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Erratum to "Hadamard spaces with isolated flats"

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The purpose of this erratum is to correct the proof of Theorem A.0.1 in the appendix to [4], which was jointly authored by Mohamad Hindawi, Hruska and Kleiner. In that appendix, many of the results of [4] about $\mathrm{CAT}(0)$ spaces with isolated flats are extended to a more general setting in which the isolated subspaces are not necessarily flats. However, one step of that extension does not follow from the argument we used the isolated flats setting. We provide a new proof that fills this gap.

In addition, we give a more detailed account of several other parts of Theorem A.0.1, which were sketched in [4].

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1 Introduction

The purpose of this erratum is to explain a gap in the proof of [4, Theorem A.0.1] and to explain how to fill it. The arguments used to fill this gap use ideas not present in [4].

In addition, we present details for several other portions of the proof of [4, Theorem A.0.1], which were briefly sketched in [4]. The new details for these portions are easy modifications of arguments from [4]. Nevertheless, the exact nature of these modifications was, perhaps, not described as explicitly as it could have been.

The main results of [4] explain the structure of a $\mathrm{CAT}(0)$ space X with isolated flats and the structure of a group Γ acting properly, cocompactly and isometrically on such a space. In a short appendix, written jointly by the authors and Mohamad Hindawi, we extend those results to a more general setting in which the isolated subspaces are not necessarily flats. Many of the details of this extension are nearly identical to the details presented for isolated flats. As a result, the details of the extension were not given explicitly.

However, one step of the proof of [4, Theorem A.0.1] does not follow from the same reasoning given in the isolated flats case. Specifically, the proof of [4, Lemma 3.3.1] is correct in the setting of isolated flats but does not extend directly to the more general setting required by the appendix. The gap occurs at the point where we prove that two

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flats F, F' in the asymptotic cone obtained as ultralimits of flats in X cannot intersect in more than one point. The argument assumes the existence of many nondegenerate triangles in F'. In particular, if $x, y \in F \cap F'$ we use that the set of points $z \in F'$ with $\Delta(x, y, z)$ nondegenerate is a dense set of F'. This conclusion is certainly true in the isolated flats case, since the only degenerate triangles in a flat are those for which x, y and z are colinear. However, this fact is not necessarily true in the more general setting of the appendix. For example, if the isolated subspaces of X are δ–hyperbolic, their ultralimits are trees, which do not contain nondegenerate triangles.

We fill this gap by proving Proposition 10, which states that two different ultralimits of isolated subspaces cannot intersect in more than one point. As mentioned above, we also give a more detailed account of several other parts of the proof of [4, Theorem A.0.1].

We begin by recalling the statement of [4, Theorem A.0.1].

Theorem 1 [4, Theorem A.0.1] *Let X be a* $\mathrm{CAT}(0)$ *space and $\Gamma \curvearrowright X$ be a geometric action. Suppose $\mathcal{F}$ is a Γ–invariant collection of unbounded, closed, convex subsets. Assume the following:*

(A) *There is a constant $D < \infty$ such that each flat $F \subseteq X$ lies in a D–tubular neighborhood of some $C \in \mathcal{F}$.*

(B) *For each positive $r < \infty$ there is a constant $\rho = \rho(r) < \infty$ so that for any two distinct elements $C, C' \in \mathcal{F}$ we have*

$$\operatorname{diam}\bigl(\mathcal{N}_r(C) \cap \mathcal{N}_r(C')\bigr) < \rho.$$

Then we conclude:

(1) *The collection $\mathcal{F}$ is locally finite, there are only finitely many Γ–orbits in $\mathcal{F}$, and each $C \in \mathcal{F}$ is Γ–periodic.*

(2) *Every connected component of $\partial_T X$ containing more than one point is contained in $\partial_T C$ for a unique $C \in \mathcal{F}$.*

(3) *Let X_ω be an asymptotic cone $\mathrm{Cone}_\omega(X, \star_n, \lambda_n)$. Let $\mathcal{F}_\omega$ denote the set of all subspaces $C_\omega \subseteq X_\omega$ of the form $C_\omega = \omega\text{–}\lim C_n$ where $C_n \in \mathcal{F}$ and $\omega\text{–}\lim \lambda_n^{-1}\, d(C_n, \star_n) < \infty$.*
Then for every $x \in X_\omega$, each connected component of $\Sigma_x X_\omega$ containing more than one point is contained in $\Sigma_x C_\omega$ for a unique $C_\omega \in \mathcal{F}_\omega$. Furthermore, if a direction $\overrightarrow{xy}$ lies in a nontrivial component of $\Sigma_x C_\omega$ then an initial segment of $[x, y]$ lies in C_ω.

