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A simply connected surface of general type with $p_g = 0$ and $K^2 = 4$

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As a continuation of the recent results of Y Lee and the second author [5] and the authors [6], we construct a simply connected minimal complex surface of general type with $p_g = 0$ and $K^2 = 4$ by using a rational blow-down surgery and $\mathbb{Q}$–Gorenstein smoothing theory.

14J29; 14J10, 14J17, 53D05

1 Introduction

A rational surface satisfies $p_g = q = 0$ and it has Kodaira dimension $\kappa = -\infty$. Around 1894 Castelnuovo conjectured that a surface with $p_g = q = 0$ is rational. However the conjecture was soon shown to be false by the examples of Enriques. Castelnuovo also found another counterexample. Enriques' example has Kodaira dimension 0 while Castelnuovo's example has Kodaira dimension 1. Hence smooth surfaces of general type (ie Kodaira dimension 2) with $p_g = q = 0$ are very interesting from the point of view of the history of surfaces with $p_g = q = 0$.

Nowadays a large number of examples of surfaces of general type with $p_g = q = 0$ are known due to Godeaux, Campedelli and so on; cf Barth et al [3]. However it was only in 1983 that the first example of a *simply connected* surface of general type with $p_g = 0$ appeared, the so-called Barlow surface [2]. The Barlow surface has $K^2 = 1$. The second examples were discovered just recently. Motivated by a result of the second author [7], Y Lee and the second author [5] constructed a family of simply connected minimal complex surfaces of general type with $p_g = 0$ and $K^2 = 1, 2$ by using rational blow-down surgery and $\mathbb{Q}$–Gorenstein smoothing theory. After this construction, the authors [6] constructed a family of simply connected minimal complex surfaces of general type with $p_g = 0$ and $K^2 = 3$ by similar methods.

In this paper we extend the results of Lee and Park [5] and Park–Park–Shin [6] to the case of $K^2 = 4$. That is, we construct a new simply connected minimal surface of

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general type with $p_g = 0$ and $K^2 = 4$ by using a rational blow-down surgery and $\mathbb{Q}$–Gorenstein smoothing theory. This is the first example of such complex surfaces.

The key ingredient of this paper is to find an elliptic surface Y equipped with a special *bisection*, ie an irreducible curve on an elliptic surface whose intersection number with a fiber is 2. Blowing up Y several times appropriately, we get a rational surface Z which makes it possible to get such a complex surface. Once we have the right candidate Z with $K^2 = 4$, the remaining argument is similar to that of the $K^2 = 1, 2, 3$ cases appearing in Lee and Park [5] and Park–Park–Shin [6]. That is, by applying a rational blow-down surgery and $\mathbb{Q}$–Gorenstein smoothing theory developed in Lee and Park [5] to Z, we obtain a minimal complex surface of general type with $p_g = 0$ and $K^2 = 4$. Then we show that the surface is simply connected. Since almost all the proofs are parallel to the case of the main construction in Park–Park–Shin [6, Section 3], we only explain how to construct such a minimal complex surface and we prove that the surface is simply connected. The main result of this paper is the following theorem.

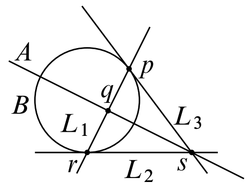
Theorem 1.1 *There exists a simply connected minimal complex surface of general type with $p_g = 0$ and $K^2 = 4$.*

Remark Răsdeaconu and Şuvaina [9] proved that the complex surfaces constructed in Lee and Park [5] and Park–Park–Shin [6] admit Kähler–Einstein metrics of negative scalar curvature. By applying their method to the complex surface constructed in this paper, one may prove that it also admits a Kähler–Einstein metric of negative scalar curvature; see Section 4.

