A simply connected surface of general type with \( p_g = 0 \) and \( K^2 = 4 \)

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As a continuation of the recent results of Y Lee and the second author [5] and the authors [6], we construct a simply connected minimal complex surface of general type with \( p_g = 0 \) and \( K^2 = 4 \) by using a rational blow-down surgery and \( \mathbb{Q} \)–Gorenstein smoothing theory.

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1 Introduction

A rational surface satisfies \( p_g = q = 0 \) and it has Kodaira dimension \( \kappa = -\infty \). Around 1894 Castelnuovo conjectured that a surface with \( p_g = q = 0 \) is rational. However the conjecture was soon shown to be false by the examples of Enriques. Castelnuovo also found another counterexample. Enriques’ example has Kodaira dimension 0 while Castelnuovo’s example has Kodaira dimension 1. Hence smooth surfaces of general type (ie Kodaira dimension 2) with \( p_g = q = 0 \) are very interesting from the point of view of the history of surfaces with \( p_g = q = 0 \).

Nowadays a large number of examples of surfaces of general type with \( p_g = q = 0 \) are known due to Godeaux, Campedelli and so on; cf Barth et al [3]. However it was only in 1983 that the first example of a simply connected surface of general type with \( p_g = 0 \) appeared, the so-called Barlow surface [2]. The Barlow surface has \( K^2 = 1 \). The second examples were discovered just recently. Motivated by a result of the second author [7], Y Lee and the second author [5] constructed a family of simply connected minimal complex surfaces of general type with \( p_g = 0 \) and \( K^2 = 1, 2 \) by using rational blow-down surgery and \( \mathbb{Q} \)–Gorenstein smoothing theory. After this construction, the authors [6] constructed a family of simply connected minimal complex surfaces of general type with \( p_g = 0 \) and \( K^2 = 3 \) by similar methods.

In this paper we extend the results of Lee and Park [5] and Park–Park–Shin [6] to the case of \( K^2 = 4 \). That is, we construct a new simply connected minimal surface of
general type with \( p_g = 0 \) and \( K^2 = 4 \) by using a rational blow-down surgery and \( \mathbb{Q} \)-Gorenstein smoothing theory. This is the first example of such complex surfaces.

The key ingredient of this paper is to find an elliptic surface \( Y \) equipped with a special \textit{bisection}, ie an irreducible curve on an elliptic surface whose intersection number with a fiber is 2. Blowing up \( Y \) several times appropriately, we get a rational surface \( Z \) which makes it possible to get such a complex surface. Once we have the right candidate \( Z \) with \( K^2 = 4 \), the remaining argument is similar to that of the \( K^2 = 1, 2, 3 \) cases appearing in Lee and Park \cite{5} and Park–Park–Shin \cite{6}. That is, by applying a rational blow-down surgery and \( \mathbb{Q} \)-Gorenstein smoothing theory developed in Lee and Park \cite{5} to \( Z \), we obtain a minimal complex surface of general type with \( p_g = 0 \) and \( K^2 = 4 \). Then we show that the surface is simply connected. Since almost all the proofs are parallel to the case of the main construction in Park–Park–Shin \cite[Section 3]{6}, we only explain how to construct such a minimal complex surface and we prove that the surface is simply connected. The main result of this paper is the following theorem.

**Theorem 1.1** There exists a simply connected minimal complex surface of general type with \( p_g = 0 \) and \( K^2 = 4 \).

**Remark** Răsdeaconu and Șuvaina \cite{9} proved that the complex surfaces constructed in Lee and Park \cite{5} and Park–Park–Shin \cite{6} admit Kähler–Einstein metrics of negative scalar curvature. By applying their method to the complex surface constructed in this paper, one may prove that it also admits a Kähler–Einstein metric of negative scalar curvature; see Section 4.