(4) *Every asymptotic cone X_ω is tree-graded with respect to the collection $\mathcal{F}_\omega$.*

(5) Γ *is hyperbolic relative to any collection* $\mathcal{P}$ *of representatives of the finitely many conjugacy classes of stabilizers of elements of* $\mathcal{F}$.

(6) *Suppose the stabilizer of each* $C \in \mathcal{F}$ *is a* CAT(0) *group with very well-defined boundary. Then* Γ *has a very well-defined boundary.*

In the sequel we will always assume that X, Γ and $\mathcal{F}$ satisfy the hypotheses of Theorem 1 (except in Lemma 6 and Corollary 7).

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2 Proof of Theorem 1 assertions (1) and (2)

The proofs in this section are all easy modifications of arguments from [4].

Lemma 2 (cf [4, Lemma 3.1.1]) *The collection* $\mathcal{F}$ *is locally finite; in other words, only finitely many elements of* $\mathcal{F}$ *intersect any given compact set.*

Proof It suffices to show that only finitely many elements of $\mathcal{F}$ intersect each closed metric ball $\overline{B}(x,r)$. Let $\mathcal{F}_0$ be the collection of all $C \in \mathcal{F}$ intersecting this ball. By hypothesis (B) of Theorem 1, there exists $\rho = \rho(1)$ such that for any distinct elements $C, C' \in \mathcal{F}$ we have

$$\operatorname{diam}\bigl(\mathcal{N}_1(C) \cap \mathcal{N}_1(C')\bigr) < \rho.$$

If we let $\kappa := r + \rho$ then for each $C \in \mathcal{F}_0$ the set $C \cap \overline{B}(x,\kappa)$ has diameter at least ρ since C is connected and unbounded.

If $\mathcal{F}_0$ is infinite then it contains a sequence of distinct elements (C_i) such that the compact sets $C_i \cap \overline{B}(x,\kappa)$ converge in the Hausdorff metric. In particular, whenever i and j are sufficiently large, the Hausdorff distance between $C_i \cap \overline{B}(x,\kappa)$ and $C_j \cap \overline{B}(x,\kappa)$ is less than 1. But $C_i \cap \overline{B}(x,\kappa)$ has diameter at least ρ and lies in $\mathcal{N}_1(C_i) \cap \mathcal{N}_1(C_j)$, contradicting our choice of ρ. □

Proof of Theorem 1(1) The collection $\mathcal{F}$ is Γ–invariant by hypothesis and is locally finite by Lemma 2. Now [4, Lemma 3.1.2] implies that such a collection of subspaces contains only finitely many Γ–orbits and that each $C \in \mathcal{F}$ is Γ–periodic, provided that each $C \in \mathcal{F}$ is a flat. However the hypothesis that elements of $\mathcal{F}$ are flats is never used in the proof of Lemma 3.1.2. Thus the same conclusion holds in the present setting. □

The following three results were proved in [4] under the additional hypothesis that the elements of $\mathcal{F}$ are flats. Again this hypothesis is never used in the proofs.

Lemma 3 (cf [4, Lemma 3.2.2]) *There is a decreasing function $D_1 = D_1(\theta) < \infty$ such that if $S \subset X$ is a flat sector of angle $\theta > 0$ then $S \subset \mathcal{N}_{D_1(\theta)}(C)$ for some $C \in \mathcal{F}$.* □

Lemma 4 (cf [4, Lemma 3.2.3]) *For all $\theta_0 > 0$ and $R < \infty$, there exist $\delta_1 = \delta_1(\theta_0, R)$ and $\rho_1 = \rho_1(\theta_0, R)$ such that if $p, x, y \in X$ satisfy $d(p,x), d(p,y) > \rho_1$ and*

$$\theta_0 < \angle_p(x,y) \le \widetilde{\angle}_p(x,y) < \angle_p(x,y) + \delta_1 < \pi - \theta_0$$

then there exists $C \in \mathcal{F}$ such that

$$\big([p,x] \cup [p,y]\big) \cap B(p,R) \subset \mathcal{N}_{D_1(\theta_0)}(C).$$ □

