Flats and the flat torus theorem in systolic spaces

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We prove the Systolic Flat Torus Theorem, which completes the list of basic properties that are simultaneously true for systolic geometry and $\mathrm{CAT}(0)$ geometry.

We develop the theory of minimal surfaces in systolic complexes, which is a powerful tool in studying systolic complexes. We prove that flat minimal surfaces in a systolic complex are almost isometrically embedded and introduce a local condition for flat surfaces which implies minimality. We also prove that minimal surfaces are stable under small deformations of their boundaries.

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1 Introduction

Systolic complexes were introduced in Januszkiewicz and Świątkowski [8] and, independently, in Haglund [5]. They are connected, simply connected simplicial complexes satisfying certain local combinatorial conditions (see Definition 2.1 for details) which are simplicial analogues of nonpositive curvature. Systolic complexes have many properties similar to properties of $\mathrm{CAT}(0)$–spaces. However, systolicity neither implies, nor is implied by nonpositive curvature of the complex equipped with the piecewise Euclidean metric for which the simplices are regular Euclidean simplices.

In the study of $\mathrm{CAT}(0)$–spaces it is often important to study their flat subspaces, ie isometrically embedded Euclidean spaces $\mathbb{E}^n$, $n \geq 2$. In the present paper we study flat subspaces of systolic complexes. A 2–dimensional flat in a systolic complex X is the equilaterally triangulated Euclidean plane (denoted $\mathbb{E}^2_\Delta$) whose 1–skeleton is isometrically embedded into $X^{(1)}$. One does not need to consider higher dimensional flats, since systolic complexes do not contain flats of dimension larger than 2, ie there are no systolic triangulations of $\mathbb{E}^n$ for $n \geq 3$ and there are no properly discontinuous actions of $\mathbb{Z}^n$ on a systolic complex for $n \geq 3$ (see Januszkiewicz–Świątkowski [9]; in Section 6 we give an alternative proof of the latter fact).

One of the main results of this paper is the Systolic Flat Torus Theorem, which completes the list of basic properties that are simultaneously true for systolic geometry and $\mathrm{CAT}(0)$ geometry.

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Systolic Flat Torus Theorem (See Theorem 6.1.) *Let G be a noncyclic free abelian group acting properly discontinuously by simplicial automorphisms on a systolic complex X. Then:*

(1) *G is isomorphic to $\mathbb{Z}^2$.*

(2) *There is a G–invariant flat in X. Any two such flats are at Hausdorff distance 1.*

(3) *A vertex $v \in X$ is contained in a G–invariant flat if and only if it satisfies the minimal displacement condition:*

$$d(v, g(v)) = \min_{x \in X^{(0)}} d(x, g(x)) \quad \text{for any } g \in G.$$

Part (2) of the theorem is elaborated in Theorem A, which characterizes flats at finite Hausdorff distance from one another. It states that not only the flats (the images of the embeddings of $\mathbb{E}^2_\triangle$ into X) are at Hausdorff distance 1, but also the embeddings themselves are at distance 1. Hence the G–invariant flat given by the Systolic Flat Torus Theorem is in some sense unique.

Theorem A (See Theorem 5.4.) *Let F and F' be flats in a systolic complex X at finite Hausdorff distance. Then there is a simplicial isometry $f\colon F \to F'$ such that*

$$d_X(v, f(v)) \le 1 \quad \text{for any vertex } v \in F.$$

In particular, F and F' are at Hausdorff distance at most 1.

The main tool used in the proof of the Systolic Flat Torus Theorem is the theory of minimal surfaces, developed in the first part of the paper (Sections 2–4). Given a cycle γ in the 1–skeleton of X, a surface spanning γ is a simplicial map $S\colon \triangle \to X$ such that $\triangle$ is a triangulation of a 2–disc and S maps $\partial\triangle$ isomorphically onto γ. The surface S is minimal if $\triangle$ has the minimal number of triangles. Since we are mainly interested in studying flats in X, the surfaces of special interest to us are *flat surfaces*, ie those whose domains are simplicial discs $\triangle \subset \mathbb{E}^2_\triangle$ such that the 1–skeleta $\triangle^{(1)}$ are isometrically embedded into the 1–skeleton of $\mathbb{E}^2_\triangle$.

We answer the following questions that naturally arise when considering flat minimal surfaces:

(1) Is it possible to characterize flat minimal surfaces in local terms?

(2) Is a flat minimal surface an isometric embedding?

(3) Is a flat minimal surface spanning a given cycle γ unique?

(4) If cycles γ_1 and γ_2 are close to each other, then are minimal surfaces spanning them close to each other?

The following theorems summarize more precise, but more technical results from the main text, pertaining to the discussion above. Theorem B presents a local characterization of flat minimal surfaces (condition (a) in the theorem) and provides the positive answer to a slightly weakened version of question (2) (the interior of a flat minimal surface is isometrically embedded).

Theorem B (See Theorem 4.12.) *Let $\triangle \subset \mathbb{E}^2_\triangle$ be a simplicial disc such that $\triangle^{(1)}$ is isometrically embedded into the 1–skeleton of $\mathbb{E}^2_\triangle$ and $\partial\triangle$ has no diagonals (ie non-consecutive vertices of $\partial\triangle$ are not connected by an edge in $\triangle$). Then for an arbitrary simplicial map $S\colon \triangle \to X$ to a systolic complex X the following are equivalent:*

(a) *The restriction of S to any simplicial disc $D \subset \triangle$ such that $\operatorname{diam} D \leq 3$ is an isometric embedding.*

(b) *The restriction of S to the subcomplex spanned by the internal vertices of $\triangle$ is an isometric embedding.*

(c) *The map S is a minimal surface.*

The answer to question (3) is negative – typically, there is a lot of minimal surfaces spanning given cycle in a systolic complex. However, we proved that if one of the surfaces is flat, then they are pairwise at Hausdorff distance 1. Moreover, they are equivalent in the following sense:

Theorem C (See Theorem 4.12.) *Let $S\colon \triangle \to X$ be a flat minimal surface in a systolic complex X and let $\partial\triangle$ have no diagonals. Then for any minimal surface $S'\colon \triangle' \to X$ spanning the same cycle as S we have $\triangle' = \triangle$ and $d_X(S(v), S'(v)) \leq 1$ for any vertex $v \in \triangle = \triangle'$.*

Theorem D describes the stability of flat minimal surfaces under small deformations of their boundaries. This is a simplified version of Theorem 4.16, where we do not assume that S and S' are flat and do not use the assumption that γ and γ' have equal lengths.

Theorem D (See Theorem 4.16.) *Let γ and γ' be cycles of equal lengths in a systolic complex X such that they have no diagonals. Denote by $\varphi\colon \gamma \to \gamma'$ a simplicial isomorphism. If S and S' are flat minimal surfaces spanning γ and γ', respectively, then:*

$$\operatorname{hdist}_X(\operatorname{Im} S, \operatorname{Im} S') \leq \max_{v \in \gamma^{(0)}} d_X(v, \varphi(v)) + 1$$

The techniques developed in this paper have more applications in the theory of systolic spaces.

As a consequence of Theorem B we obtain the result proved by Piotr Przytycki in [10]: a systolic complex admitting a geometric group action is Gromov-hyperbolic if and only if it does not contain a flat (Corollary 4.14).

For systolic spaces one has a natural modification of the Isolated Flats Property (studied by G C Hruska [6; 7] for CAT(0)–spaces). In the next paper [4] we examine systolic spaces with the Isolated Flats Property admitting a geometric action of a group G. As a consequence of the Systolic Flat Torus Theorem we obtain a bijective correspondence between the equivalence classes of flats in X (two flats are equivalent if they are at finite Hausdorff distance) and the maximal virtually abelian rank 2 subgroups in G. We use Theorem B and Theorem D to prove that such a group G is relatively hyperbolic with respect to its maximal virtually abelian rank 2 subgroups and to characterize cocompact systolic complexes with the Isolated Flats Property as the complexes which do not contain isometrically embedded triplanes (this is a systolic analogue of the 2–dimensional CAT(0) result of D Wise, contained in [6]).

In [3] we apply Theorem B to obtain a classification of individual simplicial isometries of systolic complexes. We show that such an isometry is either elliptic (ie fixes a simplex) or hyperbolic (ie fixes a "thick axis" – certain subcomplex contained in 1–neighbourhood of a bi-infinite geodesic in $X^{(1)}$).

In the forthcoming paper [2] we use Theorem B to prove the δ–thin tetrahedra property for systolic spaces, which is a higher dimensional analogue of the δ–thin triangles property. It states that given any 4 vertices in a systolic complex X, a tetrahedron obtained by joining the vertices pairwise by geodesics in $X^{(1)}$ and then spanning minimal surfaces on the four arising geodesic triangles satisfies the following property: any of its 2–dimensional faces (ie any of the minimal surfaces) is in δ–neighbourhood of the union of the remaining three faces.

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2 Systolic complexes and groups

In this section we recall the definition and main properties of systolic complexes and systolic groups, proved in Januszkiewicz–Świątkowski [8; 9]. Theorem 2.4 is a variation of [9, Theorem 9.2] and is crucial for the present paper. The remaining material here is just for reader's convenience.

Let X be a simplicial complex and σ a simplex of X. The *link* of X at σ, denoted X_σ, is the subcomplex of X consisting of all simplices that are disjoint from σ and together with σ span a simplex of X. The (closed) star of σ is the union of all (closed) simplices containing σ. A simplex σ is the *join* of its faces $\tau_1, \tau_2 \subset \sigma$ (what we denote $\sigma = \tau_1 * \tau_2$) if τ_1 and τ_2 are disjoint and their union spans σ. A complex X is the join of its disjoint subcomplexes $K, L \subset X$ (denoted $X = K * L$) if X consists of all simplices of the form $\sigma * \tau$, where σ and τ are simplices of K and L, respectively.

A simplicial complex X is *flag* if every finite set of its vertices pairwise connected by edges spans a simplex of X. A subcomplex $Y \subset X$ is *full* if any simplex $\sigma \subset X$ with all vertices in Y is contained in Y.

A *cycle* in X is a subcomplex γ isomorphic to a triangulation of a circle. The *length* of γ (denoted $|\gamma|$) is the number of its edges. A *diagonal* of a cycle is an edge joining its two nonconsecutive vertices.

Whenever we refer to a metric on a simplicial complex, we actually mean the 1–skeleton of the complex equipped with the combinatorial metric (ie the geodesic metric in which all edges have lengths 1). Thus for a simplicial complex X the symbol d_X denotes the combinatorial metric on $X^{(1)}$. Moreover, when we refer to a "geodesic" in a simplicial complex X, we mean a geodesic in $X^{(1)}$ having both endpoints in $X^{(0)}$.

Definition 2.1 (See Januszkiewicz–Świątkowski [9, Section 2].) A simplicial complex X is called:

- 6–*large* if it is flag and every cycle γ in X of length $4 \leq |\gamma| < 6$ has a diagonal;
- *locally* 6–*large* if the link of X at every (nonempty) simplex is 6–large;
- *systolic* if it is locally 6–large, connected and simply connected.

A group acting simplicially, properly discontinuously and cocompactly on a systolic complex is called a *systolic group*.

As the following fact shows, an equivalent definition of systolicity can be obtained by replacing the words "locally 6–large" with "6–large".

Fact 2.2 (See Januszkiewicz–Świątkowski [8, Proposition 1.4].) *Every systolic complex is 6–large. In particular, a cycle of length smaller than 6 in a systolic complex bounds a triangulated disc with no internal vertices.*

The original definition of systolicity by Januszkiewicz and Świątkowski introduces notions of k–largeness and k–systolicity for $k \geq 6$, obtained by the natural modification of Definition 2.1 (then *systolic complex* means 6–*systc complex*). However, k–systolic complexes for $k \geq 7$ are Gromov-hyperbolic by [8, Theorem 2.1], so they do not contain either flats, or even wide flat surfaces (see Definition 4.1) and do not admit properly discontinuous actions of $\mathbb{Z}^2$. From our point of view they are therefore not interesting.

Theorem 2.3 *Let X be a finite dimensional systolic complex. Then:*

- [8, Theorem 4.1(1)] *X is contractible.*
- [9, Corollary 2.3(4)] *Every full subcomplex of X is aspherical.*

It was proved in Januszkiewicz–Świątkowski [8, Theorem 6.1] that every connected, locally 6–large, finite dimensional complex of groups is developable. Using this result many constructions of compact complexes with systolic universal coverings were presented (see [8, Corollaries 19.2 and 19.3]).

The next theorem follows from the proof of Januszkiewicz–Świątkowski [9, Theorem 9.2]. However, as it is an important result for the present paper, we provide its proof below.

Theorem 2.4 *Let X be a systolic complex and S a triangulation of a 2–sphere. Then any simplicial map $f\colon S \to X$ can be extended to a simplicial map $F\colon B \to X$, where B is a triangulation of a 3–ball such that $\partial B = S$ and B has no internal vertices.*

Proof We proceed by induction on the area (the number of triangles) of S. The smallest possible area is 4 – then S is the 2–skeleton of a tetrahedron and the statement follows from the flagness of X. The case when S has area greater than 4 we divide into two subcases:

Case 1 *S is not flag.*

As the case of the 2–skeleton of a tetrahedron has already been discussed, there exists a cycle γ of length 3 in S not bounding a triangle. Thus γ disconnects S into two discs D_1 and D_2 ($\partial D_1 = \partial D_2 = \gamma$). We glue a single triangle to D_i, $i = 1, 2$ along γ, obtaining a simplicial sphere S_i of area smaller than the area of S (we assume $S_1 \cap S_2$ is the added triangle) and define $f_i\colon S_i \to X$ to be the simplicial map whose restriction to the 1–skeleton coincide with the restriction of f (f_i is well-defined by the flagness of X). By the inductive assumption, f_i can be extended to $F_i\colon B_i \to X$, where B_i is such a triangulation of a ball that has no internal vertices and $\partial B_i = S_i$. Finally, we put $B = B_1 \cup B_2$ and $F = F_1 \cup F_2$.