## 2 Main construction

We start with a special elliptic fibration \( Y := \mathbb{P}^2 \# 9 \mathbb{P}^2 \) which is used in the main construction of this paper. Let \( L_1, L_2, L_3 \) and \( A \) be lines in \( \mathbb{P}^2 \) and let \( B \) be a smooth conic in \( \mathbb{P}^2 \) intersecting as in Figure 1(a). We consider a pencil of cubics \( \{ \lambda(L_1 + L_2 + L_3) + \mu(A + B) \mid [\lambda : \mu] \in \mathbb{P}^1 \} \) in \( \mathbb{P}^2 \) generated by two cubic curves \( L_1 + L_2 + L_3 \) and \( A + B \), which has 4 base points, say, \( p, q, r \) and \( s \). In order to obtain an elliptic fibration over \( \mathbb{P}^1 \) from the pencil, we blow up three times at \( p \) and \( r \), respectively, and twice at \( s \), including infinitely near base-points at each point. We perform one further blow-up at the base point \( q \). By blowing up nine times in total, we resolve all base points (including infinitely near base-points) of the pencil and we then get an elliptic fibration \( Y = \mathbb{P}^2 \# 9 \mathbb{P}^2 \) over \( \mathbb{P}^1 \) (Figure 2).

There are four sections of the elliptic fibration \( Y \) corresponding to the four base points \( p, q, r \) and \( s \). Among these sections we use only two sections corresponding to \( p \)
and $q$, say $S_1$ and $S_2$ respectively, for the main construction. Furthermore, the elliptic fibration $Y$ has an $I_8$–singular fiber consisting of the proper transforms $\tilde{L}_i$ of $L_i$ ($i = 1, 2, 3$). Also $Y$ has an $I_2$–singular fiber consisting of the proper transforms $\tilde{A}$ and $\tilde{B}$ of $A$ and $B$, respectively. According to the list of Persson [8], we may assume that $Y$ has only two more nodal singular fibers $F_1$ and $F_2$ by choosing generally the $L_i$’s, $A$ and $B$ (Figure 2). For example the pencil used in Park–Park–Shin [6] works:

\[(2-1) \{ \lambda(y - \sqrt{3}x)(y + \sqrt{3}x)(2y - 3z) + \mu x (x^2 + (y - 2z)^2 - z^2) \mid [\lambda : \mu] \in \mathbb{P}^1 \} .\]

This pencil has singular fibers at $[\lambda : \mu] = [1 : 0], [0 : 1], [2 : 3\sqrt{3}]$ and $[2 : -3\sqrt{3}]$. Furthermore, setting

\[F_1 = \{ 2(y - \sqrt{3}x)(y + \sqrt{3}x)(2y - 3z) + 3\sqrt{3}x(x^2 + (y - 2z)^2 - z^2) = 0 \},\]

\[F_2 = \{ 2(y - \sqrt{3}x)(y + \sqrt{3}x)(2y - 3z) - 3\sqrt{3}x(x^2 + (y - 2z)^2 - z^2) = 0 \},\]

one can easily check that $F_1$ and $F_2$ are nodal cubic curves with one node at $[\sqrt{3} : 0 : -1]$ and $[\sqrt{3} : 0 : 1]$, respectively.

Let $M$ be the line in $\mathbb{P}^2$ passing through the point $q$ and the node of the nodal cubic curve $F_1$. The node of $F_1$ does not lie on any $L_i$’s, $A$ or $B$. Hence it satisfies that $M \neq L_1, M \neq A$ and $\tilde{M} \cdot \tilde{M} = 0$, where $\tilde{M}$ is the proper transform of $M$ in $Y$ (Figure 1(b)). We may assume further that $M$ does not pass through the node of the other nodal cubic curve $F_2$ by choosing generally the $L_i$’s, $A$ and $B$. For example, the pencil in (2-1) works: We have $q = [0 : 3 : 2]$. Hence the line $M$ passing through $q$ and the node of $F_1$ is $\{s[0 : 3 : 2] + t[\sqrt{3} : 0 : -1] \mid [s : t] \in \mathbb{P}^1 \}$. It is obvious that the node $[\sqrt{3} : 0 : 1]$ does not lie on the line $M$. Since $M$ meets every member in the pencil at three points, $\tilde{M}$ is a bisection of the elliptic fibration $Y \to \mathbb{P}^1$. Furthermore, since $q \in M$, the section $S_2$ meets $\tilde{M}$ at one point (Figure 2).
Next, by blowing up nine times on $Y$, we construct a rational surface $Z$ which contains a special configuration of linear chains of $\mathbb{P}^1$'s. At first we blow up twice at the marked point $\odot$ on $F_1$. We then blow up seven times in total at the six marked points $\bullet$ on each fiber and at the intersection point $\bullet$ of $\tilde{M}$ and $S_2$. We then get a rational surface $Z = Y \# 9\mathbb{P}^2$. We also denote by $\tilde{F}_i$ ($i = 1, 2$) the proper transforms of $F_i$. Then there exists a linear chain of $\mathbb{P}^1$'s in $Z$,

