that the right boundary of $\mathcal{C}$ is generated by (n,q).

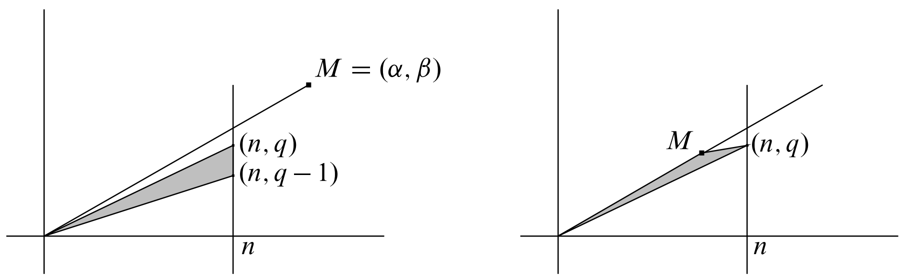

Figure 3: Computing the multi-index using Pick's formula

If $n \geq \alpha$ then the triangle with vertices $(0,0)$, (α,β) and (n,q) is inside $\mathcal{C}$ except for the edge from $(0,0)$ to (α,β) which does not contain any integer point but its vertices; see the right-hand side of Figure 3. The area of this triangle is $(n\beta - q\alpha)/2$ so that according to Pick's formula we have $n\beta - q\alpha = 1$ so $n\beta \equiv 1 \mod \alpha$.

In particular we get that if $\alpha > 1$ then $n \neq \alpha$ and $\alpha \nmid n\beta$ so $q = \lfloor n\beta/\alpha \rfloor$ and $x = \lceil n\beta/\alpha \rceil$.

If $n < \alpha$ then the condition above on integer points in $\mathcal{C}$ is exactly the best lower approximation condition.

The fact that the third condition implies the second one is analogous. □

4 Characterization of transverse contact structures

In this section we describe geodesible contact structures on torus bundles and we prove Theorem A from the introduction except for the statement that universally tight contact structures with negative twisting number on Seifert manifolds are isotopic to transverse ones which is deferred to Section 6.

4.1 Torus bundles

Proposition 4.1 *Let A be a matrix in $\mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{tr}(A) > 2$.*

A contact structure on T^3_A is isotopic to one which is transverse to the foliation of Example 1.7 if and only if it is universally tight and has zero torsion. There is exactly one isotopy class of such contact structures on T^3_A.

Every geodesible contact structure on T^3_A is symplectically fillable.

This classification is a direct consequence of the previously known classification results contained in [16] (see also [22]) which we will recall. The fillability comes from the classification and a theorem by F Ding and H Geiges.

For any function $\theta\colon \mathbb{R} \to \mathbb{R}$, the Pfaff equation

$$\cos\theta(t) dx_1 + \sin\theta(t) dx_2 = 0$$

defines a contact structure on T^3_A if and only if $\theta' < 0$ and

$$A^*(\cos\theta(t+1)dx_1 + \sin\theta(t+1)dx_2) \wedge (\cos\theta(t)dx_1 + \sin\theta(t)dx_2) = 0.$$

This contact structure is denoted by $\zeta(\theta)$. The following proposition explains the special role of these $\zeta(\theta)$. It is proved in [16, Corollary 3.8] for the first part and the second part is obtained from the proof of [16, Lemma 3.2].

Proposition 4.2 *Let A be a matrix in $\mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{tr}(A) > 2$. On T^3_A, any contact structure whose lift $\widetilde{\xi}$ to $T^2 \times \mathbb{R}$ prints a suspension on each T_t is isotopic to some $\zeta(\theta)$.*

If $\bigcup_{t\in[0,1]} D(\widetilde{\xi}T_t)$ is not the whole projective line then $\theta(1) - \theta(0) > -2\pi$.

The theory of normal forms explained in Section 2 can also be used to study tight contact structures on torus bundles over the circle. In this text we only need to consider the case of universally tight contact structures on torus bundles with hyperbolic monodromies. The following theorem comes from [16, Theorem 1.3] and its proof in Section 4.D.

Theorem 4.3 *Let A be a matrix in $\mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{tr}(A) > 2$.*

a) *A contact structure on T^3_A is universally tight if and only if it is isotopic to some $\zeta(\theta)$.*
b) *If ξ is conjugated to $\zeta(\theta)$ then $\mathrm{Tor}(\xi) = \max\left(\mathbb{N} \cap \left[0, \frac{\theta(0)-\theta(1)}{2\pi}\right)\right)$.*
c) *Two universally tight contact structures on T^3_A are isotopic if and only if they have the same torsion.*

Proof of Proposition 4.1 Let ξ be a contact structure transverse to the foliation $\mathfrak{F}_A$ of Example 1.7. Because each T_t is foliated by $\mathfrak{F}_A$, each ξT_t is a suspension and the same is true for the lift $\widetilde{\xi}$ of ξ to $T^2 \times \mathbb{R}$.

Also $\bigcup_{t\in[0,1]} D(\widetilde{\xi} T_t)$ is not the whole projective line because $\widetilde{\xi}$ is transverse to a fixed direction. According to Proposition 4.2, ξ is isotopic to some $\zeta(\theta)$ with $\theta(1) - \theta(0) > -2\pi$ so it is universally tight and has zero torsion according to Theorem 4.3.

Conversely one can construct a function θ such that $\zeta(\theta)$ is transverse to $\mathfrak{F}_A$ and all universally tight contact structures with zero torsion on T^3_A are isotopic to this $\zeta(\theta)$ according to Theorem 4.3.

The symplectic filling is constructed in [5, Theorem 1]. □

4.2 Symplectic fillings for transverse contact structures

Given any Seifert manifold V there is a symplectic manifold (W, ω) such that $\partial W = V$ (as oriented manifolds) and $\ker \omega$ on V is tangent to the fibers. This symplectic manifold is a (weak convex) filling of (V, ξ) for any transverse contact structure ξ on V.

The existence of (W, ω) can be deduced from [32, Theorem 2.1]—as observed in [29]—or, more elementarily, from the main theorem of [35]. Indeed, if (B, ω_B) is a symplectic orbifold with isolated cyclic singularities and $V \to B$ is a circle bundle over B (in the sense of Thurston [42]) then one can consider the associated disk bundle $D \to B$ where each fiber is equipped with (a quotient of) the symplectic form $d(\frac{1}{2} r^2 d\theta)$. Following the construction explained eg in [33][Theorem 6.3] we get a symplectic orbifold which can be resolved using [35] to get the desired filling. If B is a 2–orbifold like in our case then the resolution is completely explicit.

4.3 Universal tightness

Proposition 4.4 *If (V, ξ) is a Seifert manifold with a transverse contact structure then its universal cover $(\widetilde{V}, \widetilde{\xi})$ is $\mathbb{R}^3$ or S^3 with its standard tight contact structure.*

Proof The universal cover $\widetilde{V}$ of V is either $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ or $S^2 \times \mathbb{R}$ or S^3 where the Seifert structure lifts to the foliation by $\mathbb{R}$ in the first two cases and S^3 can have any of its Seifert structures (see Example 1.8).

Here $\widetilde{V}$ cannot be $S^2 \times \mathbb{R}$ since V would be covered by $S^2 \times S^1$ which has no transverse contact structure. This fact is contained in Theorem B but there is a direct argument.

Indeed, $S^2 \times S^1$ is the quotient of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times [0,1]$ by $(x,y,t) \sim (x',y,t)$ when $t = 0$ or 1 which is foliated by $\{\cdot\} \times \mathbb{R}/\mathbb{Z} \times \{\cdot\}$. If ξ is a transverse contact structure then it can be lifted to a contact structure $\widetilde{\xi}$ which prints a suspension on each torus T_t. The foliations $\widetilde{\xi}T_0$ and $\widetilde{\xi}T_1$ are conjugated to the suspension of the same rotation so the asymptotic direction of $\widetilde{\xi}T_t$ in $H_1(T^2, \mathbb{R}) = \mathbb{R} \times \mathbb{R}$ is not constant according to Remark 2.1 and goes from $\mathbb{R} \times \{0\}$ to itself. This contradicts the fact that it never is $\{0\} \times \mathbb{R}$ which is the fiber direction.