Case 2 *S is flag.*

Since the Euler characteristic of a sphere is positive, by the Gauss–Bonnet Lemma there is a vertex $v \in S$ adjacent to less than 6 triangles. The link at v is a cycle γ of length 4 or 5 (length 3 is impossible by the flagness of S). Thus $S = D_1 \cup D_2$, where $D_1 = v * \gamma$ is the closed star of v and D_2 is obtained from S by removing the open star of v. Notice that by the flagness of S the cycle $\gamma = \partial D_2 = \partial D_1$ has no diagonals.

By Fact 2.2 the map $f|_\gamma$ can be simplicially extended over some triangulated disc C ($\gamma = \partial C$) with no internal vertices. Define $B_1 = v * C$ and let $F_1\colon B_1 \to X$ be the simplicial map whose restriction to the 0–skeleton coincides with the restriction of f (it is well-defined by the flagness of X). Then $S_2 = D_2 \cup C$ is a simplicial sphere (as $\gamma = \partial D_2 = \partial C$ has no diagonals in D_2) of area smaller than the area of S. Let $f_2\colon S_2 \to X$ be the simplicial map whose restriction to the 0–skeleton coincides with the restriction of f. Applying the inductive assumption we extend it to $F_2\colon B_2 \to X$, where B_2 is a triangulation of a ball with no internal vertices satisfying $\partial B_2 = S_2$. Finally, we put $B = B_1 \cup B_2$ and $F = F_1 \cup F_2$. □

3 Systolic triangulations of a disc

The simplest example of a systolic complex is the equilaterally triangulated Euclidean plane – it will be called *the flat systolic plane* and denoted $\mathbb{E}^2_\triangle$. As we have written before, we equip it with the combinatorial metric on the 1–skeleton and do not use any metric on the whole complex. We define a *systolic disc* to be a systolic triangulation of a 2–disc and a *flat disc* – a systolic disc $\triangle$ such that $\triangle^{(1)}$ can be isometrically embedded into $\mathbb{E}^2_\triangle$. For any vertex $v \in \triangle$ the *defect at* v is defined by the following formula:

$$\operatorname{def}_\triangle(v) = \begin{cases} 6 - \#\{\text{triangles in } \triangle \text{ containing } v\} & \text{if } v \notin \partial\triangle \\ 3 - \#\{\text{triangles in } \triangle \text{ containing } v\} & \text{if } v \in \partial\triangle \end{cases}$$

It is clear that internal vertices of a systolic disc have nonpositive defects. Boundary vertices will be called, for brevity, *(non)positive*, *zero* or *(non)negative* if their defects are such. Throughout the paper we use the term “the sum of the defects along a polygonal line” to mean the sum of the defects at all of its vertices but at the endpoints.

Now we state a few facts on systolic discs, frequently used in this paper.

Fact 3.1 *If $\triangle$ is a systolic disc and g is a geodesic in $\triangle$ contained in $\partial\triangle$, then the sum of the defects along g is at most 1.*

Proof The geodesic g does not pass through any boundary vertex of defect 2. Moreover, if g passes through vertices $u, v \in g \subset \partial\triangle$ of defects 1, at least one of the vertices on g between u and v has a negative defect (by the geodesity of g). Thus the positive vertices on g have defects 1 and are separated by negative vertices, so the sum of the defects is at most 1. □

Lemma 3.2 (Gauss–Bonnet Lemma) *If $\triangle$ is any simplicial disc, then:*

$$\sum_{v \in \triangle^{(0)}} \operatorname{def}(v) = 6$$

In particular, if $\triangle$ is a systolic disc, then the sum of the defects at its boundary vertices is greater than or equal to 6, with the equality if and only if $\triangle$ has no internal vertices with negative defects.

Lemma 3.3 (Pick's Formula) *Let $\triangle$ be any simplicial disc. Denote its area (ie the number of triangles) by S, its perimeter by l and the numbers of its internal and boundary vertices by V_i and V_b, respectively. Then:*

$$S = 2V_i + V_b - 2 = l + 2(V_i - 1)$$

In particular, the area of a simplicial disc depends only on the numbers of its internal and boundary vertices.

Proof Denoting by E_i the number of internal edges of $\triangle$, we obtain $3S = 2E_i + l$. The Euler characteristic of $\triangle$ is equal to $1 = S - (E_i + l) + (V_i + l)$, hence $E_i = S + V_i - 1$. Substituting the latter equation into the first one we obtain the lemma. □

It follows from the biautomaticity of systolic groups (proved in Januszkiewicz-Świątkowski [8, Theorem 13.1]) that the systolic complexes admitting a geometric group action satisfy a quadratic isoperimetric inequality. In the subsequent lemma we prove this fact for any systolic complex and provide explicit constants, presenting the optimal estimate on the area of a systolic disc.

Lemma 3.4 *Let $\triangle$ be a systolic disc of perimeter l and area S. Then:*

(1) $S \leq \frac{1}{6}l^2$.

(2) $\operatorname{dist}(v, \partial\triangle) \leq \frac{1}{6}l$ *for any vertex* $v \in \triangle$.

The inequalities are optimal if $l \not\equiv \pm 1 \pmod 6$. In the remaining cases the optimal isoperimetric inequality is $S \le \frac{1}{6}l^2 - 1$ (since by Pick's Formula $S \equiv l \pmod 2$). If $l = 6k + r$, where k, r are natural numbers and $r < 6$ the estimate is realized by the equilaterally triangulated regular hexagon of side length $k + 1$ with cut off triangles adjacent to its $6 - r$ consecutive sides.

Proof Denote by $\lambda_\triangle(d)$ the number of vertices $v \in \triangle$ satisfying $\operatorname{dist}(v, \partial\triangle) = d$. We prove by induction on l that the inequality

$$\lambda_\triangle(d) \le \begin{cases} l - 6d & \text{if } 0 < d < \frac{1}{6}l \\ 1 & \text{if } d = \frac{1}{6}l \\ 0 & \text{if } d > \frac{1}{6}l \end{cases} \tag{3-1}$$

holds for any systolic disc $\triangle$ of perimeter l.

This is trivial when $l < 6$, as by Fact 2.2 in such a case $\triangle$ has no internal vertices. The case $l \ge 6$ will be divided into three subcases.

Case 1 *$\triangle$ has a disconnecting edge e.*

Then e disconnects $\triangle$ into two systolic discs $\triangle_1$ and $\triangle_2$ of perimeters l_1 and l_2, where $l_1 + l_2 = l + 2$ and $3 \le l_1 \le l_2 < l$. If $0 < d \le \frac{1}{6}l_1$, then $d < \frac{1}{6}l$ and by the inductive assumption:

$$\lambda_\triangle(d) = \lambda_{\triangle_1}(d) + \lambda_{\triangle_2}(d) \le (l_1 - 6d + 1) + (l_2 - 6d + 1) = l + 4 - 12d \le l - 6d$$

If $d > \frac{1}{6}l_1$, then by the inductive assumption $\lambda_\triangle(d) = \lambda_{\triangle_2}(d)$ and (3-1) follows immediately.

Case 2 *The closed star of some internal vertex $v \in \triangle$ disconnects $\triangle$ and there are no disconnecting edges in $\triangle$.*

Then there exists a geodesic line of length 2 in $\triangle$, whose middle vertex is v, disconnecting $\triangle$ into systolic discs $\triangle_1$ and $\triangle_2$ so that each of them contain an internal vertex. Therefore, by Fact 2.2 their perimeters are not smaller than 6. If $\operatorname{def}_{\triangle_i}(v) < 0$, then we glue 2 triangles at v obtaining a systolic disc $\triangle_i'$ (as in Figure 1), otherwise we put $\triangle_i' := \triangle_i$. Thus for any internal vertex $w \in \triangle_i'$ different from v we have

$$\operatorname{dist}_\triangle(w, \partial\triangle) = \operatorname{dist}_{\triangle_i'}(w, \partial\triangle_i').$$

Denoting by l_1' and l_2' the perimeters of $\triangle_1'$ and $\triangle_2'$ we have $l_1' + l_2' = l + 4$ and

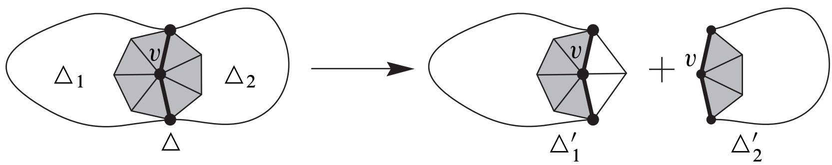

Figure 1

$6 \le l'_1 \le l'_2 < l$, hence $l'_1, l'_2 < l$. If $1 < d \le \frac{1}{6} l'_1$, then $d < \frac{1}{6} l$ and by the inductive assumption:

$$\lambda_\triangle(d) = \lambda_{\triangle'_1}(d) + \lambda_{\triangle'_2}(d) \le (l'_1 - 6d + 1) + (l'_2 - 6d + 1) = l + 6 - 12d \le l - 6d$$

Notice that by the Gauss–Bonnet Lemma a systolic disc of perimeter 6 either has a diagonal or is the join of a vertex and a cycle of length 6, so the case $l'_1 = l'_2 = 6$ is impossible. Thus by the inductive assumption:

$$\lambda_\triangle(1) \le \lambda_{\triangle'_1}(1) + \lambda_{\triangle'_2}(1) + 1 \le (l'_1 - 6 + 1) + (l'_2 - 6) + 1 = l - 6$$

In the case when $d > \frac{1}{6} l'_1$ by the inductive assumption we have $\lambda_\triangle(d) = \lambda_{\triangle'_2}(d)$ and (3-1) follows immediately.

Case 3 *$\triangle$ cannot be disconnected either by an edge, or by a closed star of an internal vertex.*

Then the subcomplex $\triangle' \subset \triangle$ spanned by all internal vertices of $\triangle$ is a deformation retract of $\triangle$ and has no disconnecting vertices. Therefore $\triangle'$ is either a systolic disc or a single vertex v or a single edge vw. Since $\triangle$ has no disconnecting edges, in the last two cases $\triangle$ is the closed star of v (and $S = l \ge 6$) or the union of the closed stars of v and w (and $S = l + 2 \ge 10$), respectively, whence (3-1) immediately follows.

Suppose $\triangle'$ is a systolic disc of perimeter l'. Since for every vertex $v \in \partial\triangle' \subset \triangle$ the intersection $\triangle_v \cap \partial\triangle = \alpha_v$ is an arc in $\partial\triangle$, we have

$$l + l' = \sum_{v \in \partial\triangle'} \big(|\alpha_v| + 1\big) = \sum_{v \in \partial\triangle} \big(2 - \mathrm{def}(v)\big) \le 2l - 6,$$

where the sums are equal to the number of edges in $\triangle$ having exactly one endpoint on $\partial\triangle$, and the inequality is by the Gauss–Bonnet Lemma. Thus $l' \le l - 6$ and applying the inductive assumption to $\lambda_{\triangle'}(d-1) = \lambda_\triangle(d)$ we complete the proof of (3-1). Part (2) of the lemma is an immediate corollary.

To prove part (1) we estimate the number V_i of internal vertices of $\triangle$:

$$V_i = \sum_{d=1}^{\infty} \lambda_\triangle(d) \le \delta + \sum_{d=1}^{[\frac{l}{6}]} (l - 6d) = \delta + \left[\frac{l}{6}\right]\Big(l - 3\left[\frac{l}{6}\right] - 3\Big) \le \frac{1}{12}l^2 - \frac{1}{2}l + 1,$$

where $\delta = 1$ if l is divisible by 6 or $\delta = 0$ otherwise. Now we apply Pick's Formula to obtain:

$$S = 2\,(V_i - 1) + l \le 2\Big(\Big(\frac{1}{12}l^2 - \frac{1}{2}l + 1\Big) - 1\Big) + l \le \frac{1}{6}l^2 \qquad \square$$

Recall that a simplicial disc $\triangle$ is flat if $\triangle^{(1)}$ can be isometrically embedded into $\mathbb{E}^2_\triangle$. Below we present an intrinsic characterization of flatness.

Lemma 3.5 *A simplicial disc $\triangle$ is flat if and only if it satisfies the following three conditions:*

(i) *Every internal vertex of $\triangle$ has defect 0.*

(ii) *$\triangle$ has no boundary vertices of defect less than -1.*

(iii) *Any two negative vertices on $\partial\triangle$ are separated by a positive one.*

Proof We prove the "if" part (the "only if" part is trivial). If $\triangle$ has a boundary vertex of defect -1, then $\triangle^{(1)}$ can be isometrically embedded into a simplicial disc satisfying (i)–(iii) having the same perimeter as $\triangle$ and larger area (by gluing two triangles onto $\triangle$ at the negative vertex). By the isoperimetric inequality (Lemma 3.4) the procedure terminates. Therefore, without loss of generality, we can assume that $\triangle$ has no negative vertices.

By induction on the number of positive vertices on $\partial\triangle$ we claim that $\triangle^{(1)}$ can be isometrically embedded into the 1–skeleton of a simplicial disc $\triangle'$ such that $\triangle'$ still satisfies (i)–(iii), has no negative vertices and, furthermore, any path in $\partial\triangle'$ joining two distinct vertices of defects 1 passes through a vertex of defect 2. Indeed, for any path $[u, v]$, such that $u, v \in \partial\triangle$ have defects 1 and $[u, v]$ does not pass through any positive vertex we glue an equilaterally triangulated equilateral triangle along $[u, v]$ and $\triangle^{(1)}$ can be isometrically embedded into the 1–skeleton of the resulting simplicial disc $\triangle'$, which still satisfies (i)–(iii), has no negative vertices and has less positive vertices on its boundary.

