Geometry & Topology 12 (2008) 2173–2201

Covering link calculus and iterated Bing doubles

JAE CHOON CHA
TAEHEE KIM

We give a new geometric obstruction to the iterated Bing double of a knot being a slice link: for $n > 1$ the $(n+1)$–st iterated Bing double of a knot is rationally slice if and only if the n–th iterated Bing double of the knot is rationally slice. The main technique of the proof is a covering link construction simplifying a given link. We prove certain similar geometric obstructions for $n \le 1$ as well. Our results are sharp enough to conclude, when combined with algebraic invariants, that if the n–th iterated Bing double of a knot is slice for some n, then the knot is algebraically slice. Also our geometric arguments applied to the smooth case show that the Ozsváth–Szabó and Manolescu–Owens invariants give obstructions to iterated Bing doubles being slice. These results generalize recent results of Harvey, Teichner, Cimasoni, Cha and Cha–Livingston–Ruberman. As another application, we give explicit examples of algebraically slice knots with nonslice iterated Bing doubles by considering von Neumann ρ–invariants and rational knot concordance. Refined versions of such examples are given, that take into account the Cochran–Orr–Teichner filtration.

57M25, 57N70

1 Introduction

The *Bing double* $\mathrm{BD}(K)$ of a knot K is defined to be the 2–component link obtained by taking two zero-linking parallel copies of K and introducing positive and negative clasps, as in Figure 1. Taking the Bing double of each component of $\mathrm{BD}(K)$, we obtain the second iterated Bing double $\mathrm{BD}_2(K)$ of K. Iterating this process, we define the *n–th iterated Bing double* $\mathrm{BD}_n(K)$ of K, which is a link with 2^n components. As our convention, for $n = 0$, $\mathrm{BD}_n(K)$ designates K itself.

The problem of deciding whether $\mathrm{BD}_n(K)$ is slice for some $n \ge 1$ has been studied actively, partly motivated by the relationship with the 4–dimensional surgery theory. We recall that a link L in the 3–sphere S^3 is a *slice link* if the components of L bound disjoint locally flat 2–disks in the 4–ball B^4. A fact that makes the problem more interesting is that many previously known obstructions to being a slice link vanish for any (iterated) Bing double. For an excellent discussion on this, the reader is referred to Cimasoni's paper [8].

Published: 8 August 2008 DOI: 10.2140/gt.2008.12.2173

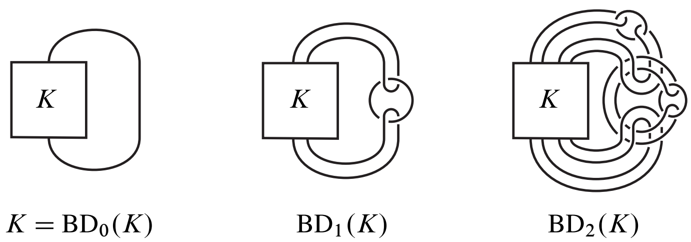

Figure 1: Iterated Bing doubles $\mathrm{BD}_n(K)$ of a knot K

It can be seen easily that if $\mathrm{BD}_n(K)$ is slice, then so is $\mathrm{BD}_{n+1}(K)$. Consequently if K is slice then all iterated Bing doubles of K are slice. The converse is a well known open problem. Recently, there has been significant progress that enables us to extract obstructions for (iterated) Bing doubles to be slice, and consequently partial results on the converse.

A first remarkable result in this direction has been proved by Harvey [20] and Teichner independently (unpublished), using von Neumann ρ–invariants: if $\mathrm{BD}_n(K)$ is slice for some n, then the integral of the Levine–Tristram signature of K over the unit circle is zero. In [8] Cimasoni proved that K is algebraically slice if $\mathrm{BD}(K)$ is "boundary" slice in the sense of Cappell and Shaneson [1], Ko [21], Mio [25] and Duval [18].

As an application of his Hirzebruch-type intersection form defect invariants, the first author found a new technique to detect nonslice iterated Bing doubles which is effective even for knots of finite order in the knot concordance group [3]. Using this, he generalized the result of Harvey and Teichner by proving that for any n the Levine–Tristram signature function of K is determined by (the concordance class of) $\mathrm{BD}_n(K)$, and also found infinitely many amphichiral knots with nonslice iterated Bing doubles [3, Theorems 1.5 and 1.6]. In particular he gave the first proof that any iterated Bing double of the figure eight knot is not slice. Subsequent to this, Livingston, Ruberman and the first author proved that if $\mathrm{BD}(K)$ is slice, then K is algebraically slice [7, Theorem 1]. Recently, Cochran, Harvey and Leidy showed that there are algebraically slice knots with nonslice iterated Bing doubles using higher-order L^2–signatures [10].

In this paper, we extend the aforementioned results on slicing iterated Bing doubles. First we prove a geometric result that the converse of the fact "$\mathrm{BD}_n(K)$ is slice $\Longrightarrow$ $\mathrm{BD}_{n+1}(K)$ is slice" is *rationally* true for higher n:

Theorem 1.1 *For any $n > 1$, $\mathrm{BD}_{n+1}(K)$ is rationally slice if and only if $\mathrm{BD}_n(K)$ is rationally slice.*

Here, as in [6; 5], a link L is said to be a *rationally slice link* if its ambient space is the boundary of some rational homology 4–ball W and there are disjoint locally flat 2–disks in W with boundary L. For a prime p, a $\mathbb{Z}_{(p)}$*–slice link* is defined similarly, namely slicing disks exist in a $\mathbb{Z}_{(p)}$–homology ball instead. (Here $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at the prime p.) A slice link is $\mathbb{Z}_{(p)}$–slice for every prime p. A link is $\mathbb{Z}_{(p)}$–slice for some prime p if and only if it is rationally slice.

In fact, we prove the $\mathbb{Z}_{(p)}$–analogue of Theorem 1.1, from which Theorem 1.1 follows immediately. As the main technique of the proof, we perform certain *iterated covering link calculus* for iterated Bing doubles. Given a link L in a $\mathbb{Z}_{(p)}$–homology sphere, the p^a–fold cyclic cover of the ambient space branched over a component of L becomes another $\mathbb{Z}_{(p)}$–homology sphere and the preimage of L can be regarded as a new link. The idea of taking such a branched cover was first applied to (noniterated) Bing doubles in the work of Cha, Livingston and Ruberman [7]. We perform a more sophisticated covering link calculus, by iterating the process of taking branched coverings and taking sublinks; we call links obtained in this way *p–covering links*. (See Section 2.) The essential part of the proof of Theorem 1.1 is the following: for $n > 1$, $\mathrm{BD}_n(K)$ is a p–covering link of a more complicated link, namely $\mathrm{BD}_{n+1}(K)$. (See Proposition 3.1.)

For the case of $n \le 1$, we do not know whether or not $\mathrm{BD}_n(K)$ is a p–covering link of $\mathrm{BD}_{n+1}(K)$. However, similarly to results for $n = 0$ in [7], our iterated covering link technique can be used to show that certain band sums of (parallel copies of) K and its reverse K^r are $\mathbb{Z}_{(p)}$–slice if $\mathrm{BD}_{n+1}(K)$ is $\mathbb{Z}_{(p)}$–slice for $n \le 1$. (For example, see Proposition 3.3 and its use in Section 4.) The following result is a simple special case:

Proposition 1.2 *If $\mathrm{BD}_n(K)$ is $\mathbb{Z}_{(p)}$–slice for some $n \ge 0$, then $2K\#2K^r$ is $\mathbb{Z}_{(p)}$–slice.*

We remark that our covering link calculus argument works in both topological and smooth cases, so that Theorem 1.1 and Proposition 1.2 hold in the smooth case as well.

Combining our geometric results with previously known facts on algebraic invariants of the $\mathbb{Z}_{(2)}$–concordance group [14; 5], we can deduce the following second main theorem of this paper:

Theorem 1.3 *For any n, if $\mathrm{BD}_n(K)$ is slice, then K is algebraically slice.*

This generalizes the result for $\mathrm{BD}_1(K)$ due to Cha, Livingston and Ruberman [7, Theorem 1] and generalizes the first author's Levine–Tristram signature obstruction for $\mathrm{BD}_n(K)$ to be a slice link [3]. Theorem 1.3 can also be used to show the following result which was first shown in [3]: there exist infinitely many knots K such that K is amphichiral (so that it has order 2 in the knot concordance group) but $\mathrm{BD}_n(K)$ is not slice for any n.

Our geometric results can also be applied to investigate (non)sliceness of iterated Bing doubles of algebraically slice knots. Recently Cochran, Harvey and Leidy [10] showed the existence of algebraically slice knots K with nonslice $\mathrm{BD}_n(K)$. In this paper, using techniques different from the ones in [10], we construct *explicit* examples:

Theorem 1.4 *The knot K illustrated in Figure 2 is algebraically slice but $\mathrm{BD}_n(K)$ is not slice for any n.*

In fact, our method gives infinitely many explicit examples. For example, for any odd prime q, the knot obtained from K in Figure 2 by replacing the ± 3 full twists on the leftmost and rightmost bands with $\pm q$ full twists satisfies the conclusion of Theorem 1.4. (More examples are given in Section 6.1.)

We remark that [10] does not give an explicit single knot with this property; they construct a family of knots such that all but possibly one in the family should have the desired property, but it is unknown which ones have the property. (Subsequent to our work, in [12] they find certain explicit examples using the method of [10].)

