The rational homotopy type of a blow-up in the stable case

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Suppose $f\colon V \to W$ is an embedding of closed oriented manifolds whose normal bundle has the structure of a complex vector bundle. It is well known in both complex and symplectic geometry that one can then construct a manifold $\widetilde{W}$ which is the blow-up of W along V. Assume that $\dim W \geq 2\dim V + 3$ and that $H^1(f)$ is injective. We construct an algebraic model of the rational homotopy type of the blow-up $\widetilde{W}$ from an algebraic model of the embedding and the Chern classes of the normal bundle. This implies that if the space W is simply connected then the rational homotopy type of $\widetilde{W}$ depends only on the rational homotopy class of f and on the Chern classes of the normal bundle.

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1 Introduction

The blow-up construction comes from complex algebraic geometry; Gromov [12] and McDuff [20] constructed the blow-up for symplectic manifolds. McDuff used it to construct the first examples of simply connected non-Kähler symplectic manifolds. In this paper we study the rational homotopy type of the blow-up construction.

It is well known that all Kähler manifolds are symplectic and a fundamental problem is to find closed symplectic manifolds which cannot be given a Kähler structure. A non-simply connected example was first found by Thurston [28], but to find simply connected examples proved more difficult. In fact McDuff introduced the blow-up construction to resolve this problem. She showed that the blow-up of Thurston's example in a complex projective space has an odd third Betti number and is thus not Kähler. This leads to more precise structural questions.

A space is formal if its rational cohomology algebra serves as a rational model of it. In particular this implies its cohomology algebra determines its rational homotopy type and that it has no nontrivial Massey products in its cohomology. Deligne, Griffiths, Morgan and Sullivan [5] showed that all closed Kähler manifolds are formal. Thurston's example was known to be nonformal. By constructing nontrivial Massey products,

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Babenko and Taimanov [1] showed that the McDuff example is not formal. Rudyak and Tralle [25] extended these results for the blow-ups along many other manifolds. One of the applications of our model is to prove [17] that the blow-up along a manifold M symplectically embedded in a large enough $\mathbb{C}P(n)$ is formal if and only if M is formal.

Although McDuff was interested in the blow-up for symplectic manifolds, the construction is more general. Suppose $f\colon V \to W$ is an embedding of smooth closed oriented manifolds and that the normal bundle of the embedding has been given the structure of a complex vector bundle. This is enough data to construct the blow-up $\widetilde{W}$ of W along V as described by McDuff [20] and outlined in Section 2 below. If f is an embedding of complex manifolds with the canonical complex structure on the normal bundle then this blow-up is homeomorphic to the classical blow-up of W along V.

Under some restrictions we give a complete description of the rational homotopy type of the blow-up $\widetilde{W}$ using only the rational homotopy class of the map $f\colon V \to W$ and the Chern classes of the normal bundle of V. The rational homotopy type of a space is defined as long as all the spaces involved are nilpotent (see Section 3.3). Note that simply connected closed manifolds are always nilpotent.

Theorem 7.8 *Let $f\colon V \to W$ be an embedding of closed orientable smooth manifolds and suppose that the normal bundle ν is equipped with the structure of a complex vector bundle. Assume that $\dim W \geq 2\dim V + 3$, that W is simply connected and that V is nilpotent. Then the rational homotopy type of the blow-up of W along V, $\widetilde{W}$ can be explicitly determined from the rational homotopy type of f and from the Chern classes $c_i(\nu) \in H^{2i}(V;\mathbb{Q})$.*

More generally our description holds as long as V, W and $\widetilde{W}$ are nilpotent, $H^1(f;\mathbb{Q})$ is injective and $\dim(W) \geq 2\dim(V) + 3$ (Corollary 7.7). It is not so easy to determine if $\widetilde{W}$ is nilpotent, so the usual case is W simply connected. We also note that without the dimension restriction, $\dim(W) \geq 2\dim(V) + 3$, there exist manifolds V and W and homotopic embeddings $V \to W$ with isomorphic complex normal bundles whose blow-ups are not rationally equivalent by our earlier paper [19]. Thus our dimension restriction, $\dim(W) \geq 2\dim(V) + 3$, cannot be discarded.

Sullivan [26] studies rational homotopy using his piecewise linear forms $A_{\mathrm{PL}}(\_)$ functor, which is analogous to the de Rham differential forms functor, $\Omega^*(\_)$. In particular it is contravariant and for a topological space X, $A_{\mathrm{PL}}(X)$ is a commutative differential graded algebra or CDGA (defined in Section 3.2). Another way to think of $A_{\mathrm{PL}}(X)$ is as a commutative version of the cochains on X with coefficients in $\mathbb{Q}$, $C^*(X;\mathbb{Q})$. A *model* of X is any CDGA weakly equivalent to $A_{\mathrm{PL}}(X)$ (see Section 3.1). Similarly a model of a map $f\colon X \to Y$ is any map of CDGAs weakly equivalent to

$A_{\mathrm{PL}}(f)\colon A_{\mathrm{PL}}(Y) \to A_{\mathrm{PL}}(X)$. Under some finiteness conditions any model of X completely determines the rational homotopy type of X (see Section 3.3). It is in this sense that we determine the rational homotopy type of $\widetilde{W}$. In fact we determine the homotopy type of the CDGA $A_{\mathrm{PL}}(\widetilde{W})$ without any finiteness restrictions.

In Section 7.2 and Theorem 7.6 using only a model for f and the Chern classes of the normal bundle we construct an explicit model for $\widetilde{W}$. This is our main result and Theorem 7.8 follows directly from it. There are a number of interesting byproducts produced along the way to the proof. For example we derive algebraic models for the compliment of the embedding (see also our paper [18]) and for the projectivization of a complex bundle.

In the last section of the paper we give a few applications of the model of the blow-up. First we study the special case of a blow-up of $\mathbb{C}P(n)$, and looking at McDuff's example of the blow-up of $\mathbb{C}P(n)$ along the Kodaira–Thurston manifold we prove the existence of nontrivial Massey products by direct calculation. Our next application is to calculate the cohomology algebra of the blow-up along $f\colon V \to W$ under our dimension restrictions (Section 8.4). This is complementary to work of Gitler [10] who gave a different description of this algebra when $H^*(f)$ is surjective. In Section 8.5 we use this calculation to show that there are infinitely many distinct rational homotopy types of symplectic manifolds that can be constructed as the blow-up of $\mathbb{C}P(5)$ along $\mathbb{C}P(1)$.

1.1 Contents

The sections break down as follows. Section 1: Introduction, Section 2: Modelling the blow-up, Section 3: Background and notation, Section 4: Thom class and the shriek map, Section 5: Model of the complement of a submanifold, Section 6: Model of the projectivization of a complex bundle, Section 7: The model of the blow-up, Section 8: Applications.

A more detailed list of contents appears at the beginning of each section.

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2 Modelling the blow-up

In this section we first describe the topology of the blow-up construction and then describe the model of the blow-up. The precise statement may be found in Theorem 7.6.

Again suppose $f\colon V \to W$ is an embedding of connected closed oriented manifolds and suppose the normal bundle ν of f has been given a complex structure. Let T be a tubular neighborhood of V in W. Let ∂T be the boundary of T and $B = \overline{W \setminus T}$. Then $T \cup B = W$ and $T \cap B = \partial T$. Hence we have a pushout

(1)
$$\begin{tikzcd} \partial T \arrow[r, "k"] \arrow[d] & B \arrow[d] \\ T \arrow[r] & W. \end{tikzcd}$$

By the Tubular Neighborhood Theorem [22, Theorem 11.1] there is a diffeomorphism between T and the disc bundle $D\nu$ that sends V to the zero section of $D\nu$ and sends ∂T to the sphere bundle $S\nu$. Since ν is a complex bundle we can quotient by the $S^1 \subset \mathbb{C}^*$–action on $S\nu \cong \partial T$. We obtain a complex projective bundle $P\nu$ over V and a commutative diagram:

$$\begin{tikzcd} \partial T \arrow[r, "q"] \arrow[d] & P\nu \arrow[dl] \\ V & \end{tikzcd}$$

Next we can remove T from W and instead of putting it back as in (1) we can replace it by $P\nu$. This gives us a pushout

(2)
$$\begin{tikzcd} \partial T \arrow[r, "k"] \arrow[d, "q"'] & B \arrow[d] \\ P\nu \arrow[r] & \widetilde{W}. \end{tikzcd}$$

The space $\widetilde{W}$ is called the *blow-up of* W *along* V. This actually only gives us the homeomorphism type of the blow-up but with slightly more care we can get the diffeomorphism type. Since we are only studying the rational homotopy type, homeomorphism type is more than enough. An important point is that (2) is also a homotopy pushout.

In the next few paragraphs we analyze Diagram (2) and give an idea of how we construct the model $\mathcal{B}(R,Q)$ of $A_{\mathrm{PL}}(\widetilde{W})$. We begin with the fact that $A_{\mathrm{PL}}(\_)$ takes homotopy pushouts to homotopy pullbacks. So we are left with analyzing the homotopy pullback of the diagram

$$\begin{tikzcd} A_{\mathrm{PL}}(\partial T) & A_{\mathrm{PL}}(B) \arrow[l, "A_{\mathrm{PL}}(k)"'] \\ A_{\mathrm{PL}}(P\nu) \arrow[u, "A_{\mathrm{PL}}(q)"] & \end{tikzcd}$$

which we get from applying $A_{\mathrm{PL}}(\_)$ to Diagram (2). To do this we construct models of the two "legs" of the diagram, the maps $A_{\mathrm{PL}}(B) \to A_{\mathrm{PL}}(\partial T)$ and $A_{\mathrm{PL}}(P\nu) \to A_{\mathrm{PL}}(\partial T)$. The pullback of these models should be a model of the pullback of the diagram. However we need to be careful how the models are "glued" together because different gluings correspond to homotopy automorphisms of $A_{\mathrm{PL}}(\partial T)$ and can lead to pullbacks that are not homotopy equivalent.

To construct a model of k we make use of our cochain level version of the classical shriek map $f^!\colon H^{*-r}(V) \to H^*(W)$, where r is the codimension of V in W. Under our hypothesis that $\dim(W) \geq 2\dim(V)+3$, a model of k can be constructed from any model of f and so the rational homotopy type of ∂T and B and the rational homotopy class of k depend only on the rational homotopy class of f. This is to be expected since we are in the stable range where homotopic maps are isotopic. We showed in [18] that without our dimension restriction the rational homotopy class of k can depend on more than just the rational homotopy type of f, and so it is at this point that the dimension restriction for the model of the blow-up arises. In [19] we showed that the blow-up along homotopic embeddings with the same Chern classes on their normal bundles can have different homotopy types.

Next we more fully describe the model of B. Suppose

$$\phi\colon R \to Q$$

is a model of $A_{\mathrm{PL}}(f)\colon A_{\mathrm{PL}}(W) \to A_{\mathrm{PL}}(T)$. We can consider Q as a differential graded R–module (or R–dgmodule, see Section 3.2). Suppose $u_V \in H^*(Q)$ and $u_W \in H^*(R)$ are orientation classes, where $m = \dim(V)$ and $n = \dim(W)$. Let

$$\phi^!\colon s^{-r}Q \to R$$

be a map of R–dgmodules such that $H(\phi^!_*)(u_V) = u_W$, where $r = n - m$. (Recall that the suspension operator gives $(s^{-r}Q)^{j+r} = Q^j$, so $u_V \in H^n(s^{-r}Q)$.) Such a map will be called a *shriek map*. A shriek map always exists (Proposition 4.5) and is unique up to homotopy (Proposition 4.4). Let $m = \dim V$, $n = \dim W$ and $r = n - m$. We describe a model of $k\colon \partial T \to B$, and refer the reader to Lemma 5.8 for the details.

Lemma 2.1 *Assume $n \geq 2m+3$. If $R^{\geq n+1} = 0$ and $Q^{\geq m+2} = 0$ then there exists explicit CDGA structures on $R \oplus ss^{-r}Q$ and $Q \oplus ss^{-r}Q$ determined by the CDGA structures on R and Q and by the shriek map $\phi^!$ such that the map*

$$\phi \oplus \mathrm{id}\colon R \oplus ss^{-r}Q \to Q \oplus ss^{-r}Q \tag{3}$$

is a CDGA model of $A_{\mathrm{PL}}(k)\colon A_{\mathrm{PL}}(B) \to A_{\mathrm{PL}}(\partial T)$.

From any model $\phi\colon R \to Q$, one satisfying the degree restrictions can always be constructed (Proposition 4.5). The differential on $R \oplus ss^{-r}Q$ comes from the fact that it is actually the mapping cone on $\phi^!$ (see Section 3.8) and the CDGA structure is what we call the *semitrivial* CDGA structure (see Definition 3.19).

Constructing a model of $q\colon \partial T \to P\nu$ is more straightforward. The cohomology algebra of the projective bundle can be used to define the Chern classes $c_i(\nu)$ of the normal bundle ν [2, IV.20]. This description allows us to construct a model of $P\nu$. The free graded commutative algebra on the graded generators a_i is denoted by $\Lambda(a_1, \dots, a_n)$. Next we describe a model of $q\colon \partial T \to P\nu$, and refer to Proposition 6.9 for the details.

Theorem 2.2 *Assume $n \geq 2m+3$.Let $2k = n - m$ and let $\gamma_0 = 1$ and $\gamma_i \in Q$ be representatives of the Chern classes $c_i(\nu)$. Suppose $|x| = 2$ and $|z| = 2k-1$. Define CDGAs $(Q \otimes \Lambda(x,z); D)$ by $Dx = 0$ and $Dz = \sum_{i=0}^{k-1} \gamma_i x^{k-i}$ with and $(Q \otimes \Lambda z; \bar{D})$ by $\bar{D}(z) = 0$. Then the projection map*

$$\mathrm{proj}\colon Q \otimes \Lambda(x,z) \to Q \otimes \Lambda z \tag{4}$$

sending x to 0 is a CDGA model of $A_{\mathrm{PL}}(q)\colon A_{\mathrm{PL}}(P\nu) \to A_{\mathrm{PL}}(\partial T)$.

In the last theorem the dimension restriction $n \geq 2m+3$ arises since we require that $\bar{D}z = 0$, which represents the Euler class of the bundle, but other than that are not really needed and probably a similar theorem without the dimension restrictions is true. Having constructed models for both k and q of Diagram (2) we can use them to construct a model $\mathcal{B}(R,Q)$ of the pushout $\widetilde{W}$ by taking the pullback of these two models. It is described precisely in Section 7.2. The form of $\mathcal{B}(R,Q)$ is:

$$\mathcal{B}(R,Q) = \left(R \oplus Q \otimes \Lambda^+(x,z), D\right).$$

As mentioned before, to get $\mathcal{B}(R, Q)$ we have to be careful how the models of k and q fit together. In the model (3) of k the model of $A_{\mathrm{PL}}(\partial T)$ is $Q \oplus ss^{-r} Q$ whereas in the model (4) of q it is $Q \otimes \Lambda(z)$. These are always isomorphic as Q–modules and our dimension restrictions imply that they are isomorphic as CDGAs. The problem is to make sure we pick the correct isomorphism. While we are constructing our models of k and q, we keep track of the isomorphism with the help of orientation classes on both of our manifolds and on the normal bundle. In our special situation, this orientation information together with the Q–dgmodule structures is enough to determine the isomorphism. Once we have this nailed down it is straightforward to construct a model of the blow-up from our models of k and q using a pullback.

3 Background and notation

For this paper the ground field and the coefficient ring will be the rational numbers, unless otherwise stated. For a topological space X, $H^*(X)$ refers to its singular cohomology. We denote also by $H(\_)$ the functor taking homology of a differential complex. We will use $H^n(\_)$ to refer both to the n–th singular cohomology group and to the homology in degree n of any differential graded object.

3.1 Categorical preliminaries

For a small category I and any category $\mathcal{D}$, let $\mathcal{D}^I$ be the *diagram category* defined as follows: the objects of $\mathcal{D}^I$ are the functors $I \to \mathcal{D}$ and the morphisms are natural transformations. We often refer to the objects in such a category as diagrams. For example our category I can consist of two objects with exactly one nonidentity map joining them (this category can be depicted as $\bullet \to \bullet$) and each diagram in $\mathcal{D}^I$ corresponds to a single map in $\mathcal{D}$. Similarly I could be the category: $\bullet \longleftarrow \bullet \longrightarrow \bullet$ which corresponds to the data of a pushout in $\mathcal{D}$ or its dual $\bullet \longrightarrow \bullet \longleftarrow \bullet$ (data for a pullback). We also use squares of objects which correspond to the category

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

If

$$\mathbf{A} = \begin{array}{ccc} A_1 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_3 & \longrightarrow & A_4 \end{array}$$

and
$$\mathbf{B} = \begin{tikzcd} B_1 \arrow[r] \arrow[d] & B_2 \arrow[d] \\ B_3 \arrow[r] & B_4 \end{tikzcd}$$
are squares of objects and
$$\alpha_i\colon A_i \to B_i$$
are maps, then
$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}\colon \mathbf{A} \to \mathbf{B}$$
denotes the map between the squares that would traditionally be denoted as a commutative cube:
$$\begin{tikzcd} A_1 \arrow[rr] \arrow[dr] \arrow[dd, "\alpha_1"'] & & A_2 \arrow[dr] \arrow[dd, "\alpha_2"] & \\ & A_3 \arrow[rr] \arrow[dd, "\alpha_3"] & & A_4 \arrow[dd, "\alpha_4"] \\ B_1 \arrow[rr] \arrow[dr] & & B_2 \arrow[dr] & \\ & B_3 \arrow[rr] & & B_4. \end{tikzcd}$$
If $\mathcal{D}$ is the category of topological spaces and $F, G \in \mathcal{D}^I$ then a morphism $\eta\colon F \to G$ is a *homeomorphism* if for each $i \in I$, $\eta(i)\colon F(i) \to G(i)$ is a homeomorphism for each element. If $\mathcal{D}$ has weak equivalences then η is called a *weak equivalence* if $\eta(i)$ is a weak equivalence for each $i \in I$. Two diagrams are weakly equivalent if they are connected by a chain of weak equivalences. So $F, G \in \mathcal{D}^I$ would be weakly equivalent if there exists a diagram in $\mathcal{D}^I$ as below with all the maps weak equivalences:
$$F \xleftarrow{\simeq} F_1 \xrightarrow{\simeq} F_2 \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} F_n \xrightarrow{\simeq} G$$
Notice that there may not be a direct map between F and G.

3.2 Homotopy theory of CDGA and R–DGMod

A good reference for the categories CDGA and R–DGmod is the book of Felix–Halperin–Thomas [9]. Next we review the notion of CDGA. Let $A = \bigoplus_{i=0}^{\infty} A^i$ be a graded vector space together with an associative multiplication $\mu\colon A \otimes A \to A$, a unit $1 \in A^0$ and a linear map of degree $+1$, $d\colon A \to A$, called the differential. We denote $\mu(a,b)$ by ab or sometimes $a \cdot b$. If $a \in A^n$ we write $|a| = n$ and say that

a has degree n. We require that the multiplication μ is graded commutative, that $d^2 = 0$ and that d is a derivation . In other words we require that $|ab| = |a| + |b|$, that $ab = (-1)^{|a||b|}ba$ and that the Leibnitz law $d(ab) = (da)b + (-1)^{|a|}a(db)$ holds. Such an (A, d) is called a commutative differential graded algebra or CDGA. Notice that the multiplication is suppressed. Sometimes we just write A also suppressing the differential. We call A *connected* if $A^0 = \mathbb{Q} \cdot 1$. Maps between CDGAs are graded vector space homomorphisms that commute with the multiplication and the differential. Clearly we get a category which we will also denote CDGA. A special case of CDGA are Sullivan algebras $(\Lambda V, d)$ where ΛV is the free graded commutative algebra on a graded vector space V and d is a differential on ΛV satisfying a certain nilpotence condition as described in [9, Part II]. However we will relax their condition that V should be concentrated in positive degrees and just demand that V be concentrated in nonnegative degrees. If $V = \mathbb{Q}\langle a_i, b_j\rangle$ is a graded vector space with basis $\{a_i\} \cup \{b_j\}$ where the a_i are homogeneous in even degrees and the b_j are homogeneous in odd degree then as a graded algebra $\Lambda V \cong P(a_i) \otimes E(b_i)$ where $P(a_i)$ is the free graded polynomial algebra on the a_i and $E(b_i)$ is the free graded exterior algebra on the b_i.

We can also define a category of differential graded modules over some fixed CDGA. Let (R, d) be a CDGA. Let $M = \bigoplus_{i=-\infty}^{\infty} M^i$ be a differential graded R–module with structure map μ: $R \otimes M \to M$ and differential of degree $+1$, d': $M \to M$. We require that $d'(rm) = (dr)m + (-1)^{|r|}d'm$. Of course M is an R–module in the ungraded sense as well and $|rm| = |r| + |m|$. We call such an object a differential graded R–module or an R–dgmodule. Maps are R–module maps that preserve the grading and commute with the differential. The category of R–dgmodules and R–dgmodule maps is denoted by R–DGmod.

Weakly equivalent diagrams of CDGA or R–dgmodules are referred to as *models* of each other. For example let f: $X \to Y$ be a map of spaces. If a diagram ϕ: $A \to B$ of CDGA is weakly equivalent to the diagram $A_{\mathrm{PL}}(f)$: $A_{\mathrm{PL}}(Y) \to A_{\mathrm{PL}}(X)$, then we will say that ϕ: $A \to B$ is a model of $A_{\mathrm{PL}}(f)$. In this case, as is common, we also call ϕ a model of f. Similarly if A is a model of $A_{\mathrm{PL}}(X)$ we call A a model of X. The concept of a CDGA model of a space or a map is well established. Note however that many authors reserve the term model to refer to free CGDA whereas we use it more generally. For our purposes we also need models of more general diagrams. In fact since we will be gluing diagrams together we will sometimes need that the equivalences between our diagrams preserve certain extra structure.

Both CDGA and R–DGMod are closed model categories (see Dwyer and Spaliński [6] for a review of model categories and Bousfield and Gugenheim [3] for a proof of the fact that CDGA satisfies the axioms of a closed model category). It is not necessary for the reader to be familiar with closed model categories since all of the relevant results can be proved directly in these categories. In both categories the closed model structure is determined by the following families of maps:

- fibrations are surjections,
- weak equivalences are quasi-isomorphisms.

We use the terms fibration and surjection interchangeably when working in CDGA or R–DGMod. By a *cellular cofibration* we mean:

- in CDGA, a relative Sullivan algebra $B \to B \otimes \Lambda V$ as defined in [9, Chapter 14] except that we allow nonnegatively graded V instead of just positively graded V.
- in R–DGMod, a semifree extension $M \to M \oplus R \otimes V$ as defined in [8, Section 2].

We will generally deal with cofibrations that are cellular ones. Note that not all cofibrations are of this form (see Bousfield and Gugenheim [3, Section 4.4]) but at least all cofibrations between connected CDGA are. A map that is both a cofibration and a weak equivalence is called an acyclic cofibration. Similarly a map that is both a fibration and a weak equivalence is called an acyclic fibration. If $\varnothing$ denotes the initial object of the category (it is $\mathbb{Q}$ concentrated in dimension 0 in CDGA and 0 in R–DGMod) then a *cellular cofibrant object* is an object X such that the map $\varnothing \to X$ is a cellular cofibration. In CDGA cellular cofibrant objects are the Sullivan algebras [9, Section 12] and in R–DGMod they are the semifree R–dgmodules [8, Section 2]. Dually if $*$ denotes the terminal object then an object X is called *fibrant* if $X \to *$ is a fibration. Note that all objects in CDGA and R–DGMod are fibrant.

The following lemma is a slight modification of one of the axioms of a closed model category.

Lemma 3.1 *Suppose A and B are CDGAs such that $H^0(A) = H^0(B) = 0$. Any CDGA map $f\colon A \to B$ can be factored into a cellular cofibration follow by an acyclic fibration.*

Proof This is proved as part of [13, Theorem 6.1] with the added condition that A is augmented. However the augmentation is not used to get the result of the lemma. □

The analogous result in R–DGmod is also true and follows by standard arguments.

Lemma 3.2 *Any map in R–DGmod can be factored into an acyclic cellular cofibration followed by a fibration and also into a cellular cofibration followed by an acyclic fibration.*

Homotopies in a closed model category can be defined with the help of a cylinder object which we describe now. Let $\_\coprod\_$ denote the coproduct, and $\nabla : A\coprod A \to A$ the fold map. Factor ∇ into a cellular cofibration $i_0 + i_1\colon A\coprod A \to \mathrm{Cyl}\, A$ followed by an acyclic fibration. This implies in particular that the maps $i_0, i_1\colon A \to \mathrm{Cyl}\, A$ are weak equivalences. The object $\mathrm{Cyl}\, A$ is called the *cylinder object* of A, it is unique up to homotopy. Two morphisms $f_0, f_1\colon A \to X$ are *homotopic* if there exists a map $H\colon \mathrm{Cyl}\, A \to X$ such that $Hi_0 = f_0$ and $Hi_1 = f_1$. We write $f_0 \simeq f_1$, or $f_0 \simeq_R f_1$ if we wish to emphasize the fact that the homotopy is in the category R–DGMod. Then it is easy to check from the definition of a homotopy in R–DGMod:

Lemma 3.3 *Two morphisms $f_0, f_1\colon A \to X$ in R–DGMod with A cofibrant are homotopic if there exists an R–module degree -1 morphism*

$$h\colon A \to X$$

such that $d_X h + h d_A = f_0 - f_1$. Note that such a homotopy can also be seen as an R–module degree 0 morphism $h\colon sA \to X$ where s is the suspension (see Definition 3.10).

In CDGA our notion of homotopy is also equivalent to the more traditional one [9, Chapter 12(b)].