Applying the Gauss–Bonnet Lemma we see that $\triangle'$ either has 3 nonzero vertices (each of defect 2), or has 4 nonzero vertices (of defects 2, 1, 2, 1, in this order). It follows that $\triangle$ is an equilateral triangulation of an equilateral triangle or of a parallelogram. This is proved by induction on the perimeter – we cut off triangles touching one side of the triangle or the parallelogram and apply the inductive assumption. Therefore $\triangle^{(1)}$ can be isometrically embedded into $\mathbb{E}^2_\triangle$. □

4 Flat surfaces in systolic complexes

Let X be a systolic complex. Any simplicial map $S\colon \triangle_S \to X$, where $\triangle_S$ is a triangulation of a 2–disc will be called a *surface*. We often use the symbol $\triangle_S$ to denote the domain of a surface S. Given a cycle γ in X, we say that a surface S *is spanning* γ if it maps $\partial\triangle_S$ isomorphically onto γ. By the area of a simplicial disc we mean the number of triangles in the triangulation.

Definition 4.1 A surface $S\colon \triangle_S \to X$ in a systolic complex X is:

- *minimal* if $\triangle_S$ has the minimal area among surfaces extending $S|_{\partial\triangle_S}$;
- *systolic* if $\triangle_S$ is a systolic disc;
- *flat* if $\triangle_S$ is a flat disc, ie $\triangle_S^{(1)}$ can be isometrically embedded into $\mathbb{E}^2_\triangle$;
- *wide* if $\partial\triangle_S$ is a full subcomplex of $\triangle_S$.

This section is devoted to the study of flat minimal surfaces. By Januszkiewicz–Świątkowski [8, Lemma 1.7] a minimal surface S spanning a cycle γ is nondegenerate, ie is injective on any simplex. Thus if the complex $\triangle_S$ has the smallest area, then also the map $S\colon \triangle_S \to X$ has the smallest area (the area of the map S is the number of triangles of $\triangle_S$ on which S is injective). The existence of minimal surfaces is given by the following lemma:

Lemma 4.2 *Let X be a systolic complex and S^1 a triangulated circle. Then any simplicial map $f\colon S^1 \to X$ can be extended to a simplicial map $F\colon \triangle \to X$, where $\triangle$ is a systolic disc such that $\partial\triangle = S^1$. Moreover, any minimal surface extending f is systolic.*

Proof Since X is simply connected, f can be extended to a map $f'\colon D^2 \to X$, where D^2 is a 2–disc. Hence, by the relative Simplicial Approximation Theorem [11, page 126], we obtain a simplicial disc $\triangle$ such that $\partial\triangle = S^1$ and a simplicial map $F\colon \triangle \to X$ extending f. We choose $\triangle$ and F so that the area of $\triangle$ is minimal.

If $\triangle$ was not systolic, then it would have an internal vertex v adjacent to less than 6 triangles. Then we could cut out the open star of v and glue in a triangulated disc with no internal vertices (extending the triangulation of $\triangle_v$) so that F could be extended over the new triangulation (Fact 2.2). This would result in a simplicial disc $\triangle'$ of area smaller than the area of $\triangle$ and a simplicial map $F'\colon \triangle' \to X$ extending f, contradicting the minimality of the area of $\triangle$. □

One of the main results in this section is the characterization of wide flat minimal surfaces in local terms (Theorem 4.12). To state it we need the following local conditions:

Definition 4.3 A surface $S\colon \triangle_S \to X$ in a systolic complex X is:

- a *locally isometric immersion* if for any internal vertex $v \in \triangle_S$, the restriction of S to the 1–skeleton of $N(v)$ is an isometric embedding;
- a *strong locally isometric immersion* if for any internal vertex $v \in \triangle_S$ and for any edge $e \subset \triangle_S$ with endpoints at internal vertices of $\triangle_S$, the restrictions of S to the 1–skeleta of $N(v)$ and $N(e)$ are isometric embeddings.

Here and subsequently $N(K)$ denotes the subcomplex equal to the union of all (closed) simplices that intersect K.

4.1 Equivalent surfaces

It is natural to study flat surfaces up to some equivalence relation, defined below. We show that if there exists a wide flat minimal surface spanning a cycle γ, then it is unique up to this equivalence (Theorem 4.12).

Definition 4.4 We call surfaces S and S' *v–equivalent* and write $S \cong_v S'$ if $\triangle_S = \triangle_{S'}$ and $S(x) = S'(x)$ for all vertices $x \neq v$, where $v \in \triangle_S$ is a fixed internal vertex.

Surfaces S and S' are *equivalent* if there exist surfaces $S = S_0, S_1, \ldots, S_n = S'$ and internal vertices $v_1, \ldots, v_n \in \triangle_S$ such that $\triangle_S = \triangle_{S_0} = \cdots = \triangle_{S_n}$ and $S_{i-1} \cong_{v_i} S_i$ for $i = 1, \ldots, n$.

Informally, two surfaces are equivalent if one of them can be obtained from the other by a sequence of small modifications. Surprisingly, such surfaces are always Hausdorff 1–close (Lemma 4.6). It is also important that this equivalence preserves the condition of being a strong locally isometric immersion (Lemma 4.5).

Lemma 4.5 *If a flat surface S in a systolic complex X is a strong locally isometric immersion, then any surface equivalent to S also has this property.*

Lemma 4.6 *If flat surfaces S and S' in a systolic complex X are equivalent and are locally isometric immersions, then:*

$$d_X(S(v), S'(v)) \leq 1 \quad \text{for any internal vertex } v \in \triangle_S = \triangle_{S'}$$

In particular, the Hausdorff distance between $\operatorname{Im} S$ and $\operatorname{Im} S'$ is at most 1.

Before proving the lemmas, we need certain characterization of locally isometric immersions in terms of local minimality.

Proposition 4.7 *Let S be a flat surface in a systolic complex X. Then:*

(1) *S is a locally isometric immersion if and only if for every internal vertex $v \in \triangle_S$ the surface $S|_{N(v)}$ is minimal.*

(2) *S is a strong locally isometric immersion if and only if for every internal vertex $v \in \triangle_S$ and for every edge $e \subset \triangle_S$ with endpoints at internal vertices the surfaces $S|_{N(v)}$ and $S|_{N(e)}$ are minimal.*

Proof (1) Let $v \in \triangle_S$ be an internal vertex. Then $H = N(v)$ is a hexagon triangulated with 6 triangles and by Pick's Formula $S|_H$ is not minimal if and only if $S|_{\partial H}$ can be extended to a surface S' so that $\triangle_{S'}$ has no internal vertices. In such an extension the cycle $\partial H = \partial \triangle_{S'}$ has a diagonal, which implies that $S|_{H^{(1)}}$ is not an isometric embedding.

If $S|_H$ is a minimal surface, then $S|_{\partial H}$ cannot be simplicially extended over $\partial H \cup \alpha$ for any diagonal α, as otherwise it could be extended over some simplicial disc with no internal vertices (Fact 2.2), contradicting the minimality of $S|_H$. Thus $S|_{\partial H}$ is injective (if $S(p) = S(q)$ for some vertices $p \neq q \in \partial H$ we define α to be the diagonal joining p with q if they are nonconsecutive vertices of ∂H or a diagonal joining p with the other neighbour of q otherwise) and the cycle $S(\partial H)$ has no diagonals. Since $\mathrm{diam}(H) = 2$, this proves that $S|_{H^{(1)}}$ is an isometric embedding.

(2) Let uv be an edge of $\triangle_S$ with both endpoints at internal vertices of $\triangle_S$. Then $P = N(uv)$ is an octagon triangulated as in Figure 2. By Pick's Formula, the restriction of S to P is not a minimal surface if and only if $S|_{\partial P}$ can be extended over some simplicial disc $\triangle'$ bounded by ∂P having at most one internal vertex. Then ∂P either has a diagonal or is contained in the link of the only internal vertex of $\triangle'$. In both cases $S|_{P^{(1)}}$ is not an isometric embedding.

If $S|_P$ is a minimal surface, then $S|_{\partial P}$ cannot be extended over $\partial P \cup \alpha$ for any diagonal α of ∂P, as otherwise it could be extended over some simplicial disc with at most one internal vertex (by Lemma 3.4 and Pick's Formula cycles of length 6 and 7 have fillings with at most 1 internal vertex and cycles of length smaller than 6 have fillings with no internal vertices), contrary to the minimality of $S|_P$. Hence, similarly as in the proof of (1), we see that $S|_{\partial P}$ is injective and the cycle $S(\partial P)$ has no diagonals.

As by (1) the restrictions of S to the 1–skeleta of $N(u)$ and $N(v)$ are isometric embeddings, $S|_P$ is an injection onto a full subcomplex of X. Suppose $S|_{P^{(1)}}$ it is not an isometric embedding. Thus there are vertices $t, w \in P$ such that $d_P(t,w) = 3$ and $d_X(S(t), S(w)) = 2$ (so there exists a vertex $x \in X$ connected by edges with $S(t)$ and $S(w)$). There are three subcases, depicted in Figure 2.

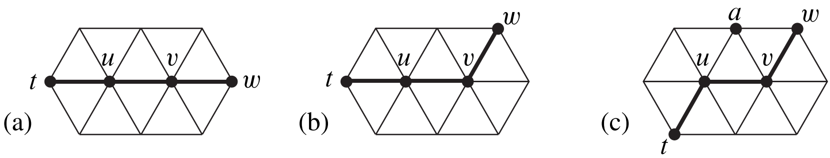

Figure 2

The pentagon $S(t)S(u)S(v)S(w)x$ has 2 diagonals (Fact 2.2) and they are $xS(u)$ and $xS(v)$ (as the restrictions of S to the 1–skeleta of $N(v)$ and $N(w)$ are isometric embeddings), so the images of t, u, v, w are in the link X_x. In case (c) we have also $S(a) \in X_x$ (as by Fact 2.2 the square $S(u)S(a)S(w)x$ has the diagonal $xS(a)$). We see that in any case the whole image of P is contained in X_x, contrary to the minimality of $S|_P$. It follows from the following fact, an argument which will be used many times in the paper.

Fact 4.8 *Let H be a minimal surface in a systolic complex X such that $\triangle_H = p * \partial\triangle_H$ (where $p \in \triangle_H$ is the only internal vertex) and $|\triangle_H| = 6$. If two opposite vertices of $\partial\triangle_H$ are mapped by H into some link X_y, then* $\operatorname{Im} H \subset X_y$.

In any case depicted in Figure 2 we apply the remark to $S|_{N(u)}$. It follows that $S|_{N(v)}$ also satisfies the assumptions of the above fact and we apply the fact to $S|_{N(v)}$, obtaining $S(P) \subset X_x$. This contradicts the minimality of $S|_P$. $\square$

Proof of Fact 4.8 Denote consecutive boundary vertices of $\triangle_H$ by $a_1, \ldots, a_6$. Suppose $H(a_1), H(a_4) \in X_y$. Since H is minimal, $H|_{\triangle_H^{(1)}}$ is an isometric embedding. By Fact 2.2 the pentagon $H(a_1)H(a_2)H(a_3)H(a_4)y$ has the diagonals $yH(a_2)$ and $yH(a_3)$. Similarly, there exist edges $yH(a_5)$ and $yH(a_6)$. The square $H(a_1)H(p)H(a_4)y$ has the diagonal $yH(p)$. $\square$

We now give proofs of the lemmas stated at the beginning of this subsection.

Proof of Lemma 4.5 Since any surface equivalent to S is a flat surface, it suffices to prove the statement for w–equivalent surfaces (for any internal vertex $w \in \triangle_S$). Thus assume $S' \cong_w S$ and denote $\triangle := \triangle_S = \triangle_{S'}$.

By Proposition 4.7 we need to prove the minimality of $S'|_{N(u)}$ for any internal vertex $u \in \triangle$ and the minimality of $S'|_{N(uv)}$ for any edge uv with endpoints at internal vertices $u, v \in \triangle$. The minimality of $S'|_{N(u)}$ follows directly from the minimality of $S|_{N(u)}$, unless $d_\triangle(u,w) = 1$, but then it follows from the minimality of $S|_{N(uw)}$. Thus by Proposition 4.7(1), S' is a locally isometric immersion.

What is left to prove is the minimality of $S'|_{N(uv)}$. The only nontrivial case is when $w \in \partial N(uv)$. Then we can assume, not losing generality, that w is connected by an edge with v and consider three subcases: two depicted in Figure 2 (a) and (b) and the third one when w is connected to both v and u. Inspecting the three subcases, case by case, we see (using the fact that S' and S restricted to the 1–skeleta of $N(vw)$ and $N(uv)$, respectively, are isometric embeddings) that the map $S'|_{N(uv)}$ is injective and the cycle $\gamma = S'(\partial N(uv))$ has no diagonals. If $S'|_{N(uv)}$ was not minimal, then γ would bound a simplicial disc with at most 1 internal vertex (by Pick's Formula). As we have just proved that γ has no diagonals, the disc would be the join of some vertex $x \in X$ and γ. However, this would contradict the fact that the restriction of S to the 1–skeleton of $N(uv)$ is an isometric embedding. $\square$

Proof of Lemma 4.6 Denote $\triangle := \triangle_S = \triangle_{S'}$. Let $S = S_0, S_1, \dots, S_n = S'$ be a sequence of surfaces such that $S_{i-1} \cong_{v_i} S_i$ for some internal vertices $v_1, \dots, v_n \in \triangle$. The proof is divided into two steps. First we prove that for internal vertices $v, w \in \triangle$ the relations $\cong_v$ and $\cong_w$ "commute" in the following sense:

Step 1 *If a flat surface S is a locally isometric immersion and $S \cong_v S' \cong_w S''$, then there exists a surface $\overline{S}$ such that $S \cong_w \overline{S} \cong_v S''$.*

Define a map $\overline{S}_0\colon \triangle^{(0)} \to X$ by:

$$\overline{S}_0(x) = \begin{cases} S(x) & \text{for } x \neq w \\ S''(w) & \text{for } x = w \end{cases}$$

It extends to the simplicial map $\overline{S}\colon \triangle \to X$ if $d_X(\overline{S}_0(x), \overline{S}_0(w)) \le 1$ for any vertex $x \in \triangle_w$. As S'' and $\overline{S}_0$ coincide at all vertices but at v it suffices to check this condition for $x = v$. Then either $S(v) = S''(w)$ or (denoting $\partial N(v) \cap \partial N(w) = \{a, b\}$) we obtain the square $S(a)S(v)S(b)S''(w)$ in X, which by Fact 2.2 has the diagonal $S(v)S''(w)$ (since S is a locally isometric immersion). In both cases we have $d_X(\overline{S}_0(v), \overline{S}_0(w)) \le 1$.