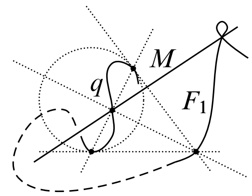
2 Main construction

We start with a special elliptic fibration $Y := \mathbb{P}^2 \sharp 9\overline{\mathbb{P}}^2$ which is used in the main construction of this paper. Let L_1, L_2, L_3 and A be lines in $\mathbb{P}^2$ and let B be a smooth conic in $\mathbb{P}^2$ intersecting as in Figure 1(a). We consider a pencil of cubics $\{\lambda(L_1 + L_2 + L_3) + \mu(A + B) \mid [\lambda : \mu] \in \mathbb{P}^1\}$ in $\mathbb{P}^2$ generated by two cubic curves $L_1 + L_2 + L_3$ and $A + B$, which has 4 base points, say, p, q, r and s. In order to obtain an elliptic fibration over $\mathbb{P}^1$ from the pencil, we blow up three times at p and r, respectively, and twice at s, including infinitely near base-points at each point. We perform one further blow-up at the base point q. By blowing up nine times in total, we resolve all base points (including infinitely near base-points) of the pencil and we then get an elliptic fibration $Y = \mathbb{P}^2 \sharp 9\overline{\mathbb{P}}^2$ over $\mathbb{P}^1$ (Figure 2).

There are four sections of the elliptic fibration Y corresponding to the four base points p, q, r and s. Among these sections we use only two sections corresponding to p



(a) Two generators



(b) A bisection

Figure 1: A pencil of cubics

and q, say S_1 and S_2 respectively, for the main construction. Furthermore, the elliptic fibration Y has an I_8–singular fiber consisting of the proper transforms $\widetilde{L_i}$ of L_i ($i = 1, 2, 3$). Also Y has an I_2–singular fiber consisting of the proper transforms $\widetilde{A}$ and $\widetilde{B}$ of A and B, respectively. According to the list of Persson [8], we may assume that Y has only two more nodal singular fibers F_1 and F_2 by choosing generally the L_i's, A and B (Figure 2). For example the pencil used in Park–Park–Shin [6] works:

$$\{\lambda(y-\sqrt{3}x)(y+\sqrt{3}x)(2y-3z)+\mu x(x^2+(y-2z)^2-z^2) \mid [\lambda:\mu]\in\mathbb{P}^1\}. \tag{2-1}$$

This pencil has singular fibers at $[\lambda:\mu] = [1:0]$, $[0:1]$, $[2:3\sqrt{3}]$ and $[2:-3\sqrt{3}]$. Furthermore, setting

$$F_1 = \big\{2(y-\sqrt{3}x)(y+\sqrt{3}x)(2y-3z)+3\sqrt{3}x(x^2+(y-2z)^2-z^2)=0\big\},$$
$$F_2 = \big\{2(y-\sqrt{3}x)(y+\sqrt{3}x)(2y-3z)-3\sqrt{3}x(x^2+(y-2z)^2-z^2)=0\big\},$$

one can easily check that F_1 and F_2 are nodal cubic curves with one node at $[\sqrt{3}:0:-1]$ and $[\sqrt{3}:0:1]$, respectively.

Let M be the line in $\mathbb{P}^2$ passing through the point q and the node of the nodal cubic curve F_1. The node of F_1 does not lie on any L_i's, A or B. Hence it satisfies that $M \neq L_1$, $M \neq A$ and $\widetilde{M}\cdot\widetilde{M} = 0$, where $\widetilde{M}$ is the proper transform of M in Y (Figure 1(b)). We may assume further that M does not pass through the node of the other nodal cubic curve F_2 by choosing generally the L_i's, A and B. For example, the pencil in (2-1) works: We have $q = [0:3:2]$. Hence the line M passing through q and the node of F_1 is $\{s[0:3:2]+t[\sqrt{3}:0:-1] \mid [s:t]\in\mathbb{P}^1\}$. It is obvious that the node $[\sqrt{3}:0:1]$ does not lie on the line M. Since M meets every member in the pencil at three points, $\widetilde{M}$ is a bisection of the elliptic fibration $Y \to \mathbb{P}^1$. Furthermore, since $q \in M$, the section S_2 meets $\widetilde{M}$ at one point (Figure 2).

