A simply connected surface of general type with $p_g = 0$ and $K^2 = 3$

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Motivated by a recent result of Y Lee and the second author [9], we construct a simply connected minimal complex surface of general type with $p_g = 0$ and $K^2 = 3$ using a rational blow-down surgery and $\mathbb{Q}$–Gorenstein smoothing theory. In a similar fashion, we also construct a new simply connected symplectic 4–manifold with $b_2^+ = 1$ and $K^2 = 4$.

14J29; 14J10, 14J17, 53D05

1 Introduction

One of the fundamental problems in the classification of complex surfaces is to find a new family of simply connected surfaces of general type with $p_g = 0$. Although a large number of non–simply connected complex surfaces of general type with $p_g = 0$ are known (see Barth et al [5]), few simply connected surfaces of general type with $p_g = 0$ are known except the Barlow surface; see Barlow [4].

Recently, the second author [18] constructed a simply connected symplectic 4–manifold with $b_2^+ = 1$ and $K^2 = 2$ using a rational blow-down surgery, and then Y Lee and the second author [9] constructed a family of simply connected, minimal, complex surfaces of general type with $p_g = 0$ and $1 \leq K^2 \leq 2$ by modifying Park's symplectic 4–manifold. After this construction, it has been a natural question whether one can find a new family of surfaces of general type with $p_g = 0$ and $K^2 \geq 3$ using the same technique.

The aim of this article is to give an affirmative answer to this question. Precisely, we are able to construct a simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 = 3$ using a rational blow-down surgery and a $\mathbb{Q}$–Gorenstein smoothing theory developed in Lee and Park [9]. The key ingredient for the construction of $K^2 = 3$ case is to find a rational surface Z which contains several disjoint chains of curves representing the resolution graphs of special quotient singularities. Once

Published: 5 January 2009 DOI: 10.2140/gt.2009.13.743

we have the right candidate Z for $K^2 = 3$, the remaining argument is parallel to that of the $K^2 = 2$ case which appeared in Lee and Park [9]. That is, we contract these chains of curves from the rational surface Z to produce a projective surface X with special quotient singularities. We then prove that the singular surface X has a $\mathbb{Q}$–Gorenstein smoothing and the general fiber X_t of the $\mathbb{Q}$–Gorenstein smoothing is a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 3$. The main result of this article is the following.

Theorem 1.1 *There exists a simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 = 3$.*

By using different pencils and fibrations, we provide more examples of simply connected minimal complex surfaces of general type with $p_g = 0$ and $K^2 = 3$ in Section 6. Furthermore, we also construct a new simply connected closed symplectic 4–manifold with $b_2^+ = 1$ and $K^2 = 4$ using the same technique as in Section 7.

Theorem 1.2 *There exists a simply connected symplectic 4–manifold with $b_2^+ = 1$ and $K^2 = 4$ which is homeomorphic, but not diffeomorphic, to a rational surface $\mathbb{P}^2 \sharp 5\overline{\mathbb{P}}^2$.*

It is a very intriguing question whether the symplectic 4–manifold constructed in Theorem 1.2 above admits a complex structure. One way to approach this problem is to use $\mathbb{Q}$–Gorenstein smoothing theory as above. But since the cohomology $H^2(T_X^0)$ is not zero in this case, it is hard to determine whether there exists a $\mathbb{Q}$–Gorenstein smoothing. Therefore we need to develop more $\mathbb{Q}$–Gorenstein smoothing theory in order to investigate the existence of a complex structure on the symplectic 4–manifold constructed in Theorem 1.2. We leave this question for future research.

Acknowledgements The authors would like to thank Yongnam Lee for his careful reading the first draft of this article. The authors also thank Ki-Heon Yun for providing some configurations appearing in Section 6. He also spent his time with the authors during the course of this work. Jongil Park was supported by SBS Foundation Grant in 2007 and he also holds a joint appointment in the Research Institute of Mathematics, Seoul National University. Dongsoo Shin was supported by Korea Research Foundation Grant funded by the Korean Government (KRF-2005-070-C00005).

2 $\mathbb{Q}$–Gorenstein smoothing

In this section we briefly review a theory of $\mathbb{Q}$–Gorenstein smoothing for projective surfaces with special quotient singularities and we quote some basic facts developed in Lee and Park [9].

Definition Let X be a normal projective surface with quotient singularities. Let $\mathcal{X} \to \Delta$ (or $\mathcal{X}/\Delta$) be a flat family of projective surfaces over a small disk Δ. The one-parameter family of surfaces $\mathcal{X} \to \Delta$ is called a *$\mathbb{Q}$–Gorenstein smoothing* of X if it satisfies the following three conditions:

(i) The general fiber X_t is a smooth projective surface.

(ii) The central fiber X_0 is X.

(iii) The canonical divisor $K_{\mathcal{X}/\Delta}$ is $\mathbb{Q}$–Cartier.

A $\mathbb{Q}$–Gorenstein smoothing for a germ of a quotient singularity $(X_0, 0)$ is defined similarly. A quotient singularity which admits a $\mathbb{Q}$–Gorenstein smoothing is called a *singularity of class T*.

Proposition 2.1 (Kollár and Shepherd-Barron [8], Manetti [12], Wahl [24]) *Let $(X_0, 0)$ be a germ of two dimensional quotient singularity. If $(X_0, 0)$ admits a $\mathbb{Q}$–Gorenstein smoothing over the disk, then $(X_0, 0)$ is either a rational double point or a cyclic quotient singularity of type $(1/(dn^2))(1, dna-1)$ for some integers a, n, d with a and n relatively prime.*

Proposition 2.2 (Kollár and Shepherd-Barron [8], Manetti [12], Wahl [23])

(1) *The singularities* $\overset{-4}{\circ}$, $\overset{-3}{\circ} - \overset{-3}{\circ}$, $\overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}$ *and* $\overset{-3}{\circ} - \overset{-2}{\circ} - \cdots - \overset{-2}{\circ} - \overset{-3}{\circ}$ *are of class T.*

(2) *If the singularity* $\overset{-b_1}{\circ} - \cdots - \overset{-b_r}{\circ}$ *is of class T, then so are*

$$\overset{-2}{\circ} - \overset{-b_1}{\circ} - \cdots - \overset{-b_{r-1}}{\circ} - \overset{-b_r-1}{\circ} \quad \text{and} \quad \overset{-b_1-1}{\circ} - \overset{-b_2}{\circ} - \cdots - \overset{-b_r}{\circ} - \overset{-2}{\circ}.$$

(3) *Every singularity of class T that is not a rational double point can be obtained by starting with one of the singularities described in* (1) *and iterating the steps described in* (2).

Let X be a normal projective surface with singularities of class T. Due to the result of Kollár and Shepherd-Barron [8], there is a $\mathbb{Q}$–Gorenstein smoothing locally for each singularity of class T on X. The natural question arises whether this local $\mathbb{Q}$–Gorenstein smoothing can be extended over the global surface X or not. Roughly geometric interpretation is the following: Let $\bigcup V_\alpha$ be an open covering of X such that each V_α has at most one singularity of class T. By the existence of a local $\mathbb{Q}$–Gorenstein smoothing, there is a $\mathbb{Q}$–Gorenstein smoothing $\mathcal{V}_\alpha/\Delta$. The question is if these families glue to a global one. The answer can be obtained by figuring out the obstruction map of the sheaves of deformation $T^i_X = \mathrm{Ext}^i_X(\Omega_X, \mathcal{O}_X)$ for $i = 0, 1, 2$. For example, if X is a smooth surface, then T^0_X is the usual holomorphic tangent sheaf T_X and $T^1_X = T^2_X = 0$. By applying the standard result of deformations in Lichtenbaum and Schlessinger [10] and Palamodov [16] to a normal projective surface with quotient singularities, we get the following:

Proposition 2.3 (Wahl [24, Section 4]) *Let X be a normal projective surface with quotient singularities. Then:*

(1) *The first order deformation space of X is represented by the global Ext 1–group $\mathbb{T}^1_X = \mathrm{Ext}^1_X(\Omega_X, \mathcal{O}_X)$.*
(2) *The obstruction lies in the global Ext 2–group $\mathbb{T}^2_X = \mathrm{Ext}^2_X(\Omega_X, \mathcal{O}_X)$.*

Furthermore, by applying the general result of local-global spectral sequence of ext sheaves (see Palamodov [16, Section 3]) to deformation theory of surfaces with quotient singularities so that $E_2^{p,q} = H^p(T^q_X) \Rightarrow \mathbb{T}^{p+q}_X$, and by $H^j(T^i_X) = 0$ for $i, j \geq 1$, we also get

Proposition 2.4 (Manetti [12], Wahl [24]) *Let X be a normal projective surface with quotient singularities. Then:*

(1) *We have the exact sequence*
$$0 \to H^1(T^0_X) \to \mathbb{T}^1_X \to \ker\big(H^0(T^1_X) \to H^2(T^0_X)\big) \to 0$$
where $H^1(T^0_X)$ represents the first order deformations of X for which the singularities remain locally a product.
(2) *If $H^2(T^0_X) = 0$, every local deformation of the singularities may be globalized.*

The vanishing $H^2(T^0_X) = 0$ can be obtained via the vanishing of $H^2(T_V(-\log E))$, where V is the minimal resolution of X and E is the reduced exceptional divisor.