By Lemma 4.5 the surfaces $S_0, \dots, S_n$ are strong locally isometric immersions. Thus by Step 1 and by the transitivity of $\cong_{v_i}$, we may assume that the vertices $v_1, \dots, v_n$ are pairwise different. To complete the proof we need the following:

Step 2 *If flat surfaces S and S' are locally isometric immersions and are v–equivalent, where $v \in \triangle$ is an internal vertex, then $d_X(S'(v), S(v)) \leq 1$.*

Let a and b be two opposite vertices of $\partial N(v)$. Then $S(a)$ and $S(b)$ are not connected by an edge in X. Thus either $S(v) = S'(v)$ or the square $S(a)S(v)S(b)S'(v)$ has a diagonal (Fact 2.2), so $S(v)$ and $S'(v)$ are at distance at most 1. □

4.2 The fundamental theorem on flat surfaces

In Theorem 4.12 we answer questions (1)–(3) stated in the introduction. The answer to question (2) (if a minimal surface is an isometric embedding) is negative, but we prove a slightly weaker statement: every minimal surface is an almost isometric embedding.

Definition 4.9 Let S be a surface in a systolic complex X. We say that S is an *almost isometric embedding* if

$$d_{\triangle_S}(u, v) = d_X(S(u), S(v))$$

holds for all pairs of vertices $u, v \in \triangle_S$ such that either one of the vertices is internal or the vertices can be connected by a neat geodesic (where a *neat geodesic* in $\triangle_S$ is a geodesic intersecting $\partial \triangle_S$ at most at the endpoints).

Theorem 4.10 *Let S be a wide flat surface in a systolic complex X. If S is a strong locally isometric immersion, then it is an almost isometric embedding.*

Proof Put $\triangle := \triangle_S$. Recall that a neat geodesic in $\triangle$ is a geodesic intersecting $\partial\triangle$ at most at the endpoints.

Step 1 *If $u, v \in \triangle$ can be joined by a neat geodesic, then $d_\triangle(u, v) = d_X(S(u), S(v))$.*

Suppose there exists a surface $\overline{S}$ equivalent to S and vertices $u, v \in \triangle$ which can be joined by a neat geodesic and satisfy:

$$d' := d_X(\overline{S}(u), \overline{S}(v)) < d_\triangle(u, v) =: d. \tag{4-1}$$

Choose u, v and $\overline{S}$ minimizing d. The surface $\overline{S}$ is a strong locally isometric immersion (Lemma 4.5), so $d > 3$. Let $g\colon [0, d] \to \triangle$ be a neat geodesic with endpoints $g(0) = u$ and $g(d) = v$.

Since S is flat, we have $\triangle \subset \mathbb{E}^2_\triangle$, so we can set $g(0) = u$, $g(1) = p$ as in Figure 3 and assume that $g(d) = v$ lies in the shaded sector with vertex q (v obviously can be

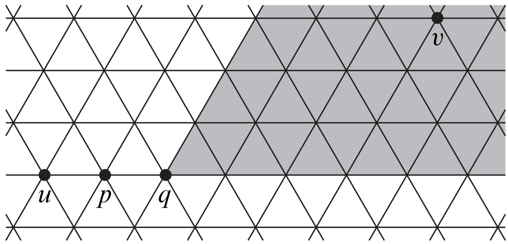

Figure 3

set in the larger sector with vertex p and if v was outside the shaded area, then we would interchange u with v, reversing the direction of the geodesic, as $d > 3$).

By the minimality of d:

$$d_{\triangle}(p, v) = d_X(\overline{S}(p), \overline{S}(v)) = d - 1 \tag{4-2}$$

Since $p \in \triangle$ is an internal vertex (g is neat), we have $N(p) \subset \triangle$, in particular $q \in \triangle$. As $\triangle^{(1)} \subset \mathbb{E}^2_{\triangle}$ is an isometric embedding, q and v can be joined by a geodesic in $\mathbb{E}^2_{\triangle}$ contained in $\triangle$ (but not necessarily neat). Since v lies in the shaded area, we can lengthen this geodesic to a geodesic $\overline{g}\colon [0, d] \to \triangle$ such that $\overline{g}(0) = u$, $\overline{g}(1) = p$, $\overline{g}(2) = q$, $\overline{g}(d) = v$.

Consider the polygonal line $\xi = \overline{S} \circ \overline{g}$ joining the vertices $\overline{S}(u), \overline{S}(v) \in X$ and a geodesic $\zeta\colon [0, d'] \to X$ with the same endpoints. By (4-2) $\xi|_{[1,d]}$ is a geodesic and as $\overline{S}$ is a locally isometric immersion, the vertices $\xi(0) = \overline{S}(u)$, $\xi(1) = \overline{S}(p)$, $\xi(2) = \overline{S}(q)$ are pairwise different. Thus ξ and ζ are injective. The concatenation $\zeta * \xi^{-1}$ need not be injective, but as $\xi|_{[1,d]}$ is a geodesic, the geodesic ζ can be chosen so that for certain $l > l' > 0$ we have $\xi|_{[l,d]} = \zeta|_{[l',d']}$ and $\gamma = \zeta|_{[0,l']} * \xi^{-1}|_{[0,l]}$ is a cycle in X (as in Figure 4). Let D be a minimal surface spanning γ. Choose ζ so that the area of $\triangle_D$ is minimal.

Consider the systolic disc $\triangle_D$. Any vertex on $\zeta([0, l'])$ different from the endpoints has a nonpositive defect (the defect cannot be 2 by the geodesity of ζ and cannot be 1 by the minimality of the area of $\triangle_D$). The sum of the defects along ζ is therefore nonpositive, the sum of the defects at its endpoints is at most 4 and the sum of defects along $\xi([1, l])$ is at most 1 by Fact 3.1. Thus the defect at $\xi(1)$ is not smaller than 1 (by the Gauss–Bonnet Lemma) and is different from 2 (since $\overline{S}$ is a strong locally isometric immersion), so it is equal to 1. Thus by the Gauss–Bonnet Lemma the defects at the vertices of $\zeta([0, l'])$ are equal to 0 and the defect at $\xi(0)$ is equal to 2, as in Figure 4.

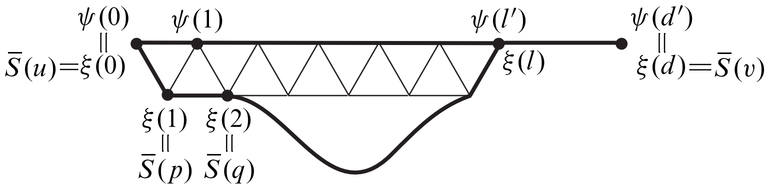

Figure 4

Therefore $\overline{S}(u), \overline{S}(q) \in X_{\zeta(1)}$, so by Fact 4.8 we have $\overline{S}(N(p)) \subset X_{\zeta(1)}$. Thus the simplicial map $S' \colon \triangle \to X$ defined on the 0–skeleton by the formula

$$S'(x) = \begin{cases} \overline{S}(x) & \text{for } x \neq p \\ \zeta(1) & \text{for } x = p \end{cases}$$

is a well-defined surface. However, by (4-1)

$$d_X(S'(p), S'(v)) = d' - 1 < d - 1 = d_\triangle(p, v),$$

contrary to the minimality of d.

Step 2 *If $u, v \in \triangle$ are internal vertices, then they are joined by a neat geodesic in $\triangle$.*

Let g be any geodesic in $\triangle$ joining u and v and let $g_0, \dots, g_d$ be its consecutive vertices. We modify g to obtain a geodesic joining u and v which is disjoint from $\partial\triangle$. For any $g_i \in \partial\triangle$ we apply (if possible) the modification depicted in Figure 5.

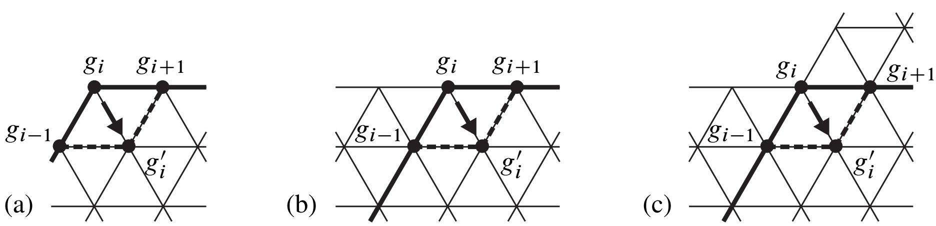

Figure 5: We apply (a) if $\operatorname{def}(g_i) = 1$, (b) if $\operatorname{def}(g_i) = 0$ and either g_{i-1} or g_{i+1} is an internal vertex and (c) if $\operatorname{def}(g_i) = -1$ and both g_{i-1} and g_{i+1} are internal vertices.

Since $\triangle$ is wide, in any of the three cases g'_i is an internal vertex, so every modification decreases the number of vertices in $\operatorname{Im} g \cap \partial\triangle$. Hence we perform finitely many modifications and arrive at the situation, when $\operatorname{Im} g \cap \partial\triangle$ is the union of disjoint segments in $\partial\triangle$ containing no positive vertices and having their endpoints at negative vertices. Since $\triangle$ is flat, it follows from Lemma 3.5 that the intersection is empty.

Step 3 *If $u \in \triangle$ is an internal vertex and $v \in \partial\triangle$, then $d_\triangle(u,v) = d_X(S(u),S(v))$.*

Let $g\colon [0,d] \to \triangle$ be a geodesic joining u and v. Since $\triangle$ is wide, there is an internal vertex $p \in \triangle$ connected with v by an edge. There are 3 cases:

(a) If $d_\triangle(u,p) = d_\triangle(u,v)+1 = d+1$, then we lengthen g to a geodesic g' with both endpoints at internal vertices, by adding the edge vp. By Step 2 and Step 1 of the proof, g' is mapped by S to a geodesic in X and so is g.

(b) If $d_\triangle(u,p) = d-1$, then by Step 2 we can join u and p by a geodesic g' disjoint from $\partial\triangle$. Adding the edge pv we obtain a neat geodesic with endpoints u and v, which by Step 1 is mapped by S to a geodesic in X and so is g.

(c) If $d_\triangle(u,p) = d$, then the link of $\triangle$ at the edge vp consists of two vertices (since p is an internal vertex). One of them is at distance $d-1$ from u and the other, say a, is at distance $d+1$. Adding the edge va we obtain a longer geodesic g', which still has one endpoint at internal vertex u, and we repeat the argument. Since $\triangle$ is a finite complex, after finitely many steps we arrive at the situation from case (a) or (b).

This concludes Step 3 and completes the proof of the theorem. □

It is an important observation that any wide flat surface which is an almost isometric embedding, is an injective map onto a full subcomplex of X (see the corollary below). Therefore, we can treat such surfaces simply as full subcomplexes of X.

Corollary 4.11 *Let S be a wide flat surface (in a systolic complex X) which is an almost isometric embedding. Then:*

(1) *For every pair of vertices $u,v \in \triangle_S$ the following holds:*

$$d_{\triangle_S}(u,v) - 1 \le d_X(S(u),S(v)) \le d_{\triangle_S}(u,v),$$

with equality on the right side if one of u and v is an internal vertex.

(2) *Any geodesic line in $\triangle_S$ which is contained in $\partial\triangle_S$ is mapped by S onto a geodesic in X.*

(3) *The map $S\colon \triangle_S \to X$ is injective and $\operatorname{Im} S \subset X$ is a full subcomplex.*

Proof (1) By Definition 4.9 it suffices to consider the case when $u, v \in \partial \triangle_S$. Denote $d := d_{\triangle_S}(u, v)$. We need to prove that $d_X(S(u), S(v)) \geq d - 1$. Since S is wide, there is an internal vertex $v' \in \triangle_S$ connected with v by an edge. If $d_{\triangle_S}(u, v') \geq d$, then by the triangle inequality and Definition 4.9:

$$d_X(S(u), S(v)) + 1 \geq d_X(S(u), S(v')) = d_{\triangle_S}(u, v') \geq d$$

Otherwise $d_{\triangle_S}(u, v') = d - 1$, so there is a geodesic $g\colon [0, d] \to \triangle_S$ joining u with v such that $g(d-1) = v'$. By the triangle inequality $d_X(S(u), S(v)) \geq d - 2$. If equality holds, then (since by Definition 4.9 we have $d_X(S(u), S(v')) = d - 1$) there is a geodesic ξ in X joining $S(u)$ with $S(v')$ and passing through $S(v)$. The geodesics ξ and $S \circ g|_{[0,d-1]}$ have common endpoints $S(u)$ and $S(v') = S(g(d-1))$. We span a minimal surface D on the concatenation $S \circ g|_{[0,d-1]} * \xi^{-1}$. By Lemma 4.2 the defects at the internal vertices of $\triangle_D$ are nonpositive and by Fact 3.1 the sum of the defects along any of the two boundary geodesics of $\triangle_D$ is at most 1, so by the Gauss–Bonnet Lemma the defect at $D^{-1}(S(v')) \in \partial \triangle_D$ is equal to 2. Thus the vertices $S(v) = S(g(d)) \in X$ and $S(g(d-2)) \in X$ either coincide or are connected by an edge, whereas $g(d-2) \in \triangle_S$ and $g(d) \in \triangle_S$ are at distance 2. This contradicts the fact that the restriction of S to the 1–skeleton of $N(v') = N(g(d-1))$ is an isometric embedding. Thus $d_X(S(u), S(v)) \geq d - 1$.