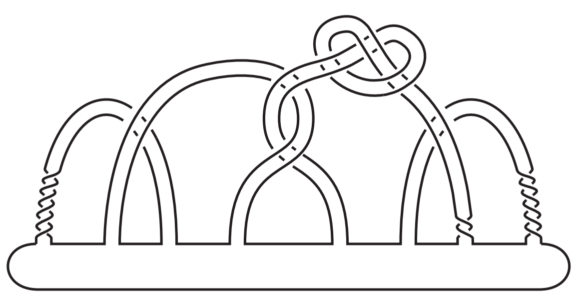

Figure 2: An algebraically slice knot K with $\mathrm{BD}_n(K)$ nonslice

It is known that the subgroup of algebraically slice knots in the knot concordance group has a very rich structure. In [16], Cochran, Orr and Teichner constructed a filtration of the knot concordance group $\mathcal{C}$,

$$0 \subset \cdots \subset \mathcal{F}_{(n.5)} \subset \mathcal{F}_{(n)} \subset \cdots \subset \mathcal{F}_{(1.5)} \subset \mathcal{F}_{(1)} \subset \mathcal{F}_{(0.5)} \subset \mathcal{F}_{(0)} \subset \mathcal{C},$$

where $\mathcal{F}_{(h)}$ is the subgroup of (h)–*solvable* knots. The subgroup of algebraically slice knots is exactly $\mathcal{F}_{(0.5)}$, the subgroup of (0.5)–solvable knots. Regarding this filtration, our covering link method can also be used to produce examples which satisfy the following refined statement:

Theorem 1.5 [10] *For any integer $h \geq 1$, there are (h)–solvable knots K such that for any n, $\mathrm{BD}_n(K)$ is not slice.*

Our examples and proofs are different from those given in [10]. To prove Theorems 1.4 and 1.5, appealing to Proposition 1.2 stated above, it suffices to find an algebraically slice or (h)–solvable knot K for which $2K\#2K^r$ is not rationally slice. For this purpose we use von Neumann ρ–invariants, which were used in [5] to give an obstruction for algebraically slice knots to being rationally slice (and to being linearly independent in the rational knot concordance group). For the highly solvable case of Theorem 1.5, we show that the examples in [13] satisfy our rational nonslice condition of $2K\#2K^r$. For this purpose, in Section 7 we generalize some results on integral knot concordance in [13] to the rational case. Some arguments are essentially the same as the ones in [13] but some results in Section 7 are not immediate consequences of [13]. (Probably Theorem 7.2 and Proposition 7.5 are of independent interest.)

In fact using this approach we show a further generalization of Theorem 1.5: there are highly solvable knots K whose iterated Bing doubles are not only nonslice but also nonsolvable. (For a precise statement, refer to Theorem 6.11.) For this purpose we use a previous result of the first author called Covering Solution Theorem [4, Theorem 3.5], which estimates solvability of covering links.

As well as the above results that hold in both topological and smooth cases, our covering link calculus method also gives results peculiar to the smooth case: using Proposition 1.2, we show that if $\mathrm{BD}_n(K)$ is smoothly slice for some $n \geq 0$, then the Heegaard Floer homology theoretic concordance invariants of Ozsváth and Szabó [26] and Manolescu and Owens [24] of K vanish (see Theorem 5.1). This generalizes the special case of $n = 1$ proved in [7].

The paper is organized as follows. In Section 2 we define p–covering links and show their properties. We prove Theorem 1.1 and Proposition 1.2 in Section 3 and Theorem 1.3 in Section 4. Our results on the Heegaard Floer invariants are proved in Section 5. Theorem 1.4 and their refinements are proved in Section 6, and in Section 7 we investigate rational knot concordance and von Neumann ρ–invariants.

Acknowledgements The authors thank an anonymous referee for helpful comments. This work was supported by the Korea Science and Engineering Foundation(KOSEF) grant funded by the Korea government(MOST) (No. R01–2007–000–11687–0).

2 Covering links

Let p be a prime and Σ a $\mathbb{Z}_{(p)}$–homology 3–sphere. Note that a manifold is a $\mathbb{Z}_{(p)}$–homology sphere if and only if it is a $\mathbb{Z}_p$–homology sphere (and this is equivalent to that it is a $\mathbb{Z}_{p^a}$–homology sphere for all/some a). Given a link L in Σ, we think of the following two operations producing new links from L:

(C1) Taking a sublink of L, a link in the same ambient space Σ is obtained.

(C2) Choose a component K of L and a positive integer a. From the homology long exact sequence for $(\Sigma, \Sigma - K)$ with $\mathbb{Z}_{p^a}$–coefficients and Alexander duality, we have

$$H_1(\Sigma - K; \mathbb{Z}_{p^a}) \cong H_2(\Sigma, \Sigma - K; \mathbb{Z}_{p^a}) \cong H^1(K; \mathbb{Z}_{p^a}) \cong \mathbb{Z}_{p^a}.$$

Therefore there is a canonical map $\phi\colon H_1(\Sigma - K) \to \mathbb{Z}_{p^a}$ sending a meridian to a generator. If the $(\mathbb{Q}/\mathbb{Z})$–valued self-linking of K in Σ is trivial, then there is a "preferred longitude" of K which is mapped to zero under the map ϕ, due to [5]. Therefore in this case the p^a–fold cyclic branched cover, say $\widetilde{\Sigma}$, of Σ branched along K is defined. By results of Casson and Gordon [2] or more generally of Levine [23], $\widetilde{\Sigma}$ is a $\mathbb{Z}_{(p)}$–homology sphere and the preimage of L can be viewed as a new link in $\widetilde{\Sigma}$.

Definition 2.1 A link $\widetilde{L}$ obtained from L by applying (C1) and/or (C2) above repeatedly is called a *p–covering link of L of height $\le h$*, where h is the number of (C2) applied.

We remark that a different exponent a can be used for each (C2) applied. As an abuse of terminology, we will often say that $\widetilde{L}$ in Definition 2.1 is of *height h*, although the precise definition of the height should be the minimal number of (C2) applied.

It can be seen easily that if L is a link in S^3, the $(\mathbb{Q}/\mathbb{Z})$–linking number condition in (C2) above is automatically satisfied. Moreover, if a component K of L in a $\mathbb{Z}_{(p)}$–homology sphere satisfies the condition as in (C2), then the condition also holds for any component of the preimage of L in $\widetilde{\Sigma}$ which projects to K; for, due to [5], K satisfies the $(\mathbb{Q}/\mathbb{Z})$–linking number condition if and only if there is a "generalized Seifert surface" F, namely, an embedded oriented surface F in Σ which is bounded by the union of $c > 0$ parallel copies of K taken along the zero-framing. Considering a component of the preimage of F in $\widetilde{\Sigma}$, the claim easily follows. These observations enable us to iterate (C2) in Definition 2.1 above in many cases.

Using the following well-known fact, we investigate the sliceness of a link via its p–covering links:

Theorem 2.2 *Let p be a prime and L a link in a $\mathbb{Z}_{(p)}$–homology sphere Σ. If L is $\mathbb{Z}_{(p)}$–slice, then any p–covering link of L is $\mathbb{Z}_{(p)}$–slice.*

Proof A sublink L' of L is obviously a $\mathbb{Z}_{(p)}$–slice link. Suppose L bounds slice disks in a $\mathbb{Z}_{(p)}$–homology 4–ball W. Let $\widetilde{L}$ be the preimage of L in $\widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is a $\mathbb{Z}_{(p)}$–homology sphere obtained by taking a p^a–fold cyclic branched cover of Σ branched along a component of L, say K. By taking a p^a–fold cyclic branched cover of W branched along the slice disk for K in W, we obtain a 4–manifold $\widetilde{W}$ such that $\widetilde{\Sigma} = \partial \widetilde{W}$. Due to [2], $\widetilde{W}$ is a $\mathbb{Z}_{(p)}$–homology ball, and the preimages of the slice disks for L are slice disks in $\widetilde{W}$ for $\widetilde{L}$. □

The following construction of covering links will play a crucial role for our purpose.

Lemma 2.3 *Suppose p is a prime and let L_0, L_1, and L_2 be the links in S^3 illustrated in Figures 3, 4 and 5, respectively. Then the following conclusions hold:*

(1) *L_1 is a p–covering link of L_0 of height 1.*

(2) *L_2 is a p–covering link of L_0 of height 2.*

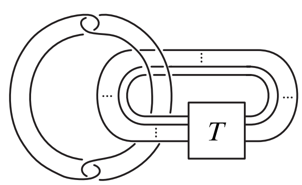

Figure 3: Link L_0

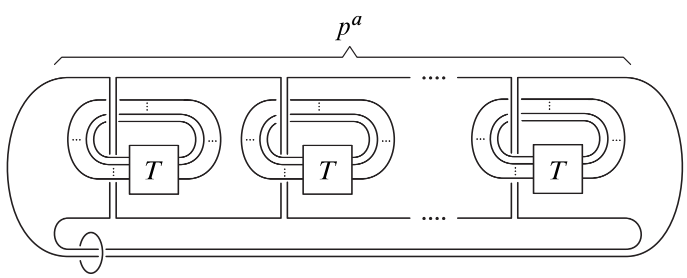

Figure 4: Link L_1

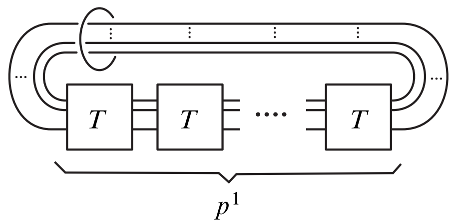

Figure 5: Link L_2

Proof (1) The link L_1 is obtained by taking the p^a–fold cyclic branched cover of S^3 along the leftmost component of L_0.

(2) Forgetting appropriate components of L_1, we obtain the link L_1' in Figure 6. Taking the p^a–fold cyclic branched cover of S^3 branched along the leftmost component of L_1', we obtain L_2. □

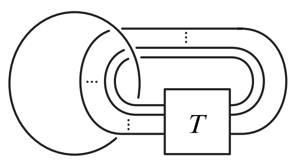

Figure 6: Link L_1'

3 Covering link construction relating iterated Bing doubles

For clarity, we describe how the iterated Bing doubles are constructed and fix notation. In what follows a solid torus is always embedded in S^3, so that its preferred longitude is defined. Let BD be the 2–component link contained in an unknotted solid torus illustrated in Figure 7. For a link L, we define the Bing double $\mathrm{BD}_1(L) = \mathrm{BD}(L)$ to be the link L obtained by replacing a tubular neighborhood of each component with a solid torus containing BD in such a way that a preferred longitude and a meridian of the solid torus for BD are matched up with those of the component of L. The n–th iterated Bing double $\mathrm{BD}_n(L)$ is defined to be $\mathrm{BD}_n(L) = \mathrm{BD}(\mathrm{BD}_{n-1}(L))$. For convenience, we denote $\mathrm{BD}_0(L) = L$.