Theorem 2.5 (Lee and Park [9]) *Let X be a normal projective surface with singularities of class T. Let $\pi\colon V \to X$ be the minimal resolution and let E be the reduced exceptional divisor. Suppose that $H^2(T_V(-\log\ E)) = 0$. Then $H^2(T^0_X) = 0$ and there is a $\mathbb{Q}$–Gorenstein smoothing of X.*

Note that Theorem 2.5 above can be easily generalized to any log resolution of X by keeping the vanishing of cohomologies under blowing up at the points. It is obtained by the following well-known result.

Proposition 2.6 (Flenner and Zaidenberg [7, Section 1]) *Let V be a nonsingular surface and let D be a simple normal crossing divisor in V. Let $f\colon V' \to V$ be a blowing up of V at a point p of D. Set $D' = f^{-1}(D)_{\mathrm{red}}$. Then $h^2(T_{V'}(-\log\ D')) = h^2(T_V(-\log\ D))$.*

3 The main construction

We begin with a special elliptic fibration $g\colon E(1) = \mathbb{P}^2 \sharp 9\overline{\mathbb{P}}^2 \to \mathbb{P}^1$ which is constructed as follows: Let A be a line and B be a smooth conic in $\mathbb{P}^2$ such that A and B meet at two different points. Choose a tangent line L_1 to B at a point $p \in B$ so that L_1 intersects with A at a different point $q \in A$, and draw a tangent line L_2 from q to B which tangents at the point $r \in B$. Let L_3 be the line connecting p and r which meets A at s. We may assume that $p, r \notin A \cap B$ and $s \notin B$ (Figure 1).

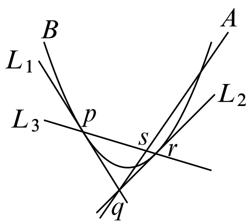


Figure 1: A cubic pencil

We now consider a cubic pencil in $\mathbb{P}^2$ induced by $A + B$ and $L_1 + L_2 + L_3$, ie $\lambda(A + B) + \mu(L_1 + L_2 + L_3)$, for $[\lambda : \mu] \in \mathbb{P}^1$. In order to obtain an elliptic fibration over $\mathbb{P}^1$ from the pencil, we blow up three times at p and r, respectively, and twice at q, including infinitely near base points at each point. We perform one further blow-up at the base point s. By blowing-up totally nine times, we resolve all base points (including infinitely near base points) of the pencil and we then get an elliptic fibration $Y = \mathbb{P}^2 \sharp 9\overline{\mathbb{P}}^2$ over $\mathbb{P}^1$. Let $\{e_1, e_2, e_3\}$, $\{e_4, e_5\}$, $\{e_6, e_7, e_8\}$, $\{e_9\}$ be the exceptional divisors lying over p, q, r and s, respectively, where e_3, e_5, e_8, e_9 are the exceptional divisors corresponding to the last blow-ups at p, q, r and s, respectively (Figure 2).

The elliptic fibration Y has an I_8–singular fiber consisting of the proper transforms $\widetilde{L_i}$ of L_i $(i = 1, 2, 3)$. Also Y has an I_2–singular fiber consisting of the proper transforms $\widetilde{A}$ and $\widetilde{B}$ of A and B, respectively. According to the list of Persson [19],

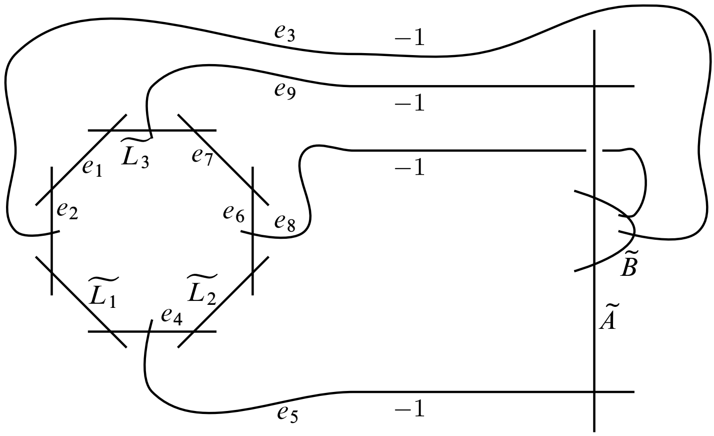


Figure 2: The proper transform of the pencil with nine exceptional curves

there exist only two more nodal singular fibers on $g\colon Y \to \mathbb{P}^1$. For example, the pencil used above can be chosen explicitly as follows:

$$\lambda x(x^2 + (y-2z)^2 - z^2) + \mu(y - \sqrt{3}x)(y + \sqrt{3}x)(2y - 3z).$$

Note that this pencil has singular fibers at $[\lambda : \mu] = [1 : 0]$, $[0 : 1]$, $[3\sqrt{3} : 2]$ and $[3\sqrt{3} : -2]$. Hence the fibration $g\colon Y \to \mathbb{P}^1$ has one I_8–singular fiber, one reducible I_2–singular fiber and two nodal singular fibers. Note that e_3, e_5, e_8, e_9 are sections of the elliptic fibration $g\colon Y \to \mathbb{P}^1$. Among these sections, we will use only three sections, e_3, e_5, e_9, in the following main construction. We denote the three sections e_3, e_5, e_9 by S_1, S_2, S_3, respectively (Figure 3).

Main construction

Let $Z' := Y \sharp 2\overline{\mathbb{P}}^2$ be the surface obtained by blowing up at two singular points of two nodal fibers on Y, and denote this map by τ. Then there are two fibers such that each consists of two $\mathbb{P}^1$'s, say E_i and F_i, satisfying $E_i^2 = -1$, $F_i^2 = -4$ and $E_i \cdot F_i = 2$ for $i = 1, 2$. Note that each E_i is an exceptional curve and F_i is the proper transform of a nodal fiber. We blow up twice at the intersection points between S_1 and F_i for $i = 1, 2$. We also blow up twice at the intersection points between S_3 and F_2, $\widetilde{A}$. And then blow up three times at the intersection points of between S_2 and F_1, F_2, $\widetilde{A}$. Finally, blowing up at the marked point $\odot$ on the I_8–singular fiber, we then get $Z'' := Y \sharp 10\overline{\mathbb{P}}^2$ (Figure 4). Note that the self-intersection numbers of proper transforms are as follows: $[S_1]^2 = -3$, $[S_2]^2 = -4$, $[S_3]^2 = -3$, $[F_1]^2 = -6$,

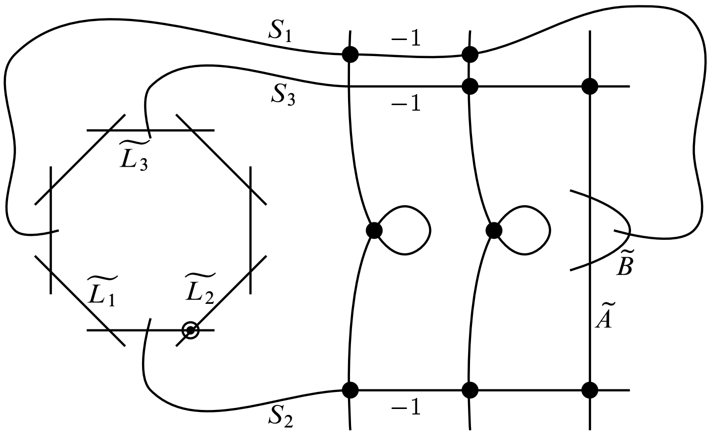


Figure 3: An elliptic fibration $Y = E(1)$

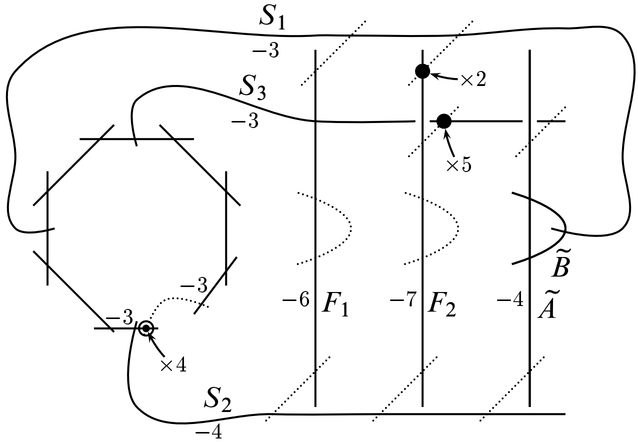


Figure 4: A rational surface $Z'' = Y \sharp 10\overline{\mathbb{P}}^2$

$[F_2]^2 = -7$ and $[\widetilde{A}]^2 = -4$. Here we denote the proper transforms of S_i, F_j, $\widetilde{A}$, $\widetilde{B}$ again by the same notation.