Proposition 5 (cf [4, Proposition 5.2.1]) *For each $\theta_0 > 0$ there is a positive constant $\delta_4 = \delta_4(\theta_0)$ such that whenever $p \in X$ and $\xi, \eta \in \partial_T X$ satisfy*

$$(\dagger) \qquad \theta_0 \le \angle_p(\xi,\eta) \le \angle_T(\xi,\eta) \le \angle_p(\xi,\eta) + \delta_4 \le \pi - \theta_0$$

then there exists $C \in \mathcal{F}$ so that

$$[p,\xi] \cup [p,\eta] \subset \mathcal{N}_{D_1(\theta_0)}(C).$$ □

Proof of Theorem 1(2) The proof is essentially the same as for the forward implication of [4, Theorem 5.2.5]. By (B), it is clear that if $C, C' \in \mathcal{F}$ are distinct then $\partial_T C \cap \partial_T C' = \varnothing$. If $\xi, \eta \in \partial_T X$ and $0 < \angle_T(\xi,\eta) < \pi$ then we can find $\theta_0 > 0$ and $p \in X$ such that $(\dagger)$ holds for $\delta_4 = \delta_4(\theta_0)$. Hence by Proposition 5 we have $[p,\xi] \cup [p,\eta] \subset \mathcal{N}_{D_1(\theta_0)}(C)$ for some $C \in \mathcal{F}$, which means that $\{\xi,\eta\} \subset \partial_T C$.

More generally, suppose ξ, η are distinct points in the same component of $\partial_T X$. Then there is a sequence $\xi = \xi_0, \dots, \xi_\ell = \eta$ such that $0 < \angle_T(\xi_i, \xi_{i+1}) < \pi$. By the previous paragraph, it follows that $\{\xi,\eta\} \subset \partial_T C$ for some $C \in \mathcal{F}$. □

3 Filling the gap

The goal of this section is to prove Proposition 10 using new arguments not found in [4]. We will use the following result due to Ballmann.

Lemma 6 [1, Lemma III.3.1] *Let X be any proper $\mathrm{CAT}(0)$ space. Let c be a geodesic line in X which does not bound a flat strip of width $R > 0$. Then there are neighborhoods U of $c(\infty)$ and V of $c(-\infty)$ in $\overline{X}$ such that for any $\zeta \in U$ and $\eta \in V$ there is a geodesic from ζ to η, and for any such geodesic c' we have $d\big(c', c(0)\big) < R$.*

The following corollary of Ballmann's result was observed by Hindawi in the setting of Hadamard manifolds.

Corollary 7 (cf [3, Proposition 3.3]) *Let X be any proper* CAT(0) *space. Suppose $p \in X$ and let $[x_n, y_n]$ be a sequence of geodesic segments in X such that x_n and y_n converge respectively to ξ_x and $\xi_y \in \partial X$. If $d_T(\xi_x, \xi_y) > \pi$ then the distances $d\big(p, [x_n, y_n]\big)$ are bounded above as $n \to \infty$.*

Proof If $d_T(\xi_x, \xi_y) > \pi$ then there exists a geodesic line c in X with endpoints ξ_x and ξ_y that does not bound a half-plane (see for instance Ballmann [1, Theorem II.4.11]). In particular, there exists R such that c does not bound a flat strip of width R. Once n is sufficiently large, $x_n \in V$ and $y_n \in U$, where U and V are the neighborhoods given by Lemma 6. Therefore for all but finitely many n, we have $d\big(c(0), [x_n, y_n]\big) < R$. Thus $d\big(p, [x_n, y_n]\big)$ remains bounded as $n \to \infty$. □

The next result shows that the convex hull of $C \cup C'$ lies within a uniformly bounded neighborhood of $C \cup C' \cup [p,q]$ where $[p,q]$ is any geodesic of shortest length from C to C'.