For a knot K, we can construct $\mathrm{BD}_n(K)$ using the process called *infection*. A precise description is as follows. Fix an unknotted solid torus V in S^3, and let BD_0 be

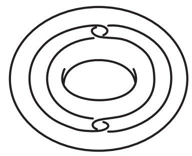

Figure 7: The link BD in a solid torus

the core of V. Let $\mathrm{BD}_n = \mathrm{BD}_n(\mathrm{BD}_0)$ be the 2^n–component link in V. Let α be a meridional curve of V. See Figure 8 for a picture of $\mathrm{BD}_n \cup \alpha$. As a simple closed

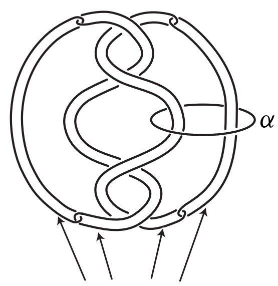

Figure 8: $\mathrm{BD}_n \cup \alpha$

curve in S^3, α is unknotted. We take the union of the exterior of $\alpha \subset S^3$ and that of the given knot $K \subset S^3$, glued along the boundaries such that a longitude and a meridian for α are identified with a meridian and a longitude for K, respectively. Then the resulting manifold is homeomorphic to S^3, and $\mathrm{BD}_n(K)$ is the image of BD_n in this new ambient manifold.

Proposition 3.1 *Let K be a knot in S^3. For any prime p and any $n \geq 3$, $\mathrm{BD}_{n-1}(K)$ is a p–covering link of $\mathrm{BD}_n(K)$ of height 2.*

From Proposition 3.1 and Theorem 2.2, Theorem 1.1 follows immediately.

Proof of Proposition 3.1 We regard $\mathrm{BD}_n \cup \alpha$ as a link in S^3, and will show that $\mathrm{BD}_{n-1} \cup \alpha$ is a p–covering link of $\mathrm{BD}_n \cup \alpha$ by constructing a sequence of (C1) and (C2) operations. In addition, we will observe that these operations behave in such a way that by performing infection along (preimages of) α, it follows that $\mathrm{BD}_{n-1}(K)$ is a p–covering link of $\mathrm{BD}_n(K)$ for any knot K.

We define V_k, $1 \le k \le n$, to be a link in an unknotted solid torus as follows: first, V_n is the core denoted by α, as in the left in Figure 9. For $k \le n$, we inductively define V_{k-1} as in the right in Figure 9. Note that each V_k has a component denoted by α.

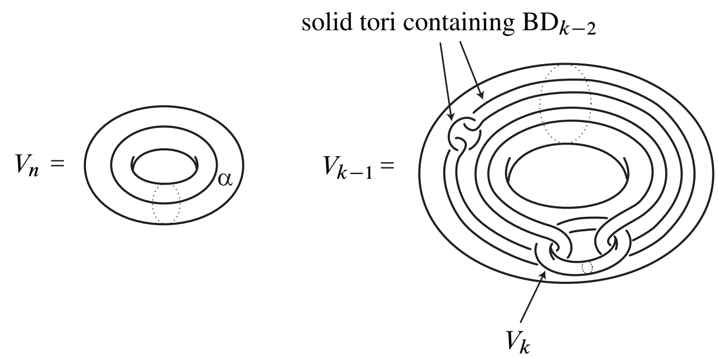

Figure 9: V_k for $1 \le k \le n$

$\mathrm{BD}_n \cup \alpha$ can be illustrated as in the left in Figure 10. (For convenience, the solid torus labeled as V_k represents our link V_k contained in the solid torus.) It can be seen that this is isotopic to the right diagram in Figure 10. Note that we may denote this diagram by $\mathrm{BD}_{n-1} \cup V_{n-1}$, by comparing it with Figure 8.

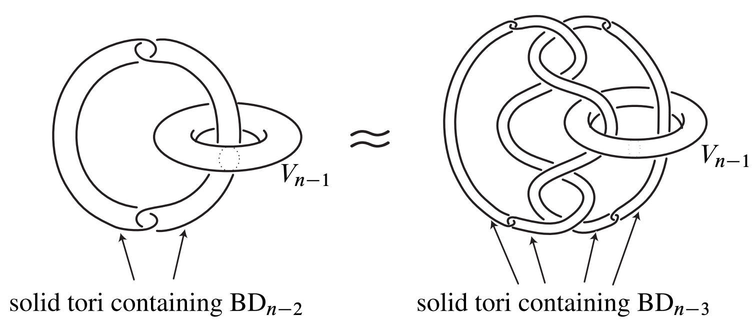

Figure 10: $\mathrm{BD}_n \cup \alpha$ isotoped to $\mathrm{BD}_{n-1} \cup V_{n-1}$

Repeatedly applying this process, we have

$$\mathrm{BD}_n \cup \alpha \approx \mathrm{BD}_n \cup V_n \approx \mathrm{BD}_{n-1} \cup V_{n-1} \approx \cdots \mathrm{BD}_1 \cup V_1$$

where $\mathrm{BD}_1 \cup V_1$ is illustrated in Figure 11.

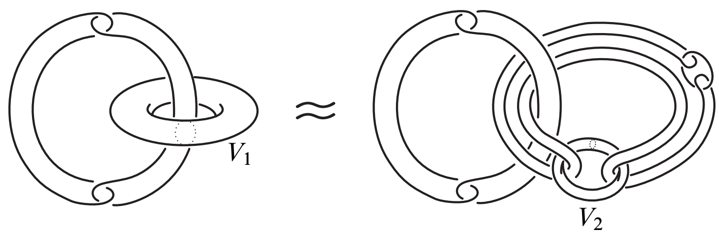

Figure 11: $\mathrm{BD}_1 \cup V_1$

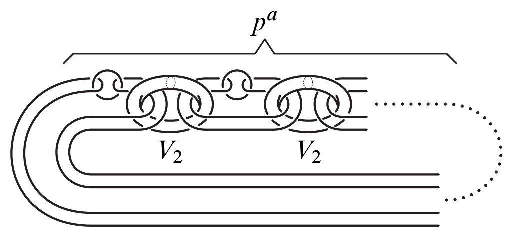

Figure 12: A p–covering link of $\mathrm{BD}_n \cup \alpha$

By Lemma 2.3 (2), the link in Figure 12 is a p–covering link of the link in the right in Figure 11 (of height 2), hence of $\mathrm{BD}_n \cup \alpha$.

Forgetting some components of the link in Figure 12, we obtain the link in Figure 13.

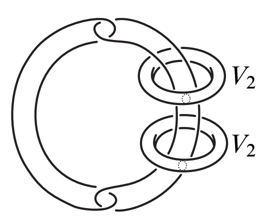

Figure 13: Another p–covering link of $\mathrm{BD}_n \cup \alpha$

Furthermore, since $n \geq 3$, $V_2 \neq V_n = \alpha$. Therefore we can forget all components of the link in (the solid torus for) the second copy of V_2 in Figure 13, in order to obtain $\mathrm{BD}_1 \cup V_2$. In order to be precise, we need to be more careful with the component labeled as α in the second copy of V_2, since it is used as an infection curve. Nonetheless, forgetting all components in the second V_2 but the concerned α, one completely splits the α from the other remaining components, so that infection along α changes nothing. We also note that one could not eliminate the second copy of V_2 in Figure 13 if V_2 were α.

Now we have that $\mathrm{BD}_1 \cup V_2$ as a p–covering link of $\mathrm{BD}_n \cup \alpha$. Performing isotopies which were described above, we obtain

$$\mathrm{BD}_1 \cup V_2 \approx \mathrm{BD}_2 \cup V_3 \approx \cdots \approx \mathrm{BD}_{n-1} \cup V_n = \mathrm{BD}_{n-1} \cup \alpha.$$

It follows that $\mathrm{BD}_{n-1} \cup \alpha$ is a p–covering link of $\mathrm{BD}_n \cup \alpha$. □

For $n = 2$, the proof of Proposition 3.1 shows the following proposition:

Proposition 3.2 *For any prime p, $\mathrm{BD}(K \# K^r)$ is a p–covering link of $\mathrm{BD}_2(K)$ of height 2.*

Proof As in the proof of Proposition 3.1, the link in Figure 13 is a p–covering link of $\mathrm{BD}_2 \cup \alpha$ of height 2. Since $n = 2$, one sees that $V_2 = \alpha$. By carefully following the transform from Figure 12 to Figure 13, one can see that the two copies of $\alpha (= V_2)$ in Figure 13 are with opposite string orientations. Performing infection by K along the two copies of α, the proposition follows. □

By arguments in [7] or by applying Lemma 2.3 (1), it can be seen easily that $K \# K^r$ is a p–covering link of $\mathrm{BD}(K)$. Consequently, by Proposition 3.2, the knot $2K \# 2K^r$ is a p–covering link of $\mathrm{BD}_2(K)$. The following statement is a generalization of this observation, which will be useful in investigating algebraic invariants of iterated Bing doubles in Section 4:

Proposition 3.3 *Let K be a knot in S^3. For every prime p, the link $\widetilde{L}$ in Figure 14 is a p–covering link of $\mathrm{BD}_2(K)$ of height 4.*

Proof As in the proof of Theorem 1.1, we start with the link $\mathrm{BD}_2 \cup \alpha$ which is illustrated in Figure 15.

By Lemma 2.3 (2), the link in the left in Figure 16 is a p–covering link of $\mathrm{BD}_2 \cup \alpha$ of height 2. Forgetting some components, we obtain the link in the right in Figure 16.

Applying Lemma 2.3 (1), it follows that the link in the left in Figure 17 is a p–covering link of $\mathrm{BD}_2 \cup \alpha$ of height 3. Forgetting some components, we obtain the link in the right in Figure 17.

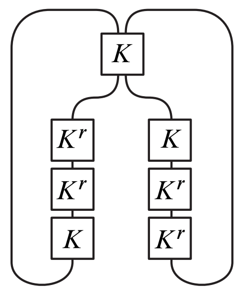

Figure 14: A p–covering link $\widetilde{L}$ of $\mathrm{BD}_2(K)$.