$$C_{252,145} = \frac{-2}{u_{13}} - \frac{-4}{u_{12}} - \frac{-6}{u_{11}} - \frac{-2}{u_{10}} - \frac{-6}{u_{9}} - \frac{-2}{u_{8}} - \frac{-4}{u_{7}} - \frac{-2}{u_{6}} - \frac{-2}{u_{5}} - \frac{-3}{u_{4}} - \frac{-2}{u_{3}} - \frac{-3}{u_{2}} - \frac{-3}{u_{1}},$$

which contains $\tilde{A}$, $S_2$, $\tilde{F}_2$, $S_1$, $\tilde{F}_1$, $\tilde{M}$, $\tilde{L}_2$, $\tilde{L}_1$ and $\tilde{L}_3$, where $u_i$ represents an

embedded rational curve (Figure 3).
Finally, by applying \( \mathbb{Q} \)-Gorenstein smoothing theory to \( Z \) as in Lee and Park [5] and Park–Park–Shin [6], we construct a minimal complex surface with \( p_g = 0 \) and \( K^2 = 4 \). That is, we first contract the chain \( C_{252,145} \) of \( \mathbb{P}^1 \)'s from \( Z \) so that it produces a normal projective surface \( X \) with one permissible singular point. It then follows by a similar technique to one in Lee and Park [5] and Park–Park–Shin [6] that \( X \) has a \( \mathbb{Q} \)-Gorenstein smoothing. Let \( X_t \) be a general fiber of the \( \mathbb{Q} \)-Gorenstein smoothing of \( X \). Since \( X \) is a (singular) surface with \( p_g = 0 \) and \( K^2 = 4 \), by applying general results of complex surface theory and \( \mathbb{Q} \)-Gorenstein smoothing theory, one may conclude that a general fiber \( X_t \) is a complex surface of general type with \( p_g = 0 \) and \( K^2 = 4 \).

The minimality of \( X_t \) follows from the nefness of the canonical divisor \( K_X \) of \( X \). Let \( f: Z \to X \) be the contraction of the chain \( C_{252,145} \) of \( \mathbb{P}^1 \)'s from \( Z \) to the singular surface \( X \). By using a similar technique to one in Lee and Park [5] and Park–Park–Shin [6], it follows that the pullback \( f^* K_X \) of the canonical divisor \( K_X \) of \( X \) is effective and nef, hence \( K_X \) is also nef, which shows the minimality of \( X_t \).

It remains to prove that \( X_t \) is simply connected.

**Proposition 2.1** \( X_t \) is simply connected.

**Proof** Let \( Z_{252} \) be a rational blow-down 4–manifold obtained from \( Z \) by replacing the configuration \( C_{252,145} \) with the corresponding rational ball \( B_{252,145} \). Since a general fiber \( X_t \) of a \( \mathbb{Q} \)-Gorenstein smoothing of \( X \) is diffeomorphic to the rational blow-down 4–manifold \( Z_{252} \), it suffices to show that \( Z_{252} \) is simply connected. We decompose the surface \( Z \) into \( Z = Z_0 \cup C_{252,145} \). Then we have \( Z_{252} = Z_0 \cup B_{252,145} \). Furthermore, since \( \pi_1(\partial B_{252,145}) \to \pi_1(B_{252,145}) \) is surjective, by van Kampen’s theorem, it suffices to show that \( \pi_1(Z_0) = 1 \).