Proposition 8 *There is a constant $\epsilon_0 > 0$ such that the following holds. Choose $C \neq C'$ in $\mathcal{F}$, and let $[p,q]$ be a geodesic of shortest length from C to C'. Then every geodesic from C to C' comes within a distance ϵ_0 of both p and q.*

Proof Suppose by way of contradiction that there were a sequence of counterexamples, ie, subspaces $C_i \neq C_i'$ in $\mathcal{F}$, points $p_i, x_i \in C_i$ and $q_i, y_i \in C_i'$ such that $[p_i, q_i]$ is a shortest path from C_i to C_i' and such that $d\big(p_i, [x_i, y_i]\big)$ tends to infinity. We have two cases depending on whether $d(C_i, C_i')$ remains bounded as $i \to \infty$.

Case 1 Suppose $d(C_i, C_i')$ remains bounded. By Theorem 1(1), the C_i lie in finitely many orbits. Pass to a subsequence and translate by the group action so that $C_i = C$ is constant. Translating by $\mathrm{Stab}(C)$, we can also assume that $C_i' = C'$ is constant. After passing to a further subsequence, the points p_i, q_i, x_i and y_i converge respectively to $p \in C$, $q \in C'$, $\xi_x \in \partial C$ and $\xi_y \in \partial C'$. Since $d\big(p, [x_i, y_i]\big)$ tends to infinity, it follows from Corollary 7 that $d_T(\xi_x, \xi_y) \leq \pi$, contradicting Theorem 1(2).

Case 2 Now suppose the distances $d(C_i, C_i')$ are unbounded. After passing to a subsequence and applying elements of Γ, we can assume that $C_i = C$ is constant and that the points p_i, q_i, x_i and y_i converge respectively to $p \in C$, $\xi_q \in \partial X$, $\xi_x \in \partial C$ and $\xi_y \in \partial X$. Furthermore, $\xi_q \notin \partial C$ since the ray from p to ξ_q meets C orthogonally. By hypothesis, $d\big(p, [x_i, y_i]\big)$ tends to infinity. Since $d(C, C_i') = d(p, C_i') \to \infty$, we also have $d\big(p, [y_i, q_i]\big) \to \infty$. Therefore, by Corollary 7 the points ξ_x, ξ_y and ξ_q all lie in the same component of $\partial_T X$, contradicting Theorem 1(2). □

Corollary 9 *There is a constant ϵ_1 such that the following holds. Suppose $C \neq C' \in \mathcal{F}$ and we have $a, b \in C$ and $a', b' \in C'$. Then*

$$d(a,b) + d(a',b') \leq d(a,a') + d(b,b') + \epsilon_1.$$

Proof Choose a geodesic $[p,q]$ of shortest length from C to C'. By Proposition 8, there are points $x, x' \in [a,a']$ and $y, y' \in [b,b']$ such that x and y are within a distance ϵ_0 of p and x' and y' are within a distance ϵ_0 of q. Therefore

$$\begin{aligned} d(a,b) + d(a',b') &\leq d(a,x) + d(x,y) + d(y,b) + d(a',x') + d(x',y') + d(y',b') \\ &\leq d(a,a') + d(b,b') + 4\epsilon_0. \end{aligned}$$ □

Proposition 10 *Suppose $C_\omega, C'_\omega \in \mathcal{F}_\omega$. If $C_\omega \neq C'_\omega$ then $C_\omega \cap C'_\omega$ contains at most one point.*

Proof Suppose $C_\omega \neq C'_\omega$. Then $C = \omega\text{–}\lim C_n$ and $C'_\omega = \omega\text{–}\lim C'_n$, where $C_n \neq C'_n$ for ω–almost all n. If $a, b \in C$, they are represented by sequences (a_n) and (b_n) such that $a_n, b_n \in C_n$. If a, b are also in C', they can also be represented by sequences (a'_n) and (b'_n) with $a'_n, b'_n \in C'_n$. Furthermore

$$\omega\text{–}\lim \lambda_n^{-1} d(a_n, a'_n) = \omega\text{–}\lim \lambda_n^{-1} d(b_n, b'_n) = 0.$$

By Corollary 9 we see that

$$d(a,b) = \omega\text{–}\lim \lambda_n^{-1} d(a_n, b_n) \leq \omega\text{–}\lim \lambda_n^{-1} \big(d(a_n, a'_n) + d(b_n, b'_n) + \epsilon_1 \big) = 0.$$

Thus $a = b$. □

4 Proofs of Theorem 1 assertions (3), (4) and (5)

The proofs in this section are modeled closely on arguments from [4]. Indeed, the reader will not find any substantially new ideas in this section. However, in many places minor modifications are necessary to adapt the proofs from the isolated flats setting to the present level of generality. In these places we have provided the detailed arguments for the benefit of the reader.