We recall the notion of sets of homotopy classes of maps in CDGA and R–DGmod. Let $X, Y \in$ CDGA or R–DGMod. Factor $\varnothing \to X$ as a cofibration followed by an acyclic fibration $\varnothing \to \widehat{X} \to X$. We define the set of homotopy classes of maps from X to Y, $[X, Y]$ as the set of equivalence classes of $\mathrm{Hom}(\widehat{X}, Y)$ under the homotopy relation in either CDGA or R–DGMod:

$$[X, Y] = \mathrm{Hom}(\widehat{X}, Y)/\simeq .$$

We will write $[X, Y]_R$ if we wish to emphasize we are looking at homotopy classes of R–dgmodule maps. The Lifting Lemma (Lemma 3.4 below) implies that $[X, Y]$ is independent of the choice of $\widehat{X}$ (any two choices give naturally isomorphic sets) and there is an obvious map $\mathrm{Hom}(X, Y) \to [X, Y]$. Weak equivalences in the range and domain of $[\_,\_]$ induce bijections of sets. To those already familiar with closed model categories we point out that we don't have to replace the range by a fibrant object since all objects in our category are already fibrant.

3.3 The functors $A_{\mathrm{PL}}(\_)$ and $|\_|$

A connected space X is *nilpotent* if $\pi_1(X)$ is a nilpotent group and $\pi_n(X)$ is a nilpotent $\pi_1(X)$–module for each $n \geq 2$ (See Hilton, Mislin and Roitberg [14, II.2]).

Note that all simply connected spaces are nilpotent. A nilpotent space X is *rational* if $\pi_n(X)$ is a rational vector space for each $n \geq 1$. A nilpotent space X has *finite $\mathbb{Q}$–type* if $H_n(X;\mathbb{Q})$ is finite dimensional for each $n \geq 1$.

Let Top be the category of topological spaces. There are adjoint functors

$$A_{\mathrm{PL}}\colon Top \to CDGA$$
$$|\_|\colon CDGA \to Top$$

introduced by Sullivan [26]. We refer the reader to [9, II 10 and II 17] for their definitions (see also [3, Section 8] and [26, Section 7 and 8]). These functors induce an equivalence between the homotopy types of finite type CDGA and of nilpotent rational topological spaces of finite $\mathbb{Q}$–type (see Bousfield and Gugenheim [3, 9.4] for a more precise statement). It is in this sense that the homotopy type of $A_{\mathrm{PL}}(X)$ as a CDGA determines the rational homotopy type of X. A key property of A_{PL} that we will use repeatedly is the $\mathbb{Q}$–version of the natural algebra de Rham isomorphism

$$HA_{\mathrm{PL}}(\_) \cong H^*(\_;\mathbb{Q}). \tag{5}$$

This isomorphism holds for all spaces and will be fixed throughout the paper.

In order to reduce clutter in our equations, for a map f of spaces we will often write f^* for $A_{\mathrm{PL}}(f)$.

3.4 Standard abuses of notation

Because of the isomorphism (5), to any cohomology class $\mu \in H^n(X;\mathbb{Z})$, using the change of coefficients morphism $H^*(\_,\mathbb{Z}) \to H^*(\_,\mathbb{Q})$, we can associate a cohomology class in $H^n(A_{\mathrm{PL}}(X))$. Abusing notation we also denote this class by $\mu \in H^n(A_{\mathrm{PL}}(X))$.

Also if R is a CDGA model of another CDGA R' through a fixed chain of weak equivalences

$$R \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} R'$$

and if $\mu \in H^n(R)$ is a cohomology class we will denote the corresponding class $\mu \in H^n(R')$ by the same name. Note that which classes correspond only depends on which element of $[R, R']$ is represented by the chain of weak equivalences.

We will often abuse "inc" to denote any inclusion map and "proj" to denote any projection map. The abuse is twofold since not only will we often not explicitly define our inclusions but also we will have many different maps referred to by the same notation. We will also sometimes use "$\hookrightarrow$" to denote cofibrations or inclusions.

3.5 Closed model category facts

The following result is standard. Let $\mathcal{C}$ be a closed model category.

Lemma 3.4 (Lifting Lemma) *Suppose the following solid arrow diagram in $\mathcal{C}$ is given:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "v"] \\ C \arrow[r, "u"'] \arrow[ur, dashed, "l"] & D. \end{tikzcd}$$

(i) *If the diagram is commutative, g is a cofibration, v is a fibration and g or v is a weak equivalence then there exists a lift l making each triangle commutative. Moreover such a lift l is unique up to homotopy.*

(ii) *If the diagram is commutative (respectively, commutative up to homotopy), g is a cofibration and v is a weak equivalence then there exists a lift l such that $l \circ g = f$ (respectively, $l \circ g \simeq f$) and $v \circ l \simeq u$; that is, the upper triangle is commutative and the lower triangle is commutative up to homotopy (respectively, both triangles are commutative up to homotopy.) Moreover such a lift l is unique up to homotopy.*

Proof See Félix–Halperin–Thomas [9, Chapter 14] for the CDGA case, and [7, Lemma A.3] for the special case of $A = 0$ in R–DGmod. The case for general A follows by standard techniques. $\square$

The following next two lemmas allow us to convert certain homotopy commutative diagrams into strictly commutative diagrams.

Lemma 3.5 *Let $\mathcal{C}$ be a closed model category and suppose*

$$\begin{tikzcd} & \widehat{A} \arrow[dl, "f"'] \arrow[d, "\bar{f}"] \arrow[dr, "f'"] & \\ B & \widehat{B} \arrow[l, "\beta"] \arrow[r, "\beta'"'] & B' \end{tikzcd}$$

is a homotopy commutative diagram in $\mathcal{C}$. If $\widehat{A}$ is a cofibrant object and if $(\beta, \beta')\colon \widehat{B} \to B \oplus B'$ is a fibration (in other words a surjection in CDGA or R–DGMod), then there *exists a morphism $\widehat{f}\colon \widehat{A}\to\widehat{B}$ such that $\overline{f}\simeq\widehat{f}$ and the diagram*

$$\begin{tikzcd} & \widehat{A} \arrow[dl, "f"'] \arrow[d, "\widehat{f}"] \arrow[dr, "f'"] & \\ B & \widehat{B} \arrow[l, "\beta"] \arrow[r, "\beta'"'] & B' \end{tikzcd}$$

is strictly commutative.

Proof Consider the following solid arrow diagram

$$\begin{tikzcd} \widehat{A} \arrow[r, "\overline{f}"] \arrow[d, "i_0"'] & \widehat{B} \arrow[d, "(\beta,\beta')"] \\ \mathrm{Cyl}\,\widehat{A} \arrow[r, "H"'] \arrow[ur, dashed, "G"] & B\oplus B' \end{tikzcd}$$

where H is a homotopy between $(\beta,\beta')\overline{f}$ and (f,f'). The map i_0 is always a weak equivalence and here is a cofibration since $\widehat{A}$ is cofibrant. Also (β,β') is a fibration by hypothesis and hence there is a lift G by Lemma 3.4(i). Let $\widehat{f}=Gi_1$. This makes the diagram of the conclusion commute since $Hi_1=(f,f')$ and $Gi_1\simeq Gi_0=\overline{f}$. □

Lemma 3.6 *Suppose in CDGA or R–DGMod that $f\colon A\to B$ is a model of $f'\colon A'\to B'$. If we are in CDGA then also assume that $H^0(A)=H^0(B)=H^0(A')=H^0(B')=\mathbb{Q}$. Then there exists a cellularly cofibrant $\widehat{A}$, a cellular cofibration $\widehat{f}\colon \widehat{A}\longrightarrow\widehat{B}$, and weak equivalences $\alpha\colon \widehat{A}\xrightarrow{\simeq}A$, $\alpha'\colon \widehat{A}\xrightarrow{\simeq}A'$ $\beta\colon \widehat{B}\xrightarrow{\simeq}B$ and $\beta'\colon \widehat{B}\xrightarrow{\simeq}B'$ such that $(\beta,\beta')\colon \widehat{B}\to B\oplus B'$ and $(\alpha,\alpha')\colon \widehat{A}\to A\oplus A'$ are surjective and the following diagram is strictly commutative*

$$\begin{tikzcd} A \arrow[d, "f"'] & \widehat{A} \arrow[l, "\alpha"', "\simeq"] \arrow[r, "\alpha'", "\simeq"'] \arrow[d, "\widehat{f}"] & A' \arrow[d, "f'"] \\ B & \widehat{B} \arrow[l, "\beta"', "\simeq"] \arrow[r, "\beta'", "\simeq"'] & B'. \end{tikzcd}$$

In addition, the two isomorphisms $H^*(\alpha')H^*(\alpha)^{-1}\in\mathrm{Hom}(H^*(A),H^*(A'))$ and $H^*(\beta')H^*(\beta)^{-1}\in\mathrm{Hom}(H^*(B),H^*(B'))$ are the same as the one determined by the original string of weak equivalences making f a model of f'.

Proof If we let I be the category $(\bullet\to\bullet)$ with two objects and one nonidentity map, then $\mathcal{C}^{\mathcal{I}}$ is the category of maps in $\mathcal{C}$. According to [16, Section 5.1] for any model category $\mathcal{C}$, $\mathcal{C}^{\mathcal{I}}$ can be given a model structure such that if $A_1 \to B_1$ and $A_2 \to B_2$ are in $\mathcal{C}^{\mathcal{I}}$ then a map between them

$$\begin{tikzcd} A_1 \arrow[r, "\alpha"] \arrow[d] & A_2 \arrow[d] \\ B_1 \arrow[r, "\beta"'] & B_2 \end{tikzcd}$$

is a weak equivalence if α and β are both weak equivalences and a fibration if both α and β are fibrations.

Since f is a model of f' they are connected by a sequence of weak equivalences,

$$f = f_0 \xleftarrow{\simeq} g_0 \xrightarrow{\simeq} f_1 \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} g_{n-1} \xrightarrow{\simeq} f_n = f'$$

with each $g_i, f_i \in \mathcal{C}$. We can factor $g_i \to f_i \oplus f_{i+1}$ as an acyclic cofibration followed by a fibration. and therefore we can assume that all of the maps in the sequence are fibrations (since they are surjections when evaluated at each object of I). Also if g_i was cofibrant we replaced it with something cofibrant.

Now let $\hat{f} \xrightarrow{\simeq} f_0$ be a weak equivalence such that $\hat{f}$ is cofibrant. Since they are fibrations, we can lift along the maps of the sequence to get a diagram

$$f \xleftarrow{\simeq} \hat{f} \xrightarrow{\simeq} f'.$$

As before we can assume that the map $\hat{f} \to f \oplus f'$ is a fibration. So we get a diagram

$$\begin{tikzcd} A \arrow[d, "f"'] & \hat{A} \arrow[l, "\alpha"', "\simeq"] \arrow[r, "\alpha'", "\simeq"'] \arrow[d, "\hat{f}"] & A' \arrow[d, "f'"] \\ B & \hat{B} \arrow[l, "\beta"', "\simeq"] \arrow[r, "\beta'", "\simeq"'] & B'. \end{tikzcd}$$

Since $\hat{f} \to f \oplus f'$ is a fibration, $(\alpha, \alpha')\colon \hat{A} \to A \oplus A'$ and $(\beta, \beta')\colon \hat{B} \to B \oplus B'$ are surjections. Finally, by a diagram chase, the isomorphisms $H^*(\alpha')H^*(\alpha)^{-1} \in \mathrm{Hom}(H^*(A), H^*(A'))$ and $H^*(\beta')H^*(\beta)^{-1} \in \mathrm{Hom}(H^*(B), H^*(B'))$ are the same as the one determined by the original string of weak equivalences making f a model of f'. □

3.6 Homotopy pullbacks

Homotopy pullbacks are homotopy invariant versions of pullbacks. They exist in any closed model category [6, Section 10]. We now give a description that is adapted to our categories CDGA and R–DGMod. Let $f\colon D \to C$ be a map in CDGA or R–DGMod and

$$D \xrightarrow{i} D' \xrightarrow{f'} C$$

be a factorization into an acyclic cofibration followed by a fibration. Then the *homotopy pullback* of the diagram

$$B \xrightarrow{g} C \xleftarrow{f} D$$

is the pullback of

$$B \xrightarrow{g} C \xleftarrow{f'} D'.$$

There is a map induced by id_B, id_C and i from the pullback to the homotopy pullback. Both the homotopy pullback and this induced map are unique up to homotopy. A homotopy pullback in CGDA or RDG-Mod gives rise to a Mayer–Vietoris sequence, corresponding to the fact that pushouts of spaces have such sequences and $A_{\mathrm{PL}}(\_)$ takes homotopy pushouts to homotopy pullbacks. The following lemma states some standard facts about homotopy pullbacks.

Lemma 3.7 *Let*

$$\begin{tikzcd} B_1 \arrow[r] \arrow[d] & C_1 \arrow[d] & D_1 \arrow[l] \arrow[d] \\ B_2 \arrow[r] & C_2 & D_2 \arrow[l] \end{tikzcd} \tag{6}$$

be a commutative diagram in CDGA or R–DGMod. Let E_i be the pullback of $B_i \longrightarrow C_i \longleftarrow D_i$ and E_i' its homotopy pullback.

(i) *There is an induced map $E_1' \to E_2'$ such that the following diagram of induced maps commutes:*

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] & E_1' \arrow[d] \\ E_2 \arrow[r] & E_2'. \end{tikzcd}$$

(ii) *If the vertical maps in* (6) *are weak equivalences then the induced map $E_1' \to E_2'$ is a weak equivalence.*

Proof Part (i) follows from the Lifting Lemma and Part (ii) follows from Mayer–Vietoris. □

The following lemma follows directly from Lemma 3.7.

Lemma 3.8 *Suppose that in CDGA or R–DGMod, the diagram*

$$A_1 \to B_1 \to C_1 \leftarrow D_1$$

is a model of the diagram

$$A_2 \to B_2 \to C_2 \leftarrow D_2.$$

Let E_i, F_i be the homotopy pullbacks of $A_i \to C_i \leftarrow D_i$ and $B_i \to C_i \leftarrow D_i$ respectively. Then the induced map $E_1 \to F_1$ is a model of the induced map $E_2 \to F_2$.

3.7 Making a homotopy commutative diagram strictly commutative

In general if we have a homotopy commutative diagram we cannot always replace it with a strictly commuting one. However in the following particularly simple case we can. Recall that weakly equivalent diagrams are connected by a sequence of weak equivalences and weak equivalences between diagrams are strictly commuting diagrams (see Section 3.1).

Lemma 3.9 *Assume that we have a homotopy commutative diagram in CDGA or R–DGMod*

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "\simeq"] & B' \arrow[d, "\simeq"] & C' \arrow[l, "g'"'] \arrow[d, "\simeq"] & D' \arrow[l, "h'"'] \arrow[d, "\simeq"] \\ A \arrow[r, "f"'] & B & C \arrow[l, "g"] & D \arrow[l, "h"] \end{tikzcd} \tag{7}$$

such that all vertical arrows are weak equivalences. Then the diagrams that make up the top and bottom row of (7) *are weakly equivalent.*

Proof Replacing the top row by something weakly equivalent we can assume that the objects in the top row are cofibrant and the middle two vertical arrows are fibrations. Consider first the left square of Diagram (7):

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "\alpha"'] & B' \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

Since β is a surjection and A' is cofibrant the map $f\alpha$ can be lifted through β to get a new map $\bar{f}\colon A' \to B'$. Also by Lemma 3.4 there is a homotopy $H\colon \mathrm{Cyl}(A') \to B'$

between f' and $\overline{f}$. So the following diagram is strictly commutative with all horizontal arrows weak equivalences:

(8)
$$
\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d] & B' \arrow[d, "="] \\
\mathrm{Cyl}(A') \arrow[r, "H"'] & B' \\
A' \arrow[r, "\overline{f}"'] \arrow[u] \arrow[d, "\alpha"'] & B' \arrow[u, "="'] \arrow[d, "\beta"] \\
A \arrow[r, "f"'] & B.
\end{tikzcd}
$$

Similarly, we can replace the center and right squares by a strictly commuting diagram

(9)
$$
\begin{tikzcd}
B' & C' \arrow[l, "g'"'] \arrow[d] & D' \arrow[l, "h'"'] \arrow[d] \\
B' \arrow[u, leftarrow, "="] & \mathrm{Cyl}(C') \arrow[l, "H'"'] & \mathrm{Cyl}\, D' \arrow[l, "\mathrm{Cyl}\, h'"'] \\
B' \arrow[u, "="] \arrow[d, "="'] & C' \arrow[l, "\overline{g}"'] \arrow[u] \arrow[d, "="'] & D' \arrow[l, "h'"] \arrow[u] \arrow[d] \\
B' & C' \arrow[l, "\overline{g}"'] & \mathrm{Cyl}\, D' \arrow[l, "H''"'] \\
B' \arrow[u, "="] \arrow[d, "\beta"'] & C' \arrow[l, "\overline{g}"'] \arrow[u, "="] \arrow[d] & D' \arrow[l, "\overline{h}"'] \arrow[u] \arrow[d] \\
B & C \arrow[l, "g"] & D \arrow[l, "h"]
\end{tikzcd}
$$

where $\mathrm{Cyl}\, h'$ is any lift in the following solid arrow diagram:

$$
\begin{tikzcd}
D' \coprod D' \arrow[r] \arrow[d] & \mathrm{Cyl}(C') \arrow[d] \\
\mathrm{Cyl}\, D' \arrow[r, "H''"'] \arrow[ur, dashed, "\mathrm{Cyl}\, h'"] & C'
\end{tikzcd}
$$

Notice that a lift exists since in this diagram the left map is a cofibration and the right an acyclic fibration. Now glue the bottom of Diagram (8) to the top of Diagram (9) to get a sequence of weak equivalences connecting the left and right columns of the new diagram. This completes the proof of the lemma. □

3.8 Mapping cones

Definition 3.10 Let $k \in \mathbb{Z}$. The *k–th suspension* of an R–dgmodule M is the R–dgmodule $s^k M$ defined by:

- $(s^k M)^j \cong M^{k+j}$ as vector spaces for $j \in \mathbb{Z}$ and this isomorphism is denoted by s^k,
- $r \cdot (s^k x) = (-1)^{|r||k|} s^k (r \cdot x)$ for $x \in M$ and $r \in R$,
- $d(s^k x) = (-1)^k s^k (dx)$ for $x \in M$.

Definition 3.11 Let R be a CDGA. The *mapping cone* of an R–dgmodule morphism $f\colon A \to B$ is the R–dgmodule $(B \oplus_f sA, d)$ defined as follows:

- $B \oplus_f sA = B \oplus sA$ as graded R–modules,
- $d(b, sa) = (d_B(b) + f(a), -sd_A(a))$ for $a \in A$ and $b \in B$.

Recall that a mapping cone gives rise to a long exact cohomology sequence, therefore the following two lemmas follow easily from the five lemma.

Lemma 3.12 *Let*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

be a short exact sequence of R–dgmodules. Then there is a quasi-isomorphism of R–dgmodules

$$p \oplus 0\colon B \oplus_i sA \xrightarrow{\simeq} C.$$

Lemma 3.13 *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A' & \xrightarrow{f'} & B' \end{array} \tag{10}$$

be a homotopy commutative diagram of R–dgmodules and let

$$h\colon sA \to B'$$

be an R–dgmodule homotopy between βf and $f'\alpha$ as in Lemma 3.3. Then there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, hook] \arrow[d, "\beta"] & B \oplus_f sA \arrow[r] \arrow[d, "\beta\oplus_h s\alpha"] & sA \arrow[r] \arrow[d, "s\alpha"] & 0 \\
0 \arrow[r] & B' \arrow[r, hook] & B' \oplus_{f'} sA' \arrow[r] & sA' \arrow[r] & 0
\end{tikzcd}$$

of R–dgmodules in which each line is a short exact sequence and where $\beta \oplus_h s\alpha$ is the R–dgmodule morphism defined by

$$(\beta \oplus_h s\alpha)(b, sa) = (\beta(b) + h(sa), s\alpha(a)).$$

Moreover if α and β are quasi-isomorphisms then so is $\beta \oplus_h s\alpha$. If the diagram (10) *is strictly commutative then we can take $h = 0$ and $\beta \oplus_h s\alpha = \beta \oplus s\alpha$.*

3.9 Sets of homotopy classes in R–DGMod

Let R be a CDGA and M and N be R–dgmodules with differentials d_M and d_N respectively. Forgetting about the differentials we denote by $\mathrm{Hom}^i_R(M, N)$ the vector space of R–module maps from M to N raising degree by i. We can put a differential D on $\mathrm{Hom}_R(M, N) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^i_R(M, N)$ as follows. For $f \in \mathrm{Hom}^i_R(M, N)$ and $m \in M$ define

$$(Df)(m) = d_N(f(m)) + (-1)^i f(d_M(m)).$$

The differential D is of degree $+1$ and turns $\mathrm{Hom}_R(M, N)$ into a chain complex. It is easy to check that the cycles are exactly the chain maps. Let

$$\rho\colon H^0(\mathrm{Hom}_R(M, N)) \to [M, N]_R \tag{11}$$

be any map which when restricted to cycles in $\mathrm{Hom}^0(M, N)$ gives their equivalence class in $[M, N]_R$. Since a map is a cycle if and only if it is a chain map and a boundary if and only if it is homotopic to 0, if M is cofibrant then ρ is an isomorphism.

We consider R and $H^n(R)$ as chain complexes with 0 differential, with R concentrated in degree 0 and $H^n(R)$ concentrated in degree n. Let $\overline{\epsilon}\colon R \to s^{-n}H^n(R)$ be any chain map that induces the identity isomorphism after taking H^n. In other words such that $\overline{\epsilon}$ maps dimension n cocycles to their equivalence classes in cohomology. The condition that $\overline{\epsilon}$ is a chain map of course implies that all other dimensions go to 0. Any such $\overline{\epsilon}$ gives us a map

$$\epsilon\colon R \to \mathrm{Hom}_{\mathbb{Q}}(R, s^{-n}H^n(R)),\ r \mapsto \overline{\epsilon}(r \cdot \_). \tag{12}$$

It is straightforward to check that ϵ is a degree 0 map of R–dgmodules. Note that in the general case where we don't assume commutativity, for $s \in R$ we would have $\epsilon(r)(s) = (-1)^{|r||s|}\overline{\epsilon}(sr)$. Since we are working in the graded commutative situation $(-1)^{|r||s|}sr = rs$. Thus we get the same formula in our situation but with fewer signs.

Definition 3.14 A connected graded algebra R is a *Poincaré duality algebra* of formal dimension n if ϵ is a quasi-isomorphism.

For M a left R–dgmodule, N a right R–dgmodule and L a chain complex, define

$$\phi\colon \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Q}}(N, L)) \to \mathrm{Hom}_{\mathbb{Q}}(N \otimes_R M, L) \tag{13}$$

by $\phi(f)(n \otimes m) = (-1)^{|n||m|} f(m)(n)$. Clearly ϕ is a degree 0 isomorphism of chain complexes.

Lemma 3.15 *Let R be a CDGA such that $H^*(R)$ is a Poincaré duality algebra of formal dimension n. Let P be an R–dgmodule. Then the map*

$$H^n\colon [P, R]_R \to \mathrm{Hom}(H^n(P), H^n(R)),\ [f] \mapsto H^n(f).$$

is an isomorphism of $\mathbb{Q}$–modules.

Proof By the definition of $[P, R]_R$ we may assume that P is cofibrant. Since R is a Poincaré duality algebra and P is cofibrant we have a quasi-isomorphism $\epsilon_*\colon H^0(\mathrm{Hom}_R(P, R)) \to H^0(\mathrm{Hom}_R(P, \mathrm{Hom}_{\mathbb{Q}}(R, s^{-n}H^n(R)))$ induced by Equation (12). Also ϕ and ρ from Equations (11) and (13) are isomorphisms. So we get a string of isomorphisms:

$$\begin{aligned}
[P, R]_R &\overset{\rho^{-1}}{\to} H^0(\mathrm{Hom}_R(P, R)) \\
&\overset{\epsilon_*}{\to} H^0(\mathrm{Hom}_R(P, \mathrm{Hom}_{\mathbb{Q}}(R, s^{-n}H^n(R))) \\
&\overset{\phi_*}{\to} H^0(\mathrm{Hom}_{\mathbb{Q}}(R \otimes_R P, s^{-n}H^n(R))) \\
&= H^0(\mathrm{Hom}_{\mathbb{Q}}(P, s^{-n}H^n(R))) \\
&= \mathrm{Hom}_{\mathbb{Q}}(H^n(P), H^n(R))
\end{aligned}$$

It is straightforward to check that the composition of these maps is the same as the map induced by taking H^n. □

Lemma 3.16 *Let P and X be R–dgmodules. If $H^{<r}(P) = 0$ and $H^{\geq r}(X) = 0$ for some $r \in \mathbb{Z}$ then $[P, X]_R = 0$.*

Proof This is a straightforward obstruction argument. See Lemma 3.3. □

3.10 Turning R–dgmodule structure into CDGA structure

Definition 3.17 Let R be a commutative graded algebra (CGA) and let X be a right R–module. Then, the *semitrivial CGA* structure on $R \oplus X$ is the multiplication

$$\mu\colon (R \oplus X) \otimes (R \oplus X) \to (R \oplus X)$$

defined, for homogeneous elements $r, r' \in R$ and x, x' in X, by:

- $\mu(r \otimes r') = r.r'$
- $\mu(x \otimes r') = (-1)^{|r'||x|} r' \cdot x$
- $\mu(r \otimes x') = r \cdot x'$
- $\mu(x \otimes x') = 0$.

Note that if d is a differential of $R-$modules on $R \oplus X$, it is in general not true that the multiplication μ defined above defines a CDGA structure on $(R \oplus X, d)$ because the Leibniz rule is not necessarily satisfied. However, we have the following:

Proposition 3.18 *Let R be a CDGA, let Q be an R–dgmodule and let $f\colon Q \to R$ be a morphism of R–dgmodules. If either*

(1) *$f\colon Q \to R$ is the inclusion of an ideal with the R module structure on Q given by multiplication in R, or*
(2) *there exist $p \in \mathbb{N}$ such that $Q^i = 0$ for $i < p$ and $i \geq 2p$*

hold then the semitrivial CGA structure on the mapping cone $R \oplus_f sQ$ defines a CDGA structure.