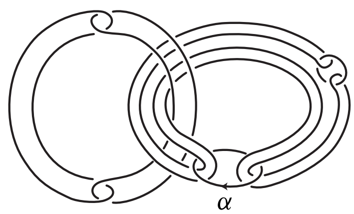

Figure 15: Link $\mathrm{BD}_2 \cup \alpha$

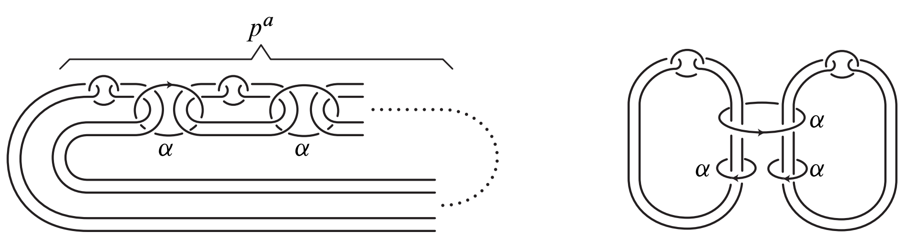

Figure 16: p–covering links of $\mathrm{BD}_2 \cup \alpha$ of height 2

Again applying Lemma 2.3 (1), it follows that the link in the left in Figure 18 is a p–covering link of $\mathrm{BD}_2 \cup \alpha$ of height 4. Forgetting some components, we obtain the link in the right in Figure 18 as a p–covering link of $\mathrm{BD}_2 \cup \alpha$ of height 4.

Finally performing infection by K along α, it follows that the link $\widetilde{L}$ in Figure 14 is a p–covering link of $\mathrm{BD}_2(K)$ of height 4. □

Proof of Proposition 1.2 For $n = 1$, the conclusion is known by arguments in [7]. (Or alternatively, apply Lemma 2.3 (1) and Theorem 2.2.) Suppose $n \geq 2$ and $\mathrm{BD}_n(K)$

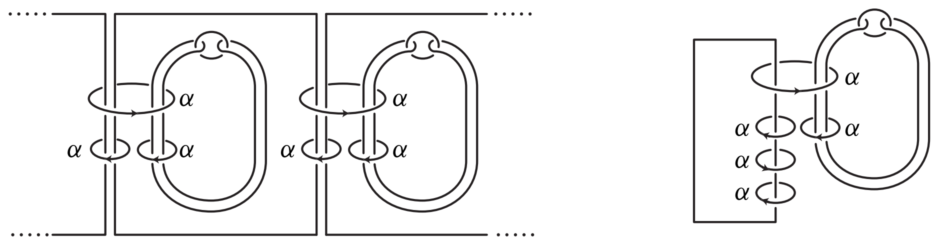

Figure 17: p–covering links of $\mathrm{BD}_2 \cup \alpha$ of height 3

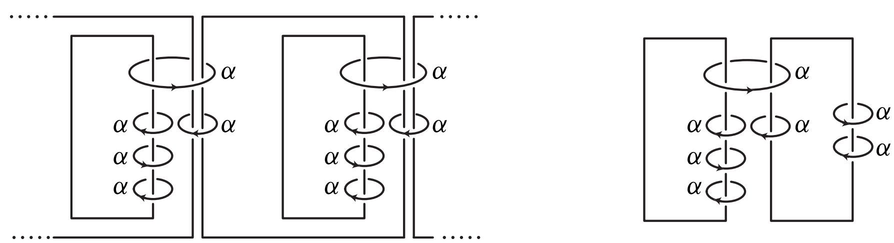

Figure 18: p–covering links of $\mathrm{BD}_2 \cup \alpha$ of height 4

is $\mathbb{Z}_{(p)}$–slice. By Theorem 1.1, we may assume that $n = 2$. Note that $2K \# 2K^r$ is a p–covering link of $\mathrm{BD}_2(K)$ by forgetting one component of the link $\widetilde{L}$ in Proposition 3.3. (Or alternatively, apply the paragraph above Proposition 3.3.) Therefore $2K \# 2K^r$ is $\mathbb{Z}_{(p)}$–slice by Theorem 2.2. □

4 Algebraic invariants and iterated Bing doubles

In this section we apply our geometric method to investigate algebraic invariants of knots with slice iterated Bing doubles. Recall that in Section 3 we showed that $2K \# 2K^r$ is $\mathbb{Z}_{(p)}$–slice if $\mathrm{BD}_n(K)$ is slice for some n (Proposition 1.2). It can be seen that this conclusion is strong enough to detect interesting examples of K with nonslice $\mathrm{BD}_n(K)$ when the Levine–Tristram signature of K is nontrivial, and furthermore when a certain von Neumann ρ–invariant of K is nontrivial. However, it gives no conclusion when K is 2–torsion in the (integral or $\mathbb{Z}_{(p)}$) knot concordance group, in particular when K is amphichiral. The first successful result on the nonsliceness of $\mathrm{BD}_n(K)$ for amphichiral K was obtained in [3] using invariants from iterated p–covers. Our Proposition 3.3 enables us to extract further information when K is amphichiral, via algebraic invariants of K, as shown in the proof of Theorem 1.3 below.

Proof of Theorem 1.3 Suppose $\mathrm{BD}_n(K)$ is slice. Let A be a Seifert matrix of K and $[A]$ be the element in the Levine's algebraic concordance group [22] represented by A. Our goal is to show that $[A] = 0$. For this purpose, we need the following facts on $\mathbb{Z}_{(p)}$–concordance: in [5] the algebraic $\mathbb{Z}_{(p)}$–concordance group and a canonical homomorphism from the algebraic concordance group to the algebraic $\mathbb{Z}_{(p)}$–concordance group are defined. If a knot is $\mathbb{Z}_{(p)}$–slice, then its Seifert matrix represents a trivial element in the algebraic $\mathbb{Z}_{(p)}$–concordance group. For $p = 2$, it is known that the homomorphism of the algebraic concordance group to the algebraic $\mathbb{Z}_{(2)}$–concordance group is injective. (For a detailed discussion on the necessary facts on $\mathbb{Z}_{(p)}$–concordance, see Cha [5].)

The map sending a knot J to its $(c,1)$–cable $i_c(J)$ induces an endomorphism on the algebraic (integral and $\mathbb{Z}_{(p)}$) concordance group, and we denote the image of $[A]$ under this homomorphism by $i_c[A]$, following [5]. Consider the link $\widetilde{L}$ in Figure 14, which is $\mathbb{Z}_{(p)}$–slice by Proposition 3.1 and Proposition 3.3 and Theorem 2.2. Taking one and c parallel copies of the left and right components of L, respectively, and then attaching appropriate bands joining distinct components, we obtain a knot which is $\mathbb{Z}_{(2)}$–slice and has a Seifert matrix identical to that of the following connected sum:

$$J = i_c(K) \# 2i_c(K^r) \# i_{c-1}(K) \# 2K \# K^r.$$

Since K and K^r give the same element in the algebraic concordance group,

$$3i_c[A] + i_{c-1}[A] + 3[A] = 0$$

in the algebraic $\mathbb{Z}_{(2)}$ concordance group and thus in the algebraic concordance group.

For $c = 1$, we have $6[A] = 0$. Since $4[A] = 0$ whenever $[A]$ is torsion [22], it follows that $2[A] = 0$. Therefore we have

$$i_c[A] + i_{c-1}[A] + [A] = 0.$$

By the arguments of [7, Proof of Theorem 1], it follows that $[A] = 0$ in the (integral) algebraic concordance group. □

5 Heegaard Floer homology theoretic concordance invariants and iterated Bing doubles

In this section we consider two concordance invariants obtained from Heegaard Floer homology theory, namely the Ozsváth–Szabó τ–invariant [26] and Manolescu–Owens δ–invariant [24]. Our result can be stated in a general form as follows:

Theorem 5.1 *Suppose ϕ is a torsion-free-abelian-group-valued knot invariant with the following properties:*

(1) *ϕ is an invariant of unoriented knots, ie, $\phi(K) = \phi(K^r)$.*

(2) *ϕ is additive under connected sum, ie, $\phi(K_1 \# K_2) = \phi(K_1) + \phi(K_2)$.*

(3) *ϕ is invariant under (smooth or topological) $\mathbb{Z}_{(p)}$–concordance for some prime p, ie, $\phi(K) = 0$ if K is (smoothly or topologically) $\mathbb{Z}_{(p)}$–slice.*

If $\mathrm{BD}_n(K)$ is (smoothly or topologically) slice for some n, then $\phi(K) = 0$.

Proof It follows immediately from Proposition 1.2. □

As mentioned in [7, Section 4], τ and δ satisfy the above (1), (2), and (3) (for any p and for $p = 2$, respectively) in the smooth case. Therefore, if $\mathrm{BD}_n(K)$ is smoothly slice for some n, then $\tau(K) = 0$ and $\delta(K) = 0$.

6 Von Neumann ρ–invariants and iterated Bing doubles

In this section we construct algebraically slice knots with nonslice iterated Bing doubles. By Proposition 1.2 the knot $2K \# 2K^r$ is $\mathbb{Z}_{(p)}$–slice for any prime p if $\mathrm{BD}_n(K)$ is $\mathbb{Z}_{(p)}$–slice for some n. Therefore for our purpose we will construct algebraically slice knots K such that $2K \# 2K^r$ is not rationally slice.

6.1 Explicit examples

In [5, Section 5], it was shown that there exist concrete and explicit examples of algebraically slice knots K_i, $i \geq 1$, which are linearly independent in the rational knot concordance group. In particular, it was shown that for each i, the knot $2K_i \# 2K_i$ ($= 4K_i$) is not rationally slice. Using the same argument we will show that $2K_i \# 2K_i^r$ is not rationally slice, in order to obtain the following theorem:

Theorem 6.1 *For the algebraically slice knots K_i in [5, Section 5], $\mathrm{BD}_n(K_i)$ is not slice for any $n \geq 1$.*

Proof First we describe the construction of $K = K_i$. We choose a "seed knot" K_0 which is slice and has the rational Alexander module $\mathbb{Q}[t^{\pm 1}]/\langle p(t)^2 \rangle$ where $p(t)$ is a Laurent polynomial such that $p(t^{-1})$ equals $p(t)$ up to multiplication by a unit in $\mathbb{Q}[t^{\pm 1}]$, $p(1) = \pm 1$, and $p(t^c)$ is irreducible for any integer $c > 0$. The existence of such $p(t)$ and K_0 was shown in [5, Section 5]. We choose a simple closed curve η in $S^3 - K_0$ which is unknotted in S^3 and satisfies the following: (1) $\ell k(\eta, K_0) = 0$, (2) the homology class $[\eta]$ in the rational Alexander module for K_0 equals $1 + \langle p(t)^2\rangle \in \mathbb{Q}[t^{\pm 1}]/\langle p(t)^2\rangle$. In particular, $[\eta]$ generates the rational Alexander module for K_0. Let J be a knot such that $\rho(J) \neq 0$ where $\rho(J)$ denotes the integral of the Levine–Tristram signature function of J over the unit circle normalized to length one. For example, one can take J to be the connected sum of copies of the trefoil knot. Then our $K = K_i$ is the knot $K_0(\eta, J)$, which denotes K_0 infected by a knot J along the curve η.