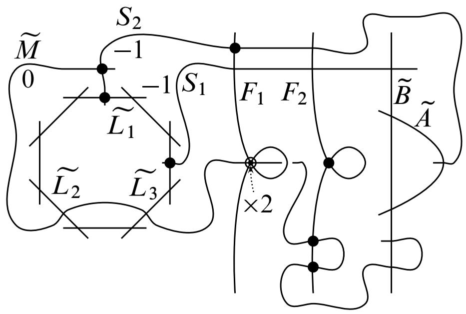


Figure 2: An elliptic fibration Y

Next, by blowing up nine times on Y, we construct a rational surface Z which contains a special configuration of linear chains of $\mathbb{P}^1$'s. At first we blow up twice at the marked point $\odot$ on F_1. We then blow up seven times in total at the six marked points $\bullet$ on each fiber and at the intersection point $\bullet$ of $\widetilde{M}$ and S_2. We then get a rational surface $Z = Y \sharp 9\overline{\mathbb{P}}^2$. We also denote by $\widetilde{F_i}$ $(i = 1, 2)$ the proper transforms of F_i. Then there exists a linear chain of $\mathbb{P}^1$'s in Z,

$$C_{252,145} = \underset{u_{13}}{\overset{-2}{\circ}} - \underset{u_{12}}{\overset{-4}{\circ}} - \underset{u_{11}}{\overset{-6}{\circ}} - \underset{u_{10}}{\overset{-2}{\circ}} - \underset{u_{9}}{\overset{-6}{\circ}} - \underset{u_{8}}{\overset{-2}{\circ}} - \underset{u_{7}}{\overset{-4}{\circ}} - \underset{u_{6}}{\overset{-2}{\circ}} - \underset{u_{5}}{\overset{-2}{\circ}} - \underset{u_{4}}{\overset{-2}{\circ}} - \underset{u_{3}}{\overset{-3}{\circ}} - \underset{u_{2}}{\overset{-2}{\circ}} - \underset{u_{1}}{\overset{-3}{\circ}},$$

which contains $\widetilde{A}$, S_2, $\widetilde{F_2}$, S_1, $\widetilde{F_1}$, $\widetilde{M}$, $\widetilde{L_2}$, $\widetilde{L_1}$ and $\widetilde{L_3}$, where u_i represents an embedded rational curve (Figure 3).

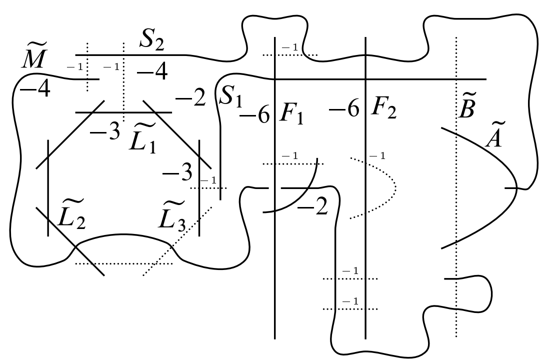


Figure 3: A rational surface $Z = Y \sharp 9\overline{\mathbb{P}}^2$

Finally, by applying $\mathbb{Q}$–Gorenstein smoothing theory to Z as in Lee and Park [5] and Park–Park–Shin [6], we construct a minimal complex surface with $p_g = 0$ and $K^2 = 4$. That is, we first contract the chain $C_{252,145}$ of $\mathbb{P}^1$'s from Z so that it produces a normal projective surface X with one permissible singular point. It then follows by a similar technique to one in Lee and Park [5] and Park–Park–Shin [6] that X has a $\mathbb{Q}$–Gorenstein smoothing. Let X_t be a general fiber of the $\mathbb{Q}$–Gorenstein smoothing of X. Since X is a (singular) surface with $p_g = 0$ and $K^2 = 4$, by applying general results of complex surface theory and $\mathbb{Q}$–Gorenstein smoothing theory, one may conclude that a general fiber X_t is a complex surface of general type with $p_g = 0$ and $K^2 = 4$.

The minimality of X_t follows from the nefness of the canonical divisor K_X of X. Let $f\colon Z \to X$ be the contraction of the chain $C_{252,145}$ of $\mathbb{P}^1$'s from Z to the singular surface X. By using a similar technique to one in Lee and Park [5] and Park–Park–Shin [6], it follows that the pullback f^*K_X of the canonical divisor K_X of X is effective and nef, hence K_X is also nef, which shows the minimality of X_t.

It remains to prove that X_t is simply connected.