Proof To see that the semitrivial CGA structure defines a CGDA structure we only have to check that the Leibniz law holds. By definition $(r, sq) \cdot (r', sq') = (rr', s(rq' + (-1)^{|q||r'|} r'q))$. Since the product is bilinear and the differential is linear, we can check Leibniz on terms of the form $(r, 0) \cdot (r', 0)$, $(r, 0) \cdot (0, sq')$, $(0, sq') \cdot (r', 0)$ and $(0, sq) \cdot (0, sq')$. Since R satisfies Leibniz so do terms of the first type. Since Q is an R–module and f is an R–dgmodule map, it follows directly from the definition of the differential that Leibniz is satisfied for terms of the second and third type.

Next consider terms of the fourth type. On one side of the Leibniz equation we have $d[(0, sq) \cdot (0, sq')] = d(0) = 0$. On the other side we have,

$$\begin{aligned} d(0, sq) \cdot (0, sq') + (-1)^{|sq|} (0, sq) \cdot d(0, sq') \\ &= (f(q), -sdq) \cdot (0, sq') + (-1)^{|q|-1} (0, sq) \cdot (f(q'), -sdq') \\ &= f(q) sq' + (-1)^{|q|-1} (-1)^{|q'|(|q|-1)} f(q') sq. \end{aligned}$$

In Case 2 this term is zero for degree reasons. In Case 1 we continue using the fact that the multiplication is given by that in R,

$$\begin{aligned}
&= (-1)^{|q|} s(qq') + (-1)^{|q'|}(-1)^{|q|-1}(-1)^{|q'|(|q|-1)} s(q'q) \\
&= (-1)^{|q|} s(qq') + (-1)^{|q||q'|}(-1)^{|q'|}(-1)^{|q|-1}(-1)^{|q'|(|q|-1)} s(qq') \\
&= (-1)^{|q|} s(qq') - (-1)^{|q|} s(qq') = 0.
\end{aligned}$$

So in both cases the product is zero and Leibniz holds. □

Definition 3.19 The CDGA structure of the last proposition is called the *semitrivial CDGA structure* on the mapping cone $R \oplus_f sQ$.

4 Thom class and the shriek map

Throughout this section we fix a connected oriented smooth manifold W of dimension n and an oriented connected closed smooth submanifold $V \subset W$ of dimension m and codimension $r = n - m$. We denote this embedding by

$$f \colon V \hookrightarrow W,$$

and the orientation classes by $u_V \in H^m(V;\mathbb{Z})$ and $u_W \in H^n(W;\mathbb{Z})$.

In this section we introduce the ingredients needed for describing a model of $W \setminus V$ and we build a first version of such a model. In Section 4.1 we review the description of the Thom isomorphism and the normal bundle of V in W as a tubular neighborhood T [22, Chapters 10 and 11]. We also describe W as a pushout (Diagram (14)), for which we find a rational model in Section 6. We associate a Thom class $\overline{\theta} \in H^*(T, \partial T;\mathbb{Z})$ to the normal bundle of the embedding f compatible in a certain way with the orientations of V and W (Lemma 4.1). In Section 4.2 and Section 4.3 we introduce our chain level version of the shriek map and prove its existence and uniqueness. Finally in Section 4.4 we give a first CDGA model of Diagram (14) below using mapping cones (Lemma 4.7). This is a very specific model but an intermediate step in constructing the CDGA model of this diagram based on any model of f, which is done in the Lemma 5.8.

4.1 Thom class and orientations

For this subsection alone our coefficients for cohomology are $\mathbb{Z}$. By the tubular neighborhood theorem [22, Theorem 11.1] the normal bundle ν of the embedding f is diffeomorphic to some open neighborhood T' of V in W. The associated normal disk bundle $D\nu$ can be identified with a compact manifold $T \subset T'$ such that the inclusion makes V a strong deformation retract of T. Let $B := \overline{W \setminus T}$ be the closure of the complement of T. Then B is a compact manifold with boundary $\partial B = \partial T = B \cap T$ and $B \cup T = W$. In other words we have the following pushout:

$$\begin{tikzcd} \partial T \arrow[r, hook, "k"] \arrow[d, hook, "i"'] & B \arrow[d, hook, "l"] \\ T \arrow[r, hook, "j"] & W \end{tikzcd} \tag{14}$$

This pushout is also a homotopy pushout since the inclusion $\partial T \to T$ is a cofibration [15, Theorem 13.1.10 and Proposition 13.5.4]. It is clear that the projection of the normal bundle $D\nu$ on V defines a homotopy inverse

$$\pi \colon T \xrightarrow{\simeq} V$$

of the inclusion such that the composition

$$V \subset T \overset{j}{\hookrightarrow} W$$

is equal to f. Also $B \simeq W \setminus V$.

The projection map $(T, \partial T) \to V$ is an orientable (D^r, S^{r-1})–bundle. Let $* \in V$ be some point and consider $(D^r, S^{r-1}) = (T, \partial T)|_*$ as the restriction of the bundle over that point. We thus have an associated inclusion inc: $(D^r, S^{r-1}) \to (T, \partial T)$. Let τ_W and τ_V be the tangent bundles of W and V respectively. We can write

$$\tau_W|_V \cong \tau_V \oplus \nu$$

Because W and V are orientable so is ν. Let $D^m \subset V$ be a closed neighborhood of $*$ with boundary S^{m-1} and D^n be a neighborhood of $*$ in W with boundary S^{n-1}. The given orientations u_V and u_W induce orientations

$$u_{D^m} \in H^m(D^m, S^{m-1}) \cong H^m(V, V \setminus *) \cong H^m(V)$$

and

$$u_{D^n} \in H^n(D^n, S^{n-1}) \cong H^n(W, W \setminus *) \cong H^n(W).$$

By [22, Lemma 11.6] these correspond to orientations on τ_W and τ_V. Suppose an orientation

$$u_{D^r} \in H^r(D^r, S^{r-1}) \tag{15}$$

on ν has also been given. We say that u_V, u_W and u_{D^r} are *compatible* if they satisfy the formula

$$u_{D^n} = u_{D^m} \times u_{D^r} \in H^{m+r}((D^m, S^{m-1}) \times (D^r, S^{r-1})) = H^n(D^n, S^{n-1}). \tag{16}$$

By [22, Theorem 9.1] there exists a Thom class $\overline{\theta} \in H^r(T, \partial T)$ such that $\overline{\theta}|_* = u_{D^r}$, in other words such that

$$\mathrm{inc}^*(\overline{\theta}) = u_{D^r}. \tag{17}$$

It also induces a Thom isomorphism:

$$\_ \cup \overline{\theta}\colon H^*(T) \xrightarrow{\cong} H^{*+r}(T, \partial T). \tag{18}$$

The following formula summarizes the link between the Thom class $\overline{\theta}$ and the fixed orientation classes u_V and u_W.

Lemma 4.1 *Suppose that u_v, u_W and u_{D^r} are compatible. Then the following sequence of arrows sends u_V to u_W:*

$$H^*(V) \xrightarrow[\pi^*]{\cong} H^*(T) \xrightarrow{\_\cup\overline{\theta}} H^{*+r}(T, \partial T) \xleftarrow[\text{restriction}]{\cong} H^{*+r}(W, B) \longrightarrow H^{*+r}(W).$$

Proof The lemma follows from Equation (16) and the commutativity of the following diagram:

$$\begin{tikzcd}
H^*(T) \arrow[r, "_\cup\overline{\theta}"] \arrow[d] & H^{*+r}(T, \partial T) \arrow[d] & H^{*+r}(W, B) \arrow[l, "\cong"'] \arrow[r] \arrow[d] & H^{*+r}(W) \arrow[d] \\
H^*(D^m, S^{m-1}) \arrow[r, "_\times u_{D^r}"'] & H^{*+r}(D^n, S^{n-1}) \arrow[r, equal] & H^{*+r}(D^n, S^{n-1}) \arrow[r, equal] & H^{*+r}(D^n, S^{n-1}).
\end{tikzcd}$$
□

4.2 The shriek map

To construct a CDGA-model of $W \setminus V$ we will need an analogue of the shriek map (or Gysin map, or transfer map) which we recall now. Let $\mu_V \in H_m(V, \mathbb{Z})$ and $\mu_W \in H_n(W, \mathbb{Z})$ be homology classes dual to u_V and u_W. The classical *cohomological shriek map* (see Bredon [4, VI.11.2], also called Gysin map, pushforward map or umkehr map) is a map

$$f^!\colon s^{-r} H^*(V, \mathbb{Z}) \cong H^{*-r}(V, \mathbb{Z}) \to H^*(W, \mathbb{Z}).$$

By [4, VI.14.1] $f^!$ is a morphism of $H^*(W, \mathbb{Z})$–modules and induces an isomorphism in degree n sending $s^{-r} u_V$ to u_W. These two facts characterize $f^!$ as can also be seen using Lemma 3.15.

An important ingredient in our model of $W \setminus V$ will be the following analogue of the shriek map at the level of models. Following our standard abuse of notation (Section 3.4) denote also by $u_V \in HA_{\mathrm{PL}}(V)$ and $u_W \in HA_{\mathrm{PL}}(W)$ the images of the orientation classes through the isomorphism between $HA_{\mathrm{PL}}(\_)$ and $H^*(\_)$ (Equation (5)).

Definition 4.2 Let $\phi\colon R \to Q$ be a CDGA-model of $A_{\mathrm{PL}}(f)\colon A_{\mathrm{PL}}(W) \to A_{\mathrm{PL}}(V)$. A *shriek map* associated to ϕ is any R–dgmodule morphism

$$\phi^!\colon s^{-r}Q \to R$$

such that $H^n(\phi^!)(s^{-r}u_V) = u_W$.

Proposition 4.3 *Let $\phi\colon R \to Q$ be a CDGA-model of $A_{\mathrm{PL}}(f)\colon A_{\mathrm{PL}}(W) \to A_{\mathrm{PL}}(V)$. An R–dgmodule morphism $\widetilde{\phi}\colon s^{-r}Q \to R$ is a shriek map if and only if*

$$H(\widetilde{\phi})\colon s^{-r}H^*(V) \cong H(s^{-r}Q) \to H(R) \cong H^*(W)$$

is the classical cohomological shriek map $f^!$.

Proof Assume that $H(\widetilde{\phi})$ is the cohomological shriek map $f^!$. We know that $H_*(f)(u_V \cap \mu_V) = H_*(f)(1) = u_W \cap \mu_W$. Thus by [4, VI.14.1] $f^!(s^{-r}u_V) = u_W$ and so $\widetilde{\phi}$ is a shriek map.

Next assume that $\widetilde{\phi}$ is a shriek map, then $H^n(\phi^!)(s^{-r}u_V) = u_W$ by definition. We also know that the cohomological shriek map $f^!$ satisfies $f^!(s^{-r}u_V) = u_W$. Letting $(H(R), 0)$ be the CGDA and $H^*(Q)$ the $H^*(R)$–dgmodule in Lemma 3.15 we see that $H^r(\phi^!) \simeq f^!$ in the category $H^*(R)$–DGMod. Hence they must be equal on homology, which completes the proof of the other direction. □

4.3 Uniqueness and existence of the shriek map

Proposition 4.4 *The shriek map is unique up to homotopy. More precisely let $\phi\colon R \to Q$ be a CDGA model of $A_{\mathrm{PL}}(f)\colon A_{\mathrm{PL}}(W) \to A_{\mathrm{PL}}(V)$. Let $\phi^!, \overline{\phi}^!\colon s^{-r}Q \to R$ be shriek maps associated to ϕ, then $\phi^! = \overline{\phi}^!$ in $[s^{-r}Q, R]_R$.*

Proof We know that $H^n(\phi^!)(s^{-r}u_V) = H^n(\overline{\phi}^!)(s^{-r}u_V)$ since both $\phi^!$ and $\overline{\phi}^!$ are shriek maps. Thus as $H^n(s^{-r}Q)$ is one dimensional, $H^n(\phi^!) = H^n(\overline{\phi}^!)$ in $\mathrm{Hom}(H^n(s^{-r}Q), H^n(R))$ and so Lemma 3.15 implies that $\phi^! = \overline{\phi}^!$ in $[s^{-r}Q, R]_R$. □

The next proposition and its proof show how to associate to any CDGA model of an embedding $f\colon V \to W$, a suitable CDGA model together with a shriek map which will be used in our model for $W \setminus V$ and for the blow-up.

Proposition 4.5 *Assume that $H^1(f)$ is injective and that $n \geq m+2$. Suppose $\phi'\colon R' \to Q'$ is a CDGA-model of the embedding $f : V \hookrightarrow W$. Then we can construct from ϕ' another CDGA model $\phi\colon R \to Q$ of f together with a shriek map $\phi^!\colon s^{-r}Q \to R$, such that*

- *R and Q are connected, that is, $R^0 = Q^0 = \mathbb{Q}$,*
- *$R^{\geq n+1} = 0$, and*
- *$Q^{\geq m+2} = 0$.*

Proof By taking a minimal Sullivan model of R', β: $\widetilde{R} = \Lambda X \xrightarrow{\simeq} R'$ and a minimal relative Sullivan model $\widetilde{\phi}$: $\widetilde{R} \to \widetilde{Q} = \widetilde{R} \otimes \Lambda Y$ of the composition $\phi'\beta$ we get a new model $\widetilde{\phi}$ of f. Since $H^0(V) = H^0(W) = \mathbb{Q}$ and $H^1(f)$ is injective, $\widetilde{R}$ and $\widetilde{Q}$ are connected.

By [9, Lemma 14.1], $\widetilde{Q}$ is a semifree $\widetilde{R}$–dgmodule and so is $s^{-r}\widetilde{Q}$, and hence they are cofibrant. Since $H^*(\widetilde{R}) \cong H^*(W;\mathbb{Q})$ is a connected Poincaré duality algebra of formal dimension n, Lemma 3.15 implies that

$$[s^{-r}\widetilde{Q}, \widetilde{R}]_{\widetilde{R}} \cong \mathrm{Hom}(H^n(s^{-r}\widetilde{Q}), H^n(\widetilde{R})).$$

From the definition of $[\_,\_]$, $\mathrm{Hom}(A,B) \to [A,B]$ is surjective if A is cofibrant, so since $s^{-r}\widetilde{Q}$ is cofibrant we can take a representative $\widetilde{\phi}^!$: $s^{-r}\widetilde{Q} \to \widetilde{R}$ of a homotopy class such that $H^n(\widetilde{\phi}^!)(u_V) = u_W$. Then $\widetilde{\phi}^!$ is a shriek map associated to $\widetilde{\phi}$.

We next adjust this shriek map so that the dimension conditions at the end of the proposition are satisfied. Set

$$\begin{aligned} I &= \widetilde{R}^{\geq n+1} \oplus (\text{a complement of the cocycles in } \widetilde{R}^n) \\ J &= \widetilde{Q}^{\geq m+2} \oplus (\text{a complement of the cocycles in } \widetilde{Q}^{m+1}). \end{aligned}$$

Since $\widetilde{R}$ is connected, I is an ideal and it is acyclic because $H^{>n}(\widetilde{R}) = 0$. Similarly J is an acyclic ideal in $\widetilde{Q}$. Define $R = \widetilde{R}/I$ and $Q = \widetilde{Q}/J$. Since $n \geq m+2$, $\widetilde{\phi}$ induces a map ϕ: $R \to Q$ which is a model of f and since $m+1+r \geq n+1$, $\widetilde{\phi}^!$ induces an R–dgmodule morphism $\phi^!$: $s^{-r}Q \to R$ which is a shriek map. Obviously R and Q are connected and $R^{\geq n+1} = Q^{\geq m+2} = 0$. □

4.4 Preliminaries with A_{PL}

Recall the notation of Diagram (14) at the beginning of the section. Consider the ladder of maps between short exact sequences of pairs:

$$\begin{array}{ccccc} A_{\mathrm{PL}}(W,B) & \xrightarrow{\iota} & A_{\mathrm{PL}}(W) & \xrightarrow{l^*} & A_{\mathrm{PL}}(B) \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle k^*} \\ A_{\mathrm{PL}}(T,\partial T) & \xrightarrow{\iota'} & A_{\mathrm{PL}}(T) & \xrightarrow{i^*} & A_{\mathrm{PL}}(\partial T) \end{array}$$

where ι is the kernel of l^* and ι' is the kernel of i^*. Also ρ is the induced map which is well defined since Diagram (14) commutes and A_{PL} is a functor. As each is a kernel of a CDGA map, each is a differential ideal – $A_{\mathrm{PL}}(W,B)$ inherits an $A_{\mathrm{PL}}(W)$–dgmodule structure and $A_{\mathrm{PL}}(T,\partial T)$ inherits an $A_{\mathrm{PL}}(T)$–dgmodule structure. Since j^* is an algebra map, ρ is an $A_{\mathrm{PL}}(W)$–dgmodule map. Note that $l\colon B\to W$ and $i\colon \partial T\to T$ are cofibrations and so l^* and i^* are surjections [9, Proposition 10.4 and Lemma 10.7]. This will be used later.

Lemma 4.6 *The restriction map*

$$\rho\colon A_{\mathrm{PL}}(W,B)\to A_{\mathrm{PL}}(T,\partial T)$$

is a surjective weak equivalence of $A_{\mathrm{PL}}(W)$*–dgmodules.*

Proof As observed above ρ is an $A_{\mathrm{PL}}(W)$–dgmodule map. It is a weak equivalence by excision. To see that it is surjective, let $\alpha\in A_{\mathrm{PL}}(T,\partial T)$. Since j is a cofibration we know by [9, Proposition 10.4 and Lemma 10.7] that j^* is a surjection. Let $\beta\in A_{\mathrm{PL}}(W)$ be such that $j^*(\beta)=\iota'(\alpha)$. Since $i^*\iota'=0$, $k^*l^*(\beta)=0$. So we can extend $l^*(\beta)$ by 0 on simplices contained in T and arbitrarily to the rest of the singular simplices in W (the ones whose image is contained in neither B or T) to get $\beta'\in A_{\mathrm{PL}}(W)$. We know that $l^*(\beta')=l^*(\beta)$ and $j^*(\beta')=0$. Thus $l^*(\beta-\beta')=0$ and $j^*(\beta-\beta')=j^*(\beta)=\iota'(\alpha)$. So $\beta-\beta'$ lifts to $\widetilde{\beta}\in A_{\mathrm{PL}}(W,B)$ such that $\rho(\widetilde{\beta})=\alpha$. Since α was arbitrary ρ is surjective. □

See Section 3.1 for notation concerning squares.

Lemma 4.7 *Consider the following squares:*

$$\mathbf{F}' = \begin{tikzcd} A_{\mathrm{PL}}(W)_{\mathrm{inc}} \arrow[r, "j^*"] \arrow[d] & A_{\mathrm{PL}}(T)_{\mathrm{inc}} \arrow[d] \\ A_{\mathrm{PL}}(W)\oplus_\iota sA_{\mathrm{PL}}(W,B) \arrow[r, "j^*\oplus s\rho"] & A_{\mathrm{PL}}(T)\oplus_{\iota'} sA_{\mathrm{PL}}(T,\partial T) \end{tikzcd}$$

$$\mathbf{F} = \begin{tikzcd} A_{\mathrm{PL}}(W) \arrow[r, "j^*"] \arrow[d, "l^*"] & A_{\mathrm{PL}}(T) \arrow[d, "i^*"] \\ A_{\mathrm{PL}}(B) \arrow[r, "k^*"] & A_{\mathrm{PL}}(\partial T) \end{tikzcd}$$

(i) *With the semitrivial CDGA structure (see Definition 3.19) on the mapping cones,* $\mathbf{F}'$ *is a CDGA square.*

(ii) $\Theta_6=\begin{pmatrix}\mathrm{id} & \mathrm{id}\\ l^*+0 & i^*+0\end{pmatrix}\colon \mathbf{F}'\to\mathbf{F}$ *is a weak equivalence of CDGA squares.*

Proof (i) By Proposition 3.18 (i) there are semitrivial CDGA-structures on the mapping cones:

$$A_{\mathrm{PL}}(W) \oplus_\iota sA_{\mathrm{PL}}(W,B) \quad \text{and} \quad A_{\mathrm{PL}}(T) \oplus_{\iota'} sA_{\mathrm{PL}}(T,\partial T).$$

It is then easy to see that $\mathbf{F}'$ is a CDGA squares.

(ii) It is immediate that the following diagram is one in CDGA and is commutative:

$$\begin{tikzcd} A_{\mathrm{PL}}(W) \oplus_\iota sA_{\mathrm{PL}}(W,B) \arrow[r, "j^*\oplus s\rho"] \arrow[d, "l^*+0"'] & A_{\mathrm{PL}}(T) \oplus_{\iota'} sA_{\mathrm{PL}}(T,\partial T) \arrow[d, "i^*+0"'] \\ A_{\mathrm{PL}}(B) \arrow[r, "k^*"] & A_{\mathrm{PL}}(\partial T). \end{tikzcd} \tag{19}$$

We need only to prove that the vertical arrows in Diagram (19) are quasi-isomorphisms. As noted above Lemma 4.6, l^* is surjective. Thus we have a short exact sequence

$$0 \longrightarrow A_{\mathrm{PL}}(W,B) \xrightarrow{\iota} A_{\mathrm{PL}}(W) \xrightarrow{l^*} A_{\mathrm{PL}}(B) \longrightarrow 0.$$

By Lemma 3.12 this implies that the vertical map l^*+0 is a quasi-isomorphism. The proof that i^*+0 is a quasi-isomorphism is similar. □

Lemma 4.8 *Let $\theta \in A^r_{\mathrm{PL}}(T,\partial T) \cap \ker d$ be a representative of the Thom class $\overline{\theta}$ chosen in Section 4.1. Then multiplication by θ,*

$$\_ \cdot \theta \colon s^{-r} A_{\mathrm{PL}}(T) \xrightarrow{\simeq} A_{\mathrm{PL}}(T,\partial T)$$

is a quasi-isomorphism of left $A_{\mathrm{PL}}(T)$–dgmodules.

Proof This follows from the Thom isomorphism of Equation (18). □

Lemma 4.9 *The following sequence of arrows*

$$s^{-r} A_{\mathrm{PL}}(V) \longrightarrow s^{-r} A_{\mathrm{PL}}(T) \underset{\simeq}{\xrightarrow{\_\cdot\theta}} A_{\mathrm{PL}}(T,\partial T) \underset{\simeq}{\xleftarrow{\rho}} A_{\mathrm{PL}}(W,B) \xrightarrow{\iota} A_{\mathrm{PL}}(W)$$

induces an isomorphism

$$H^{n-r}(V) \cong H^n(W)$$

that takes $s^{-r}u_V$ to u_W.

Proof Apply Lemma 4.1 and the natural equivalence of Equation (5) between $H^*(\_)$ and $H(A_{\mathrm{PL}}(\_))$. □

5 A model of the complement of a submanifold

As in Section 4 we suppose we are given an embedding of closed manifolds $f\colon V \hookrightarrow W$ as well as orientation classes $u_V \in H^m(V;\mathbb{Z})$ and $u_W \in H^n(W;\mathbb{Z})$. Again W is of dimension n and V of dimension m and codimension $r = n - m$. We also use the notation from Diagram (14), the CDGA maps ι, ι', and ρ from Section 4.4 and the representative θ of the Thom class from Lemma 4.8.

Assume that $H^1(f)$ is injective and $\dim W \geq 2\dim V + 3$. We fix a CDGA-model

$$\phi\colon R \to Q$$

of $j^*\colon A_{\mathrm{PL}}(W) \to A_{\mathrm{PL}}(T)$ such that R and Q are connected, $R^{\geq n+1} = 0$ and $Q^{\geq m+2} = 0$. Notice that ϕ is also a model of $A_{\mathrm{PL}}(f)$. By our standard abuse of notation (see Section 3.4) this determines orientation classes $u_W \in H(R)$ and $u_V \in H(Q)$. We suppose also that we have been given an associated shriek map

$$\phi^!\colon s^{-r}Q \to R.$$

Notice that by Proposition 4.5 we can always build such a CDGA model ϕ and shriek map $\phi^!$.

Our aim in this section is to describe a CDGA-model of the map $k\colon \partial T \to B$ using a CDGA model of the embedding f under the hypotheses that $n \geq 2m + 3$. In fact we will give a CDGA model of the Diagram (14) of the last section (Lemma 5.8) using only the model ϕ and the shriek map $\phi^!$. This extra precision is necessary to get the model of the blow-up.

The section is organized as follows. In Section 5.1 we fix a common model $\widehat{\phi}\colon \widehat{R} \to \widehat{Q}$ of ϕ and j^*. In Section 5.2 we construct a common model $\widehat{\phi}^!$ of $\phi^!$ and ι. The common model $\widehat{\phi}\widehat{\phi}^!$ of $\phi\phi^!$ and $j^*\iota$ comes with a $\widehat{R}$–dgmodule structure. In Section 5.3 we show that it is homotopic to a $\widehat{Q}$–dgmodule map χ. This extension of structure would not be necessary if we just wanted a model of $A_{\mathrm{PL}}(B)$, but it is necessary to obtain a model of $A_{\mathrm{PL}}(k)$. In Section 5.4 (Lemma 5.4) we show that the cone on $\phi^!$ is a $\widehat{R}$–dgmodule model of the cone on ι and that the cone on $\phi\phi^!$ is a $\widehat{Q}$–dgmodule model of the cone on $j^*\iota$. We already know from the last section (Lemma 4.7) that the cones on ι and $j^*\iota$ are CDGA models of $A_{\mathrm{PL}}(B)$ and $A_{\mathrm{PL}}(\partial T)$ respectively. We also construct the maps between our models that we will need. In Section 5.5 (Lemma 5.6) we give conditions under which a map between diagrams with certain dgmodule structure can be extended to a CDGA map. Next we put everything together and construct a CDGA model of Diagram (14), (Lemma 5.8). Since we are constructing the model of the blow-up we also keep track of the weak equivalences connecting the models.