The proof of the following proposition is identical to that of [4, Proposition 3.2.5].

Proposition 11 *For all $\theta_0 > 0$ there are $\delta_2 = \delta_2(\theta_0) > 0$ and $\rho_2 = \rho_2(\theta_0)$ such that if $x, y, z \in X$, all vertex angles and comparison angles of $\Delta(x,y,z)$ lie in $(\theta_0, \pi - \theta_0)$,* *each vertex angle is within δ_2 of the corresponding comparison angle, and all three sides of $\Delta(x,y,z)$ have length greater than ρ_2, then*

$$[x,y]\cup[x,z]\cup[y,z]\subset\mathcal{N}_{D_1(\theta_0)}(C)$$

for some $C\in\mathcal{F}$. □

Lemma 12 (cf [4, Lemma 3.3.1]) *For all $\theta_0>0$ there is a $\delta_3=\delta_3(\theta_0)>0$ such that if $x,y,z\in X_\omega$ are distinct, all vertex angles and comparison angles of $\Delta(x,y,z)$ lie in $(\theta_0,\pi-\theta_0)$, and each vertex angle is within δ_3 of the corresponding comparison angle, then*

$$[x,y]\cup[x,z]\cup[y,z]\subset C_\omega$$

for some $C_\omega\in\mathcal{F}_\omega$.

Proof The proof is essentially the same as the first part of [4, Lemma 3.3.1]. Choose θ_0, and let δ_2 and ρ_2 be the constants provided by Proposition 11. Set $\delta_3:=\delta_2$. Choose $x,y,z\in X_\omega$ as above, and apply [4, Corollary 2.4.2] to get sequences (x_k), (y_k) and (z_k) representing x, y and z such that each vertex angle of $\Delta(x,y,z)$ is the ultralimit of the corresponding angle of $\Delta(x_k,y_k,z_k)$. Then $\Delta(x_k,y_k,z_k)$ satisfies the hypothesis of Proposition 11 for ω–almost all k. Consequently x, y and z lie in some $C_\omega\in\mathcal{F}_\omega$. □

Proof of Theorem 1(3) The proof is similar to the proof of [4, Proposition 3.3.2]. If $\overrightarrow{xy},\overrightarrow{xz}\in\Sigma_x X_\omega$ and $0<\angle_x(y,z)<\pi$ then $\angle_x(y,z)\in(\theta,\pi-\theta)$ for some positive θ. Let $\delta_3=\delta_3(\theta/8)$ be the constant given by Lemma 12, and let $\delta:=\min\{\delta_3,\theta/4\}$. By [4, Proposition 2.2.3] there exist points $y'\in[x,y]$ and $z'\in[x,z]$ such that the angles of $\Delta(x,y',z')$ are within δ of their respective comparison angles and also such that $d(x,y')=d(x,z')$. Since $\delta\le\theta/4$, the angles of $\Delta(x,y',z')$ at y' and z' lie in the interval $(\theta/8,\pi/2)$. Lemma 12 now implies that

$$[x,y']\cup[x,z']\cup[y',z']\subset C_\omega$$

for some $C_\omega\in\mathcal{F}_\omega$. Thus the directions $\overrightarrow{xy}$ and $\overrightarrow{xz}$ both lie in $\Sigma_x C_\omega$ and any geodesic representing either direction has an initial segment that lies in C_ω. The uniqueness of C_ω is an immediate consequence of Proposition 10.