Proposition 2.1 X_t *is simply connected.*

Proof Let Z_{252} be a rational blow-down 4–manifold obtained from Z by replacing the configuration $C_{252,145}$ with the corresponding rational ball $B_{252,145}$. Since a general fiber X_t of a $\mathbb{Q}$–Gorenstein smoothing of X is diffeomorphic to the rational blow-down 4–manifold Z_{252}, it suffices to show that Z_{252} is simply connected. We decompose the surface Z into $Z = Z_0 \cup C_{252,145}$. Then we have $Z_{252} = Z_0 \cup B_{252,145}$. Furthermore, since $\pi_1(\partial B_{252,145}) \to \pi_1(B_{252,145})$ is surjective, by van Kampen's theorem, it suffices to show that $\pi_1(Z_0) = 1$.

Let α_i be a normal circle of u_i. First, note that Z and the configuration $C_{252,145}$ are all simply connected. Hence, applying van Kampen's theorem on Z, we get

$$1 = \pi_1(Z_0)/\langle N_{i_*(\alpha_1)}\rangle, \tag{2-2}$$

where i_* is the induced homomorphism by the inclusion $i\colon \partial C_{252,145} \to Z_0$.

We write $a \sim b$ if a and b are conjugate to each other in $\pi_1(Z_0)$. From Figure 4, one can easily show that $1 = i_*(\alpha_6) \sim i_*(\alpha_1)^{26}$, ie $i_*(\alpha_1)^{26} = 1$ and $i_*(\alpha_1)^5 \sim i_*(\alpha_3) \sim i_*(\alpha_{12}) \sim i_*(\alpha_1)^{9574}$. Since $9574 \equiv 6 \pmod{26}$, we have $i_*(\alpha_1)^5 \sim i_*(\alpha_1)^6$. Hence $i_*(\alpha_1)^{5\cdot 13} \sim i_*(\alpha_1)^{26} = 1$, which implies that $i_*(\alpha_1)^{5\cdot 13} = 1$. Since $\alpha_1^{5\cdot 13}$ is also a generator of $\pi_1(\partial C_{252,145})$, we have $i_*(\alpha_1) = 1$. Therefore $\pi_1(Z_0) = 1$ by (2-2). □

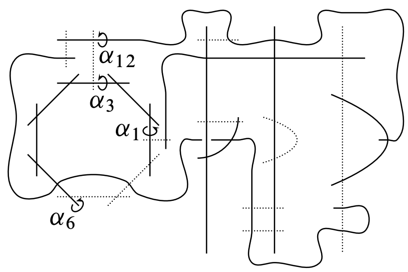


Figure 4: Normal circles

3 More examples

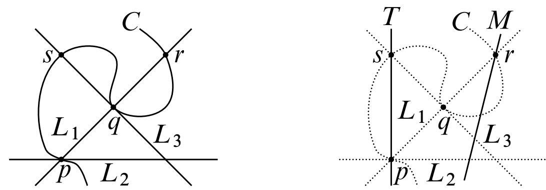
In this section we describe another rational surface Z which makes it possible to get a simply connected surface of general type with $p_g = 0$ and $K^2 = 4$.

Construction

Let C be a smooth cubic curve in $\mathbb{P}^2$ and p its inflection point. Let L_1 be a line passing through p which intersects C at two more different points q and r. Let L_2 be the tangent line to C at p and L_3 the tangent line to C at one of the intersection points of L_1 and C, say q. Let s be the other intersection point of L_3 and C (Figure 5(a)). We consider a pencil of cubics $\{\lambda(L_1+L_2+L_3)+\mu C \mid [\lambda:\mu]\in\mathbb{P}^1\}$ in $\mathbb{P}^2$ generated by two cubic curves $L_1 + L_2 + L_3$ and C. According to Persson [8], if we choose a general C, we may assume that the pencil of cubics contains four nodal singular curves. Let T be a line joining p and s and M a line through r and the node of a nodal singular member of the pencil of cubics. We may assume that M does not pass through the other nodes (Figure 5(b)).

In order to obtain an elliptic fibration over $\mathbb{P}^1$ from the pencil above, we blow up 9 times in total at the base points of the pencil of cubics including infinitely near base-points at each base point. We then get an elliptic fibration $Y = \mathbb{P}^2 \sharp 9\overline{\mathbb{P}}^2$ over $\mathbb{P}^1$ (Figure 6). Note that the proper transform $\widetilde{T}$ of T is a section of Y and the proper transform $\widetilde{M}$ of M is a bisection of Y (Figure 6). Here the section S in Y is an exceptional curve induced by the blow-up at the point s.