If $\widetilde{V}$ is $\mathbb{R}^3$ then (V, ξ) is covered by $\mathbb{R}^2 \times S^1$ equipped with a contact structure transverse to the S^1 factor. So we can decompose $\mathbb{R}^3$ as $\mathbb{R}^2 \times \mathbb{R}$ so that the lifted contact structure $\widetilde{\xi}$ is transverse to the $\mathbb{R}$ factor and invariant under integral translations in this direction. According to [17, Section 2.B.c], $\widetilde{\xi}$ is tight (this is proved using explicit contact embedding of any ball inside the standard $\mathbb{R}^3$ where the Bennequin theorem [2] is available). According to a theorem of Y Eliashberg [8], $\widetilde{\xi}$ is the standard contact structure on $\mathbb{R}^3$.

Suppose now that $\widetilde{V}$ is S^3. We have to prove that any transverse contact structure ξ on S^3 is isotopic to the standard contact structure on S^3, which is tight according to the Bennequin theorem.

Let $p\colon T^2 \times [0,1] \to S^3$ be the projection introduced in Example 1.8 and $\widehat{\xi}$ the lifted contact structure on $T^2 \times [0,1]$. Up to an isotopy of ξ among transverse contact structures, we can assume that ξ coincides with the standard contact structure ξ_0 of S^3 in a neighborhood of $L_0 \cup L_1$. According to Theorem 2.6 d) applied to $\widehat{\xi}$ and $p^*\xi_0$, ξ is isotopic to the standard contact structure of S^3. □

4.4 Twisting number

Proposition 4.5 *Transverse contact structures on Seifert manifolds have negative twisting numbers.*

Proof Let V be a Seifert manifold and ξ a transverse contact structure on V. We first remark that a contact structure which is covered by a contact structure with negative twisting number has negative twisting number. Indeed, suppose $\widehat{V} \to V$ is a k–fold fibered covering map, $\widehat{\xi}$ is lifted from ξ, f is a regular fiber in V and $\widehat{f}$ the lifted regular fiber of $\widehat{V}$. For every isotopy bringing f to a ξ–Legendrian curve L in V there is a lifted isotopy which brings $\widehat{f}$ to a $\widehat{\xi}$–Legendrian curve $\widehat{L}$ in $\widehat{V}$ and $t(\widehat{L}) = k\,t(L)$ so if L has nonnegative twisting number then so has $\widehat{L}$. Note that in general $t(\widehat{\xi})$ can nonetheless be higher than $t(\xi)$ because of Legendrian curves which are not lifted from curves in V so Theorem C cannot be deduced from the circle bundle case and indeed exhibits a much richer behavior in the general case.

If the base orbifold B of V is covered by a smooth surface then V is covered by a circle bundle $\widehat{V}$ and we conclude using the preliminary remark and [17, Proposition 2.4c]. If B is not covered by a smooth surface then V is a lens space whose universal cover is S^3. If this universal cover is a circle bundle then we can use Giroux's result again but in general S^3 will have one or two exceptional fibers; see Example 1.8. According to Proposition 4.4 the lifted contact structure coincides with the standard contact structure on S^3. Regular fibers of S^3 are then positive torus knots and we can conclude using Bennequin's inequality. Indeed, if the regular fibers are (p,q) torus knots then the Seifert framing and the fibration framing differ by pq (this is the linking number of two disjoint (p,q) torus knots contained in the same torus) and the Seifert genus of a (p,q) torus knot is easily seen to be at most $(p-1)(q-1)/2$ (this is actually the exact Seifert genus); see eg Rolfsen [38, Chapter 5]. Bennequin's inequality applied to any vertical Legendrian curve L then gives $t(L) = tb(L) - pq \leq 2g - 1 - pq \leq -p - q < 0$. □

Remark 4.6 If V is a Seifert manifold such that there exists an isotopy relative to a regular fiber which does not preserve the canonical framing of this fiber then every contact structure has infinite twisting number. Because of the above proposition we are not interested in those Seifert manifolds in this paper so we can safely forget about the isotopy when we consider vertical curves and still have a canonical framing.

In some papers, the twisting number is defined to be zero whenever it is nonnegative. This discrepancy has no impact in the present paper since we will study almost exclusively negative twisting numbers and we stick to the definition of E Giroux in [17] because of the following lemma.

Lemma 4.7 *Overtwisted contact structures on Seifert manifolds have infinite twisting number.*

This is [17, Proposition 2.4.b] which was written in the context of circle bundles but the statement and proof are the same (take the connected sum of a vertical Legendrian curve and a Legendrian unknot with positive Thurston–Bennequin invariant near an overtwisted disk to increase twisting number arbitrarily).

This lemma tells us that in order to prove that a contact structure ξ is tight, it is sufficient to prove that $t(\xi) < 0$. Also the original proof of Bennequin's theorem that the standard contact structure on S^3 is tight consists in proving that any Legendrian unknot has nonpositive Thurston–Bennequin invariant and this is completely equivalent to the fact that this contact structure has negative twisting number for the circle bundle structure of S^3. However there exist tight (and even universally tight) contact structures on circle bundles with zero twisting number; see [17].

5 Existence criterion on Seifert manifolds

In this section we prove Theorem B and its corollary from the introduction.

5.1 Arithmetic criteria

We first explain how the corollary stated in the introduction as well as the following one are deduced from Theorem B.

Corollary 5.1 *Let V be a Seifert manifold. The following statements are equivalent:*

- *V has a transverse contact structure ξ with $t(\xi) = -1$;*
- *V has a contact structure ξ with $t(\xi) = -1$;*
- *$e_0(V) \leq 2g-2$.*

In order to get concise statements, we recall a definition coming from [26] and [34]: a tuple $(\gamma_1, \dots, \gamma_r) \in (\mathbb{Q} \cap (0,1))^r$ is *realizable* if $r \geq 3$ and if there exist relatively prime integers $0 < a < m$ such that—possibly after reordering the tuple—we have $\gamma_1 < a/m$, $\gamma_2 < (m-a)/m$ and $\gamma_i < 1/m$ for every $i \geq 3$. If V is a Seifert manifold we set $\Gamma(V) = (1-\beta_1/\alpha_1, \dots, 1-\beta_r/\alpha_r)$.

Using the equivalence of (i) and (iii) in Theorem B we only have to prove the following purely arithmetic fact:

Proposition 5.2 *Let g and r be nonnegative integers, b an integer and*

$$(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$$

pairs of integers such that $1 \leq \beta_i < \alpha_i$ for every i. We set $e = -b - \sum \beta_i/\alpha_i$ and $e_0 = -b - r$. The following statements are equivalent:

(a) *There exist a positive integer n and integers $x_0, \dots, x_r$ such that*

$$(x_0 - 1)/n < b, \quad (x_i - 1)/n < \beta_i/\alpha_i \text{ for } i > 0 \text{ and } \sum x_i = 2-2g;$$

(b) *One of the following holds:*
  (i) *$e_0 \leq 2g-2$;*
  (ii) *$g = 0$, $r \leq 2$ and $e < 0$;*
  (iii) *$g = 0$, $r \geq 3$, $e_0 = -1$ and $(1-\beta_1/\alpha_1, \dots, 1-\beta_r/\alpha_r)$ is realizable.*

In addition, (a) *is true with $n = 1$ if and only if $e_0 \leq 2g-2$.*

Proof Remark first that if (a) is true then $x_0 \leq nb$ and $x_i \leq n$ for every $i \geq 1$. Also $2-2g = \sum x_i$ so $2-2g \leq n(b+r) = -ne_0$ so that $ne_0 \leq 2g-2$.

Nonzero genus case Suppose $g > 0$. If (a) is true then the above remark gives $e_0 \leq (2g-2)/n \leq 2g-2$.

Conversely if (b) is true then $e_0 \leq 2g-2$ and we can choose $n = 1$, $x_0 \leq b$ and $x_1, \dots x_r \leq 1$ such that $\sum x_i = 2-2g$.

Zero genus case We now consider the case $g = 0$. The same argument as above shows (a) is true with $n = 1$ if and only if $e_0 \leq 2g-2 = -2$.

Suppose that (a) is true with $n > 1$. The preliminary remark gives us $e_0 \leq (2g-2)/n$ which is negative so $e_0 \leq -1$ because e_0 is an integer.