5.1 A common model of ϕ and j^*

Solely for purposes of the proof we use Lemma 3.6 to fix a commuting diagram of CDGAs

(20)
$$\begin{tikzcd}
R \arrow[d, "\phi"'] & \widehat{R} \arrow[l, "\alpha"', "\simeq"] \arrow[r, "\alpha'", "\simeq"'] \arrow[d, "\widehat{\phi}"'] & A_{\mathrm{PL}}(W) \arrow[d, "j^*"'] \\
Q & \widehat{Q} \arrow[l, "\beta"', "\simeq"] \arrow[r, "\beta'", "\simeq"'] & A_{\mathrm{PL}}(T)
\end{tikzcd}$$

such that α, α', β and β' are quasi-isomorphisms, $\widehat{R}$ is cellular, $\widehat{\phi}$ is a cellular cofibration and the maps

$$(\alpha, \alpha')\colon\ \widehat{R} \to R \oplus A_{\mathrm{PL}}(W)$$
$$(\beta, \beta')\colon\ \widehat{Q} \to Q \oplus A_{\mathrm{PL}}(T)$$

are surjective. As a consequence R and $A_{\mathrm{PL}}(W)$ are $\widehat{R}$–dgmodules and Q and $A_{\mathrm{PL}}(T)$ are $\widehat{Q}$–dgmodules. Notice also that since $\widehat{\phi}$ is a cellular cofibration and $\widehat{R}$ and $\widehat{Q}$ are connected, $\widehat{Q}$ is a semifree $\widehat{R}$–dgmodule. By the second part of Lemma 3.6 the homology classes $u_W \in H(R)$ and $u_V \in H(Q)$ corresponding to the orientations $u_W \in H^*(W) = H(A_{\mathrm{PL}}(W))$ and $u_V \in H^*(V) \cong H^*(T) = H(A_{\mathrm{PL}}(T))$ (where the isomorphism is π^*) using the quasi-isomorphisms of Diagram (20) are the same as those given by the original string of weak equivalences that made ϕ a model of j^*.

5.2 A common model of $\phi^!$ and ι

Next we construct a common model $\widehat{\phi}^!$ of $\phi^!$ and of $\iota\colon A_{\mathrm{PL}}(W, B) \to A_{\mathrm{PL}}(W)$ defined in Section 4.4. Recall also from Section 4.4 the definition ρ and the cocycle θ from Lemma 4.8. We consider the following commutative solid arrow diagram of $\widehat{R}$–dgmodules

(21)
$$\begin{tikzcd}
 & & s^{-r} A_{\mathrm{PL}}(T) \arrow[dr, "_\cdot\theta", "\simeq"'] & \\
s^{-r} Q \arrow[d, "\phi^!"'] & s^{-r}\widehat{Q} \arrow[l, "s^{-r}\beta"', "\simeq"] \arrow[ur, "s^{-r}\beta'", "\simeq"'] \arrow[r, dashed, "\gamma'", "\simeq"'] \arrow[d, dashed, "\widehat{\phi}^!"'] & A_{\mathrm{PL}}(W, B) \arrow[r, "\simeq", "\rho"'] \arrow[d, "\iota"'] & A_{\mathrm{PL}}(T, \partial T) \\
R & \widehat{R} \arrow[l, "\alpha"', "\simeq"] \arrow[r, "\alpha'", "\simeq"'] & A_{\mathrm{PL}}(W). &
\end{tikzcd}$$

In the next lemma we construct $\widehat{R}$–dgmodule maps γ' and $\widehat{\phi}'$ making the diagram commute. Notice that all horizontal and diagonal maps are weak equivalences. Some of the maps above are more than just $\widehat{R}$–dgmodule maps and this extra structure will be used when we enhance our $\widehat{R}$–dgmodule structure to get a CDGA structure in Section 5.5.

Lemma 5.1 *There exists an $\widehat{R}$–dgmodule weak equivalence $\gamma'\colon s^{-r}\widehat{Q}\to A_{\mathrm{PL}}(W,B)$ and a $\widehat{R}$–dgmodule map $\widehat{\phi}'\colon s^{-r}\widehat{Q}\to\widehat{R}$ making Diagram* (21) *above commute.*

Proof The quasi-isomorphism γ' is just any lift in the following solid arrow diagram:

$$\begin{tikzcd} & A_{\mathrm{PL}}(W,B) \arrow[d, "\rho", "\simeq"'] \\ s^{-r}\widehat{Q} \arrow[ur, dashed, "\gamma'"] \arrow[r, "\simeq", "(\_\cdot\theta)s^{-r}\beta'"'] & A_{\mathrm{PL}}(T,\partial T). \end{tikzcd}$$

The Lifting Lemma (Lemma 3.4) implies the lift γ' exists since by Lemma 4.6 ρ is an acyclic fibration and $s^{-r}\widehat{Q}$ is a cofibrant $\widehat{R}$–dgmodule.

By the same argument there exists a lift $\widetilde{\phi}^!$ in the solid arrow diagram:

$$\begin{tikzcd} & \widehat{R} \arrow[d, "\alpha'"] \\ s^{-r}\widehat{Q} \arrow[ur, dashed, "\widetilde{\phi}^!"] \arrow[r, "\iota\gamma'"'] & A_{\mathrm{PL}}(W) \end{tikzcd}$$

It follows from Lemma 4.9 that $H(\alpha)H(\widetilde{\phi}^!)H(s^{-r}\beta)^{-1}(s^{-r}u_V)=u_W$. So since by Proposition 4.4 $\phi^!$ represents the unique homotopy class of maps with that property, we have that $\phi^! s^{-r}\beta\simeq\alpha\widetilde{\phi}^!$. Since $(\alpha,\alpha')\colon\widehat{R}\to R\oplus A_{\mathrm{PL}}(W)$ is a surjection we can use Lemma 3.5 to replace $\widetilde{\phi}'$ by a map $\widehat{\phi}^!\colon s^{-r}\widehat{Q}\to\widehat{R}$ making Diagram (21) commute on the nose. $\square$

5.3 Replacing $\widehat{\phi}\widehat{\phi}^!$ by a $\widehat{Q}$–dgmodule morphism

In this subsection we suppose fixed Diagrams (20) and (21) with the maps γ' and $\widehat{\phi}^!$ constructed in Lemma 5.1. Here we show that the $\widehat{R}$–dgmodule map $\widehat{\phi}\widehat{\phi}^!$ can be replaced by a $\widehat{Q}$–dgmodule map χ which is homotopic to $\widehat{\phi}\widehat{\phi}^!$ in a controlled way.

Lemma 5.2 (i) $\rho\gamma'\colon s^{-r}\widehat{Q}\to A_{\mathrm{PL}}(T,\partial T)$ *is a morphism of $\widehat{Q}$–dgmodules.*

(ii) $\phi\phi^!=0$ *and is thus a morphism of $\widehat{Q}$–dgmodules.*

Proof (i) Lemma 5.1 says that γ' makes Diagram (21) commute so we have that

$$\rho\gamma' = (\_\cdot\theta)s^{-r}(\beta').$$

Of course $(\_\cdot\theta)$ is an $A_{\mathrm{PL}}(T)$–dgmodule map and hence a $\widehat{Q}$–dgmodule map. Also β' and thus $s^{-r}\beta'$ are $\widehat{Q}$–dgmodule maps. Hence the composition

$$(\_\cdot\theta)s^{-r}\beta' = \rho\gamma'$$

is a $\widehat{Q}$ dgmodule map.

(ii) We have assumed that $n \geq 2m+3$, hence $r = n-m \geq m+3$. Since $Q^{\geq m+2} = 0$ the statement follows. □

In general $\widehat{\phi}\widehat{\phi}^!\colon s^{-r}\widehat{Q} \to \widehat{Q}$ is not a $\widehat{Q}$–dgmodule map but it can be adjusted as described in the following lemma.

Lemma 5.3 *There exists a $\widehat{Q}$–dgmodule map $\chi\colon s^{-r}\widehat{Q} \to \widehat{Q}$ and a $\widehat{R}$–dgmodule homotopy $h\colon ss^{-r}\widehat{Q} \to \widehat{Q}$ from χ to $\widehat{\phi}\widehat{\phi}^!$ such that $\beta h = \beta' h = 0$. In particular $(\beta,\beta')\chi = (\beta,\beta')\widehat{\phi}\widehat{\phi}^!$.*

Proof Since $n \geq 2m+3$, $r = n-m \geq m+1$, so we know that $H^{<r}(s^{-r}\widehat{Q}) = 0$ and $H^{\geq r}(T;\mathbb{Q}) \subset H^{\geq m+1}(T;\mathbb{Q}) = H^{\geq m+1}(V;\mathbb{Q}) = 0$. Thus Lemma 3.16 implies that $[s^{-r}\widehat{Q}, A_{\mathrm{PL}}(T)]_{\widehat{Q}} = 0$. Similarly $[s^{-r}\widehat{Q}, Q]_{\widehat{Q}} = 0$. As $s^{-r}\widehat{Q}$ is a free $\widehat{Q}$–dgmodule on one generator, it is semifree as a $\widehat{Q}$–dgmodule. Thus using Lemma 5.2 we see that the following diagram is homotopy commutative in the category of $\widehat{Q}$–dgmodules

$$\begin{tikzcd}
s^{-r}Q \arrow[d, "\phi\phi^!"'] & s^{-r}\widehat{Q} \arrow[l, "s^{-r}\beta"', "\simeq"] \arrow[r, "\rho\gamma'", "\simeq"'] \arrow[d, "0"'] & A_{\mathrm{PL}}(T,\partial T) \arrow[d, "\iota'"'] \\
Q & \widehat{Q} \arrow[l, "\beta"', "\simeq"] \arrow[r, "\beta'", "\simeq"'] & A_{\mathrm{PL}}(T).
\end{tikzcd}$$

Since (β,β') is surjective, Lemma 3.5 asserts that we can replace the zero-map in the previous diagram by a $\widehat{Q}$–dgmodule morphism $\chi\colon s^{-r}\widehat{Q} \to \widehat{Q}$ making the diagram strictly commutative, in other words such that $(\beta,\beta')\chi = (\beta,\beta')\widehat{\phi}\widehat{\phi}^!$.

Next we construct the homotopy h. Let $H'\colon \mathrm{Cyl}(s^{-r}\widehat{Q}) \to Q \oplus A_{\mathrm{PL}}(T)$ be the map corresponding to the constant homotopy $h' = 0\colon ss^{-r}\widehat{Q} \to Q \oplus A_{\mathrm{PL}}(T)$. We get the following commutative solid arrow diagram

$$\begin{tikzcd} s^{-r}\widehat{Q}\oplus s^{-r}\widehat{Q} \arrow[r, "\chi+\widehat{\phi}\widehat{\phi}^!"] \arrow[d, "i_0+i_1"'] & \widehat{Q} \arrow[d, "(\beta,\beta')"] \\ \mathrm{Cyl}\, s^{-r}\widehat{Q} \arrow[r, "H'"'] \arrow[ur, dashed, "H"] & Q\oplus A_{\mathrm{PL}}(T). \end{tikzcd}$$

The map i_0+i_1 is a cellular $\widehat{R}$–dgmodule cofibration such that $H^{<r}(i_0+i_1)$ is an isomorphism. Also since $n\geq 2m+3$, $r-1=n-m-1\geq m+1>\dim(V)$ and so $H^{\geq r-1}(Q\oplus A_{\mathrm{PL}}(T))=0=H^{\geq r-1}(\widehat{Q})$. Thus a lift H exists. Our desired $h\colon ss^{-r}\widehat{Q}\to\widehat{Q}$ is the homotopy corresponding to H. □

5.4 A dgmodule model of $j^*\oplus s\rho$

Recall from Lemma 4.7 that the map of mapping cones

$$j^*\oplus s\rho\colon A_{\mathrm{PL}}(W)\oplus_\iota sA_{\mathrm{PL}}(W,B)\to A_{\mathrm{PL}}(T)\oplus_{\iota'} sA_{\mathrm{PL}}(T,\partial T)$$

is a CDGA model of $k^*\colon A_{\mathrm{PL}}(B)\to A_{\mathrm{PL}}(\partial T)$. Our aim in this subsection is to give another model (as dgmodules) of that map.

We consider Diagrams (20) and (21) with the maps $\widehat{\phi}^!$ and γ' from Lemma 5.1 as well as the $\widehat{Q}$–dgmodule map $\chi\colon s^{-r}Q\to\widehat{Q}$ and the $\widehat{R}$–dgmodule homotopy $h\colon \chi\simeq\widehat{\phi}\widehat{\phi}$ from Lemma 5.3. Recall the notation from Lemma 3.13.

Lemma 5.4 *The map*

$$\widehat{\phi}\oplus_h \mathrm{id}\colon \widehat{R}\oplus_{\widehat{\phi}^!} ss^{-r}\widehat{Q}\to\widehat{Q}\oplus_\chi ss^{-r}\widehat{Q}$$

is a morphism of $\widehat{R}$–dgmodules making the following diagram commutative in the category of $\widehat{R}$–dgmodules:

$$\begin{tikzcd} R\oplus_{\phi^!} ss^{-r}Q \arrow[d, "\phi\oplus\mathrm{id}"'] & \widehat{R}\oplus_{\widehat{\phi}^!} ss^{-r}\widehat{Q} \arrow[l, "\alpha\oplus ss^{-r}\beta", "\simeq"'] \arrow[r, "\alpha'\oplus s\gamma'", "\simeq"'] \arrow[d, "\widehat{\phi}\oplus_h\mathrm{id}"] & A_{\mathrm{PL}}(W)\oplus_\iota sA_{\mathrm{PL}}(W,B) \arrow[d, "j^*\oplus s\rho"] \\ Q\oplus_{\phi\phi^!} ss^{-r}Q & \widehat{Q}\oplus_\chi ss^{-r}\widehat{Q} \arrow[l, "\beta\oplus ss^{-r}\beta", "\simeq"'] \arrow[r, "\beta'\oplus s\rho\gamma'", "\simeq"'] & A_{\mathrm{PL}}(T)\oplus_{\iota'} sA_{\mathrm{PL}}(T,\partial T) \end{tikzcd} \tag{22}$$

Moreover the horizontal maps in this diagram are quasi-isomorphisms and the bottom row consists of $\widehat{Q}$–dgmodule maps.

Proof Since Lemma 5.3 says that $\beta h = \beta' h = 0$, the diagram in the statement of the lemma is commutative.

We see that the horizontal maps of the diagram are quasi-isomorphisms by Lemma 3.13. The fact that the bottom line of the diagram is of $\widehat{Q}$–dgmodules follows from the construction of β, β' and γ'. □

5.5 Extending dgmodule structure to CDGA structure

For this subsection we suppose fixed Diagrams (20) and (21) with the maps γ' and $\widehat{\phi}^!$ constructed in Lemma 5.1 and χ and h from Lemma 5.3.

Here we use CDGA squares. At first sight this may seem clumsy but it keeps track of the $\widehat{R}$ and $\widehat{Q}$ module structures in a convenient way. Notation concerning squares is described in Section 3.1.

Lemma 5.5 *Assume that r is even. There exists a unique CDGA structure on $\widehat{Q} \oplus_\chi ss^{-r}\widehat{Q}$ (respectively, $Q \oplus ss^{-r}Q$) extending its $\widehat{Q}$–dgmodule (respectively, Q–dgmodule) structure. Moreover we can find CDGA isomorphisms $\widehat{e}$ and e, such that $e(z) = ss^{-r}1$, $\widehat{e}(z) = (\widehat{q}, ss^{-r}1)$ for some $q \in \widehat{Q}^{r-1}$, $e|_Q = \mathrm{id}$, $\widehat{e}|_{\widehat{Q}} = \mathrm{id}$, and which make the following diagram commute*

$$\begin{tikzcd}
\widehat{Q} \otimes \Lambda(z) \arrow[r, "\widehat{e}"] \arrow[d, "\beta\otimes\mathrm{id}"'] & \widehat{Q} \oplus_\chi ss^{-r}\widehat{Q} \arrow[d, "\beta\oplus ss^{-r}\beta"] \\
Q \otimes \Lambda(z) \arrow[r, "e"'] & Q \oplus ss^{-r}Q
\end{tikzcd}$$

where $|z| = r-1$ and $dz = 0$.

Proof Recall that $\phi\phi^! = 0$ for dimension reasons. The map e is determined by the conditions that $e(z) = ss^{-r}1$, $e|_Q = \mathrm{id}$ and the fact that it is a Q–module map. It is clearly an isomorphism. Since $\chi(ss^{-r}1)$ is a cocycle in $\widehat{Q}$ and since $H^{\geq r}(\widehat{Q}) = 0$, there exists $\widehat{q} \in \widehat{Q}^{r-1}$ such that $d(\widehat{q}) = -\chi(ss^{-r}1)$. For degree reasons $\beta(\widehat{q}) = 0$. We see also that $(\widehat{q}, ss^{-r}1)$ is a cocycle in $\widehat{Q} \oplus_\chi ss^{-r}\widehat{Q}$. Then $\widehat{e}$ is determined by the formula $\widehat{e}(z) = (\widehat{q}, ss^{-r}1)$ and is an isomorphism. Clearly the diagram commutes. The CDGA structures are unique since all products are determined by $(ss^{-r}(1))^2$ and the $\widehat{Q}$ and Q module structures and $ss^{-r}(1)$ must square to 0 since it is in odd dimension. □

Lemma 5.6 *For r even, let $\mathbf{D}$ denote the commutative square*

$$\mathbf{D} = \begin{tikzcd} \widehat{R} \arrow[r, "\widehat{\phi}"] \arrow[d, hook] & \widehat{Q} \arrow[d, hook] \\ \widehat{R} \oplus_{\widehat{\phi}^!} ss^{-r}\widehat{Q} \arrow[r, "\widehat{\phi}\oplus_h \mathrm{id}"'] & \widehat{Q} \oplus_\chi ss^{-r}\widehat{Q} \end{tikzcd}$$

and let $(\widehat{Q} \otimes \Lambda z, dz = 0)$ be the relative Sullivan algebra from Lemma 5.5 together with the CDGA isomorphism $\widehat{e}\colon \widehat{Q} \otimes \Lambda z \to \widehat{Q} \oplus_\chi ss^{-r}\widehat{Q}$.

(i) *There exists a CDGA square*

$$\mathbf{D}' = \begin{tikzcd} \widehat{R} \arrow[r, "\widehat{\phi}"] \arrow[d] & \widehat{Q} \arrow[d, hook] \\ \widehat{A} \arrow[r, "\psi"] & \widehat{Q} \otimes \Lambda z \end{tikzcd}$$

and a weak equivalence between $\widehat{R}$–dgmodule squares $\mathbf{\Theta} = \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \theta_3 & (\widehat{e})^{-1} \end{pmatrix} \colon \mathbf{D} \to \mathbf{D}'$.

(ii) *Suppose*

$$\mathbf{C} = \begin{tikzcd} C_1 \arrow[r] \arrow[d] & C_2 \arrow[d] \\ C_3 \arrow[r] & C_4 \end{tikzcd}$$

is another CDGA square and there is a map

$$\mathbf{\Theta}' = \begin{pmatrix} \theta_1' & \theta_2' \\ \theta_3' & \theta_4' \end{pmatrix} \colon \mathbf{D} \to \mathbf{C}$$

such that θ_1' and θ_2' are CGDA maps, θ_3' is a $\widehat{R}$–dgmodule map and θ_4' is a $\widehat{Q}$–dgmodule maps where C_3 (respectively, C_4) has been given the $\widehat{R}$–dgmodule (respectively, $\widehat{Q}$–dgmodule) induced by θ_1' (respectively, θ_2'). If $H^i(C_3) = H^i(C_4) = 0$ for $i \geq 2r-3$ and $C_3 \to C_4$ is a fibration, then there exists a CDGA map $\overline{\theta}_3$ such that

$$\overline{\mathbf{\Theta}} = \begin{pmatrix} \theta_1' & \theta_2' \\ \overline{\theta}_3 & \theta_4'\widehat{e} \end{pmatrix} \colon \mathbf{D}' \to \mathbf{C}$$

is a map of CDGA squares making the following diagram commute:

$$\begin{tikzcd} \mathbf{D} \arrow[r, "\mathbf{\Theta}"] \arrow[d, "\mathbf{\Theta}'"'] & \mathbf{D}' \arrow[dl, "\bar{\mathbf{\Theta}}"] \\ \mathbf{C} & \end{tikzcd}$$

Proof In view of Lemma 5.5 to construct $\mathbf{D}'$ and $\mathbf{\Theta}$, and prove part (i), it is enough to construct a relative Sullivan model $\widehat{R} \to \widehat{A}$ (thus also giving $\widehat{A}$ a $\widehat{R}$–dgmodule structure), a CDGA map $\psi\colon \widehat{A} \to \widehat{Q} \otimes \Lambda(z)$ and an $\widehat{R}$–dgmodule map $\theta_3\colon \widehat{R} \oplus_{\widehat{\phi}^!} ss^{-r}\widehat{Q} \to \widehat{A}$ so that the following diagram commutes

$$\begin{tikzcd} \widehat{R} \oplus_{\widehat{\phi}^!} ss^{-r}\widehat{Q} \arrow[r, "\theta_3"] \arrow[dr, "\widehat{e}^{-1}(\widehat{\phi} \oplus_h \mathrm{id})"'] & \widehat{A} \arrow[d, "\psi"] \\ & \widehat{Q} \otimes \Lambda z \end{tikzcd}$$

and θ_3 is an equivalence. Since $\widehat{\phi}$ is a cellular cofibration of CDGAs, $\widehat{Q} = \widehat{R} \otimes \Lambda U$ and so $ss^{-r}\widehat{Q} = ss^{-r}\widehat{R} \otimes \Lambda U$. There is an isomorphism of $\widehat{R}$–dgmodules $\rho\colon ss^{-r}\widehat{R} \otimes \Lambda U \to \widehat{R} \otimes ss^{-r}\Lambda U$ given by $\rho(ss^{-r}\alpha \otimes \beta) = (-1)^{(1-r)|\alpha|}\alpha \otimes ss^{-r}\beta$ where $\alpha \in \widehat{R}$ and $\beta \in \Lambda U$. Thus $ss^{-r}\widehat{Q} \cong \widehat{R} \otimes ss^{-r}\Lambda U$ as $\widehat{R}$–dgmodules and so $\widehat{R} \oplus_{\widehat{\phi}^!} ss^{-r}\widehat{Q} \cong \widehat{R} \oplus \widehat{R} \otimes ss^{-r}\Lambda U$. Setting $U_0 = ss^{-r}\Lambda U$, $\widehat{R} \oplus_{\widehat{\phi}^!} ss^{-r}\widehat{Q}$ is isomorphic as an $\widehat{R}$–dgmodule to $\widehat{R} \otimes (\mathbb{Q} \oplus U_0)$. The inclusion $\mathbb{Q} \oplus U_0 \to \Lambda U_0$ induces an inclusion $R \otimes (\mathbb{Q} \oplus U_0) \to R \otimes \Lambda U_0$. There is a unique differential on $R \otimes \Lambda U_0$ satisfying the Leibniz law such that the inclusion $f_0\colon \widehat{R} \oplus_{\widehat{\phi}^!} ss^{-r}\widehat{Q} \cong R \otimes (\mathbb{Q} \oplus U_0) \to \widehat{R} \otimes \Lambda U_0$ is an $\widehat{R}$–dgmodule map. Clearly then we get a commuting diagram

$$\begin{tikzcd} \widehat{R} \arrow[r] \arrow[dr] & \widehat{R} \oplus_{\widehat{\phi}^!} ss^{-r}\widehat{Q} \arrow[d, "f_0"] \\ & \widehat{R} \otimes \Lambda U_0. \end{tikzcd}$$

Moreover any $\widehat{R}$–dgmodule map from $\widehat{R} \oplus_{\widehat{\phi}^!} ss^{-r}\widehat{Q}$ into a CDGA extends uniquely to a CDGA map out of $\widehat{R} \otimes \Lambda U_0$. In particular we get a CDGA map $g_0\colon \widehat{R} \otimes \Lambda U_0 \to$

$\widehat{Q}\oplus_\chi ss^{-r}\widehat{Q}$ and a commutative diagram

$$\begin{tikzcd} \widehat{R}\oplus_{\widehat{\phi}^!} ss^{-r}\widehat{Q} \arrow[r, "f_0"] \arrow[dr, "\widehat{\phi}\oplus_h \mathrm{id}"'] & \widehat{R}\otimes\Lambda(U_0) \arrow[d, "g_0"] \\ & \widehat{Q}\oplus_\chi ss^{-r}\widehat{Q}. \end{tikzcd}$$

The elements of the cokernel of f_0 are elements in ΛU_0 of product length at least two. So they are in degree at least $2r-2$. We add a minimal set of generators U_1 of degree at least $2r-3$ to kill the cohomology of the cokernel of f_0. We then get a CDGA extension $h_1\colon \widehat{R}\otimes\Lambda U_0\to\widehat{R}\otimes\Lambda U_0\otimes\Lambda U_1$ and an $\widehat{R}$–dgmodule map $f_1=h_1f_0\colon \widehat{R}\oplus_{\widehat{\phi}^!}ss^{-r}\widehat{Q}\to\widehat{R}\otimes\Lambda U_0\otimes\Lambda U_1$. We have assumed that $n\geq 2m+3$ and $r=n-m$, so $2r-2=n-m+r-2\geq m+r+1>m+r-1$. Recalling that $\widehat{Q}$ models $A_{\mathrm{PL}}(V)$ and $\widehat{Q}\oplus_\chi ss^{-r}\widehat{Q}$ models $A_{\mathrm{PL}}(\partial T)$ we know that $\widehat{Q}\oplus_\chi ss^{-r}\widehat{Q}$ has trivial cohomology in degrees greater than $n-1=m+r-1$ and so also in degrees greater than or equal to $2r-2$ where the boundaries of U_1 lie. Therefore the map g_0 can be extended over these new elements to a CDGA map $g_1\colon \widehat{R}\otimes\Lambda U_0\otimes\Lambda U_1\to\widehat{Q}\oplus_\chi ss^{-r}\widehat{Q}$.