We blow up 7 times at the marked points $\bullet$ on Y and blow up two more times at the marked point $\odot$ on Y. We finally obtain a rational surface $Z = Y \sharp 9\overline{\mathbb{P}}^2$ which



(a) Two generators (b) Two lines T and M

Figure 5: A pencil of cubics

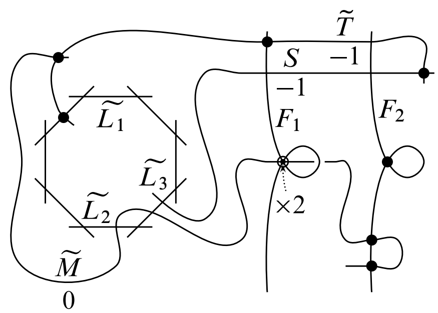


Figure 6: An elliptic fibration Y

contains the following linear chain of $\mathbb{P}^1$'s (Figure 7):

$$C_{183,38} = \underset{u_{13}}{\overset{-5}{\circ}} - \underset{u_{12}}{\overset{-6}{\circ}} - \underset{u_{11}}{\overset{-2}{\circ}} - \underset{u_{10}}{\overset{-6}{\circ}} - \underset{u_{9}}{\overset{-2}{\circ}} - \underset{u_{8}}{\overset{-4}{\circ}} - \underset{u_{7}}{\overset{-2}{\circ}} - \underset{u_{6}}{\overset{-2}{\circ}} - \underset{u_{5}}{\overset{-2}{\circ}} - \underset{u_{4}}{\overset{-3}{\circ}} - \underset{u_{3}}{\overset{-2}{\circ}} - \underset{u_{2}}{\overset{-2}{\circ}} - \underset{u_{1}}{\overset{-2}{\circ}}.$$

Finally, by applying $\mathbb{Q}$–Gorenstein smoothing theory to Z as in Lee and Park [5] and Park–Park–Shin [6], we are able to construct a minimal complex surface with $p_g = 0$ and $K^2 = 4$, say X_t, which is a general fiber of a $\mathbb{Q}$–Gorenstein smoothing of X.

Proposition 3.1 *The complex surface X_t is simply connected.*

Proof Let us decompose the surface $Z = Y \sharp 9\overline{\mathbb{P}}^2$ into $Z = Z_0 \cup C_{183,38}$. Then, as in the proof of Proposition 2.1, it is enough to show that $\pi_1(Z_0) = 1$.

Let E be an exceptional curve intersecting $\widetilde{F_2}$ at two points. The intersection of a boundary of a tubular neighborhood of $\widetilde{F_2}$ and E consists of two normal circles of $\widetilde{F_2}$,

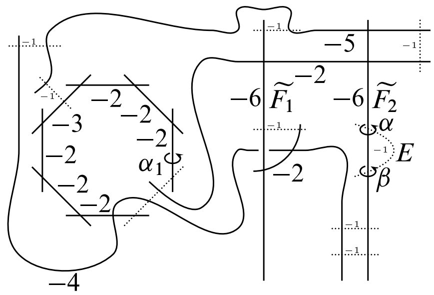


Figure 7: A rational surface $Z = Y \sharp 9\overline{\mathbb{P}}^2$

say α and β, which are contained in Z_0. We choose a point $x_0 \in \alpha$ as a base point for the homotopy group of Z_0. Let $x_1 \in \beta$ be any point.

Since $\widetilde{F_2}$ and E intersect positively at each intersection point, α and β have the same orientation induced by the orientation of the exceptional curve E. Therefore, as circles on the punctured sphere $E \setminus C_{183,38}$, they are the boundaries of the cylinder $E \setminus C_{183,38}$ and, furthermore, they have the opposite orientation in the cylinder $E \setminus C_{183,38}$. Let i_* be the induced homomorphism by the inclusion $i\colon \partial C_{183,38} \to Z_0$. Then we have

$$[i_*(\alpha)] = [\lambda \cdot i_*(\beta)^{-1} \cdot \lambda^{-1}] \quad \text{in } \pi_1(Z_0, x_0), \tag{3-1}$$

where λ is a path connecting x_0 and x_1 which lies on E.