(2) Since S is a wide flat surface, it has no boundary vertices with defects 2 and any two negative boundary vertices are separated by a positive one (Lemma 3.5). Thus any geodesic g contained in $\partial \triangle_S$ can be lengthened to a geodesic g' contained in $\partial \triangle_S$ so that the sum of defects along g' is equal to 1 (ie the first and the last nonzero vertices on g' have defects 1). Applying to g' the procedure from Step 2 of the proof of Theorem 4.10 (see Figure 5) we obtain a neat geodesic g'' with the same endpoints as g'. Since g'' is mapped by S to a geodesic in X (Definition 4.9), so are g' and g.

(3) By (1) we only need to prove $d_{\triangle_S}(u, v) = d_X(S(u), S(v))$ for any vertices $u, v \in \partial \triangle_S$ such that $d_{\triangle_S}(u, v) \leq 2$. Since S is wide, from (2) and Definition 4.9, any geodesic connecting such vertices either is neat or is contained in $\partial \triangle_S$. □

Theorem 4.12 (Fundamental theorem on flat surfaces) *For a wide flat surface S in a systolic complex X, the following are equivalent:*

(1) *S is a strong locally isometric immersion.*

(2) *S is an almost isometric embedding.*

(3) *S is a minimal surface.*

Moreover, if S is a wide flat minimal surface spanning a cycle γ, then any minimal surface M spanning γ is equivalent to S. In particular, the Hausdorff distance between $\operatorname{Im} S$ *and* $\operatorname{Im} M$ *is at most 1.*

Proof In Theorem 4.10 we proved $(1) \Rightarrow (2)$. The implication $(2) \Rightarrow (1)$ follows immediately from the definitions (Definition 4.3 and Definition 4.9) by using Corollary 4.11(2). By Proposition 4.7 we have $(3) \Rightarrow (1)$. We only need to prove $(1) \Rightarrow (3)$.

Let S be a strong locally isometric immersion (thus by Theorem 4.10, an almost isometric embedding and by Corollary 4.11(3) an injective map). Denote by M a minimal surface spanning the cycle $S(\partial \triangle_S)$. Let $(v_i)_{i=1}^n$ be a permutation of all internal vertices of $\triangle_S$. We construct a sequence $S_0, \dots S_n$ of wide flat surfaces such that:

$$\begin{aligned} S_0 &= S \\ S_i &\cong_{v_i} S_{i-1} \quad \text{for } i = 1, \dots, n \\ S_i(v_i) &\in \operatorname{Im} M \quad \text{for } i = 1, \dots, n \end{aligned} \tag{4-3}$$

Denote $\triangle = \triangle_S = \triangle_{S_i}$ for $i = 1, \dots, n$. Suppose S_i has already been constructed. By Lemma 4.5 S_i is a strong locally isometric immersion, so by Theorem 4.10 and Corollary 4.11(3) it is an injective map onto the full subcomplex $\operatorname{Im} S_i \subset X$. Gluing S_i and M along $S_i|_{\partial\triangle} = M|_{\partial\triangle_M}$ we obtain a map $f\colon P \to X$ from a triangulation P of a sphere (it is simplicial, since $\triangle$ is wide). By Theorem 2.4 f can be extended to $F\colon B \to X$, where B is a triangulation of a ball that has no internal vertices and satisfies $\partial B = P$. The link $P_{v_{i+1}}$ is a cycle of length 6 and the link $B_{v_{i+1}}$ is a simplicial disc (not necessarily systolic) such that $\partial B_{v_{i+1}} = P_{v_{i+1}}$. Since S_i is injective and $\operatorname{Im} S_i \subset X$ is a full subcomplex, any internal vertex $w \in B_{v_{i+1}}$ lies in $\triangle_M \subset P$, so is mapped by F into $\operatorname{Im} M$.

To complete the proof we need the following lemma, which will be proved later:

Lemma 4.13 *Let X be a systolic complex and $\triangle$ a simplicial disc (not necessarily systolic) of perimeter 6. If there is a simplicial map $f\colon \triangle \to X$ such that $f|_{\partial\triangle}$ is an isomorphism onto a cycle in X having no diagonals, then there exists an internal vertex $w \in \triangle$ such that $f(\partial\triangle) \subset X_{f(w)}$.*

Applying the lemma to $F|_{B_{v_{i+1}}}$ (the assumptions are satisfied, since S_i is a locally isometric immersion) we obtain an internal vertex $w \in B_{v_{i+1}}$ such that the simplicial map defined on the 0–skeleton by

$$S_{i+1}(x) = \begin{cases} S_i(x) & \text{for } x \neq v_{i+1} \\ F(w) & \text{for } x = v_{i+1} \end{cases}$$

extends to a surface. Clearly S_{i+1} satisfies (4-3).

The last surface in the sequence, S_n, by (4-3) maps the set of internal vertices of $\triangle$ injectively into the set

$$\{M(w) : \ w \in \triangle_M \text{ is an internal vertex}\}$$

(S_n is an almost isometric embedding by Lemma 4.5 and Theorem 4.10 and is injective by Corollary 4.11(3)). Thus, since M is a minimal surface spanning the same cycle as S_n, by Pick's Formula $\triangle_M$ has not more internal vertices than $\triangle$ has. It follows that S_n maps $\triangle^{(0)}$ bijectively onto $(\operatorname{Im} M)^{(0)}$ and M is injective. As $\operatorname{Im} S_n \subset X$ is a full subcomplex (Corollary 4.11(3)), we have $\operatorname{Im} M \subset \operatorname{Im} S_n$. But both $\operatorname{Im} M$ and $\operatorname{Im} S_n$ are simplicial discs and they have the common boundary, so $\operatorname{Im} M = \operatorname{Im} S_n$. Moreover, $\triangle_M \cong \operatorname{Im} M = \operatorname{Im} S_n \cong \triangle$. Therefore, identifying $\triangle_M$ with $\triangle$ we obtain $M = S_n$, so S_n is a minimal surface and so is S.

As the above construction shows, if S is a wide flat minimal surface spanning a cycle γ, then any minimal surface M spanning γ is equivalent to S. In particular, the Hausdorff distance between $\operatorname{Im} S$ and $\operatorname{Im} M$ is at most 1 (Lemma 4.6). $\square$

Proof of Lemma 4.13 We modify f to $f'\colon \triangle' \to X$, where $\triangle'$ is a systolic disc such that $\partial\triangle = \partial\triangle'$, the internal vertices of $\triangle'$ are vertices of $\triangle$, and $f|_{\partial\triangle} = f'|_{\partial\triangle'}$. If $\triangle$ contains a cycle γ of length 3 not bounding a triangle in $\triangle$, then we cut out the disc of $\triangle$ bounded by γ and glue in a single triangle. By the flagness of X we modify f. If $\triangle$ does not contain such cycles and is not systolic, then there is an internal vertex $v \in \triangle$ adjacent to 4 or 5 triangles. Then we modify $\triangle$ by cutting out the open star of v and gluing a simplicial disc with no internal vertices, such that f can be extended over the new triangulation (this is possible by the systolicity of X). These operations decrease the number of internal vertices of $\triangle$, so the procedure terminates, producing a systolic disc $\triangle'$ such that $\partial\triangle = \partial\triangle'$ and a simplicial map $f'\colon \triangle' \to X$ which extends $f|_{\partial\triangle}$.

Nonconsecutive vertices of $\partial\triangle = \partial\triangle'$ are not connected by an edge, because $f(\partial\triangle)$ has no diagonals. Moreover, by the isoperimetric inequality (Lemma 3.4) the area of $\triangle'$ is at most 6, so by Pick's Formula $\triangle'$ has at most one internal vertex. Therefore $\triangle' = \partial\triangle' * w$, where $w \in \triangle'$ is the only internal vertex. As our procedure did not add any new vertices, w is an internal vertex of $\triangle$ and $f(w) = f'(w)$. Moreover, since $f'(\partial\triangle') = f(\partial\triangle)$ is a cycle in X with no diagonals, we have $f'(w) \notin f(\partial\triangle')$, so $f(\partial\triangle) \subset X_{f(w)}$. $\square$

Theorem 4.12 gives an alternative proof of the next theorem, proved in Przytycki [10] (for the definition of a flat see Section 5):

Corollary 4.14 *Let X be a systolic complex, admitting a simplicial, cocompact and properly discontinuous action of a group G. Then X is Gromov-hyperbolic if and only if it does not contain a flat.*

To prove the corollary we need the following lemma:

Lemma 4.15 *Let $\triangle$ be a systolic disc and $\gamma \subset \partial\triangle$ a geodesic in $\triangle$.*

(1) *Denote by $\triangle' \subset \triangle$ the subcomplex obtained by cutting out the open stars of all the vertices $v \in \gamma$. Then $\mathrm{hdist}_{\triangle}(\triangle, \triangle') = 1$ and either $\triangle'$ has a disconnecting vertex or it is a systolic disc such that $\gamma' := \partial\triangle' \setminus \partial\triangle$ is a geodesic.*

(2) *If $\partial\triangle$ is the concatenation of geodesics α, β and γ, then for any natural c holds $\gamma \subset \mathcal{N}_{2c}(\alpha \cup \beta \cup \triangle_c)$, where $\triangle_c \subset \triangle$ is the subcomplex spanned by all the vertices $v \in \triangle$ which satisfy $\mathrm{dist}(v, \partial\triangle) \geq c$.*

Proof Suppose $\triangle'$ has no disconnecting vertices and γ' is not a geodesic in $\triangle'$. Let $v', w' \in \gamma' \subset \triangle'$ be the endpoints of the shortest segment of γ' which is not a geodesic in $\triangle'$. Connect v' and w' by a geodesic g' in $\triangle'$, choose vertices $v, w \in \gamma$ connected by edges with v' and w', respectively. Let v, w and g' be such that the subcomplex $D \subset \triangle$ bounded by the loop $g^{-1} * vv' * g' * ww'$, where g is the segment of γ with endpoints v and w, has the minimal area. As D is a systolic disc, by Fact 3.1 the sum of its defects along g is at most 1, by the minimality of its area the sum of the defects along g' is nonpositive, the defects at v, w, v' and w' are at most 1 (by the minimality of the length of $[v', w'] \subset \gamma'$ and the minimality of the area of D), which gives a contradiction to the Gauss–Bonnet Lemma. This proves (1).

To prove (2) it suffices to show that $\gamma \subset \mathcal{N}_c(\alpha \cup \beta \cup \triangle_c^\gamma)$, where $\triangle_c^\gamma \subset \triangle$ denotes the subcomplex spanned by all the vertices $v \in \triangle$ satisfying $\mathrm{dist}(v, \gamma) \geq c$. We proceed by induction on c using (1) and applying the inductive assumption to maximal subcomplexes of $\triangle'$ having no disconnecting vertices. □

Proof of Corollary 4.14 Suppose X is not Gromov-hyperbolic. Then for every n there exists a loop being the concatenation of three geodesics α_n, β_n, γ_n such that $\gamma_n \not\subset \mathcal{N}_n(\alpha_n \cup \beta_n)$. Let S_n be a minimal surface spanning this loop. Thus by Lemma 4.15(2) there exists a vertex $v \in \triangle_{S_n}$ such that $\mathrm{dist}_{\triangle_{S_n}}(v, \partial\triangle_{S_n}) \geq \frac{n}{2}$. Since by the Gauss–Bonnet Lemma and Fact 3.1, there are at most 3 negative internal vertices in $\triangle_{S_n}$, there is a vertex w on a geodesic joining v with the closest vertex on $\partial\triangle_{S_n}$, such that $\mathcal{N}_{\frac{1}{8}\cdot\frac{n}{2}}(w)$ does not contain a negative internal vertex, so it is an equilaterally triangulated regular hexagon of side length $[\frac{1}{8}\cdot\frac{n}{2}]$. The 1–skeleton of the hexagon is isometrically embedded into $\triangle_{S_n}$, so by Theorem 4.12 is isometrically embedded into X. Thus by the cocompactness of the action of G and by the standard diagonal argument (X is uniformly locally finite, since G acts cocompactly and properly discontinuously), there is a flat in X. □

4.3 The stability of minimal surfaces

Now we answer question (4) from the introduction, proving the stability of minimal surfaces under small modifications of their boundaries. The next theorem concerns more general situation than wide flat surfaces, namely injective maps whose images are full subcomplexes of X (by Corollary 4.11(3) any wide flat surface is such). We expect the stability of minimal surfaces holds in full generality, ie that the assumption on S and S' to be injective maps onto full subcomplexes is unnecessary.

To formulate the theorem we need to define the function measuring how much one of the cycles has to be deformed to obtain the other cycle. Given cycles γ and γ' in a systolic complex X, we denote by $d(\gamma,\gamma')$ the minimum of

$$\max\{d_X(f(v), f'(v)),\ v \in C^{(0)}\}$$

taken over all triangulations C of a circle and over all simplicial maps $f\colon C \to \gamma$ and $f'\colon C \to \gamma'$ that are surjective and monotonous (ie the counterimages of the vertices on γ or γ' are segments in C).