Let \( \alpha_t \) be a normal circle of \( u_t \). First, note that \( Z \) and the configuration \( C_{252,145} \) are all simply connected. Hence, applying van Kampen’s theorem on \( Z \), we get

\[
1 = \pi_1(Z_0)/\langle N_{i_*}(\alpha_t) \rangle,
\]

where \( i_* \) is the induced homomorphism by the inclusion \( i: \partial C_{252,145} \to Z_0 \).

We write \( a \sim b \) if \( a \) and \( b \) are conjugate to each other in \( \pi_1(Z_0) \). From Figure 4, one can easily show that \( 1 = i_*(\alpha_6) \sim i_*(\alpha_1)^{26} \), \( i_*(\alpha_1)^{26} = 1 \) and \( i_*(\alpha_1)^5 \sim i_*(\alpha_3) \sim i_*(\alpha_2) \sim i_*(\alpha_1)^{9574} \). Since \( 9574 \equiv 6 \pmod{26} \), we have \( i_*(\alpha_1)^5 \sim i_*(\alpha_1)^6 \). Hence \( i_*(\alpha_1)^{5:13} \sim i_*(\alpha_1)^{26} = 1 \), which implies that \( i_*(\alpha_1)^{5:13} = 1 \). Since \( \alpha_1^{5:13} \) is also a generator of \( \pi_1(\partial C_{252,145}) \), we have \( i_*(\alpha_1) = 1 \). Therefore \( \pi_1(Z_0) = 1 \) by (2-2).
3 More examples

In this section we describe another rational surface $Z$ which makes it possible to get a simply connected surface of general type with $p_g = 0$ and $K^2 = 4$.

Construction

Let $C$ be a smooth cubic curve in $\mathbb{P}^2$ and $p$ its inflection point. Let $L_1$ be a line passing through $p$ which intersects $C$ at two more different points $q$ and $r$. Let $L_2$ be the tangent line to $C$ at $p$ and $L_3$ the tangent line to $C$ at one of the intersection points of $L_1$ and $C$, say $q$. Let $s$ be the other intersection point of $L_3$ and $C$ (Figure 5(a)). We consider a pencil of cubics $\lambda(L_1 + L_2 + L_3) + \mu C \mid [\lambda : \mu] \in \mathbb{P}^1$ in $\mathbb{P}^2$ generated by two cubic curves $L_1 + L_2 + L_3$ and $C$. According to Persson [8], if we choose a general $C$, we may assume that the pencil of cubics contains four nodal singular curves. Let $T$ be a line joining $p$ and $s$ and $M$ a line through $r$ and the node of a nodal singular member of the pencil of cubics. We may assume that $M$ does not pass through the other nodes (Figure 5(b)).

In order to obtain an elliptic fibration over $\mathbb{P}^1$ from the pencil above, we blow up 9 times in total at the base points of the pencil of cubics including infinitely near base-points at each base point. We then get an elliptic fibration $Y = \mathbb{P}^2 \# 9\mathbb{P}^2$ over $\mathbb{P}^1$ (Figure 6). Note that the proper transform $\tilde{T}$ of $T$ is a section of $Y$ and the proper transform $\tilde{M}$ of $M$ is a bisection of $Y$ (Figure 6). Here the section $S$ in $Y$ is an exceptional curve induced by the blow-up at the point $s$.