On the other hand, since α and β are normal circles of $\widetilde{F_2}$, we also have

$$[i_*(\alpha)] = [\mu \cdot i_*(\beta) \cdot \mu^{-1}] \quad \text{in } \pi_1(Z_0, x_0) \tag{3-2}$$

where μ is a path connecting x_0 and x_1 which is contained in the boundary of a tubular neighborhood of $\widetilde{F_2}$. Note that we may choose λ and μ so that they are homotopically equivalent. Therefore it follows by (3-1) and (3-2) that

$$[i_*(\alpha)^2] = 1 \quad \text{in } \pi_1(Z_0, x_0). \tag{3-3}$$

It is not difficult to show that $i_*(\alpha)^2$ is conjugate to $i_*(\alpha_1)^{2552}$, where α_1 is a generator of $\pi_1(\partial Z_0 = L(183^2, -6953), x_0) = \mathbb{Z}_{183^2}$. Since $2552 = 8 \cdot 11 \cdot 29$ is relatively prime to $183^2 = (3 \cdot 61)^2$, it implies that α^2 is also a generator of $\pi_1(\partial Z_0)$. By applying van Kampen's theorem on Z, we get

$$1 = \pi_1(Z_0, x_0)/\langle N_{i_*(\alpha)^2} \rangle.$$

Therefore $\pi_1(Z_0, x_0) = 1$ by (3-3). □

Remark (1) One can find more examples of simply connected surfaces of general type with $p_g = 0$ and $K^2 = 4$ using different configurations. For example, using an elliptic fibration on $E(1)$ with one I_7–singular fiber, one I_2–singular fiber and two nodal fibers, we can find the following linear chain of $\mathbb{P}^1$'s in $E(1)\sharp 9\overline{\mathbb{P}}^2$:

$$C_{252,145} = \overset{-2}{\circ} - \overset{-4}{\circ} - \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-4}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}.$$

It is a very intriguing question whether all these configurations above produce the same deformation equivalent type of simply connected surfaces with $p_g = 0$ and $K^2 = 4$. We leave this problem for future research.

(2) It is also a natural question whether one can find an appropriate configuration in a rational surface which produces a surface of general type with $p_g = 0$ and $K^2 \geq 5$. Note that the basic scheme used in this paper as well as in Lee and Park [5] and Park–Park–Shin [6] is the following: We chose a delicate configuration in a certain rational surface Z so that its induced singular surface X obtained by contracting linear chains of curves in Z satisfies the cohomology condition $H^2(T^0_X) = 0$, which guarantees automatically the existence of a $\mathbb{Q}$–Gorenstein smoothing of X. In this respect, it seems impossible to find a configuration satisfying $H^2(T^0_X) = 0$ for $K^2 \geq 5$. But, without the hypothesis $H^2(T^0_X) = 0$, there might still be a chance to find a configuration for $K^2 \geq 5$. Of course, if such a configuration exists, it will be another problem to determine whether the induced singular surface X admits a $\mathbb{Q}$–Gorenstein smoothing or not.

4 Einstein metrics on $\mathbb{CP}^2 \sharp 5\overline{\mathbb{CP}}^2$

In this section we show that the complex surface X_t constructed in the main construction admits a Kähler–Einstein metric of negative scalar curvature, which implies the following theorem.

Theorem 4.1 *The topological 4–manifold $\mathbb{CP}^2 \sharp 5\overline{\mathbb{CP}}^2$ has a smooth structure which admits an Einstein metric with negative scalar curvature.*

Recently Răsdeaconu and Şuvaina [9] proved the existence of a smooth structure on each of the topological 4–manifolds $\mathbb{CP}^2 \sharp k\overline{\mathbb{CP}}^2$, for $k = 6, 7$, which has an Einstein metric of negative scalar curvature. By applying their method on the surface X_t constructed in Section 2, we can easily prove the existence of a Kähler–Einstein metric on X_t with negative scalar curvature. We explain it in a detail in the rest of this section.