More generally, suppose $\overrightarrow{xy}$ and $\overrightarrow{xz}$ are distinct directions in the same component of $\Sigma_x X_\omega$. Then there is a sequence $y=y_0,\dots,y_\ell=z$ such that $0<\angle_x(y_{i-1},y_i)<\pi$ for $i=1,\dots,\ell$. By the previous paragraph, $\overrightarrow{xy_{i-1}}$ and $\overrightarrow{xy_i}$ both lie in $\Sigma_x C_i$ for a unique $C_i\in\mathcal{F}_\omega$, and $[x,y_{i-1}]$ and $[x,y_i]$ have initial segments in C_i. Since $[x,y_i]$ has initial segments in both C_i and C_{i+1}, it follows from Proposition 10 that $C_1=\cdots=C_{\ell-1}$. □

Corollary 13 (cf [4, Corollary 3.3.3]) *Let $\pi_{C_\omega}\colon X_\omega \to C_\omega$ be the nearest point projection. Then π_{C_ω} is locally constant on $X_\omega \setminus C_\omega$.*

Proof Choose $s \in X_\omega \setminus C_\omega$, and let $x := \pi_{C_\omega}(s)$. Then $\angle_x(S, F)$ is at least $\pi/2$. In particular, the direction $\overrightarrow{xs} \notin \Sigma_x C_\omega$. By continuity of $\log_x$, if U is any connected set containing s in $X_\omega \setminus \{x\}$ then $\log_x(U)$ is a connected set containing $\overrightarrow{xs}$. Since $\log_x(U)$ is not contained in $\Sigma_x C_\omega$, it follows from Theorem 1(3) that $\log_x(U)$ is disjoint from $\Sigma_x C_\omega$, and that each point of $\log_x(U)$ is at an angular distance π from $\Sigma_x C_\omega$. Hence for each $s' \in U$, we have $\pi_{C_\omega}(s') = x$. □

Lemma 14 (cf [4, Lemma 3.3.4]) *If p lies in the interior of the geodesic $[x, y] \subset X_\omega$ and x and y lie in the same component of $X_\omega \setminus \{p\}$ then p is contained in an open subarc of $[x, y]$ that lies in C_ω for some $C_\omega \in \mathcal{F}_\omega$.*

Proof By continuity of $\log_p$, the directions $\overrightarrow{px}$ and $\overrightarrow{py}$ lie in the same component of $\Sigma_x F_\omega$, which is therefore nontrivial. Theorem 1(3) now implies that initial segments of $[p, x]$ and $[p, y]$ lie in C_ω for some $C_\omega \in \mathcal{F}_\omega$. □

Lemma 15 (cf [4, Lemma 3.3.5]) *Every embedded loop in X_ω lies in some $C_\omega \in \mathcal{F}_\omega$.*

Proof Let γ be an embedded loop containing points $x \neq y$. For each $p \in [x, y]$, the loop γ provides a path from x to y that avoids p. By Lemma 14, an open subarc of $[x, y]$ containing p lies in some $C_p \in \mathcal{F}_\omega$. The interior of $[x, y]$ is covered by these open intervals. By Proposition 10, it follows that $C_p = C_\omega$ is independent of the choice of p. Let β be a maximal open subpath of γ in the complement of C_ω. It follows from Corollary 13 that β projects to a constant under π_{C_ω}. Hence the endpoints of β coincide, which is absurd. □

Proof of Theorem 1(4) Each $C \in \mathcal{F}$ is closed and convex in X. Therefore each $C_\omega \in \mathcal{F}_\omega$ is closed and convex in X_ω. By Proposition 10, distinct subspaces $C_\omega, C'_\omega \in \mathcal{F}_\omega$ intersect in at most one point. Furthermore, Lemma 15 implies that every embedded geodesic triangle in X_ω lies in some $C_\omega \in \mathcal{F}_\omega$. □

Proof of Theorem 1(5) Let $\mathcal{P}$ be a set of representatives of the finitely many conjugacy classes of stabilizers of elements of $\mathcal{F}$. The action of Γ on X induces a quasi-isometry $\Gamma \to X$ that induces a one-to-one correspondence between the left cosets of elements of $\mathcal{P}$ and the elements of $\mathcal{F}$. It follows from Druţu–Sapir [2, Theorem 5.1] that every asymptotic cone of Γ is tree graded with respect to ultralimits of sequences of left cosets of elements of $\mathcal{P}$. Now [2, Theorem 1.11] implies that Γ is relatively hyperbolic with respect to $\mathcal{P}$. □

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