We blow up 7 times at the marked points $\bullet$ on $Y$ and blow up two more times at the marked point $\circ$ on $Y$. We finally obtain a rational surface $Z = Y \# 9\mathbb{P}^2$ which
A simply connected surface of general type with $p_g = 0$ and $K^2 = 4$ contains the following linear chain of $\mathbb{P}^1$'s (Figure 7):

$$C_{183,38} = \frac{-5}{u_1} \frac{-6}{u_2} \frac{-2}{u_3} \frac{-6}{u_4} \frac{-2}{u_5} \frac{-4}{u_6} \frac{-2}{u_7} \frac{-2}{u_8} \frac{-3}{u_9} \frac{-2}{u_{10}} \frac{-2}{u_{11}} \frac{-2}{u_{12}} \frac{-2}{u_{13}} \frac{-2}{u_{14}} \frac{-2}{u_{15}} \frac{-2}{u_{16}} \frac{-2}{u_{17}}.$$

Finally, by applying $\mathbb{Q}$–Gorenstein smoothing theory to $Z$ as in Lee and Park [5] and Park–Park–Shin [6], we are able to construct a minimal complex surface with $p_g = 0$ and $K^2 = 4$, say $X_t$, which is a general fiber of a $\mathbb{Q}$–Gorenstein smoothing of $X$.

**Proposition 3.1** The complex surface $X_t$ is simply connected.

**Proof** Let us decompose the surface $Z = Y \# 9 \mathbb{P}^2$ into $Z = Z_0 \cup C_{183,38}$. Then, as in the proof of Proposition 2.1, it is enough to show that $\pi_1(Z_0) = 1$.

Let $E$ be an exceptional curve intersecting $\tilde{F}_2$ at two points. The intersection of a boundary of a tubular neighborhood of $\tilde{F}_2$ and $E$ consists of two normal circles of $\tilde{F}_2,$
say $\alpha$ and $\beta$, which are contained in $Z_0$. We choose a point $x_0 \in \alpha$ as a base point for the homotopy group of $Z_0$. Let $x_1 \in \beta$ be any point.

Since $\tilde{F}_2$ and $E$ intersect positively at each intersection point, $\alpha$ and $\beta$ have the same orientation induced by the orientation of the exceptional curve $E$. Therefore, as circles on the punctured sphere $E \setminus C_{183,38}$, they are the boundaries of the cylinder $E \setminus C_{183,38}$ and, furthermore, they have the opposite orientation in the cylinder $E \setminus C_{183,38}$. Let $i_*$ be the induced homomorphism by the inclusion $i: \partial C_{183,38} \to Z_0$. Then we have

\[(3-1) \quad [i_*(\alpha)] = [\lambda \cdot i_*(\beta)^{-1} \cdot \lambda^{-1}] \quad \text{in} \quad \pi_1(Z_0,x_0),\]

where $\lambda$ is a path connecting $x_0$ and $x_1$ which lies on $E$.

On the other hand, since $\alpha$ and $\beta$ are normal circles of $\tilde{F}_2$, we also have

\[(3-2) \quad [i_*(\alpha)] = [\mu \cdot i_*(\beta) \cdot \mu^{-1}] \quad \text{in} \quad \pi_1(Z_0,x_0)\]

where $\mu$ is a path connecting $x_0$ and $x_1$ which is contained in the boundary of a tubular neighborhood of $\tilde{F}_2$. Note that we may choose $\lambda$ and $\mu$ so that they are homotopically equivalent. Therefore it follows by (3-1) and (3-2) that

\[(3-3) \quad [i_*(\alpha)^2] = 1 \quad \text{in} \quad \pi_1(Z_0,x_0).\]

It is not difficult to show that $i_*(\alpha)^2$ is conjugate to $i_*(\alpha)^{2552}$, where $\alpha_1$ is a generator of $\pi_1(\partial Z_0 = L(183^2,-6953),x_0) = \mathbb{Z}_{1832}$. Since $2552 = 8 \cdot 11 \cdot 29$ is relatively prime to $183^2 = (3 \cdot 61)^2$, it implies that $\alpha^2$ is also a generator of $\pi_1(\partial Z_0)$. By applying van Kampen’s theorem on $Z$, we get

\[1 = \pi_1(Z_0,x_0)/\langle N_{i_*(\alpha)^2} \rangle.\]