If $e_0 \leq -2$ then we conclude using the first remark. Suppose now that $e_0 = -1$, so that $b = 1-r$.

If $r = 0$ then $b = 1$ so $e = -b < 0$. If $r \in \{1, 2\}$

$$2 = \sum_{i=0}^{r} x_i < nb + \sum_{i=1}^{r} \left(\frac{n\beta_i}{\alpha_i} + 1 \right) = -ne + r$$

so $e < (r-2)/n \leq 0$ and $e < 0$.

Conversely if $e < 0$ then $b \geq 1-r$ and if $r = 0$ then we can choose $n = 2$, $x_0 = 2$, if $r = 1$ then $e < 0$ implies the existence of n such that $e < -1/n$ and we choose $x_0 = nb$ and $x_1 = 2-x_0$. If $r = 2$ then $e < 0$ means $-b-\beta_2/\alpha_2 < \beta_1/\alpha_1$ so there exist positive k and n such that $-b-\beta_2/\alpha_2 < k/n < \beta_1/\alpha_1$ and we choose $x_0 = nb$, $x_1 = k+1$ and $x_2 = 2-x_0-x_1$.

The only remaining case is $g = 0$, $e_0 = -1$ and $r \geq 3$. If (b) is true then we choose $n = m$, $x_0 = m(1-r)$, $x_1 = m-a+1$, $x_2 = a+1$ and $x_i = m$ for every $i \geq 3$.

Conversely if (a) is true then $x_0 \leq nb$ and we can replace x_0 by nb and x_1 by $x_1 - (nb - x_0)$ without losing anything so we can assume that $x_0 = nb$. We claim that $x_i \geq 2$ for every positive i. Indeed if this is not true then

$$2 = \sum_{i \geq 0} x_i < nb + 2 + n(r-1) = 2$$

which is absurd. So $x_i \geq 2$ and we conclude using the following lemma. □

Lemma 5.3 *If $r \geq 3$, $n \geq 2$ and $2 \leq x_1 \leq \cdots \leq x_r \leq n$ are integers such that $\sum x_i = 2 + n(r-1)$ then there exist relatively prime integers $0 < a < m$ such that*

$$\frac{x_1 - 1}{n} \geq \frac{m-a}{m}, \qquad \frac{x_2 - 1}{n} \geq \frac{a}{m}, \qquad \forall i \geq 3,\ \frac{x_i - 1}{n} \geq \frac{m-1}{m}.$$

Proof First remark that we do not have to care about a and m being relatively prime because we can always divide them by their greatest common divisor while retaining their relations to the x_i.

Set $k = x_1 + x_2 - 2 - n$. The integer k is nonnegative because

$$x_1 + x_2 = 2 + n(r-1) - \sum_{i=3}^{r} x_i \geq 2 + n(r-1) - n(r-2) = 2 + n$$

(we replaced the x_i by n in the sum).

Set $l = n - x_3$. By assumption l is nonnegative and it is not larger than k since

$$2 + n(r-1) = \sum x_i = 2 + n + k + n - l + \sum_{i \geq 4} x_i \leq 2 + n + k + n - l + n(r-3).$$

Since the x_i form a nondecreasing sequence, we only need to show that there exist a and m meeting the conditions related to x_1 and x_2 and such that $(n-l-1)/n \geq (m-1)/m$. The latter condition is equivalent to $m \leq n/(l+1)$. Put $m = \lfloor n/(l+1) \rfloor$. We know $m \geq 2$ because $x_1 \leq n$ so $x_2 \geq 2 + k$ and $n - l = x_3 \geq x_2$ so $n - l \geq 2 + k \geq 2 + l$ so $n/(l+1) \geq 2$. We only need to check the existence of $0 < a < m$ meeting the conditions related to x_1 and x_2.

In the Euclidean plane $\mathbb{R}^2$ we consider points $A = (n, x_1 - 1)$, $B = (n, -(x_2 - 1))$, $H = (n, 0)$ and $H' = (m, 0)$ (see Figure 4).

Denote by A' (resp. B') the intersection point between the line $x = m$ and the line (OA) (resp. (OB)). According to the intercept theorem the segment $[A'B']$ has length $(n+k)m/n \geq m$ therefore it contains a segment $[A''B'']$ with length m whose extremities have integer coordinates. Let $m - a$ be the ordinate of A'' so that the ordinate of B'' is $-a$. The integer a is such that $(x_1 - 1)/n \geq (m-a)/m$ and $(x_2 - 1)/n \geq a/m$ and we have $a > 0$ because if $a = 0$ then $x_1 \geq n + 1$, which is absurd. □

Using the Eliashberg–Thurston perturbation theorem in [9] we can recover the following result about foliations:

Corollary 5.4 [6; 25; 26; 34] *A Seifert manifold V with a C^2 transverse foliation satisfies one of the following conditions:*

- $e_0(V) \leq 2g - 2$ *and* $e_0(-V) \leq 2g - 2$;
- $g = 0$ *and* $e(V) = e(-V) = 0$;
- $g = 0$, $e_0(V) = -1$ *and* $\Gamma(V)$ *is realizable;*
- $g = 0$, $e_0(-V) = -1$ *and* $\Gamma(-V)$ *is realizable.*

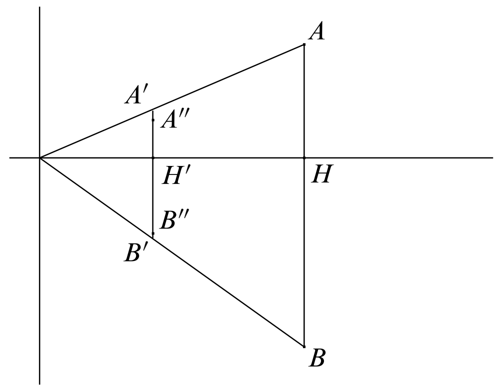

Figure 4: Intercept theorem and realizability

5.2 Normal forms

Let R be a compact orientable surface of genus g with $r+1$ boundary components and J be a complex structure on R defining its orientation. The Seifert manifold V with invariants

$$(g, b, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$$

is the union of $R \times \mathrm{S}^1$ and $r+1$ solid tori $W_0, \dots, W_r$; see Section 1.3.

Recall from the introduction that a contact structure ξ on V is in normal form if its restriction to $R \times \mathrm{S}^1$ has an equation $\cos(n\theta)\lambda + \sin(n\theta)\lambda \circ J$ where λ is some nonsingular 1–form on R and $n = -t(\xi)$. Such a ξ is denoted by $\xi(\lambda, n, \xi_0, \dots, \xi_r)$ where $\xi_i = \xi_{|W_i}$. Also, the multi-index of this contact structure is the collection of the indices of λ along the boundary components of R.

Proposition 5.5 *Every contact structure having negative twisting number on a Seifert manifold $V = (R \times \mathrm{S}^1) \cup W_0 \cup \dots \cup W_r$ is isotopic to a contact structure in normal form.*

This proposition follows from ideas in [17] which we briefly recall. The following lemma is a consequence of the flexibility lemma:

Lemma 5.6 *Let ξ be a contact structure having negative twisting number on a Seifert manifold. Any ξ–convex vertical torus which contains a maximally twisting vertical Legendrian curve is isotopic through ξ–convex surfaces to a torus ruled by maximally twisting vertical Legendrian curves. Any ξ–convex annulus whose boundary consists* *of two vertical Legendrian curves with maximal twisting number is divided by curves going from one boundary component to the other one and is ∂–isotopic through ξ–convex surfaces to an annulus ruled by maximally twisting vertical Legendrian curves.*

Using this lemma as in [17, Lemma 2.8] one proves that any contact structure having negative twisting number $-n$ is isotopic to a contact structure such that every fiber in $R \times S^1$ is Legendrian with twisting number $-n$. These contact structures over $R \times S^1$ are all pulled back from ξ_R on ST^*R by fibered covering maps as explained by (the oriented version of) [17, Proposition 3.3] (which is stated in a slightly generalized form as Proposition 8.9 below). These covering maps can be deformed to get a contact structure in normal form.

5.3 Criterion

Proof of Theorem B According to Proposition 4.5, transverse contact structures have negative twisting number so (i) $\Rightarrow$ (ii).