Of course f_1 has a new cokernel but since $H^0(\widehat{R})=\mathbb{Q}$ (of course the homology is 0 in negative degrees) and U_1 was chosen minimally the cohomology of this cokernel is still in degrees greater than $2r-3$. So again we can add generators to kill the cohomology of the cokernel. We continue this process countably many times to get our CDGA, $\widehat{A}=\widehat{R}\otimes\Lambda(\oplus_{i\geq 0}U_i)$ and our map $\theta_3=f_\infty\colon \widehat{R}\oplus_{\widehat{\phi}^!}ss^{-r}\widehat{Q}\to\widehat{A}$. Since each element of the cohomology of the cokernel of θ_3 was killed at the next stage, $H(\theta_3)$ is surjective. Since $2r-2>n$, $H^{\geq 2r-2}(\widehat{R}\oplus_{\widehat{\phi}^!}ss^{-r}\widehat{Q})=0$ and we have not killed anything in $H(\widehat{R}\oplus_{\widehat{\phi}^!}ss^{-r}\widehat{Q})$; so $H(\theta_3)$ is injective. Thus θ_3 is a quasi-isomorphism of $\widehat{R}$–dgmodules. The original map g_0 extends to a CDGA map $g_\infty\colon \widehat{A}\to\widehat{Q}\oplus_\chi ss^{-r}\widehat{Q}$ since $H^{\geq 2r-2}(\widehat{Q}\oplus_\chi ss^{-r}\widehat{Q})=0$ and so at each stage all obstructions are trivial for degree reasons. So we can set $\psi=\widehat{e}^{-1}g_\infty$ and we have completed the construction of $\mathbf{D}'$ and $\mathbf{\Theta}$.

We now proceed to construct the map $\bar{\mathbf{\Theta}}$ and prove part (ii). Any $\widehat{Q}$–dgmodule map out of $\widehat{Q}\oplus_\chi ss^{-r}\widehat{Q}$ is automatically a CDGA map since $(ss^{-r}1)\cdot(ss^{-r}1)=0$ because $ss^{-r}1$ is of odd degree. Thus θ_4' and $\theta_4'\widehat{e}$ are CDGA maps. So we only have to construct a CDGA map $\overline{\theta}_3\colon \widehat{A}\to C_3$ extending θ_3' and making the following diagram commute:

$$\begin{tikzcd} \widehat{A} \arrow[r, "\widehat{e}\psi"] \arrow[d, "\overline{\theta}_3"'] & \widehat{Q}\oplus_\chi ss^{-r}\widehat{Q} \arrow[d, "\theta_4'"] \\ C_3 \arrow[r] & C_4. \end{tikzcd}$$

Clearly the $\widehat{R}$–dgmodule map θ_3' extends uniquely to a CDGA map $h_0\colon \widehat{R}\otimes\Lambda U_0\to C_3$ making the following diagram commute:

$$\begin{tikzcd} \widehat{R}\otimes\Lambda U_0 \arrow[r, "g_0"] \arrow[d, "h_0"'] & \widehat{Q}\oplus_\chi ss^{-r}\widehat{Q} \arrow[d, "\theta_4"] \\ C_3 \arrow[r] & C_4. \end{tikzcd}$$

Since $H^i(C_3)=0$ for $i>2r-2$, we can further extend to a CDGA map $h_1'\colon \widehat{R}\otimes\Lambda U_0\otimes\Lambda U_1\to C_3$. We show that the following CDGA diagram

$$\begin{tikzcd} \widehat{R}\otimes\Lambda U_0\otimes\Lambda U_1 \arrow[r, "g_1"] \arrow[d, "h_1'"'] & \widehat{Q}\oplus_\chi ss^{-r}\widehat{Q} \arrow[d, "\theta_4"] \\ C_3 \arrow[r] & C_4 \end{tikzcd}$$

commutes up to CDGA homotopy. Indeed it already commutes on $\widehat{R}\otimes\Lambda U_0$ and hence the two ways of going around differ by an element of $[\Lambda U_1, C_4]$ and this group is 0 since $H^i(C_4)=0$ for $i\geq 2r-3$ and U_1 has no elements in degrees $\leq 2r-3$. Because $C_3\to C_4$ is a surjection we can replace h_1' by a map $h_1\colon \widehat{R}\otimes\Lambda U_0\otimes\Lambda U_1\to C_3$ making the last diagram commute exactly. Using this same method at each stage and taking the direct limit we get our desired CDGA map $\overline{\theta}_3\colon \widehat{A}\to C_3$ making the diagram commute. □

The approach taken in the last lemma gives a hint of how to approach the problem of extending an $\widehat{R}$–dgmodule structure to a CDGA structure when there are no dimension restrictions. At each stage one would have to choose which representatives of the cokernel to kill. Now we describe a CDGA model of Diagram (14)

Lemma 5.7 *Assume that r is even. With the semitrivial CDGA on the mapping cones, the square*

$$\mathbf{E} = \begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, hook] & Q \arrow[d, hook] \\ R \oplus_{\phi^!} ss^{-r} Q \arrow[r, "\phi\oplus\mathrm{id}"'] & Q \oplus_{\phi\phi^!} ss^{-r} Q. \end{tikzcd}$$

is a commuting square of CDGAs.

Also there exists a CDGA square

$$\mathbf{E}' = \begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, hook] & Q \arrow[d, hook] \\ A \arrow[r, "\kappa"'] & Q \otimes \Lambda(z). \end{tikzcd}$$

such that κ is a fibration and there exists a CDGA quasi-isomorphism $\xi\colon R \oplus_{\phi^!} ss^{-r} Q \to A$ such that

$$\mathbf{\Theta}_1 = \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \xi & e^{-1} \end{pmatrix} : \mathbf{E} \to \mathbf{E}'$$

is a weak equivalence of CDGA squares.

Proof That $\mathbf{E}$ is a CDGA square follows using Proposition 3.18. To construct $\mathbf{E}'$ and $\mathbf{\Theta}_1$ we factor the CDGA map $\phi \oplus \mathrm{id}\colon R \oplus_{\phi^!} ss^{-r} Q \to Q \oplus_{\phi\phi^!} ss^{-r} Q$ as an acyclic cofibration $\xi\colon R \oplus_{\phi^!} ss^{-r} Q \to A$ followed by a fibration $\kappa\colon A \to Q \oplus_{\phi\phi^!} ss^{-r} Q$. The lemma follows easily. □

In the next lemma $\mathbf{D}$ and $\mathbf{D}'$ are the diagrams of Lemma 5.6, $\mathbf{E}$ and $\mathbf{E}'$ are the diagrams of the previous lemma and $\mathbf{F}$, $\mathbf{F}'$ and $\mathbf{\Theta}_6$ are the diagrams and map from Lemma 4.7.

Lemma 5.8 *Assume that r is even. Let ρ be the map from Section 4.4, γ' from Section 5.2, $\widehat{e}$ from Lemma 5.5 and $\mathbf{\Theta}_1$ from Lemma 5.7. Set*

$$\mu' = (i^* + 0)(\beta' \oplus s\rho\gamma')\widehat{e} : \widehat{Q} \otimes \Lambda z \to A_{\mathrm{PL}}(\partial T)$$

which is a quasi-isomorphism. There exist two CDGA maps $\zeta\colon \widehat{A} \xrightarrow{\simeq} A$ and $\zeta'\colon \widehat{A} \xrightarrow{\simeq} A_{\mathrm{PL}}(B)$ such that if we set

$$\mathbf{\Theta}_2 = \begin{pmatrix} \alpha & \beta \\ \zeta & \beta \otimes \mathrm{id} \end{pmatrix} \quad \textit{and} \quad \mathbf{\Theta}_3 = \begin{pmatrix} \alpha' & \beta' \\ \zeta' & \mu' \end{pmatrix}$$

then we have a chain of quasi-isomorphisms of CDGA squares

$$\mathbf{E} \xrightarrow{\mathbf{\Theta}_1} \mathbf{E}' \xrightarrow{\mathbf{\Theta}_2} \mathbf{D}' \xrightarrow{\mathbf{\Theta}_3} \mathbf{F}.$$

Proof Set

$$\mathbf{\Theta}_4 = \begin{pmatrix} \alpha & \beta \\ \alpha \oplus ss^{-r}\beta & \beta \oplus_h ss^{-r}\beta \end{pmatrix} \colon \mathbf{D} \to \mathbf{E}$$

and

$$\mathbf{\Theta}_5 = \begin{pmatrix} \alpha' & \beta' \\ \alpha' \oplus s\gamma' & \beta' \oplus s\rho\gamma' \end{pmatrix} \colon \mathbf{D} \to \mathbf{F}'.$$

By Lemmas 4.7, 5.4 and 5.7 we have a weak equivalences of $\widehat{R}$–dgmodule squares $\mathbf{\Theta}_4\mathbf{\Theta}_1 \colon \mathbf{D} \to \mathbf{E}'$ and $\mathbf{\Theta}_6\mathbf{\Theta}_5 \colon \mathbf{D} \to \mathbf{F}$. Also the maps between the objects in the right hand column of each square are $\widehat{Q}$–dgmodule maps. We can apply Lemma 5.6 to these maps and get the string of quasi-isomorphisms of squares as stated in the lemma. □

6 Model of the projectivization of a complex bundle

In this section we suppose that $f \colon V \to W$ is a smooth embedding of closed manifolds of codimension $2k$. We will assume that $k > 1$. We also suppose that the normal bundle ν of the embedding has some fixed structure of a complex vector bundle,

$$\nu \colon \mathbb{C}^k \to E \xrightarrow{\pi} V.$$

Let T be a compact tubular neighborhood of V in W. By the Tubular Neighborhood Theorem, we can identify T with the disk bundle $D\nu$ and ∂T with its sphere bundle $S\nu$ in such a way that the zero section of $D\nu$ corresponds to the inclusion

$$\sigma \colon V \to T.$$

We fix such an identification. We also suppose we have been given some CDGA model Q of $A_{\mathrm{PL}}(T)$ and a common model $\widehat{Q}$

$$Q \xleftarrow{\beta} \widehat{Q} \xrightarrow{\beta'} A_{\mathrm{PL}}(T)$$

of Q and $A_{\mathrm{PL}}(V)$ such that (β, β') is surjective and β and β' are quasi-isomorphisms as in Section 5.1. Clearly $\sigma^*\beta' \colon \widehat{Q} \to A_{\mathrm{PL}}V$ is also a quasi-isomorphism. By Lemma 3.6 such a common model can be constructed from any CDGA model Q of $A_{\mathrm{PL}}(T)$.

The aim of this section is to describe the projective bundle $P\nu$ associated to ν and give a CDGA model for this projective bundle. In Section 6.1 we review the definition of the projective bundle and of Chern classes and prove the triviality of a certain line bundle (Lemma 6.1). Next in Section 6.2 we consider the pullback over a point of the sphere bundle and of its projectivization. Using orientation information we show that the models of these pullbacks and the model of the sphere bundle $P\nu$ can be chosen in a compatible way (Lemma 6.4 and Lemma 6.6). Then in Section 6.3 we construct a model of the projectivization of the sphere bundle and use the results from Section 6.2 to show this model is compatible with the model of the boundary of the normal bundle (Lemma 6.8 and Proposition 6.9).

6.1 The projective bundle and the Chern classes

Next we recall the definition of the Chern classes of a complex bundle using the associated projective bundle as described in [2, IV.20]. Consider the universal complex line bundle γ^1 over $\mathbb{C}P(\infty)$

$$\mathbb{C} \to E\gamma^1 \to \mathbb{C}P(\infty)$$

where $\mathbb{C}P(\infty) = \{l : l \text{ is a } \mathbb{C} \text{ line in } \mathbb{C}^\infty\}$ and $E\gamma^1 = \{(l, v) : l \in \mathbb{C}P(\infty), v \in l\}$. This complex line bundle can also be viewed as an oriented real vector bundle of rank 2. Therefore we have an associated Euler class $e(\gamma^1)$ and we set

$$a_\infty := -e(\gamma^1) \in H^2(\mathbb{C}P(\infty), \mathbb{Z}).$$

This is our preferred generator of the cohomology algebra $H^*(\mathbb{C}P(\infty), \mathbb{Z})$. Note that if $j\colon S^2 = \mathbb{C}P(1) \to \mathbb{C}P(\infty)$ is the obvious inclusion then $j^*(a_\infty) \in H^2(S^2; \mathbb{Z})$ is the orientation class corresponding to the orientation coming from its complex structure.

To any complex vector bundle ν we can associate its *projective bundle* which is defined as follows (see Bott and Tu [2, page 269]). Set $E_0 = E \setminus \{\text{zero section}\}$ and consider the bundle

$$\nu_0\colon \mathbb{C}^k \setminus \{0\} \to E_0 \xrightarrow{\pi_0} V$$

Then $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ acts on each fibre in E_0 by complex multiplication and we can define the orbit space $P\nu = E_0/\mathbb{C}^*$. In other words,

$$P\nu = \{(v, \ell) : v \in V, \ell \text{ is a complex line in the fibre } E_v := \pi^{-1}(v)\}$$

and we have the projective bundle

$$\mathbb{C}P(k-1) \xrightarrow{\text{inc}} P\nu \xrightarrow{\pi'} V.$$

Denote by $q\colon E_0 \to P\nu$ the quotient map and consider the commutative diagram

(23)
$$\begin{tikzcd} E_0 \arrow[rr, "q"] \arrow[dr, "\pi_0"'] & & P\nu \arrow[dl, "\pi'"] \\ & V. & \end{tikzcd}$$

Since the inclusion $\partial T \cong S\nu \hookrightarrow E_0$ is a homotopy equivalence, we also use q to denote the composition $\partial T \cong S\nu \to E_0 \xrightarrow{q} P\nu$.

We now come to the definition of the Chern classes $c_i(\nu) \in H^{2i}(V;\mathbb{Z})$ in terms of the projective bundle. The pullback of ν along π' is a $\mathbb{C}^k-$bundle $\pi'^*(\nu)$ over $P\nu$ containing a tautological line bundle defined by

$$\lambda = \{(v,\ell,x) : v \in V, \ell \text{ is a line in } E_v, x \in \ell\}. \tag{24}$$

The complex line bundle λ is classified by some map $\alpha\colon P\nu \to \mathbb{C}P(\infty) \simeq BU(1)$, that is $\lambda \cong \alpha^*(\gamma^1)$. We define the *canonical class* of ν as the cohomology class

$$a = \alpha^*(a_\infty) \in H^2(P\nu,\mathbb{Z}). \tag{25}$$

Since the restriction of that class to each fibre $\mathbb{C}P(k-1)$ of π' is a generator of the cohomology (because the pullback of λ to that fibre is the universal line bundle over $\mathbb{C}P(k-1)$), the Leray–Hirsch Theorem gives an isomorphism of $H^*(V,\mathbb{Z})$–algebras

$$H^*(P\nu,\mathbb{Z}) \cong H^*(V,\mathbb{Z})[a] / \left(\sum_{i=0}^{k} c_i(\nu) a^{k-i} \right) \tag{26}$$

where $c_0(\nu) = 1$ and by definition the $c_i(\nu) \in H^{2i}(V,\mathbb{Z})$ for $i > 0$ are the Chern classes. Notice a straightforward calculation shows that

$$a = -c_1(\lambda). \tag{27}$$

Lemma 6.1 $q^*(a) = 0$.

Proof Consider the following diagram of vector bundles:

$$\begin{tikzcd}
q^*\lambda \arrow[r] \arrow[d] & \lambda \arrow[r, hook] \arrow[dr] & \pi'^*(\nu) \arrow[rr] \arrow[d] & & \nu \arrow[d] \\
E_0 \arrow[rr, "q"] & & P\nu \arrow[rr, "\pi'"] \arrow[drr, "\alpha"'] & \text{pullback} & V \\
& & & & \mathbb{C}P(\infty) = BU(1)
\end{tikzcd}$$

The cohomology class $q^*(a)$ is the classifying class of the line bundle $q^*(\lambda)$. We will prove that this bundle is trivial, so that $q^*(a) = 0$.

By (24) we have

$$\begin{aligned} q^*\lambda &= \{(e,v,\ell,x) : e \in E_0, v \in V, v = \pi_0(e), \ell \text{ is a line in } E_v, x \in \ell\} \\ &\cong \{(e,x) : e \in E_0, x \in \mathbb{C}.e \subset E_{\pi_0(e)}\}. \end{aligned}$$

Therefore we have the trivialization

$$E_0 \times \mathbb{C} \xrightarrow{\cong} q^*\lambda\ ,\ \ (e,z) \mapsto (e, e.z). \qquad \square$$

6.2 Orientations

As in Section 4.1 we suppose $* \in V$ is some fixed point. Consider the map

$$\eta\colon S^{2k-1} \to \mathbb{C}P(k-1)$$

obtained as the restriction of the map

$$q\colon S\nu \to P\nu$$

to the fibres over our point $* \in V$. The aim of this subsection is to give an explicit CDGA model of the map η.

First we define some more cohomology classes. Recall the class $a \in H^2(P\nu)$ defined in (25). We also denote by $a \in H^2(\mathbb{C}P(k-1))$ its restriction to the fibre. Notice that the restriction of λ to $\mathbb{C}P(k-1)$ is the tautological line bundle γ_{k-1} and hence by Equation (27) and the naturality of the Chern classes

$$a = -c_1(\gamma_{k-1}). \tag{28}$$

Equivalently we can take Equation (28) to be the definition of $a \in H^2(\mathbb{C}P(k-1))$. The fibre D^{2k} of the disk bundle $D\nu$ over our point $* \in V$ has a canonical orientation given by its complex structure $D^{2k} \subset \mathbb{C}^k$. This determines the cohomology class

$$u_{D^{2k}} \in H^{2k}(D^{2k}, S^{2k-1}; \mathbb{Z}) \tag{29}$$

of Equation (15) which through the connecting homomorphism

$$\delta\colon H^{2k-1}(S^{2k-1};\mathbb{Z}) \xrightarrow{\cong} H^{2k}(D^{2k}, S^{2k-1};\mathbb{Z})$$

determines an orientation class

$$u_S \in H^{2k-1}(S^{2k-1}, \mathbb{Z}). \tag{30}$$

Next we extend η to some map $\tilde{\eta}\colon D^{2k} \to \mathbb{C}P(k)$ and describe $\mathbb{C}P(k)$ as a suitable pushout. For $z = (z_1, \dots, z_k) \in \mathbb{C}^k$ we set $||z||_2 = (\sum_{i=1}^k |z_i|^2)^{1/2}$. Consider the disc and the sphere

$$D^{2k} = \{z \in \mathbb{C}^k : ||z||_2 \le 1\}$$
$$S^{2k-1} = \{z \in \mathbb{C}^k : ||z||_2 = 1\}.$$

Define a map

$$\widetilde{\eta}\colon D^{2k} \to \mathbb{C}P(k), z \mapsto [z_1 : \cdots : z_k : 1 - ||z||_2].$$

Then $\eta = \widetilde{\eta}|_{S^{2k-1}}$ where $\mathbb{C}P(k-1)$ is considered as the hyperplane with last coordinate 0.

We need to replace the map η by an equivalent map that is a cofibration. This is done using the standard mapping cylinder construction. We will describe it explicitly so that it is easy to see how it is compatible with $\widetilde{\eta}$. Define the contraction

$$\rho\colon D^{2k} \to D^{2k}, z \mapsto z/2.$$

Set
$$X = \{\widetilde{\eta}(z) : z \in D^{2k}, 1/2 \leq ||z||_2 \leq 1\} \subset \mathbb{C}P(k)$$

and denote by $l\colon S^{2k-1} \to D^{2k}$, $\widetilde{l}\colon X \to \mathbb{C}P(k)$ and $s\colon \mathbb{C}P(k-1) \to X$ the obvious inclusions. Then the composite

$$\widetilde{h} = \widetilde{\eta}\rho\colon D^{2k} \to \mathbb{C}P(k)$$

is a homeomorphism onto its image and a cofibration. Also we can think of X as a tubular neighbourhood of $\mathbb{C}P(k-1)$ in $\mathbb{C}P(k)$ although we will not use this fact.

We summarize a few facts about these maps in the following lemma. Its proof is straightforward.

Lemma 6.2 (i) *The restriction of $\widetilde{h}$ to S^{2k-1} induces a cofibration*

$$h\colon S^{2k-1} \to X.$$

(ii) *There is a pushout:*

$$\begin{tikzcd} S^{2k-1} \arrow[r, "h"] \arrow[d, "l"'] & X \arrow[d, "\widetilde{l}"] \\ D^{2k} \arrow[r, "\widetilde{h}"'] & \mathbb{C}P(k) \end{tikzcd}$$

(iii) *The inclusion map $s\colon \mathbb{C}P(k-1) \to X$ is a homotopy equivalence.*

(iv) *The following diagram commutes up to homotopy:*

$$\begin{tikzcd} S^{2k-1} \arrow[r, "h"] \arrow[dr, "\eta"'] & X \\ & \mathbb{C}P(k-1). \arrow[u, "s"'] \end{tikzcd}$$

By abuse of notation we also use $a \in H^2(X)$ to denote the preimage through the isomorphism s^* of the element $a \in H^2(\mathbb{C}P(k-1))$ defined in (28).

Lemma 6.3 *Consider the Sullivan algebra $(\Lambda(x,z); dx = 0, dz = x^k)$ with $|x| = 2$ and $|z| = 2k-1$. There exists a homotopy commutative diagram of CDGA*

$$\begin{tikzcd} \Lambda(x,z) \arrow[r, "\mathrm{proj}"] \arrow[d, "g_0"'] & \Lambda(z) \arrow[d, "f_0"] \\ A_{\mathrm{PL}}(X) \arrow[r, "A_{\mathrm{PL}}(h)"'] & A_{\mathrm{PL}}(S^{2k-1}). \end{tikzcd}$$

such that $[g_0(x)] = a$ (defined in (28)) and $[f_0(z)] = -u_S$ (defined in (30)).

Proof Since $k > 1$, the restriction map

$$\iota\colon H^2(X, S^{2k-1}) \to H^2(X)$$

is an isomorphism and so the class a lifts to a unique class $a_0 \in H^2(X, S^{2k-1})$.

Denote by $\widetilde{a} \in H^2(\mathbb{C}P(k))$ the opposite of the first Chern class of the tautological line bundle γ_k over $\mathbb{C}P(k)$. Since γ_{k-1} is the restriction of γ_k over $\mathbb{C}P(k-1)$, Equation (28) and the naturality of the Chern classes implies that $\widetilde{l}^*(\widetilde{a}) = a$. In view of the pushout of Lemma 6.2(ii), $\widetilde{a}$ lifts to some class $\widetilde{a}_0 \in H^2(\mathbb{C}P(k), D^{2k})$ such that $(\widetilde{l}, l)^*(\widetilde{a}_0) = a_0$.

Let $\alpha_0 \in A^2_{\mathrm{PL}}(X, S^{2k-1}) \cap \ker d$ be a representative of a_0 and α be its image in $A^2_{\mathrm{PL}}(X) \cap \ker d$. Since $a^k = 0$ in $H^2(X)$, there exists $\zeta \in A^{2k-1}_{\mathrm{PL}}(X)$ such that $d\zeta = \alpha^k$. Define

$$g_0\colon (\Lambda(x,z); dz = x^k) \to A_{\mathrm{PL}}(X)$$

by $g_0(x) = \alpha$ and $g_0(z) = \zeta$.

Since $h^*(\alpha) = 0$, $h^*(\zeta)$ is a cocycle in $A_{\mathrm{PL}}(S^{2k-1})$ and so we can define

$$f_0\colon \Lambda(z) \to A_{\mathrm{PL}}(S^{2k-1}),\ z \mapsto h^*(\zeta).$$

This definition makes the diagram of the lemma commutative.

We proceed to prove that $[h^*(\zeta)] = -u_S$. This will imply that $[f_0(z)] = -u_S$ and thus complete the proof of the lemma.

Denote the cohomology connecting homomorphisms of the pairs (D^{2k}, S^{2k-1}) and (X, S^{2k-1}) by δ and δ' respectively. Consider the following diagram in cohomology:

$$\begin{tikzcd}
H^{2k-1}(S^{2k-1};\mathbb{Z}) \arrow[rr, bend left, "\delta'"] \arrow[dd, bend right, "\delta"'] & H^{2k}(X;\mathbb{Z}) & H^{2k}(X, S^{2k-1};\mathbb{Z}) \arrow[l, "\iota"'] \\
H^{2k}(D^{2k};\mathbb{Z}) & H^{2k}(\mathbb{C}P(k);\mathbb{Z}) \arrow[l, "\tilde{h}^*"'] \arrow[u, "\tilde{l}^*"'] & H^{2k}(\mathbb{C}P(k), D^{2k};\mathbb{Z}) \arrow[l, "\tilde{\iota}"'] \arrow[u, "(\tilde{l},l)^*"'] \\
H^{2k}(D^{2k}, S^{2k-1};\mathbb{Z}) \arrow[u] & H^{2k}(\mathbb{C}P(k), X;\mathbb{Z}). \arrow[l, "(\tilde{h},h)^*"'] \arrow[u, "\tilde{\iota}'"'] &
\end{tikzcd}$$

A diagram chase at the chain level gives us the formula

$$\tilde{\iota}'((\tilde{h}, h)^*)^{-1}\delta = -\tilde{\iota}((\tilde{l}, l)^*)^{-1}\delta', \tag{31}$$

the minus sign corresponding to the fact that in the Mayer–Vietoris sequence there is a minus sign on one of the maps.