Next, we blow up two times successively at the intersection point between F_2 and the exceptional curve meeting F_2 and S_1. It makes a chain of $\mathbb{P}^1$

$$\overset{-9}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$$

lying in the total transform of F_2. We then blow up five times successively at the intersection point between S_3 and the exceptional curve meeting S_3 and F_2, so that it produces a chain of $\mathbb{P}^1$

$$\overset{-8}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$$

lying in the total transform of S_3. Finally, we blow up four times successively at the marked point $\odot$ of the proper transform of the I_8–singular fiber, so that it produces a chain of $\mathbb{P}^1$

$$\overset{-7}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}$$

lying in the proper transform of the I_8–singular fiber. Note that the self-intersection numbers of proper transforms, denoted again by the same notation, are as follows: $[S_1]^2 = -3$, $[S_2]^2 = -4$, $[S_3]^2 = -8$, $[F_1]^2 = -6$, $[F_2]^2 = -9$ and $[\widetilde{A}]^2 = -4$.

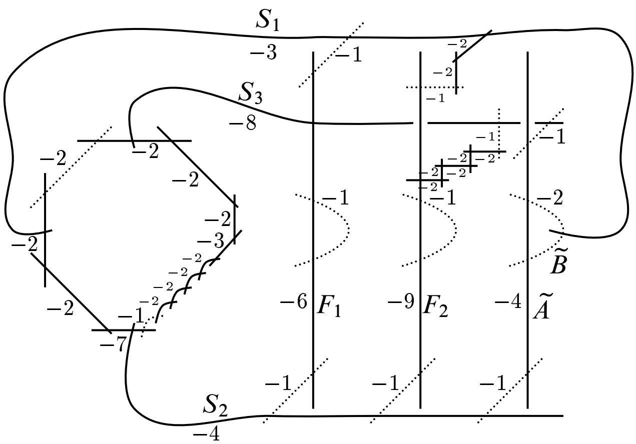


Figure 5: A rational surface $Z = Y \sharp 21\overline{\mathbb{P}}^2$

In summary, we get a rational surface $Z := Y \sharp 21\overline{\mathbb{P}}^2$, which contains four disjoint linear chains of $\mathbb{P}^1$:

$$C_{2,1} = \overset{-4}{\circ}(\widetilde{A}),$$

$$C_{7,1} = \overset{-9}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$$

(which contains the proper transform of F_2),

$$C_{19,5} = \overset{-4}{\circ} - \overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$$

(which contains the proper transforms of S_1, S_2, and a part of proper transforms of I_8–singular fibers) and

$$C_{35,6} = \overset{-6}{\circ} - \overset{-8}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$$

(which contains the proper transforms of S_3, F_1 and a part of proper transforms of I_8–singular fibers) (Figure 5).

Finally, we follow the same procedures as in Lee and Park [9]. That is, we contract these four disjoint chains of $\mathbb{P}^1$ from Z. Since it satisfies the Artin's criterion, it produces a projective surface with four singularities of class T (see Artin [2, Section 2]). We denote this surface by X.

In the next sections, we are going to prove that X has a $\mathbb{Q}$–Gorenstein smoothing and a general fiber of the $\mathbb{Q}$–Gorenstein smoothing is a simply connected minimal complex surface of general type with $p_g = 0$ and $K^2 = 3$.

In the remaining of this section, we investigate a rational blow-down manifold of the surface Z obtained in the main construction above. First we describe topological aspects of a rational blow-down surgery (see Fintushel and Stern [6] and Park [17] for details): For any relatively prime integers p and q with $p > q > 0$, we define a configuration $C_{p,q}$ as a smooth 4–manifold obtained by plumbing disk bundles over the 2–sphere instructed by the following linear diagram

$$\underset{u_k}{\overset{-b_k}{\circ}} - \underset{u_{k-1}}{\overset{-b_{k-1}}{\circ}} - \cdots - \underset{u_2}{\overset{-b_2}{\circ}} - \underset{u_1}{\overset{-b_1}{\circ}}$$

where $p^2/(pq-1) = [b_k, b_{k-1}, \dots, b_1]$ is the unique continued fraction with all $b_i \geq 2$, and each vertex u_i represents a disk bundle over the 2–sphere whose Euler number is $-b_i$. Orient the 2–spheres in $C_{p,q}$ so that $u_i \cdot u_{i+1} = +1$. Then the configuration $C_{p,q}$ is a negative definite simply connected smooth 4–manifold whose boundary is the lens space $L(p^2, 1 - pq)$.

Definition Suppose M is a smooth 4–manifold containing a configuration $C_{p,q}$. Then we construct a new smooth 4–manifold M_p, called a *(generalized) rational blow-down* of M, by replacing $C_{p,q}$ with the rational ball $B_{p,q}$. Note that this process is well-defined, that is, a new smooth 4–manifold M_p is uniquely determined (up to diffeomorphism) from M because each diffeomorphism of $\partial B_{p,q}$ extends over the rational ball $B_{p,q}$. We call this a *rational blow-down surgery*.

Furthermore, M Symington proved that a rational blow-down manifold M_p admits a symplectic structure in some cases. For example, if M is a symplectic 4–manifold containing a configuration $C_{p,q}$ such that all 2–spheres u_i in $C_{p,q}$ are symplectically embedded and intersect positively, then the rational blow-down manifold M_p also admits a symplectic structure; cf Symington [21; 22].

Now we perform a rational blow-down surgery of the surface Z obtained in the main construction. Note that the surface Z contains four disjoint configurations $C_{35,6}$, $C_{19,5}$, $C_{7,1}$ and $C_{2,1}$. Let us decompose the surface Z into

$$Z = Z_0 \cup \{C_{35,6} \cup C_{19,5} \cup C_{7,1} \cup C_{2,1}\}.$$

Then the 4–manifold, say $Z_{35,19,7,2}$, obtained by rationally blowing down along the four configurations can be decomposed into

$$Z_{35,19,7,2} = Z_0 \cup \{B_{35,6} \cup B_{19,5} \cup B_{7,1} \cup B_{2,1}\},$$

where $B_{35,6}$, $B_{19,5}$, $B_{7,1}$ and $B_{2,1}$ are the corresponding rational balls. We claim:

Theorem 3.1 *The rational blow-down $Z_{35,19,7,2}$ of the surface Z in the main construction is a simply connected closed symplectic 4–manifold with $b_2^+ = 1$ and $K^2 = 3$.*

Proof Since all the curves lying in the configurations $C_{35,6}$, $C_{19,5}$, $C_{7,1}$ and $C_{2,1}$ are symplectically (in fact, holomorphically) embedded 2–spheres, Symington's result [21; 22] guarantees the existence of a symplectic structure on the rational blow-down 4–manifold $Z_{35,19,7,2}$. Furthermore, it is easy to check that $b_2^+(Z_{35,19,7,2}) = b_2^+(Z) = 1$ and $K^2(Z_{35,19,7,2}) = K^2(Z) + 24 = 3$.

It remains to prove the simple connectivity of $Z_{35,19,7,2}$: Since $\pi_1(\partial B_{p,q}) \to \pi_1(B_{p,q})$ is surjective (see Looijenga and Wahl [11, Section 5]), by Van Kampen's theorem, it suffices to show that $\pi_1(Z_0) = 1$. First, note that Z and all four configurations $C_{35,6}$, $C_{19,5}$, $C_{7,1}$ and $C_{2,1}$ are all simply connected. Hence, applying Van Kampen's theorem on Z inductively, we get

$$1 = \pi_1(Z_0)/\langle N_{i_*(\alpha)}, N_{j_{1*}(\beta_1)}, N_{j_{2*}(\beta_2)}, N_{j_{3*}(\beta_3)} \rangle. \tag{3-1}$$