We now prove that (ii) $\Rightarrow$ (iii). Let $\xi = \xi(\lambda, n, \xi_0, \dots, \xi_r)$ be a contact structure in normal form in the isotopy class of ξ given by (ii) and Proposition 5.5 and let $(x_0, \dots, x_r)$ be its multi-index. Since normal forms are completely explicit, we easily see that each $\xi\partial W_i$ is divided by curves whose total homology class (when they are all oriented to intersect positively F_i) is $2nS_i + 2(x_i - 1)F_i$.

According to the Poincaré–Hopf theorem applied to λ, $\sum x_i = 2 - 2g$, and according to Proposition 3.6, $(x_i - 1)/n < \beta_i/\alpha_i$.

We now prove that (iii) $\Rightarrow$ (i). Since $\sum x_i = 2 - 2g$, there exists a nonsingular 1–form λ on R whose indices along the boundary components ∂D_i of R are the x_i's. The corresponding contact structure $\cos(n\theta)\lambda + \sin(n\theta)\lambda \circ J$ on $R \times S^1$ can be extended by contact structures ξ_i inside each W_i which are positively transverse, first in a neighborhood of ∂W_i using the flexibility lemma and then explicitly by a contact structure printing suspensions on concentric tori around the central fiber of W_i because $(x_i - 1)/n < \beta_i/\alpha_i$ so the asymptotic directions of these suspensions can go from the half-line spanned by M_i to the one spanned by $nS_i + (x_i - 1)F_i$ without crossing the fiber direction. We then push slightly the resulting contact structure on V using a Legendrian vector field which is orthogonal to the fibers over R and zero inside the W_i's and then use a small isotopy near $\partial R \times S^1$ to get a transverse contact structure ξ. Of course $t(\xi) \geq -n$ and $t(\xi) < 0$ according to Proposition 4.5. □

6 Universally tight normal forms

In this section we end the proof of Theorem A from the introduction with the following proposition:

Proposition 6.1 *Let $V = (R \times S^1) \cup W_0 \cup \cdots \cup W_r$ be a Seifert manifold. A contact structure $\xi = \xi(\lambda, n, \xi_0, \ldots, \xi_r)$ in normal form on V is universally tight if and only if it can be cooriented such that each ξ_i is ∂–isotopic to a positively transverse contact structure.*

Proof We have already explained in the proof of Theorem B that if all the ξ_i's are positively transverse then ξ is isotopic to a (positively) transverse contact structure hence universally tight according to Proposition 4.4.

Conversely, suppose that ξ is universally tight. According to Proposition 3.6, all the $\xi_i \partial W_i$ are divided by two circles whose homology class—when they are oriented to intersect the fibers positively—will be denoted by d_i.

According to Lemma 3.4, there are two things to check. The first step is to prove that all the ξ_i's are universally tight and this implies that they are all ∂–isotopic to transverse contact structures. Then if $d_i \cdot M_i = 1$ for every i there is nothing more to prove since all the ξ_i's are isotopic to positively transverse contact structures. Else if there is some i_0 such that $d_{i_0} \cdot M_{i_0} > 1$ then we can coorient ξ such that ξ_{i_0} is positively transverse and the second step is to prove that for every j such that ξ_j is negatively transverse we have $d_j \cdot M_j = 1$.

In the following we will use normal forms of contact structures on toric annuli and solid tori from Section 2 and Section 3. There should not be any confusion with normal forms defined in the introduction since they do not live on the same manifolds. For each i we put ξ_i in normal form using Theorem 3.1 a) and then, according to Theorem 3.1 b) ξ_i is universally tight if and only if every d in its flip locus satisfies either $d \cdot M_i = 1$ or $d = d_i$.

Let $(\widetilde{V}, \widetilde{\xi})$ be the universal cover of (V, ξ) and ρ the covering map. According to Theorem B, $\widetilde{V}$ is S^3 or $\mathbb{R}^3$—this can be seen using (ii) $\Rightarrow$ (iii) and the fact that a Seifert manifold is covered by $S^2 \times \mathbb{R}$ if and only if its base is spherical and $e = 0$ or using (ii) $\Rightarrow$ (i) and Proposition 4.4. We will explain in detail the case where $\widetilde{V} = S^3$ with no exceptional fiber—which the subtlest—and indicate briefly how to deal with the other cases.

Preliminary observations If a Seifert manifold Y is the union of tubular neighborhoods of two fibers K and K' then it can be seen as a quotient of a toric annulus

$T\times[0,1]$—where T_0 projects to K and T_1 to K'—and if Y has some contact structure ξ with negative twisting number then there are meridian classes M and M' coming from K and K' in $H_1(T)$ such that $M'\cdot M$, $M'\cdot F$ and $M\cdot F$ are positive. If ξ is transverse to K and K' then it can be lifted to a contact structure $\widehat{\xi}$ on $T\times[0,1]$. If Y is S^3 then $M'\cdot M=1$ and the Giroux polygon of $\widehat{\xi}$ is $\{M'+M\}$ so every d in the flip locus of a normal form of $\widehat{\xi}$ satisfies $M'\cdot d=1$ or $d\cdot M=1$.

First case Suppose first that $\widetilde{V}=S^3$ with no exceptional fiber.

For the first step we fix some i and we consider a connected component $\widetilde{W}$ of $\rho^{-1}(W_i)$. The complement $\widetilde{W}'$ of $\widetilde{W}$ in S^3 is also a solid torus and we are in the situation of the preliminary observations.

For every d in the flip locus of ξ_i, $\widehat{\xi}$ has an orbit flip with homology class ρ^*d so ρ^*d is simple according to Lemma 2.5 and $\widetilde{\xi}|_{\widetilde{W}}$ is in normal form and then we put $\widehat{\xi}$ in normal form using Theorem 2.6 a).

According to Proposition 3.6, the triangle with vertices $0, M_i, d_i$ doesn't contain any integer point $d\notin\{0,M_i\}$ with $d\cdot F<d_i\cdot F$. In particular such a d cannot be in the triangle Δ with vertices $0,F,d_i$ because the triangle with vertices $0,F,d$ would then be included in Δ so it would have area $d\cdot F/2$ smaller than the area $d_i\cdot F/2$ of Δ. So for every $d\neq d_i$ in the flip locus of ξ_i, $d\notin\Delta$ and $\rho^*d\notin\rho^*\Delta$—because ρ^* is linear. The point is that the latter triangle contains all the integer points a such that $M'\cdot a=1$, $\rho^*d_i\cdot a>0$ and $a\cdot F>0$ so $M'\cdot\rho^*d>1$ and $\rho^*d\cdot M=1$. This implies that $d\cdot M_i=1$ and ξ_i is universally tight.

For the second step, let $\widetilde{W}$ be a connected component of $\rho^{-1}(W_{i_0})$ and N be a toric annulus around $\widetilde{W}$ containing one connected component of $\rho^{-1}(W_j)$ and no other connected component of a $\rho^{-1}(W_i)$. The torus $\partial\widetilde{W}$ is divided by curves with total homology class $2\rho^*d_{i_0}$ and the other component of ∂N is divided by curves with total homology class $2d''$ such that $d''-\rho^*d_{i_0}=kF$ because these two tori are ruled by Legendrian fibers with the same twisting number and $k\leq 0$ because $d''\cdot\rho^*d_{i_0}\geq 0$. Because of the additivity property of the indices of λ along curves, $d_j\cdot M_j=1$ if and only if $k=0$.

We isotop $\widetilde{\xi}$ in N such that it is negatively transverse in the interior of N. The contact structure $\widetilde{\xi}_{|\widetilde{W}\cup N}$ is then in normal form with flip locus $\{\rho^*d_{i_0}\}$. The complement of $\widetilde{W}\cup N$ in S^3 is a solid torus $\widetilde{W}''$, again the situation of the preliminary observations. We put $\widetilde{\xi}_{|\widetilde{W}''}$ in normal form and then $\widehat{\xi}$ is in normal form and—because $\rho^*d_{i_0}\cdot M>1$—we get $M''\cdot\rho^*d_{i_0}=1$. So we have $0<M''\cdot d''=M''\cdot\rho^*d_{i_0}+kM''\cdot F=1+kM''\cdot F$ hence $-1<k\leq 0$ so $k=0$.