By the definition of the pushout of Lemma 6.2(ii) and the construction of ζ we have

$$\delta'([h^*\zeta]) = [\alpha^k] = a_0^k \in H^{2k}(X, S^{2k-1}).$$

Since $(\tilde{l}, l)^*$ and $\tilde{\iota}$ are multiplicative we get

$$\tilde{\iota}((\tilde{l}, l)^*)^{-1}\delta'[h^*\zeta] = \tilde{a}^k \in H^{2k}(\mathbb{C}P(k)). \tag{32}$$

On the other hand since $\tilde{a} = -c_1(\lambda)$, Milnor and Stasheff [22, page 170] says that the orientation of $\mathbb{C}P(k)$ induced by its complex structure corresponds to the class

$$\tilde{a}^k \in H^{2k}(\mathbb{C}P(k)).$$

Since $\tilde{h}$ preserves the given orientations, we have $\tilde{\iota}'((\tilde{h}, h)^*)^{-1}u_D = \tilde{a}^k$, and hence

$$\tilde{\iota}'((\tilde{h}, h)^*)^{-1}\delta(u_S) = \tilde{a}^k. \tag{33}$$

Equations (31), (32) and (33) imply that

$$\tilde{\iota}'((\tilde{h}, h)^*)^{-1}\delta(u_S) = -\tilde{\iota}'((\tilde{h}, h)^*)^{-1}\delta[h^*\zeta].$$

Thus since $\tilde{\iota}'((\tilde{h}, h)^*)^{-1}\delta\colon H^{2k-1} \to H^{2k}\mathbb{C}P(n)$ is an isomorphism we deduce that

$$[h^*\zeta] = -u_S.$$

This completes the proof of the lemma. □

Lemma 6.4 *Let $|x| = 2$ and $|z| = 2k-1$. Suppose we are given a CDGA diagram*

$$\begin{tikzcd}
(\Lambda(x,z); dz = x^k) \arrow[r, "\mathrm{proj}"] \arrow[d, "g"'] & \Lambda(z) \arrow[d, "f"] \\
A_{\mathrm{PL}}(\mathbb{C}P(k-1)) \arrow[r, "A_{\mathrm{PL}}(\eta)"'] & A_{\mathrm{PL}}(S^{2k-1}).
\end{tikzcd}$$

such that $[g(x)] = a$ and $[f(z)] = -u_S$. Then the diagram is homotopy commutative and f and g are quasi-isomorphisms.

Proof Observe that since $H^{2k-1}(\mathbb{C}P(k-1)) = 0$ the homotopy class of the map g is determined by the fact that $[g(x)] = a$. This equation also implies that g is a quasi-isomorphism. Similarly the equation $[f(z)] = -u_S$ determines the homotopy class of the map and implies that it is a quasi-isomorphism. The lemma then follows easily from Lemma 6.2 (iii) and (iv) and Lemma 6.3. □

Lemma 6.5 *Consider the map $\mu'\colon \widehat{Q} \otimes \Lambda(z) \to A_{\mathrm{PL}}(\partial T)$ from Lemma 5.8.*

(i) *Consider the connecting homomorphism*

$$\delta\colon H^{2k-1}(\partial T) \to H^{2k}(T, \partial T)$$

and the Thom class $\overline{\theta}$ from Section 4.1. Then

$$\delta[\mu'(z)] = -\overline{\theta}.$$

(ii) *Let* $\mathrm{inc}\colon S^{2k-1} \to \partial T$ *be the inclusion. Then*

$$\mathrm{inc}^*[\mu'(z)] = -u_S.$$

Proof By the definition of μ' in Lemma 5.8, the following diagram commutes

$$\begin{tikzcd}
\widehat{Q} \otimes \Lambda(z) \arrow[r, "\widehat{e}"] \arrow[ddr, "\mu'"'] & \widehat{Q} \oplus ss^{-2k}\widehat{Q} \arrow[d, "\beta' \oplus s\rho\gamma'"] \\
& A_{\mathrm{PL}}(T) \oplus_{\iota'} sA_{\mathrm{PL}}(T, \partial T) \arrow[d, "i^* + 0"] \\
& A_{\mathrm{PL}}(\partial T).
\end{tikzcd}$$

By the construction of $\widehat{e}$ in Lemma 5.5 there exists $\widehat{q} \in \widehat{Q}^{2k-1}$ such that $\widehat{e}(z) = (\widehat{q}, ss^{-2k}1)$. By Lemma 5.1 $\rho\gamma'(ss^{2k}(1)) = \theta$ and we see that

$$(\beta' \oplus s\rho\gamma')(\widehat{q}, ss^{-2k}1) = (\beta'(\widehat{q}), s\theta),$$

so it follows that

$$\mu'(z) = i^*(\beta'(\hat{q})). \tag{34}$$

Since z is a cocycle, $(\beta'(\hat{q}), s\theta) \in A_{\mathrm{PL}}(T) \oplus_{\iota'} sA_{\mathrm{PL}}(T, \partial T)$ is also a cocycle which implies that in $A_{\mathrm{PL}}(T)$

$$d(\beta'(\hat{q})) = -\iota'(\theta).$$

The definition of the connecting homomorphism then implies that

$$\delta[i^*\beta'(\hat{q})] = -[\theta] = -\overline{\theta}.$$

Combining this formula with Equation (34) proves (i).

(ii) This is a consequence of (i), of the fact that $u_{D^{2k}} = \mathrm{inc}^*(\overline{\theta})$ (see Equation (17)) and of the naturality of the connecting homomorphism in the following diagram:

$$\begin{tikzcd} H^{2k-1}(\partial T) \arrow[r, "\delta"] \arrow[d, "\mathrm{inc}^*"'] & H^{2k}(T, \partial T) \arrow[d, "\mathrm{inc}^*"] \\ H^{2k-1}(S^{r-1}) \arrow[r, "\cong", "\delta"'] & H^{2k}(D^{2k}, S^{2k-1}). \end{tikzcd}$$

□

Lemma 6.6 *The inclusion of the point $* \in V$ determines an augmentation on $\hat{Q}$ using the composition*

$$\hat{Q} \xrightarrow{\sigma^*\beta'} A_{\mathrm{PL}}(V) \to A_{\mathrm{PL}}(*) = \mathbb{Q}.$$

In turn the augmentation determines the projection map proj: $\hat{Q} \otimes \Lambda(z) \to \Lambda(z)$. *Suppose $f\colon (\Lambda(z); dz=0) \to A_{\mathrm{PL}}(S^{2k-1})$ is any CDGA map such that $[f(z)] = -u_S$. Then the following diagram commutes up to CDGA homotopy:*

$$\begin{tikzcd} \hat{Q} \otimes \Lambda(z) \arrow[r, "\mathrm{proj}"] \arrow[d, "\mu'"'] & \Lambda(z) \arrow[d, "f"] \\ A_{\mathrm{PL}}(\partial T) \arrow[r, "\mathrm{inc}^*"'] & A_{\mathrm{PL}}(S^{2k-1}) \end{tikzcd}$$

Proof In this proof we denote by $*$ an augmentation map followed by a unit map. For fixed domain and range this composition is unique up to homotopy since all of our CDGAs are homologically connected. Since $\mathrm{proj}|_{\hat{Q}} = *$, $f\,\mathrm{proj}|_{\hat{Q}} = *$. Also $\mu'|_{\hat{Q}}$ factors up to homotopy through $A_{\mathrm{PL}}(V)$ so $\mathrm{inc}^*\mu'|_{\hat{Q}} \simeq *$. Because $dz = 0$ in $\hat{Q} \otimes \Lambda(z)$ (see Lemma 5.5) the diagram commutes if and only if $\mathrm{inc}^*[\mu'(z)] = [f\,\mathrm{proj}(z)]$ in $HA_{\mathrm{PL}}(S^{2n-1})$. Lemma 6.5 says that $\mathrm{inc}^*[\mu'(z)] = -u_S$ and by assumption $[f(z)] = -u_S$. Thus the diagram homotopy commutes as required. □

6.3 A model of the projective and the sphere bundles

Recall from the start of Section 6 that $V \hookrightarrow W$ has codimension $2k$ and so this is also the real rank of the normal bundle ν, which also has a complex structure of rank k. Recall also the maps σ, β and β' from the start of Section 6.

We describe the Sullivan models that we will prove are models for the projective bundle $P\nu$. Set $\gamma_0 = 1$ and for $1 \le i \le k$ let $\gamma_i \in Q^{2i} \cap \ker d$ be representatives of the images of the Chern classes

$$c_i(\nu) \in H^{2i}(V, \mathbb{Z}) \to H^{2i}(V, \mathbb{Q}) \overset{(\sigma\beta')^*}{\cong} H^{2i}(\widehat{Q}) \overset{\beta^*}{\cong} H^{2i}(Q).$$

Since β is a surjective quasi-isomorphism there exists $\widehat{\gamma}_i \in \widehat{Q}^{2i} \cap \ker d$ such that $\beta(\widehat{\gamma}_i) = \gamma_i$. Also we can take $\widehat{\gamma}_0 = 1$. Let $|x| = 2$ and $|z| = 2k-1$ and define relative Sullivan models

$$(Q \otimes \Lambda(x,z); Dx = 0,\ Dz = \textstyle\sum_{i=0}^{k} \gamma_i x^{k-i}) \tag{35}$$

and similarly

$$(\widehat{Q} \otimes \Lambda(x,z); \widehat{D}x = 0,\ \widehat{D}z = \textstyle\sum_{i=0}^{k} \widehat{\gamma}_i x^{k-i}). \tag{36}$$

These models are motivated by Equation (26) for the Chern classes. For the next lemma recall that $\sigma\colon V \to T$ corresponds to the inclusion of the zero section. In addition recall the classes $a \in H^2(P\nu)$ defined in (25) and its restriction to $\mathbb{C}P(k-1)$ also denoted by $a \in H^2(\mathbb{C}P(k-1))$.

Lemma 6.7 *Consider the projective bundle associated to* ν

$$\mathbb{C}P(k-1) \xrightarrow{\text{inc}} P\nu \xrightarrow{\pi'} V.$$

Let $\widehat{D}$ *be the differential given in Equation* (36)*. Suppose* $g\colon (\Lambda(x,z); dz = x^k) \to A_{\mathrm{PL}}(\mathbb{C}P(k-1))$ *is any map such that* $[g(x)] = a$*. Then there exists a quasi-isomorphism* $\theta'\colon (\widehat{Q} \otimes \Lambda(x,z); \widehat{D}) \to A_{\mathrm{PL}}(P\nu)$ *making the following diagram commute up to homotopy:*

$$\begin{tikzcd}
\widehat{Q} \arrow[r] \arrow[d, "\sigma^*\beta'"'] & (\widehat{Q} \otimes \Lambda(x,z), \widehat{D}) \arrow[r, "\text{proj}"] \arrow[d, "\theta'"', "\simeq"] & (\Lambda(x,z), dz = x^k) \arrow[d, "g"] \\
A_{\mathrm{PL}}(V) \arrow[r, "\pi'^*"] & A_{\mathrm{PL}}(P\nu) \arrow[r, "\text{inc}^*"] & A_{\mathrm{PL}}(\mathbb{C}P(k-1)).
\end{tikzcd}$$

Proof As observed at the start of the section $\sigma^*\beta'$ is a quasi-isomorphism. Define $\theta'|_{\hat{Q}\otimes\Lambda(x)}$ so that $\theta'|_{\hat{Q}} = (\sigma\pi')^*\beta'$ and $\theta'(x)$ is any representative of the image of a under the map.

$$H^*(P\nu,\mathbb{Z}) \to H^*(P\nu,\mathbb{Q}) \cong H(A_{\mathrm{PL}}(P\nu))$$

Then from the definition of the Chern classes (see Section 6.1), $\sum_{i=0}^{k} c_i(\nu)a^{k-i} = 0$ in $H^*(P\nu,\mathbb{Z})$ and so $[\sum_{i=0}^{k}\pi'^*\sigma^*\beta'(\hat{\gamma}_i)(\theta'(x))^{k-i}] = 0$ in $H(A_{\mathrm{PL}}(P\nu))$. Thus an extension over z of $\theta'|_{\hat{Q}\otimes\Lambda(x)}$ exists. Let θ' be any such extension. It is clear using Equation (26) that θ' is a quasi-isomorphism and makes the left hand square of our diagram commute. Since $(\mathrm{inc}^*\theta')|_{\hat{Q}} = 0$ the right hand square commutes when restricted to $\hat{Q}$. Since $g(x)$ represents $\mathrm{inc}^*(a)$ we see that the right hand square commutes up to homotopy when restricted to $\Lambda(x)$. Thus it commutes up to homotopy when restricted to $\hat{Q}\otimes\Lambda(x)$. We know that $[\Lambda(z), A_{\mathrm{PL}}\mathbb{C}P(k-1)] = H^{2k-1}(\mathbb{C}P(k-1)) = 0$. This implies that, up to homotopy, there is a unique extension of the map $\hat{Q}\otimes\Lambda(x) \to A_{\mathrm{PL}}(\mathbb{C}P(k-1))$ over $\hat{Q}\otimes\Lambda(x,z)$. Thus the right hand square in the diagram commutes up to homotopy and the lemma has been proven. □

In this section θ' yielded a model of $A_{\mathrm{PL}}(P\nu)$. Lemma 6.8 will show that the two are compatible. This lemma controls the automorphism of the model of $A_{\mathrm{PL}}(\partial T)$. If Q is our model of $A_{\mathrm{PL}}(T)$ then the model of $A_{\mathrm{PL}}(\partial T)$ is $Q\otimes\Lambda z$. An automorphism $\phi\in[A_{\mathrm{PL}}(\partial T), A_{\mathrm{PL}}(\partial T)]$ can be considered as an element of $[Q\otimes\Lambda z, Q\otimes\Lambda z] \cong [Q, Q\otimes\Lambda z]\times[\Lambda z, Q\otimes\Lambda z]$. We control the first factor by working in the category of Q–dgmodules. Another way to think of this is that we work with objects together with maps from Q. To handle the second factor we can observe that for dimension reasons $[\Lambda z, Q\otimes\Lambda z]\cong[\Lambda z,\Lambda z]$ and in turn we have the general isomorphism $[\Lambda z,\Lambda z]\cong \mathrm{Hom}(H^{2k-1}\partial T, H^{2k-1}\partial T)$. So we only have to control a single element in homology and this is the reason we have been keeping track of orientation classes. In the last section μ' gave us a model of $A_{\mathrm{PL}}(\partial T)$.

Recall β from the start of the section, μ' from Lemma 5.8, θ' from Lemma 6.7, $q\colon \partial T\to P\nu$ defined in Section 6.1 and the CDGAs $Q\otimes\Lambda(x,z)$ and $\hat{Q}\otimes\Lambda(x,z)$ defined in Equations (35) and (36).

Lemma 6.8 *Assume that* $2k \geq \dim(V)+2$.

The following diagram commutes up to homotopy:

(37)
$$\begin{tikzcd}
Q\otimes\Lambda(x,z) \arrow[r, "\mathrm{proj}"] & Q\otimes\Lambda(z) \\
\widehat{Q}\otimes\Lambda(x,z) \arrow[u, "\beta\otimes\mathrm{id}"] \arrow[r, "\mathrm{proj}"] \arrow[d, "\theta'"'] & \widehat{Q}\otimes\Lambda(z) \arrow[u, "\beta\otimes\mathrm{id}"'] \arrow[d, "\mu'"] \\
A_{\mathrm{PL}}(P\nu) \arrow[r, "q^*"'] & A_{\mathrm{PL}}(\partial T)
\end{tikzcd}$$

Proof The top square clearly commutes. Since we will be using different inclusions, to avoid confusion for the rest of the proof set $I=\mathrm{inc}^*\colon A_{\mathrm{PL}}(\partial T)\to A_{\mathrm{PL}}(S^{2k-1})$. To avoid too much clutter in the equations we will also sometimes write I instead of I_* for induced maps. Recall the classes $a\in H^2(P\nu)$ defined in (25) and also its restriction to $\mathbb{C}P(k-1)$ also denoted by $a\in H^2(\mathbb{C}P(k-1))$. Let $g\colon \Lambda(x,z)\to A_{\mathrm{PL}}(\mathbb{C}P(k-1))$ be any map such that $[g(x)]=a$ and $f\colon \Lambda(z)\to A_{\mathrm{PL}}(S^{2k-1})$ be any map such that $[f(z)]=-u_S$. Note that these equations determine g and f up to homotopy. To see that the bottom square of (37) commutes consider the following cube:

$$\begin{tikzcd}
\widehat{Q}\otimes\Lambda(x,z) \arrow[rr, "\mathrm{proj}"] \arrow[dr, "\mathrm{proj}"'] \arrow[dd, "\theta'"'] & & \widehat{Q}\otimes\Lambda(z) \arrow[dr, "\mathrm{proj}"] \arrow[dd, "\mu'"] & \\
& \Lambda(x,z) \arrow[rr, "\mathrm{proj}"] \arrow[dd, "g"] & & \Lambda(z) \arrow[dd, "f"] \\
A_{\mathrm{PL}}(P\nu) \arrow[rr, "q^*"'] \arrow[dr, "\mathrm{inc}^*"'] & & A_{\mathrm{PL}}(\partial T) \arrow[dr, "I"] & \\
& A_{\mathrm{PL}}(\mathbb{C}P(k-1)) \arrow[rr, "A_{\mathrm{PL}}(\eta)"'] & & A_{\mathrm{PL}}(S^{2k-1}).
\end{tikzcd}$$

The back face is the one we wish to show is homotopy commutative. The top and bottom faces clearly commute. The front, right and left faces are homotopy commutative by Lemmas 6.4, 6.6 and 6.7. So we get that $I\mu'(\mathrm{proj})\simeq Iq^*\theta'$. Next consider the coaction sequence [23, Chapter I, Section 3, Proposition 4] associated to the map from the cofibration sequence of CDGA

$$\Lambda(s^{-1}z)\to\widehat{Q}\otimes\Lambda(x)\to\widehat{Q}\otimes\Lambda(x,z)\to\Lambda(z)$$

into $A_{\mathrm{PL}}(\partial T) \to A_{\mathrm{PL}}(S^{2k-1})$. We get a commutative diagram of sets:

$$\begin{tikzcd}
{[\Lambda(z), A_{\mathrm{PL}}(\partial T)]} \arrow[r, "p^*"] \arrow[d, "I_*"] & {[\widehat{Q} \otimes \Lambda(x,z), A_{\mathrm{PL}}(\partial T)]} \arrow[r, "\mathrm{inc}^*"] \arrow[d, "I_*"] & {[\widehat{Q} \otimes \Lambda(x), A_{\mathrm{PL}}(\partial T)]} \arrow[d, "I_*"] \\
{[\Lambda(z), A_{\mathrm{PL}}(S^{2k-1})]} \arrow[r, "p^*"'] & {[\widehat{Q} \otimes \Lambda(x,z), A_{\mathrm{PL}}(S^{2k-1})]} \arrow[r, "\mathrm{inc}^*"'] & {[\widehat{Q} \otimes \Lambda(x), A_{\mathrm{PL}}(S^{2k-1})]}
\end{tikzcd}$$

By [23, Chapter I, Section 3, Proposition 4'] the rows are exact in the sense that, if $\mathrm{inc}^* f = \mathrm{inc}^* g$ then there exists $\alpha \in [\Lambda(z), \_]$ such that $\alpha \cdot f = g$, where $\_ \cdot \_$ denotes the action of the group $[\Lambda(z), \_]$ on $[\widehat{Q} \otimes \Lambda(x,z), \_]$. Note that our cofibration sequence is a model of a fibration sequence

$$S^{2k-1}_{\mathbb{Q}} = K(\mathbb{Q}, 2k-1) \to P\nu_{\mathbb{Q}} \to V_{\mathbb{Q}} \times K(\mathbb{Q}, 2) \to K(\mathbb{Q}, 2k)$$

and at the space level the coaction sequence is the mapping class sequence (see Switzer [27, Chapter 2]). For any cofibration sequence $A \to B \to C \to \Sigma A$ and group object G in a pointed model category the coaction $\_ \cdot \_ \colon [\Sigma A, G] \times [C, G] \to [C, G]$ from the cofibration sequence is compatible with the group action $\phi\colon [C, G] \times [C, G] \to [C, G]$ induced by the multiplication on G. In particular for any $\alpha \in [\Sigma A, G]$ and $f, g \in [C, G]$:

$$\alpha \cdot \phi(f, g) = \phi(\alpha \cdot f, g). \tag{38}$$

Consider the diagram:

$$\begin{tikzcd}
\widehat{Q} \arrow[d, "\beta'"'] \arrow[drr, "\theta'|_{\widehat{Q}}"] & & \\
A_{\mathrm{PL}}(T) \arrow[r, "\sigma^*"] \arrow[d, "i^*"'] & A_{\mathrm{PL}}(V) \arrow[r, "(\pi')^*"] \arrow[dl, "(\pi|_{\partial T})^*"'] & A_{\mathrm{PL}}(P\nu) \arrow[dll, "(q|_{\partial T})^*"] \\
A_{\mathrm{PL}}(\partial T) & &
\end{tikzcd}$$

The top triangle commutes up to homotopy by Lemma 6.7. The bottom left triangle commutes up to homotopy since $\sigma\colon V \to T$ is the zero section of π and the bottom right triangle commutes since it is A_{PL} of Diagram (23). Thus Diagram (37) commutes up to homotopy when restricted to $\widehat{Q}$.

Since $\theta'(x) = a$ and by Lemma 6.1 $q^*(a) = 0$, Diagram (37) restricted to $\Lambda(x)$ commutes and so restricted to $\widehat{Q} \otimes \Lambda(x)$ commutes up to homotopy. Thus as homotopy classes $\mathrm{inc}^*\mu'(\mathrm{proj}) = \mathrm{inc}^* q^*\theta'$, and there exists $\alpha \in [\Lambda(z), A_{\mathrm{PL}}(\partial T)]$ such that $\alpha\cdot\mu'\mathrm{proj} = q^*\theta'$. Since the action is natural in the second variable $I\alpha\cdot I\mu'\mathrm{proj} = Iq^*\theta'$. As observed above $I\mu'(\mathrm{proj}) = Iq^*\theta'$, so $I\alpha\cdot I\mu'(\mathrm{proj}) = I\mu'(\mathrm{proj})$ and so

$$\begin{aligned} I\alpha\cdot 0 &= I\alpha\cdot((I\mu'(\mathrm{proj}))(I\mu'(\mathrm{proj}))^{-1}) = (I\alpha\cdot I\mu'(\mathrm{proj}))(I\mu'(\mathrm{proj}))^{-1} \\ &= (I\mu'(\mathrm{proj}))(I\mu'(\mathrm{proj}))^{-1} = 0 \end{aligned}$$

with the first equality following from Equation (38) since $A_{\mathrm{PL}}(S^{2k-1})$ is a group object in CDGA. This implies that $p^*(I\alpha) = 0$.

Since $H^{2k}(\hat{Q}) = 0$ by looking at models we get that

$$\pi_{2k-1}(S^{2k-1})\otimes\mathbb{Q} \to \pi_{2k-1}(P\nu)\otimes\mathbb{Q}$$

is injective. Thus $p^*\colon [\Lambda(z), A_{\mathrm{PL}}(S^{2k-1})] \to [\hat{Q}\otimes\Lambda(x,z), A_{\mathrm{PL}}(S^{2k-1})]$ is injective, and so $I\alpha = 0$.

Because $Q\otimes\Lambda(z) \to \Lambda(z)$ with $dz = 0$ models $A_{\mathrm{PL}}(\partial T) \to A_{\mathrm{PL}}(S^{2k-1})$ and $H^{2k-1}(Q) = 0$, I induces an isomorphism on H^{2k-1} and so $I_*\colon [\Lambda(z), A_{\mathrm{PL}}(\partial T)] \to [\Lambda(z), A_{\mathrm{PL}}(S^{2k-1})]$ is an isomorphism. Thus $\alpha = 0$ and so $\mu'\mathrm{proj} = q^*\theta'$ as homotopy classes. □

Proposition 6.9 *Let $\nu\colon \mathbb{C}^k \to E \to V$ be a complex vector bundle of rank k such that $2k \geq \dim V + 2$ and Q be a CDGA weakly equivalent to $A_{\mathrm{PL}}(V)$. Set $\gamma_0 = 1$, and for $1 \leq i \leq k-1$ let $\gamma_i \in Q^{2i}\cap\ker d$ be cocycle representatives of the Chern classes $c_i(\xi) \in H^{2i}(V;\mathbb{Q})$. Using the same notation as at the beginning of Section 6 the diagram*

$$\begin{tikzcd} (Q\otimes\Lambda z, Dz=0) & & (Q\otimes\Lambda(x,z); Dx=0, Dz=\sum_{i=0}^k \gamma_i x^{k-i}) \arrow[ll, "\phi"'] \\ & Q \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

with $\phi|_Q = \mathrm{inc}$, $\phi(x) = 0$, $\phi(z) = z$, $|x| = 2$ and $|z| = 2k-1$ is a CDGA model of the diagram

$$\begin{tikzcd} E_0 \arrow[rr, "q"] \arrow[dr, "\pi_0"'] & & P\nu \arrow[dl, "\pi'"] \\ & V. & \end{tikzcd}$$

Proof Observe that Lemma 3.9 also works in the case that f and f' are identity maps. Thus the proposition follows from Lemma 6.8 by applying Lemma 3.9 twice. □

Notice that since $2k \geq \dim V + 2$, $\gamma_k = 0$, and so $Dz = \sum_{i=0}^{k-1} \gamma_i x^{k-i}$. Note that in this section we never used the hypothesis that V is a manifold; only paracompactness is needed to define the classifying map of the tautological bundle λ and we also had some dimension restrictions.