Here i_*, j_{1*}, j_{2*} and j_{3*} are induced homomorphisms by inclusions $i\colon \partial C_{19,5} \to Z_0$, $j_1\colon \partial C_{35,6} \to Z_0$, $j_2\colon \partial C_{7,1} \to Z_0$ and $j_3\colon \partial C_{2,1} \to Z_0$ respectively. We may also choose the generators, say α, β_1, β_2 and β_3, of $\pi_1(\partial C_{19,5}) \cong \mathbb{Z}_{19^2}$, $\pi_1(\partial C_{35,6}) \cong \mathbb{Z}_{35^2}$, $\pi_1(\partial C_{7,1}) \cong \mathbb{Z}_{7^2}$ and $\pi_1(\partial C_{2,1}) \cong \mathbb{Z}_{2^2}$, so that α, β_1, β_2 and β_3 are represented by circles $\partial C_{19,5} \cap E_1'$ (equivalently $\partial C_{19,5} \cap E_2'$ or $\partial C_{19,5} \cap E_3'$), $\partial C_{35,6} \cap E_1'$, $\partial C_{7,1} \cap E_2'$ and $\partial C_{2,1} \cap E_3'$, respectively, where E_1', E_2' and E_3' are exceptional curves connecting the last 2–spheres in the configurations $C_{19,5}$ and $C_{35,6}$, $C_{19,5}$ and $C_{7,1}$, $C_{19,5}$ and $C_{2,1}$, respectively. Note that the circle cut out by a 2–sphere which intersects transversely one of the two end 2–spheres in the configurations $C_{p,q}$ is a generator of π_1 of the lens space, and other circles cut out by a 2–sphere which intersects transversely one of the middle 2–spheres in the configurations $C_{p,q}$, is a power of the generator by Mumford [13]. Finally $N_{i_*(\alpha)}$ and $N_{j_{i*}(\beta_i)}$ denote the least normal subgroups of $\pi_1(Z_0)$ containing $i_*(\alpha)$ and $j_{i*}(\beta_i)$ respectively. Note that there is a relation between $i_*(\alpha)$ and $j_{i*}(\beta_i)$ when we restrict them to Z_0. That is, they satisfy either $i_*(\alpha) = \gamma^{-1} \cdot j_{i*}(\beta_i) \cdot \gamma$ or $i_*(\alpha) = \gamma^{-1} \cdot j_{i*}(\beta_i)^{-1} \cdot \gamma$ (depending on orientations) for some path γ, because one is homotopic to the other in $E'_i \setminus \{\text{two open disks}\} \subset Z_0$. Hence, by combining two facts above, for example, we get $i_*(\alpha)^{19^2} = (\gamma^{-1} \cdot j_{1*}(\beta_1)^{\pm 1} \cdot \gamma)^{19^2} = \gamma^{-1} \cdot j_{1*}(\beta_1)^{\pm 19^2} \cdot \gamma = 1 = j_{1*}(\beta_1)^{35^2}$. Since the two numbers 19^2 and 35^2 are relatively prime, the element $j_{1*}(\beta_1)$ should be trivial. So the relation $i_*(\alpha) = \gamma^{-1} \cdot j_{1*}(\beta_1)^{\pm 1} \cdot \gamma$ implies the triviality of $i_*(\alpha)$.

Furthermore, since $i_*(\alpha)$ and $j_{i*}(\beta_i)$ are also conjugate to each other for $i = 2, 3$, the triviality of $i_*(\alpha)$ implies that $j_{2*}(\beta_2)$ and $j_{3*}(\beta_3)$ are trivial. Hence, all normal subgroups $N_{i_*(\alpha)}$ and $N_{j_{i*}(\beta_i)}$ are trivial, so that relation (1) implies $\pi_1(Z_0) = 1$. □

4 Existence of smoothing

In this section we prove the existence of a $\mathbb{Q}$–Gorenstein smoothing for the singular projective surface X which is obtained by contracting from the rational surface Z in the main construction in Section 3. The procedure is parallel to the $K^2 = 2$ case appeared in Lee and Park [9]. For the completeness of this article, we repeat the procedure here. First we need the following two essential lemmas.

Lemma 4.1 (Lee and Park [9]) *Let Y be a rational elliptic surface. Let C be a general fiber of the elliptic fibration $g\colon Y \to \mathbb{P}^1$. Then the global sections $H^0(Y, \Omega_Y(kC))$ come from the global sections $H^0(Y, g^*\Omega_{\mathbb{P}^1}(k))$. In particular, $h^0(Y, \Omega_Y(kC)) = k-1$ for $k \geq 1$.*

Lemma 4.2 *Suppose $Z' = Y \sharp 2\overline{\mathbb{P}}^2$ is the rational elliptic surface in the main construction and F_i is the proper transform of a nodal fiber in Z' for $i = 1, 2$. Let $\widetilde{A}$ and $\widetilde{B}$ be the proper transforms of the line A and the conic B respectively. Let D be the reduced subscheme of the I_8–singular fiber. Assume that D is not whole I_8–singular fiber as a reduced scheme. Then $H^2(Z', T_{Z'}(-F_1 - F_2 - \widetilde{A} - D)) = 0$.*

Proof By Serre duality, it is equivalent to prove

$$H^0(Z', \Omega_{Z'}(K_{Z'} + F_1 + F_2 + \widetilde{A} + D)) = 0.$$

Let C be a general fiber in the elliptic fibration $g\colon Y \to \mathbb{P}^1$. Since $K_{Z'} = \tau^*(-C) + E_1 + E_2$ and $\tau^*(C) = F_1 + 2E_1 = F_2 + 2E_2$,

$$H^0(Z', \Omega_{Z'}(K_{Z'} + F_1 + F_2 + \widetilde{A} + D)) \subseteq H^0(Z', \Omega_{Z'}(\tau^*(C) + \widetilde{A} + D)).$$

Furthermore, since $\widetilde{A}$ and D are not changed by the map τ, we have the same curves in Y. So $H^0(Z', \Omega_{Z'}(\tau^*(C)+\widetilde{A}+D)) = H^0(Y, \Omega_Y(C+\widetilde{A}+D))$ by the projection formula. Note that $\tau_*\Omega_{Z'} = \Omega_Y$. Then the cohomology $H^0(Y, \Omega_Y(C + \widetilde{A} + D))$ vanishes: Note that $H^0(Y, \Omega_Y(C + \widetilde{A} + D)) = H^0(Y, \Omega_Y(3C - \widetilde{B} - G))$ with $G + D = I_8$–singular fiber. By Lemma 4.1 above, all global sections of $\Omega_Y(3C)$ are coming form the global sections of $g^*(\Omega_{\mathbb{P}^1}(3)) = g^*(\mathcal{O}_{\mathbb{P}^1}(1))$. But, if this global section vanishes on $\widetilde{B}$ and G which lie on two different fibers, then it should be zero. Note that the dualizing sheaf of each fiber of the elliptic fibration is the structure sheaf of the fiber by using the adjunction formula. Therefore we have $H^2(Z', T_{Z'}(-F_1 - F_2 - \widetilde{A} - D)) = 0$. □

Theorem 4.3 *The projective surface X with four singularities of class T in the main construction has a $\mathbb{Q}$–Gorenstein smoothing.*

Proof Let D be the reduced scheme of the I_8–singular fiber minus the rational (-2)–curve G in the main construction in Section 3. Note that the curve G is not contracted from Z to X. By Lemma 4.2 above, we have $H^2(Z', T_{Z'}(-F_1 - F_2 - \widetilde{A} - D)) = H^2(Z', T_{Z'}(-\log(F_1 + F_2 + \widetilde{A} + D))) = 0$. Let $D_{Z'} = F_1 + F_2 + \widetilde{A} + D + S_1 + S_2 + S_3$. Since the self-intersection number of the section is -1, we still have the vanishing $H^2(Z', T_{Z'}(-\log\ D_{Z'})) = 0$. Remind that the surface $Z'' = Y \sharp 10\overline{\mathbb{P}}^2$ is obtained by blowing up eight times from Z': We blow up twice at the intersection points between S_1 and F_i for $i = 1, 2$. We also blow up twice at the intersection points between S_3 and F_2, $\widetilde{A}$. And then blow up three times at the intersection points of between S_2 and F_1, F_2, $\widetilde{A}$. Finally, blowing up at the marked point $\odot$ on the I_8–singular fiber, we then get a rational surface Z'' (Figure 4). Now choose the exceptional curve in the total transform of I_8–singular fiber which intersects the proper transform of I_8, and choose two exceptional curves in the total transform of F_2 which intersect the proper transform of S_1 and S_3. Let $D_{Z''}$ be the reduced scheme of $F_1 + F_2 + \widetilde{A} + D + S_1 + S_2 + S_3 +$ these three exceptional divisors. Then, by Lemma 4.2, Proposition 2.6 and the self-intersection number, -1, of each exceptional divisor, we have $H^2(Z'', T_{Z''}(-\log\ D_{Z''})) = 0$. Finally, by using the same argument finite times through blowing up, we have the vanishing $H^2(Z, T_Z(-\log\ D_Z)) = 0$, where D_Z are the four disjoint linear chains of $\mathbb{P}^1$ which are the exceptional divisors from the contraction from Z to X. Hence there is a $\mathbb{Q}$–Gorenstein smoothing for X by Theorem 2.5. □

5 Properties of X_t

We showed in Section 4 that the projective surface X has a $\mathbb{Q}$–Gorenstein smoothing. We denote a general fiber of the $\mathbb{Q}$–Gorenstein smoothing by X_t. In this section, we prove that X_t is a simply connected and minimal surface of general type with $p_g = 0$ and $K^2_{X_t} = 3$ by using a standard argument.

We first prove that X_t satisfies $p_g = 0$ and $K^2 = 3$: Since Z is a nonsingular rational surface and X has only rational singularities, X is a projective surface with $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Then the upper semicontinuity implies that $H^2(X_t, \mathcal{O}_{X_t}) = 0$, so that the Serre duality implies that $p_g(X_t) = 0$. And $K^2_X = 3$ can be computed by using the explicit description of f^*K_X (refer to (5-1) below). Then we have $K^2_{X_t} = 3$ by the property of the $\mathbb{Q}$–Gorenstein smoothing.