Second case If $\widetilde{V}$ is S^3 with at least one exceptional fiber then V is a Lens space. We see V as $W \cup N \cup W'$ where all pieces are fibered, W and W' contain at most one exceptional fiber each, $W_0 \subset N$ and N intersects no other W_i. We are in the situation of the preliminary observations and we denote by $2d$ and $2d'$ the total homology classes of the dividing sets of ∂N seen in $H_1(T)$ and oriented such that $d \cdot F > 0$ and $d' \cdot F > 0$.

For the first step we can use the same argument as above except when $i = 0$. If $n > 1$ then according to Proposition 3.6 and Lemma 3.4 ξ_0 is universally tight (and isotopic to a tangent contact structure). If $n = 1$ then the Giroux polygon of $\xi_{|N}$ is $[d, d']$ and is included in the Giroux polygon of $\widehat{\xi}$. But, still because of the Giroux polygon associated to S^3 in the preliminary observations, $\widehat{\xi}$ is universally tight so $\xi_{|N}$ cannot have an orbit flip in the interior of its polygon so it is universally tight and so is ξ_0 because any cover of W_0 is contained in a cover of N.

For the second step we can directly use that every $\widehat{d}$ in the flip locus of $\widehat{\xi}$ satisfies $M' \cdot \widehat{d} = 1$ or $\widehat{d} \cdot M = 1$ and that ξ_0 is isotopic to a tangent contact structure if and only if $d = d'$.

Third case If $\widetilde{V} = \mathbb{R}^3$ then all the W_i's are covered by some $D^2 \times \mathbb{R} \subset \mathbb{R}^3$ so all the ξ_i's are universally tight.

The second step is analogous to the first case but we only go to a cover by $\mathbb{R}^2 \times S^1$ and use that all fibered tori in it have universally tight lifted contact structures. Using that the analogous of $\widetilde{\xi}_{|\widetilde{W} \cup N}$ is universally tight we directly get that $\rho^* d_{i_0} = d''$.

Note that in the $\mathbb{R}^3$ case we don't need $\xi(\lambda, n, \xi_0, \dots, \xi_r)$ to have twisting number $-n$ to get the result. □

7 Maximal twisting number calculations

In this section we prove Theorem C from the introduction. We also note the following corollary of Theorem C and of the proof of (iii) ⇒ (i) in Theorem B which makes more precise the equivalence of (i) and (ii) in Theorem B.

Corollary 7.1 *Let n be a positive integer. A Seifert manifold whose base is not a sphere has a contact structure ξ with $t(\xi) = -n$ if and only if it has a transverse contact structure ξ' with $t(\xi') = -n$.*

In this section and the following one, we will use frequently the idea of topological discretization which was first used in contact geometry in [4]. Recall that, for a surface

F, the pinched product $F \times_\partial [0,1]$ is obtained from $F \times [0,1]$ by collapsing $\{x\} \times [0,1]$ for every $x \in \partial F$. Two embedded surfaces in a 3–manifold are said to be parallel if they bound a pinched product. Let F be embedded in V and φ be an isotopy of V relative to ∂F. Any time $t \in [0,1]$ has a neighborhood J such that $\varphi_{|J}$ moves all connected components of F in disjoint pinched products. Using this remark, one can show that φ is homotopic to a concatenation of isotopies φ^i which are relative to ∂F and to all connected components of the image of F under the preceding φ^j but one denoted by F_i and $\varphi^i_0(F_i)$ and $\varphi^i_1(F_i)$ bound a pinched product.

If V carries a contact structure ξ then we can assume that all the intermediate surfaces arising in the preceding process are ξ–convex using the genericity of ξ–convex surfaces.

This discretization process will be called topological discretization to avoid confusion with the more elaborate contact discretization first used systematically in [24] which goes further by imposing—using [18, Lemma 15]—pinched products which are as simple as possible from a contact point of view.

Lemma 7.2 *Let V be a Seifert manifold with base an orbifold disk and $W_1, \dots, W_r$ be fibered solid tori such that $V \setminus (W_1 \cup \dots \cup W_r)$ contains only regular fibers. If ξ is a contact structure which is tangent to the fibers outside the W_i's with twisting number $-n$ and $t(\xi_{|W_i}) = -n$ for every i then $t(\xi) = -n$ (in particular ξ is tight).*

Proof Let $A_1, \dots, A_r$ be fibered annuli with boundary in ∂V such that

$$V \setminus (A_1 \cup \dots \cup A_r) = W'_0 \cup \dots \cup W'_r$$

where $W_i \subset W'_i$ for every $i \geq 1$.

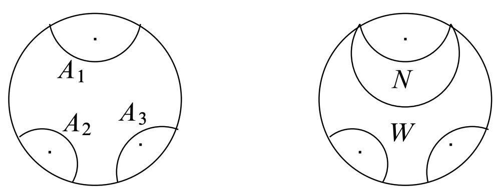

Figure 5: Computing $t(\xi)$ over an orbifold disk

Let L_0 be a fiber in W'_0, L a vertical Legendrian curve and φ a ∂–isotopy such that $L = \varphi_1(L_0)$.

Using topological discretization, there exists a sequence of annuli A^j_i, $0 \leq j \leq K$ such that:

- for every i, $A_i^0 = A_i$, $A_i^K = \varphi_1(A_i)$;
- for every j the A_i^j's are disjoint ξ–convex annuli;
- for every j there exists i_j such that $A_i^{j+1} = A_i^j$ for all $i \neq i_j$;
- $A_{i_j}^j$ and $A_{i_j}^{j+1}$ bound a pinched product.

At each step j we equip V with a Seifert fibration ∂–isotopic to the original one such that all the A_i^j are fibered.

We now prove by induction on j that for every j the complement of the A_i^j doesn't contain any vertical Legendrian curve with $t > -n$ and each A_i^j is divided by $2n$ curves going from one boundary component to the other one. This will prove the lemma since L is in the complement of the A_i^K.

This statement is true at the initial step by assumption. Let's assume that it holds at step j, set $A = A_{i_j}^j$, $A' = A_{i_j}^{j+1}$ and let N be the pinched product they bound. The annulus A' is contained either in the solid torus that A splits from V or in the solid torus obtained from V by removing all the solid tori split from V by the A_i^j. Let's denote this solid torus containing A' by W and denote by $W' \subset W$ the complement of N in W.

By the induction hypothesis, the dividing set of A consists of $2n$ curves going from one boundary component to the other one. Since A' is contained in W we know by induction hypothesis that it does not contain any vertical Legendrian curve with twisting number greater than $-n$ and since $\partial A' = \partial A$, Lemma 5.6 guarantees that the dividing set of A' consists of $2n$ curves going from one boundary component of A to the other one.

If $n > 1$ then according to Proposition 3.6 the dividing set of W' is isotopic to that of W and the classification of tight contact structures on solid tori proves that there is a ∂–isotopy which brings A to A' through ξ–convex surfaces so nothing changes.

If $n = 1$ then according to Proposition 3.6 we only have to prove that ξ restricted to all connected components of the complement of the A_i^{j+1}'s is tight. All of these components but one are contained in solid tori which are known to be tight by the induction hypothesis. The last component is in the union of a tight solid torus and a tight toric annulus whose boundary components are divided by two curves with homology classes $d = S + kF$ and $d' = S + k'F$ respectively so this component is tight according to Lemma 3.5. □

Proof of Theorem C Let L be a vertical Legendrian curve in V. There exists a finite cover of the base B coming from a cover of the underlying topological surface which induces a fibered covering $\widehat{V} \to V$ such L can be lifted to a Legendrian curve $\widehat{L}$ which is contained in a Seifert manifold with boundary $\widehat{V}'$ over an orbifold disk and $\widehat{L}$ is ∂–isotopic to a regular fiber. Using Lemma 7.2 we get $t(L) = t(\widehat{L}) \leq -n$. □

We end this part with some remarks about the genus zero case. The theorem above can be extended easily to spherical bases when there is less than three exceptional fibers or when each W_i has a unique tight contact structure; see Section 8.3. Concerning small Seifert manifolds with three exceptional fibers which are the subjects of many recent papers, there are three cases: when $e_0(V) \geq 0$ then according to Theorem B there exist no contact structure with negative twisting number. When $e_0(V) \leq -3$ then condition (iii) of Theorem B can be met only when $n = 1$ and [43] (or Theorem D) combined with [44] proves that the theorem is true in this case. The remaining cases $e_0(V) \in \{-2, -1\}$ are much more difficult, very few results are known especially when there are contact structures with different negative twisting numbers.