In [5, Proof of Theorem 5.25] it was shown that if $2K \# 2K'(\eta', J')$ is rationally slice for some slice knot K' and a simple closed curve η', then

$$\rho(J) + \epsilon \cdot \rho(J) + \epsilon' \cdot \rho(J') = 0,$$

for some nonnegative integers ϵ and ϵ'. Note that $K^r = K_i^r = K_0^r(\eta, J)$. Therefore from the above equation, if $2K \# 2K^r$ were rationally slice, then

$$\rho(J) + \epsilon'' \cdot \rho(J) = 0,$$

for some nonnegative integer ϵ''. Since $\rho(J) \neq 0$ by our choice of J, one can conclude that $2K \# 2K^r$ is not rationally slice, and the theorem follows. $\square$

Theorem 1.4 stated in the introduction is a special case of our construction in the above proof. In fact, using a trefoil knot as J, a genus 2 slice knot with $p(t) = 3t^2 - 7t + 3$ as K_0, we can obtain the knot illustrated in Figure 2. The required irreducibility condition is satisfied by Lemma 5.20 of [5].

6.2 Examples of higher solvability

In [16], Cochran, Orr and Teichner introduced a filtration on the knot concordance group $\mathcal{C}$

$$0 \subset \cdots \subset \mathcal{F}_{(n.5)} \subset \mathcal{F}_{(n)} \subset \cdots \subset \mathcal{F}_{(1)} \subset \mathcal{F}_{(0.5)} \subset \mathcal{F}_{(0)} \subset \mathcal{C},$$

where $\mathcal{F}_{(h)}$ is the subgroup of (h)*–solvable* knots for each nonnegative half-integer h. The subgroup $\mathcal{F}_{(0.5)}$ is exactly the subgroup of algebraically slice knots [16, Remark 1.3.2]. It is known that the filtration is nonstable. For example, $\mathcal{F}_{(h)}/\mathcal{F}_{(h.5)}$ is nontrivial for any integer $h \geq 0$ [15; 17].

We will show that for each integer $h > 0$, there are (h)–solvable knots K such that $\mathrm{BD}_n(K)$ is nonslice for any n. As we discussed above, by Proposition 1.2, it suffices to find (h)–solvable knots K such that $2K \# 2K^r$ is not rationally slice. We will show that certain examples of knots K considered by Cochran and the second author in [13] have the desired property. In [13], these knots K were shown to have the property that

$4K = 2K\#2K$ is not (integrally) slice, and it can be easily seen that their argument also shows that $2K\#2K^r$ is not (integrally) slice. We will show that it generalizes to that $2K\#2K^r$ is not rationally slice.

Our argument is best described in terms of rational (or $\mathbb{Z}_{(p)}$–) analogues of (h)–solvability and a relative version called "rational (h)–solvequivalence", which are due to Cochran, Orr and Teichner [16], Cha [5; 4], and Cochran and Kim [13]. Although we will not use the definitions excessively, we give precise definitions below for the convenience of the reader. For a group G, the *n–th derived group* $G^{(n)}$ of G is defined inductively as follows: $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$. Let R be a subring of $\mathbb{Q}$. The examples to keep in mind are $R = \mathbb{Z}, \mathbb{Z}_{(p)}$, and $\mathbb{Q}$ where p is a prime.

Definition 6.2 Let n be a nonnegative integer and W a 4–manifold with boundary components $M_1, \dots, M_s$ such that $H_1(M_i; R) \cong R$ for each i. The 4–manifold W is called an *R–coefficient (n)–cylinder* if the following hold:

(1) $H_1(M_i; R) \to H_1(W; R)$ is an isomorphism for each i.

(2) There exist elements
$$u_1, \dots u_m, v_1, \dots, v_m \in H_2\big(W; R[\pi/\pi^{(n)}]\big),$$
where $m = (1/2)\dim_{\mathbb{Q}} \operatorname{Coker}\{H_2(M;\mathbb{Q}) \to H_2(W;\mathbb{Q})\}$ and $\pi = \pi_1(W)$, such that the $R[\pi/\pi^{(n)}]$–valued intersection form $\lambda_W^{(n)}$ on $H_2\big(W; R[\pi/\pi^{(n)}]\big)$ satisfies $\lambda_W^{(n)}(u_i, u_j) = 0$ and $\lambda_W^{(n)}(u_i, v_j) = \delta_{ij}$ (the Kronecker symbol).

In addition, if the following holds then W is called an *R–coefficient $(n.5)$–cylinder*:

(3) There exist $\widetilde{u}_1, \dots, \widetilde{u}_m \in H_2\big(W; R[\pi/\pi^{(n+1)}]\big)$ such that $\lambda_W^{(n+1)}(\widetilde{u}_i, \widetilde{u}_j) = 0$ and u_i is the image of $\widetilde{u}_i$ for each i.

Here the submodules generated by $\{u_i\}$, $\{v_i\}$, and $\{\widetilde{u}_i\}$ are called an *(n)–Lagrangian*, an *(n)–dual*, and an *$(n+1)$–Lagrangian*, respectively.

As a special case, for a nonnegative half-integer h, an R–coefficient (h)–cylinder W with connected boundary M is called an *R–coefficient (h)–solution for M*.

Definition 6.3 Let h be a nonnegative half-integer. Two 3–manifolds M and M' are *R–coefficient (h)–solvequivalent* if there exists an R–coefficient (h)–cylinder W such that $\partial W = M \coprod -M'$. A 3–manifold M is *R–coefficient (h)–solvable* if there is a R–coefficient (h)–solution for M.

Two links in an R–homology 3–sphere are *R–coefficient (h)–solvequivalent* if the zero surgeries on the links are R–coefficient (h)–solvequivalent. A link in an R–homology 3–sphere is *R–coefficient (h)–solvable* if the zero surgery on the link is R–coefficient (h)–solvable.

In the above definitions, when $R = \mathbb{Q}$, we often use "rationally" in place of "$\mathbb{Q}$–coefficient". Note that if a link is (R–coefficient) slice then it is R–coefficient (h)–solvable for any subring R of $\mathbb{Q}$ and for any h.

In this subsection we only need a couple of facts on solvability and solvequivalence (Proposition 6.4 and Lemma 6.5). First, the following is an R–coefficient version of [13, Proposition 2.7]. The proof is identical to the argument in [13], and therefore we omit details.

Proposition 6.4 *For two knots J and K, if $J - K$ is R–coefficient (h)–solvable, then J is R–coefficient (h)–solvequivalent to K. In particular, if $J - K$ is rationally slice, then J is rationally (h)–solvequivalent to K for all h.*

Here $-K$ is the mirror image with reversed orientation (ie, a concordance inverse) and $J - K$ denotes the connected sum of J and $-K$.

Denote the zero surgery manifold of a knot K by $M(K)$. In [13], for any given integer $h > 0$, they constructed an infinite family of certain knots K_i such that for any $i > j$, $K_i - K_j$ is (h)–solvable and $\coprod^k M(K_i)$ is not $(h.5)$–solvequivalent to $\coprod^k M(K_j)$ whenever $k > 0$. The only property of the K_i we need is the following rational analogue:

Lemma 6.5 *For any $i > j$ and $k > 0$, $\coprod^k M(K_i)$ and $\coprod^k M(K_j)$ are not rationally $(h.5)$–solvequivalent.*

The proof of Lemma 6.5 is postponed to Section 7. (A precise description of the K_i is also given in Section 7.)

Proof of Theorem 1.5 We will show that for the knot $K = K_i - K_j$ (with $i > j$), $\mathrm{BD}_n(K)$ is not slice for any n. Since $2K \# 2K^r = 2(K_i \# K_i^r) - 2(K_j \# K_j^r)$, by Propositions 1.2 and 6.4, it suffices to show that $2(K_i \# K_i^r)$ is not rationally $(h.5)$–solvequivalent to $2(K_j \# K_j^r)$.

Suppose that there is a rational $(h.5)$–cylinder, say U, between $M(2(K_i \# K_i^r))$ and $M(2(K_j \# K_j^r))$. Note that for any finite collection $\{J_\ell\}$ of knots, there is a "standard" cobordism between $M(\#^\ell J_\ell)$ and $\coprod^\ell M(J_\ell)$ (eg, see [16, p 113]). Attaching such standard cobordisms to U, we obtain a rational $(h.5)$–cylinder between $\coprod^2 \big(M(K_i) \coprod M(K_i^r)\big)$ and $\coprod^2 \big(M(K_j) \coprod M(K_j^r)\big)$. Since $M(J) = M(J^r)$ for any knot J, we have actually obtained a rational $(h.5)$–cylinder between $\coprod^4 M(K_i)$ and $\coprod^4 M(K_j)$. This contradicts Lemma 6.5. □

6.3 Further refinement

In this subsection, we investigate relationships between $\mathbb{Z}_{(p)}$–coefficient solvability of a link L and that of a p–covering link of L. The first interesting result along this line is the *Covering Solution Theorem* obtained by the first author in [4, Theorem 3.5]. For a space X and a group homomorphism $\pi_1(X) \to \Gamma$, let X_Γ denote the induced Γ–cover of X.

Theorem 6.6 [4, Covering Solution Theorem] *Let p be a prime and $h \geq 1$ be a half-integer. Let M be a closed 3–manifold. Suppose W is a $\mathbb{Z}_{(p)}$–coefficient (h)–solution for M, $\phi\colon \pi_1(M) \to \Gamma$ is a homomorphism onto an abelian p–group Γ, and both $H_1(M)$ and $H_1(M_\Gamma)$ are p–torsion free. Then ϕ extends to $\pi_1(W)$, and W_Γ is an $(h-1)$–solution for M_Γ.*

It immediately follows that (C2) in Definition 2.1 reduces solvability of a link by at most one:

Corollary 6.7 *Let p be a prime and h a half-integer with $h \geq 1$. Suppose L is a $\mathbb{Z}_{(p)}$–coefficient (h)–solvable link in a $\mathbb{Z}_{(p)}$–homology 3–sphere and $\widetilde{L}$ is a p–covering link of L obtained by applying (C2) in Definition 2.1 once. Then $\widetilde{L}$ is $\mathbb{Z}_{(p)}$–coefficient $(h-1)$–solvable.*

On the other hand, the following theorem and its corollary show that (C1) in Definition 2.1 preserves solvability of a link.