Theorem 4.16 *Let γ and γ' be cycles in a systolic complex X with $d(\gamma,\gamma') = c$ and let S and S' be minimal surfaces spanning them. If S and S' are injections and* $\operatorname{Im} S$, $\operatorname{Im} S'$ *are full subcomplexes of X, then:*

(1) $\operatorname{hdist}_X(\operatorname{Im} S, \operatorname{Im} S') \le c+1$.

(2) *If S is a flat surface and $w \in \triangle_S^{(0)}$ satisfies* $\operatorname{dist}(w, \partial\triangle_S) > c+1$, *then for some surface $\overline{S}$ which is w–equivalent to S we have* $\overline{S}(w) \in \operatorname{Im} S'$. *In particular,* $S(w) \in \mathcal{N}_1(\operatorname{Im} S')$.

Proof Choose C, f and f' realizing $d(\gamma,\gamma')$ and denote consecutive vertices of C by $t_1,\dots,t_n$. Choose geodesics g_i, $i=1,\dots,n$ in X joining $f(t_i) = v_i \in \gamma$ with $f'(t_i) = v_i' \in \gamma'$ (we allow g_i to be a single vertex). The concatenation $v_{i+1}v_i * g_i * v_i'v_{i+1}' * g_{i+1}^{-1}$ (we use the cyclic order of indices) is a closed path in X, so by Lemma 4.2 there is a simplicial map $s_i\colon D_i \to X$ from a systolic disc D_i mapping ∂D_i onto this path.

Step 1 *For any vertex $w \in \triangle_S$ we have $S(w) \in \mathcal{N}_1(\operatorname{Im} S' \cup \operatorname{Im} s_1 \cup \dots \cup \operatorname{Im} s_n)$. Moreover, if S is flat and $w \in \triangle_S$ is an internal vertex, then there is a surface $\overline{S} \cong_w S$ such that $\overline{S}(w) \in \operatorname{Im} S' \cup \operatorname{Im} s_1 \cup \dots \cup \operatorname{Im} s_n$.*

We glue maps S, S' and $s_1, \dots, s_n$ to obtain a simplicial map $f\colon P \to X$, where P is a triangulation of a sphere. It can be extended to $F\colon B \to X$ for some triangulation B of a ball such that $\partial B = P$ and B has no internal vertices (Theorem 2.4). In the case $w \in \partial \triangle_S$ the statement is immediate. Thus consider the case when w is an internal vertex of $\triangle_S$. As $w \in \triangle_S \subset P \subset B$, consider the link B_w – it is a filling of the cycle P_w. Since S is injective and $\operatorname{Im} S \subset X$ is a full subcomplex, B_w has at least one internal vertex and internal vertices of B_w are disjoint from $\triangle_S \subset P$. Thus $S(w) \in \mathcal{N}_1(\operatorname{Im} S' \cup \operatorname{Im} s_1 \cup \dots \cup \operatorname{Im} s_n)$. If S is a flat surface and $w \in \triangle_S$ an internal vertex, then by Lemma 4.13 there is a surface $\overline{S}$ which is w–equivalent to S such that $\overline{S}(w) \in \operatorname{Im} S' \cup \operatorname{Im} s_1 \cup \dots \cup \operatorname{Im} s_n$.

Step 2 *Let D be a systolic disc and let $a_1, a_2, b_1, b_2 \in \partial D$ be such vertices that ∂D is the concatenation of the edge $a_1 a_2$ (or the vertex a_1, if $a_1 = a_2$), the edge $b_1 b_2$ (or the vertex b_1, if $b_1 = b_2$) and geodesics $[a_1, b_1]$ and $[a_2, b_2]$. Then D is spanned by all geodesics joining a_i with b_j for $i, j = 1, 2$.*

We proceed by induction on the area of D. The statement is trivial when D is a single triangle. If there is a vertex $v \in (a_i, b_i) \subset \partial D$ of positive defect (ie of defect 1, by the geodesity of $[a_i, b_i]$), then we cut out two triangles adjacent to v obtaining either a smaller disc D' or two discs D' and D'' intersecting at a single vertex and apply the inductive assumption.

If $a_1 \neq a_2$ and the defect at a_i is 2, then we cut out the only triangle adjacent to a_i and apply the inductive assumption. We proceed similarly with b_1 and b_2.

If none of the above cases occur, then the defects at a_1 and a_2 are not greater than 1 (if $a_1 \neq a_2$) or the defect at $a_1 = a_2$ is not greater than 2 and similarly with b_1 and b_2, and the sum of the defects along the geodesic $[a_i, b_i]$ is nonpositive, for $i = 1, 2$. Thus the sum of the defects at vertices on ∂D does not exceed 4, contrary to the Gauss–Bonnet Lemma.

Step 3 $\operatorname{Im} S \subset \mathcal{N}_{c+1}(\operatorname{Im} S')$ *and* $\operatorname{Im} S' \subset \mathcal{N}_{c+1}(\operatorname{Im} S)$. *If S is flat and $w \in \triangle_S$ is an internal vertex such that $\operatorname{dist}(w, \partial \triangle_S) > c+1$, then $\overline{S}(w) \in \operatorname{Im} S'$ for some surface $\overline{S}$ w–equivalent to S.*

By Step 2, $\operatorname{Im} s_i \subset \mathcal{N}_c(\gamma')$ for $i = 1, \dots, n$, so by Step 1 we have $\operatorname{Im} S \subset \mathcal{N}_{c+1}(\operatorname{Im} S')$. Similarly we obtain $\operatorname{Im} S' \subset \mathcal{N}_{c+1}(\operatorname{Im} S)$. The second statement from Step 3 follows from Lemma 4.13 applied to $S|_{B_w}$ and the fact that $\operatorname{Im} s_i \subset \mathcal{N}_c(\gamma) \subset \mathcal{N}_c(S(\partial \triangle_S))$. □

The following corollary provides the answer to question (3) from the introduction in a more general case than Theorem 4.12 does.

Corollary 4.17 *If S and S' are minimal surfaces which are injections onto full subcomplexes spanning the same cycle, then the Hausdorff distance between them is at most 1.*

5 Flats in systolic complexes

A *flat* in a systolic complex X is a simplicial map $F: \mathbb{E}^2_\triangle \to X$ which when restricted to the 1–skeleton of $\mathbb{E}^2_\triangle$ is an isometric embedding. Sometimes we will identify F with its image and treat F as a subcomplex of X.

Definition 5.1 Two flats F and F' in a systolic complex X are called *equivalent* if they are at finite Hausdorff distance.

The above definition is different from the one for flat surfaces (Definition 4.4). However, in Lemma 5.3 we provide a characterization of the flat equivalence similar to the flat surfaces equivalence. In Theorem 5.4 we show that the Hausdorff distance between equivalent flats is actually at most 1 and there is a unique simplicial retraction onto F of the subcomplex of X spanned by all flats equivalent to F.

Now we restate the main theorem from Section 4, namely Theorem 4.12, for flats. In order to do it we generalize the notions of a *locally isometric immersion* and a *strong locally isometric immersion* for flats by replacing the triangulated disc $\triangle_S$ with the flat systolic plane $\mathbb{E}^2_\triangle$ in Definition 4.3.

Theorem 5.2 *Let X be a systolic complex and $F: \mathbb{E}^2_\triangle \to X$ a simplicial map.*

(1) *If F is a strong locally isometric immersion, then F is a flat.*

(2) *If F is a locally isometric immersion and* $\operatorname{diam}(\operatorname{Im} F) \geq 3$, *then F is a flat.*

Proof Part (1) of the theorem follows from Theorem 4.12 applied to $F|_{\triangle_n}$ for a sequence of regular hexagons $\triangle_n \subset \mathbb{E}^2_\triangle$. To prove (2) we need to show that under the additional assumption $\operatorname{diam}(\operatorname{Im} F) \geq 3$, a locally isometric immersion is actually a strong locally isometric immersion.

Suppose F is a locally isometric immersion, but not a strong locally isometric immersion. Then by Proposition 4.7 there is an edge $uv \subset \mathbb{E}^2_\triangle$ such that $F|_{\partial N(uv)}$ can be extended to a surface S ($\partial \triangle_S = \partial N(uv)$) so that $\triangle_S$ has at most one internal vertex (Pick's Formula). Thus either $\partial \triangle_S$ has a diagonal joining two nonconsecutive vertices (which contradicts the fact that F is a locally isometric immersion), or $\triangle_S = w * \partial \triangle_S$, where $w \in \triangle_S$ is the only internal vertex. Define $x = S(w) \in X$ and put:

$$\triangle_n = \begin{cases} N(uv) & \text{if } n = 0 \\ N(\triangle_{n-1}) & \text{if } n \geq 1 \end{cases}$$

Proceeding by induction we prove that $F(\triangle_n) \subset X_x$, for every $n \geq 0$.

(i) We already know that $F(\partial \triangle_0) \subset X_x$, so applying Fact 4.8 to hexagons $N(u)$ and $N(v)$ we obtain $F(\triangle_0) \subset X_x$.

(ii) Suppose $F(\triangle_{n-1}) \subset X_x$. Denote successive vertices of $\partial \triangle_n$ by $b_1, \ldots, b_k$ so that b_1 has defect 0. By induction on i we obtain $b_i \in X_x$ for $i = 1, \ldots, k$. It follows from Fact 4.8 applied to a hexagon with the centre and two opposite vertices on $\partial \triangle_{n-1}$ (in the case $i = 1$) or to a hexagon with the centre on $\partial \triangle_{n-1}$, vertex b_{i-1} and the opposite vertex in $\triangle_{n-1}$. (in the case $i > 1$). Thus the image of $\triangle_n$ is contained in X_x.

It follows that $\operatorname{Im} F \subset X_x$, hence the diameter of $\operatorname{Im} F$ is not greater than 2, contrary to the assumption. $\square$

We define for two flats a relation $\cong_v$, similar to that from Definition 4.4: flats F and F' are v–equivalent if $F(x) = F'(x)$ for all vertices $x \in \mathbb{E}^2_\triangle$ different from v.

Lemma 5.3 *Let F and F' be equivalent flats in a systolic complex X. Then there exist a sequence of vertices $v_1, v_2, \dots \in \mathbb{E}^2_\triangle$ and a sequence of flats $F = F_0, F_1, F_2, \ldots$ such that:*

- *$F_i \cong_{v_i} F_{i-1}$ for $i = 1, 2, 3, \ldots$;*
- *the flat $F'' = \lim_{n\to\infty} F_n$ (pointwise convergence) has the same image as F'.*

Moreover, we can choose $(v_i)_{i=1}^\infty$ to be an arbitrary permutation of the vertices of $\mathbb{E}^2_\triangle$.

Since $X^{(0)}$ is a discrete space, the pointwise convergence of flats is equivalent to the condition that the sequence $F_0(v), F_1(v), \dots \in X$ stabilizes for any vertex $v \in \mathbb{E}^2_\triangle$. We therefore prove that equivalent flats are obtained from each other by a (possibly infinite) sequence of small deformations such that on every compact subcomplex $K \subset \mathbb{E}^2_\triangle$ only finitely many of them are applied.

Proof Let $(v_i)_{i=1}^{\infty}$ be any permutation of the vertices of $\mathbb{E}^2_\Delta$. We construct a sequence of flats $(F_i)_{i=0}^{\infty}$ such that:

$$\begin{aligned} F_0 &= F \\ F_i &\cong_{v_i} F_{i-1} \qquad \text{for } i = 1, 2, \dots \\ F_i(v_i) &\in \operatorname{Im} F' \qquad \text{for } i = 1, 2, \dots \end{aligned} \tag{5-1}$$

Suppose we have already constructed F_n. Denote $c = \operatorname{hdist}_X(F_n, F') < \infty$ (it is finite, since F_n and F' are equivalent). If $F_n(v_{n+1}) \in \operatorname{Im} F'$, then we put $F_{n+1} := F_n$. Otherwise, consider the regular hexagon $H \subset \mathbb{E}^2_\Delta$ of side length $40c$ with centre v_{n+1}. Denote by $a_1, \dots, a_6$ the images by F_n of the vertices of H and by $\sigma_1, \dots, \sigma_6$ the images by F_n of its sides. Let ξ_i be the shortest geodesic joining a_i with F' and denote its second endpoint by $b_i \in F'$ (we allow $b_i = a_i$). Since flats are isometric embeddings, we can join b_i with b_{i+1} by a geodesic τ_i contained in F' for $i = 1, \dots, 6$ (we use the cyclic order of indices). By Lemma 4.2 there exist simplicial maps:

- $h'\colon H' \to X$, where H' is a systolic disc and h' maps $\partial H'$ onto the closed path $\tau_1 * \cdots * \tau_6$ so that $\operatorname{Im} H' \subset \operatorname{Im} F'$;
- $s_i\colon D_i \to X$, where D_i is a systolic disc and s_i maps ∂D_i onto the closed path $\xi_i * \tau_i * \xi_{i+1}^{-1} * \sigma_i^{-1}$ for $i = 1, \dots, 6$.