As an example of what we can still get using our techniques, let's consider the Brieskorn homology sphere $V = -\Sigma(2, 3, 6k-1)$ which has a Seifert structure with invariants $(0, -2, (2, 1), (3, 2), (6k-1, 5k-1))$ and has been studied for instance in [28]. Our results prove that for every contact structure ξ on V, if $t(\xi) < 0$ then

$$t(\xi) \in \{-(6l + 5),\ 0 \leq l \leq k - 2\}.$$

There is exactly one isotopy class of contact structures with $t = -(6(k-2) + 5)$, its elements are universally tight and isotopic to a tangent contact structure; see Section 8.3. Theorem B gives at least one transverse contact structure with $t \geq -5$ and -5 is the maximal possible negative twisting number so there is at least one isotopy class of universally tight contact structures with $t = -5$. If the hypothesis $g > 0$ can be removed from Theorem C and Theorem E then the predicted number of isotopy classes of tight contact structures with $t = -(6l + 5)$ is $k - l - 1$ with two consisting of universally tight contact structures when $t > -(6(k-2) + 5)$. In any cases this is an upper bound on the number of such isotopy classes.

Using Eliashberg–Gompf surgery [7; 19] and the slice Thurston–Bennequin inequality [28, Corollary 4.2] it can be proved that the predicted number is correct for $t = -5$ and that they are all Stein fillable and distinguished by their Stein fillings (I thank Paolo Lisca for a very instructive conversation which led to this result). A similar phenomenon occurs in a paper by J Kim [27] where it is shown that the Seifert manifolds with invariants $(0, -2, (2, 1), (3, 2), (6k-1, 6k-3))$, $k \geq 2$ have $3k-5$ isotopy classes of tight contact structures. Using the above arguments, we can show that all these contact structures have twisting number -5, as expected using the results of the present paper.

In this case the fact that $t = -5$ is the only possible twisting number seems to be what makes the classification feasible using such techniques.

It seems that there is no known counterexample to the following statement: if a Stein fillable contact structure on a Seifert manifold V has twisting number $t_0 < 0$ then there is no contact structure on V with twisting number $t_0 < t < 0$. This can be checked for instance on Lens spaces and on T^3 and we can note that, according to P Ghiggini in [12], "Stein" cannot be replaced by "strongly" in the above sentence.

8 Classification

8.1 Flexible case

In this subsection we prove Theorem D from the introduction.

This result was announced by K Honda for circle bundles ($r = 0$) in [23] but proved using the incorrect claim that these contact structures can always be distinguished by their homotopy classes as oriented plane fields. Indeed, when V has a contact structure which is tangent to the fibers (ie when there exists n such that $ne(V) = -\chi(B)$; see Giroux [17] or Theorem 8.7) there is only one homotopy class of (oriented) plane fields transverse to the fibers because the tangent contact structure is isotopic to contact structures transverse to the fibers with either orientation. If $-n < -1$ and B has genus at least two then $e < 2g-2$. The mistake in [23] arose from overlooking the fact that fibers of V have finite order in $H_1(V, \mathbb{Z})$ when $e \neq 0$.

Lemma 8.1 *Let V be a Seifert manifold with $e_0(V) < 2g-2$. Two contact structures on V with twisting number -1 always have normal forms with multi-index*

$$(2-2g-r, 1, \dots, 1)$$

and the same R–class.

Proof We first prove that any such contact structure ξ has a normal form with multi-index $(2-2g-r, 1, \dots, 1)$. Let ξ' be a normal form of ξ (obtained using Proposition 5.5). According to Proposition 3.6, the multi-index of ξ' satisfies $x_i \leq 1$ for every $i \geq 1$. So, according to Corollary 3.2 and the flexibility lemma, for every $i \geq 1$, W_i contains a torus T_i parallel to the boundary, ruled by vertical Legendrian curves with twisting number -1 and divided by two curves whose homology class is S_i. Consider a vertical Legendrian curve L_0 outside the W_i's with $t(L_0) = -1$ and, for each i, a ξ–convex annulus A_i such that ∂A_i is the union of L_0 and a vertical Legendrian curve on T_i. According to Lemma 5.6, these annuli are isotopic—relative to L_0 and the T_i's—to annuli intersecting only along L_0 and ruled by vertical Legendrian curves with twisting number -1. One can then follow the proof of the existence of normal forms to get a normal form with $x_i = 1$ for every $i \geq 1$. According to the Poincaré–Hopf theorem, this normal form has $x_0 = 2-2g-r$.

The proof that if ξ and ξ' are in normal form with this multi-index then ξ has a normal form with 1–form homotopic to that of ξ' is exactly as in [17, Lemma 3.8] using the fact that $x_0 < b$ so that $d_0 = M_0 + (x_0 - b - 1)F_0$ with $x_0 - b - 1 \leq -2$. □

Note that neither the above lemma nor Theorem D claims that every normal form with twisting number -1 has the given multi-index—this would be false—but only that there is a normal form with this multi-index in every isotopy class of ξ with $t(\xi) = -1$. This contrasts with Theorem E where the multi-index is fixed by the Seifert invariants and the twisting number.

Proof of Theorem D The preceding lemma proves that the number of contact structures on V with $t = -1$ is at most the number of contact structures one can obtain by fixing a tangent contact structure over R with multi-index $(2-2g-r, 1, \dots, 1)$ and extending it by a tight contact structure ξ_i in each W_i. The proof that this upper bound is the exact count follows a well-known strategy—see eg Ghiggini [13] or Wu [43]—so we only indicate the steps.

We use the Eliashberg–Gompf construction of Stein fillable contact structures [7; 19] to construct the right number of diffeomorphic Stein fillings of V with different first Chern classes. The induced contact structures are nonisotopic according to [28, Corollary 4.2]. If ξ is one of them then the Gompf diagram shows that $t(\xi) \geq -1$, and $t(\xi) < 0$ according to the slice Thurston–Bennequin inequality [28, Theorem 3.4] so $t(\xi) = -1$.

The count of transverse contact structures follows from this, Proposition 6.1 and Lemma 3.4 using $d_0 \cdot M_0 > 1$. Suppose that ξ and ξ' are in normal form with $t = -1$, isotopic to transverse contact structures and coincide on $R \times S^1$ but are not isotopic. The rotation of angle π on $R \times S^1$ pushes ξ to $-\xi'$ and normal forms in the W_i's show that it can be extended to a diffeomorphism of V still isotopic to the identity and pushing ξ to $-\xi'$. □

8.2 Rigid case

In this subsection we prove Theorem E from the introduction.

Proposition 8.2 *Let n be a positive integer and V be a Seifert manifold whose base has genus g. If $e_0 = 2g-2$ or $n > 1$ then every contact structure $\xi(\lambda, n, \xi_0, \dots, \xi_r)$ with $t(\xi_i) \leq -n$ for every i has multi-index $(nb, \lceil n\beta_1/\alpha_1 \rceil, \dots, \lceil n\beta_r/\alpha_r \rceil)$.*

Proof If $n > 1$ then the proposition follows directly from Proposition 3.6. If $n = 1$ then we need the additional remark that the multi-index $(x_0, \dots, x_r)$ of ξ satisfies $x_0 - 1 < b$ and $x_i - 1 < \beta_i/\alpha_i$ if $i > 0$ so $x_0 \le b$ and $x_i \le 1$ if $i > 0$ but also $\sum x_i = 2 - 2g$ and $2 - 2g = b + r$ by hypothesis so all the above inequalities are equalities. □

The following proposition will be proved later in this subsection. It is the only point where we need g to be positive. A proof in the $g = 0$ case would lead to the extension of Theorem E to $g = 0$.

Proposition 8.3 *Let n be a positive integer and V a Seifert manifold whose base has genus $g > 0$. Assume that $n > 1$ or $e_0(V) = 2g - 2$. Let ξ be a contact structure on V with $t(\xi) = -n$. If L_0 and L_1 are vertical Legendrian curves with $t(L_0) = t(L_1) = -n$ then every isotopy L_t between L_0 and L_1 is homotopic to a Legendrian isotopy.*

We will also use the following flexibility lemma which is a special case of the general flexibility lemma for families [18, Lemma 7].

