Dehn twists have roots

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We show that every Dehn twist in the mapping class group of a closed, connected, orientable surface of genus at least two has a nontrivial root.

Let $S_g$ denote a closed, connected, orientable surface of genus $g$. We denote by $\text{Mod}(S_g)$ its mapping class group: the group of homotopy classes of orientation preserving homeomorphisms of $S_g$. In this note, we demonstrate:

**Fact** If $g \geq 2$, then every Dehn twist in $\text{Mod}(S_g)$ has a nontrivial root.

It follows from the classification of elements in $\text{Mod}(S_1) \cong \text{SL}(2, \mathbb{Z})$ that Dehn twists are primitive in the mapping class group of the torus.

For Dehn twists about separating curves, the fact is well-known: if $c$ is a separating curve then a square root of the left Dehn twist $T_c$ is obtained by twisting one side of $c$ through an angle of $\pi$. We construct roots of Dehn twists about nonseparating curves in two ways.

**Geometric construction** Fix $g \geq 2$. Let $P$ be a regular $(4g-2)$-gon. Glue opposite sides to obtain a surface $Q \cong S_{g-1}$. The rotation of $P$ about its center through angle $2\pi k/(2g-1)$ induces a periodic map $f_k$ of $Q$. Notice that $f_k$ fixes the points $x, y \in Q$ that are the images of the vertices of $P$ and induces a rotation through angle $-4\pi k/(2g-1)$ about each. Let $R$ be the surface obtained from $Q$ by removing small open disks centered at $x$ and $y$. Define $f = f_{g^2}^{-1}|R$.

Modify $f$ by an isotopy supported in a collar of $\partial R$ so that $f|\partial R$ is the identity and $f$ restricts to a $(g-1)/(2g-1)$–right Dehn twist in each annulus. Identify the components of $\partial R$ to obtain a surface $S \cong S_g$. The image of $\partial R$ in $S$ is a nonseparating curve; call it $d$. We see that $(f T_d)^{2g-1} = T_d$, as desired.
Algebraic construction  Let $c_1, \ldots, c_k$ be curves in $S_g$ where $c_i$ intersects $c_{i+1}$ once for each $i$, and all other pairs of curves are disjoint. If $k$ is odd, then a regular neighborhood of $\bigcup c_i$ has two boundary components, say $d_1$ and $d_2$, and we have a relation in $\text{Mod}(S_g)$:

$$T_{c_1}^2 T_{c_2} \cdots T_{c_k} = T_{d_1} T_{d_2}.$$ 

This relation comes from the Artin group of type $B_n$, in particular, the factorization of the central element in terms of standard generators. The relation also follows from the $D_{2g}$ case of [2, Proposition 2.12(i)]. If $k = 2g - 1$ the curves $d_1$ and $d_2$ are isotopic nonseparating curves; call this isotopy class $d$. Using the fact that $T_d$ commutes with each $T_{c_i}$, we see that

$$[(T_{c_1}^2 T_{c_2} \cdots T_{c_{2g-1}})^{1-g} T_d]^{2g-1} = T_d.$$ 

In the remainder of the paper, we find roots for several analogues of Dehn twists.

Roots of half-twists  We denote by $S_{0,n}$ a two-sphere with $n$ punctures (or cone points). Let $d$ be a curve in $S_{0,2g+2}$ with two punctures on one side and $2g$ on the other. On the side of $d$ with two punctures we perform a left half-twist. On the other side of $d$ with two punctures perform a right half-twist by arranging the punctures so that one puncture is in the middle and the other punctures rotate around this central puncture. The $(2g-1)$–st power of the composition is a left half-twist about $d$. Thus, we have roots of half-twists in $\text{Mod}(S_{0,2g+2})$ for $g \geq 2$. Forgetting the central puncture gives roots of half-twists in $\text{Mod}(S_{0,2g+1})$.

In the geometric construction, reflection through the center of the polygon $P$ induces a hyperelliptic involution of the surface $S$. In the algebraic construction there is a hyperelliptic involution preserving each curve $c_i$. In either case there is an induced orbifold double covering $S_g \to S_{0,2g+2}$ and the root of the Dehn twist descends to the given root of the half-twist in $\text{Mod}(S_{0,2g+2})$ [1, Theorem 1 plus Corollary 7.1].

Roots of elementary matrices  If we consider the map $\text{Mod}(S_g) \to \text{Sp}(2g, \mathbb{Z})$ given by the action of $\text{Mod}(S_g)$ on $H_1(S_g, \mathbb{Z})$ we also see that elementary matrices have roots in $\text{Sp}(2g, \mathbb{Z})$:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

By stabilizing we obtain cube roots of elementary matrices in $\text{Sp}(2g, \mathbb{Z})$ for $g \geq 2$. 

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Roots of Nielsen transformations Let $F_n$ denote the free group generated by elements $x_1, \ldots, x_n$. Let $\text{Aut}(F_n)$ denote the group of automorphisms of $F_n$, and assume $n \geq 2$. A Nielsen transformation is an element of $\text{Aut}(F_n)$ conjugate to the one given by $x_1 \mapsto x_1 x_2$ and $x_k \mapsto x_k$ for $2 \leq k \leq n$. The following automorphism is the square root of a Nielsen transformation in $\text{Aut}(F_n)$ for $n \geq 3$:

$$
\begin{align*}
    x_1 & \mapsto x_1 x_3 \\
    x_2 & \mapsto x_3^{-1} x_2 x_3 \\
    x_3 & \mapsto x_3^{-1} x_2
\end{align*}
$$

Passing to quotients, this gives a square root of a Nielsen transformation in $\text{Out}(F_n)$ and, multiplying by $-\text{Id}$, a square root of an elementary matrix in $\text{SL}(n, \mathbb{Z})$, $n \geq 3$. Also, our roots of Dehn twists in $\text{Mod}(S_g)$ can be modified to work for once-punctured surfaces, thus giving “geometric” roots of Nielsen transformations in $\text{Out}(F_n)$.

Other roots If $f \in \text{Mod}(S_g)$ is a root of a Dehn twist $T_d$, then $f$ commutes with $T_d$. Since $f T_c f^{-1} = T_{f(c)}$ for any curve $c$, we see that $f$ fixes $d$. In the complement of $d$, the class $f$ must be periodic. This line of reasoning translates to $\text{GL}(n, \mathbb{Z})$ and $\text{Aut}(F_n)$: roots correspond to torsion elements in $\text{GL}(n-1, \mathbb{Z})$ and $\text{Aut}(F_{n-1})$, respectively. In all cases, one can show that the degree of the root is equal to the order of the torsion element.

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References