Therefore $\pi_1(Z_0,x_0) = 1$ by (3-3). \qed
Remark  (1) One can find more examples of simply connected surfaces of general type with \(p_g = 0\) and \(K^2 = 4\) using different configurations. For example, using an elliptic fibration on \(E(1)\) with one \(I_7\)-singular fiber, one \(I_2\)-singular fiber and two nodal fibers, we can find the following linear chain of \(\mathbb{P}^1\)'s in \(E(1)\# 9\mathbb{P}^2\):

\[ C_{252,145} = \frac{-2}{0} - \frac{-4}{-6} - \frac{-6}{0} - \frac{-6}{-2} - \frac{-2}{-4} - \frac{-4}{-2} - \frac{-2}{-3} - \frac{-3}{-2} - \frac{-2}{0}. \]

It is a very intriguing question whether all these configurations above produce the same deformation equivalent type of simply connected surfaces with \(p_g = 0\) and \(K^2 = 4\). We leave this problem for future research.

(2) It is also a natural question whether one can find an appropriate configuration in a rational surface which produces a surface of general type with \(p_g = 0\) and \(K^2 \geq 5\). Note that the basic scheme used in this paper as well as in Lee and Park [5] and Park–Park–Shin [6] is the following: We chose a delicate configuration in a certain rational surface \(Z\) so that its induced singular surface \(X\) obtained by contracting linear chains of curves in \(Z\) satisfies the cohomology condition \(H^2(T_X^0) = 0\), which guarantees automatically the existence of a \(\mathbb{Q}\)-Gorenstein smoothing of \(X\). In this respect, it seems impossible to find a configuration satisfying \(H^2(T_X^0) = 0\) for \(K^2 \geq 5\). But, without the hypothesis \(H^2(T_X^0) = 0\), there might still be a chance to find a configuration for \(K^2 \geq 5\). Of course, if such a configuration exists, it will be another problem to determine whether the induced singular surface \(X\) admits a \(\mathbb{Q}\)-Gorenstein smoothing or not.

4 Einstein metrics on \(\mathbb{C}\mathbb{P}^2 \# 5\mathbb{C}\mathbb{P}^2\)

In this section we show that the complex surface \(X_t\) constructed in the main construction admits a Kähler–Einstein metric of negative scalar curvature, which implies the following theorem.

**Theorem 4.1** The topological 4–manifold \(\mathbb{C}\mathbb{P}^2 \# 5\mathbb{C}\mathbb{P}^2\) has a smooth structure which admits an Einstein metric with negative scalar curvature.

Recently Răsdeaconu and Șuvaina [9] proved the existence of a smooth structure on each of the topological 4–manifolds \(\mathbb{C}\mathbb{P}^2 \# k\mathbb{C}\mathbb{P}^2\), for \(k = 6, 7\), which has an Einstein metric of negative scalar curvature. By applying their method on the surface \(X_t\) constructed in Section 2, we can easily prove the existence of a Kähler–Einstein metric on \(X_t\) with negative scalar curvature. We explain it in detail in the rest of this section.
First, note that there is a criterion for the existence of a Kähler–Einstein metric on a compact complex 4–manifold with \( c_1(M) < 0 \), which was found independently by Aubin [1] and Yau [10]:

**Theorem 4.2** (Aubin–Yau) A compact complex 4–manifold \( (M, J) \) admits a compatible Kähler–Einstein metric with negative scalar curvature if and only if its canonical line bundle \( K_M \) is ample. When such a metric exists, it is unique, up to an overall multiplicative constant.

**Proof of Theorem 4.1** Based on the idea Rașdeaconu and Şuvaina [9], we show that the surface \( X_t \) has an ample canonical bundle. Then it follows from Theorem 4.2 of Aubin–Yau that there exists a Kähler–Einstein metric on \( X_t \) of negative scalar curvature.