Theorem 6.8 *Let M be a closed 3–manifold and h a nonnegative half-integer. Suppose W is an R–coefficient (h)–solution for M. Suppose α is a simple closed curve in M such that the homology class $[\alpha] \in H_1(M;R)$ is of infinite order. Moreover, suppose that for the meridian μ_α for α, the homology class $[\mu_\alpha] = 0$ in $H_1(M-\alpha;R)$. If M' is the manifold obtained by surgery on M along (any framing of) α, then M' is R–coefficient (h)–solvable.*

Proof Suppose h is an integer. Let W' be the manifold obtained from W by attaching a 2–handle along α in M. Then $\partial W' = M'$ and

$$\begin{aligned}
H_1(W';R) &\cong H_1(W;R)/\langle\alpha\rangle \cong H_1(M;R)/\langle\alpha\rangle,\\
H_1(M;R) &\cong H_1(M-\alpha;R)/\langle\mu_\alpha\rangle \cong H_1(M-\alpha;R),\\
H_1(M';R) &\cong H_1(M-\alpha;R)/\langle\lambda_\alpha\rangle \cong H_1(M;R)/\langle\alpha\rangle,
\end{aligned}$$

where λ_α is the longitude for α. Therefore $H_1(M';R) \to H_1(W';R)$ is an isomorphism. By Mayer–Vietoris, we have the exact sequence

$$0 \to H_2(W;R) \to H_2(W';R) \to H_1(S^1;R) \xrightarrow{i_*} H_1(W;R).$$

Since $[\alpha]$ generates $H_1(S^1;R)$ and it is of infinite order in $H_1(M;R) \cong H_1(W;R)$, the map i_* is injective. It follows that $H_2(W;R) \cong H_2(W';R)$. Therefore the images of the (h)–Lagrangian and its (h)–dual for W are an (h)–Lagrangian and its (h)–dual for W'. Hence W' is an R–coefficient (h)–solution for M'. When h is a nonintegral half-integer, the theorem is similarly proved. □

Corollary 6.9 *Suppose L is an R–coefficient (h)–solvable link in an R–homology 3–sphere such that each component of L has vanishing $R/\mathbb{Z}$–valued self linking number and any two distinct components have vanishing R–valued linking number. Then, any sublink of L is R–coefficient (h)–solvable.*

Proof It suffices to prove the theorem for the sublink $L' = L - K$ where K is a component of L. Let M_L and $M_{L'}$ be the zero surgeries on L and L', respectively. Let α be the meridian for K. Then $M_{L'}$ is homeomorphic to the manifold obtained from M_L by surgery along α.

Let μ_α be the meridian for α. From the self-linking number condition, it follows that there is a properly embedded oriented surface F in the exterior of K such that ∂F is c parallel copies of a preferred longitude of K, where c is an integer such that $1/c \in R$, due to [6; 5, Theorem 2.6(2)]. (In [6; 5], such a surface F is called a *generalized Seifert surface for K with complexity c.*) Since the mutual linking number is zero, we may assume that F is disjoint from $L - K$. It follows that $c\mu_\alpha$ is homologous to $c \cdot (\text{preferred longitude for } K)$, which is null-homologous in $M_L - \alpha$. Thus $[\mu_\alpha] = 0$ in $H_1(M_L - \alpha; R)$. Since $H_1(M_L;R)$ is freely generated by meridians for L, $[\alpha]$ is of infinite order in $H_1(M_L;R)$. Applying Theorem 6.8, the corollary follows. □

Corollary 6.10 *Let p be a prime and h a nonnegative half-integer. Let r be a nonnegative integer such that $r \le h$. Suppose L is a $\mathbb{Z}_{(p)}$–coefficient (h)–solvable link in a $\mathbb{Z}_{(p)}$–homology 3–sphere and the linking number conditions in Corollary 6.9 are satisfied (here $R = \mathbb{Z}_{(p)}$). Then any p–covering link of L of height r is $\mathbb{Z}_{(p)}$–coefficient $(h-r)$–solvable.*

Proof It easily follows from Corollaries 6.7 and 6.9. □

Using Corollary 6.10 we can prove the following theorem which strengthens Theorem 1.5. We remark that Cochran, Harvey, and Leidy first proved (a more refined version of) the following theorem in [10] using a different method and examples.

Theorem 6.11 *For any positive integers h and r, there exists an (h)–solvable knot K such that $\mathrm{BD}_r(K)$ is not $(h+2r-0.5)$–solvable.*

Proof Let K be the knot $K_i - K_j$ considered in the proof of Theorem 1.5. Then K is (h)–solvable. Suppose that $\mathrm{BD}_r(K)$ is $(h+2r-0.5)$–solvable. Then it is $\mathbb{Z}_{(p)}$–coefficient $(h+2r-0.5)$–solvable. By Proposition 3.1, $\mathrm{BD}_2(K)$ is a p–covering link of $\mathrm{BD}_r(K)$ of height $2r-4$. Then by Proposition 3.2 $\mathrm{BD}(K\# K^r)$ is a p–covering link of $\mathrm{BD}_r(K)$ of height $2r-2$. Recall that Lemma 2.3 (1) can be used to show that $J\# J^r$ is a p–covering link of $\mathrm{BD}(J)$ of height 1. Therefore it follows that $2K\# 2K^r$ is a p–covering link of $\mathrm{BD}_r(K)$ of height $2r-1$. By Corollary 6.10 it follows that $2K\# 2K^r$ is $\mathbb{Z}_{(p)}$–coefficient $(h.5)$–solvable. Since $\mathbb{Q}$ is flat over $\mathbb{Z}_{(p)}$, it follows that $2K\# 2K^r$ is rationally $(h.5)$–solvable. But then as was shown in the proof of Theorem 1.5, it leads us to a contradiction. □

7 Rational concordance and von Neumann ρ–invariants

The purpose of this section is twofold: we extend results on *integral* concordance and solvability obtained by using the von Neumann ρ–invariants in [17; 13] to the *rational* case, and give a proof of Lemma 6.5 which was needed in the previous section. If the reader is more interested in the latter, we would recommend to read the last subsection first, assuming Theorem 7.6.

Essentially we follow the strategy of [13], focusing on what differs from the integral case. Details will be omitted when arguments are almost identical to those of the integral case.

7.1 Homology of rational cylinders with PTFA coefficients

To investigate rational (n)–cylinders more systematically, we need the following notion of multiplicity given in [13, Definition 2.1]. (It is often called the "complexity"; eg, see Cochran and Orr [14] or Cha [6; 5].)

Definition 7.1 Let h be a nonnegative half-integer. A boundary component M of a rational (h)–cylinder W with $H_1(W;\mathbb{Q})\cong\mathbb{Q}$ is said to be *of multiplicity m* if a generator in $H_1(M)/\text{torsion}\cong\mathbb{Z}$ is sent to $m\in H_1(W)/\text{torsion}\cong\mathbb{Z}$.

We consider homology modules of 3–manifolds and rational cylinders with coefficients in a certain Laurent polynomial ring $\mathbb{K}[t^{\pm 1}]$ over a skew field $\mathbb{K}$, following the idea of [15] and subsequent works. Details are as follows. Let Γ be a poly-torsion-free-abelian (henceforth PTFA) group such that $\Gamma/\Gamma_r^{(1)}\cong\mathbb{Z}$, where $\Gamma_r^{(n)}$ denotes the n–th rational derived group of Γ. ($\Gamma_r^{(0)} = \Gamma$ and $\Gamma_r^{(n)}$ is inductively defined to be the minimal normal subgroup of $\Gamma_r^{(n-1)}$ such that $\Gamma_r^{(n-1)}/\Gamma_r^{(n)}$ is abelian and torsion free; for more details, see [19, Section 3].) Let t be the generator of $\mathbb{Z}$. Then $\Gamma \cong \Gamma_r^{(1)} \rtimes \langle t\rangle$. Let $\mathcal{K}_\Gamma$ be the (skew) quotient field of $\mathbb{Z}\Gamma$. The subgroup $\Gamma_r^{(1)}$ is also PTFA, and hence $\mathbb{Z}\Gamma_r^{(1)}$ embeds in its (skew) quotient field, say $\mathbb{K}$. Therefore $\mathbb{Z}\Gamma = \mathbb{Z}[\Gamma_r^{(1)} \rtimes \langle t\rangle]$ embeds in $\mathbb{Q}\Gamma(\mathbb{Q}\Gamma_r^{(1)} - \{0\})^{-1}$, which is a Laurent polynomial ring $\mathbb{K}[t^{\pm 1}]$. Note $\mathbb{Z}\Gamma \subset \mathbb{K}[t^{\pm 1}] \subset \mathcal{K}_\Gamma$ and $\mathbb{K}[t^{\pm 1}]$ is a PID.

Suppose K is a knot with zero surgery M and W is a rational (n)–cylinder which has M as a boundary component of multiplicity c. Suppose ψ is a homomorphism of $\pi_1(W)$ into our Γ described above, which induces an isomorphism

$$\pi_1(W)/\pi_1(W)_r^{(1)} \to \Gamma/\,\Gamma_r^{(1)} = \langle t\rangle.$$

Then $H_*(W;\mathbb{K}[t^{\pm 1}])$ is defined. On the other hand, the composition

$$\pi_1(M) \to \pi_1(W) \xrightarrow{\psi} \Gamma = \Gamma_r^{(1)} \rtimes \langle t\rangle$$

factors through $\Gamma_r^{(1)} \rtimes \langle s\rangle$, where $s = t^c$. As we did for $\Gamma_r^{(1)} \rtimes \langle t\rangle$, $\mathbb{Z}[\Gamma_r^{(1)} \rtimes \langle s\rangle]$ embeds into $\mathbb{K}[s^{\pm 1}]$, and the homology module $H_*(M;\mathbb{K}[s^{\pm 1}])$ is defined. Viewing $\mathbb{K}[s^{\pm 1}]$ as a subring of $\mathbb{K}[t^{\pm 1}]$, there is a natural map

$$j_*\colon H_*(M;\mathbb{K}[s^{\pm 1}]) \to H_*(M;\mathbb{K}[t^{\pm 1}]).$$

The following is a rational cylinder analogue of [13, Theorem 3.8].