Lemma 8.4 *Let F be a closed surface in a contact 3–manifold (V, ξ). Suppose ξF is divided by a multi-curve Γ and φ is an isotopy such that $\varphi_t(F)$ is divided by $\varphi_t(\Gamma)$ for every t. If $\xi(\varphi_1(F)) = \varphi_1(\xi F)$ then φ is homotopic to an isotopy ψ such that $\xi(\psi_t(F)) = \psi_t(\xi F)$ for every t.*

Proof of Theorem E If two contact structures ξ, ξ' in normal forms with $t = -n$ have the same R–class then ξ' is isotopic through contact structures in normal form to ξ'' with the same Euler classes as ξ' and $\lambda'' = \lambda$. Because of Proposition 3.6, each characteristic foliation of a ∂W_i is divided by two curves so ξ'' is isotopic to ξ according to Theorem 3.1 c).

Conversely, let ξ and ξ' be contact structures in normal form on V with twisting number $-n$ and suppose there is an isotopy ϕ pushing ξ' on ξ. We will simplify ϕ in three steps. We make it relative to a fiber $L_0 \in R$ in step one, to a system of tori intersecting along L_0 with a regular neighborhood isotopic to $R \times S^1$ in step two and to the ∂W_i's in step three.

Step 1 According to Proposition 8.2, we can assume—up to an isotopy of ξ' among contact structures in normal form preserving its R–class and the Euler classes of the ξ'_i's—that $\xi'_{|\partial W_i} = \xi_{|\partial W_i}$ for every i.

Let L_0 be a fiber in R. The fiber L_0 and its image L_1 by ϕ_1 are ξ–Legendrian with maximal twisting number so we can apply Proposition 8.3 and the Legendrian isotopy we get is induced by an isotopy ψ such that $\psi_1 = \phi_1$.

Since $\psi_t(L_0)$ is ξ–Legendrian for all t, there exists an isotopy θ preserving ξ which coincides with ψ on L_0 for all time. Denote by $\overline{\phi}$ the isotopy given by $\theta_t^{-1} \circ \psi_t$. This isotopy pushes ξ' on ξ, is relative to L_0, and we have

$$\xi\overline{\phi}_1(\partial W_i) = \xi\phi_1(\partial W_i) = \phi_1(\xi'\partial W_i) = \phi_1(\xi\partial W_i) = \overline{\phi}_1(\xi\partial W_i).$$

Step 2 Let $T_1, \dots, T_{2g+r}$ be fibered tori in $R \times \mathrm{S}^1$ intersecting along L_0 such that a regular neighborhood of $T_1 \cup \dots \cup T_{2g+r}$ is isotopic to $R \times \mathrm{S}^1$.

We now prove that $\overline{\phi}$ is homotopic to an isotopy which moves the T_i's through ξ–convex surfaces. The key is Proposition 8.2 which gives the multi-index independently of the normal form.

Using topological discretization, it is sufficient to prove that if $\overline{\phi}$ is relative to all the T_i's but one denoted by T and if $T' := \overline{\phi}_1(T)$ bounds a pinched product N with T then $\overline{\phi}$ is homotopic to an isotopy which, in addition, moves T through ξ–convex surfaces. Using the flexibility lemma and Lemma 5.6 we can assume that T' is ruled by vertical Legendrian curves with twisting number $-n$.

The complement of $T_1 \cup \dots \cup T_{2g+r}$ is the union of $r+1$ solid tori. Let W be the one that contains T' and set $W' = W \setminus N$. The solid tori W and W' are nested and they have isotopic dividing sets according to Proposition 8.2. The classification of tight contact structures on solid tori then implies that $\overline{\phi}$ is homotopic to an isotopy moving T through convex surfaces.

This already proves that λ is homotopic to λ'. Moreover, because $\xi\overline{\phi}_1(T_i) = \overline{\phi}_1(\xi' T_i)$ the tori $\overline{\phi}_1(T_i)$ are ruled by ξ–Legendrian curves so, using Lemma 8.4, we get an isotopy φ homotopic to $\overline{\phi}$ and such that $\xi\varphi_t(T_i) = \varphi_t(\xi T_i)$ for all t and every i.

Step 3 At all times, there is a regular neighborhood of $\bigcup \varphi_t(T_i)$ foliated by vertical Legendrian curves so φ is homotopic to $\widetilde{\psi}$ such that $\widetilde{\psi}_t(\partial W_i)$ is ruled by vertical curves for all t and every i. Using Lemma 8.4, we get an isotopy $\overline{\psi}$ homotopic to $\widetilde{\psi}$ such that

$$\xi\overline{\psi}_t(\partial W_i) = \overline{\psi}_t(\xi\partial W_i)$$

for all t and every i.

Let $\overline{\theta}$ be an isotopy preserving ξ which coincides with $\overline{\psi}$ on every ∂W_i, and denote by $\widetilde{\phi}$ the isotopy given by $\widetilde{\phi}_t = \overline{\theta}_t^{-1} \circ \overline{\psi}_t$. This isotopy pushes ξ' on ξ and is relative to the ∂W_i's so that each ξ_i is isotopic to the corresponding ξ_i' so they have the same relative Euler class.

The count of transverse contact structures goes as in the proof of Theorem D using Proposition 6.1, Lemma 3.4 and Proposition 3.6. There is only one isotopy class of transverse contact structure in each R–class if and only if $d_i \cdot M_i = 1$ for every i and then this isotopy class contains a tangent contact structure; see Section 8.3. □

Uniqueness of the maximally twisting vertical curve In this paragraph we prove Proposition 8.3. As a first approach to this proposition, one can try to consider a vertical ξ–convex torus containing L_0, discretize the isotopy and prove that all bifurcations can be eliminated. However this can not be straightforward since one can always have bifurcations increasing the number of dividing curves by folding inside an invariant neighborhood and it is difficult to prove that a sequence of folding and unfolding can be unraveled. Here we use Ghiggini's trick introduced in [13, Proposition 5.4] and consider two tori intersecting along L_0, discretize the motion of one of them while constructing an isotopy of the second one through ξ–convex surfaces. Here again, everything is based on the fact that the involved Giroux cones contain only one integer point having the right intersection with the fiber class.

Lemma 8.5 *Let V be a Seifert manifold and ξ be a contact structure with negative twisting number $t(\xi) = -n$. Let T and F be transverse vertical ξ–convex tori such that $T \cap F$ is a Legendrian vertical curve which intersects the dividing set of T efficiently. If $n > 1$ or $e_0(V) = 2g - 2$ then every isotopy relative to T which sends F to a ξ–convex torus F' is homotopic to an isotopy relative to T and moving F through ξ–convex surfaces.*

Proof Up to a change of Seifert structure by isotopy, we can assume that T is fibered. Let $-n'$ be the maximal twisting number of Legendrian curves isotopic to the regular fibers relative to T. Using topological discretization and the flexibility lemma, we can assume that F contains a vertical Legendrian curve L with $t(L) = -n'$ and that F and F' bound a pinched product N intersecting T only along $T \cap F$. Let Σ and Σ' be ξ–convex tori bounding regular neighborhoods of $T \cup F$ and $T \cup N$ respectively, chosen so that they both contain a vertical Legendrian curve L_0 with $t(L_0) = -n'$.

Claim the tori Σ and Σ' have isotopic dividing sets with total homology class $2n'S - 2F$.

Using this claim we can cut V along Σ', keep the regular neighborhood of $T \cup N$ and fill it with a solid torus W with meridian class S and a tight contact structure—there is no choice here, up to isotopy—to get a contact manifold V' diffeomorphic to T^3. Cutting V' along T we get a toric annulus with a tight contact structure having the same dividing set on both boundary components. Using [16, Theorem 4.5] we see that F and F' are isotopic through convex surfaces in V' relative to $T \cup W$ so relative to the boundary and the lemma is proved.

We now prove the claim. By maximality of $-n'$, we know that Σ and Σ' have dividing sets with total homology classes $2n'S+2kF$ and $2n'S+2k'F$ respectively. We will first prove that $k=k'$ and then that $k=-1$. We consider two cases:

If $n'>1$ then Corollary 3.2 and Pick's formula prove that $k\neq k'$ contradicts the maximality of $-n'$ as in the proof of Proposition 3.6.