As we showed in the main construction, the pullback \( f^* K_X \) of the canonical divisor \( X \) onto the rational surface \( Z \) is effective and nef; hence \( K_X \) is also nef. Let \( E_1, \ldots, E_8 \) be the \((-1)\)-curves on the rational surface \( Z \) and set

\[
C_{252,145} = -\frac{2}{G_{13}} - \frac{4}{G_{12}} - \frac{6}{G_{11}} - \frac{2}{G_{10}} - \frac{6}{G_9} - \frac{2}{G_8} - \frac{4}{G_7} - \frac{2}{G_6} - \frac{2}{G_5} - \frac{3}{G_4} - \frac{2}{G_3} - \frac{3}{G_2} - \frac{3}{G_1}.
\]

Then one may write

\[
f^* K_X = \sum_{i=1}^{8} a_i E_i + \sum_{j=1}^{13} b_j G_j
\]

for some rational numbers \( a_i, b_j \geq 0 \).

We first show that \( K_X \) is ample. Suppose on the contrary that \( K_X \) is not ample. Since \( K_X \) is already nef and \( K_X^2 = 4 > 0 \), according to the Nakai–Moishezon criterion, there exists an irreducible curve \( C \subset X \) such that \( (K_X \cdot C) = 0 \). Let \( \bar{C} \subset Z \) be the proper transform of \( C \). Then we have

\[
(K_X \cdot C) = (f^* K_X \cdot f^* C) = (f^* K_X \cdot \bar{C}) = \sum_{i=1}^{8} a_i (E_i \cdot \bar{C}) + \sum_{j=1}^{13} b_j (G_j \cdot \bar{C}) = 0.
\]

Since \( G_j \)'s are irreducible components of the exceptional divisors of \( f \), it is obvious that \( (G_j \cdot \bar{C}) \geq 0 \) for \( j = 1, \ldots, 13 \) with equality if and only if \( C \) does not pass through the singular point of \( X \). Hence it follows that

\[
\sum_{i=1}^{8} a_i (E_i \cdot \bar{C}) \leq 0.
\]
A simply connected surface of general type with $p_g = 0$ and $K^2 = 4$

Then either $(E_{i_0} \cdot \widetilde{C}) < 0$ for some $i_0$, or $(E_i \cdot \widetilde{C}) = 0$ for all $i = 1, \ldots, 8$ and $(G_j \cdot \widetilde{C}) = 0$ for all $j = 1, \ldots, 13$. In the first case $\widetilde{C}$ must coincide with $E_{i_0}$. However, by using a similar technique to one in Lee and Park [5] and Park–Park–Shin [6], one may show that $(f^* K_X \cdot E_i) > 0$ for all $i = 1, \ldots, 8$, which is a contradiction to our assumption $(K_X \cdot \widetilde{C}) = 0$. Therefore we have $(E_i \cdot \widetilde{C}) = 0$ for all $i = 1, \ldots, 8$ and $(G_j \cdot \widetilde{C}) = 0$ for all $j = 1, \ldots, 13$. On the other hand, note that the Poincaré duals of the irreducible components $G_j$ and of the $(-1)$-curves $E_i$ generate $H_2(Z, \mathbb{Q})$; hence $\widetilde{C}$ must be numerically trivial on $Z$. Then, for any ample divisor $H$ on $X$, we have

$$0 = (\widetilde{C} \cdot f^* H) = (f^* C \cdot f^* H) = (C \cdot H),$$

which is again a contradiction. Therefore $K_X$ is ample.

Note that ampleness is an open property; cf Kollár and Mori [4]. So the canonical divisor $K_{X_t}$ of a general fiber $X_t$ of $\mathbb{Q}$–Gorenstein smoothing is automatically ample. Therefore, by Aubin and Yau’s criterion, $X_t$ has a Kähler–Einstein metric of negative scalar curvature.

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**References**


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[9] R Răsdeaconu, I Șuvaina, Smooth structures and Einstein metrics on $\mathbb{CP}^2 \# 5, 6, 7 \mathbb{CP}^2$ arXiv:0806.1424


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