First, note that there is a criterion for the existence of a Kähler–Einstein metric on a compact complex 4–manifold with $c_1(M) < 0$, which was found independently by Aubin [1] and Yau [10]:

Theorem 4.2 (Aubin–Yau) *A compact complex 4–manifold (M, J) admits a compatible Kähler–Einstein metric with negative scalar curvature if and only if its canonical line bundle K_M is ample. When such a metric exists, it is unique, up to an overall multiplicative constant.*

Proof of Theorem 4.1 Based on the idea Răsdeaconu and Şuvaina [9], we show that the surface X_t has an ample canonical bundle. Then it follows from Theorem 4.2 of Aubin–Yau that there exists a Kähler–Einstein metric on X_t of negative scalar curvature.

As we showed in the main construction, the pullback f^*K_X of the canonical divisor X onto the rational surface Z is effective and nef; hence K_X is also nef. Let $E_1, \dots, E_8$ be the (-1)-curves on the rational surface Z and set

$$C_{252,145} = \underset{G_{13}}{\overset{-2}{\circ}} - \underset{G_{12}}{\overset{-4}{\circ}} - \underset{G_{11}}{\overset{-6}{\circ}} - \underset{G_{10}}{\overset{-2}{\circ}} - \underset{G_9}{\overset{-6}{\circ}} - \underset{G_8}{\overset{-2}{\circ}} - \underset{G_7}{\overset{-4}{\circ}} - \underset{G_6}{\overset{-2}{\circ}} - \underset{G_5}{\overset{-2}{\circ}} - \underset{G_4}{\overset{-2}{\circ}} - \underset{G_3}{\overset{-3}{\circ}} - \underset{G_2}{\overset{-2}{\circ}} - \underset{G_1}{\overset{-3}{\circ}}.$$

Then one may write

$$f^*K_X \equiv_{\mathbb{Q}} \sum_{i=1}^{8} a_i E_i + \sum_{j=1}^{13} b_j G_j$$

for some rational numbers $a_i, b_j \geq 0$.

We first show that K_X is ample. Suppose on the contrary that K_X is not ample. Since K_X is already nef and $K_X^2 = 4 > 0$, according to the Nakai–Moishezon criterion, there exists an irreducible curve $C \subset X$ such that $(K_X \cdot C) = 0$. Let $\bar{C} \subset Z$ be the proper transform of C. Then we have

$$(K_X \cdot C) = (f^*K_X \cdot f^*C) = (f^*K_X \cdot \bar{C}) = \sum_{i=1}^{8} a_i (E_i \cdot \bar{C}) + \sum_{j=1}^{13} b_j (G_j \cdot \bar{C}) = 0.$$

Since G_j's are irreducible components of the exceptional divisors of f, it is obvious that $(G_j \cdot \bar{C}) \geq 0$ ($j = 1, \dots, 13$) with equality if and only if C does not pass through the singular point of X. Hence it follows that

$$\sum_{i=1}^{8} a_i (E_i \cdot \bar{C}) \leq 0.$$

Then either $(E_{i_0} \cdot \overline{C}) < 0$ for some i_0, or $(E_i \cdot \overline{C}) = 0$ for all $i = 1, \dots, 8$ and $(G_j \cdot \overline{C}) = 0$ for all $j = 1, \dots, 13$. In the first case $\overline{C}$ must coincide with E_{i_0}. However, by using a similar technique to one in Lee and Park [5] and Park–Park–Shin [6], one may show that $(f^* K_X \cdot E_i) > 0$ for all $i = 1, \dots, 8$, which is a contradiction to our assumption $(K_X \cdot \overline{C}) = 0$. Therefore we have $(E_i \cdot \overline{C}) = 0$ for all $i = 1, \dots, 8$ and $(G_j \cdot \overline{C}) = 0$ for all $j = 1, \dots, 13$. On the other hand, note that the Poincaré duals of the irreducible components G_j and of the (-1)-curves E_i generate $H_2(Z, \mathbb{Q})$; hence $\overline{C}$ must be numerically trivial on Z. Then, for any ample divisor H on X, we have

$$0 = (\overline{C} \cdot f^* H) = (f^* C \cdot f^* H) = (C \cdot H),$$

which is again a contradiction. Therefore K_X is ample.

Note that ampleness is an open property; cf Kollár and Mori [4]. So the canonical divisor K_{X_t} of a general fiber X_t of $\mathbb{Q}$–Gorenstein smoothing is automatically ample. Therefore, by Aubin and Yau's criterion, X_t has a Kähler–Einstein metric of negative scalar curvature. □

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