Theorem 7.2 *Suppose K, M, W, and Γ are as above, and Γ is $(n-1)$–solvable. Let d denote the degree of the Alexander polynomial of K. Then for the inclusion $i\colon M \hookrightarrow W$ we have*

$$\operatorname{rank}_{\mathbb{K}} \operatorname{Im}\left\{i_*\colon H_1(M;\mathbb{K}[t^{\pm 1}]) \to H_1(W;\mathbb{K}[t^{\pm 1}])\right\} \geq \begin{cases} |c|(d-2)/2 & \text{if } n>1, \\ |c|d/2 & \text{if } n=1. \end{cases}$$

Proof Let $P = \operatorname{Ker}(i_*)$ and $Q = \operatorname{Im}(i_*)$. Let $\mathcal{A}' = H_1(M;\mathbb{K}[t^{\pm 1}])$ and $\mathcal{A} = H_1(M;\mathbb{K}[s^{\pm 1}])$. It is known that the Blanchfield linking form

$$\mathcal{A}' \to \operatorname{Hom}_{\mathbb{K}[t^{\pm 1}]}(\mathcal{A}', \mathcal{K}_\Gamma/\mathbb{K}[t^{\pm 1}])$$

is nonsingular, and with respect to the Blanchfield linking form, $P \subset P^\perp$ [15, Theorem 2.13] [13, Proposition 3.6]. Using these, one can show that

$$\operatorname{rank}_{\mathbb{K}} Q \geq \frac{1}{2} \operatorname{rank}_{\mathbb{K}} \mathcal{A}',$$

as done in the proof of [13, Theorem 3.8].

Since $\mathbb{K}[s^{\pm 1}]$ is a (noncommutative) PID, we have a $\mathbb{K}[s^{\pm 1}]$–module isomorphism

$$\mathcal{A}' \cong \mathcal{A} \underset{\mathbb{K}[s^{\pm 1}]}{\otimes} \mathbb{K}[t^{\pm 1}] \cong \bigoplus^{|c|} \mathcal{A}$$

as in [5, Theorem 5.16(1)]. Thus $\operatorname{rank}_{\mathbb{K}} \mathcal{A}' = |c| \cdot \operatorname{rank}_{\mathbb{K}} \mathcal{A}$, and it suffices to show that

$$\operatorname{rank}_{\mathbb{K}} \mathcal{A} \geq \begin{cases} d-2 & \text{if } n>1, \\ d & \text{if } n=1. \end{cases}$$

Suppose $n > 1$ and let $X = S^3 - K$. Since $\pi_1(X) \to \langle s \rangle$ is surjective, we have $H_1(X;\mathbb{K}[s^{\pm 1}]) \cong H_1(X_\infty;\mathbb{K})$, where X_∞ denotes the connected infinite cyclic cover of X. Therefore $\operatorname{rank}_{\mathbb{K}} H_1(X;\mathbb{K}[s^{\pm 1}]) \geq d-1$ by [9, Corollary 4.7]. Since the longitude for K in $H_1(X;\mathbb{K}[s^{\pm 1}])$ is annihilated by $s-1 \in \mathbb{K}[s^{\pm 1}]$ and generates a $\mathbb{K}[s^{\pm 1}]$–submodule which is isomorphic to $\mathbb{K}$, we have $\operatorname{rank}_{\mathbb{K}} \mathcal{A} \geq d-2$.

If $n = 1$, Γ is abelian and torsion free, and hence $\Gamma \cong \mathbb{Z}$. Therefore $\mathbb{K} = \mathbb{Q}$. Since $\pi_1(M)$ surjects to $\langle s \rangle$, $H_1(M;\mathbb{Q}[s^{\pm 1}])$ is the rational Alexander module. It follows that $\operatorname{rank}_{\mathbb{K}} \mathcal{A} = d$. □

Corollary 7.3 *Suppose that K, M, W, and Γ are as in Theorem 7.2. Let*

$$j_*\colon H_1(M;\mathbb{K}[s^{\pm 1}]) \to H_1(M;\mathbb{K}[t^{\pm 1}])$$

be the map induced by the inclusion $j\colon \mathbb{K}[s^{\pm 1}] \to \mathbb{K}[t^{\pm 1}]$. *Then we have*

$$\operatorname{rank}_{\mathbb{K}} \operatorname{Im}\left\{i_* j_*\colon H_1(M;\mathbb{K}[s^{\pm 1}]) \to H_1(W;\mathbb{K}[t^{\pm 1}])\right\} \geq \begin{cases} (d-2)/2 & \text{if } n > 1, \\ d/2 & \text{if } n = 1. \end{cases}$$

Proof As in the proof of Theorem 7.2,

$$H_1(M;\mathbb{K}[t^{\pm 1}]) \cong \bigoplus^{|c|} H_1(M;\mathbb{K}[s^{\pm 1}])$$

as $\mathbb{K}[s^{\pm 1}]$–modules. The images of the $|c|$ copies of $H_1(M;\mathbb{K}[s^{\pm 1}])$ under i_* have the same $\mathbb{K}$–rank since multiplication by t^m $(m \in \mathbb{Z})$ in $H_1(W;\mathbb{K}[t^{\pm 1}])$ is an automorphism of $H_1(W;\mathbb{K}[t^{\pm 1}])$ permuting the images of those copies of $H_1(M;\mathbb{K}[s^{\pm 1}])$. Now the conclusion follows from Theorem 7.2. □

7.2 Rational cylinders and algebraic solutions

In [17], the notion of an algebraic (n)–solution was first introduced in order to investigate the behavior of $\pi_1(M) \to \pi_1(W) \to \pi_1(W)/\pi_1(W)^{(n)}_r$ for a (integral) solution W of M. In [13], Cochran and the second author extended it to (integral) cylinders. For the convenience of the reader, the definition of an algebraic (n)–solution [13] is given below: for a group G, let $G_k = G/G^{(k)}_r$. Then G_k is a $(k-1)$–solvable PTFA group, hence $\mathbb{Z}G_k$ embeds in its (skew) quotient field denoted by $\mathbb{K}(G_k)$.

Definition 7.4 Let S be a group such that $H_1(S;\mathbb{Q}) \neq 0$. Let F be a free group and $i\colon F \to S$ a homomorphism. A homomorphism $r\colon S \to G$ is called an *algebraic* (n)*–solution* $(n \geq 0)$ *for* $i\colon F \to S$ if the following hold:

(1) For each $0 \leq k \leq n-1$, the image of the following composition, after tensoring with $\mathbb{K}(G_k)$, is nontrivial:

$$H_1(S;\mathbb{Z}G_k) \xrightarrow{r_*} H_1(G;\mathbb{Z}G_k) \cong G^{(k)}_r/[G^{(k)}_r, G^{(k)}_r] \to G^{(k)}_r/G^{(k+1)}_r.$$

(2) For each $0 \leq k \leq n$, the map $H_1(F;\mathbb{Z}G_k) \xrightarrow{i_*} H_1(S;\mathbb{Z}G_k)$, after tensoring with $\mathbb{K}(G_k)$, is surjective.

The following is a generalization of [13, Proposition 6.3] to the case of *rational* (n)–cylinders:

Proposition 7.5 *Suppose $n > 0$ is an integer, K is a knot with zero surgery M, and the Alexander polynomial of K has degree $d > 2$. (If $n = 1$, $d = 2$ is also allowed.) Suppose that W is a* rational *(n)–cylinder with M as one of its boundary* *components (of any multiplicity). Let Σ be a capped-off Seifert surface for K. Suppose $F \to \pi_1(M - \Sigma)$ is a homomorphism of a free group F inducing an isomorphism on $H_1(-;\mathbb{Q})$. Let $S = \pi_1(M)^{(1)}$, $G = \pi_1(W)_r^{(1)}$, and i be the composition $F \to \pi_1(M - \Sigma) \to S$. Then the map $j\colon S \to G$ induced by inclusion is an algebraic (n)–solution for $i\colon F \to S$.*

Proof We follow the lines in the proof of [13, Proposition 6.3]. Let $\mathbb{K} = \mathbb{K}(G_k)$ be the (skew) quotient field of $\mathbb{Z}G_k$. First, we will prove that Definition 7.4 (1) holds. The map $G_r^{(k)}/[G_r^{(k)}, G_r^{(k)}] \to G_r^{(k)}/G_r^{(k+1)}$ becomes an isomorphism after tensoring with $\mathbb{K}$, since its kernel is $\mathbb{Z}$–torsion. Since $\mathbb{K}$ is flat over $\mathbb{Z}G_k$, it suffices to show that $j_*\colon H_1(S;\mathbb{K}) \to H_1(G;\mathbb{K})$ is nontrivial.

Let c denote the multiplicity of M for W. Let $\Gamma = \pi_1(W)/\pi_1(W)_r^{(k+1)}$. Then as in the previous subsection, $\Gamma \cong \Gamma/\Gamma_r^{(1)} \rtimes \langle t\rangle$, the composition $\pi_1(M) \to \pi_1(W) \to \Gamma$ factors through $\Gamma/\Gamma_r^{(1)} \rtimes \langle s\rangle$ where $s = t^c$, and $H_*(M;\mathbb{K}[s^{\pm 1}])$ and $H_*(W;\mathbb{K}[t^{\pm 1}])$ are defined. Since $\pi_1(M)/\pi_1(M)^{(1)} = \langle s\rangle$ and $S = \pi_1(M)^{(1)}$, we have $H_1(S;\mathbb{K}) \cong H_1(M;\mathbb{K}[s^{\pm 1}])$. Similarly, $H_1(G;\mathbb{K}) \cong H_1(W;\mathbb{K}[t^{\pm 1}])$. Therefore j_* is identical to $H_1(M;\mathbb{K}[s^{\pm 1}]) \to H_1(W;\mathbb{K}[t^{\pm 1}])$. By Corollary 7.3, it is nontrivial.

One can prove that Definition 7.4 (2) holds using the argument of the proof in [13, Proposition 6.3]; one only needs to replace $\mathbb{K}[t^{\pm 1}]$ by $\mathbb{K}[s^{\pm 1}]$. □

We have the following theorem which generalizes [13, Theorem 5.13] to the case of *rational* (n)–cylinders.