Gluing $F_n|_H$, h' and $s_1, \dots, s_6$ we obtain a simplicial map $p\colon S \to X$ from certain triangulation S of a sphere. By Theorem 2.4 we extend it to $P\colon B \to X$, where B is a triangulation of a ball that has no internal vertices and satisfies $\partial B = S$. Thus $B_{v_{n+1}}$ is a simplicial disc of perimeter 6 (as the link $S_{v_{n+1}}$ is a cycle of length 6). Applying Lemma 4.13 to $P|_{B_{v_{n+1}}}$ we obtain an internal vertex $y \in B_{v_{n+1}}$ such that $P(\partial B_{v_{n+1}}) \subset X_{P(y)}$. We put $F_{n+1}\colon \mathbb{E}^2_\Delta \to X$ to be the simplicial map defined on the 0–skeleton by:

$$F_{n+1}(x) = \begin{cases} F_n(x) & \text{for } x \neq v_{n+1} \\ P(y) & \text{for } x = v_{n+1} \end{cases}$$

The map F_{n+1} coincides with the flat F_n at all vertices but v_{n+1} and for any vertex $w \in \mathbb{E}^2_\Delta$ there is a vertex $w' \in \mathbb{E}^2_\Delta$ and a geodesic joining w with w' that passes through v_{n+1}, so:

$$\begin{aligned} d(w, w') &= d(F_{n+1}(w), F_{n+1}(w')) \\ &\leq d(F_{n+1}(w), F_{n+1}(v_{n+1})) + d(F_{n+1}(v_{n+1}), F_{n+1}(w')) \\ &\leq d(w, v_{n+1}) + d(v_{n+1}, w') = d(w, w') \end{aligned}$$

Thus all inequalities are actually equalities, so F_{n+1} is a flat.

To see that F_{n+1} satisfies (5-1) we need to prove that $P(y) \in \operatorname{Im} F''$. Since $F_n|_H$ is an isometric embedding, $y \in B_{v_{n+1}}$ is contained in $H' \cup D_1 \cup \dots \cup D_6$. Moreover, by Lemma 3.4:

$$D_i \subset \mathcal{N}_{\frac{1}{6}|\xi_i * \tau_i * \xi_{i+1}^{-1} * \sigma_i^{-1}|}(\partial D_i) \subset \mathcal{N}_{\frac{1}{6}(c+42c+c+40c)}(\partial D_i) = \mathcal{N}_{14c}(\partial D_i) \subset \mathcal{N}_{36c}(\partial H)$$

so $P(D_i) \subset \mathcal{N}_{36c}(P(\partial H))$, while

$$\operatorname{dist}(y, P(\partial H)) \geq \operatorname{dist}(F_n(v_n), F_n(\partial H)) - 1 = 40c - 1.$$

Thus $y \notin D_i$, for $i = 1, \dots, 6$, so $y \in H'$ and therefore $P(y) \in \operatorname{Im} F''$.

The flat $F'' = \lim F_n$ satisfies $\operatorname{Im} F'' \subset \operatorname{Im} F'$, hence $\operatorname{Im} F'' = \operatorname{Im} F'$ (as $\mathbb{E}^2_\triangle$ is not isomorphic to a proper subcomplex). □

5.1 Thickenings of flats

Theorem 5.4 *Let F be a flat in a systolic complex X. Denote by $\operatorname{Th}(F) \subset X$ (the thickening of F) the full subcomplex spanned by all flats at finite Hausdorff distance from F. Then:*

1. *Every maximal simplex of $\operatorname{Th}(F)$ has nonempty intersection with F.*
2. *There is a unique simplicial retraction $r\colon \operatorname{Th}(F) \to F$. Moreover, the restriction of r to any flat $F' \subset \operatorname{Th}(F)$ is an isometry.*
3. *Every map $s\colon F^{(0)} \to \operatorname{Th}(F)$ such that $r \circ s = \mathrm{id}_{F^{(0)}}$ extends to a flat and every flat in $\operatorname{Th}(F)$ is of this form. Moreover, $r^{-1}(v)$ is a simplex in X for any vertex $v \in F$.*

Proof For a vertex $v \in \mathbb{E}^2_\triangle$ we denote by σ_v the simplex spanned by the vertices $F'(v)$ for all flats F' that are v–equivalent to F (these vertices span a simplex by Fact 2.2). Clearly $\sigma_v \subset \operatorname{Th}(F)$. Notice that by Lemma 5.3 for any flat $F' \subset \operatorname{Th}(F)$ and for any vertex $v \in \mathbb{E}^2_\triangle$ there is a flat F_1 such that $F_1 \cong_v F$ and $F_1(v) = F'(v)$. Hence $\operatorname{Th}(F)$ is spanned by σ_v for $v \in \mathbb{E}^2_\triangle$.

If $F' \cong_v F \cong_w F''$ for some distinct vertices $v, w \in \mathbb{E}^2_\triangle$, then by Lemma 5.3 (applied to F' and F'') there exists a flat $\overline{F}$ such that $F' \cong_w \overline{F} \cong_v F''$. Thus:

$$\overline{F}(x) = \begin{cases} F(x) & \text{for } x \neq v, w \\ F'(v) & \text{for } x = v \\ F''(w) & \text{for } x = w \end{cases}$$

Since $\overline{F}$ when restricted to the 1–skeleton of $\mathbb{E}^2_\triangle$ is an isometric embedding, we have

$$d_X\big(F'(v), F''(w)\big) = d_{\mathbb{E}^2_\triangle}(v, w).$$

Hence there are no edges joining σ_v with σ_w if $v \neq w \in \mathbb{E}^2_\Delta$ are not connected by an edge and every vertex of σ_v is connected by an edge with every vertex of σ_w if $v, w \in \mathbb{E}^2_\Delta$ are connected by an edge. Thus:

$$\begin{aligned}
\big(\mathrm{Th}(F)\big)^{(0)} &= \bigcup_{v\in\mathcal{V}} (\sigma_v)^{(0)} \\
\big(\mathrm{Th}(F)\big)^{(1)} &= \bigcup_{uv\in\mathcal{E}} (\sigma_u * \sigma_v)^{(1)} \qquad \text{(5-2)}\\
\mathrm{Th}(F) &= \bigcup_{uvw\in\mathcal{T}} (\sigma_u * \sigma_v * \sigma_w)
\end{aligned}$$

where $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{T}$ denote the sets of vertices, edges and triangles of $\mathbb{E}^2_\Delta$, respectively. This implies (1), as the maximal simplices of $\mathrm{Th}(F)$ are $\sigma_u * \sigma_v * \sigma_w$, where $uvw \in \mathcal{T}$.

Any map $s\colon F^{(0)} \to \mathrm{Th}(F)$ such that $s(v) \in \sigma_v$ for any $v \in F^{(0)}$ extends to an injective map $S\colon F \to \mathrm{Th}(F)$ such that $\operatorname{Im} S \subset X$ is a full subcomplex. By Theorem 5.2(2) S is a flat and by Lemma 5.3 every flat in $\mathrm{Th}(F)$ has this form.

Let $r\colon \mathrm{Th}(F) \to F$ be a simplicial retraction. For every vertex $p \in \sigma_v$ there is a flat $\overline{F} \cong_v F$ such that $\overline{F}(v) = p$. Since $r|_F = \mathrm{id}_F$ and r is simplicial, $r(p) = v$. Thus $r(\sigma_v) = v$, for every $v \in F$. Clearly the function mapping σ_v to v for any $v \in F$ has a simplicial extension to the unique simplicial retraction $r\colon \mathrm{Th}(F) \to F$, which when restricted to any flat is an isometry. This completes the proof of (2) and (3). $\square$

Corollary 5.5 *The action of any group G on the thickening $\mathrm{Th}(F)$ induces an action of G on $\mathbb{E}^2_\Delta$. Moreover, if $\mathrm{Th}(F)$ is locally finite and the action is properly discontinuous, then so is the induced action on $\mathbb{E}^2_\Delta$.*

Proof Denote by $r_F\colon \mathrm{Th}(F) \to F$ the retraction constructed in Theorem 5.4 and by $a_g\colon \mathrm{Th}(F) \to \mathrm{Th}(F)$ the action of $g \in G$ on $\mathrm{Th}(F)$. Notice that by (5-2) the 1–skeleta of σ_v (for $v \in \mathbb{E}^2_\Delta$) are precisely the connected components of the subgraph of $\mathrm{Th}(F)$ consisting of the edges that cannot be extended to geodesics of length 2 inside $\mathrm{Th}(F)$. Thus a_g permutes the simplices σ_v and we can define the action of G on $\mathbb{E}^2_\Delta$ by:

$$G \ni g \mapsto (r_F \circ g)|_F \in \mathrm{Aut}(F) \cong \mathrm{Aut}\big(\mathbb{E}^2_\Delta\big).$$

We need to show that $(r_F \circ g') \circ (r_F \circ g) = r_F \circ (g'g)$ for any $g, g' \in G$. Both maps restrict to the same isometry $\varphi\colon g^{-1}(F) \to F$. Thus:

$$\varphi^{-1} \circ (r_F \circ g') \circ (r_F \circ g) = \varphi^{-1} \circ r_F \circ (g'g)$$

as by Theorem 5.4 there is a unique simplicial retraction of $\mathrm{Th}(F) = \mathrm{Th}(g^{-1}(F))$ onto the flat $g^{-1}(F)$, which completes the proof of the main part of the corollary. The second part follows from the fact that $r_F^{-1}(v)$ is a simplex in $\mathrm{Th}(F)$ for any vertex $v \in F$ and $\mathrm{Th}(F)$ is locally finite. □

6 Flat Torus Theorem

In this section we study virtually abelian subgroups of rank at least 2 in systolic groups. Actually, systolic groups do not contain abelian subgroups of rank greater than 2 (see Januszkiewicz–Świątkowski [9, Corollary 5.5]; we present an alternative proof in Theorem 6.1(1)), so we are mainly interested in actions of $\mathbb{Z}^2$ on systolic complexes. Actions of finite extensions of $\mathbb{Z}^2$ are described in Corollary 6.2.

Let X be a simplicial complex and G a group acting on X by simplicial automorphisms. Recall that G acts *cocompactly* if there is a compact subset $K \subset X$ intersecting every orbit of the action, and *properly discontinuously* if the stabilizer of any vertex $v \in X$ is finite (this is a weaker condition than the usual definition for metric spaces, but for simplicial complexes it is equivalent to the standard one).

If X admits a cocompact, properly discontinuous action of a group, then it is uniformly locally finite (ie there is a finite upper bound for the valences of its vertices). Thus the action of G is cocompact if and only if there are finitely many orbits of vertices.

For any $g \in G$ we define $\mathrm{Min}(g)$ to be the subcomplex spanned by the vertices $x \in X$ realizing the minimal displacement of g, ie satisfying $d(x, g(x)) = \min_{y \in X} d(y, g(y))$. We also define:

$$\mathrm{Min}(G) = \bigcap_{g \in G} \mathrm{Min}(g)$$

We show that $\mathrm{Min}(G)$ is nonempty for $G \cong \mathbb{Z}^2$ acting properly discontinuously on a systolic complex X. In fact, we prove that $\mathrm{Min}(G)$ is the thickening of a G–invariant flat. This result is a systolic analogue of the Flat Torus Theorem for CAT(0)–spaces (see Bridson–Haefliger [1]).

Theorem 6.1 (Flat Torus Theorem) *Let G be a noncyclic free abelian group acting simplicially, properly discontinuously on a uniformly locally finite systolic complex X. Then:*

(1) *G is isomorphic to $\mathbb{Z}^2$.*

(2) *There is a G–invariant flat $F \subset X$, unique up to the flat equivalence.*

(3) *$\mathrm{Min}(G)$ is nonempty and is equal to the thickening of the G–invariant flat.*

Proof Since G is torsion-free and acts properly discontinuously, the action is also free. In Steps 1–4 we prove the theorem for $G \cong \mathbb{Z}^2$. In Step 5 we complete the proof.

Step 1 *There exists an H–invariant flat in X for a certain finite-index subgroup $H < G$.*

Choose a vertex $x \in X$ and elements $g, h \in G$ generating G. Connect x with $g(x)$ and $h(x)$ by geodesics α and β, respectively, and denote by γ the closed path being the concatenation $\alpha * g(\beta) * h(\alpha^{-1}) * \beta^{-1}$. By Lemma 4.2 there is a map $f\colon \triangle \to X$, where $\triangle$ is a systolic disc, mapping $\partial\triangle$ onto γ.

Denote by Y the full subcomplex of X spanned by the orbits of all vertices of $f(\triangle)$. Then Y is G–invariant and G acts freely and cocompactly on Y. Thus by the local finiteness of Y there is a finite-index subgroup $H < G$ generated by g^n and h^n for some n such that:

$$\min\{d_Y(y, p(y)) : \ p \in H \setminus \{1\}, \ y \in Y^{(0)}\} > 3, \tag{6-1}$$

so the quotient space Y/H is a flag simplicial complex. Since the links of Y/H are isomorphic to the links of Y, the quotient complex is locally 6–large.

By the construction of Y, $x \in Y$ and there are such geodesics α' and β' joining x with $g^n(x)$ and $h^n(x)$, respectively, that there exists a simplicial map $f\colon \triangle' \to Y$, where $\triangle'$ is a simplicial disc mapping $\partial\triangle'$ to the concatenation $\alpha' * g^n(\beta') * h^n(\alpha'^{-1}) * \beta'^{-1}$. This gives us a map $f'\colon T \to Y/H$, where T is a triangulation of a torus. The following diagram of simplicial maps commutes:

$$\begin{tikzcd}
\widetilde{T} \arrow[r, "\widetilde{f'}"] \arrow[dd] & \widetilde{Y} \arrow[dd] \arrow[dr] & & \\
 & & Y \arrow[r, hook] \arrow[dl] & X \\
T \arrow[r, "f'"] & Y/H & &
\end{tikzcd}$$

where $\widetilde{T}$ is the universal covering of T, ie a triangulation of a plane (not necessarily systolic). Now we modify T to a systolic triangulation, by applying three types of modifications:

(a) If there exists in T a cycle ξ of length 3 not bounding a triangle in T, then by (6-1) $f(\xi)$ is a homotopically trivial loop in Y/H and since $f_*\colon \pi_1(T) \to \pi_1(Y/H)$ is injective, ξ is homotopically trivial in T. Therefore, it disconnects

T into two components, one of them being a simplicial disc. Replacing the disc with a single triangle we obtain another triangulation T' of the torus. The map f can be extended over the new triangulation, since Y/H is a flag complex.