Next, let us show the minimality of X_t: As we noticed in Section 3, the surface Z contains the following four chains of $\mathbb{P}^1$ including the proper transforms of three sections. We denote them by the following dual graphs:

$$C_{35,6} = \underset{G_1}{\overset{-6}{\circ}} - \underset{G_2}{\overset{-8}{\circ}} - \underset{G_3}{\overset{-2}{\circ}} - \underset{G_4}{\overset{-2}{\circ}} - \underset{G_5}{\overset{-2}{\circ}} - \underset{G_6}{\overset{-3}{\circ}} - \underset{G_7}{\overset{-2}{\circ}} - \underset{G_8}{\overset{-2}{\circ}} - \underset{G_9}{\overset{-2}{\circ}} - \underset{G_{10}}{\overset{-2}{\circ}}$$

$$C_{19,5} = \underset{H_1}{\overset{-4}{\circ}} - \underset{H_2}{\overset{-7}{\circ}} - \underset{H_3}{\overset{-2}{\circ}} - \underset{H_4}{\overset{-2}{\circ}} - \underset{H_5}{\overset{-3}{\circ}} - \underset{H_6}{\overset{-2}{\circ}} - \underset{H_7}{\overset{-2}{\circ}}$$

$$C_{7,1} = \underset{I_1}{\overset{-9}{\circ}} - \underset{I_2}{\overset{-2}{\circ}} - \underset{I_3}{\overset{-2}{\circ}} - \underset{I_4}{\overset{-2}{\circ}} - \underset{I_5}{\overset{-2}{\circ}} - \underset{I_6}{\overset{-2}{\circ}}, \quad C_{2,1} = \underset{\widetilde{A}}{\overset{-4}{\circ}}$$

and we also denote the four special fibers by the following dual graphs:

$$\begin{array}{ccccc} & & \widetilde{B},-2 & & \\ & & \circ & & \\ & & \| & & \\ \circ & - & \circ & - & \circ \\ E_4,-1 & & \widetilde{A},-4 & & E'_4,-1 \end{array} \qquad \begin{array}{ccccc} & & E_1,-1 & & \\ & & \circ & & \\ & & \| & & \\ \circ & - & \circ & - & \circ \\ E'_1,-1 & & G_1,-6 & & E''_1,-1 \end{array}$$

$$\begin{array}{ccccccccccc} & & E_2,-1 & & & & & & & & \\ & & \circ & & & & & & & & \\ & & \| & & & & & & & & \\ \circ & - & \circ & - & \circ & - & \circ & - & \circ & & \\ E'''_2,-1 & & I_1,-9 & & E'_2,-1 & & H_7,-2 & & H_6,-2 & & \\ & & | & & & & & & & & \\ & & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ & & I_2,-2 & & I_3,-2 & & I_4,-2 & & I_5,-2 & & I_6,-2 & & E''_2,-1 \end{array}$$

$$\begin{tikzcd}[arrows=dash]
\underset{\circ}{G,-2} \arrow[r] \arrow[dr] & \underset{\circ}{G_3,-2} \arrow[r] & \underset{\circ}{G_4,-2} \arrow[r] & \underset{\circ}{G_5,-2} \arrow[r] & \underset{\circ}{G_6,-3} \arrow[r] & \underset{\circ}{G_7,-2} \arrow[r] & \underset{\circ}{G_8,-2} \arrow[d] \\
 & \underset{\circ}{H_4,-2} \arrow[r] & \underset{\circ}{H_3,-2} \arrow[r] & \underset{\circ}{H_2,-7} \arrow[r] & \underset{\circ}{E_3,-1} \arrow[r] & \underset{\circ}{G_{10},-2} \arrow[r] & \underset{\circ}{G_9,-2}
\end{tikzcd}$$

Note that the final one indicates the total transform of I_8–singular fiber and G denotes the rational (-2)–curve which is not contracted from Z to X. The numbers indicate the self-intersection numbers of curves. Let $f\colon Z \to X$ and let $h\colon Z \to Y$. Then we have:

$$\begin{aligned}
K_Z \equiv f^*K_X - \Big(&\frac{29}{35}G_1 + \frac{34}{35}G_2 + \frac{33}{35}G_3 + \frac{32}{35}G_4 + \frac{31}{35}G_5 + \frac{30}{35}G_6 + \frac{24}{35}G_7 \\
&+ \frac{18}{35}G_8 + \frac{12}{35}G_9 + \frac{6}{35}G_{10} + \frac{14}{19}H_1 + \frac{18}{19}H_2 + \frac{17}{19}H_3 + \frac{16}{19}H_4 \\
&+ \frac{15}{19}H_5 + \frac{10}{19}H_6 + \frac{5}{19}H_7 + \frac{6}{7}I_1 + \frac{5}{7}I_2 + \frac{4}{7}I_3 + \frac{3}{7}I_4 \\
&+ \frac{2}{7}I_5 + \frac{1}{7}I_6 + \frac{1}{2}\widetilde{A}\Big),
\end{aligned}$$

$$\begin{aligned}
K_Z \equiv h^*K_Y &+ E_1 + E_1' + E_1'' + E_2 + 3E_2' + 2H_7 + H_6 \\
&+ 6E_2'' + 5I_6 + 4I_5 + 3I_4 + 2I_3 + I_2 + E_2''' \\
&+ 5E_3 + 4G_{10} + 3G_9 + 2G_8 + G_7 + E_4 + E_4'.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
h^*K_Y \equiv &-\frac{1}{2}\left(2E_1 + E_1' + E_1'' + G_1\right) \\
&-\frac{1}{2}\big(2E_2 + 3E_2' + 2H_7 + H_6 + 6E_2'' + 5I_6 + 4I_5 + 3I_4 \\
&\qquad\qquad\qquad\qquad\qquad\qquad + 2I_3 + I_2 + E_2''' + I_1\big)
\end{aligned}$$

Hence, combining these relations, we get

$$\begin{aligned}
f^*K_X \equiv &\frac{1}{2}E_1' + \frac{1}{2}E_1'' + \frac{3}{2}E_2' + 3E_2'' + \frac{1}{2}E_2''' + 5E_3 + E_4 + E_4' \\
&+ \frac{23}{70}G_1 + \frac{34}{35}G_2 + \frac{33}{35}G_3 + \frac{32}{35}G_4 + \frac{31}{35}G_5 + \frac{6}{7}G_6 \\
&+ \frac{59}{35}G_7 + \frac{88}{35}G_8 + \frac{117}{35}G_9 + \frac{146}{35}G_{10} + \frac{14}{19}H_1 \\
&+ \frac{18}{19}H_2 + \frac{17}{19}H_3 + \frac{16}{19}H_4 + \frac{15}{19}H_5 + \frac{39}{38}H_6 + \frac{24}{19}H_7 \\
&+ \frac{5}{14}I_1 + \frac{17}{14}I_2 + \frac{11}{7}I_3 + \frac{27}{14}I_4 + \frac{16}{7}I_5 + \frac{37}{14}I_6 + \frac{1}{2}\widetilde{A}.
\end{aligned} \tag{5-1}$$

Since all the coefficients are positive in the expression of f^*K_X, the $\mathbb{Q}$–divisor f^*K_X is nef if

$$f^*K_X \cdot E_i \geq 0 \quad \text{for } i = 3, 4, \qquad f^*K_X \cdot E_i' \geq 0 \quad \text{for } i = 1, 2, 4,$$
$$f^*K_X \cdot E_i'' \geq 0 \quad \text{for } i = 1, 2, \qquad f^*K_X \cdot E_2''' \geq 0.$$

We have $f^*K_X \cdot E_3 = \frac{79}{665}$, $f^*K_X \cdot E_4 = \frac{33}{70}$, $f^*K_X \cdot E_1' = \frac{411}{665}$, $f^*K_X \cdot E_2' = \frac{16}{133}$, $f^*K_X \cdot E_4' = \frac{9}{38}$, $f^*K_X \cdot E_1'' = \frac{376}{665}$, $f^*K_X \cdot E_2'' = \frac{43}{70}$ and $f^*K_X \cdot E_2''' = \frac{79}{133}$. Note that other divisors are contracted under the map f. The nefness of f^*K_X implies the nefness of K_X. Since all coefficients are positive in the expression of f^*K_X, we get the vanishing $h^0(-K_X) = 0$. Hence, by the upper semicontinuity property, ie the vanishing $h^0(-K_X) = 0$ implies that $h^0(-K_{X_t}) = 0$, we conclude that X_t is not a rational surface: If X_t is a rational surface with $h^0(-K_{X_t}) = 0$, then $\chi(2K_{X_t}) \leq 0$. But $\chi(2K_{X_t}) = \chi(\mathcal{O}_{X_t}) + K_{X_t}^2 = 4$, which is a contradiction. Since $K_{X_t}^2 = 3$, X_t is a surface of general type by the classification theory of surfaces. Let $\pi\colon \mathcal{X} \to \Delta$ be a $\mathbb{Q}$–Gorenstein smoothing of X. Since the $\mathbb{Q}$–Cartier divisor $K_{\mathcal{X}/\Delta}$ is π–big over Δ and π–nef at the point 0, the nefness of K_{X_t} is also obtained by shrinking Δ if it is necessary; cf Nakayama [14]. Therefore we have:

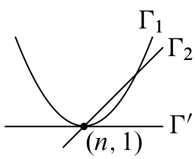
Proposition 5.1 *X_t is a minimal surface of general type with $p_g = 0$ and $K_{X_t}^2 = 3$.*

Finally, applying the standard arguments about Milnor fibers (see Looijenga and Wahl [11, Section 5]), we conclude that X_t is diffeomorphic to the rational blow-down 4–manifold $Z_{35,19,7,2}$ constructed in Theorem 3.1 (see Lee and Park [9] for details). Hence the simple connectivity of X_t follows from the fact that $Z_{35,19,7,2}$ is simply connected.