7 The model of the blow-up

For all this section we have the following assumptions and notation. Let $f\colon V \hookrightarrow W$ be an embedding of connected closed manifolds of codimension $2k$ with a fixed complex structure on its normal bundle ν. Set $n = \dim(W)$ and $m = \dim(V)$ as before and $r = n - m = 2k$. Assume that $n \geq 2m+3$ and that $H^1(f)$ is injective. Let $\phi\colon R \to Q$ be a CDGA-model of the embedding f such that $R^{\geq n+1} = 0$ and $Q^{\geq m+2} = 0$. Suppose we have fixed some shriek map $\phi^!\colon s^{-2k}Q \to R$ of R–dgmodules. Recall that all this can be done using Proposition 4.5. Set $\gamma_0 = 1$ and for $1 \leq i \leq k-1$ let $\gamma_i \in Q^{2i} \cap \ker d$ be representatives of the Chern classes $c_i(\nu) \in H^{2i}(V;\mathbb{Z})$. We will also use the notation of Diagram (39) of Section 7.1 below.

In this section we pull together what we have done up to now and give our model of the blow-up. In Section 7.1 we recall a definition of the blow-up $\widetilde{W}$ and of the map $\tilde{\pi}\colon \widetilde{W} \to W$ which are suitable for studying homotopy theoretic questions. In Section 7.2 we define a CDGA model $\mathcal{B}(R,Q)$ which depends on the model $\phi\colon R \to Q$ of the embedding $f\colon V \to W$, the shriek map $\phi^!\colon s^{-2k}Q \to R$ and the representatives γ_i of the Chern classes. In Section 7.3 we prove the equivalence of two diagrams of CDGAs (Lemma 7.3). The first is constructed from our models R and Q, ϕ, $\phi^!$ and the γ_i and the second comes from taking A_{PL} of Diagram (39). In Section 7.4 we take pullbacks derived from the equivalent diagrams of Lemma 7.3 and show that they are equivalent to $\mathcal{B}(R,Q)$ (Lemma 7.4) and $A_{\mathrm{PL}}(\widetilde{W})$(Lemma 7.5). Putting these together we prove our main theorem (Theorem 7.6) that $\mathcal{B}(R,Q)$ is a model for $A_{\mathrm{PL}}(\widetilde{W})$. Under a certain nilpotence condition this also implies that the rational homotopy type of the blow-up along $f\colon V \to W$ depends only on the rational homotopy class of the map f and the Chern classes of the normal bundle of V in W (Corollary 7.7).

7.1 The homotopy type of the blow-up

Consider the following cubical diagram (39) with a triangle added to the bottom face. As in Section 5, T is a tubular neighborhood of V in W and $B = \overline{W \setminus T}$. The projective bundle $P\nu$ was defined in Section 6.1 as were the projection maps q, π_0 and π'. The maps i, k, l and π were first seen in Section 4.1. So we have come across all the maps in the diagram except $\widetilde{\pi}$. The space $\widetilde{W}$ is defined as the pushout of the top face.

(39)
$$\begin{tikzcd}
\partial T \arrow[rr, "k"] \arrow[dd, equal] \arrow[dr, "q"] & & B \arrow[dd, equal] \arrow[dr] & \\
& P\nu \arrow[rr] \arrow[dd, "\pi'"] & & \widetilde{W} \arrow[dd, "\widetilde{\pi}"] \\
\partial T \arrow[rr, "k"] \arrow[d, hook, "i"'] \arrow[dr, "\pi_0"] & & B \arrow[dr, "l"] & \\
T \arrow[r, "\pi"'] & V \arrow[rr, "f"] & & W
\end{tikzcd}$$

We know that when we replace $f\pi$ by j the outside bottom quadrilateral is a pushout (see Section 4.1). In fact the bottom face of the cube is also a pushout since π is a deformation retraction. However the map between these pushouts induced by π is a homotopy equivalence and not a homeomorphism.

Definition 7.1 The *blow-up of W along V* is the pushout $\widetilde{W}$ and the map $\widetilde{\pi}\colon \widetilde{W} \to W$ is the map induced by π' between pushouts comprising the top and bottom faces of the above cube.

This definition of blow-up is equivalent to those of Griffiths and Harris [11] and McDuff and Salamon [21, 7.1].

7.2 Description of the model $\mathcal{B}(R, Q)$ for the blow-up

We now construct a CDGA, $\mathcal{B}(R, Q)$, which will be a CDGA model of the blow-up $\widetilde{W}$ of W along V. We also define a morphism $\iota(R, Q)\colon R \to \mathcal{B}(R, Q)$ that will model the projection $\widetilde{\pi}\colon \widetilde{W} \to W$.

Let x and z be generators such that $|x| = 2$ and $|z| = 2k-1$ and denote by $\Lambda^+(x, z)$ the augmentation ideal of the free graded commutative algebra $\Lambda(x, z)$. The CDGA $\mathcal{B}(R, Q)$ is of the form:

$$\mathcal{B}(R, Q) = \big(R \oplus Q \otimes \Lambda^+(x, z), D\big).$$

The graded commutative algebra structure on $\mathcal{B}(R, Q)$ is induced by the multiplications on R and $Q \otimes \Lambda^+(x, z)$, and by the R–module structure on the free Q–module

$Q \otimes \Lambda^+(x, z)$ induced by the algebra map $\phi\colon R \to Q$. More explicitly for $r \in R$, $q \in Q$ and $w \in \Lambda^+(x, z)$,

$$r \cdot (q \otimes w) = (\phi(r) \cdot q) \otimes w,$$
$$(q \otimes w) \cdot r = (-1)^{(|w|+|q|)|r|}(\phi(r) \cdot q) \otimes w.$$

Let d_R and d_Q denote the differentials on R and Q respectively, $r \in R$ and $q \in Q$. The differential D on $\mathcal{B}(R, Q)$ is determined by the Leibnitz law and the formulas

$$\begin{aligned} D(r) &= d_R r \\ D(q \otimes x) &= d_Q q \otimes x \\ D(q \otimes z) &= d_Q q \otimes z + (-1)^{|q|}\left(\phi^!(s^{-2k} q) + \sum_{i=0}^{k-1} (q \cdot \gamma_i) \otimes x^{k-i}\right). \end{aligned}$$

There is an obvious inclusion morphism

$$\iota(R, Q)\colon R \to \mathcal{B}(R, Q).$$

Notice that $\mathcal{B}(R, Q)$ actually depends not only on R and Q but also on the γ_i, ϕ and $\phi^!$. These are implicit and not included in the notation.

Lemma 7.2 *$\mathcal{B}(R, Q)$ is a CDGA and $\iota(R, Q)\colon R \to \mathcal{B}(R, Q)$ is a CDGA morphism.*

Proof For $q, q' \in Q$ we have $(q \otimes z) \cdot (q' \otimes z) = 0$, therefore we need to check that the Leibnitz law applied to $D((q \otimes z) \cdot (q' \otimes z))$ gives zero. This follows because, for degree reasons, $\phi^! \phi = 0$. To check the rest of the definition of CDGA is straightforward. □

7.3 Two equivalent diagrams

Lemma 7.3 *Consider the map e from Lemma 5.5 and the relative Sullivan algebra $(Q \otimes \Lambda(x, z), D)$ from Equation* (35)*. The CDGA diagram*

$$Q \xrightarrow{\text{inc}} Q \otimes \Lambda(x, z) \xrightarrow{e\text{proj}} Q \oplus ss^{-r} Q \xleftarrow{\phi \oplus \text{id}} R \oplus_{\phi^!} ss^{-r} Q$$

is weakly equivalent to the diagram

$$A_{\text{PL}}(V) \xrightarrow{(\pi')^*} A_{\text{PL}}(P\nu) \xrightarrow{q^*} A_{\text{PL}}(\partial T) \xleftarrow{k^*} A_{\text{PL}}(B).$$

Proof We begin by fixing a common model $\hat{\phi}\colon \hat{R} \to \hat{Q}$ as in Section 5.1. All the notation is given in the lemmas we refer to. Lemma 5.8 gives us a commutative diagram of CDGAs

(40)
$$\begin{tikzcd}
Q \oplus ss^{-r} Q \arrow[d, "e^{-1}"'] & R \oplus_{\phi^!} ss^{-r} Q \arrow[l, "\phi\oplus\mathrm{id}"'] \arrow[d, "\xi"] \\
Q \otimes \Lambda(z) & A \arrow[l, "\kappa"'] \\
\widehat{Q} \otimes \Lambda(z) \arrow[u, "\beta\otimes\mathrm{id}"] \arrow[d, "\mu'"'] & \widehat{A} \arrow[l, "\psi"'] \arrow[u, "\zeta"'] \arrow[d, "\zeta'"] \\
A_{\mathrm{PL}}(\partial T) & A_{\mathrm{PL}}(B) \arrow[l, "k^*"']
\end{tikzcd}$$

with all vertical arrows being weak equivalences. Choose representatives $\widehat{\gamma}_i \in \widehat{Q}$ of $c_i(\nu)$ such that $\beta(\widehat{\gamma}_i) = \gamma_i$. This can be done since β is an acyclic fibration. Next Lemma 6.7 and Lemma 6.8 imply that we have a homotopy commutative diagram of CDGA

(41)
$$\begin{tikzcd}
Q \arrow[r, "\mathrm{inc}"] \arrow[d, "="'] & Q \otimes \Lambda(x,z) \arrow[r, "e\mathrm{proj}"] \arrow[d, "="'] & Q \oplus_{\phi\phi^!} ss^{-r} Q \arrow[d, "e^{-1}"'] \\
Q \arrow[r, "\mathrm{inc}"] & Q \otimes \Lambda(x,z) \arrow[r, "\mathrm{proj}"] & Q \otimes \Lambda(z) \\
\widehat{Q} \arrow[r, "\mathrm{inc}"] \arrow[u, "\beta"] \arrow[d, "\sigma^*\beta'"'] & \widehat{Q} \otimes \Lambda(x,z) \arrow[r, "\mathrm{proj}"] \arrow[u, "\beta\otimes\mathrm{id}"] \arrow[d, "\theta'"'] & \widehat{Q} \otimes \Lambda(z) \arrow[u, "\beta\otimes\mathrm{id}"] \arrow[d, "\mu'"'] \\
A_{\mathrm{PL}}(V) \arrow[r, "(\pi')^*"'] & A_{\mathrm{PL}}(P\nu) \arrow[r, "q^*"'] & A_{\mathrm{PL}}(\partial T)
\end{tikzcd}$$

with all vertical arrows being weak equivalences. Note that $Q \oplus_{\phi\phi^!} ss^{-r} Q = Q \oplus ss^{-r} Q$ since $Q^{\geq m+2} = 0$ entails $\phi\phi^! = 0$, so we can glue Diagram (40) to the right of Diagram (41) and apply Lemma 3.9 three times to get the desired result. $\square$

7.4 Two pairs of pullbacks and the proof the main theorem

Lemma 7.4 *In CDGA, the pullback of*

(42)
$$\begin{tikzcd} & Q \arrow[d, "\mathrm{inc}"] \\ R \oplus_{\phi^!} ss^{-r} Q \arrow[r, "\phi\oplus\mathrm{id}"'] & Q \oplus ss^{-r} Q \end{tikzcd}$$

is R *and the pullback of*

(43)
$$\begin{tikzcd} & Q \otimes \Lambda(x,z) \arrow[d, "e\mathrm{proj}"] \\ R \oplus_{\phi^!} ss^{-r} Q \arrow[r, "\phi\oplus\mathrm{id}"'] & Q \oplus ss^{-r} Q \end{tikzcd}$$

is $\mathcal{B}(R,Q)$.

The map between the pullbacks induced by the inclusion $Q \to Q \otimes \Lambda(x,z)$ *is* $\iota(R,Q)\colon R \to \mathcal{B}(R,Q)$. *For both of the diagrams above the map from the homotopy pullback to the pullback is a weak equivalence. Thus the map between the homotopy pullbacks induced by the inclusion* $Q \to Q \otimes \Lambda(x,z)$ *is equivalent to* $\iota(R,Q)\colon R \to \mathcal{B}(R,Q)$.

Proof Consider the maps

$$g\colon \mathcal{B}(R,Q) \to R \oplus_{\phi^!} ss^{-2k} Q$$

determined by the equations

$$\begin{aligned} g(r,0) &= (r,0) \\ g(0, q\otimes z) &= (0, (-1)^{|q|} ss^{-2k} q) \\ g(0, q \otimes x^i \otimes z^{\epsilon}) &= (0,0) \text{ if } i > 0 \end{aligned}$$

and
$$\begin{aligned} g'\colon \mathcal{B}(R,Q) &\to Q \otimes \Lambda(x,z) \\ (r, q\otimes x^i \otimes z^{\epsilon}) &\mapsto \phi(r) + q \otimes x^i \otimes z^{\epsilon}. \end{aligned}$$

Since $(\phi\oplus\mathrm{id})g = e(\mathrm{proj})g'$, g and g' determine a map

$$h\colon \mathcal{B}(R,Q) \to \text{pullback of Diagram (43)}.$$

Because the forgetful functor from CDGA to graded modules commutes with taking pullbacks it is easy to check that h is an isomorphism. Similarly R is the pullback of Diagram (42) and $\iota(R,Q)\colon R\to\mathcal{B}(R,Q)$ is the induced map between the pullbacks.

Next we will show that these pullbacks are indeed homotopy pullbacks. The short exact sequence of differential graded modules

$$0\to R\to Q\oplus(R\oplus_{\phi^!} ss^{-r}Q)\to Q\oplus ss^{-r}Q\to 0$$

gives rise to a Mayer–Vietoris long exact sequence on homology. This maps into the corresponding long exact sequence for the homotopy pullback. Thus the map from R to the homotopy pullback of Diagram (42) is an equivalence by the five lemma. The map from $\mathcal{B}(R,Q)$ into the homotopy pullback of Diagram (43) is similarly a weak equivalence. The fact that the induced map between the homotopy pullbacks is equivalent to $\iota(R,Q)\colon R\to\mathcal{B}(R,Q)$ follows by naturality. □

Lemma 7.5 *Recalling the cube* (39)*, the homotopy pullback of*

(44)
$$\begin{tikzcd} & A_{\mathrm{PL}}(V) \arrow[d, "(\pi i)^*"] \\ A_{\mathrm{PL}}(B) \arrow[r, "k^*"'] & A_{\mathrm{PL}}(\partial T) \end{tikzcd}$$

is quasi-isomorphic to $A_{\mathrm{PL}}(W)$ *and the homotopy pullback of*

(45)
$$\begin{tikzcd} & A_{\mathrm{PL}}(P\nu) \arrow[d, "q^*"] \\ A_{\mathrm{PL}}(B) \arrow[r, "k^*"'] & A_{\mathrm{PL}}(\partial T) \end{tikzcd}$$

is quasi-isomorphic to $A_{\mathrm{PL}}(\widetilde{W})$*. The map between the homotopy pullbacks induced by* $\pi'^*\colon A_{\mathrm{PL}}(V)\to A_{\mathrm{PL}}(P\nu)$ *is weakly equivalent to* $\tilde{\pi}^*\colon A_{\mathrm{PL}}(W)\to A_{\mathrm{PL}}(\widetilde{W})$.

Proof Recall (39) the following cube in which the bottom and the top faces are pushouts :

$$\begin{tikzcd}
\partial T \arrow[rr] \arrow[dr, "q"] \arrow[dd, equal] & & B \arrow[dr] \arrow[dd, equal] & \\
& P\nu \arrow[rr] \arrow[dd, "\pi'"'] & & \widetilde{W} \arrow[dd, "\widetilde{\pi}"] \\
\partial T \arrow[rr] \arrow[dr, "\pi_0"'] & & B \arrow[dr] & \\
& V \arrow[rr] & & W
\end{tikzcd}$$

Let U be the homotopy pullback of (44) and $f\colon A_{\mathrm{PL}}(W) \to U$ be the induced map. We have maps between Mayer–Vietoris sequences

$$\begin{tikzcd}
H^{*-1}(A_{\mathrm{PL}}(\partial T)) \arrow[r] \arrow[d, "="] & H^*(A_{\mathrm{PL}}(W)) \arrow[r] \arrow[d, "H(f)"] & H^*(A_{\mathrm{PL}}(V)) \oplus H^*(A_{\mathrm{PL}}(B)) \arrow[r] \arrow[d, "="] & H^*(A_{\mathrm{PL}}(\partial T)) \arrow[d, "="] \\
H^{*-1}(A_{\mathrm{PL}}(\partial T)) \arrow[r] & H^*(U) \arrow[r] & H^*(A_{\mathrm{PL}}(V)) \oplus H^*(A_{\mathrm{PL}}(B)) \arrow[r] & H^*(A_{\mathrm{PL}}(\partial T)),
\end{tikzcd}$$

hence f is a weak equivalence by the five lemma. Similarly $A_{\mathrm{PL}}(\widetilde{W})$ is weakly equivalent to the homotopy pullback of (45). The fact that the induced map is weakly equivalent to $\widetilde{\pi}^*$ follows by naturality. □

7.5 Proof of main theorem

Here is the main result of the paper.

Theorem 7.6 *Let $f\colon V \hookrightarrow W$ be an embedding of connected closed manifolds of codimension $2k$ with a fixed complex structure on its normal bundle ν. Set $n = \dim(W)$ and $m = \dim(V)$ as before and $r = n - m = 2k$. Assume that $n \geq 2m + 3$ and that $H^1(f)$ is injective. Let $\phi\colon R \to Q$ be a CDGA-model of the embedding f such that $R^{\geq n+1} = 0$ and $Q^{\geq m+2} = 0$. Let $\phi^!\colon s^{-2k}Q \to R$ be a shriek map of R–dgmodules. Set $\gamma_0 = 1$ and for $1 \leq i \leq k-1$ let $\gamma_i \in Q^{2i} \cap \ker d$ be representatives of the Chern classes $c_i(\nu) \in H^{2i}(V;\mathbb{Z})$.*

The CDGA

$$\mathcal{B}(R,Q) = \big(R \oplus Q \otimes \Lambda^+(x,z), D\big)$$

defined in Section 7.2 is a CDGA model of $A_{\mathrm{PL}}(\widetilde{W})$ where $\widetilde{W}$ is the blow-up of W along V. Also $\iota(R,Q)\colon R \hookrightarrow \mathcal{B}(R,Q)$ is a CDGA model of $A_{\mathrm{PL}}(\widetilde{\pi})\colon A_{\mathrm{PL}}(W) \to A_{\mathrm{PL}}(\widetilde{W})$.

Proof The theorem follows directly from Lemmas 3.8, 7.3, 7.4 and 7.5. □

Note that if $n \geq 2m+3$ and $H^1(f)$ is injective, by Proposition 4.5 any model of f can be replaced by one satisfying the hypotheses of the theorem.

Corollary 7.7 *With the hypotheses of Theorem 7.6, if we assume that V, W and the blow-up $\widetilde{W}$ are nilpotent spaces then the rational homotopy type of $\widetilde{W}$ is determined by the rational homotopy class of f and by the rational Chern classes $c_i(\nu) \in H^{2i}(V;\mathbb{Q})$ of the normal bundle.*

Proof This is an immediate application of Sullivan's theory [26] to Theorem 7.6. In particular (see [3, Section 9]) since V and W are nilpotent if $f, g\colon V \to W$ are rationally homotopic then $A_{\mathrm{PL}}(f)$ and $A_{\mathrm{PL}}(g)$ have the same models $R \to Q$. Since the Chern classes are also the same, $\mathcal{B}(R,Q)$ will model $A_{\mathrm{PL}}(\widetilde{W})$ along both embeddings. Since $\widetilde{W}$ is nilpotent $A_{\mathrm{PL}}(\widetilde{W})$ determines the rational homotopy type of $\widetilde{W}$. □

We have a homotopy pushout

$$\begin{array}{ccc} P\mu & \longrightarrow & \widetilde{W} \\ \downarrow & & \downarrow \\ V & \longrightarrow & W \end{array}$$

and so by the Van Kampen theorem $\pi_1(\widetilde{W}) \to \pi_1(W)$ is an isomorphism. So if W is nilpotent then $\pi_1(\widetilde{W}) = \pi_1(W)$ is a nilpotent group but the action of π_1 on the homotopy groups of the universal cover of $\widetilde{W}$ may not be nilpotent. In certain cases Rao [24] has determined when a homotopy pushout is nilpotent, however it seems difficult to see if they apply in our situation. However notice that the nilpotence condition in the corollary is automatically satisfied if V is nilpotent and W is simply connected because then $\widetilde{W}$ is also simply connected. Hence we have proved our first theorem from the introduction:

Theorem 7.8 *Let $f\colon V \to W$ be an embedding of smooth closed orientable manifolds such that W is simply connected and V is nilpotent and oriented. Suppose that the normal bundle ν is equipped with the structure of a complex vector bundle and assume that $\dim W \geq 2\dim V + 3$. Then the rational homotopy type of the blow-up of W along V, $\widetilde{W}$ can be explicitly determined from the rational homotopy type of f and from the Chern classes $c_i(\nu) \in H^{2i}(V;\mathbb{Q})$.*

Even without the nilpotence condition we can still determine a model for $A_{\mathrm{PL}}(\widetilde{W})$.

Corollary 7.9 *With the hypotheses in the first paragraph of Section 7, a CDGA model of $A_{\mathrm{PL}}(\widetilde{W})$ is determined by any model $A_{\mathrm{PL}}(f)$ and by the rational Chern classes $c_i(\nu) \in H^{2i}(V;\mathbb{Q})$ of the normal bundle.*

As we will see in the next section this is enough to determine $H^*(\widetilde{W})$ (Theorem 8.6).

8 Applications

In this section we apply our model of the blow-up to three situations. First, after some preliminaries in Section 8.1 we describe in Section 8.2 the model of the blow-up of $\mathbb{C}P(n)$ along a submanifold. This is somewhat simpler than our general model since here the shriek map is more easily described. Secondly, Section 8.3 examines the model of the example of McDuff of the blow-up of $\mathbb{C}P(n)$ along the Kodaira–Thurston manifold. We recover directly the Babenko–Taimanov result that this $\widetilde{\mathbb{C}P}(n)$ has a nontrivial Massey product and is thus nonformal. Finally in Section 8.4 we describe the cohomology algebra of certain blow-ups. Some other special cases have been studied by Gitler [10]. In Section 8.5 we calculate the rational homotopy type of blow-ups of $\mathbb{C}P(5)$ along $\mathbb{C}P(1)$. It turns out that there are infinitely many rationally inequivalent ones.

8.1 Preliminaries on symplectic manifolds

A *symplectic form* on a $2n$–dimensional manifold M is a nondegenerate closed differential 2–form ω. Thus ω^n is a volume form. The pair (M,ω) is called a *symplectic manifold* (see McDuff and Salamon [21]). The form ω is called *integral* if it belongs to the image $H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R}) \cong H^2(\Omega^*(M))$ where $\Omega^*(M)$ is the de Rham complex of differential forms.

It is well known that to any symplectic form ω we can associate an almost complex structure on the tangent bundle of M. This almost complex structure in turn induces a preferred orientation on M and hence a generator $u_M \in H^{2n}(M,\mathbb{Z})$.

Definition 8.1 Define the real number l_M by the equation

$$[\omega^n] = l_M \cdot u_M \in H^{2n}(M;\mathbb{R}).$$

Since ω^n is a volume form and u_M is induced from the almost complex structure associated to ω we know that l_M is a positive real number. If ω is integral then l_M is a positive integer. We cannot always choose an integral ω so that $l_M = 1$. For example if $[\omega] \in H^2(S^2 \times S^2, \mathbb{Z})$ then $[\omega]^2 \in H^4(S^2 \times S^2, \mathbb{Z})$ is always divisible by 2. The number l_M will appear in the formula when we blow-up of $\mathbb{C}P(n)$ along M.

An example of a symplectic manifold is given by the complex projective space $\mathbb{C}P(n)$ equipped with the 2–form ω_0 associated to the Fubini–Study metric [21, Example 4.21; 11, page 31]. It is classical (see for example Griffiths and Harris [11, pages 30–32 and 144–150]) that

$$[\omega_0] = -c_1(\gamma^1) \in H^2(\mathbb{C}P(n); \mathbb{Z}).$$

In particular ω_0 is integral and $l_{\mathbb{C}P(n)} = 1$.

8.2 Blow-ups of $\mathbb{C}P(n)$

Let (M, ω) be a symplectic manifold. By a *symplectic embedding*

$$f\colon (M, \omega) \to (\mathbb{C}P(n), \omega_0) \tag{46}$$

we mean a smooth embedding such that $f^*(\omega_0) = \omega$. It can be proved that f respects the almost complex structures associated to ω and ω_0, therefore the normal bundle of f admits a natural complex structure and we can consider the blow-up $\widetilde{\mathbb{C}P}(n)$ of $\mathbb{C}P(n)$ along M. Notice also that, since ω_0 is integral, if a symplectic manifold (M, ω) symplectically embeds in $(\mathbb{C}P(n), \omega_0)$ then ω is integral. Moreover the converse is almost true thanks to a theorem of Tischler [29] and Gromov [12]: if M is a manifold equipped with an integral symplectic form ω then, for n large enough, there exists a symplectic embedding as in Equation (46).