Theorem 7.6 *Suppose n, K, and M are as in Proposition 7.5. For any given Seifert surface for K, there exists an oriented trivial link $\{\eta_1, \eta_2, \dots, \eta_m\}$ in S^3 which is disjoint from the Seifert surface and satisfies the following:*

(1) *$\eta_i \in \pi_1(M)^{(1)}$ for all i. Furthermore, the η_i bound (smoothly embedded) symmetric capped gropes of height n, disjointly embedded in $S^3 - K$ (except for the caps, which may intersect K).*

(2) *For every* rational *(n)–cylinder W with M as one of its boundary components (of any multiplicity), there is some η_i such that $j_*(\eta_i) \notin \pi_1(W)_r^{(n+1)}$ where $j_*\colon \pi_1(M) \to \pi_1(W)$ is induced by the inclusion. The number of such η_i is at least $(d-2)/2$ if $n > 1$ or at least $d/2$ if $n = 1$ where d denotes the degree of the Alexander polynomial for K.*

Proof One can proceed exactly in the same way as the proof of the integral version [13, Theorem 5.13], except that one should use $\pi_1(W)_r^{(1)}$, $H_1(M;\mathbb{K}(G_{n-1})[s^{\pm 1}])$ and Corollary 7.3 and Proposition 7.5, instead of $\pi_1(W)^{(1)}$, $H_1(M;\mathbb{K}(G_{n-1})[t^{\pm 1}])$, and the integral analogues used in [13]. (Here t, s are as in the proof of Proposition 7.5.) □

We remark that the η_i in Theorem 7.6 are the same as those used in [13, Theorem 5.13].

7.3 Rational knot concordance and the Cochran–Orr–Teichner filtration

For a given positive integer n, we consider a family of knots K_i which was given in [13, Theorem 5.1]. For the convenience of the reader, we briefly describe how the K_i are constructed. Choose K and $\{\eta_1, \dots, \eta_m\}$ satisfying (the conclusion of) Theorem 7.6. Then we let $K_0 = K$ and for $i \geq 1$, $K_i = K(\eta_1, \dots, \eta_m, J_1^i, \dots, J_m^i)$, the knot obtained from K by infection along the η_ℓ, where the infection knots J_ℓ^i are chosen so that (the integrals of) the Levine–Tristram signatures of the J_ℓ^i satisfy certain inequalities described in [13, p 1429, Proof of Theorem 5.1].

For the K_i, we prove Lemma 6.5 used in the previous section: $\coprod^k M(K_i)$ and $\coprod^k M(K_j)$ are not *rationally* $(n.5)$–solvequivalent.

Proof of Lemma 6.5 We follow the arguments of the proof of [13, Theorem 5.1(5)], which shows that $\coprod^k M(K_i)$ and $\coprod^k M(K_j)$ are not *integrally* $(n.5)$–solvequivalent. All the arguments of [13] proving their integral statement work verbatim in our case except that Theorem 7.6 should be applied instead of [13, Theorem 5.13], in order to guarantee that whenever W is a *rational* (n)–cylinder with $M(K)$ as one of its boundary components, $j_*(\eta_\ell) \notin \pi_1(W)_r^{(n+1)}$ for some η_ℓ. $\square$

Consider the Cochran–Orr–Teichner filtration

$$0 \subset \cdots \subset \mathcal{F}^{\mathbb{Q}}_{(n.5)} \subset \mathcal{F}^{\mathbb{Q}}_{(n)} \subset \cdots \subset \mathcal{F}^{\mathbb{Q}}_{(1)} \subset \mathcal{F}^{\mathbb{Q}}_{(0.5)} \subset \mathcal{F}^{\mathbb{Q}}_{(0)} \subset \mathcal{C}^{\mathbb{Q}}$$

of the rational knot concordance group $\mathcal{C}^{\mathbb{Q}}$ [5], where $\mathcal{F}^{\mathbb{Q}}_{(h)}$ is the subgroup of rationally (h)–solvable knots.

Theorem 7.7 *For the K_i, the following hold:*

1. *If $i \neq j$, K_i is not* rationally *$(n.5)$–solvequivalent to K_j. In particular, $K_i - K_j$ is not* rationally *$(n.5)$–solvable.*
2. *For each $i > j$, $K_i - K_j$ is of infinite order in $\mathcal{F}^{\mathbb{Q}}_{(n)}/\mathcal{F}^{\mathbb{Q}}_{(n.5)}$.*

The corollary below, which was first proved in [10, Theorem 4.3], easily follows from Theorem 7.7. We remark that it is further generalized to an infinite rank result in [11].

Corollary 7.8 *For each positive integer n, $\mathcal{F}^{\mathbb{Q}}_{(n)}/\mathcal{F}^{\mathbb{Q}}_{(n.5)}$ has positive rank.*

Proof of Theorem 7.7 The first part is a special case of Lemma 6.5 (when $k=1$). For the second part, suppose that for a positive integer k, the connected sum $\#^k(K_i - K_j)$ is $(n.5)$–solvable. Then by Proposition 6.4, $M(\#^k K_i)$ and $M(\#^k K_j)$ are $(n.5)$–solvequivalent. Let U denote an $(n.5)$–cylinder between $M(\#^k K_i)$ and $M(\#^k K_j)$. As we did in the proof of Theorem 1.5, by attaching standard cobordisms to U we obtain an $(h.5)$–cylinder between $\coprod^k M(K_i)$ and $\coprod^k M(K_j)$. This contradicts Lemma 6.5. $\square$

References

[1] **S E Cappell**, **J L Shaneson**, *Link cobordism*, Comment. Math. Helv. 55 (1980) 20–49 MR569244

[2] **A J Casson**, **C M Gordon**, *Cobordism of classical knots*, from: "À la recherche de la topologie perdue", Progr. Math. 62, Birkhäuser, Boston (1986) 181–199 MR900252 With an appendix by P M Gilmer

[3] **J C Cha**, *Link concordance, homology cobordism, and Hirzebruch-type defects from iterated p-covers*, to appear in J. Eur. Math. Soc. (JEMS) `arXiv:math.GT/0705.0088`

[4] **J C Cha**, *Structure of the string link concordance group and Hirzebruch-type invariants*, to appear in Indiana Univ. Math. J. `arXiv:0709.2968`

[5] **J C Cha**, *The structure of the rational concordance group of knots*, Mem. Amer. Math. Soc. 189 (2007) x+95 MR2343079

[6] **J C Cha**, **K H Ko**, *Signatures of links in rational homology spheres*, Topology 41 (2002) 1161–1182 MR1923217

[7] **J C Cha**, **C Livingston**, **D Ruberman**, *Algebraic and Heegaard Floer invariants of knots with slice Bing doubles*, Math. Proc. Cambridge Philos. Soc. 144 (2008) 403–410 MR2405897

[8] **D Cimasoni**, *Slicing Bing doubles*, Algebr. Geom. Topol. 6 (2006) 2395–2415 MR2286030

[9] **T D Cochran**, *Noncommutative knot theory*, Algebr. Geom. Topol. 4 (2004) 347–398 MR2077670

[10] **T D Cochran**, **S L Harvey**, **C Leidy**, *Knot concordance and Blanchfield duality* `arXiv:math.GT/0705.3987`

[11] **T D Cochran**, **S L Harvey**, **C Leidy**, *Knot concordance and higher-order Blanchfield duality* `arXiv:0710.3082`

[12] **T D Cochran**, **S L Harvey**, **C Leidy**, *Link concordance and generalized doubling operators*, to appear in Alg. Geom. Top. `arXiv:0801.3677`

[13] **T D Cochran**, **T Kim**, *Higher-order Alexander invariants and filtrations of the knot concordance group*, Trans. Amer. Math. Soc. 360 (2008) 1407–1441 MR2357701

[14] **T D Cochran**, **K E Orr**, *Not all links are concordant to boundary links*, Ann. of Math. (2) 138 (1993) 519–554 MR1247992

[15] **T D Cochran**, **K E Orr**, **P Teichner**, *Knot concordance, Whitney towers and L^2–signatures*, Ann. of Math. (2) 157 (2003) 433–519 MR1973052

[16] **T D Cochran**, **K E Orr**, **P Teichner**, *Structure in the classical knot concordance group*, Comment. Math. Helv. 79 (2004) 105–123 MR2031301

[17] **T D Cochran**, **P Teichner**, *Knot concordance and von Neumann ρ–invariants*, Duke Math. J. 137 (2007) 337–379 MR2309149

[18] **J Duval**, *Forme de Blanchfield et cobordisme d'entrelacs bords*, Comment. Math. Helv. 61 (1986) 617–635 MR870709

[19] **S L Harvey**, *Higher-order polynomial invariants of 3–manifolds giving lower bounds for the Thurston norm*, Topology 44 (2005) 895–945 MR2153977

[20] **S L Harvey**, *Homology cobordism invariants and the Cochran–Orr–Teichner filtration of the link concordance group*, Geom. Topol. 12 (2008) 387–430 MR2390349

[21] **K H Ko**, *Seifert matrices and boundary link cobordisms*, Trans. Amer. Math. Soc. 299 (1987) 657–681 MR869227

[22] **J P Levine**, *Invariants of knot cobordism*, Invent. Math. 8 (1969), 98–110; addendum, ibid. 8 (1969) 355 MR0253348

[23] **J P Levine**, *Link invariants via the eta invariant*, Comment. Math. Helv. 69 (1994) 82–119 MR1259607

[24] **C Manolescu**, **B Owens**, *A concordance invariant from the Floer homology of double branched covers*, Int. Math. Res. Not. IMRN (2007) Art. ID rnm077, 21 MR2363303

[25] **W Mio**, *On boundary-link cobordism*, Math. Proc. Cambridge Philos. Soc. 101 (1987) 259–266 MR870597

[26] **P Ozsváth**, **Z Szabó**, *Knot Floer homology and the four-ball genus*, Geom. Topol. 7 (2003) 615–639 MR2026543

Department of Mathematics and Pohang Mathematics Institute
Pohang University of Science and Technology
Pohang, Gyungbuk 790–784, Republic of Korea

Department of Mathematics, Konkuk University
Seoul 143–701, Republic of Korea

jccha@postech.ac.kr, tkim@konkuk.ac.kr

Proposed: Peter Teichner Received: 8 January 2008
Seconded: Ron Stern, Cameron Gordon Revised: 23 June 2008