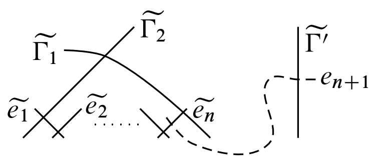
6 More examples

In this section we present more examples of simply connected and minimal complex surfaces of general type with $p_g = 0$ and $K^2 = 3$ using different configurations coming from different elliptic pencils in $\mathbb{P}^2$ (cf Figures 7–10 below). Since all the proofs are basically the same as the case of the main construction in Section 3, we only list elliptic pencils in $\mathbb{P}^2$, elliptic fibrations Y obtained from the pencils and the rational surfaces Z obtained by blowing up Y several times appropriately. In the rational surfaces Z, we indicate the configurations of linear chains of $\mathbb{P}^1$ which will be contracted so that we obtain a singular surface X which has a $\mathbb{Q}$–Gorenstein smoothing.

In Figures 7–10 below, the index, for example $I_6 + 2I_2 + 2I_1$ (Figure 7), denotes the type of singular fibers of elliptic fibrations. In part (a) of the figures, the pencil of cubics, we present cubic curves in the pencil giving rise to the singular fibers and we describe how they intersect. In the figure of the pencil, the pair $(n, 1)$ denotes the intersection numbers of a curve with the two branches of another curve at a node. We briefly explain how to get an elliptic fibration from a pencil of cubics. Let Γ and Γ' be two cubic curves in the pencil of cubics. Suppose that Γ' has the intersection numbers $(n, 1)$ with the two branches of Γ at a node of Γ. We blow up Γ $(n+1)$-times at the node to resolve base points of the pencil. Then we acquire n new curves $\widetilde{e_1}, \dots, \widetilde{e_n}$, which are proper transforms of the exceptional curves $e_1, \dots, e_n$, on the blow-up of Γ and a section e_{n+1} of an elliptic fibration $g\colon E(1) \to \mathbb{P}^1$, which is the exceptional curve of the final blow-up and connects $\widetilde{e_n}$ and $\widetilde{\Gamma'}$ (Figure 6). Blowing up several times at each intersection points of two cubic curves, we get an elliptic fibration $E(1)$.



(a) Before



(b) After

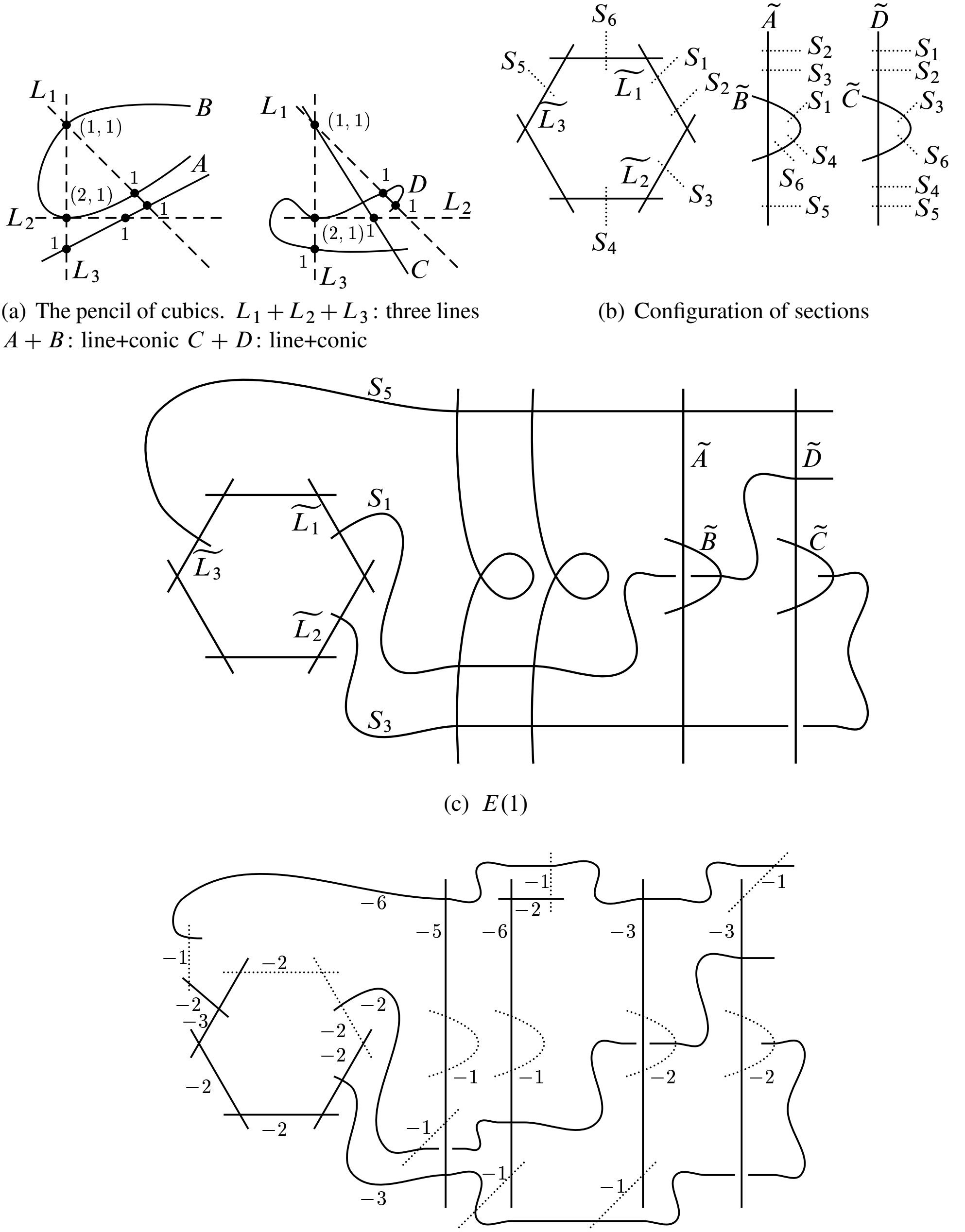
Figure 6: Blowing up at nodes

In part (b) of Figures 7–10, configuration of sections, we describe how sections S_i of $E(1)$ intersect with special singular fibers of the elliptic fibration. Since we don't use all sections, on $E(1)$, we indicate which sections are used to construct the rational surface Z. Finally, the number n in $Z = E(1) \sharp n\overline{\mathbb{P}^2}$ indicates how many times we blow up to get Z from $E(1)$. The linear chain $C_{p,q} = \overset{-a_1}{\circ} - \cdots - \overset{-a_n}{\circ}$ of $\mathbb{P}^1$ is denoted by $(-a_1, \dots, -a_n)$ for convenience.

7 A simply connected symplectic 4–manifold with $b_2^+ = 1$ and $K^2 = 4$

In this section we construct a new simply connected symplectic 4–manifold with $b_2^+ = 1$ and $K^2 = 4$ using a rational blow-down surgery, and then we discuss the existence of a complex structure on it by using $\mathbb{Q}$–Gorenstein smoothing theory.

We first consider the elliptic fibration $g\colon Y = E(1) \to \mathbb{P}^1$ used in the main construction in Section 3, which has one I_8–singular fiber, one I_2–singular fiber and three sections



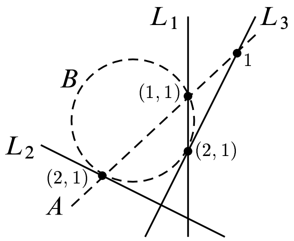
(a) The pencil of cubics. $L_1 + L_2 + L_3$: three lines
$A + B$: line+conic $C + D$: line+conic

(b) Configuration of sections

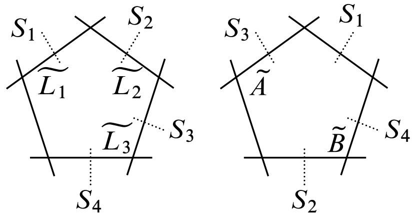
(c) $E(1)$

(d) $Z = E(1) \sharp 10\overline{\mathbb{P}^2}$. $C_{7,4} = (-2, -6, -2, -3)$.
$C_{48,17} = (-3, -6, -5, -3, -2, -2, -2, -3, -2)$.