The first application of our model is to the blow-up of $\mathbb{C}P(n)$ along a symplectically embedded submanifold. To simplify the description we will use:

Lemma 8.2 *Let Q be a CDGA-model of some closed symplectic manifold of dimension $2m$ with symplectic form $\omega \in Q^2 \cap \ker d$ and suppose that $Q^{\geq 2m+2} = 0$. Consider the $(\Lambda(\omega)/(\omega^{m+1}), 0)$–dgmodule structure on Q induced by multiplication by ω. Then there exists a sub–$(\Lambda(\omega)/(\omega^{m+1}), 0)$–dgmodule $I \subset Q$ such that $Q = I \oplus \Lambda(\omega)/(\omega^{m+1})$ as $(\Lambda(\omega)/(\omega^{m+1}), 0)$–dgmodules. Moreover if $Q^{2m} = \mathbb{Q}\cdot\omega^m$ then this sub–$(\Lambda(\omega)/(\omega^{m+1}), 0)$–dgmodule is unique.*

Proof Take a complementary vector subspace S of $\mathbb{Q}\cdot\omega^m \oplus dQ^{2m-1}$ in Q^{2m} and set $I_{(2m)} = S \oplus Q^{2m+1}$. We know $I_{(2m)}$ is a sub–$(\Lambda(\omega)/(\omega^{m+1}), 0)$–dgmodule of Q since $Q^{\geq 2m+2} = 0$. Suppose that for some $k \leq m$ and for each $j > k$ we have already defined a differential submodule $I_{(2j)} \subset Q$ such that $Q^{\geq 2j} = \mathbb{Q}\{\omega^j, \omega^{j+1}, \dots, \omega^m\} \oplus I_{(2j)}$. We define $I_{(2k)}$ as follows. Consider the morphism

$$\lambda_k\colon Q^{2k} \stackrel{\cdot\omega}{\to} Q^{2k+2} \stackrel{\mathrm{pr}}{\to} Q^{2k+2}/I_{(2k+2)} \cong \mathbb{Q}\cdot\omega^{k+1}.$$

Set $I_{(2k)} = \ker(\lambda_k) \oplus Q^{2k+1} \oplus I_{(2k+2)}$. Since $\alpha \in \ker \lambda_k$ implies that $\alpha \cdot \omega \in I_{(2k+2)}$, it is straightforward to check that $I_{(2k)}$ is a sub–$\big(\Lambda(\omega)/(\omega^{m+1}), 0\big)$–dgmodule of Q such that $Q^{\geq 2j} = \mathbb{Q}\{\omega^j, \omega^{j+1}, \dots, \omega^m\} \oplus I_{(2j)}$. Finally $I = I_{(0)}$ is the desired submodule of Q.

If $Q = I \oplus \Lambda(\omega)/(\omega^{m+1})$ as $(\Lambda(\omega)/(\omega^{m+1}), 0)$–dgmodules then $\ker \lambda_j$ must be included in I. Since $Q^{\geq 2m+2} = 0$, we must have that $I_{(2m)} = S \oplus Q^{2m+2}$ and then we can show by induction that $I^{\geq 2k} = I_{(2k)} = \ker(\lambda_k) \oplus Q^{2k+1} \oplus I_{2k+2}$ for all $k < m$. This implies that I is completely determined by the choice of S. Therefore when $Q^{2m} = \mathbb{Q} \cdot \omega^m$ the ideal I is unique. □

Let (M, ω) be a symplectic manifold of dimension $2m$, $f\colon M \to \mathbb{C}P(n)$ be a symplectic embedding and ν be the normal bundle of the embedding. Let (Q, d) be a CDGA-model of M such that $Q^{\geq 2m+2} = 0$. Let $\omega \in Q^2 \cap \ker d$ be a representative of the symplectic form and $\gamma_i \in Q^{2i} \cap \ker d$ be representatives of the Chern classes $c_i(\nu) \in H^{2i}(M)$ of the normal bundle of M in $\mathbb{C}P(n)$. Then $R = (\Lambda(a)/(a^{n+1}), 0)$ is a CDGA model of $\mathbb{C}P(n)$ with $[a] = [\omega_0] \in H^2(\mathbb{C}P(n))$ and a^n represents the orientation class of $\mathbb{C}P(n)$. A model of the embedding $f\colon M \subset \mathbb{C}P(n)$ is given by the map $\phi\colon R \to Q$ defined by $\phi(a^j) = \omega^j$ which is indeed a CDGA morphism since $Q^{\geq 2m+2} = 0$. This induces an R–dgmodule structure on Q. Let I be a differential submodule of Q complementary to the image of ϕ which exists by Lemma 8.2. Let $2k = 2n - 2m$, the codimension of M inside $\mathbb{C}P(n)$.

Lemma 8.3 *The map $\phi^!\colon s^{-2k} Q \to R$ defined by*

$$\phi^!(\omega^j) = l_M a^{j+k} \quad \text{and} \quad \phi^!(I) = 0$$

is a shriek map, where l_M is given by $\omega^m = l_M u_M$ for the orientation class u_M determined by the almost complex structure. (See Definition 8.1.)

Proof This follows directly from the definition of a shriek map (Definition 4.2). □

From Section 7.2 we get

$$\mathcal{B}(\Lambda(a)/(a^{n+1}), Q) = \left(\frac{\Lambda(a)}{(a^{n+1})} \oplus Q \otimes \Lambda^+(x, z), D \right) \tag{47}$$

a CDGA with $|x| = 2$, $|z| = 2k - 1$. The algebra structure extends the algebra structure on Q and the $\Lambda(a)/(a^{n+1})$–module structure and the differential is determined by the Leibnitz law and the following equations

$$\begin{aligned} D(a) &= 0 \\ D(q\otimes x^j) &= dq\otimes x^j \\ D(q\otimes z) &= dq\otimes z + (-1)^{|q|}\Big(\phi^!(s^{-2k}q)+\sum_{i=0}^{k-1} q\gamma_i\otimes x^{k-i}\Big). \end{aligned}$$

Notice that taking the product of a with an element of Q gives multiplication by ω. Note that for the next theorem our standard dimension hypothesis would be $2n\geq 4m+3$ or $n\geq 2n+3/2$ but since n is an integer this is equivalent to $n\geq 2m+2$.

Theorem 8.4 *Let (M,ω) be a symplectic manifold of dimension $2m$ and suppose we have been given a symplectic embedding $f\colon (M,\omega)\to(\mathbb{C}P(n),\omega_0)$. Let Q be a CDGA model of M such that Q is connected and $Q^{\geq 2m+1}=0$. Let $\omega\in Q^2\cap\ker d$ be a representative of the symplectic form. If $n\geq 2m+2$ then the CDGA $\mathcal{B}(\Lambda(a)/(a^{n+1}),Q)$ of Equation (47) is a model of the blow-up $\widetilde{\mathbb{C}P}(n)$.*

Proof Since $R=\Lambda(a)/(a^{n+1})$ and $Q^{\geq 2n+1}=0$, ϕ defined above is the only homotopy class of a CDGA-morphism such that $\phi(a)=\omega$.

Therefore it is a model of the embedding f. The morphism $\phi^!$ is clearly a shriek map. Therefore all the hypotheses of Theorem 7.6 are fulfilled and the theorem follows. □

8.3 McDuff's example

Next we look at the model of the example of McDuff [20; 21, Exercise 6.55] that we now review. We start with the Kodaira–Thurston manifold (see Thurston [28] or Oprea and Tralle [30, Example II.2.1]) which is a closed symplectic nilmanifold V of dimension 4 and is defined as the product of the circle S^1 with the orbit space $\mathbb{R}^3/\Gamma$ where Γ is the uniform lattice of the integral upper triangular $3\times 3-$matrices in the Heisenberg group. A CDGA-model of that manifold is given by the following exterior algebra on four generators of degree 1 (see Oprea and Tralle [30, Example II.1.7 (2)]):

$$(Q,d)=(\Lambda(u_1,y_1,v_1,t_1),du=dy=dt=0,dv=uy).$$

The symplectic form is represented in this model by $\omega=uv+yt$. By the symplectic embedding theorem of Tischler [29] and Gromov [12, 3.4.2] for $n\geq 5$ there exists a symplectic embedding $f\colon V\hookrightarrow\mathbb{C}P(n)$ such that $f^*(a)=[\omega]$. To fulfill the hypotheses of Theorem 8.4 we will suppose that $n\geq 6$. Then $m=2$ and $k=n-2\geq 4$. In [20, page 271] we see that the Chern classes of V are trivial and [22, Theorem 14.10] the Chern classes of $\mathbb{C}P(n)$ are $c(\mathbb{C}P(n))=\sum_{i=0}^{n} c_i(\mathbb{C}P(n))=(1+a)^{n+1}$ where

$a \in H^2(\mathbb{C}P(n), \mathbb{Z})$ represents the form ω_0 described in Section 8.1. Therefore the Chern classes of the normal bundle ν of the embedding are given by the equation $c(\nu) \cdot c(V) = f^*(c(\mathbb{C}P(n)))$ which yields

$$c(\nu) = 1 + (n+1)\omega + \frac{n(n+1)}{2}\omega^2.$$

The morphism $\phi^!\colon s^{4-2n}Q \to \Lambda(a)/(a^{n+1})$ is characterized by Lemma 8.3 and satisfies, $\phi^!(s^{4-2n}1) = 2a^{n-2}$, $\phi^!(s^{4-2n}uv) = \phi^!(s^{4-2n}yt) = a^{n-1}$, $\phi^!(s^{4-2n}uvyt) = a^n$, and $\phi^!(s^{4-2n}\zeta) = 0$ for any other monomial ζ in $Q = \Lambda(u, y, v, t)$. Using these data in the definition of the CDGA

$$B = \mathcal{B}(\Lambda(a)/(a^{n+1}), Q) = \Big(\Lambda(a)/(a^{n+1}) \oplus \Lambda(u, y, v, t \otimes \Lambda^+(x, z), D\Big)$$

above gives a model of the McDuff example.

From this CDGA-model of the McDuff example we recover the Babenko–Taimanov [1] result.

Theorem 8.5 *There exist nontrivial Massey products in $\widetilde{\mathbb{C}P}(n)$.*

Proof In B we have $D(v\otimes x^2) = (u\otimes x)\cdot(y\otimes x)$ and $(y\otimes x)\cdot(y\otimes x) = 0$, so $D(0) = (y\otimes x)\cdot(y\otimes x)$. Thus $vy\otimes x^3$ is in the triple Massey product $\langle[u\otimes x], [y\otimes x], [y\otimes x]\rangle$. Also $[vy \otimes x^3]$ is not in the ideal of $H^*(\widetilde{\mathbb{C}P}(n))$ generated by $[u \otimes x]$ and $[y \otimes x]$ so $0 \notin \langle[u \otimes x], [y \otimes x], [y \otimes x]\rangle$. □

In this example the Massey product $\langle u, y, y\rangle$ in V became $\langle[u \otimes x], [y \otimes x], [y \otimes x]\rangle$ in $\widetilde{\mathbb{C}P}(n)$. This is the general way in which Massey products propagate to the blow-up. More generally we show in [17] that any obstruction to formality in any manifold propagates to an obstruction in its blow-up in $\mathbb{C}P(n)$.

8.4 The cohomology algebra of a blow-up

In the next theorem $f^!$ is the classical cohomological shriek map (see Section 4.2). Also the algebra structure on $H^*(W) \oplus H^*(V) \otimes \Lambda^+(x)$ is determined by the formula on basic tensors

$$(w, v \otimes x^i) \cdot (w', v' \otimes x^j) = (ww', wv' \otimes x^j + (-1)^{|w'||v|}w'v \otimes x^i + vv' \otimes x^{i+j})$$

where $w, w' \in H^*(W)$ and $v, v' \in H^*(V)$.

Theorem 8.6 *Let $f\colon V \to W$ be an embedding of closed oriented manifolds of codimension $2k$ with a fixed complex structure on the normal bundle ν. Assume that $\dim W \geq 2\dim V + 3$. Let $c_i(\nu) \in H^{2i}(V)$ be the Chern classes. Let I be the ideal in $H^*(W) \oplus H^*(V) \otimes \Lambda^+(x)$ generated by the set*

$$\{f^!(v) + \sum\_i = 0^{k-1} v \cdot c_i(\nu) \otimes x^{k-i} : v \in H^*(V)\}.$$

Then we have an isomorphism of algebras

$$H^*(\widetilde{W}) \cong (H^*(W) \oplus H^*(V) \otimes \Lambda^+(x))/I.$$

Proof Recall the model of $A_{\mathrm{PL}}(\widetilde{W})$ obtained in Theorem 7.6:

$$B = \mathcal{B}(R,Q) = \big(R \oplus Q \otimes \Lambda^+(x,z), D\big)$$

We define an increasing filtration on that CDGA by

$$F^0 B = R \oplus Q \otimes \Lambda^+(x)$$
$$F^p B = B \qquad \text{for } p \geq 1.$$

This filtration is compatible with the CDGA structure. The E_1–term of the associated spectral sequence is

$$E_1^{0,*} = H(R) \oplus H(Q) \otimes \Lambda^+(x)$$
$$E_1^{1,*} = H(Q) \otimes \Lambda(x) \cdot z$$
$$E_1^{p,*} = 0 \text{ for } p \neq 0,1$$

and the d_1–differential is nontrivial only on $E_1^{1,*}$, being there

$$d_1([q] \otimes x^r \otimes z) = (-1)^{|q|}\left(f^!([q]) + \sum_{i=0}^{k-1}[q] \cdot c_i(\nu) \otimes x^{r+k-i}\right)$$

hence $\qquad d_1([q] \otimes x^r \otimes z) = \pm[q] \otimes x^{r+k} + (\text{terms with lower powers of } x).$

Thus d_1 is injective and therefore

$$E_2 = H(E_1, d_1) = \frac{H(R) \oplus H(Q) \otimes (\Lambda^+(x))}{\operatorname{im} d_1}$$

with $$\operatorname{im} d_1 = \left(\{f^!(v) + \sum_{i=0}^{k-1} v \cdot c_i(\nu) \otimes x^{k-i} : v \in H^*(V)\}\right).$$

The E_2–term is concentrated in column $p = 0$, so the spectral sequence collapses at E_2 where $E_2 = E^{0,*}$ as algebras. Therefore there is no possibility for extensions and $H^*(\widetilde{W}) \cong E_\infty = E_2$ as algebras. □

Theorem 8.6 determines the algebra structure of the cohomology of the blow-up under the "stable" condition: $\dim(W) \geq 2\dim(V) + 3$. This result is complementary to a theorem of Gitler [10, Theorem 3.11] that determines the cohomology algebra $H^*(\widetilde{W})$ under the hypothesis that $f^*\colon H^*(W) \to H^*(V)$ is surjective.

8.5 A simple example

As noted in Section 8.1 the Fubini–Study metric $\omega_0 \in H^2(\mathbb{C}P(n))$ is an integral symplectic form. Thus for any $l \in \mathbb{Z}$ with $l > 0$, $l\omega_0 \in H^2(\mathbb{C}P(n))$ is also one. So by [29] there exists a symplectic embedding

$$f_l\colon \mathbb{C}P(1) \to \mathbb{C}P(5)$$

such that $f_l^*(\omega_0) = l\omega_0'$, where ω_0 denotes the Fubini–Study metric in $H^2(\mathbb{C}P(5))$ and ω_0' denotes it in $H^2(\mathbb{C}P(1))$. Identifying a with $\omega_0 \in H^2(\mathbb{C}P(5))$ and a' with $\omega_0 \in H^2(\mathbb{C}P(1))$, we get isomorphisms $H^*(\mathbb{C}P(5)) \cong \Lambda(a)/(a^6)$ and $H^*(\mathbb{C}P(1)) \cong \Lambda(a')/((a')^2)$. Also $f_l^*(a) = la'$ and

$$f_l^!\colon s^{-8}\Lambda(a')/((a')^2) \to \Lambda(a)/(a^6)$$

is given by the formulas

$$\phi^!(s^{-8}1) = la^4$$
$$\phi^!(s^{-8}a') = a^5.$$

Next we calculate $c_1(\nu)$, the first Chern class of the normal bundle. Using the following three formulas

$$f^*(c(\mathbb{C}P(5)) = 1 + 6f^*(a) = 1 + 6la'$$
$$c(\mathbb{C}P(1)) = 1 + 2a'$$

and
$$c(\nu)c(\mathbb{C}P(1)) = f^*(c(\mathbb{C}P(5))) \in H^*(\mathbb{C}P(1)),$$

a simple calculation gives us that

$$c(\nu) = 1 + (6l - 2)a'$$

and hence
$$c_1(\nu) = (6l - 2)a'.$$

Theorem 8.6 tells us that the cohomology of the blow-up $\widetilde{\mathbb{C}P}_l(5)$ of $\mathbb{C}P(5)$ along f_l is

$$(48)\quad H^*(\widetilde{\mathbb{C}P}_l(5)) \cong \big(\Lambda(a)/(a^6) \oplus (\Lambda(a')/((a')^2) \otimes \Lambda^+(x))\big)/\big(la^4 + (6l-2)a'x^3 + x^4\big).$$

Since $f_l(a) = la'$, $a \cdot (1 \otimes x) = la' \otimes x$ so we can write $ax = la'x$ or $(1/l)ax = a'x$. Thus $a^2x = l^2(a')^2x = 0$, and also $la^4 + (6l-2)a'x^3 + x^4 = (1/l)(l^2a^4 + (6l-2)ax^3 + lx^4)$. So it is straightforward to check that we also get an isomorphism

$$(49)\qquad H^*(\widetilde{\mathbb{C}P}_l(5)) \cong \Lambda(a,x)/\big(a^6, a^2x, l^2a^4 + (6l-2)ax^3 + lx^4\big)$$

with the isomorphism in (49) realized by mapping a to a and x to x on the right hand side of (48).

Proposition 8.7 *If $\widetilde{\mathbb{C}P}_l(5)$ and $\widetilde{\mathbb{C}P}_r(5)$ are rationally homotopy equivalent then $l = r$.*

Proof We have just seen that

$$H^*(\widetilde{\mathbb{C}P}_l(5)) \cong \Lambda(a,x)/\big(a^6, a^2x, l^2a^4 + (6l-2)ax^3 + lx^4\big)$$

and

$$H^*(\widetilde{\mathbb{C}P}_r(5)) \cong \Lambda(b,y)/\big(b^6, b^2y, r^2b^4 + (6r-2)by^3 + ry^4\big)$$

Assume that $\widetilde{\mathbb{C}P}_l(5)$ and $\widetilde{\mathbb{C}P}_r(5)$ are rationally homotopy equivalent. So there is an isomorphism

$$g\colon H^*(\widetilde{\mathbb{C}P}_l(5)) \to H^*(\widetilde{\mathbb{C}P}_r(5)).$$

Write

$$g(a) = \alpha_1 b + \beta_1 y$$
$$g(x) = \alpha_2 b + \beta_2 y.$$

We will first show that $\beta_1 = \alpha_2 = 0$. We know that $a^2x = 0$ in $H^*(\widetilde{\mathbb{C}P}_l(5))$, so $g(a^2x) = 0$. This implies that $\alpha_1^2\alpha_2 = 0$, $\beta_1^2\beta_2 = 0$ and $2\alpha_1\beta_1\beta_2 + \alpha_2\beta_1^2 = 0$. Since g is an isomorphism it is straightforward to check that the only possibility is $\beta_1 = 0 = \alpha_2$.

We know that $l^2a^4 + (6l-2)ax^3 + lx^4 = 0 \in H^*(\widetilde{\mathbb{C}P}_l(5))$. So looking at the coefficients of b^4, by^3 and y^4 in $g(l^2a^4 + (6l-2)ax^3 + lx^4) = 0$ we get that for some fixed $\delta \in \mathbb{Q}$,

$$l^2\alpha_1^4 = \delta r^2 \quad \text{so } \delta = \frac{l^2}{r^2}\alpha_1^4 \tag{50}$$

$$(6l-2)\alpha_1\beta_2^3 = \delta(6r-2) \tag{51}$$

$$l\beta_2^4 = \delta r. \tag{52}$$

Observe that none of l, r, δ, β_2 and α_1 can be 0 so we can divide by them at will. Equations (50) and (52) imply that

$$\frac{l}{r} = \left(\frac{\beta_2}{\alpha_1}\right)^4. \tag{53}$$

Subbing (50) into (51), taking fourth powers and gathering α_1 to one side we get

$$(6l-2)^4\left(\frac{\beta_2}{\alpha_1}\right)^{12} = \frac{l^8}{r^8}(6r-2)^4$$

so using (53) we get

$$\frac{(6l-2)^4}{l^5} = \frac{(6r-2)^4}{r^5}. \tag{54}$$

Let $h(x) = (6x-2)^4/x^5$. Calculus tells us that $h(x)$ is decreasing for $x \geq 2$, also $h(1) \neq h(2)$, $h(1) \neq h(3)$ and $h(1) > h(4)$. Together these imply that $h(x)$ takes distinct values on each positive integer. Thus we must have $r = l$ as required. $\square$

Since all of the $\widetilde{\mathbb{C}P}_l(5)$ are not rationally homotopy equivalent they are not integrally homotopy equivalent and hence not diffeomorphic. Using the $H^*(\mathbb{C}P(5);\mathbb{Z})$ module structure on $H^*(\widetilde{\mathbb{C}P}_l(5);\mathbb{Z})$ gives another way to see that all of the $\widetilde{\mathbb{C}P}_l(5)$ are not integrally homotopy equivalent. If we look at blow-ups of $\mathbb{C}P(1)$ in $\mathbb{C}P(4)$ we would have difficulty proving the last theorem because two of the relations in the cohomology algebra would be in the same degree. This leads to some interesting algebraic equations which seem difficult to solve. However it can still be shown using the module structure that the blow-ups have different integral homotopy type. This leads us to the following question which seems more likely to have a positive answer if the codimension is large.

Question Let (M,ω) be an integral symplectic manifold, l a positive integer and $f_l\colon M \to \mathbb{C}P(n)$ be an embedding such that $f_l^*(\omega_0) = l\omega$. Let $\widetilde{\mathbb{C}P}_l(n)$ be the blowup along f_l. Are all of the $\widetilde{\mathbb{C}P}_l(n)$ rationally inequivalent?

References

[1] **I K Babenko**, **I A Taĭmanov**, *On the existence of nonformal simply connected symplectic manifolds*, Uspekhi Mat. Nauk 53 (1998) 225–226 MR1691189

[2] **R Bott**, **L W Tu**, *Differential forms in algebraic topology*, Graduate Texts in Math. 82, Springer, New York (1982) MR658304

[3] **A K Bousfield**, **V K A M Gugenheim**, *On* PL *de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. 8 (1976) ix+94 MR0425956

[4] **G E Bredon**, *Topology and geometry*, Graduate Texts in Math. 139, Springer, New York (1997) MR1700700 Corrected third printing of the 1993 original

[5] **P Deligne**, **P Griffiths**, **J Morgan**, **D Sullivan**, *Real homotopy theory of Kähler manifolds*, Invent. Math. 29 (1975) 245–274 MR0382702

[6] **W G Dwyer**, **J Spaliński**, *Homotopy theories and model categories*, from: "Handbook of algebraic topology", North-Holland, Amsterdam (1995) 73–126 MR1361887

[7] **Y Félix**, **S Halperin**, **J-C Thomas**, *Gorenstein spaces*, Adv. in Math. 71 (1988) 92–112 MR960364

[8] **Y Félix**, **S Halperin**, **J-C Thomas**, *Differential graded algebras in topology*, from: "Handbook of algebraic topology", (I James, editor), North-Holland, Amsterdam (1995) 829–865 MR1361901

[9] **Y Félix**, **S Halperin**, **J-C Thomas**, *Rational homotopy theory*, Graduate Texts in Math. 205, Springer, New York (2001) MR1802847

[10] **S Gitler**, *The cohomology of blow ups*, Bol. Soc. Mat. Mexicana (2) 37 (1992) 167–175 MR1317571Papers in honor of José Adem (Spanish)

[11] **P Griffiths**, **J Harris**, *Principles of algebraic geometry*, Pure and Applied Math., Wiley-Interscience [John Wiley & Sons], New York (1978) MR507725

[12] **M Gromov**, *Partial differential relations*, Ergebnisse der Math. und ihrer Grenzgebiete (3) [Results in Math. and Related Areas (3)] 9, Springer, Berlin (1986) MR864505

[13] **S Halperin**, *Lectures on minimal models*, Mém. Soc. Math. France (N.S.) (1983) 261 MR736299

[14] **P Hilton**, **G Mislin**, **J Roitberg**, *Localization of nilpotent groups and spaces*, North-Holland Math. Studies 15, Notas de Matemática (55), North-Holland Publishing Co., Amsterdam (1975) MR0478146

[15] **P S Hirschhorn**, *Model categories and their localizations*, Math. Surveys and Monographs 99, Amer. Math. Soc. (2003) MR1944041

[16] **M Hovey**, *Model categories*, Math. Surveys and Monographs 63, Amer. Math. Soc. (1999) MR1650134

[17] **P Lambrechts**, **D Stanley**, *Blowing up in $\mathbb{C}P(n)$ preserves non-formality*, in preparation

[18] **P Lambrechts**, **D Stanley**, *Algebraic models of Poincaré embeddings*, Algebr. Geom. Topol. 5 (2005) 135–182 MR2135550

[19] **P Lambrechts**, **D Stanley**, *Examples of rational homotopy types of blow-ups*, Proc. Amer. Math. Soc. 133 (2005) 3713–3719 MR2163611

[20] **D McDuff**, *Examples of simply-connected symplectic non-Kählerian manifolds*, J. Differential Geom. 20 (1984) 267–277 MR772133

[21] **D McDuff**, **D Salamon**, *Introduction to symplectic topology*, second edition, Oxford Math. Monographs, The Clarendon Oxford University Press, New York (1998) MR1698616

[22] **J W Milnor**, **J D Stasheff**, *Characteristic classes*, Annals of Math. Studies 76, Princeton University Press (1974) MR0440554

[23] **D G Quillen**, *Homotopical algebra*, Lecture Notes in Math. 43, Springer, Berlin (1967) MR0223432

[24] **V Rao**, *Nilpotency of homotopy pushouts*, Proc. Amer. Math. Soc. 87 (1983) 335–341 MR681844

[25] **Y Rudyak**, **A Tralle**, *On Thom spaces, Massey products, and nonformal symplectic manifolds*, Internat. Math. Res. Notices (2000) 495–513 MR1759504

[26] **D Sullivan**, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. (1977) 269–331 (1978) MR0646078

[27] **R M Switzer**, *Algebraic topology—homotopy and homology*, Die Grundlehren der math. Wissenschaften 212, Springer, New York (1975) MR0385836

[28] **W P Thurston**, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. 55 (1976) 467–468 MR0402764

[29] **D Tischler**, *Closed 2-forms and an embedding theorem for symplectic manifolds*, J. Differential Geometry 12 (1977) 229–235 MR0488108

[30] **A Tralle**, **J Oprea**, *Symplectic manifolds with no Kähler structure*, Lecture Notes in Math. 1661, Springer, Berlin (1997) MR1465676

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