(b) If any cycle of length 3 in T bounds a triangle and there is a vertex $v \in T$ adjacent to 4 or 5 triangles, we cut out the open star of v and glue a filling without internal vertices instead so that f' can be extended over the new triangulation (it is possible, since Y/H is locally 6–large), obtaining another simplicial triangulation T' of the torus.

(c) If any cycle of length 3 in T bounds a triangle and there exists a vertex v adjacent to 6 or more triangles such that $f'(T_v)$ can be filled without internal vertices, then we apply the procedure from (b) also in this case.

As we modify T, we modify f'. Since each operation (a), (b), (c) decreases the number of vertices in T, the procedure terminates. Therefore, without loss of generality, we can assume that any vertex in T is adjacent to at least 6 triangles and $f'|_{\partial N(v)}$ cannot be extended over a simplicial disc with boundary $\partial N(v)$ and with no internal vertices, for any vertex $v \in T$. Since the Euler characteristic of a torus is 0, that implies that any vertex is adjacent to exactly 6 triangles, so the universal covering $\widetilde{T}$ is isomorphic to $\mathbb{E}^2_\triangle$ and $\widetilde{f'}\colon \widetilde{T} \to \widetilde{Y}$ is a locally isometric immersion (Proposition 4.7(1)). The composition of f' and the covering map $\widetilde{Y} \to Y$ is a locally isometric immersion $p\colon \mathbb{E}^2_\triangle \to Y$, whose image is H–invariant. Since $Y \subset X$ is a full subcomplex, p treated as a map into X is also a locally isometric immersion, so by Theorem 5.2 it is an H–invariant flat (the diameter of its image is greater than 3 by the local finiteness of X and by the freedom of the action of G).

Step 2 *If there exists in X an H–invariant flat F, where $H < G$ is a finite-index subgroup, then there exists a G–invariant flat F'. Moreover, any vertex $v \in \mathrm{Th}(F)$ is contained in some G–invariant flat.*

Let $g_1, \dots, g_n \in G$ be representatives of all cosets of H. Since G is abelian, $F_i = g_i(F)$ are H–invariant flats. As $F^{(0)}$ consists of a finite number of H–orbits, there is a constant c such that $\mathrm{hdist}_X(Hx, F) \le c$ and similarly $\mathrm{hdist}_X(Hg_i(x), F_i) \le c$. As any two H–orbits are at finite Hausdorff distance, F_i is at finite Hausdorff distance from F, so by Theorem 5.4 we have $F_i \subset \mathrm{Th}(F)$ for $i = 1, \dots, n$. For every $g \in G$ there is an i such that $g(F) = F_i$, so $g(\mathrm{Th}(F)) = \mathrm{Th}(F_i) = \mathrm{Th}(F)$ (the latter equality follows from the fact that the Hausdorff distance between F_i and F is finite) and $\mathrm{Th}(F)$ is therefore G–invariant. By Corollary 5.5 the retraction $r\colon \mathrm{Th}(F) \to F \cong \mathbb{E}^2_\triangle$ defined in Theorem 5.4 induces an action of G on $\mathbb{E}^2_\triangle$, which is free, as G is torsion-free. We choose equivariantly vertices $F'(v) \in r^{-1}(v) \subset X$ for $v \in \mathbb{E}^2_\Delta$ and by Theorem 5.4(3) extend it to a G–invariant flat $F' \colon \mathbb{E}^2_\Delta \to X$.

Step 3 *If F is a G–invariant flat, then $F \subset \mathrm{Min}(G)$. In particular, $\mathrm{Min}(G)$ is nonempty.*

Let $g \in G$, $v \in F$ and $y \in \mathrm{Min}(g)$. There is a g–invariant geodesic in F passing through v, on which g acts by a translation. By the triangle inequality:

$$\begin{aligned} n \cdot d(v, g(v)) = d(v, g^n(v)) &\le d(v, y) + d(y, g^n(y)) + d(g^n(y), g^n(v)) \\ &\le 2 \cdot d(v, y) + n \cdot d(y, g(y)) \end{aligned}$$

for any natural n, so $d(v, g(v)) \le d(y, g(y))$, hence $v \in \mathrm{Min}(g)$. As this holds for any $g \in G$ and for any vertex $v \in F$, we have $F \subset \mathrm{Min}(G)$.

Step 4 *If F is a G–invariant flat, then $\mathrm{Min}(G) = \mathrm{Th}(F)$.*

By Step 2 and Step 3 we have $\mathrm{Th}(F) \subset \mathrm{Min}(G)$. Now we prove the opposite inclusion. Choose an arbitrary vertex $v \in \mathrm{Min}(G)$. It suffices to find a G–invariant flat containing v.

Choose in $F^{(1)}$ two convex half-lines k and l with a common endpoint x intersecting at the angle $\frac{2}{3}\pi$. Since the action of G on $F \cong \mathbb{E}^2_\Delta$ is cocompact, there are nontrivial elements $g, h \in G$ such that $g(x) \in k$ and $h(x) \in l$. Replacing g and h by some powers we can assume that $d(x, g(x)) = d(x, h(x)) > 3$. Therefore the vertices x, $g(x)$, $h(x)$, $g^2h(x)$, $gh^2(x)$, $g^2h^2(x)$ and the geodesics α (joining x with $g(x)$), β (joining x with $h(x)$), γ (joining $h(x)$ with $gh^2(x)$), $gh^2(\alpha)$, $g^2h(\beta)$, $gh^{-1}(\gamma)$ bound a regular hexagon in F (as in Figure 6(a)).

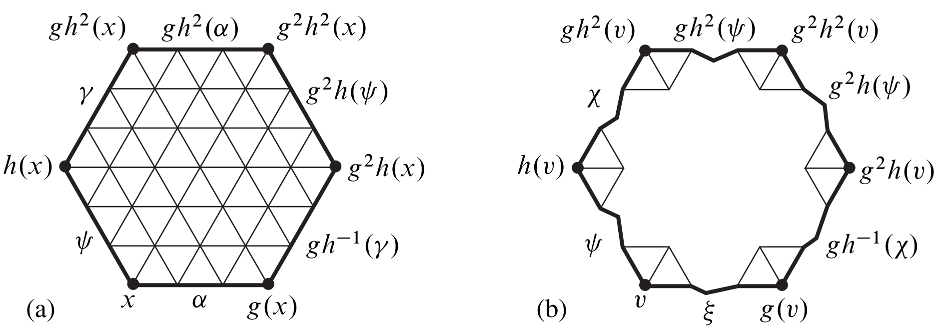

Figure 6

Join the vertices v, $g(v)$, $h(v)$, $g^2h(v)$, $gh^2(v)$, $g^2h^2(v)$ in X by geodesics ξ, ζ, χ and $gh^2(\xi)$, $g^2h(\zeta)$ and $hg^{-1}(\chi)$ (as in Figure 6(b)). Since $x, v \in \mathrm{Min}(G)$, for any elements $p, q \in G$ we have:

$$d(p(x), q(x)) = d(p(v), q(v)) \tag{6-2}$$

Notice that any two consecutive sides of the hexagon in Figure 6(a) form a geodesic in X and by (6-2) so do consecutive sides of the hexagon in Figure 6(b) – thus they intersect only at the endpoints. Since the distance between opposite vertices of the hexagon in (a) is twice the length of its side, the nonconsecutive sides are also disjoint in (b) (again by (6-2)). Thus the closed path being the concatenation $\xi * \zeta * \chi * gh^2(\xi^{-1}) * g^2h(\zeta^{-1}) * gh^{-1}(\chi^{-1})$ is a cycle in X. Let S be a minimal surface spanning this cycle and denote by $y_1, \dots, y_6 \in \partial\triangle_S$ the vertices mapping to vertices of the hexagon in (b).

By Lemma 4.2 the simplicial disc $\triangle_S$ is systolic, ie any of its internal vertices has nonpositive defect. Since any two consecutive sides of the hexagon in (b) form a geodesic in X, any vertex $v \in \partial\triangle_S$ is adjacent to at least 2 triangles and $\partial\triangle_S$ is the union of three geodesic arcs: $[y_1, y_3]$, $[y_3, y_5]$, $[y_5, y_1]$. By Fact 3.1 the sum of the defects along any of the three arcs is at most 1. As the sum of the defects at internal vertices of $\triangle_S$ is nonpositive, and by the Gauss–Bonnet Lemma the sum of the defects at all vertices of $\triangle_S$ is 6, we have that any internal vertex has defect 0 (is adjacent to exactly 6 triangles) and the defects at y_1, y_3, y_5 are equal to 1. Similarly we prove that the defects at y_2, y_4, y_6 are equal to 1 (as in Figure 6(b)).

Since $[y_{i-1}, y_i] \cup [y_i, y_{i+1}]$ and $[y_i, y_{i+1}] \cup [y_{i+1}, y_{i+2}]$, $i = 1, \dots, 6$ (we use the cyclic order of indices) are geodesics in $\triangle_S$, for any vertex $w \in (y_i, y_{i+1}) \subset \partial\triangle_S$ of defect 1 there are vertices of negative defects $w' \in (y_i, w)$ and $w'' \in (w, y_{i+1})$. Moreover, any two vertices $w_1, w_2 \in (y_i, y_{i+1}) \subset \partial\triangle_S$ of defects 1 are separated by a vertex of negative defect. Thus either the sum of the defects along (y_i, y_{i+1}) is negative or there are no positive vertices (and also no negative vertices) on (y_i, y_{i+1}). As the sum of the defects at vertices of $\partial\triangle_S$ is 6 and the defect at y_i, $i = 1, \dots, 6$ is equal to 1, there are no nonzero vertices on $\partial\triangle_S$ different from $y_1, \dots, y_6$. Thus $\triangle_S$ is a regular equilaterally triangulated hexagon (isomorphic to the one in Figure 6(a)).

Let $H < G$ be the subgroup generated by g and h. As H satisfies (6-1), X/H is a locally 6–large simplicial complex. As a quotient of S we obtain a simplicial map $f\colon T \to X/H$, where T is a triangulation of a torus such that any vertex of T is adjacent to exactly 6 triangles. If there is a vertex $y \in T$ such that $f(T_y)$ can be filled with a disc with no internal vertices, we can apply the minimizing procedure from Step 1 (starting with operation (c)), resulting in a triangulation of a torus T' and a simplicial map $f'\colon T' \to X/H$ such that the universal covering $\widetilde{f}'\colon \widetilde{T}' \to X$ is a flat F' in X at finite Hausdorff distance from F. Moreover, F' has a smaller number of H–orbits of vertices than F, which is impossible, as by Corollary 5.5 retractions r_F and $r_{F'}$ induce isomorphic actions on $\mathbb{E}^2_\triangle$.

Thus the universal covering $\widetilde{T}$ is isomorphic to $\mathbb{E}^2_\triangle$ and $\widetilde{f}\colon \mathbb{E}^2_\triangle \to X$ is a locally isometric immersion, so by Theorem 5.2 it is an H–invariant flat. Moreover, by the construction $v \in \operatorname{Im} \widetilde{f}$, so by Step 2 there is a G–invariant flat passing through v, which completes the proof of the inclusion $\operatorname{Min}(G) \subset \operatorname{Th}(F)$.

Step 5 *G is a free abelian group of rank 2.*

Assume $G \cong \mathbb{Z}^n$ for $n > 2$. Let $H < G$ be a subgroup isomorphic to $\mathbb{Z}^2$. We have already proved that $\operatorname{Min}(H) = \operatorname{Th}(F)$ for an H–invariant flat $F \subset X$. Since every $g \in G$ centralizes H, it preserves $\operatorname{Min}(H)$, so the thickening $\operatorname{Th}(F)$ is G–invariant. Since G is torsion-free, the retraction $r\colon \operatorname{Th}(F) \to F \cong \mathbb{E}^2_\triangle$ defined in Theorem 5.4 induces a free action of G on $\mathbb{E}^2_\triangle$ (Corollary 5.5). However, there are no free actions of $\mathbb{Z}^n$ on $\mathbb{E}^2_\triangle$ for $n > 2$. □

Corollary 6.2 *Let a group G act simplicially, properly discontinuously on a uniformly locally finite systolic complex X.*

(1) *If G is a virtually abelian group of rank 2, then there is a flat F, unique up to the flat equivalence, such that $\operatorname{Th}(F)$ is G–invariant.*

(2) *If $H < G$ is a maximal virtually abelian rank 2 subgroup, then there is a flat F, unique up to the flat equivalence, such that $\operatorname{Stab}_G(\operatorname{Th}(F)) = H$.*

Proof There is a finite index subgroup $A < G$ isomorphic to $\mathbb{Z}^2$ and a finite index normal subgroup $N \lhd G$ isomorphic to $\mathbb{Z}^2$ (eg $N = \bigcap_{g \in G} g^{-1}Ag$). By the Flat Torus Theorem there is an N–invariant flat F in X, unique up to the flat equivalence. Let $g_1, \dots, g_k$ be representatives of all cosets of N in G. The flats $F_i = g_i(F)$, $i = 1, \dots, k$ are N–invariant (as $g_i^{-1} N g_i = N$), so by the Flat Torus Theorem they are equivalent to F. Thus G stabilizes the thickening $\operatorname{Th}(F)$, which proves (1).

To prove (2) consider the H–invariant thickening $\operatorname{Th}(F)$. By Corollary 5.5 the induced action of $\operatorname{Stab}_G(\operatorname{Th}(F))$ on $F \cong \mathbb{E}^2_\triangle$ is properly discontinuous and as the stabilizer $\operatorname{Stab}_G(\operatorname{Th}(F))$ contains a subgroup isomorphic to $\mathbb{Z}^2$ it is also cocompact. Thus $\operatorname{Stab}_G(\operatorname{Th}(F)$ is a virtually abelian rank 2 group and (2) follows from the maximality of H. □

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