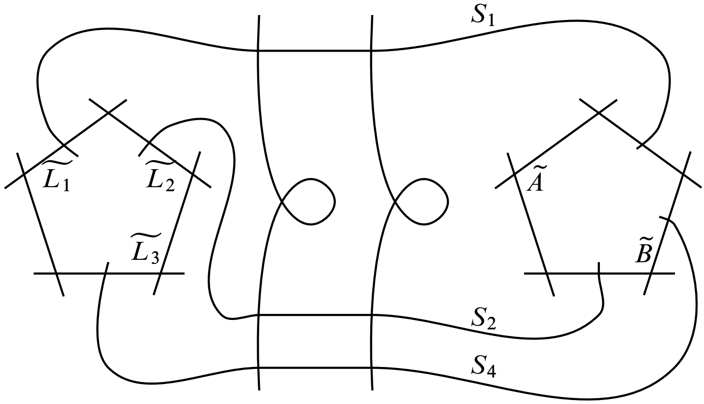
Figure 7: $I_6 + 2I_2 + 2I_1$



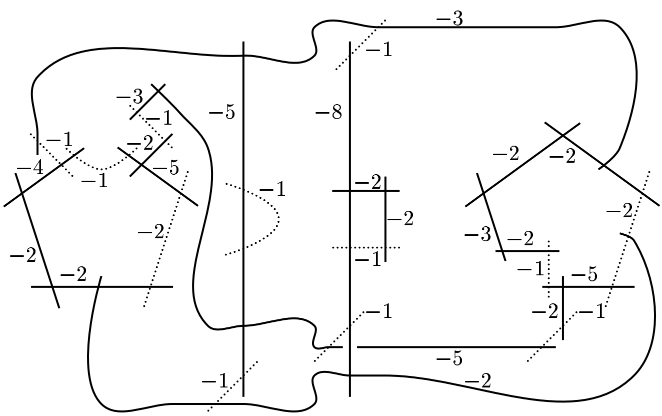
(a) The pencil of cubics.
$L_1 + L_2 + L_3$: three lines
$A + B$: line+conic



(b) Configuration of sections

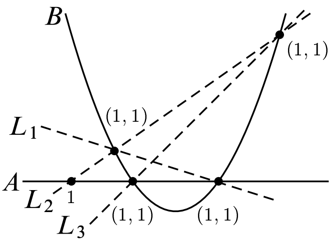


(c) $E(1)$

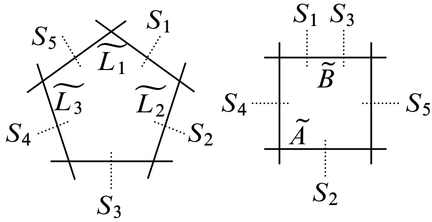


(d) $Z = E(1) \sharp 16\overline{\mathbb{P}^2}$. $C_{3,1} : (-5,-2)$. $C_{3,1} : (-5,-2)$.
$C_{16,11} : (-2,-2,-8,-2,-2,-2,-4)$.
$C_{39,14} : (-3,-5,-5,-3,-2,-2,-3,-2)$.

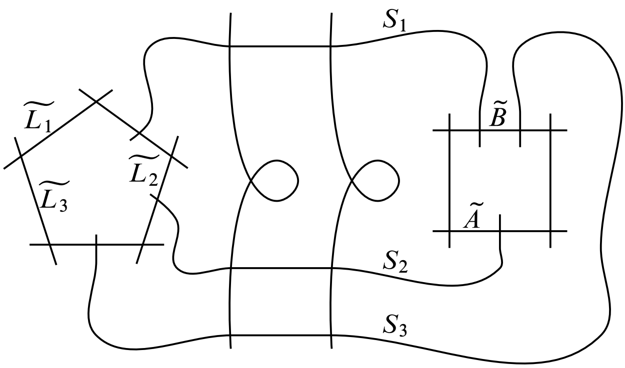
Figure 8: $I_5 + I_5 + 2I_1$



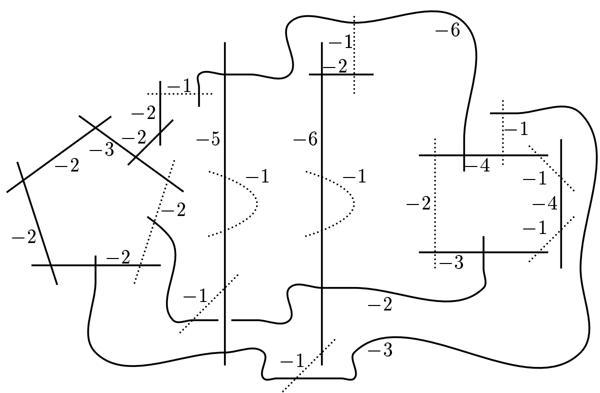
(a) The pencil of cubics.
$L_1 + L_2 + L_3$: three lines
$A + B$: line+conic



(b) Configuration of sections

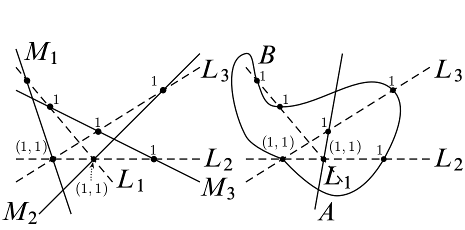


(c) $E(1)$

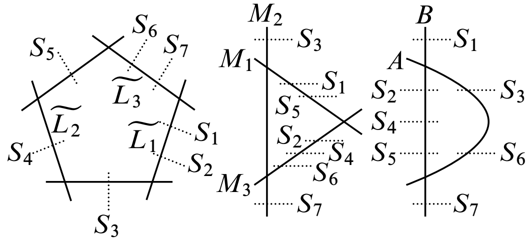


(d) $Z = E(1) \sharp 12\overline{\mathbb{P}^2}$. $C_{2,1} = (-4)$. $C_{7,4} = (-2,-6,-2,-3)$.
$C_{65,17} = (-4,-6,-5,-3,-2,-2,-2,-3,-2,-2)$.

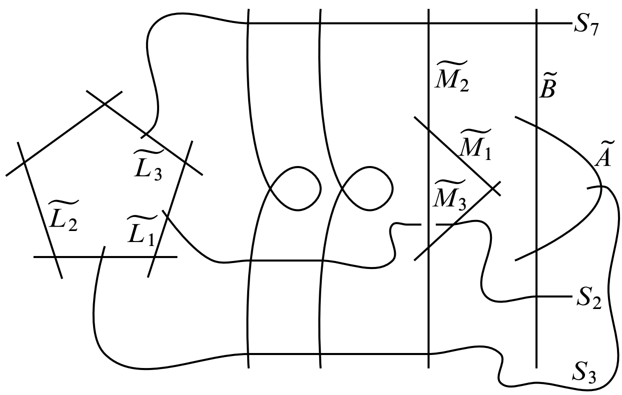
Figure 9: $I_5 + I_4 + 2I_1$



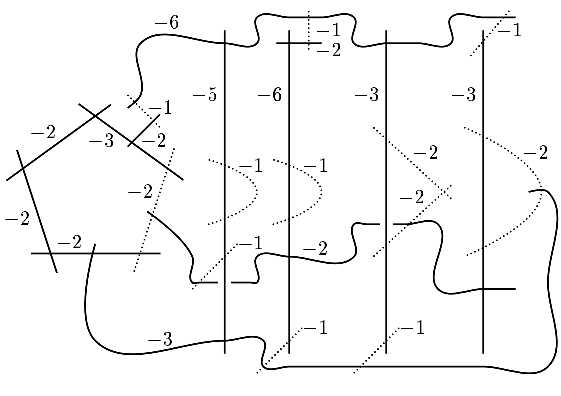
(a) The pencil of cubics.
$L_1 + L_2 + L_3$: three lines
$M_1 + M_2 + M_3$: three lines
$A + B$: line+conic



(b) Configuration of sections



(c) $E(1)$



(d) $Z = E(1) \sharp 10\overline{\mathbb{P}^2}$. $C_{7,4} = (-2, -6, -2, -3)$.
$C_{48,17} = (-3, -6, -5, -3, -2, -2, -2, -3, -2)$.

Figure 10: $I_5 + I_3 + I_2 + 2I_1$

S_1, S_2 and S_3. Among these sections, we use only S_1 and S_2 in the following construction (Figure 11).