If $n'=1$ then $n=1$ and the proof is a variation of the proof of Proposition 8.2. We can construct a tori system based at L_0—and not intersecting Σ and Σ' anywhere else—which is ruled by vertical Legendrian curves with twisting number -1. We equip V with a Seifert structure isotopic to the original one so that our tori system is fibered. The tori Σ and Σ' intersect the corresponding W_0 solid torus along annuli with common boundaries. Using Proposition 8.2 we know that the Giroux cone of W_0 contains only one integer point with abscissa 1 so that we can conclude using the classification of tight contact structures on solid tori like in the proof of Theorem E.

So in both cases we proved that $k'=k$. It remains to prove that $k=-1$. It is sufficient to construct a curve with homology class S which intersects only twice the dividing set of Σ with the correct orientations. We know that the dividing set of T is made of curves traversing $T\setminus(T\cap F)$ and that $2n'$ dividing curves of $F\setminus(T\cap F)$ are traversing. Our curve is constructed so that it intersects the dividing curves only in the rounding regions. It starts just below (for the fibers orientation) one of the traversing curves C of F, follows it then traverse T without intersecting its dividing set then traverse back F below C and traverse back T before closing up. There are two intersection points with the dividing set of Σ and the orientation is correct because all traversing curves in the two copies of $F\setminus(T\cap F)$ are oriented in the same way and because of the edge-rounding lemma [22, Lemma 3.11]. □

Proof of Proposition 8.3 Let ϕ be an isotopy such that $L_t=\phi_t(L_0)$. Let T and F be ξ–convex vertical tori intersecting transversely along L_0. Up to a modification of ϕ relative to L_0 we can assume that $T':=\phi_1(T)$ and $F':=\phi_1(F)$ are also ξ–convex. By maximality of $t(L_0)$ (resp. $t(L_1)$) and the flexibility lemma, L_0 (resp. L_1) intersects efficiently the dividing sets of T and F (resp. T' and F').

By topological discretization, ϕ is homotopic to a concatenation of isotopies moving T through a sequence $T=T_0,T_1,\dots,T_N=T'$ of ξ–convex tori such that T_i and T_{i+1} bound a product. Also, using the flexibility lemma at each step, we can assume that every T_i contains a vertical Legendrian curve C_i intersecting efficiently the dividing set of T_i, $C_0=L_0$ and $C_N=L_1$.

We now prove by induction that, for every i, there exists a torus F_i intersecting T_i along C_i which is isotopic to F through ξ–convex surfaces. For the initial step we take F_0 to be F.

Once F_i has been constructed we consider a torus F_{i+1} which is isotopic to F_i relative to T_i and intersects T_{i+1} along C_{i+1}. According to Lemma 8.5, F_{i+1} is isotopic to F_i through ξ–convex surfaces.

We apply Lemma 8.5 one last time to get an isotopy between F_N and F' and the proposition is proved. □

Remark 8.6 About Legendrian knots:

- Lemma 8.5 can also be used to prove that any vertical Legendrian curve is a stabilization of one with maximal twisting number so we have a complete classification of vertical Legendrian knots under the hypotheses of Theorem E.
- When $t(\xi) = -1$ and $e_0 = 2g - 2$, Theorem D can be used to prove that if $t(L_0) = t(L_1) = -1$ then there exists a contactomorphism which sends L_0 to L_1 and is isotopic to the identity—although possibly not through contactomorphisms. If one could prove the existence of a Legendrian isotopy bringing L_0 to L_1 then it would be possible to prove Theorem D in the spirit of the proof of Theorem C (without using Seiberg–Witten theory).

8.3 Tangent contact structures

Among transverse contact structures on a Seifert manifold are those which are isotopic to tangent contact structures such as the standard contact structure on S^3. Conversely, any tangent contact structure can be perturbed by a C^∞–small isotopy to be positively or negatively transverse. The following theorem explain their special role among contact structures with negative twisting number. It is a direct consequence of the previous results.

Theorem 8.7 *Let V be a Seifert manifold with invariants*

$$(g, b, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$$

and n be a positive integer. The following conditions are equivalent:

(1) $ne(V) = -\chi(B)$ *and* $n\beta_i \equiv 1 \mod \alpha_i$ *for all* i ;

(2) V *carries a tangent contact structure with twisting number* $-n$.

In addition, if (1) (*or* (2)) *is satisfied then every contact structure on* V *with twisting number* $-n$ *is isotopic to a tangent one.*

Also, if V *carries a contact structure with twisting number* $-n$ *and* $n > \max(\alpha_i)$ *then* (1) *and* (2) *are satisfied and this contact structure is isotopic to a tangent one.*

Proof Suppose that V has a tangent contact structure ξ and $t(\xi) = -n$. Up to isotopy among tangent contact structures, this ξ is in normal form and, according to Lemma 3.4, the multi-index $(x_0, \dots, x_r)$ of ξ satisfies $1 = d_i \cdot M_i = n\beta_i - \alpha_i(x_i - 1)$ so that $n\beta_i \equiv 1 \mod \alpha_i$ for every i. Also

$$\begin{aligned} ne(V) &= -\sum_{i=0}^{r} \frac{n\beta_i}{\alpha_i} = -\sum_{i=0}^{r} \left(\frac{1}{\alpha_i} + (x_i - 1) \right) \\ &= -\left(\sum_{i=0}^{r} x_i - (r+1) + \sum_{i=0}^{r} \frac{1}{\alpha_i} \right) = -\chi(B). \end{aligned}$$

Conversely if these conditions are met and ξ is in normal form we consider two cases. If $n > 1$ then Proposition 3.6 and Lemma 3.4 show that each ξ_i is isotopic to a tangent contact structure. If $n = 1$ then $\beta_i = 1$ for every $i \geq 1$ because $\beta_i \equiv 1 \mod \alpha_i$ and this combines with $e(V) = -\chi(B)$ to give $-b = 2g - 2 + r$ so $e_0(V) = 2g - 2$ and according to Proposition 8.2 we can apply again Lemma 3.4.

Note that this proof of $(1) \Rightarrow (2)$ gives also the second assertion.

The last assertion of the theorem follows from Proposition 3.6 and Lemma 3.4. □

These contact structures are classified up to isotopy by Theorem E since, according to the preceding theorem, if ξ is of tangent type and $t(\xi) = -1$ then $e(V) = -\chi(B)$ and $\beta_i = 1$ for every $i \geq 1$ so $e_0(V) = 2g - 2$.

It is plausible that the isotopy classes containing tangent contact structures are exactly the ones containing both positively and negatively transverse contact structures. This would follow from the fact that a positively transverse contact structure has a normal form with positively transverse ξ_i's but this is not what Propositions 5.5 and 6.1 give us.

Tangent contact structures also have the virtue that their twisting number is easy to compute: it is given by the twisting number of regular fibers; see Giroux [17, Lemma 3.6] which—using a cover by a circle bundle—only leaves the case of certain Lens spaces which can be dealt with in the spirit of the proof of Proposition 6.1. This remark and the fact expressed in the above theorem that all contact structures with sufficiently low twisting number $-n$ on a given Seifert manifold must satisfy $ne(V) = -\chi(B)$ can be used to get a list of eight Seifert manifolds—with $\chi(B) = 0$—which are exactly the Seifert manifolds having an infinite family of contact structures with distinct negative twisting numbers. They are the cooriented contact elements bundles of the parabolic orbifolds with either orientation (for T^2 and the pillowcase the two orientations give the same Seifert manifold).

A better understanding of tangent contact structures and an alternative proof of the equivalence of (1) and (2) in the previous theorem come from the two following statements which can be proved exactly as in the circle bundle case using equivariance of all the constructions. The second one is a slight generalization of [17, Proposition 3.3] which actually was first discussed by R Lutz in [30].

Lemma 8.8 *Let B be a 2–dimensional orbifold. The cooriented contact elements bundle ST^*B has a canonical contact structure ξ_B which is tangent to the fibers with twisting number -1 and is invariant under any fibered diffeomorphism lifted from B.*

Proposition 8.9 *Let V be a Seifert manifold with base B. The map which associates to each covering map $\rho\colon V \to ST^*B$ fibered over the identity the contact structure $\rho^*\xi_B$ is a bijection onto the space of tangent contact structures.*

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