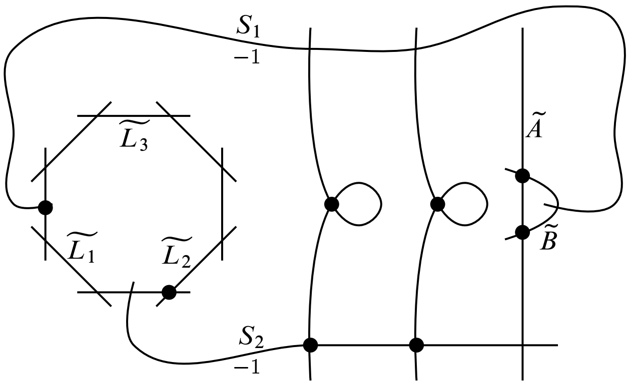


Figure 11: An elliptic fibration $Y = E(1)$

Let $Z' := Y \sharp 2\overline{\mathbb{P}}^2$ be a rational surface obtained by blowing up at two nodal points of two nodal singular fibers of Y. We denote the proper transforms of two nodal fibers by F_1 and F_2. Blowing up once at the six marked points • on S_2, the I_8–singular fiber and the I_2–singular fiber, we get a rational surface $Z'' := Y \sharp 8\overline{\mathbb{P}}^2$ (Figure 12).

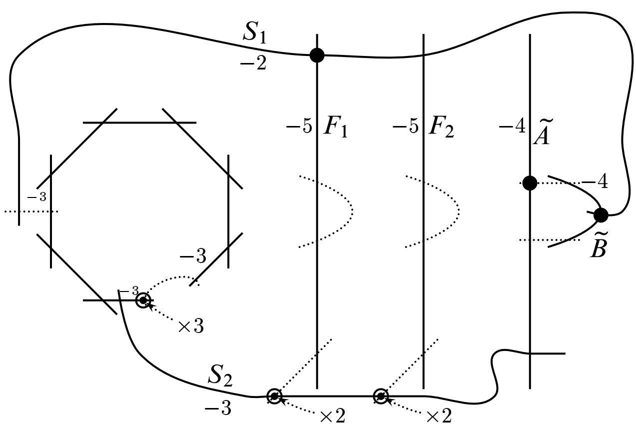


Figure 12: A rational surface $Z'' = Y \sharp 8\overline{\mathbb{P}}^2$

Finally, blow up once at the three marked points $\bullet$ on the proper transforms of S_1 and $\widetilde{A}$, and blow up again 3–times, 2–times and 2–times as indicated in Figure 12 at the three marked points $\odot$ on the proper transforms of I_9–singular fibers and S_2 respectively. Then we get a rational surface $Z := Y \sharp 18\overline{\mathbb{P}}^2$, which contains four disjoint linear chains of $\mathbb{P}^1$:

$$C_{131,27} = \overset{-5}{\circ} - \overset{-7}{\circ} - \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$$

(which contains the proper transforms of $\widetilde{A}$, S_2 and the I_8–singular fiber),

$$C_{7,2} = \overset{-4}{\circ} - \overset{-5}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$$

(which contains the proper transforms of S_1 and F_2),

$$C_{4,1} = \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$$

(which contains the proper transform of F_1) and

$$C_{3,1} = \overset{-5}{\circ} - \overset{-2}{\circ}$$

(which contains the proper transform of $\widetilde{B}$) (Figure 13).

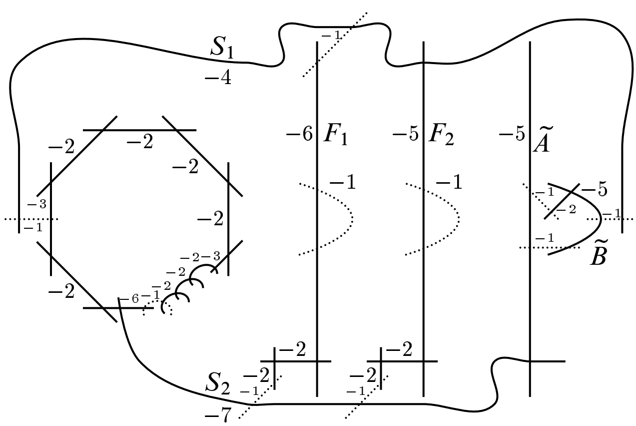


Figure 13: A rational surface $Z = Y \sharp 18\overline{\mathbb{P}}^2$

Now we perform a rational blow-down surgery of the surface $Z = Y \sharp 18\overline{\mathbb{P}}^2$ as before. Note that the surface Z contains four disjoint configurations $C_{131,27}$, $C_{7,2}$, $C_{4,1}$ and $C_{3,1}$. Let us decompose the surface Z into

$$Z = Z_0 \cup \{C_{131,27} \cup C_{7,2} \cup C_{4,1} \cup C_{3,1}\}.$$

Then the 4–manifold, say $Z_{131,7,4,3}$, obtained by rationally blowing down along the four configurations can be decomposed into

$$Z_{131,7,4,3} = Z_0 \cup \{B_{131,27} \cup B_{7,2} \cup B_{4,1} \cup B_{3,1}\},$$

where $B_{131,27}$, $B_{7,2}$, $B_{4,1}$ and $B_{3,1}$ are the corresponding rational balls. Then we have:

Theorem 7.1 *The rational blow-down $Z_{131,7,4,3}$ of the surface Z in the construction above is a simply connected symplectic 4–manifold with $b_2^+ = 1$ and $K^2 = 4$ which is homeomorphic, but not diffeomorphic, to a rational surface $\mathbb{P}^2 \sharp 5\overline{\mathbb{P}}^2$.*

Proof Since all proofs except the last statement are parallel to those of Theorem 3.1, we omit it. Furthermore, using the same method in the proof of Theorem 3 in Park [18], we show easily that the canonical class $K_{Z_{131,7,4,3}}$ and the compatible symplectic 2–form $\omega_{Z_{131,7,4,3}}$ on $Z_{131,7,4,3}$ induced from the standard canonical class K_Z and the compatible symplectic 2–form ω_Z on Z satisfy

$$\begin{aligned} &K_{Z_{131,7,4,3}} \cdot [\omega_{Z_{131,7,4,3}}] \\ &\qquad = K_Z \cdot [\omega_Z] - K_Z|_{C_{131,27} \cup C_{7,2} \cup C_{4,1} \cup C_{3,1}} \cdot [\omega_Z|_{B_{131,27} \cup B_{7,2} \cup B_{4,1} \cup B_{3,1}}] > 0. \end{aligned}$$

Hence the rational blow-down 4–manifold $Z_{131,7,4,3}$ is not diffeomorphic to a rational surface $\mathbb{P}^2 \sharp 5\overline{\mathbb{P}}^2$. Alternatively, one can get the same conclusion by showing that the standard canonical class K_Z of Z induces a nontrivial Seiberg–Witten basic class $\widetilde{K_Z}$ on $Z_{131,7,4,3}$ (refer to Stipsicz and Szabó [20, Theorem 3.1]). □

Remark (1) In fact, one can prove that the symplectic 4–manifold $Z_{131,7,4,3}$ constructed in Theorem 7.1 above is minimal by using a technique in Ozsváth and Szabó [15].

(2) Recently, several authors constructed a simply connected minimal symplectic 4–manifold with $b_2^+ = 1$ and $K^2 = 4$ using a fiber-sum technique and a Luttinger surgery (see Akhmedov [1] and Baldridge and Kirk [3]). We do not know whether the symplectic 4–manifold $Z_{131,7,4,3}$ constructed in Theorem 7.1 above is diffeomorphic to one of their constructions. Despite of the fact, there is a big difference between our construction and theirs. That is, our example $Z_{131,7,4,3}$ has more room to investigate the existence of a complex structure, which is discussed below.

We close this article by discussing the possibility of the existence of a complex structure on the symplectic 4–manifold $Z_{131,7,4,3}$. One way to approach this problem is to use $\mathbb{Q}$–Gorenstein smoothing theory as in Section 4. That is, by contracting four disjoint chains of $\mathbb{P}^1$ from Z as before, we first obtain a projective surface X with four singularities of class T. And then we investigate a $\mathbb{Q}$–Gorenstein smoothing for X. It is known that the cohomology $H^2(T^0_X)$ contains the obstruction space of a $\mathbb{Q}$–Gorenstein smoothing of X (refer to Proposition 2.4 and Theorem 2.5). That is, if $H^2(T^0_X) = 0$, then there is a $\mathbb{Q}$–Gorenstein smoothing of X. But the cohomology $H^2(T^0_X)$ is not zero in our case, so that it is hard to determine whether there exists a $\mathbb{Q}$–Gorenstein smoothing of X. Therefore we need to develop more $\mathbb{Q}$–Gorenstein smoothing theory in order to investigate the existence of a complex structure on the symplectic 4–manifold $Z_{131,7,4,3}$. We leave this question for future research.

Open Problem Determine whether the symplectic 4–manifold $Z_{131,7,4,3}$ admits a complex structure.

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Proposed: Ron Stern — Received: 10 February 2008
Seconded: Ron Fintushel, Simon Donaldson — Revised: 6 December 2008